F. Kh. Mukminov, Of the first mixed problem for the system of Navier–Stokes equations in domains with noncompact boundaries, *Russian Academy of Sciences. Sbornik. Mathematics*, 1994, Volume 78, Issue 2, 507–524

DOI: 10.1070/SM1994v078n02ABEH003482

Росс. Акад. Наук Матем. Сборник
Том 184 (1993), № 4

Russian Acad. Sci. Sb. Math.
Vol. 78 (1994), No. 2

ON THE RATE OF DECAY OF A STRONG SOLUTION OF THE FIRST MIXED PROBLEM FOR THE SYSTEM OF NAVIER-STOKES EQUATIONS IN DOMAINS WITH NONCOMPACT BOUNDARIES

UDC 517.946

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ABSTRACT. This article contains an investigation of the behavior as $t \to \infty$ of a solution of the mixed problem with Dirichlet conditions on the boundary for the system of Navier-Stokes equations in an unbounded three-dimensional domain. An estimate, determined by the geometry of the domain, is proved for the rate of decay of a solution for a compactly supported initial function satisfying a certain smallness condition. This estimate coincides in form with the sharp estimate obtained earlier by the author for the solution of the first mixed problem for the heat equation.

Bibliography: 35 titles.

1. INTRODUCTION

In the domain $D = (0, \infty) \times \Omega$, where Ω is an unbounded domain in R^n, $n \geq 2$, we consider the problem

$$(1) \qquad \mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \Delta \mathbf{u} - \nabla p, \qquad \operatorname{div}_x \mathbf{u} = 0 \quad \text{in } D;$$

$$(2) \qquad \mathbf{u}|_{x \in \partial\Omega} = 0, \qquad \mathbf{u}|_{t=0} = \boldsymbol{\varphi}(x).$$

Here for $n = 3$, $\mathbf{u}(t, x) = (u_1, u_2, u_3)$ and $p(t, x)$ are the unknown velocities of flow of a fluid and the pressure, and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \varphi_3)$ are the given initial velocities. In the sequel, instead of

$$\operatorname{div}_x \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}$$

we simply write $\operatorname{div} \mathbf{u}$. The expression

$$\mathbf{u} \cdot \nabla f = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial x_i}$$

denotes the derivative of f along the vector $\mathbf{u}$.

The existence of a weak solution of the Cauchy problem (1), (2) with $n = 3$ was proved by Leray [21], and for an arbitrary three-dimensional domain by Hopf [11]. The existence of a weak solution for $n > 3$ was proved, for example, in [24]. The uniqueness and regularity of a weak solution are both still open questions.

The existence of a strong solution of the problem (1), (2) in the domain D was proved by Kiselev and Ladyzhenskaya [16] under a certain smallness condition on the initial function $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_2^2(\Omega)$. In the same paper the uniqueness of a strong solution was proved. There the domain Ω was assumed to be bounded, while in [33] Serrin

1991 *Mathematics Subject Classification*. Primary 35Q30, 35B40; Secondary 76D05.

extended these results to the case of an unbounded domain. Strong solutions were constructed under other assumptions about the smallness of the initial function in [13] and [15].

The following question was posed as far back as 1934 by Leray [21]: *does the kinetic energy*

$$\frac{1}{2}\int_{\Omega}\mathbf{u}^2(t,x)\,dx$$

(the $\mathbf{L}_2$-norm) *of a fluid flow in an unbounded domain tend to zero as* $t\to\infty$? A positive answer was given in the case of the Cauchy problem by Kato [15] (for a strong solution) and Masuda [24] (for a weak solution). Moreover, the following estimate was obtained in [15]. If a solenoidal vector $\boldsymbol{\varphi}$ belongs to the intersection $\mathbf{L}_n(R^n)\cap\mathbf{L}_r(R^n)$, $r\in[1,n]$, and the norm $\|\boldsymbol{\varphi}\|_n$ is sufficiently small, then there exists a unique strong solution of the Cauchy problem (1), (2), and $\|\mathbf{u}(t)\|_\alpha = O(t^{-\gamma})$, $\gamma=(n/r-n/\alpha)/2$, for $a>r$ as $t\to\infty$. Here and below,

$$\|\mathbf{v}\|_{\alpha,Q}=\left[\sum_{i=1}^{n}\int_{Q}v_i^\alpha(x)\,dx\right]^{1/\alpha},$$

and the corresponding indices will be omitted for $\alpha=2$ and $Q=\Omega$. In [1] certain estimates were obtained for a strong solution of the nonhomogeneous system of Navier-Stokes equations; in the homogeneous case these estimates are a consequence of the cited estimate.

