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A. V. Egorov, S. Mondié, Критерий устойчивости линейных уравнений с одним запаздыванием в терминах матриц Ляпунова, *Вестн. С.-Петербург. ун-та. Сер. 10. Прикл. матем. Информ. Проц. упр.*, 2013, выпуск 1, 106–115

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UDK 517.929.4

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A STABILITY CRITERION FOR THE SINGLE DELAY EQUATION IN TERMS OF THE LYAPUNOV MATRIX

1. Introduction. In the case of delay systems the Lyapunov-Krasovskii functionals method plays the role of the second Lyapunov approach for the case of ordinary differential equations. The main idea of the method, proposed by Krasovskii in [1], has been developed for linear systems in [2–6].

In [6] complete type functionals, admitting a quadratic lower bounds, were proposed. The functionals have been applied both to the stability analysis and to solutions of some related problems [7–9]. The complete type functionals depend on special matrix valued functions, called, by analogy with ordinary differential equations, the Lyapunov matrices.

It is of interest to find conditions on a Lyapunov matrix, which guarantee the stability of the system. Such conditions have been established in [10] for the linear single delay equations. In [9] some necessary stability conditions have been obtained for the single delay linear systems. In our contribution it is proved that these necessary conditions become sufficient for the case of scalar single delay equation.

The organization of the paper is as follows. In the section 2 a linear delay equation is introduced and stability region for the equation is provided. In section 3 Lyapunov-Krasovskii quadratic functionals with prescribed time derivatives are presented. Section 4 is devoted to the computations of the Lyapunov auxiliary function. The main contribution, a new stability criterion, is proved in section 5. Some concluding remarks end the paper.

2. Preliminaries. We consider a linear time-delay equation of the form

$$\dot{x}(t) = ax(t) + bx(t-1), \quad t \geqslant 0, \tag{1}$$

where $a, b \in R$.

Let us introduce the following notation:

$$x_t(\varphi) : \theta \to x(t+\theta, \varphi), \quad \theta \in [-1, 0],$$

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where $x(t,\varphi)$ is the solution of (1) with the initial time instant $t_0 = 0$, and a function φ. It is supposed that the initial function belongs to the space of piecewise continuous, on the segment $[-1,0]$, functions $PC^0([-1,0],R)$,

$$x(\theta,\varphi) = \varphi(\theta), \quad \theta \in [-1,0]. \tag{2}$$

It is known [11] that the initial value problem (1), (2) has a unique solution, defined on $[-1,\infty)$.

We will use the norm

$$\|\varphi\|_h = \sup_{\theta\in[-1,0]} |\varphi(\theta)|.$$

The equation (1) is said to be exponentially stable if there exist constants $\gamma \geqslant 1$ and $\sigma > 0$, such that

$$|x(t,\varphi)| \leqslant \gamma e^{-\sigma t}\|\varphi\|_h, \quad t \geqslant 0.$$

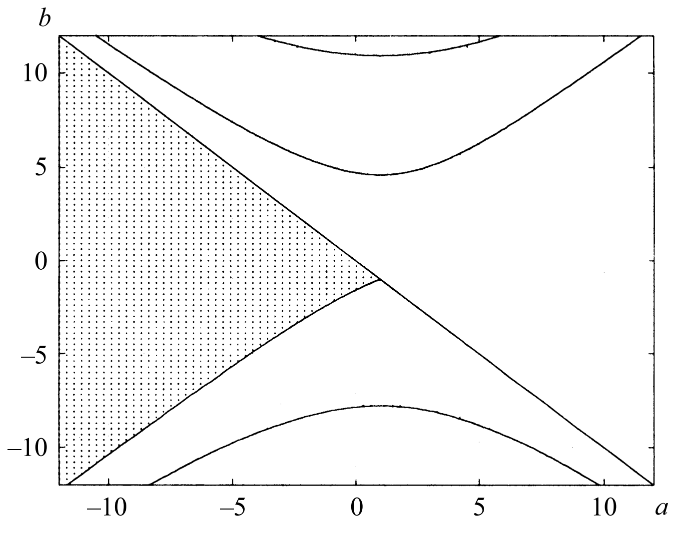
The characteristic equation for the equation (1) is

$$s - a - be^{-s} = 0. \tag{3}$$

The exact exponential stability domain in space of coefficients is [12]

$$\Omega = \Big(\Big\{a+b<0\Big\} \cap \Big\{a-b\leqslant 0\Big\}\Big) \cup \\ \cup \Big(\Big\{|a|+b<0\Big\} \cap \Big\{a+b\cos\sqrt{b^2-a^2}<0\Big\} \cap \Big\{\sqrt{b^2-a^2}<\pi\Big\}\Big).$$

The first term of this union describes the delay-independent part of the exponential stability domain, while the second term describes the delay-dependent part. The region Ω is depicted in figure with the curves, corresponding to parameter values for which the characteristic equation (3) has pure imaginary roots.



