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On a q-analogue of the Sturm–Liouville operator with discontinuity conditions

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Abstract

In this paper, a q-analogue of the Sturm–Liouville problem with discontinuity condition on a finite interval is studied. It is proved that the q-Sturm–Liouville problem with discontinuity conditions is self-adjoint in $L^2_q(0,\pi)$. The completeness theorem and the sampling theorem are proved.

Keywords: q-Sturm–Liouville operator, completeness of eigenfunctions, self-adjoint operator.

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1. Introduction

Let us consider a q-analogue of the Sturm–Liouville equation in the form

$$l(y) := -\frac{1}{q} D_{q^{-1}} D_q y(t) + u(t)y(t) = \nu y(t), \quad 0 < t < \pi, \quad \nu \in \mathbb{C}, \tag{1}$$

together with the discontinuity conditions at a point $a \in (0,\pi)$

$$y(a+0) = \alpha y(a-0), \quad D_{q^{-1}} y(a+0) = \alpha^{-1} D_{q^{-1}} y(a-0), \tag{2}$$

and boundary conditions

$$y(0) = y(\pi) = 0, \tag{3}$$

where $0 < q < 1$, $u(t) \in L^2_q(0,\pi)$ is a real function, α is real; $\alpha \neq 1$, $\alpha > 0$.

In [1], it is worth mentioning that this work is based on the q-difference operator, which is attributed to Jackson. In recent years, many papers have been

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published subject to boundary value problems consisting of a q-Jackson derivative in the classical Sturm–Liouville problem. In [2–4], the q-analogues of Sturm–Liouville problems are investigated, and a space of boundary values of the minimal operator is created, and all maximal dissipative, self-adjoint, maximal accretive operators are described, and other extensions of the q-analogue of Sturm–Liouville operators in terms of boundary conditions are raised. A theorem on completeness of the system of eigenfunctions and associated functions of dissipative operators are proved by using the Lidskii's theorem.

In the current decade, many authors have investigated the q-sampling theory of signal analysis. [6–8] are the first studies in this subject. In these studies, the construction of expansions in the q-Fourier series [5] was followed by the derivation of the q-sampling theorems. The sampling theory associated with q-type of Sturm–Liouville equations is conceived (see [9]). In [10], M. Annaby and Z. Mansour obtained asymptotic formulae for eigenvalues and eigenfunctions of q-type of Sturm–Liouville problems.

In [11–13], B. Allahverdiev and H. Tuna investigated the continuous spectrum of the singular q-Sturm–Liouville operators and established some criteria under which the q-Sturm–Liouville equation is of limit-point case at infinity. In [14], authors established a Parseval equality and an expansion formula in eigenfunctions for a singular q-Sturm–Liouville operator on the entire line. (Also, B. Allahverdiev and H. Tuna investigated the resolvent operator of a singular q-Dirac system (see [15])). In [16], the spectral properties of the eigenvalues and the eigenfunctions of the q-Sturm-Liouville boundary value problem are investigated.

Also, there are many physical models involving q-difference and their related problems in [21–23]. In these studies, several physical models involving q-functions, q-derivatives, q-integrals and their related problems are investigated. However, to our knowledge, there is no study of the general problem as we do in the present setting. At this point, it is worth mentioning that this work is based on the q-difference operator, which is attributed to Jackson and a similar study of the Stum–Liouville systems generated by the Askey–Wilson derivative.

In [24] and [25], the Sturm–Liouville problems are generalized by a fractional derivative of order α, $0 < \alpha \leqslant 1$. The numerical solutions of fractional Sturm–Liouville problems were examined.

2. Preliminaries

In this section, we give some of the q-notations and we use these q-notations throughout the paper. These standard notations are based on [18].

