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ON SOME GENERALIZATIONS OF THE RESULTS ABOUT THE DISTRIBUTION OF THE MAXIMUM OF THE CHENTSOV RANDOM FIELD ON POLYGONAL LINES

In this paper we compute the probability $\mathbf{P}\left\{\sup_{t\in[T_1,T_2]}(w(t)-h(t))<0\right\}$, where $w(t)$ is a Wiener process and h is a step-wise linear function. We use it to obtain the distribution of the maximum of the Chentsov random field on polygonal lines. We have considerably expanded a class of such polygonal lines in this paper.

1. Introduction

Let w be a Wiener process and let h be a measurable function. The main purpose of this paper is to find the probability

$$\mathbf{P}\left\{\sup_{t\in[T_1,T_2]}(w(t)-h(t))<0\right\}, \tag{1}$$

for a class of step-wise linear functions h. The probability (1) can not be found in a simple form for general function h. But for special cases useful results were obtained. For instance, Bachelier considered a case when $T_1=0$, $h(t)=b$. Another result was obtained by Malmquist for a linear function h. Earlier Doob has solved the same problem for a limiting case. That is, he found the probability of the form: $\mathbf{P}\left\{\sup_{[0,\infty)}(w(t)-at-b)<0\right\}$. An integral equation for evaluating the probability (1) were proposed in [3], [4] for a large class of differentiable functions h. Klesov and Kruglova [8] expressed probability (1) in terms of n-tuple integral of a function involving exponents and standard Gaussian density if h is a polygonal line with n changes of the direction and $T_1=0$.

We want to generalize this result for the case of arbitrary $T_1\geq 0$. Previously the probability distribution of functionals of the Wiener process like max were investigated on intervals like $[0,T]$ or $[0,\infty)$. We will prove more general result. There are few ways of obtaining this result, using Theorem 2.1 in [8]. In this article the most simple is shown.

We provide an application of this result for finding the distribution of the maximum of the Chentsov random field and the Chentsov random field with a linear drift on a polygonal line with n changes of direction. We discover the exact expressions for the probability distribution of the maximum of the Chentsov random field on polygonal lines which begin at arbitrary point (x_0,y_0) and end at point (x_{n+1},y_{n+1}). Earlier only polygonal lines which connected points $(0,1)$ and $(1,0)$ were considered.

2. Definitions and preliminaries

2.1. **The maximum of the Wiener process with a drift.** To prove our results the following lemmas and theorems are needed.

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Theorem 2.1. *(Doob's Transformation Theorem* [1]*) Let $Y(t)$ be any Gaussian process with $E[Y(t)] = 0$, $\forall t$, and covariance function*

$$R(s,t) = u(s)v(t), \qquad s \le t. \tag{2}$$

If the ratio $a(t) = u(t)/v(t)$ is continuous and strictly increasing together with its inverse $a^{-1}(t)$, then $w(t)$ and $Y(a^{-1}(t))/v(a^{-1}(t))$ are stochastically equivalent processes.

Lemma 2.1. *(*[3]*) Let $T > 0$, $a \ge 0$, $b > 0$. Denote*

$$F(T) = \boldsymbol{P}\left\{\sup_{0\leqslant t\leqslant T} w(t) - at \geqslant b\right\}.$$

Then

$$F(T) = 1 - \Phi[(aT+b)T^{-1/2}] + e^{-2ab}\Phi[(aT-b)T^{-1/2}] \tag{3}$$

The following Theorem is a generalization of Theorem 1 in [3] to a case when h is a polygonal line with n changes of the direction.

Theorem 2.2. *Let $n \ge 1$ and $T > 0$. Let $t_0 = 0$; $t_{n+1} = T$, and let*

$$t_{j-1} < t_j, \qquad 1 \le j \le n+1.$$

Put $\Delta t_j = t_j - t_{j-1}$, $1 \le j \le n+1$. Consider two sequences $\{a_j, 1 \le j \le n+1\}$ and $\{b_j, 1 \le j \le n+1\}$ such that

$$b_1 > 0, \qquad a_j \ge 0, \quad 1 \le j \le n, \qquad a_{n+1} > 0.$$

Set

$$h_j(t) \stackrel{\text{def}}{=} a_j t + b_j, 1 \le j \le n+1.$$

Assume that

$$h_j(t_j) = h_{j+1}(t_j), \qquad 1 \le j \le n,$$

and

$$h_j(t_j) > 0, 1 \le j \le n.$$

Put

$$h(t) \stackrel{\text{def}}{=} b_1 I_{\{0\}}(t) + \sum_{j=1}^{n+1} h_j(t) I_{(t_{j-1},t_j]}(t).$$

Denote $\beta_j = h_j(t_{j-1}) - u_{j-1}$, $1 \le j \le n+1$, where $u_0 = 0$. Then

$$\boldsymbol{P}\left(\sup_{0\le t\le T}(w(t) - h(t)) < 0\right)$$

$$= \int_{-\infty}^{h_1(t_1)} \cdots \int_{-\infty}^{h_n(t_n)} \prod_{j=1}^{n}\left(1 - \exp\left\{\frac{-2\beta_j(h_j(t_j) - u_j))}{\Delta t_j}\right\}\right)$$

$$\times\left(\Phi\left(\frac{a_{n+1}\Delta t_{n+1} + \beta_{n+1}}{\sqrt{\Delta t_{n+1}}}\right) - e^{-2a_{n+1}\beta_{n+1}}\Phi\left(\frac{a_{n+1}\Delta t_{n+1} - \beta_{n+1}}{\sqrt{\Delta t_{n+1}}}\right)\right)$$

$$\times \prod_{j=1}^{n} \varphi_{0,\Delta t_j}(u_j - u_{j-1}) du_1 \ldots du_n. \tag{4}$$

2.2. **The maximum of tne Chentsov random field on polygonal lines.** Let us consider a two-parameter Chentsov random field $X(s,t)$. Denote $D=[0,1]\times[0,1]$.