Exponential stability region of (1)

For a symmetric matrix Q, the notation $Q > 0$ ($Q \geqslant 0$) means that Q is positive definite (positive semidefinite).

3. The Lyapunov-Krasovskii approach. According to the second Lyapunov method, the exponential stability of delay free system

$$\dot{x}(t) = Ax(t) \tag{4}$$

is equivalent to the existence of two positive-definite quadratic forms $v(x) = x^T V x$ and $w(x) = x^T W x$, satisfying the condition

$$\frac{d}{dt} v(x(t)) = -w(x(t))$$

along the solutions $x(t)$ of system (4).

In [6, 7], a generalization of this method was given for delay linear systems. In particular, the true counterpart of the quadratic form $v(x)$ for equation (1) is the functional

$$v(\varphi) = u(0)\varphi^2(0) + 2b\varphi(0) \int_{-1}^{0} u(\theta+1)\varphi(\theta)\,d\theta + \\ + b^2 \int_{-1}^{0}\int_{-1}^{0} \varphi(\theta_1)u(\theta_1-\theta_2)\varphi(\theta_2)\,d\theta_2\,d\theta_1 + \bar{w}_1 \int_{-1}^{0} \varphi^2(\theta)\,d\theta, \tag{5}$$

where $\bar{w}_1$ is some positive constant, and $u(\tau)$ is the function, satisfying the set of equations

$$u(-\tau) = u(\tau), \quad \tau \geqslant 0, \tag{6}$$

$$\dot{u}(\tau) = au(\tau) + bu(\tau-1), \quad \tau \geqslant 0, \tag{7}$$

$$au(0) + bu(-1) = -\bar{w}, \tag{8}$$

where $\bar{w} = \dfrac{\bar{w}_0 + \bar{w}_1}{2}$, and $\bar{w}_0$ is an arbitrary positive constant. In the case of delay systems this function is named *the Lyapunov delay matrix*. In this paper we name it *the Lyapunov function.*

The derivative of the functional (5) along the solutions of equation (1) is $-w(x_t)$, where

$$w(\varphi) = \bar{w}_0\varphi^2(0) + \bar{w}_1\varphi^2(-1).$$

The proof of the following theorem can be found in [6].

Theorem 1. *Equation* (1) *is exponentially stable if and only if the set of equations* (6)–(8) *admits a unique continuous solution, and for any* (*there exists*) *number* $\bar{w}_1 \in (0, 2\bar{w})$ *there exists a constant* $\beta > 0$*, such that the following inequality holds*:

$$v(\varphi) \geqslant \beta\varphi^2(0).$$

Phrase “for any (there exists)” means that the theorem remains true if we use “for any” or “there exists”.

4. The solution of the system (6)–(8). If we have the Lyapunov function $\bar{u}(\tau)$, satisfying the system (6)–(8) with $\bar{w} = 1$, then the Lyapunov function for any $\bar{w}$ can be defined as $u(\tau) = \bar{w}\bar{u}(\tau)$. Therefore, without any loss of generality, below, we take $\bar{w} = 1$.

The Lyapunov function $u(\tau)$ on $[0,1]$ is a solution of the system

$$\dot{u}(\tau) = au(\tau) + bu(1-\tau), \tag{9}$$

$$au(0) + bu(1) = -1. \tag{10}$$

The following theorem gives exhaustive information about $u(\tau)$, satisfying (9), (10). The following identities, which are extensively used, are recalled:

$$\cosh\alpha = \frac{e^{\alpha} + e^{-\alpha}}{2} = \cos(i\alpha),$$

$$\sinh\alpha = \frac{e^{\alpha} - e^{-\alpha}}{2} = -i\sin(i\alpha).$$

Theorem 2. *Let us set* $\lambda = \sqrt{a^2 - b^2}$ *and* $\tilde{\lambda} = \sqrt{b^2 - a^2}$. *All values in the following expressions are real*:

1) if $|a| > |b|$, $a + b\cosh\lambda \neq 0$, *then*

$$u(\tau) = \frac{b\sinh\lambda(1-\tau) - a\sinh\lambda\tau - \lambda\cosh\lambda\tau}{\lambda(a + b\cosh\lambda)};$$

2) if $|a| > |b|$, $a + b\cosh\lambda = 0$, $b > 0$, *then*

$$u(\tau) = \frac{\cosh\lambda(1-\tau)}{\lambda\sinh\lambda};$$

3) if $a = b$, $b \neq 0$, *then*

$$u(\tau) = \frac{b-1}{2b} - \tau;$$

4) if $|b| > |a|$, $a + b\cos\tilde{\lambda} \neq 0$, *then*

$$u(\tau) = \frac{b\sin\tilde{\lambda}(1-\tau) - a\sin\tilde{\lambda}\tau - \tilde{\lambda}\cos\tilde{\lambda}\tau}{\tilde{\lambda}(a + b\cos\tilde{\lambda})};$$

5) if $|b| > |a|$, $a + b\cos\tilde{\lambda} = 0$, $b\sin\tilde{\lambda} > 0$, *then*

$$u(\tau) = -\frac{\cos\tilde{\lambda}(1-\tau)}{\tilde{\lambda}\sin\tilde{\lambda}}.$$

In the remaining cases, the Lyapunov function $u(\tau)$ does not exist.

P r o o f. Let us differentiate the equation (9):

$$\ddot{u}(\tau) = a\dot{u}(\tau) - b\dot{u}(1-\tau) = a(au(\tau) + bu(1-\tau)) - \\ - b(au(1-\tau) + bu(\tau)) = (a^2 - b^2)u(\tau) = \lambda^2 u(\tau) = -\tilde{\lambda}^2 u(\tau). \tag{11}$$

Consider first the third case: $a^2 - b^2 = 0$. It is evident that the general solution of (11) has the form

$$u(\tau) = C_1\tau + C_2.$$

Substitution of this expression into (9) gives

$$C_1 = a(C_1\tau + C_2) + b(C_1(1-\tau) + C_2),$$

hence,

$$(b-1)C_1 + (a+b)C_2 = 0,$$
$$(a-b)C_1 = 0.$$

The expression (10) takes the form

$$bC_1 + (a+b)C_2 = -1.$$

The obtained system is consistent if and only if $a = b \neq 0$. In this case $u(\tau)$ has the form given in the third item of the theorem.

If $|a| > |b|$, then λ is real and the Lyapunov function belongs to the family

$$u(\tau) = C_1 \cosh \lambda\tau + C_2 \sinh \lambda\tau.$$

Substitution into (9) and (10) gives us the system for determination of C_1 and C_2:

$$(\lambda + b\sinh\lambda)C_1 - (a - b\cosh\lambda)C_2 = 0, \tag{12}$$

$$(a + b\cosh\lambda)C_1 - (\lambda - b\sinh\lambda)C_2 = 0, \tag{13}$$

$$(a + b\cosh\lambda)C_1 + b\sinh\lambda C_2 = -1. \tag{14}$$

When $a + b\cosh\lambda \neq 0$,

$$C_1 = -\frac{\lambda - b\sinh\lambda}{\lambda(a + b\cosh\lambda)},$$
$$C_2 = -\frac{1}{\lambda}.$$

These satisfy (12) and give the expression for $u(\tau)$ of the first item of the theorem.

The equality $a + b\cosh\lambda = 0$ implies $a = -b\cosh\lambda$. Hence,

$$\lambda = \sqrt{a^2 - b^2} = \sqrt{b^2(\cosh^2\lambda - 1)} = |b\sinh\lambda|.$$

If $b\sinh\lambda < 0$, the equation (13) is inconsistent with (14). If $b\sinh\lambda > 0$ (equivalently, $b > 0$, because $\sinh\alpha > 0$ for $\alpha > 0$), then

$$C_1 = \frac{\cosh\lambda}{\lambda\sinh\lambda},$$
$$C_2 = -\frac{1}{\lambda}.$$

Consider now the case $a^2 - b^2 < 0$, i. e. $|a| < |b|$. Notice that $\tilde{\lambda}$ is real. The general solution of (11) is

$$u(\tau) = C_1 \cos\tilde{\lambda}\tau + C_2 \sin\tilde{\lambda}\tau.$$

The fourth theorem item is direct consequence of the first item, as $\lambda = i\tilde{\lambda}$.

