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Some of the simplest non-linear problems of the consolidation of a water-saturated earth medium[1]

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Let us consider to begin with the unidimensional problem of the consolidation of a two–phase soil medium. Assuming the material of the soil framework and the liquid (water) filling the pores to be incompressible we have equations of continuity for each of the phases:

$$\frac{\partial m}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\frac{\partial n}{\partial t} + \frac{\partial v}{\partial x} = 0, \tag{2}$$

where m and n are the contents of the liquid and of the solid phase in the unit of volume of the soil medium and u and v are the velocities of filtration of the respective phases.

In consequence of the mobility of the soil framework let us assume the Darcy-Gersanov equation in the form:

$$u - \varepsilon v = -k\frac{\partial H}{\partial x}\,, \tag{3}$$

where ε is the coefficient of porosity, k is the filtration coefficient of the liquid, and H is the hydrodynamic pressure.

Taking into account the known relationships:

$$m = \frac{\varepsilon}{1+\varepsilon} \quad \text{and} \quad n = \frac{1}{1+\varepsilon}\,,$$

we obtain from equations (1) and (2):

$$\frac{\partial}{\partial x}(u+v) = 0,$$

from which we have

$$u = u_0 - v, \tag{4}$$

where u_0 is some velocity of filtration of the liquid which is constant for all points of the medium, and is determined from the conditions of each particular problem.

[1]reported at the seminar of Mechanics Institute of AN SSSR in May of 1948

Then equation (3) may be written in the form:

$$-u_0 + (1+\varepsilon)v = k\tfrac{\partial H}{\partial x}. \tag{5}$$

Differentiating, we find

$$v\tfrac{\partial \varepsilon}{\partial x} + (1+\varepsilon)\tfrac{\partial v}{\partial x} = k\tfrac{\partial^2 H}{\partial x^2} + \tfrac{\partial k}{\partial x}\tfrac{\partial H}{\partial x}. \tag{6}$$

Let σ and p be the stresses in the soil framework and the pressures of liquid filling the pores at any given moment of time t, at a point determined by the coordinate x. Let the external loading of the boundary surface be q and the external pressure of liquid on that surface be w; let the density of the liquid be γ that of the material of the soil framework be γ_c. The thickness of the consolidated layer of soil let us call h. The axis x is vertically upwards, and coincident — for definition of the origin of the co-ordinates — with the lower boundary surface.

From the conditions of equilibrium of a prism bounded by the upper limiting (boundary) surface, with two vertical planes and at any given distance x from the origin of the co-ordinates of the lower surface, we have:

$$q + w + \gamma_c(h-x) + \tfrac{\varepsilon}{1+\varepsilon}(h-x)\gamma - \sigma - p = 0,$$

or, if the weight of the soil framework suspended in the liquid be γ_B, we obtain:

$$q + w + \gamma_B(h-x) + \gamma(h-x) - \sigma - p = 0. \tag{7}$$

From this, taking into account the known relationship

$$H = \tfrac{p}{\gamma} + x,$$

we have

$$\tfrac{\partial \sigma}{\partial x} = -\left(\tfrac{\partial p}{\partial x} + \gamma + \gamma_B\right) = -\left(\gamma\tfrac{\partial H}{\partial x} + \gamma_B\right),$$

$$\tfrac{\partial \sigma}{\partial t} = -\gamma\tfrac{\partial H}{\partial t} + \tfrac{dq}{dt} + \tfrac{dw}{dt}. \tag{8}$$

Since $\varepsilon = \varepsilon(\sigma)$ and $k = k(\varepsilon)$ we obtain:

$$\tfrac{\partial \varepsilon}{\partial x} = \tfrac{d\varepsilon}{d\sigma}\tfrac{\partial \sigma}{\partial x} = -\tfrac{d\varepsilon}{d\sigma}\left(\gamma\tfrac{\partial H}{\partial x} + \gamma_B\right), \tag{9}$$

$$\tfrac{\partial k}{\partial x} = \tfrac{dk}{d\varepsilon}\tfrac{d\varepsilon}{d\sigma}\tfrac{\partial \sigma}{\partial x} = -\tfrac{dk}{d\varepsilon}\tfrac{d\varepsilon}{d\sigma}\left(\gamma\tfrac{\partial H}{\partial x} + \gamma_B\right). \tag{10}$$

Using equation (2) we have

$$\tfrac{\partial v}{\partial x} = -\tfrac{\partial n}{\partial t} = \tfrac{1}{(1+\varepsilon)^2}\tfrac{\partial \varepsilon}{\partial t} = \tfrac{1}{(1+\varepsilon)^2}\tfrac{d\varepsilon}{d\sigma}\left(-\gamma\tfrac{\partial H}{\partial t} + \tfrac{dq}{dt} + \tfrac{dw}{dt}\right). \tag{11}$$