An estimate of the rate of decrease of the kinetic energy for a weak solution of the Cauchy problem (1), (2) was given in [31] and improved in [14] and [35]. We formulate the result in [14]. If a solenoidal vector $\boldsymbol{\varphi}$ belongs to $\mathbf{L}_2(R^n)\cap\mathbf{L}_r(R^n)$, $n\geq 2$, $r\in[1,2)$, then there exists a weak solution of the Cauchy problem (1), (2) that decays just as in the case of the heat equation: $\|\mathbf{u}(t)\|=O(t^{-\gamma})$, $\gamma=(n/r-n/2)/2$. In [35] the same estimate was established for an arbitrary weak solution satisfying the energy inequality

$$\|\mathbf{u}(t)\|^2+2\nu\int_s^t\|\nabla\mathbf{u}(\tau)\|^2\,d\tau\leq\|\mathbf{u}(s)\|^2 \tag{3}$$

for $s=0$, for a.e. $s>0$, and for all $t>s$. In the case of the problem in the exterior of a bounded domain analogous results were obtained for $r\in(1,2)$ in [22] $(n=3)$ and [5] $(n\geq 3)$.

Many papers have been devoted to the investigation of the rate of decay of the motion of a rotating fluid described by linear ([3], [8], [9], [25], [26]) and nonlinear [30] equations.

We are interested in uniform (with respect to Ω) decay of the velocities as t tends to infinity. The first estimate of this kind for a solution of the problem (1), (2) in the exterior of a bounded domain was obtained by Masuda in [23]. The order of decay $t^{-1/8}$ established there is not sharp, of course. In [13] Heywood constructed a strong solution of the problem (1), (2) in an arbitrary domain Ω, $n=3$, with boundary uniformly of class C^3, and proved the estimate $\sup_{x\in\Omega}|\mathbf{u}(t,x)|=O(t^{-1/2})$ as $t\to\infty$. We formulate this result in greater detail. The requirement that the boundary belong to the class C^3 uniformly means that there exist positive numbers d and b such that for any point $\xi\in\partial\Omega$ the intersection $\partial\Omega\cap\{|x-\xi|<d\}$ can be represented in a local Cartesian system of coordinates as the graph of a function with derivatives up to the third order bounded by the constant b. Denote by $\overset{\circ}{\mathbf{J}}(\Omega)$ the completion of the set $\overset{\bullet}{\mathbf{J}}(\Omega)$ of smooth solenoidal compactly supported vectors on

Ω with respect to the norm $\|\mathbf{v}\|$, and by $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$ the completion of the same set with respect to the norm $\|\mathbf{v}\| + \|\nabla\mathbf{v}\|$. Then it is asserted in [13] that there are positive numbers C and C', dependent on d, b, and ν, such that if the initial function $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$ satisfies the condition

$$\|\boldsymbol{\varphi}\|^2 \le \max_{a>0} \frac{2\nu \ln(a/\|\nabla\boldsymbol{\varphi}\|^2)}{C + C'a}, \tag{4}$$

then there exists a unique solution $\mathbf{u}(t, x), p(t, x) \in C^\infty(D)$ of the problem (1), (2); further, $\mathbf{u} \in C([0, \infty); \overset{\circ}{\mathbf{J}}{}^1(\Omega))$. It decays in the way indicated above. Setting $a = e\|\nabla\boldsymbol{\varphi}\|^2$ and $a = 1$, we get a simple sufficient condition for (4):

$$\|\boldsymbol{\varphi}\|^2 \le 2\nu \max\left\{\frac{1}{C + eC'\|\nabla\boldsymbol{\varphi}\|^2}, \frac{-2\ln\|\nabla\boldsymbol{\varphi}\|}{C + C'}\right\}. \tag{4'}$$

This result of Heywood does not take into account the zero boundary condition of the problem. For example, if Ω is contained between two parallel planes, then the Friedrichs inequality holds: $\|v\| \le c_\Omega\|\nabla v\|$, $v \in C_0^\infty(\Omega)$. It was established in [13] that a solution decays exponentially for such domains.

Our result concerns the maximal rate of stabilization of a solution of the problem (1), (2), which is attained on initial functions with bounded support. It is natural to expect that, for example, in the class of domains of revolution the slower the expansion of the domain at infinity, the greater the rate of decrease to zero ensured by the zero boundary condition.

Our goal is to get an estimate of the rate of decay of a solution of the problem (1), (2) that takes into account the geometry of the unbounded domain. Such a formulation of the problem was first investigated by Gushchin for a parabolic equation with the second boundary condition. A survey of the literature on this question for a second-order parabolic equation can be found in [10].

Here we formulate the results for a domain of revolution. It will be proved for a broader class of domains satisfying the conditions A and B in §3.