But the fifth theorem item could not be obtained from the second item, because hyperbolic sine is positive function of the positive argument, in contrast to trigonometric sine.

In this case the condition $\lambda - b\sinh\lambda = 0$ which follows immediately from equation (13), is equivalent to $i|b\sin\tilde{\lambda}| - ib\sin\tilde{\lambda} = 0$ or $b\sin\tilde{\lambda} > 0$. $\square$

Denote the set of parameters (a, b), for which the Lyapunov function exists, by E.

5. The exponential stability criterion. In [9] some necessary stability conditions have been obtained for the single delay linear systems.

Theorem 3. *If the system*

$$\dot{x}(t) = Ax(t) + Bx(t-1),$$

where A, $B \in R^{n\times n}$, *is exponentially stable, and the matrix* B *is non-singular, then the following inequalities hold*:

$$U(0) > 0,$$

$$\begin{pmatrix} U(0) & U^T(\tau) \\ U(\tau) & U(0) \end{pmatrix} \geqslant 0, \quad \tau \in [0,1],$$

where $U(\tau)$ *is the Lyapunov delay matrix* (*see* [7]).

The following theorem shows that these necessary conditions become sufficient for the case of scalar equation.

Theorem 4. *The equation* (1) *is exponentially stable if and only if the Lyapunov matrix for* (1) *is well defined on* $[0,1]$, *and*

$$u(0) \geqslant |u(\tau)|, \quad \tau \in [0,1],$$

which is equivalent to

$$u(0) > |u(\tau)|, \quad \tau \in (0,1]. \tag{15}$$

If the equation (1) *is non-exponentially stable, then the Lyapunov matrix is not well defined.*

P r o o f. It has been shown [7] that

- if (1) is non-exponentially stable, the Lyapunov matrix is not well defined;
- the equation (1) is not exponentially stable, if the Lyapunov matrix is not well defined.

Therefore, it remains to show two following statements:

I. If $(a,b) \in \Omega$, then $u(0) > |u(\tau)|$ for any $\tau \in (0,1]$.

II. If $(a,b) \in \Lambda = E \setminus \Omega$, then $u(0) < |u(\overline{\tau})|$ for some $\overline{\tau} \in (0,1]$.

Let us prove the first item. Divide Ω into a union of four domains:

$$\Omega_1 = \{a < -|b|\} \cap \{a + b\cosh\lambda \neq 0\},$$

$$\Omega_2 = \{a + b\cosh\lambda = 0\} \cap \{b > 0\},$$

$$\Omega_3 = \{a = b\} \cap \{a < 0\},$$

$$\Omega_4 = \left\{|a| + b < 0\right\} \cap \left\{a + b\cos\widetilde{\lambda} < 0\right\} \cap \left\{\widetilde{\lambda} < \pi\right\}.$$

1. The domain Ω_1 corresponds to the first item of theorem 2, and

$$u(0) = -\frac{\lambda - b\sinh\lambda}{\lambda(a + b\cosh\lambda)}.$$

Obviously, if $b < 0$, then $u(0) > 0$, because $a < 0$. Consider now the case $b \geqslant 0$. Notice that $\lambda + b\sinh\lambda \neq 0$, therefore,

$$u(0) = -\frac{\lambda^2 - b^2\sinh^2\lambda}{\lambda(a + b\cosh\lambda)(\lambda + b\sinh\lambda)} = -\frac{a - b\cosh\lambda}{\lambda(\lambda + b\sinh\lambda)}.$$

This expression is positive for every $b \geqslant 0$. As $u(0) > 0$, we can raise both sides of (15) to the second power. Consider the expression

$$\left(u^2(0) - u^2(\tau)\right) \lambda^2(a + b\cosh\lambda)^2 = (\lambda - b\sinh\lambda)^2 -$$

$$- ((\lambda - b\sinh\lambda)\cosh\lambda\tau + (a + b\cosh\lambda)\sinh\lambda\tau)^2 =$$

$$= -\sinh\lambda\tau\left(2(\lambda - b\sinh\lambda)(a + b\cosh\lambda)\cosh\lambda\tau +\right.$$

$$\left. + \left[(\lambda - b\sinh\lambda)^2 + (a + b\cosh\lambda)^2\right]\sinh\lambda\tau\right) =$$

$$= \frac{1}{2}\sinh\lambda\tau e^{-\lambda\tau}\left(\left(\lambda - a - be^{\lambda}\right)^2 - \left(\lambda + a + be^{-\lambda}\right)^2 e^{2\lambda\tau}\right) \geqslant$$

$$\geqslant \lambda\sinh\lambda\tau e^{-\lambda\tau}\left((\lambda - a) - 2be^{\lambda} - (a + \lambda)\, e^{2\lambda}\right). \tag{16}$$