If $q \in \mathbb{R}$ is fixed, a subset A of $\mathbb{C}$ is called q-geometric if $qt \in A$ whenever $t \in A$. Let h be a function, real or complex valued, defined on a q-geometric set A, $q \neq 1$. Let q be a positive number with $0 < q < 1$. The q-difference operator D_q is defined as

$$D_q h(t) = \frac{h(t) - h(qt)}{t(1-q)} \quad \forall t \in A \setminus \{0\}.$$

The q^{-1}-derivative at zero is defined by

$$D_{q^{-1}} h(t) = \lim_{n \to -\infty} \frac{h(tq^{-n}) - h(0)}{tq^{-n}} = D_q h(0).$$

When required, we will replace q by q^{-1}. We can demonstrate the correctness of the following facts using the definition and will often use it

$$D_{q^{-1}}h(t) = (D_q h)(q^{-1}t), \quad D_q^2 h(q^{-1}t) = qD_q[D_q h(q^{-1}t)] = D_{q^{-1}}D_q h(t).$$

Associated with this operator, there is a non-symmetric formula for the q-differentiation of a product of the functions h and g defined in the q-geometric set.

$$D_q[h(t)g(t)] = h(qt)D_q g(t) + g(t)D_q h(t). \tag{4}$$

The q-integral called Jackson integral is given by

$$\int_0^\pi h(t)d_q t = (1-q)\sum_{n=0}^\infty h(\pi q^n)\pi q^n.$$

$L_q^2(0,\pi)$ is the space of all complex-valued functions defined in $(0,\pi)$ with the norm

$$\|h\| = \left(\int_0^\pi |h(t)|^2 d_q t\right)^{1/2} < \infty$$

and it is a separable Hilbert space (see [6]) with the inner product

$$\langle h, g\rangle = \int_0^\pi h(t)\overline{g(t)}d_q t.$$

The space $C_q^2(0)$ is the space of all continuous functions with the q-derivative first order at the point zero.

Definition. A function f that is defined on a q-geometric set A, $0 \in A$, is said to be q-regular at zero if

$$\lim_{n\to\infty} f(tq^n) = f(0), \quad \forall t \in A.$$

If h and g are both q-regular at zero, there is a rule of q-integration by parts given by

$$\int_0^\pi g(t)D_q h(t)d_q t = (hg)(\pi) - (hg)(0) - \int_0^\pi D_q g(t)h(qt)d_q t.$$

An important special case, we have

$$\int_0^\pi D_q h(t)d_q t = (h)(\pi) - (h)(0). \tag{5}$$

The q-Wronskian of two functions h and g is defined as

$$W_q(h,g)(t) = h(t)D_q g(t) - g(t)D_q h(t).$$

Lemma 1 (see [2]). *Let $h(\,\cdot\,)$, $g(\,\cdot\,)$ in $L_q^2(0,\pi)$ be defined on $[0,q^{-1}\pi]$. Then, for $t \in (0,\pi]$ we have*

$$\langle D_q h, g\rangle = h(\pi)\overline{g(\pi q^{-1})} - \lim_{n\to\infty} h(\pi q^n)\overline{g(\pi q^{n-1})} + \left\langle h, -\frac{1}{q}D_{q^{-1}}g\right\rangle, \tag{6}$$

$$\left\langle -\frac{1}{q}D_{q^{-1}}h, g\right\rangle = \lim_{n\to\infty} h(\pi q^{n-1})\overline{g(\pi q^n)} - h(\pi q^{-1})\overline{g(\pi)} + \langle h, D_q g\rangle . \tag{7}$$

3. The self-adjoint problem

THEOREM 1. *The q-analogue of the Sturm–Liouville eigenvalue problem* (1)–(3) *is self-adjoint on* $C_q^2(0) \cap L_q^2(0,\pi)$.