Then, taking into account equations (5), (8), (9), (10) and (11), we can write equation (6) in the following form:

$$\frac{\partial H}{\partial t} - \frac{1}{\gamma}\frac{d}{dt}(q+w) + \left(\frac{\partial H}{\partial x}\right)^2 \left[(k - (1+\varepsilon)\frac{dk}{d\varepsilon})\right] +$$

$$+\frac{\partial H}{\partial x}\frac{\gamma_B}{\gamma}\left[k + u_0\frac{\gamma}{\gamma_B} - (1+\varepsilon)\frac{dk}{d\varepsilon}\right] + \frac{k(1+\varepsilon)}{\gamma\frac{d\varepsilon}{d\sigma}}\frac{\partial^2 H}{\partial x^2} + \frac{\gamma_B}{\gamma}u_0 = 0\,. \qquad (12)$$

If we call the stresses in the soil framework σ^* and the pressure in the pores filling with water p^* for a moment of time t and the co-ordinate x, but only assuming "momentary" consolidation of the ground during external loading q and w corresponding to that moment of times then from equation (7) we have for the state of "momentary consolidations", equation

$$q + w + \gamma_B(h-x) + \gamma(h-x) - \sigma^* - p^* = 0 \qquad (13)$$

From (7) and (13) we obtain equation

$$\sigma + p = \sigma^* + p^* \quad \text{or} \quad \sigma + \gamma H = \sigma^* + \gamma H^*. \qquad (14)$$

Differentiating equation (13) we obtain

$$\frac{1}{\gamma}\frac{d}{dt}(q+w) = \frac{1}{\gamma}\frac{\partial}{\partial t}(\sigma^* + p^*) = \frac{\partial H^*}{\partial t} + \frac{1}{\gamma}\frac{\partial \sigma^*}{\partial t}$$

As a result, in equation (12) we can substitute H^* and σ^* for q and w.

Using equations (13) and (14) we can also write equation (12) in the form:

$$\frac{\partial \sigma}{\partial t} - \frac{1}{\gamma}\left[k - (1+\varepsilon)\frac{dk}{d\varepsilon}\right]\left(\frac{\partial \sigma}{\partial x}\right)^2 - \frac{\gamma_B}{\gamma}\left[k + u_0\frac{\gamma}{\gamma_B} - (1+\varepsilon)\frac{dk}{d\varepsilon}\right]\frac{\partial \sigma}{\partial x} +$$

$$+\frac{k(1+\varepsilon)}{\gamma\frac{d\varepsilon}{d\sigma}}\frac{\partial^2 \sigma}{\partial x^2} = 0\,. \qquad (15)$$

Equation (12) like equation (15) has the form

$$\frac{\partial H}{\partial t} + \alpha\left(\frac{\partial H}{\partial x}\right)^2 + \beta\frac{\partial H}{\partial x} + \delta\frac{\partial^2 H}{\partial x^2} + \frac{\partial F}{\partial t} = 0\,, \qquad (16)$$

where, conformably with (12)

$$\alpha = k - (1+\varepsilon)\frac{dk}{d\varepsilon} = -(1+\varepsilon)^2\frac{d}{d\varepsilon}\frac{k}{1+\varepsilon}\,,$$

$$\beta = \frac{\gamma_B}{\gamma}\alpha + u_0\,,$$

$$\delta = \frac{k(1+\varepsilon)}{\gamma\frac{d\varepsilon}{d\sigma}}\,,$$

$$\frac{\partial F}{\partial t} = -\frac{1}{\gamma}\frac{d}{dt}(q+w) + \frac{\gamma_B}{\gamma}u_0\,.$$

Equation (16) is of greater importance for problems of soil consolidation and for the investigation of unceasing filtration in conditions where the soil framework is undergoing deformation. For the case of a spatial problem the analogical equation has the form:

$$\tfrac{\partial H}{\partial t} + \alpha(\operatorname{grad} H)^2 + \beta(\operatorname{grad} H, \operatorname{grad}\psi) + \delta\nabla^2 H + \tfrac{\partial F}{\partial t} = 0\,, \qquad (17)$$

where α, β, δ, ψ and $\frac{\partial F}{\partial t}$ are certain given functions from the co-ordinates and time.

It can be shown without difficulty that, for $\frac{\alpha}{\delta} = const$ and assuming

$$H = \tfrac{\delta}{\alpha}\ln(\varphi + C) + D\,, \qquad (18)$$

where C and D are arbitrary constants, equation (17) can be reduced to the linear form:

$$\tfrac{\partial\varphi}{\partial t} + \beta(\operatorname{grad}\varphi, \operatorname{grad}\psi) + \delta\nabla^2\varphi + \tfrac{\alpha}{\delta}(\varphi + C)\tfrac{\partial F}{\partial t} = 0\,. \qquad (19)$$

If, however, in addition, the function F as, e.g., in the case of the unidimensional problem of consolidation, does not depend on the co-ordinates but only on time, then, assuming

$$H = \tfrac{\delta}{\alpha}\ln(\varphi + C) - F + D\,, \qquad (20)$$

the equation (17) can be reduced to a form such as:

$$\tfrac{\partial\varphi}{\partial t} + \beta(\operatorname{grad}\varphi, \operatorname{grad}\psi) + \delta\nabla^2\varphi = 0\,. \qquad (21)$$