As $a + \lambda = -|a| + \sqrt{a^2 - b^2} \leqslant -|a| + |a| = 0$, $\lambda - a > 0$, and $\sqrt{(\lambda - a)\,|a + \lambda|} = |b|$, the chain of equalities can be continued:

$$\left(u^2(0) - u^2(\tau)\right) \lambda^2(a + b\cosh\lambda)^2 \geqslant$$

$$\geqslant \lambda\sinh\lambda\tau e^{-\lambda\tau}\left((\lambda - a) - 2\,\mathrm{sign}(b)\, be^{\lambda} + |a + \lambda|\, e^{2\lambda}\right) =$$

$$= \lambda\sinh\lambda\tau e^{-\lambda\tau}\left(\sqrt{\lambda - a} - \mathrm{sign}(b)\sqrt{|a + \lambda|}e^{\lambda}\right)^2 > 0.$$

Non-negativity of the last expression is obvious. Let us prove that the equality to zero is impossible. If $b \leqslant 0$, the inequality is evident. Consider the case $b > 0$. Let $\sqrt{\lambda - a} - \sqrt{|a + \lambda|}e^{\lambda} = 0$. Multiplication by $\sqrt{\lambda - a}$ result in $be^{\lambda} = \lambda - a$, while multiplication by $\sqrt{|a + \lambda|}$ gives $be^{-\lambda} = -\lambda - a$. As $\cosh\lambda = \dfrac{e^{\lambda} + e^{-\lambda}}{2}$,

$$a + b\cosh\lambda = a + \frac{\lambda - a}{2} + \frac{-\lambda - a}{2} = 0.$$

But $a + b\cosh\lambda \neq 0$ in Ω_1. We arrive at a contradiction. It proves inequality (15) for Ω_1.

2. As $\cosh x$ is a positive and increasable function for $x > 0$, the inequality (15) holds for Ω_2.

3. For Ω_3 the inequality is also obvious.

4. Consider Ω_4:

$$u(0) = \frac{b\sin\widetilde{\lambda} - \widetilde{\lambda}}{\widetilde{\lambda}(a + b\cos\widetilde{\lambda})}.$$

As $b < 0$, $\widetilde{\lambda} < \pi$, and $a + b\cos\widetilde{\lambda} < 0$, then $u(0) > 0$. Hence, we can raise both sides of (15) to the second power. Consider the expression

$$\left(u^2(0) - u^2(\tau)\right) \widetilde{\lambda}^2(a + b\cos\widetilde{\lambda})^2 = (\widetilde{\lambda} - b\sin\widetilde{\lambda})^2 -$$

$$- ((\widetilde{\lambda} - b\sin\widetilde{\lambda})\cos\widetilde{\lambda}\tau + (a + b\cos\widetilde{\lambda})\sin\widetilde{\lambda}\tau)^2 \geqslant$$

$$\geqslant \sin^2\widetilde{\lambda}\tau\left((\widetilde{\lambda} - b\sin\widetilde{\lambda})^2 - (a + b\cos\widetilde{\lambda})^2 -\right.$$

$$\left. - 2(\widetilde{\lambda} - b\sin\widetilde{\lambda})(a + b\cos\widetilde{\lambda})\cot\widetilde{\lambda}\right) = \sin^2\widetilde{\lambda}\tau\left(\widetilde{\lambda} - \frac{b + a\cos\widetilde{\lambda}}{\sin\widetilde{\lambda}}\right) > 0.$$

The last inequality is not so obvious. Therefore, we should show that $b+a\cos\widetilde{\lambda}<0$. Suppose $a\cos\widetilde{\lambda}\geqslant -b>0$, then $a^2\cos^2\widetilde{\lambda}\geqslant\widetilde{\lambda}^2+a^2$ or $0\geqslant\widetilde{\lambda}^2+a^2\sin^2\widetilde{\lambda}$. We arrive at a contradiction.