Proof. We first prove that $\kappa(\,\cdot\,)$, $\sigma(\,\cdot\,)$ in $L_q^2(0,\pi)$, we have the following q-Lagrange's identity

$$\int_0^\pi \left(l\kappa(t)\overline{\sigma(t)} - \kappa(t)\overline{l\sigma(t)}\right) d_q t = [\kappa,\sigma](\pi) - \lim_{n\to\infty}[\kappa,\sigma](\pi q^n), \tag{8}$$

where

$$[\kappa,\sigma](t) := \kappa(t)\overline{D_{q^{-1}}\sigma(t)} - D_{q^{-1}}\kappa(t)\overline{\sigma(t)}.$$

Applying (7) with $h(t) = D_q\kappa(t)$ and $g(t) = \sigma(t)$, we obtain the following:

$$\left\langle -\frac{1}{q}D_{q^{-1}}D_q\kappa(t), \sigma(t)\right\rangle =$$
$$= -(D_q\kappa)(\pi q^{-1})\overline{\sigma(\pi)} + \lim_{n\to\infty}(D_q\kappa)(\pi q^{n-1})\overline{\sigma(\pi q^n)} + \langle D_q\kappa, D_q\sigma\rangle =$$
$$= -D_{q^{-1}}\kappa(\pi)\overline{\sigma(\pi)} + \lim_{n\to\infty} D_{q^{-1}}\kappa(\pi q^n)\overline{\sigma(\pi q^n)} + \langle D_q\kappa, D_q\sigma\rangle .$$

Applying (6) to $h(t) = \kappa(t)$, $g(t) = D_q\sigma(t)$, we obtain

$$\langle D_q\kappa, D_q\sigma\rangle =$$
$$= \kappa(\pi)\overline{D_q\sigma(\pi q^{-1})} - \lim_{n\to\infty}\kappa(\pi q^n)\overline{D_q\sigma(\pi q^{n-1})} + \left\langle \kappa, -\frac{1}{q}D_{q^{-1}}D_q\sigma\right\rangle =$$
$$= \kappa(\pi)\overline{D_{q^{-1}}\sigma(\pi)} - \lim_{n\to\infty}\kappa(\pi q^n)\overline{D_{q^{-1}}\sigma(\pi q^n)} + \left\langle \kappa, -\frac{1}{q}D_{q^{-1}}D_q\sigma\right\rangle.$$

Therefore,

$$\left\langle -\frac{1}{q}D_{q^{-1}}D_q\kappa(t), \sigma(t)\right\rangle = [\kappa,\sigma](\pi) - \lim_{n\to\infty}[\kappa,\sigma](\pi q^n) + \left\langle \kappa, -\frac{1}{q}D_{q^{-1}}D_q\sigma\right\rangle. \tag{9}$$

The Lagrange's identity (8) is the result of (9) and the reality of $u(t)$. Letting $\kappa(\cdot)$, $\sigma(\,\cdot\,)$ in $C_q^2(0)$ and assuming that they satisfy (2), (3), we obtain the following:

$$\kappa(0) = 0, \quad \sigma(0) = 0. \tag{10}$$

The continuity of $\kappa(\,\cdot\,)$, $\sigma(\,\cdot\,)$ at zero implies that

$$\lim_{n\to\infty}[\kappa,\sigma](\pi q^n) = [\kappa,\sigma](0).$$

Then (9) will be

$$\left\langle -\frac{1}{q}D_{q^{-1}}D_q\kappa, \sigma\right\rangle = [\kappa,\sigma](\pi) - [\kappa,\sigma](0) + \left\langle \kappa, -\frac{1}{q}D_{q^{-1}}D_q\sigma\right\rangle.$$

From (10), we have

$$[\kappa,\sigma](0)=\kappa(0)\overline{D_{q^{-1}}\sigma(0)}-D_{q^{-1}}\kappa(0)\overline{\sigma(0)}=0.$$

Similarly,

$$[\kappa,\sigma](\pi)=\kappa(\pi)\overline{D_{q^{-1}}\sigma(\pi)}-D_{q^{-1}}\kappa(\pi)\overline{\sigma(\pi)}=0.$$