If in conformity with the usual conditions of the plane or the spatial problem of consolidation of the earth medium we consider the functions $\psi(x, y, z, t)$ and $F(x, y, z, t)$ as known and the relation (fractions) $\frac{\alpha}{\delta} = const$ and $\frac{\alpha}{\beta} = const$ then assuming

$$H = \tfrac{\delta}{\alpha}\ln(\varphi + C) - \tfrac{\beta}{2\alpha}\psi + D\,, \qquad (22)$$

equation (17) can be reduced to the form

$$\tfrac{\partial\varphi}{\partial t} + \delta\nabla^2\varphi + (\varphi + C)\big\{\tfrac{\alpha}{\beta}\tfrac{\partial F}{\partial t} - \tfrac{\beta^2}{4\delta}(\operatorname{grad}\psi)^2 - \tfrac{\beta}{2\delta}\tfrac{\partial\psi}{\partial t} - \tfrac{\beta}{2}\nabla^2\psi\big\} = 0\,,$$

where the expression in brackets represents some known function of the co-ordinates and time, while the function ψ is usually harmonic, in consequence of which $\nabla^2\psi = 0$.

In the particular case where $\alpha = const$, $\beta = const$, $\delta = const$, $F = F(t)$ and $\psi = ax + by + cz$, if we take

$$H = \tfrac{\delta}{\alpha}\ln(\varphi + C) - F - \tfrac{\beta}{2\alpha}(ax + by + cz)\tfrac{(a^2+b^2+c^2)\beta^2}{4\alpha}t + D\,, \qquad (23)$$

we can reduce equation (17) to its simplest forms

$$\frac{\partial \varphi}{\partial t} + \delta \nabla^2 \varphi = 0.$$

The particular values C and D in the expressions given above are choosed for reasons of convenience.

Applying, in accordance with the conditions of the special problem considered, for substitution of the dependent variable, one of the equations (18), (20), (22), or (23), we have the following expressions for the functional φ:

$$\begin{aligned} &\varphi + C = \exp\big\{\tfrac{\alpha}{\delta}(H - D)\big\}, \\ &\varphi + C = \exp\big\{\tfrac{\alpha}{\delta}(H + F - D)\big\}, \\ &\varphi + C = \exp\big\{\tfrac{\alpha}{\delta}(H + \tfrac{\beta}{2\alpha}\psi - D)\big\}, \\ &\varphi + C = \exp\big\{\tfrac{\alpha}{\delta}\big[H + \tfrac{\beta}{2\alpha}(ax + by + cz) - \tfrac{(a^2+b^2+c^2)\beta^2}{4\alpha}t - D\big]\big\}. \end{aligned} \tag{24}$$

Considering now the unidimensional problem, let us point out that the equation (16) is the particular case of equation (17).

The initial condition for the unidimensional problem for any of the considered cases of substitution of the dependent variable may be obtained, the assumption being made in the corresponding equation of series (24) that $\varphi = \varphi_0$, $H = H_0$ and $t = 0$.

The boundary conditions in the case of unidimensional problem for permeable boundary surfaces are applied, as usual, $H = H'$ or $H = H''$ and may be expressed without difficulty by the function φ.

For impermeable boundary surfaces from the condition $\frac{\partial H}{\partial x} = 0$ in conformity with the equations (18) and (20) we obtain $\frac{\partial \varphi}{\partial x} = 0$ while in conformity with equation (22) we obtain:

$$\frac{\partial \varphi}{\partial x} = \frac{\beta}{2\delta}(\varphi + c)\frac{\partial \psi}{\partial x}\,.$$

(after) The function φ having been an determined, the determination of the function H can take place without difficulty using one of the equations (18), (20), (22) or (23).

As an example, let us consider the simpler special case, it being assumed that

$$u_0 = \frac{dq}{dt} = \frac{dw}{dt} = \frac{dk}{d\varepsilon} = \gamma_B = 0\,,$$

$$1 + \varepsilon = 1 + \varepsilon_{cp} = const\,,$$

$$\frac{d\varepsilon}{d\sigma} = -a = const, \quad k = const\,.$$

That case corresponds to the usual statement of the problem by Terzaghi and Gersevanov but without neglecting the term $v\frac{\partial\varepsilon}{\partial x}$ in equation (6).