Prove the item II. The domain Λ can be divided into a union of domains:

$$\Lambda_1=\{a>|b|\}\cap\{a+b\cosh\lambda\neq 0\},$$

$$\Lambda_2=\{a=b\}\cap\{b>0\},$$

$$\Lambda_3=\left\{b>|a|\right\}\cap\left\{a+b\cos\widetilde{\lambda}\neq 0\right\},$$

$$\Lambda_4=\left\{b<-|a|\right\}\cap\left(\left\{\widetilde{\lambda}\geqslant\pi\right\}\cup\left\{a+b\cos\widetilde{\lambda}>0\right\}\right),$$

$$\Lambda_5=\left\{|b|>|a|\right\}\cap\left\{a+b\cos\widetilde{\lambda}=0\right\}\cap\left\{b\sin\widetilde{\lambda}>0\right\}.$$

It remains to find a point $\overline{\tau}\in(0,1]$ for each pair of parameters $(a,b)\in\Lambda$, such that $u(0)<|u(\overline{\tau})|$.

1. Consider the set Λ_1. Using the chain of conclusions (16), we obtain

$$\left(u^2(0)-u^2(1)\right)\lambda^2(a+b\cosh\lambda)^2=$$

$$=\lambda\sinh\lambda e^{-\lambda}\left(-2be^{\lambda}-a\left(e^{2\lambda}+1\right)-\lambda\left(e^{2\lambda}-1\right)\right)<$$

$$<\lambda\sinh\lambda e^{-\lambda}\left(2ae^{\lambda}-a\left(e^{2\lambda}+1\right)-\lambda\left(e^{2\lambda}-1\right)\right)<$$

$$<\lambda\sinh\lambda e^{-\lambda}\left(-a\left(e^{\lambda}-1\right)^2-\lambda\left(e^{2\lambda}-1\right)\right)<0.$$

This implies that $u^2(0)<u^2(1)$ and $u(0)\leqslant|u(0)|<|u(1)|$.

2. On Λ_2: $u(0)+u(1)=-\dfrac{1}{b}<0$. Hence, $u(0)<-u(1)\leqslant|u(1)|$.

3. Let us consider the set Λ_3. Take the point $\widehat{\tau}=1-\dfrac{\alpha}{\widetilde{\lambda}}$, where $\alpha=\arccos\left(\dfrac{a}{b}\right)$. This point belongs to the interval $(0,1)$, when either $\widetilde{\lambda}\geqslant\pi$ or $a-b\cos\widetilde{\lambda}>0$. Introduce the function

$$g_+(\tau)=(u(0)+u(\tau))\widetilde{\lambda}(a+b\cos\widetilde{\lambda}).$$

The value at the point $\widehat{\tau}$ is

$$g_+(\widehat{\tau})=(u(0)+u(\widehat{\tau}))\widetilde{\lambda}(a+b\cos\widetilde{\lambda})=b\sin\widetilde{\lambda}-\widetilde{\lambda}+b\sin\alpha-$$

$$-a\sin(\widetilde{\lambda}-\alpha)-\widetilde{\lambda}\cos(\widetilde{\lambda}-\alpha)=b\sin\widetilde{\lambda}-\widetilde{\lambda}+b\frac{\widetilde{\lambda}}{b}-$$

$$-\frac{a^2}{b}\sin\widetilde{\lambda}+\frac{a\widetilde{\lambda}}{b}\cos\widetilde{\lambda}-\frac{a\widetilde{\lambda}}{b}\cos\widetilde{\lambda}-\frac{\widetilde{\lambda}^2}{b}\sin\widetilde{\lambda}=0.$$

The derivative of the function at $\widehat{\tau}$ is

$$g'_+(\widehat{\tau})=(au(\widehat{\tau})+bu(1-\widehat{\tau}))\widetilde{\lambda}(a+b\cos\widetilde{\lambda})=$$

$$=ab\sin\alpha-a^2\sin(\widetilde{\lambda}-\alpha)-a\widetilde{\lambda}\cos(\widetilde{\lambda}-\alpha)+$$

$$+b^2\sin(\widetilde{\lambda}-\alpha)-ab\sin\alpha-b\widetilde{\lambda}\cos\alpha=-\widetilde{\lambda}(a+b\cos\widetilde{\lambda})\neq 0.$$

This means that at the point $\widehat{\tau} \in (0,1)$ the function has no extremum, i. e. it changes sign. So, we can find a point $\overline{\tau}$ either from the right or from the left of the point $\widehat{\tau}$ (it depends upon the sign of $a + b\cos\widetilde{\lambda}$) to get $u(0) < |u(\overline{\tau})|$.