Definition 2.1. Let $\{X(s,t): s,t\geq 0\}$ be a standard Chentsov random field of two parameters, that is a separable real Gaussian stochastic process such that:

(1) $X(0,t)=X(s,0)=0$ for all $s,t\in[0,1]$;
(2) $\mathsf{E}[X(s,t)]=0$ for all $(s,t)\in D$;
(3) $\mathsf{E}[X(s,t)X(s_1,t_1)]=\min\{s,s_1\}\min\{t,t_1\}$ for all $(s,t)\in D$ and $(s_1,t_1)\in D$.

The probability distribution of the supremum of $X(s,t)$ on a class of polygonal lines with one change of direction were obtained by Paranjape and Park [3]. Also Park and Skoug [5] found the probability that $X(s,t)$ crosses a barrier of the type $ast+bs+ct+d$ on the boundary $\partial\Lambda$, where $\Lambda=[0,S]\times[0,T]$ is a rectangle. Later I. Klesov [6] considered the probability of the form

$$P(L,g)=\mathbf{P}\left\{\sup_{L}X(s,t)-g(s,t)<0\right\}, \tag{5}$$

where X is a Chentsov random field on D, L is a polygonal line with one change of direction and g is a linear function on D. Kruglova [7] considered the probability of the form (5), where $g(s,t)=\lambda$ and L is a polygonal line with several changes of direction.

Let $n\geq 1$. Let the polygonal line L have n points of break $Q_1,\dots,Q_n$ with coordinates $(x_1,y_1),\dots,(x_n,y_n)$ respectively and be given by the formula:

$$L=\{(s,t): t=v(s), s\in[0,1]\}, \tag{6}$$

where

$$v(s)=I_{\{0\}}(s)+\sum_{i=1}^{n+1}\left(-\frac{(y_{i-1}-y_i)s}{x_i-x_{i-1}}+\frac{x_iy_{i-1}-x_{i-1}y_i}{x_i-x_{i-1}}\right)I_{(x_{i-1};x_i]}(s). \tag{7}$$

Let

$$0=x_0<x_1<\dots<x_n<x_{n+1}=1, \tag{8}$$

$$1=y_0>y_1\geq\dots\geq y_n>y_{n+1}=0. \tag{9}$$

Theorem 2.3. *Let $\{X(s,t): s,t\geq 0\}$ be a standard Chentsov random field on the unit square. Let $u_0=0$. Let L be a polygonal line, which has n points of break and which is given by the formula (6). Let the coordinates of these points satisfy the conditions (8)-(9). Put $\Delta_i=\frac{x_i}{y_i}, i=\overline{0,n}$. Then for all $\lambda>0$*

$$P_n(\lambda)=\boldsymbol{P}\left\{\sup_{(s;t)\in L}X(s;t)<\lambda\right\}=\int_{-\infty}^{\frac{\lambda}{y_1}}\dots\int_{-\infty}^{\frac{\lambda}{y_n}}\left(1-\exp\left\{-2\lambda\left(\frac{\lambda}{y_n}-u_n\right)\right\}\right)\times$$

$$\times\prod_{i=1}^{n}\left(1-\exp\left\{-\frac{2\left(\frac{\lambda}{y_{i-1}}-u_{i-1}\right)\left(\frac{\lambda}{y_i}-u_i\right)}{(\Delta_i-\Delta_{i-1})}\right\}\right)\varphi_{0,\Delta_i-\Delta_{i-1}}(u_i-u_{i-1})\,du_1\dots du_n,$$

where $\varphi_{0,\Delta}(u)=\frac{e^{-\frac{u^2}{2\Delta}}}{\sqrt{2\pi\Delta}}$ is the density of a zero mean Gaussian random variable with variance Δ.

3. Main results

3.1. **Wiener process.**

Theorem 3.1. *Let $n \geq 1$ and $T_1, T_2 > 0$. Let $t_0 = T_1$, $t_{n+1} = T_2$, and let*

$$t_{j-1} < t_j, \qquad 1 \leq j \leq n+1.$$

Put $\Delta t_j = t_j - t_{j-1}$, $1 \leq j \leq n+1$. Consider two sequences $\{a_j, 1 \leq j \leq n+1\}, \{b_j, 1 \leq j \leq n+1\}$ such that

$$b_1 > 0, \qquad a_j \geq 0, \quad 1 \leq j \leq n, \qquad a_{n+1} > 0.$$

Set

$$h_j(t) \stackrel{\text{def}}{=} a_j t + b_j, 1 \leq j \leq n+1.$$

Assume that

$$h_j(t_j) = h_{j+1}(t_j), \qquad 1 \leq j \leq n,$$

and

$$h_j(t_j) > 0, \qquad 1 \leq j \leq n.$$

Put

$$h(t) \stackrel{\text{def}}{=} b_1 I_{\{T_1\}}(t) + \sum_{j=1}^{n+1} h_j(t) I_{(t_{j-1}, t_j]}(t).$$