The equation (16) in this case has the following form:

$$\frac{\partial H}{\partial t} + \alpha\left(\frac{\partial H}{\partial x}\right)^2 + \delta\frac{\partial^2 H}{\partial x^2} = 0\,,$$

where

$$\alpha = k\,,$$

$$\delta = -\frac{k(1+\varepsilon)}{\gamma a}\,,$$

$$\frac{\alpha}{\delta} = -\frac{\gamma a}{1+\varepsilon_{cp}}\,.$$

Assuming (taking, substituting...)

$$H = \frac{\delta}{\alpha}\ln(\varphi+1)\,,$$

we find

$$\frac{\partial\varphi}{\partial t} + \delta\frac{\partial^2\varphi}{\partial x^2} = 0.$$

Assuming the initial and boundary conditions to be:

$$\text{for } t=0 \qquad H = H_0 \ \text{ or } \ \varphi = e^{\frac{\alpha}{\delta}H_0} - 1$$

$$\text{for } x=0 \text{ and } x=h \qquad H = 0 \ \text{ or } \varphi = 0\,,$$

the solution of this problem can be written in the known form:

$$\varphi = (e^{\frac{\alpha}{\delta}H_0} - 1)\frac{4}{\pi}\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i}\sin\frac{i\pi x}{h}\exp\left\{\frac{i^2\pi^2\delta}{h^2}t\right\}\,.$$

From this we obtain

$$H = \frac{\delta}{\alpha}\ln\left[1+\mu(e^{\frac{\alpha}{\delta}H_0} - 1)\right], \tag{25}$$

where

$$\mu = \frac{4}{\pi}\sum_{i=1,3,\ldots}^{\infty}\frac{1}{i}\sin\frac{i\pi x}{h}\exp\left\{\frac{i^2\pi^2\delta}{h^2}t\right\}\,.$$

We note that $0 \leqslant \mu \leqslant 1$.

One finds without difficulty that if α tends to zero then this solution takes the generally known form:

$$H_{\alpha=0} = \mu H_0\,. \tag{26}$$

One finds again without difficulty that for the limiting cases corresponding to $t=0$ and $t=\infty$ the values of μ are, respectively, $\mu=1$ and $\mu=0$; hence, we have, respectively, $H = H_{\alpha=0} = H_0$ and $H = H_{\alpha=0} = 0$, i.e., for these limiting cases the solutions (25) and (26), naturally, coincide.

For the intervening (intermediate) moments (points) of time $0 < t < \infty$ the ratio of the values of the required(desired) functions H and $H_{\alpha=0}$ for any moment of time t and of the co-ordinate x is determined(expressed) by the expression

$$r = \frac{H}{H_{\alpha=0}} = \frac{\delta}{\alpha}\frac{1}{\mu H_0}\ln\left[1+\mu(e^{\frac{\alpha}{\delta}H_0}-1)\right].$$

Taking as a numerical example big enough (sufficiently great) values $H_0 = 30m$ and $a = 0.05\frac{cm^2}{kg}$, and also $\varepsilon_{cp} = 1$ and $\gamma = 1\frac{T}{m^3}$, we obtain

$$\frac{\alpha}{\delta} = -\frac{\gamma a}{1+\varepsilon_{cp}} = -\frac{1\times 0.005}{2} = -0.0025\frac{1}{m}$$

$$\frac{\alpha}{\delta}H_0 = -0.0025\times 30 = -0.075\,.$$

Whence

$$r = -\frac{1}{0.075\mu}\ln\left[1+\mu(e^{-0.075}-1)\right].$$

Table 1

μ	0	0.1	0.2	0.4	0.6	0.8	1.0
r	1	0.97	0.97	0.98	0.98	0.99	1

The values of r for different values of μ are given in Table 1; it follows from this that these values deviate from unity by a sufficiently small amount; this illustrates objectively the practical basis for neglecting in the fundamental equation of consolidation in the ease considered, the unidimensional problem in equation (6) the terms containing the factor "velocity of filtration". Therefore, in considering the plane problem as well as the spatial problems of consolidating an earth medium, there is a sufficient reason for the same admission.

As a second example, we give the solution of the unidimensional problem of consolidation where it is assumed that the filtration properties of the soil medium vary with the variation of the stresses and the porosity of the soil framework. Neglecting in accordance with what has been said, the term $v\frac{\partial\varepsilon}{\partial x}$ in equation (6), and assuming, e.g., $u_0 = 0$ we obtain (conformably with equation (16))

$$\alpha = -(1+\varepsilon)\frac{dk}{d\varepsilon}\,,$$
$$\beta = \frac{\gamma_B}{\gamma}\alpha\,,$$
$$\delta = \frac{k(1+\varepsilon)}{\gamma\frac{d\varepsilon}{d\sigma}}\,.$$

Let the symbols for the quantities corresponding to the initial and the final states of stress have (supply), respectively, the index $'$ and $''$. Taking into account that on the basis of experiment the ratio of the coefficient of filtration k to the coefficient of porosity ε may be represented sufficiently satisfactorily by a linear function, let us assume

$$k = k' - \frac{k'-k''}{\varepsilon'-\varepsilon''}(\varepsilon' - \varepsilon) \tag{27}$$