Let Ω be a domain of revolution with boundary of class C^3 of the form

$$\Omega = \{x : x_1^2 + x_2^2 < f^2(x_3), x_3 > 0\}, \tag{5}$$

determined by a monotonically nondecreasing function $f(r) \in C^3(0, \infty)$. We require that the behavior of f be regular. Namely, there exists a constant $q \in (0, 1)$ such that

$$\lim_{r\to\infty} f(r)/f(qr) < \infty. \tag{6}$$

The condition

$$|f'| + |f''| + |f'''(r)| \le a_0, \qquad r \ge 1, \tag{7}$$

ensures that the boundary is uniformly of class C^3.

We define the function $r(t)$, $t > 0$, to be the inverse of the monotonically increasing function $rf(r)$, $r > 0$. Obviously, $r(t)$ increases monotonically to infinity and satisfies the equalities

$$\frac{t}{f^2(r(t))} = \frac{r(t)}{f(r(t))} = \frac{r^2(t)}{t}. \tag{8}$$

It is assumed everywhere that the initial function has bounded support:

$$\boldsymbol{\varphi}(x) = 0 \quad \text{for } |x| > R_0. \tag{9}$$

Theorem 1. *Let Ω be a domain of revolution of the form* (5) *satisfying the conditions described above. Suppose that the function $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$ satisfies the conditions* (4) *and* (9). *Then there exist positive constants κ and A such that the solution of the problem* (1), (2) *satisfies for all $x \in \Omega$ and $t > 1$ the estimates*

$$|\mathbf{u}(t, x)| \leq A \exp(-\kappa r^2(t)/t), \tag{10}$$

$$\|\mathbf{u}(t)\|_{\mathbf{W}_2^1(\Omega)} \leq A t^{1/2} \exp(-\kappa r^2(t)/t), \tag{11}$$

$$\|\nabla p(t)\| \leq A \exp(-\kappa r^2(t)/t). \tag{12}$$

The constant κ does not depend on the initial function $\boldsymbol{\varphi}$.

We remark that the equalities (8) imply the simple condition

$$\lim_{r \to \infty} r/f(r) = \infty, \tag{13}$$

which suffices for the exponential in the estimates (10)–(12) to converge to zero as $t \to \infty$. In particular, if a domain Ω with boundary of class C^3 has the form (5) with function $f(r) = r^\alpha$, $r > 1$, $\alpha \in (0, 1)$, then after uncomplicated computations we find that $r^2(t)/t = t^{(1-\alpha)/(1+\alpha)}$. For such a domain the estimate (10) takes the form

$$|\mathbf{u}(t, x)| \leq A \exp(-\kappa t^{(1-\alpha)/(1+\alpha)}).$$

We impose a condition stronger than (13):

$$\lim_{r \to \infty} \frac{f(r) \ln r}{r} = 0. \tag{14}$$

Then we get from the equalities (8) and the relation (7) that

$$\lim_{t \to \infty} \frac{\ln t}{r^2(t)/t} = \lim_{r \to \infty} \frac{f(r) \ln r}{r} = 0.$$

Consequently, the condition (14) suffices for the exponential in (11) to converge to zero more rapidly than any negative power of t.

An estimate of the rate of decrease analogous to (10) was proved earlier in [28] for solutions of the first mixed problem for the heat equation. Moreover, for domains of the form (5) and for nonnegative initial functions it was proved there that this estimate is sharp when the behavior of f has some regularity.

The assertion of Theorem 1 remains valid also for the strong solutions constructed by Serrin [33] and by Ladyzhenskaya [17]. Our choice of the strong solution as in Heywood [13] is due to certain simplifications in the proofs.

It is well known that for a strong solution the energy identity

$$\|\mathbf{u}(t)\|^2 + 2\nu \int_0^t \|\nabla \mathbf{u}(\tau)\|^2 \, d\tau = \|\boldsymbol{\varphi}\|^2 \tag{15}$$

holds instead of (3). The method of proof of Theorem 1 amounts to reducing the question of decay is a solution of the problem (1), (2) as $t \to \infty$ to the question of its decay as $|x| \to \infty$ with the help of the energy identity (see Theorem 4 in §4).