Denote $\beta_j^* = h_j(t_{j-1}) - v_{j-1}$, $1 \leq j \leq n+1$, where $v_0 = 0$. Then

$$\boldsymbol{P}\left(\sup_{T_1 \leq t \leq T_2} (w(t) - h(t)) < 0\right)$$

$$= \int_{-\infty}^{h_1(t_1)} \cdots \int_{-\infty}^{h_n(t_n)} \prod_{j=2}^{n} \left(1 - \exp\left\{\frac{-2\beta_j^* \beta_{j+1}^*}{\Delta t_j}\right\}\right)$$

$$\times \left(\Phi\left(\frac{t_1 h_1(T_1) - v_1 T_1}{\sqrt{T_1 t_1 (t_1 - T_1)}}\right) - e^{-\frac{2b_1\beta_2^*}{t_1}} \Phi\left(\frac{b_1 \Delta t_1 - T_1 \beta_2^*}{\sqrt{t_1 T_1 (t_1 - T_1)}}\right)\right)$$

$$\times \left(\Phi\left(\frac{a_{n+1}\Delta t_{n+1} + \beta_{n+1}^*}{\sqrt{\Delta t_{n+1}}}\right) - e^{-2a_{n+1}\beta_{n+1}^*} \Phi\left(\frac{a_{n+1}\Delta t_{n+1} - \beta_{n+1}^*}{\sqrt{\Delta t_{n+1}}}\right)\right)$$

$$\times \prod_{j=1}^{n} \varphi_{0,\Delta t_j}(v_j - v_{j-1}) dv_1 \ldots dv_n. \tag{10}$$

Proof. By the full probability formula

$$B \stackrel{\text{def}}{=} \mathbf{P}\left(\sup_{T_1 \leq t \leq T_2} (w(t) - h(t)) < 0\right)$$

$$= \int_{-\infty}^{h(T_1)} P\left(\sup_{T_1 \leq t \leq T_2} (w(t) - h(t)) < 0 / w(T_1) = u\right) \varphi_{0,T_1}(u) du.$$

Denote

$$A(u) \stackrel{\text{def}}{=} P\left(\sup_{T_1 \leq t \leq T_2} (w(t) - h(t)) < 0 / w(T_1) = u\right)$$

$$= \mathbf{P}\left(\sup_{T_1 \leq t \leq T_2} (w(t) - w(T_1) - h(t) + u) < 0 / w(T_1) = u\right)$$

$$= \mathbf{P}\left(\sup_{0 \leq t \leq T_2 - T_1} (w(t) - h(t + T_1) + u) < 0\right).$$

It is clear that

$$h(t+T_1)-u=b_1 I_{\{0\}}(t)+\sum_{j=1}^{n+1}(h_j(t)+a_jT_1-u))I_{(t_{j-1}-T_1,t_j-T_1]}(t).$$

Using Theorem 2.2:

$$T=T_2-T_1, \qquad t_i\to t_i-T_1, \quad i=\overline{0,n+1},$$

$$a_i\to a_i, \qquad b_i\to a_iT_1+b_i-u, \quad i=\overline{1,n+1},$$

$$h_i(t_i)\to h_i(t_i)-u, \qquad i=\overline{1,n+1},$$

$$\beta_1\longrightarrow b_1+a_1T_1-u, \qquad \beta_i\to\beta_i-u, \quad i=\overline{2,n+1},$$

$$\Delta t_i\to\Delta t_i, \quad i=\overline{2,n+1}, \qquad \Delta t_1=t_1-T_1.$$

After that we get

$$A(u)=\int_{-\infty}^{h_1(t_1)-u}\dots\int_{-\infty}^{h_n(t_n)-u}\prod_{j=2}^{n}\left(1-\exp\left\{\frac{-2(\beta_j-u)(h_j(t_j)-u_j-u))}{\Delta t_j}\right\}\right)$$

$$\times\left(1-\exp\left\{\frac{-2(\beta_1-u)(h_1(t_1)-u_1-u))}{\Delta t_1}\right\}\right)$$

$$\times\left(\Phi\left(\frac{a_{n+1}\Delta t_{n+1}+\beta_{n+1}-u}{\sqrt{\Delta t_{n+1}}}\right)-e^{-2a_{n+1}(\beta_{n+1}-u)}\Phi\left(\frac{a_{n+1}\Delta t_{n+1}-(\beta_{n+1}-u)}{\sqrt{\Delta t_{n+1}}}\right)\right)$$

$$\times\prod_{j=1}^{n}\varphi_{0,\Delta t_j}(u_j-u_{j-1})du_1\dots du_n.$$

So

$$B=\int_{-\infty}^{a_1T_1+b_1}\varphi_{0,T_1}(u)A(u)du.$$

After the changing the variables in this integral

$$v_j=u_j+u, \qquad j=\overline{1,n},$$

and using the fact that

$$h_j(t_j)-v_j=h_{j+1}(t_j)-v_j=\beta^*_{j+1}, \qquad j=\overline{1,n},$$

we get:

$$B=\int_{-\infty}^{h_1(t_1)}\dots\int_{-\infty}^{h_n(t_n)}\prod_{j=2}^{n}\left(1-\exp\left\{\frac{-2\beta^*_j\beta^*_{j+1}}{\Delta t_j}\right\}\right)$$

$$\times\left(\Phi\left(\frac{a_{n+1}\Delta t_{n+1}+\beta^*_{n+1}}{\sqrt{\Delta t_{n+1}}}\right)-e^{-2a_{n+1}\beta^*_{n+1}}\Phi\left(\frac{a_{n+1}\Delta t_{n+1}-\beta^*_{n+1}}{\sqrt{\Delta t_{n+1}}}\right)\right)$$

$$\times\prod_{j=2}^{n}\varphi_{0,\Delta t_j}(v_j-v_{j-1})dv_1\dots dv_n$$

$$\times\int_{-\infty}^{a_1T_1+b_1}\varphi_{0,T_1}(u)\varphi_{0,t_1-T_1}(v_1-u)\left(1-\exp\left\{\frac{-2(\beta_1-u)\beta^*_2}{\Delta t_1}\right\}\right)du.$$

The computation of the last integral reduces to the statement of Theorem. □

3.2. **The maximum of the Chentsov random field with linear drift.** Let us generalize Theorem 2.3. Denote $f_i = \frac{G}{y_i} + \frac{Ex_i}{y_i} + F$, $i = \overline{0,n}$, where $E, F > 0$ and $G \geq 0$.

Theorem 3.2. *Let $\{X(s,t) : s,t \geq 0\}$ be a standard Chentsov random field on the unit square and $g(s,t) = Es + Ft + G$. Let L be a polygonal line, which has n points of break $Q_1, ..., Q_n$ with coordinates $(x_1, y_1), ..., (x_n, y_n)$ and which is given by the formula (6). Let the coordinates of these points satisfy the conditions (8)-(9). Put $\Delta_i = \frac{x_i}{y_i}, i = \overline{0,n}$. Let $u_0 = 0$. Then*

$$P_n(g) = \boldsymbol{P}\left\{ \sup_{(s;t)\in L} \left(X\left(s;t\right) - g(s,t)\right) < 0 \right\} =$$

$$= \int_{-\infty}^{f_1} ... \int_{-\infty}^{f_n} \left(1 - \exp\left\{-2(G+E)\left(\frac{G+Ex_n}{y_n} + F - u_n\right)\right\}\right) \times$$

$$\times \prod_{i=1}^{n} \left(1 - \exp\left\{-\frac{2(f_{i-1}-u_{i-1})(f_i - u_i)}{(\Delta_i - \Delta_{i-1})}\right\}\right) \varphi_{0,\Delta_i - \Delta_{i-1}}\left(u_i - u_{i-1}\right) du_1 \dots du_n,$$

where $\varphi_{0,\Delta}(u) = \frac{e^{-\frac{u^2}{2\Delta}}}{\sqrt{2\pi\Delta}}$ is the density of a zero mean Gaussian random variable with variance Δ.

Proof. Using the notations of Theorem 2.3 we can write

$$v(s) = I_0(s) + \sum_{i=1}^{n+1} \left(-\frac{s(y_{i-1}-y_i)}{x_i - x_{i-1}} + \frac{x_i y_{i-1} - x_{i-1} y_i}{x_i - x_{i-1}}\right) I_{(x_{i-1};x_i]}(s).$$

Let us denote the restriction of the function $g(s,t)$ to L by $g_L(s)$. Then $g_L(s) = Es + Fv(s) + G$. Denote $a(s) = \frac{s}{v(s)}$. We can rewrite $a(s)$ in an explicit form:

$$a(s) = \sum_{i=1}^{n} \frac{s}{-\frac{s(y_{i-1}-y_i)}{x_i - x_{i-1}} + \frac{x_i y_{i-1} - x_{i-1}y_i}{x_i - x_{i-1}}} I_{[x_{i-1},x_i)}(s) + \frac{(1-x_n)s}{(1-s)y_n} I_{[x_n,1)}(s).$$

For $a(s)$ the inverse will be the function:

$$a^{-1}(s) = \sum_{i=1}^{n} \frac{s(x_i y_{i-1} - x_{i-1}y_i)}{s(y_{i-1}-y_i) + x_i - x_{i-1}} I_{[\Delta_{i-1},\Delta_i)}(s) + \frac{sy_n}{sy_n + 1 - x_n} I_{[\Delta_n,\infty)}(s).$$