Since $u(t)$ is real valued,

$$\begin{aligned}\langle l(\kappa),\sigma\rangle &= \left\langle -\frac{1}{q}D_{q^{-1}}D_q\kappa(t)+u(t)\kappa(t),\sigma(t)\right\rangle = \\ &= \left\langle -\frac{1}{q}D_{q^{-1}}D_q\kappa(t),\sigma(t)\right\rangle+\langle u(t)\kappa(t),\sigma(t)\rangle = \\ &= \left\langle \kappa,-\frac{1}{q}D_{q^{-1}}D_q\sigma\right\rangle+\langle \kappa(t),u(t)\sigma(t)\rangle=\langle\kappa,l(\sigma)\rangle,\end{aligned}$$

i.e. l is a self-adjoint operator. □

4. Completeness of eigenfunctions

LEMMA 2. *Let h and g be q-regular at zero. The Wronskian $W_q(h,g)(t)$ of the q-analogue of the Sturm–Liouville problem (1) does not defend on t.*

Proof. The proof can be done similarly to [9]. □

Let $\eta(t,\nu)$ be the solution of equation (1) with discontinuity conditions (2) and initial conditions

$$\eta(0,\nu)=0,\quad D_{q^{-1}}\eta(0,\nu)=1, \tag{11}$$

and $\xi(t,\nu)$ be the solution of (1) with discontinuity conditions (2) and

$$\xi(\pi,\nu)=0,\quad D_{q^{-1}}\xi(\pi,\nu)=1.$$

Since the q-Wronskian is independent of t, we can evaluate it at $t=0$ and use the above conditions at ξ in order to write

$$W_q(\eta,\xi)(\nu)=W_q(\nu)=-\xi(0,\nu). \tag{12}$$

It follows from condition (3) that $W_q(\nu)=0$ if and only if ν is an eigenvalue of the q-analogue of the Sturm–Liouville problem (1).

Denote by ν_n the eigenvalues and by α_n the normalized numbers of problem (1), (2):

$$\alpha_n=\int_0^\pi \eta^2(t,\nu_n)d_qt. \tag{13}$$

The numbers $\{\nu_n,\alpha_n\}$ are said to be the spectral date of the problem (1)–(3). λ_n^0 and α_n^0 are eigenvalues and normalized numbers, respectively, in the case of $u(t)\equiv 0$ in the equation (1), where $u(t)$ is a potential function. Then there exists a sequence β_n such that

1) $\xi(t,\nu_n)=\beta_n\eta(t,\nu_n)$, $\beta_n\neq 0$,
2) $\beta_n\alpha_n=-\dot{W}_q(\nu_n)$, where $\dot{W}_q(\nu)=D_qW_q(\nu)$ (respect to ν),

3) $\sqrt{\nu_n} = \sqrt{\nu_n^0} + \dfrac{\epsilon_n}{\sqrt{\nu_n^0}} + \dfrac{a_n}{\sqrt{\nu_n^0}}$, $\{\epsilon_n\} \in l_2$, $a_n = \dfrac{1}{\alpha_n}\displaystyle\int_0^\pi f(t)\xi(t,\lambda_n)d_qt$, (see [20]),

4) $\alpha_n = \alpha_n^0 + \dfrac{\delta_n}{\sqrt{\nu_n^0}}$, $\{\delta_n\} \in l_2$.

LEMMA 3. *The eigenvalues and eigenfunctions of the q-analogue of the Sturm–Liouville problem* (1)–(3) *have the following properties*:

i) the eigenvalues are real;

ii) eigenfunctions that belong to different eigenvalues are orthogonal;

iii) all eigenvalues are simple.

Proof.

i) Let ν_0 be an eigenvalue with an eigenfunction $\eta_0(\,\cdot\,)$. Then

$$\langle l(\eta_0), \eta_0\rangle = \langle \eta_0, l(\eta_0)\rangle\,.$$

Since $l(\eta_0) = \nu_0\eta_0$ then

$$(\nu_0 - \overline{\nu_0})\int_0^\pi |\eta_0(x)|^2 d_qx.$$

Since $\eta_0(\,\cdot\,)$ is non-trivial then $\nu_0 = \overline{\nu_0}$, which proves *i*).

ii) Let ν, μ be two distinct eigenvalues with the corresponding eigenfunctions $\eta(\,\cdot\,)$, $\xi(\,\cdot\,)$, respectively. Then

$$(\nu - \mu)\int_0^\pi \eta(t)\overline{\xi(t)}d_qt = 0.$$

Since $\nu \equiv \mu$ then $\eta(\,\cdot\,)$ and $\xi(\,\cdot\,)$ are orthogonal.