The problem of the decay of $\mathbf{u}(t, x)$ as $|x| \to \infty$ is solved by Theorem 3 in §3 on an estimate in the exterior of the ball $\Omega_r^\infty = \Omega \backslash \{|x| < r\}$. Namely, under the conditions of Theorem 1 there exist positive numbers Γ, γ, and A_1 such that for all $R \geq 1$ and $T > 0$ a solution of the problem (1), (2) satisfies the inequality

$$\int_0^T \int_{\Omega_R^\infty} \mathbf{u}_t^2(t, x) \, dx \, dt \leq A_1 \exp\left(\frac{\Gamma T}{f^2(R)} - \frac{\gamma R}{f(R)}\right). \tag{16}$$

The proof of Theorem 3 was our main difficulty. In the note [18] a class of uniqueness was described for the linearized system of Navier-Stokes equations in an unbounded domain. We have not been able to determine whether the method used there can be adapted to the nonlinear system (1) in order to get an estimate analogous to (16). The point is that the proofs are not given in [18], and the method itself is not expounded in detail.

The proof of (16) is based on an idea in the paper [20], in which the corresponding technique was used for the steady-state problems of Stokes and Navier-Stokes.

The presence of several "sleeves" of Ω is no obstruction to the use of our methods of proof. In particular, we allow domains with boundary of class C^3 that are representable as a union $\bigcup_{i=0}^{n} \Omega_i$ where Ω_0 is a bounded domain, and Ω_i, $i > 0$, are domains of revolution of the form (5) with noncollinear rays of revolution. Here f can be taken to be the function $f(r) = \max_i f_i(r)$.

2. In this section we present results from Heywood's paper in a stronger formulation.

Define the space $\mathbf{H}_0(\Omega)$ of solenoidal functions to be the completion of $\overset{\bullet}{\mathbf{J}}(\Omega)$ in the norm $\|\nabla \mathbf{v}\|$.

We formulate more completely the properties of the strong solution mentioned in the introduction. If the initial function satisfies the condition (4), then it is asserted in [13] that there is a unique strong solution $\mathbf{u}(t, x)$, $p(t, x) \in C^\infty(D)$ of the problem (1), (2) with the following properties:

$$\mathbf{u} \in C([0, \infty); \mathbf{H}_0(\Omega)), \tag{17}$$

$$\mathbf{u}_t \in L_2(0, T; \overset{\circ}{\mathbf{J}}(\Omega)) \quad \text{for } 0 < T < \infty, \tag{18}$$

$$D^2\mathbf{u}, \nabla p \in L^2(0, T; \mathbf{L}^2(\Omega)) \quad \text{for } 0 < T < \infty, \tag{19}$$

$$\nabla \mathbf{u}_t, D^2\mathbf{u} \in C((0, \infty); \mathbf{L}_2(\Omega)), \tag{20}$$

$$\|\nabla \mathbf{u}(t)\| \le a, \qquad t \ge 0, \tag{21}$$

$$\mathbf{u} \in C((0, \infty) \times \overline{\Omega}). \tag{22}$$

Here we use the notation

$$D^2\mathbf{u} = \left(\sum_{i,j,k=1}^{3} \left| \frac{\partial^2 u_i}{\partial x_j \partial x_k} \right|^2 \right)^{1/2}.$$

The constant $a(\nu, \|\varphi\|, d, b)$ in (21) ensures the maximum on the right-hand side of (4).

Remark. Instead of (18) it is asserted in [13] only that $\mathbf{u}_t \in L^2(0, T; \mathbf{L}^2(\Omega))$. We show that (18) actually holds. The solution of the problem (1), (2) is obtained as the weak limit of the Galërkin approximations $\mathbf{u}^{m,k}$ constructed for a sequence of bounded domains $\Omega_m \subset \Omega$. Since $\mathbf{H}_0(\Omega) \subset \overset{\circ}{\mathbf{J}}(\Omega)$ in a bounded domain, each Galërkin approximation satisfies (18). The relation (18) remains valid for the limit function—the solution—after passage to the weak limit.

It is well known (see, for example, [17], Chapter I, §2.2) that the orthogonal complement of $\overset{\circ}{\mathbf{J}}(\Omega)$ in $\mathbf{L}_2(\Omega)$ is

$$\mathbf{G}(\Omega) = \{\mathbf{v} : \mathbf{v} = \nabla p \text{ for some } p \in W^1_{2,\text{loc}}(\Omega) \text{ with } \nabla p \in \mathbf{L}_2(\Omega)\}.$$

The orthogonal projection of $\mathbf{L}_2(\Omega)$ onto $\overset{\circ}{\mathbf{J}}(\Omega)$ is denoted by P.