The functions $a(\cdot)$ and $v(\cdot)$ satisfy the conditions of Doob's transformation Theorem [1]. Let us denote the restriction of the Chentsov random field $X\left(s;t\right)$ to L by $X_L(s)$. Thus,

$$X^*(s) = \sum_{i=1}^{n+1} \left(\frac{s(y_{i-1}-y_i) + x_i - x_{i-1}}{x_i y_{i-1} - x_{i-1} y_i}\right) X_L\left(\frac{s(x_i y_{i-1} - x_{i-1}y_i)}{s(y_{i-1}-y_i) + x_i - x_{i-1}}\right)$$

$$\times I_{[\Delta_{i-1},\Delta_i)}(s) + \left(s + \frac{1-x_n}{y_n}\right) X_L\left(\frac{sy_n}{sy_n + 1 - x_n}\right).$$

and $w(s)$ are stochastically equivalent processes. So

$$P_n(g) = \mathbf{P}\left\{ \sup_{(s;t)\in L} \left(X\left(s;t\right) - g(s,t)\right) < 0 \right\}$$

$$= \mathbf{P}\left\{ \sup_{s\in[0,1]} \left(X\left(s;v(s)\right) - g_L(s)\right) < 0 \right\} =$$

$$= \mathbf{P}\left\{ \sup_{s\in[0,1]} \left(X_L(s) - g_L(s)\right) < 0 \right\}$$

$$= \mathbf{P}\left\{\sup_{s\in[0,\infty)} \left(X_L\left(a^{-1}(s)\right) - g_L(a^{-1}(s))\right) < 0\right\} =$$

$$= \mathbf{P}\left(\bigcap_{s>0}\left\{\frac{X_L\left(a^{-1}(s)\right)}{v\left(a^{-1}(s)\right)} - \frac{g_L(a^{-1}(s))}{v\left(a^{-1}(s)\right)} < 0\right\}\right) =$$

$$= \mathbf{P}\left\{\sup_{s\in(0,\infty)}\left(\frac{X_L\left(a^{-1}(s)\right)}{v\left(a^{-1}(s)\right)} - \frac{g_L(a^{-1}(s))}{v\left(a^{-1}(s)\right)}\right) < 0\right\} =$$

$$= \mathbf{P}\left\{\sup_{s\in(0,\infty)}\left(w(s) - \frac{g_L(a^{-1}(s))}{v\left(a^{-1}(s)\right)}\right) < 0\right\} =$$

$$= \mathbf{P}\left\{w(s) < F + \frac{G(x_i - x_{i-1})}{x_i y_{i-1} - x_{i-1}y_i} + s\left(\frac{G(y_{i-1} - y_i)}{x_i y_{i-1} - x_{i-1}y_i} + E\right);\right.$$

$$\left. s\in(\Delta_{i-1};\Delta_i], i = \overline{1,n}; w(s) < \frac{G(1-x_n)}{y_n} + F + s(G+E), s > \Delta_n\right\}$$

Using Theorem 2.2:

$$a_i = \frac{G(y_{i-1} - y_i)}{x_i y_{i-1} - x_{i-1}y_i} + E, \quad i = \overline{1,n},$$

$$b_i = \frac{G(x_{i-1} - x_i)}{x_i y_{i-1} - x_{i-1}y_i} + F, \quad i = \overline{1,n},$$

$$\beta_i = f_{i-1} - u_{i-1}, \quad i = \overline{1,n}, \qquad \beta_{n+1} = F - u_n + \frac{G + Ex_n}{y_n},$$

$$a_{n+1} = G + E, \qquad b_{n+1} = \frac{G(1-x_n)}{y_n} + F,$$

and passing to the limit as $T\to\infty$ we get:

$$P_n(g) = \mathbf{P}\left\{\sup_{(s;t)\in L}\left(X(s;t) - g(s,t)\right) < 0\right\} =$$

$$\int_{-\infty}^{f_1}\dots\int_{-\infty}^{f_n}\left(1 - \exp\left\{-2(G+E)\left(\frac{G+Ex_n}{y_n} + F - u_n\right)\right\}\right)\times$$

$$\times\prod_{i=1}^{n}\left(1 - \exp\left\{-\frac{2(f_{i-1} - u_{i-1})(f_i - u_i)}{(\Delta_i - \Delta_{i-1})}\right\}\right)\varphi_{0,\Delta_i - \Delta_{i-1}}\left(u_i - u_{i-1}\right)du_1\dots du_n.$$

□

Example 3.1. Let a polygonal line have one change of direction and be given by the formula:

$$L = \left\{(s,t)|t = I_0(s) + \left(-\frac{s(1-y_1)}{x_1} + 1\right)I_{(0;x_1]} + \left(\frac{y_1(1-s)}{1-x_1}\right)I_{(x_1;1]}\right\}. \tag{11}$$

And let $g(s,t) = s + t$. We need to find the probability:

$$P\left\{\sup_{(s,t)\in L} X(s,t) - g(s,t) < 0\right\}.$$

Therefore by Theorem 3.2 with $E = F = 1$ and $G = 0$:

$$\mathbf{P}\left\{\sup_{(s,t)\in L} X(s,t) - g(s,t) < 0\right\} = \int_{-\infty}^{\frac{x_1}{y_1}}\varphi_{0,\Delta_1}(u_1)\times$$

$$\times\left(1 - \exp\left\{-2\left(\frac{x_1}{y_1} + 1 - u1\right)\right\}\right)\left(1 - \exp\left\{-\frac{2\left(\frac{x_1}{y_1} + 1 - u1\right)}{\Delta_1}\right\}\right) =$$

$$= \Phi\left(\sqrt{\Delta_1}\right) - \Phi\left(-\Delta_1\right) e^{-\frac{1}{2\Delta_1}} - \Phi\left(\frac{\Delta_1 - 2}{\Delta_1}\right) e^{-\frac{2}{\Delta_1}} + \Phi\left(\frac{-\Delta_1 - 2}{\Delta_1}\right).$$

3.3. **The maximum of the Chentsov random field on the broadest class of polygonal lines.** Let us consider a polygonal line L with n changes of direction, which begins at point (x_0, y_0) and ends at point (x_{n+1}, y_{n+1}).