If with a wide range of varying porosity, that relationship does not occur, with a sufficiently small range practically any relationship may be replaced by a linear function; so, for example, it is assumed in the substitution of the rectilinear for the curvilinear form of the compression curve. Taking into account that the form of the compression curve may be represented much better by the exponential function than by the linear, we assume:

$$\varepsilon = \varepsilon' - \frac{\varepsilon'-\varepsilon''}{k'-k''}\left[k' - \exp\left\{-\frac{\ln k' - \ln k''}{\sigma''-\sigma'}\sigma + \frac{\sigma''\ln k' - \sigma'\ln k''}{\sigma''-\sigma'}\right\}\right]. \tag{28}$$

From equations (27) and (28) we can write equation (27) in the form

$$k = \exp\{\ \}. \tag{29}$$

Differentiating the expression (27) we obtain

$$\frac{dk}{d\varepsilon} = \frac{k'-k''}{\varepsilon'-\varepsilon''},$$

whence

$$\alpha = -(1+\varepsilon)\frac{k'-k''}{\varepsilon'-\varepsilon''},$$

$$\beta = -(1+\varepsilon)\frac{\gamma_B}{\gamma}\frac{k'-k''}{\varepsilon'-\varepsilon''}.$$

Differentiating the expression (28) and compare the result of this differentiation with (29) we have

$$\delta = \frac{k(1+\varepsilon)}{\gamma\frac{d\varepsilon}{d\sigma}} = -\frac{1+\varepsilon}{\gamma}\frac{k'-k''}{\varepsilon'-\varepsilon''}\frac{\sigma''-\sigma'}{\ln\frac{k'}{k''}},$$

whence

$$\frac{\alpha}{\delta} = \gamma\frac{\ln\frac{k'}{k''}}{\sigma''-\sigma'} = const.$$

Because of the fact that in the 1D case we have $F = F(t)$ we can reduce the original equation of consolidation (16) to the form (21) by the substitution (20).

However, for further simplification of the example let us, as a preliminary, make the usual assumption which is also assumed even to the problems with a constant coefficient of filtration, namely that $1+\varepsilon \approx 1+\varepsilon_{cp}$.

The influence of that assumption on the results obtained, as has been noted already by Gersevanov, is rather slight. In consequence of this assumption in the present case the values α, β and δ become constants.

Let us assume, for the sake of distinctness, that at a moment of time $t = 0$ to the upper, boundary surface of a soil layer in a stabilised state a load q, unvarying in time, is momentarily applied and the pressure w in the water.

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Taking into account that in the case considered we have $\frac{\partial F}{\partial t} = \beta = 0$, and assuming:

$$H = \tfrac{\delta}{\alpha} \ln(\varphi + C) + D, \tag{30}$$

we obtain the equation of consolidation in the following form:

$$\tfrac{\partial \varphi}{\partial t} + \delta \tfrac{\partial^2}{\partial x^2} \varphi = 0.$$

in accordance with the expression (30) we have

$$\varphi + C = \exp\bigl\{(H - D)\tfrac{\alpha}{\delta}\bigr\};$$

from this taking into account the initial condition[1] for function H:

$$\text{with } t = 0 \text{ and } 0 \leqslant x \leqslant h \qquad H = \tfrac{1}{\gamma}(q + w) + h = H_0,$$

we have the initial condition for the function φ (with a choice $D = 0$ and $C = 1$):

$$\text{with } t = 0 \text{ and } 0 \leqslant x \leqslant h \qquad \varphi = \exp\bigl\{\tfrac{\alpha}{\delta} H_0\bigr\} - 1 = \varphi_0.$$

The boundary conditions[1] for the functions H and φ for (at) $0 < t < \infty$:

$$\begin{aligned} &\text{for } x = 0 && H = 0 \ \text{ and } \ \varphi = 0\,, \\ &\text{for } x = h && H = \tfrac{w}{\gamma} + h = H_h \ \text{ and } \ \varphi = \exp\bigl\{\tfrac{\alpha}{\delta} H_h\bigr\} - 1 = \varphi_h\,. \end{aligned}$$

The solution of this problem, as we know, can be represented in the following form:

$$\varphi = \tfrac{x}{h}\varphi_h + \sum_{i=1,2,\ldots}^{\infty} \exp\left\{\tfrac{i^2\pi^2\delta}{h^2} t\right\} \sin\tfrac{i\pi x}{h} \tfrac{2}{h} \int_0^h \left(\varphi_0 - \tfrac{x}{h}\varphi_h\right) \sin\tfrac{i\pi x}{h} dx =$$

$$= \tfrac{x}{h}\varphi_h + \frac{2}{\pi} \sum_{i=1,2,\ldots}^{\infty} \tfrac{1}{i} \sin\tfrac{i\pi x}{h} \{\varphi_h(-1)^i - \varphi_0[(-1)^i - 1]\} \exp\left\{\tfrac{i^2\pi^2\delta}{h^2} t\right\}.$$