We need some inequalities. Let us introduce the notation

$$\nabla \mathbf{v} : \nabla \mathbf{w} = \sum_{i,j=1}^{3} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j},$$

$$\mathbf{u} \cdot \nabla \mathbf{v} \cdot \mathbf{w} = \sum_{i,j=1}^{3} u_j \frac{\partial v_i}{\partial x_j} w_i.$$

Let Ω be a domain in R^3 with boundary uniformly of class C^3. Suppose that the functions $\mathbf{w} \in \mathbf{H}_0(\Omega)$ and $\mathbf{f} \in \overset{\circ}{\mathbf{J}}(\Omega)$ satisfy the identity

$$\int_\Omega \nabla \mathbf{w} : \nabla \mathbf{v}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \tag{23}$$

for all $\mathbf{v} \in \overset{\bullet}{\mathbf{J}}(\Omega)$. Then $\mathbf{f}$ is uniquely determined by $\mathbf{w}$, hence we can introduce the operator $\mathbf{f} = \tilde{\Delta}\mathbf{w}$. Further, we have the inequalities (see [13])

$$\|D^2\mathbf{w}\| \le C_1(\|\tilde{\Delta}\mathbf{w}\| + \|\nabla \mathbf{w}\|), \tag{24}$$

$$\sup_{x\in\Omega} |\mathbf{w}(x)| \le C_2(\|\tilde{\Delta}\mathbf{w}\| + \|\nabla \mathbf{w}\|). \tag{25}$$

The constants C_1 and C_2 depend only on d, b, and ν. It follows easily from (24) and (23) and $\tilde{\Delta}\mathbf{w} = P\Delta\mathbf{w}$.

We prove an assertion that refines some results in [13].

Theorem 2. *Suppose that the boundary of Ω belongs uniformly to the class C^3 and the initial function satisfies* (4) *and* (9). *Then the solution of the problem* (1), (2) *satisfies the inequalities*

$$\|\nabla \mathbf{u}(T)\|^2 \le b_1 \|\mathbf{u}(t)\|^2/(T-t), \tag{26}$$

$$\|D^2\mathbf{u}(T)\|^2 \le b_2 \|\mathbf{u}(t)\|^2/(T-t) \quad \text{for } 1+t < T; \tag{27}$$

$$\|\nabla p(T)\|^2 \le b_3 \|\mathbf{u}(t)\|^2/(T-t) \quad \text{for } T > t+1, \tag{28}$$

$$|\mathbf{u}(T,x)| \le b_4 \|\mathbf{u}(t)\|/(T-t)^{1/2} \quad \text{for } x \in \Omega,\ T \ge t+1, \tag{29}$$

$$\int_0^\infty \|\mathbf{u}_t\|^2\, dt + \int_0^\infty \sup_{x\in\Omega} \mathbf{u}^2(t,x)\, dt < b_5. \tag{30}$$

Here and below, the letter b with indices denotes constants depending only on ν, $\|\varphi\|$, a in the inequality (21), and the numbers d and b in the definition of the uniform class C^3.

Proof. We recall the construction used in [13] to prove the basic assertions. An unbounded domain Ω can be approximated by a sequence of bounded domains Ω_m, $\bigcup_m \Omega_m = \Omega$, each of which has boundary uniformly of class C^3 with constants d and b independent of m. Under the condition (9) it is possible to choose Ω_m such that $\overline{\Omega}_m \supset \operatorname{supp} \varphi$. Then the solution of the problem (1), (2) for Ω is obtained as the weak limit (in suitable spaces) of the sequence $\mathbf{u}^m$ of solutions for the bounded domains Ω_m with the same initial function φ. In turn, each solution $\mathbf{u}^m$ is the weak limit as $k \to \infty$ of a subsequence of Galërkin approximations $\mathbf{u}^{m,k}$. We write some inequalities for the Galërkin approximations, omitting the indices for brevity:

$$\frac{d}{dt}\|\nabla \mathbf{u}\|^2 + \nu\|\tilde{\Delta}\mathbf{u}\|^2 \le C\|\nabla \mathbf{u}\|^4 + C'\|\nabla \mathbf{u}\|^6, \tag{31}$$

$$\|\mathbf{u}_t\| \le (\nu+1)\|\tilde{\Delta}\mathbf{u}\| + C_3(\|\nabla \mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^3), \tag{32}$$

$$\frac{\nu}{2}\|\tilde{\Delta}\mathbf{u}\| \le \|\mathbf{u}_t\| + C_3(\|\nabla \mathbf{u}\|^2 + \|\nabla \mathbf{u}\|^3), \tag{33}$$

$$\frac{d}{dt}\|\mathbf{u}_t\|^2 + \nu\|\nabla \mathbf{u}_t\|^2 \le C_4\|\nabla \mathbf{u}\|^4\|\mathbf{u}_t\|^2. \tag{34}$$

The constants C_3 and C_4 depend only on d, b, and ν. For the convenience of the reader we give the numbering of the corresponding relations in [13]: (31), (35), (74), (72). By Lemmas 8 and 10 of [13], the inequalities (21) and (3) are also valid for the Galërkin approximations (equality holds in (3), of course).