Consider the case with $\widetilde{\lambda} < \pi$ and $a - b\cos\widetilde{\lambda} \leqslant 0$. Take the point $\overline{\tau} = 1$:

$$(u^2(0) - u^2(1))\widetilde{\lambda}^2(a + b\cos\widetilde{\lambda})^2 =$$

$$= \left(b\sin\widetilde{\lambda} - \widetilde{\lambda}\right)^2 - \left(a\sin\widetilde{\lambda} + \widetilde{\lambda}\cos\widetilde{\lambda}\right)^2 =$$

$$= 2\widetilde{\lambda}\sin\widetilde{\lambda}(\widetilde{\lambda}\sin\widetilde{\lambda} - b - a\cos\widetilde{\lambda}).$$

If $0 < \widetilde{\lambda} < \pi$ and $b > |a|$, then $b + a\cos\widetilde{\lambda} > 0$. Show that $\widetilde{\lambda}\sin\widetilde{\lambda} < b + a\cos\widetilde{\lambda}$. Consider the difference

$$(b + a\cos\widetilde{\lambda})^2 - (\widetilde{\lambda}\sin\widetilde{\lambda})^2 =$$

$$= b^2 + a^2\cos^2\widetilde{\lambda} + 2ab\cos\widetilde{\lambda} - b^2\sin^2\widetilde{\lambda} + a^2\sin^2\widetilde{\lambda} =$$

$$= b^2\cos^2\widetilde{\lambda} + a^2 + 2ab\cos\widetilde{\lambda} = (a + b\cos\widetilde{\lambda})^2 > 0.$$

4. Take now the point $\widehat{\tau} = 1 - \dfrac{\alpha}{\widetilde{\lambda}}$, where $\alpha = \arccos\left(-\dfrac{a}{b}\right)$. In Λ_4 this point belongs to the interval $(0,1)$. The function

$$g_-(\tau) = (u(0) - u(\tau))\widetilde{\lambda}(a + b\cos\widetilde{\lambda})$$

takes the value

$$g_-(\widehat{\tau}) = (u(0) - u(\widehat{\tau}))\widetilde{\lambda}(a + b\cos\widetilde{\lambda}) = b\sin\widetilde{\lambda} - \widetilde{\lambda} - b\sin\alpha +$$

$$+ a\sin(\widetilde{\lambda} - \alpha) + \widetilde{\lambda}\cos(\widetilde{\lambda} - \alpha) = b\sin\widetilde{\lambda} - \widetilde{\lambda} + b\frac{\widetilde{\lambda}}{b} -$$

$$- \frac{a^2}{b}\sin\widetilde{\lambda} + \frac{a\widetilde{\lambda}}{b}\cos\widetilde{\lambda} - \frac{a\widetilde{\lambda}}{b}\cos\widetilde{\lambda} - \frac{\widetilde{\lambda}^2}{b}\sin\widetilde{\lambda} = 0.$$

The derivative at this point is equal to

$$g'_-(\widehat{\tau}) = -(au(\widehat{\tau}) + bu(1 - \widehat{\tau}))\widetilde{\lambda}(a + b\cos\widetilde{\lambda}) =$$

$$= -ab\sin\alpha + a^2\sin(\widetilde{\lambda} - \alpha) + a\widetilde{\lambda}\cos(\widetilde{\lambda} - \alpha) -$$

$$- b^2\sin(\widetilde{\lambda} - \alpha) + ab\sin\alpha + b\widetilde{\lambda}\cos\alpha = -\widetilde{\lambda}(a + b\cos\widetilde{\lambda}) \neq 0.$$

We can find the point $\overline{\tau}$, for which $u(0) < |u(\overline{\tau})|$, like in the item 3.

5. In the domain Λ_5

$$u^2(0) - u^2(1) = -\frac{1}{\widetilde{\lambda}^2} < 0.$$

The theorem is proved. □

Given theorem can be viewed as the generalization of the classical Lyapunov result for the scalar case.

6. Conclusion. A criterion of the exponential stability for the single delay equation is presented. The conditions of the criterion depend on the Lyapunov matrix-function of the equation. In the future we plan to extend the result for a wider class of linear time-delay systems.

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Статья рекомендована к печати проф. В. Л. Харитоновым

Статья принята к печати 25 октября 2012 г.