Whence

$$H = \frac{\delta}{\alpha} \ln\Bigl[1 + \tfrac{x}{h}\bigl[\exp\bigl\{\tfrac{\alpha}{\delta} H_h\bigr\} - 1\bigr] + \frac{2}{\pi} \sum_{i=1,2,\ldots}^{\infty} \tfrac{1}{i} \sin\tfrac{i\pi x}{h} \times$$

$$\times \Bigl\{(-1)^i \bigl[\exp\bigl\{\tfrac{\alpha}{\delta} H_h\bigr\} - 1\bigr] - [(-1)^i - 1]\bigl[\exp\bigl\{\tfrac{\alpha}{\delta} H_0\bigr\} - 1\bigr]\Bigr\} \times$$

$$\times \exp\bigl\{\tfrac{i^2\pi^2\delta}{h^2} t\bigr\}\Bigr].$$

[1]In cases where the weight of the water in the pores is neglected in the expressions for H_0 and H_h, it is necessary to neglect h (to drop(throw off/away) the additive term h) also.

The solution obtained for $\alpha = 0$ naturally agrees with the usual solution for constant soil properties.

In the same way the solution for any case of a unidimensional problem can be obtained, the variability of the soil properties being taken into account, and, in particular, when taking into account its consolidation under its own weight, the varying boundary conditions etc.

A similar method of solution may be applied also for the solution of a plane or a spatial problem of consolidation. It appears to be satisfactory to apply it in the case of a numerical solution also, e,g., by the method of finite differences.

It was pointed out by us in an earlier paper [1] that in the case of a planar or a spatial problem of consolidation of the soil medium with the variable properties of the ground used in the form of a two-phase medium, and where the terms are neglected in which the filtration velocities occur as factors, the fundamental equation may be represented in the form

$$\begin{gathered}\frac{\partial H}{\partial t} = \frac{1}{2\gamma}\frac{\partial \theta^*}{\partial t} + \frac{\partial H^*}{\partial t} - (1+\varepsilon)\frac{dk}{d\varepsilon}\Big\{\frac{1}{2\gamma}(\operatorname{grad}\theta^*, \operatorname{grad} H) + \\ +(\operatorname{grad} H, \operatorname{grad} H^*) - (\operatorname{grad} H)^2\Big\} - \frac{(1+\varepsilon)k}{2\gamma\frac{d\varepsilon}{d\theta}}\nabla^2 H\,,\end{gathered} \tag{31}$$

where θ is the sum of the principal stresses in the soil.
Denoting

$$\psi = H^* + \frac{1}{2\gamma}\theta^*$$

and taking into account that

$$\frac{1}{2\gamma}(\operatorname{grad}\theta^*, \operatorname{grad} H) + (\operatorname{grad} H^*, \operatorname{grad} H) = (\operatorname{grad} H, \operatorname{grad}\psi),$$

we can write equation (31) in the form

$$\begin{gathered}\frac{\partial H}{\partial t} - (1+\varepsilon)\frac{dk}{d\varepsilon}(\operatorname{grad} H)^2 + (1+\varepsilon)\frac{dk}{d\varepsilon}(\operatorname{grad} H, \operatorname{grad}\psi) + \\ +\frac{(1+\varepsilon)k}{2\gamma\frac{d\varepsilon}{d\theta}}\nabla^2 H - \frac{\partial \psi}{\partial t} = 0,\end{gathered}$$

or

$$\frac{\partial H}{\partial t} + \alpha(\operatorname{grad} H)^2 + \beta(\operatorname{grad} H, \operatorname{grad}\psi) + \delta\nabla^2 H + \frac{\partial F}{\partial t}\,. \tag{32}$$

Taking into account that the stress σ which occurs in the equation of the compression curve $\varepsilon = \varepsilon(\sigma)$ has the following relationship to the principal stresses: $\sigma = \frac{\theta}{1+\xi}$, where ξ is the coefficient of lateral pressure, we find:

$$\frac{d\varepsilon}{d\theta} = \frac{d\varepsilon}{d\sigma}\frac{d\sigma}{d\theta} = \frac{1}{1+\xi}\frac{d\varepsilon}{d\sigma}.$$

Then in accordance with what was said earlier with reference to equation (32) we have:

$$\alpha = -(1+\varepsilon)\frac{dk}{d\varepsilon} = -(1+\varepsilon)\frac{k'-k''}{\varepsilon'-\varepsilon''},$$

$$\beta = (1+\varepsilon)\tfrac{dk}{d\varepsilon} = (1+\varepsilon)\tfrac{k'-k''}{\varepsilon'-\varepsilon''},$$

$$\delta = \frac{(1+\varepsilon)k}{2\gamma\frac{d\varepsilon}{d\theta}} = -(1+\xi)\tfrac{(1+\varepsilon)}{2\gamma}\tfrac{k'-k''}{\varepsilon'-\varepsilon''}\tfrac{\sigma''-\sigma'}{\ln\frac{k'}{k''}},$$

$$F = -\psi$$

and consequently

$$\tfrac{\alpha}{\delta} = \tfrac{2\gamma}{1+\xi}\frac{\ln\frac{k'}{k''}}{\sigma''-\sigma'} = \text{const},$$

$$\tfrac{\beta}{\alpha} = -1.$$

Assuming as usual that, $1+\varepsilon = 1+\varepsilon_{cp}$, we obtain that not only their ratios but also the values α, β and δ are themselves constants.