Let

$$0 < x_0 < x_1 < ... < x_n < x_{n+1} < 1, \tag{12}$$

$$1 > y_0 \geq y_1 \geq ... \geq y_n \geq y_{n+1} > 0. \tag{13}$$

Theorem 3.3. *Let $\{X(s,t) : s,t \geq 0\}$ be a standard Chentsov random field on the unit square. Let $u_0 = 0$. Let L be a polygonal line, which has n points of break $Q_1, ..., Q_n$ with coordinates (x_1, y_1),..., (x_n, y_n), starts at point (x_0, y_0) and ends at point (x_{n+1}, y_{n+1}). Let the coordinates of these points satisfy the conditions (12)-(13). Then for all $\lambda > 0$*

$$P_n(\lambda) = \boldsymbol{P}\left\{\sup_{(s;t)\in L} X(s;t) < \lambda\right\} = \int_{-\infty}^{\frac{\lambda}{y_1}} ... \int_{-\infty}^{\frac{\lambda}{y_n}} \prod_{i=1}^{n} \varphi_{0,\Delta_i - \Delta_{i-1}}(u_i - u_{i-1}) \times$$

$$\times \left(\Phi\left(\frac{\frac{\lambda}{x_0} - \frac{u_1 y_1}{x_1}}{\sqrt{1/\Delta_0 - 1/\Delta_1}}\right) - \exp\left\{\frac{-2\lambda(x_1 - x_0)(\lambda - u_1 y_1)}{x_1(x_1 y_0 - x_0 y_1)}\right\} \Phi\left(\frac{\frac{\lambda}{x_0} - \frac{2\lambda}{x_1} + \frac{u_1 y_1}{x_1}}{\sqrt{1/\Delta_0 - 1/\Delta_1}}\right)\right)$$

$$\times \left(\Phi\left(\frac{\frac{\lambda}{y_{n+1}} - u_n}{\sqrt{\Delta_{n+1} - \Delta_n}}\right) - \exp\left\{\frac{-2\lambda(y_n - y_{n+1})(\lambda - u_n y_n)}{y_n(x_{n+1} y_n - x_n y_{n+1})}\right\} \Phi\left(\frac{\frac{\lambda}{y_{n+1}} - \frac{2\lambda}{y_n} + u_n}{\sqrt{\Delta_{n+1} - \Delta_n}}\right)\right)$$

$$\times \prod_{i=2}^{n}\left(1 - \exp\left\{-\frac{2\left(\frac{\lambda}{y_{i-1}} - u_{i-1}\right)\left(\frac{\lambda}{y_i} - u_i\right)}{(\Delta_i - \Delta_{i-1})}\right\}\right) du_1 \dots du_n,$$

where $\varphi_{0,\Delta}(u) = \frac{e^{-\frac{u^2}{2\Delta}}}{\sqrt{2\pi\Delta}}$.

Proof. Let us follow the line of the proof of the previous Theorem. Let the restriction of the Chentsov random field $X(s;t)$ to L be denoted as $X_L(s)$. Then the covariance function of this process is:

$$cov\left[X_L(s_1), X_L(s_2)\right] = s_1 v(s_2), 0 < s_1 \leq s_2 \leq 1,$$

where

$$v(s) = \sum_{i=1}^{n+1}\left(-\frac{s(y_{i-1} - y_i)}{x_i - x_{i-1}} + \frac{x_i y_{i-1} - x_{i-1} y_i}{x_i - x_{i-1}}\right) I_{(x_{i-1}; x_i]}(s).$$

Function $a(s) = \frac{s}{v(s)}$, $s \in (x_0, x_{n+1})$, is continuous and strictly increasing with inverse:

$$a^{-1}(s) = \sum_{i=1}^{n+1} \frac{s(x_i y_{i-1} - x_{i-1} y_i)}{s(y_{i-1} - y_i) + x_i - x_{i-1}} I_{[\Delta_{i-1}, \Delta_i)}.$$

Process

$$X^*(s) = \sum_{i=1}^{n+1}\left(\frac{s(y_{i-1} - y_i) + x_i - x_{i-1}}{x_i y_{i-1} - x_{i-1} y_i}\right) X_L\left(\frac{s(x_i y_{i-1} - x_{i-1} y_i)}{s(y_{i-1} - y_i) + x_i - x_{i-1}}\right) I_{[\Delta_{i-1}, \Delta_i)}(s)$$

and $w(s)$ are stochastically equivalent processes. Thus,

$$P_n(\lambda) = \mathbf{P}\left\{\sup_{(s;t)\in L} X(s;t) < \lambda\right\} = \mathbf{P}\left\{\sup_{s\in[x_0, x_{n+1}]} X(s; v(s)) < \lambda\right\} =$$

$$= \mathbf{P}\left\{\sup_{s\in[x_0, x_{n+1}]} X_L(s) < \lambda\right\} = \mathbf{P}\left\{\sup_{s\in[\Delta_0, \Delta_{n+1}]} X_L\left(a^{-1}(s)\right) < \lambda\right\} =$$

$$= \mathbf{P}\left\{ \sup_{s\in[\Delta_0,\Delta_{n+1}]} \left(w(s) - \frac{\lambda}{v\left(a^{-1}(s)\right)} \right) < 0 \right\} =$$

$$= \mathbf{P}\left\{ w(s) < \frac{\lambda(x_i - x_{i-1})}{x_i y_{i-1} - x_{i-1} y_i} + \frac{\lambda s(y_{i-1} - y_i)}{x_i y_{i-1} - x_{i-1} y_i}, s \in (\Delta_{i-1}; \Delta_i], i = \overline{1, n+1} \right\}.$$