iii) Let ν_0 be an eigenvalue with two eigenfunctions $\eta_1(\cdot)$ and $\eta_2(\cdot)$. From [2, Corollary 2.15] we can prove that the functions $\{\eta_1(\,\cdot\,), \eta_2(\,\cdot\,)\}$ are linearly dependent by proving that their q-Wronskian vanishes at $t = 0$. Indeed,

$$W_q(\eta_1,\eta_2)(0) = \eta_1(0)D_q\eta_2(0) - \eta_2(0)D_q\eta_1(0) = \\ = \eta_1(0)D_{q^{-1}}\eta_2(0) - \eta_2(0)D_{q^{-1}}\eta_1(0) = 0,$$

since both η_1 and η_2 satisfy (3).

□

THEOREM 1. *The system of eigenfunctions $\{\eta(t,\nu_n)\}_{n\geqslant 0}$ of the problem* (1)–(3) *is complete in $L_q^2(0,\pi)$.*

Proof. Denote

$$G(t,\tau,\nu) = -\frac{1}{W_q(\nu)}\begin{cases}\eta(t,\nu)\xi(\tau,\nu), & t \leqslant \tau,\\ \eta(\tau,\nu)\xi(t,\nu), & t \geqslant \tau,\end{cases}$$

and consider the function

$$Y(t,\nu) = \int_0^\pi G(t,\tau,\nu)h(\tau)d_q\tau =$$

$$= -\frac{1}{W_q(\nu)}\left[\xi(t,\nu)\int_0^t \eta(\tau,\nu)h(\tau)d_q\tau + \eta(t,\nu)\int_t^\pi \xi(\tau,\nu)h(\tau)d_q\tau\right].$$

The function $G(t,\tau,\nu)$ is called the q-type Green function for the q-analogue of the Sturm–Liouville problem (1)–(3). $G(t,\tau,\nu)$ is the kernel of the inverse operator for the q-analogue of the Sturm–Liouville problem, i.e. $Y(t,\nu)$ is the solution of the problem

$$-\frac{1}{q}D_{q^{-1}}D_qY(t) + \{-\nu + u(t)\}Y(t) = h(t), \quad t\in[0,\pi], \quad \nu\in\mathbb{C}, \tag{14}$$

satisfies the discontinuity condition (2) and the boundary condition (3). Furthermore, taking into account (13), we get the following:

$$\begin{aligned}
&\operatorname*{Res}_{\nu=\nu_n} Y(t,\nu) = \\
&\quad = -\frac{1}{\dot W_q(\nu_n)}\left[\xi(t,\nu_n)\int_0^t \eta(\tau,\nu_n)h(\tau)d_q\tau + \eta(t,\nu_n)\int_t^\pi \xi(\tau,\nu_n)h(\tau)d_q\tau\right] = \\
&\quad = -\frac{\beta_n}{\dot W_q(\nu)}\eta(t,\nu_n)\int_0^\pi \eta(\tau,\nu_n)h(\tau)d_q\tau = \frac{1}{\alpha_n}\eta(t,\nu_n)\int_0^\pi \eta(\tau,\nu_n)h(\tau)d_q\tau.
\end{aligned} \tag{15}$$

Let the function $h(t)\in L_q^2(0,\pi)$ be such that

$$\int_0^\pi \eta(\tau,\nu_n)h(\tau)d_q\tau = 0, \quad n=1,2,\ldots$$

Then in view of (15), $\operatorname*{Res}_{\nu=\nu_n} Y(t,\nu) = 0$ and consequently for each fixed $t\in[0,\pi]$, the function $Y(t,\nu)$ is entire in ν. Furthermore, for $\rho\in G_\delta = \{\rho : |\rho-\rho_{k,0}|\geqslant\delta,$ $k=\pm1,\pm2,\ldots\}$ and $|\rho|\geqslant\rho^*$ for sufficiently large $\rho^*=\rho^*(\delta)$, where $\nu=\rho^2$, $\rho_{k,0}$ are the zeros of the function

$$W_q^0(\rho) = \alpha^+\frac{\sin\rho\pi}{\rho} + \alpha^-\frac{\sin\rho(2a-\pi)}{\rho},$$