Introducing a new dependent variable defined by the expression

$$H = \tfrac{\delta}{\alpha}\ln(\varphi + C) + \tfrac{1}{2}\psi + D, \tag{33}$$

we obtain equation (32) in the form

$$\tfrac{\partial\varphi}{\partial t} + \delta\nabla^2\varphi - (\varphi + C)\tfrac{\alpha}{2\delta}\left[\tfrac{\partial\psi}{\partial t} + \tfrac{\alpha}{2}(\operatorname{grad}\psi)^2\right] = 0. \tag{34}$$

In accordance with the expression (33) we have

$$\varphi + C = \exp\left\{\tfrac{\alpha}{\delta}\big(H - \tfrac{1}{2}\psi - D\big)\right\}.$$

Then, starting from the initial condition for function H, that for $t = 0$, $H = H_0 = H_0^* + \frac{1}{2\gamma}\theta_0^* = \psi_0$, and assuming $C = D = 0$, we can represent the initial condition for function φ:

$$\text{for } t = 0 \qquad \varphi = \exp\left\{\tfrac{\alpha}{2\delta}\psi_0\right\}.$$

The usual boundary conditions are as follows:

a) For the permeable parts of the boundary surface the limiting(boundary) values $H = H_s$ must be given, from which

$$\varphi_s = \exp\left\{\frac{\alpha}{\delta}\big(H_s - \tfrac{1}{2}\psi_s\big)\right\};$$

b) On the impermeable parts

$$\left(\tfrac{\partial H}{\partial n}\right)_s = 0,$$

from which

$$\left(\tfrac{\partial\varphi}{\partial n}\right)_s = -\varphi_s\tfrac{\alpha}{2\delta}\left(\tfrac{\partial\psi}{\partial n}\right)_s.$$

Thus we formulate the mix initial boundary value problem for any type of the consolidated region of the soil medium, dictated by practical needs. It is appropriate to tackle such a problem by the method of finite differences. Taking, for example, in the case of the planar problem, a square grid with the intervals between the grid points equal to Δh, and assuming that the time steps are constant and equal Δt we can, passing to the finite differences, to write equation (34) in the form:

$$\begin{gathered}\tfrac{1}{\Delta t}(\varphi_{t+1,i,k}-\varphi_{t,i,k})+\tfrac{\delta}{\Delta h^2}(\Lambda_t-4\varphi_{t,i,k})-\\ -\varphi_{t,i,k}\tfrac{\alpha}{2\delta}\big[\tfrac{1}{\Delta t}\Delta\psi_{t+1,t,i,k}+\tfrac{\alpha}{2}(\operatorname{grad}\psi)^2_{t,i,k}\big]=0,\end{gathered}$$

where

$$\begin{gathered}\Lambda_t=\varphi_{t,i+1,k}+\varphi_{t,i-1,k}+\varphi_{t,i,k+1}+\varphi_{t,i,k-1}\,,\\ \Delta\psi_{t+1,t,i,k}=\psi_{t+1,i,k}-\psi_{t,i,k}\,.\end{gathered}$$

From this we have

$$\varphi_{t+1,i,k}=\varphi_{t,i,k}\big[1-4\eta+\tfrac{\alpha}{2\delta}\Delta\psi_{t+1,t,i,k}-\tfrac{\eta\alpha^2\Delta h^2}{4\delta^2}(\operatorname{grad}\psi)^2_{t,i,k}\big]+\eta\Lambda_t\,, \qquad (35)$$

where

$$\eta=-\delta\tfrac{\Delta t}{\Delta h^2}.$$

If we assume $\eta=0.25$, then

$$\varphi_{t+1,i,k}=\varphi_{t,i,k}\big[\tfrac{\alpha}{2\delta}\Delta\psi_{t+1,t,i,k}-\tfrac{\alpha^2\Delta h^2}{16\delta^2}(\operatorname{grad}\psi)^2_{t,i,k}\big]+\tfrac{1}{4}\Lambda_t\,.$$

The value $\eta=0.25$ may be taken if

$$\begin{gathered}\text{for } \varphi_{t,i,k}>\tfrac14\Lambda_t \qquad \varphi_{t+1,i,k}\geqslant\tfrac14\Lambda_t\,,\\ \text{for } \varphi_{t,i,k}<\tfrac14\Lambda_t \qquad \varphi_{t+1,i,k}\leqslant\tfrac14\Lambda_t\,.\end{gathered}$$

In the reverse case the value η should be assumed to be less in order that one of the inequalities may be maintained.