Using Theorem 3.1: $t_i = \Delta_i = \frac{x_i}{y_i}, i = \overline{0, n+1}$,

$$a_i = \frac{\lambda(y_{i-1} - y_i)}{x_i y_{i-1} - x_{i-1} y_i}, i = \overline{1, n+1},$$

$$b_i = \frac{\lambda(x_i - x_{i-1})}{x_i y_{i-1} - x_{i-1} y_i}, i = \overline{1, n+1},$$

we complete the proof. □

Let L be a polygonal line, which has n points of break $Q_1, ..., Q_n$ with coordinates (x_1, y_1),..., (x_n, y_n) respectively and which is given by the formula:

$$L = \left\{ (s,t) : t = pI_{\{0\}}(s) + \right.$$

$$\left. + \sum_{i=1}^{n+1} \left(-\frac{s(y_{i-1} - y_i)}{x_i - x_{i-1}} + \frac{x_i y_{i-1} - x_{i-1} y_i}{x_i - x_{i-1}} \right) I_{(x_{i-1};x_i]}(s), s \in [0,1] \right\}.$$

Let

$$0 = x_0 < x_1 < ... < x_n < x_{n+1} = q, \tag{14}$$

$$p = y_0 \ge y_1 \ge ... \ge y_n > y_{n+1} = 0. \tag{15}$$

Corollary 3.1. *Let $\{X(s,t) : s,t \ge 0\}$ be a standard Chentsov random field on the unit square. Let $u_0 = 0$. Let L be a polygonal line, which has n points of break $Q_1, ..., Q_n$ with coordinates (x_1, y_1),..., (x_n, y_n). Let the coordinates of these points satisfy the conditions (14)-(15). Denote $\Delta_i = \frac{x_i}{y_i}$, $i = \overline{1, n}$. Then for all $\lambda > 0$*

$$P_n(\lambda) = \boldsymbol{P}\left\{ \sup_{(s;t)\in L} X(s;t) < \lambda \right\} = \int_{-\infty}^{\frac{\lambda}{y_1}} ... \int_{-\infty}^{\frac{\lambda}{y_n}} \prod_{i=1}^{n} \varphi_{0,\Delta_i - \Delta_{i-1}}(u_i - u_{i-1}) \times$$

$$\times \left(1 - \exp\left\{ -\frac{2\lambda}{q} \left(\frac{\lambda}{y_n} - u_n \right) \right\} \right)$$

$$\times \prod_{i=1}^{n} \left(1 - \exp\left\{ -\frac{2\left(\frac{\lambda}{y_{i-1}} - u_{i-1}\right)\left(\frac{\lambda}{y_i} - u_i\right)}{(\Delta_i - \Delta_{i-1})} \right\} \right) du_1 \dots du_n,$$

where $\varphi_{0,\Delta}(u) = \frac{e^{-\frac{u^2}{2\Delta}}}{\sqrt{2\pi\Delta}}$.

Proof. Using Theorem 3.1 $x_{n+1} = q$, $y_0 = p$ and passing to the limit as $x_0 \to 0$ we get:

$$\lim_{x_0\to 0} \Phi\left(\frac{\frac{\lambda}{x_0} - \frac{u_1 y_1}{x_1}}{\sqrt{\frac{p}{x_0} - \frac{y_1}{x_1}}} \right) = 1,$$

$$\lim_{x_0\to 0} \Phi\left(\frac{\frac{\lambda}{x_0} - \frac{2\lambda}{x_1} + \frac{u_1 y_1}{x_1}}{\sqrt{\frac{p}{x_0} - \frac{y_1}{x_1}}} \right) = 1.$$

So

$$\lim_{x_0\to 0}\left(\Phi\left(\frac{\frac{\lambda}{x_0}-\frac{u_1y_1}{x_1}}{\sqrt{1/\Delta_0-1/\Delta_1}}\right)-\exp\left\{\frac{-2\lambda(x_1-x_0)(\lambda-u_1y_1)}{x_1(x_1y_0-x_0y_1)}\right\}\Phi\left(\frac{\frac{\lambda}{x_0}-\frac{2\lambda}{x_1}+\frac{u_1y_1}{x_1}}{\sqrt{1/\Delta_0-1/\Delta_1}}\right)\right)$$

$$=1-\exp\left\{-\frac{2\lambda(\lambda-u_1y_1)}{px_1}\right\}.$$

Passing to the limit as $y_{n+1}\to 0$ we get:

$$\lim_{y_{n+1}\to 0}\Phi\left(\frac{\frac{\lambda}{y_{n+1}}-u_n}{\sqrt{\frac{q}{y_{n+1}}-\frac{x_n}{y_n}}}\right)=1,$$

$$\lim_{y_{n+1}\to 0}\Phi\left(\frac{\frac{\lambda}{y_{n+1}}-\frac{2\lambda}{y_n}+u_n}{\sqrt{\frac{q}{y_{n+1}}-\frac{x_n}{y_n}}}\right)=1.$$