δ is a fixed positive number, $\alpha^\pm = \frac{1}{2}\left(\alpha\pm\frac{1}{\alpha}\right)$ (see [17]), ρ^* is rather large, the inequality

$$|W_q(\nu)| \geqslant \frac{C_\delta}{|\rho|}e^{|\operatorname{Im}\rho|\pi},$$

and consequently the inequality

$$|Y(t,\nu)| \leqslant \frac{C'_\delta}{|\rho|}, \quad \rho\in G_\delta, \quad |\rho|\geqslant\rho^*,$$

are fulfilled (see [17]). Using the maximum principle and Liouville's theorem, we conclude that $Y(t,\nu)\equiv 0$. From this and (14) it follows that $h(t)=0$ a.e. on $(0,\pi)$. Thus, the theorem is proved. □

5. The q-sampling theory

THEOREM 2. *Let $\eta(t,\nu)$ and $\xi(t,\nu)$ be the solutions of (1) selected as above. Then all functions h of the form*

$$h(\nu)=\int_0^\pi u(t)\eta(t,\nu)d_qt,\quad u\in L_q^2(0,\pi), \tag{16}$$

can be written as the Lagrange-type sampling expansion:

$$h(\nu)=\sum_{n=0}^{\infty}h(\nu_n)\frac{W_q(\nu)}{\dot{W}_q(\nu_n)(\nu-\nu_n)},$$

where $W_q(\nu)$ is the q-Wronskian of the functions $\eta(t,\nu)$ and $\xi(t,\nu)$.

Proof. We multiply equation (1) with $\eta(t,\nu_n)$. Then we again consider equation (1), but replace ν by ν_n and multiply this last equation by $\eta(t,\nu)$. Subtracting the two results yields

$$(\nu-\nu_n)\eta(t,\nu)\eta(t,\nu_n)=D_q^2\eta(q^{-1}t,\nu_n)\eta(t,\nu)-D_q^2\eta(q^{-1}t,\nu)\eta(t,\nu_n).$$

From the rule for the q-differentiation of a product (4), we can write

$$(\nu-\nu_n)\eta(t,\nu)\eta(t,\nu_n)=D_q\left[D_q\eta(q^{-1}t,\nu_n)\eta(t,\nu)-D_q\eta(q^{-1}t,\nu)\eta(t,\nu_n)\right].$$

If we apply a q-integration by means of (5) we obtain

$$\begin{aligned}(\nu-\nu_n)&\int_0^\pi\eta(t,\nu)\eta(t,\nu_n)d_qt=\\&=\int_0^\pi D_q\big[D_q\eta(q^{-1}t,\nu_n)\eta(t,\nu)-D_q\eta(q^{-1}t,\nu)\eta(t,\nu_n)\big]d_qt=\\&=D_q\eta(q^{-1}\pi,\nu_n)\eta(\pi,\nu)-D_q\eta(q^{-1}\pi,\nu)\eta(\pi,\nu_n)-\\&\qquad-\big(D_q\eta(q^{-1}0,\nu_n)\eta(0,\nu)-D_q\eta(q^{-1}0,\nu)\eta(0,\nu_n)\big).\end{aligned}$$

From conditions (3) and (11) we get the following:

$$\begin{aligned}(\nu-\nu_n)&\int_0^\pi\eta(t,\nu)\eta(t,\nu_n)d_qt=\\&=D_q\eta(q^{-1}\pi,\nu_n)\eta(\pi,\nu)-D_q\eta(q^{-1}\pi,\nu)\eta(\pi,\nu_n)=\\&=\eta(\pi,\nu)D_{q^{-1}}\eta(\pi,\nu_n)-\eta(\pi,\nu_n)D_{q^{-1}}\eta(\pi,\nu).\end{aligned}$$