If the limiting(boundary) conditions do not depend on time and the consolidating loading is constant, then

$$\begin{gathered}\psi=\psi_0\,,\\ \Delta\psi_{t+1,t,k}=0\end{gathered}$$

and equation (35) has the form

$$\begin{gathered}\varphi_{t+1,i,k}=\varphi_{t,i,k}\big[1-4\eta-\tfrac{\eta\alpha^2\Delta h^2}{4\delta^2}(\operatorname{grad}\psi_0)^2_{i,k}\big]+\eta\Lambda_t=\\ =\varphi_{t,i,k}A_{i,k}+\eta\Lambda_t\,.\end{gathered}$$

In that case a table of the values $A_{i,k}$ having been prepared, the values of the function φ are very simply found.

The values of function φ having been determined for all grid points within the region of compaction considered here, and, successively, for all moments of time used in the computations, beginning with $t=0$ to some arbitrarily selected moment of time t, the subsequent determinations of the values of the required function H is reduced without difficulty to the expression (33).

Let us consider now the case of the variation in time of the external consolidating loading of $q(x)$ and let us assume, for example, that that variation may be represented by the linear function

$$q(x) = v(x)t = (v\Delta t)\tfrac{t}{\Delta t} = q_{\Delta t}(x)m\,,$$

where $v(x)$ is the velocity of variation of the ordinates of the loading diagram, $q_{\Delta t}$ is the diagram of the increase of the ordinates of external loading during the time interval Δt, and m is the number of intervals of time in a moment of time t.

Then assuming, for example, $H^* = const$, we have

$$\Delta\psi_{t+1,t,i,k} = \tfrac{1}{2\gamma}\Delta\theta^*_{t+1,t,i,k} = \tfrac{1}{2\gamma}\theta_{i,k}(q_{\Delta t})\,,$$
$$(\operatorname{grad}\psi)^2_{t,i,k} = \tfrac{1}{4\gamma^2}(\operatorname{grad}\theta^*)^2_{t,i,k} = \tfrac{m^2}{4\gamma^2}[\operatorname{grad}\theta^*_{i,k}(q_{\Delta t})]^2\,,$$

where $m = 1, 2, 3, \ldots$

Equation (35) may, consequently be represented in the form

$$\varphi_{t+1,i,k} = \varphi_{t,i,k}[1 - 4\eta + A_{i,k} - B_{i,k}m^2] + \eta\Lambda_t\,,$$

where

$$A_{i,k} = \tfrac{\alpha}{4\delta\gamma}\theta^*_{i,k}(q_{\Delta t})$$

and

$$B_{i,k} = \tfrac{\eta\alpha^2\Delta h^2}{16\delta^2\gamma^2}[\operatorname{grad}\theta^*_{i,k}(q_{\Delta t})]^2\,,$$

where those values are constant for each grid point (i, k).

In the particular case of a load uniformly distributed along the edge of the hemi-plane of a strip $(-a, +a)$:

$$\theta^*_{i,k}(q_{\Delta t}) = \frac{2}{\pi}q_{\Delta t}\operatorname{arc\,tg}\frac{2\frac{x}{a}}{\left(\frac{x}{a}\right)^2 + \left(\frac{z}{a}\right)^2 - 1}\,,$$

$$[\operatorname{grad}\theta^*_{i,k}(q_{\Delta t})]^2 = \frac{16}{\pi^2a^2}q^2_{\Delta t}\frac{1}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{z}{a}\right)^2 + 1\right]^2 - 4\left(\frac{x}{a}\right)^2}\,,$$

where x and z are the horizontal and the vertical co-ordinates of the grid point (i, k).

In the cases where the consolidating load gradually increases, for $t = 0$ it is equal to 0 and the initial condition has the form: for $t = 0$, assuming $H^* = 0$, $\theta^* = 0$, we have $\varphi = 1$.

On permeable parts of the boundary hemi-plane of axis $z = 0$ for grid points $(i, 0)$ assuming $H_{i,0} = 0$ we find

$$\varphi_{i,0} = \exp\left\{-\tfrac{\alpha}{4\gamma\delta}\theta^*_{i,0}\right\}.$$

On non-permeable parts of axis $z = 0$ we have

$$\left(\tfrac{\partial\varphi}{\partial x}\right)_{i,0} = -\varphi_{i,0}\tfrac{\alpha}{2\delta}\left(\tfrac{\partial\psi}{\partial x}\right)_{i,0} = \varphi_{i,0}\tfrac{\alpha}{4\delta\gamma}\left(\tfrac{\partial\theta^*}{\partial x}\right)_{i,0}$$

or, converting to finite differences, we have

$$\frac{\varphi_{i,0}-\varphi_{i,1}}{\Delta h} = -\varphi_{i,0}\frac{\alpha}{4\delta\gamma}\frac{\theta^*_{i,0}-\theta_{i,1}}{\Delta h},$$

whence

$$\varphi_{i,0} = \frac{\varphi_{i,1}}{1+\frac{\alpha}{4\delta\gamma}(\theta^*_{i,0}-\theta_{i,1})}.$$

The order of the numerical solution of problems of this kind is obvious and there is no need for further consideration.

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