We first prove the inequalities (26) and (27) for the Galërkin approximations. In particular, it follows from (31) in view of (21) that

$$\nu\int_t^\infty \|\tilde{\Delta}\mathbf{u}\|^2\,ds \le \|\nabla \mathbf{u}(t)\|^2 + (C + C'a)\max_{s\ge t}\|\nabla \mathbf{u}(s)\|^2\int_t^\infty \|\nabla \mathbf{u}\|^2\,ds. \tag{35}$$

Another consequence of (31) and (21) is the differential inequality $G' \le \alpha(t)G$ for the function $G(t) = \|\nabla \mathbf{u}(t)\|^2$, in which $\alpha(t) = (C + C'a)\|\nabla \mathbf{u}(t)\|^2$. From it we get

$$G(t) \le G(s)\exp\int_s^t \alpha(\tau)\,d\tau.$$

By (3) with $s = 0$, this leads us to conclude that

$$G(t) \le b_6 G(s), \qquad t > s. \tag{36}$$

In a completely analogous way we can get from (34) the inequality

$$\|\mathbf{u}_t(t)\|^2 \le b_7\|\mathbf{u}_t(s)\|^2, \qquad t > s. \tag{37}$$

An elementary consequence of (35), (21), and (3) is the estimate

$$\int_0^\infty \|\tilde{\Delta}\mathbf{u}\|^2\,ds \le b_8. \tag{38}$$

From (38), using (32), (21), and (3), we obtain

$$\int_0^\infty \|\mathbf{u}_t\|^2\,ds \le b_9. \tag{39}$$

Let us substitute $t = T$ in (36) and integrate with respect to $s \in (t, T)$. We get $(T-t)G(T) \le b_6\int_t^T G(s)\,ds$, which together with (3) yields (26).

From (35), (3), and (26) it is easy to get the inequality

$$\int_T^\infty \|\tilde{\Delta}\mathbf{u}\|^2\,ds \le b_{10}\|\mathbf{u}(t)\|^2/(T-t). \tag{40}$$

We square (32) and integrate:

$$\int_T^\infty \|\mathbf{u}_t\|^2\,ds \le b_{11}\left(\int_T^\infty \|\tilde{\Delta}\mathbf{u}\|^2\,ds + \max_{s>T} G(s)\int_T^\infty \|\nabla \mathbf{u}\|^2\,ds\right).$$

Applying (40), (26), and (3) to the last inequality, we have

$$\int_T^\infty \|\mathbf{u}_t\|^2\,ds \le b_{12}\|\mathbf{u}(t)\|^2/(T-t). \tag{41}$$

Let us substitute $t = T + \tau$ in (37) and integrate with respect to $s \in (T, T+\tau)$. Together with (41), this yields

$$\tau \|\mathbf{u}_t(T+\tau)\|^2 \leq b_{13}\|\mathbf{u}(t)\|^2/(T-t).$$

In particular, for $\tau = T - t$ and $s = T + \tau \geq t$ this gives the inequality

$$\|\mathbf{u}_t(s)\|^2 \leq 4b_{13}\|\mathbf{u}(t)\|^2/(s-t)^2.$$

By using the relations (33), (21), and (26), it is now not hard to see that

$$\|\tilde{\Delta}\mathbf{u}(T)\| \leq b_{14}\|\mathbf{u}(t)\|/(T-t), \qquad T > t.$$

We prove that the inequalities (38), (39), (26), and (27), which were established up to now for the Galërkin approximations $\mathbf{u}^{m,k}$, remain valid also for the solution of the problem (1), (2). The constants b with indices are independent of m and k; therefore, the inequalities (38) and (39) are obviously preserved after passage to the weak limits in the corresponding spaces as $k \to \infty$ and $m \to \infty$. Then we conclude from (38) and (15) with the help of (25) that

$$\int_0^\infty \sup_{x\in\Omega} \mathbf{u}^2(t, x)\, dt < b_{15}.$$

Together with (39) this yields (30).

We proceed to a proof of the inequalities (26) and (27) for the solution of the problem (1), (2). Fix $t > 0$. We choose a subsequence $\mathbf{u}^{m,k}$ for which the limit

$$\lim_{k\to\infty} \|\mathbf{u}^{m,k}(t)\| = L$$

exists. Using the equalities (3) for the Galërkin approximations $\mathbf{u}^{m,k}$ and (15) for the solutions $\mathbf{u}^m$, we see that $L \leq \|\mathbf{u}^m(t)\|$. Indeed,

$$\begin{aligned}\lim_{k\to\infty} \|\mathbf{u}^{m,k}(t)\|^2 &= \lim_{k\to\infty} \left(\|\mathbf{u}^{m,k}(0)\|^2 - 2\nu \int_0^t \|\nabla \mathbf{u}^{m,k}(s)\|^2\, ds\right) \\ &\leq \|\varphi\|^2 - 2\nu \int_0^t \|\nabla \mathbf{u}^m(s)\|^2\, ds = \|\mathbf{u}^m(t)\|^2.\end{aligned}$$