From (12), we have the following:

$$(\nu-\nu_n)\int_0^\pi\eta(t,\nu)\eta(t,\nu_n)d_qx=W_q(\nu)D_{q^{-1}}\eta(\pi,\nu_n)-W_q(\nu_n)D_{q^{-1}}\eta(\pi,\nu).$$

From ν_n eigenvalues being zeros of the characteristic function $W_q(\nu)$ of the q-analogue of the Sturm–Liouville problem (1)–(3), we obtain $W_q(\nu_n)=0$. Then, we have

$$(\nu-\nu_n)\int_0^\pi\eta(t,\nu)\eta(t,\nu_n)d_qt=W_q(\nu)D_{q^{-1}}\eta(\pi,\nu_n).$$

As a result,

$$\int_0^\pi \eta(t,\nu)\eta(t,\nu_n)d_qt = \frac{W_q(\nu)D_{q^{-1}}\eta(\pi,\nu_n)}{(\nu-\nu_n)},$$

and taking the limit a $\nu \to \nu_n$ gives

$$\int_0^\pi |\eta(t,\nu_n)|^2 d_qt = \dot{W}_q(\nu_n)D_{q^{-1}}\eta(\pi,\nu_n).$$

Therefore, we can apply Kramer's lemma (see [19]) and write an integral transform of the form (16) as

$$h(\nu) = \sum_{n=0}^{\infty} h(\nu_n)\frac{W_q(\nu)}{\dot{W}_q(\nu_n)(\nu-\nu_n)}.$$

□

Example. Consider the following q-Sturm–Liouville problem:

$$-\frac{1}{q}D_{q^{-1}}D_q y(t) = \nu y(t), \quad 0<t<\pi, \quad \nu\in\mathbb{C},$$

together with the discontinuity conditions at a point $a\in(0,\pi)$

$$y(a+0) = \alpha y(a-0), \quad D_{q^{-1}}y(a+0) = \alpha^{-1}D_{q^{-1}}y(a-0),$$

and boundary conditions

$$y(0) = y(\pi) = 0,$$

where $0<q<1$, α is real; $\alpha\neq 1$, $\alpha>0$. The system of functions $\{\eta_0(t,\nu_n^0)\}_{n=1}^{\infty}$, where $\nu=\rho^2$

$$\eta_0(t,\nu) = \begin{cases} c\dfrac{\sin\rho t}{\rho}, & 0<t\leqslant a, \\ \alpha^+\dfrac{\sin\rho t}{\rho} + \alpha^-\dfrac{\sin\rho(2a-t)}{\rho}, & a<t\leqslant\pi, \end{cases}$$

where $\alpha^{\pm} = \frac{1}{2}\left(\alpha \pm \frac{1}{\alpha}\right)$, is complete in the space $L_q^2(0,\pi)$.

6. Conclusion

In this paper, a q-analogue of the Sturm–Liouville problem with discontinuity condition on a finite interval is studied. It is shown that the eigenfunctions of this problem are in the form of a complete system. A sampling theorem is proved for integral transforms whose kernels are basic functions and the integral is of Jackson's type. Finally, it is proved that the q-analogue of the Sturm–Liouville problem with discontinuity conditions is self-adjoint in $L_q^2(0,\pi)$.

In future studies, the main equation for the q-analogue of the Sturm–Liouville problem can be obtained. The Weyl solution and the Weyl function can be defined for the q-analogue of the Sturm–Liouville problem. Uniqueness theorems for the solution of the inverse problem according to the Weyl function and spectral date can be proved.

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Об q-аналоге оператора Штурма–Лиувилля с условиями разрыва

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Аннотация

Исследуется q-аналог задачи Штурма–Лиувилля с условием разрыва на конечном интервале. Доказано, что q-задача Штурма–Лиувилля с условиями разрыва является самосопряженной в $L_q^2(0,\pi)$. Доказаны теорема о полноте и теорема о выборке. Приводится пример, иллюстрирующий полученные результаты.

Ключевые слова: q-оператор Штурма–Лиувилля, полнота собственных функций, самосопряженный оператор.

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