So

$$\lim_{y_{n+1}\to 0}\left(\Phi\left(\frac{\frac{\lambda}{y_{n+1}}-u_n}{\sqrt{\Delta_{n+1}-\Delta_n}}\right)-\right.$$

$$\left.-\exp\left\{\frac{-2\lambda(y_n-y_{n+1})(\lambda-u_ny_n)}{y_n(x_{n+1}y_n-x_ny_{n+1})}\right\}\Phi\left(\frac{\frac{\lambda}{y_{n+1}}-\frac{2\lambda}{y_n}+u_n}{\sqrt{\Delta_{n+1}-\Delta_n}}\right)\right)=$$

$$=\left(1-\exp\left\{-\frac{2\lambda}{q}\left(\frac{\lambda}{y_n}-u_n\right)\right\}\right).$$

Using this results we get the statement of the corollary. □

Example 3.2. Let us compute the probability of the type:

$$\mathbf{P}\left\{\sup_{(s,t)\in L}X(s,t)<\lambda\right\}=P_1(\lambda),$$

considered in Theorem 3.3, in the case $n=1$.

Using the result of this Theorem we can write:

$$P_1(\lambda)=\int_{-\infty}^{\frac{\lambda}{y_1}}\left(\Phi\left(\frac{\frac{\lambda}{x_0}-\frac{u_1y_1}{x_1}}{\sqrt{1/\Delta_0-1/\Delta_1}}\right)-\right.$$

$$\left.-\exp\left\{\frac{-2\lambda(x_1-x_0)(\lambda-u_1y_1)}{x_1(x_1y_0-x_0y_1)}\right\}\Phi\left(\frac{\frac{\lambda}{x_0}-\frac{2\lambda}{x_1}+\frac{u_1y_1}{x_1}}{\sqrt{1/\Delta_0-1/\Delta_1}}\right)\right)\times$$

$$\times\varphi_{0,\Delta_1}(u_1)\left(\Phi\left(\frac{\frac{\lambda}{y_2}-u_1}{\sqrt{\Delta_2-\Delta_1}}\right)-\right.$$

$$\left.-\exp\left\{\frac{-2\lambda(y_1-y_2)(\lambda-u_1y_1)}{y_1(x_2y_1-x_1y_2)}\right\}\Phi\left(\frac{\frac{\lambda}{y_2}-\frac{2\lambda}{y_1}+u_1}{\sqrt{\Delta_2-\Delta_1}}\right)\right)du_1. \tag{16}$$

It will be interesting to consider some limiting case of this probability. Passing to the limit in (16) as $y_0\to 1$, $x_0\to 0$, $x_2\to 1$ and $y_2\to 0$ we get:

$$\lim_{\substack{x_0\to 0,\\ y_0\to 1}}\Phi\left(\frac{\frac{\lambda}{x_0}-\frac{u_1y_1}{x_1}}{\sqrt{1/\Delta_0-1/\Delta_1}}\right)=1,$$

$$\lim_{\substack{x_0 \to 0, \\ y_0 \to 1}} \Phi\left(\frac{\frac{\lambda}{x_0} - \frac{2\lambda}{x_1} + \frac{u_1 y_1}{x_1}}{\sqrt{1/\Delta_0 - 1/\Delta_1}}\right) = 1,$$

$$\lim_{\substack{x_2 \to 1, \\ y_2 \to 0}} \Phi\left(\frac{\frac{\lambda}{y_2} - u_1}{\sqrt{\Delta_2 - \Delta_1}}\right) = 1,$$

$$\lim_{\substack{x_2 \to 1, \\ y_2 \to 0}} \Phi\left(\frac{\frac{\lambda}{y_2} - \frac{2\lambda}{y_1} + u_1}{\sqrt{\Delta_2 - \Delta_1}}\right) = 1.$$

So

$$\lim_{\substack{(x_0, y_0) \to (0,1), \\ (x_2, y_2) \to (1,0)}} P_1(\lambda) = \Phi\left(\frac{\lambda}{\sqrt{x_1 y_1}}\right) -$$

$$- \exp\left\{-\frac{2\lambda^2}{x_1}(1-y_1)\right\} \Phi\left(\frac{\lambda(1-2y_1)}{\sqrt{x_1 y_1}}\right) - \exp\left\{-\frac{2\lambda^2}{y_1}(1-x_1)\right\} \Phi\left(\frac{\lambda(1-2x_1)}{\sqrt{x_1 y_1}}\right) +$$

$$+ \exp\left\{\frac{2\lambda^2}{x_1 y_1}(x_1+y_1)(x_1+y_1-1)\right\} \Phi\left(\frac{\lambda(1-2x_1-2y_1)}{\sqrt{x_1 y_1}}\right).$$

This result agree with statement of Theorem 2 in [3].

Conclusion

In this paper we have obtained an expression for the probability of Wiener process crossing stepwise linear barriers. We have applied this result to the derivation of the distribution of the maximum of the Chentsov random field and, also, of the Chentsov random field with a linear drift on a polygonal line. We have expanded a class of polygonal lines on which it is possible to find the exact distribution of the maximum of the Chentsov random field.

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