Here we have used the fact that the subsequence $\mathbf{u}^{m,k}$ converges weakly to the function $\mathbf{u}^m$ in the space $L_2(0, t; \mathbf{H}_0(\Omega_m))$ as $k \to \infty$. As is known, the norm

$$\int_0^t \|\nabla \mathbf{u}^m(s)\|^2\, ds$$

of the limit function does not exceed the limit inferior

$$\varliminf_{k\to\infty} \int_0^t \|\nabla \mathbf{u}^{m,k}(s)\|^2\, ds.$$

We prove that the inequality (26) holds for each $\mathbf{u}^m$. Suppose that it fails for some $\mathbf{u}^m$ for some $T > t$. By the continuity property (17), it fails also in some interval $[T, T+\varepsilon]$, so that

$$\int_T^{T+\varepsilon} \|\nabla \mathbf{u}^m(s)\|^2\, ds > \|\mathbf{u}^m(t)\|^2 \int_T^{T+\varepsilon} \frac{b_1}{s-t}\, ds \equiv M\|\mathbf{u}^m(t)\|^2. \tag{42}$$

On the other hand, integrating the inequality (26) written for the Galërkin approximations, we get

$$\int_T^{T+\varepsilon} \|\nabla \mathbf{u}^{m,k}(s)\|^2\, ds \leq M\|\mathbf{u}^{m,k}(t)\|^2.$$

Passage to the limit as $k \to \infty$ gives the inequality

$$\int_T^{T+\varepsilon} \|\nabla \mathbf{u}^m(s)\|^2 \, ds \le ML^2,$$

which contradicts (42). Consequently, (26) holds for the functions $\mathbf{u}^m$ for all $T > t$.

In exactly the same way we pass in (26) from the functions $\mathbf{u}^m$ to the solution of the problem (1), (2).

The proof of (27) is analogous to that of (26).

Since $\|\tilde{\Delta}\mathbf{u}\| = \|P\Delta\mathbf{u}\| \le \|D^2\mathbf{u}\|$, the inequality (29) is a consequence of (26), (27), and (25). To prove (28) note that

$$\begin{aligned} \|\mathbf{u}\cdot\nabla\mathbf{u}(T)\| &\le \max_{x\in\Omega} |\mathbf{u}(t, x)| \, \|\nabla\mathbf{u}(T)\| \\ &\le b_4 b_1^{1/2} \|\mathbf{u}(t)\|^2/(T-t), \qquad T \ge t+1. \end{aligned} \tag{43}$$

Further, from the equations (1),

$$\|\nabla p(T)\| = \|(1-P)(\nu\Delta\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u})\|.$$

Now (28) follows from (27) and (43). Theorem 2 is proved.

3. An estimate of a solution in the exterior of a ball

Here we shall not assume that Ω has the concrete form (5). Instead, we require that it satisfy the conditions formulated below.

Let $B_r = \{x \in R_3 : |x| < r\}$ be a ball of radius r, and let $\Omega_r = \Omega \cap B_r$. We say that Ω satisfies *condition* A if:

A) There exist a number $q \in (0, 1)$ and an absolutely continuous monotonically nondecreasing function $l(r)$, $r > 0$, such that

$$\varlimsup_{r\to\infty} l(r)/l(qr) < \infty, \tag{44}$$

$$r + l(r) \le r/q, \qquad r \ge R_0, \tag{45}$$

and for any $v \in C_0^\infty(\Omega)$

$$\int_{\Omega_r} v^2 \, dx \le a l^2(r) \int_{\Omega_r} |\nabla v|^2 \, dx, \qquad r \ge R_0. \tag{46}$$

It can be assumed without loss of generality that the constants R_0 in (45), (46), and (9) coincide.

It is not hard to prove (see [29]) that condition A is satisfied, for example, by domains of revolution of the form (5) with the function $l(r) = f(r)$ if the conditions (6) and (7) hold.

An important role in our method is played by estimates of the solution of the problem D:

$$\operatorname{div}\mathbf{v}(x) = f(x), \qquad \mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_2^1(Q), \qquad \int_Q f(x)\,dx = 0.$$

In general, the *problem* D has infinitely many solutions. In [20] it is proved that for a bounded domain Q with Lipschitz boundary there exists a unique solution $\mathbf{w}$ of problem D for which $\inf\|\nabla\mathbf{v}\|_Q$ is attained. It is this solution we have in mind below. For it

$$\|\nabla\mathbf{w}\|_Q \le d_1(Q)\|f\|_Q. \tag{47}$$

In order to distinguish families of domains for which we can choose the same constant d_1 in (47) for each domain in the family, it is useful to know the dependence of d_1 on Q.

Denote by $\widehat{L}_2(Q)$ the subset of functions in $L_2(Q)$ with zero mean value. In [2] an operator R_ρ: $\widehat{L}_2(R^n) \to \mathbf{W}_2^1(R^n)$ depending on the parameter $\rho > 0$ was constructed with the property that if the bounded domain Q is starlike with respect to the ball B_ρ and $\operatorname{supp} f \subset \overline{Q}$, then $\operatorname{supp} R_\rho f \subset \overline{Q}$. Further, if $\mathbf{v} = R_\rho f$, then $\mathbf{v}$ is a solution of problem D and the inequality $\|\nabla \mathbf{v}\| \leq d_2 \|f\|$ holds, with the constant d_2 dependent only on the starlikeness ratio $\operatorname{diam} Q/\rho$. Thus, the constant $d_1(Q)$ in the inequality (47) can be chosen the same for the family of starlike domains with the same starlikeness ratio.

We say that a domain Ω satisfies *condition* B if it satisfies condition A and

B) The set $\Omega \backslash B_{R_0}$ decomposes into N connected "sleeves", and there exists a single constant d_1 in (47) for the family of domains $\omega^i(r)$, $i = 1, \dots, N$, $r \geq R_0$, where the $\omega^i(r)$ are the connected components of the set

$$\omega(r) = \{x \in \Omega : r < |x| < r + l(r)\}, \qquad r \geq R_0,$$

belonging to the different "sleeves".

For a domain of revolution of the form (5) it is natural to define the sets Ω_r and $\omega(r)$ somewhat differently: $\Omega_r = \{x \in \Omega : x_3 < r\}$, $\omega(r) = \{x \in \Omega : r < x_3 < r + f(r)\}$. Then the domain Ω satisfies condition B if the requirements (6) and (7) hold. Indeed, the facts that f is monotonically nondecreasing and the derivative f' is bounded allow us to conclude that the family $\omega(r)$ can be regarded as a family of domains with the same starlikeness ratio.

In the next example Ω is a domain of revolution with a cylinder removed. For such a domain the use of the method in [29] for the case of the linearized problem seems problematical.

Let $\widehat{\Omega} = \Omega \backslash \{x \in R^3 : x_1^2 + x_2^2 \leq 1\}$, where Ω is a domain of revolution of the form (5) with the conditions (6) and (7). Then the domains $\omega(r)$ are not starlike. But each of them can be represented as the union, for example, of four domains with the same starlikeness ratios. Using results from [6], we establish that in this case it is possible to choose the same constant d_1 for the family of domains $\omega(r)$. Consequently, condition B holds for the domain $\widehat{\Omega}$.

If the domain of revolution

$$\Omega = \{x : x_1^2 + x_2^2 < f^2(x_3), x_3 \in (-\infty, \infty)\}$$

has "sleeves" expanding differently as $x_3 \to \infty$ and as $x_3 \to -\infty$, then condition B may fail. But the proofs below can be modified in such a way that they work also for such a domain. It is necessary simply to estimate the decay of a solution along each "sleeve" separately as $|x| \to \infty$.

The inequality (53) is the basic technical result to be used in proving the estimate (16) in the exterior of a ball.

It is easy to see that the value of the number ν can be reduced to 1 by dilations of the vector $\mathbf{u}$, the pressure p, and the variable x. Therefore, for the sake of some simplifications the proofs will be carried out under the assumption that $\nu = 1$.

Suppose that the conditions of Theorem 2 hold and $\mathbf{u}(t, x)$ is a solution of the problem (1), (2). Let

$$M(t) = \sup_{x \in \Omega} \mathbf{u}^2(t, x), \qquad g(t, r) = M(t) + 2/l^2(r).$$

In view of (30) we have the simple inequality

$$\int_0^t g(\tau, r)\, d\tau \leq b_{16} + 2t/l^2(r). \tag{48}$$

We define the cutoff function $\eta(x)$ by

$$\eta(x) = \xi((|x| - r)/l(r)),$$

where $\xi(r)$ is a continuous function equal to 0 for $r < 0$, equal to 1 for $r > 1$, and linear on the remaining interval. Then the support of the gradient η lies in the closure of the set $\omega(r)$, and

$$\nabla\eta = \frac{x}{l(x)|x|}. \tag{49}$$

Moreover, for $r \geq R_0$

$$-\frac{\partial\eta}{\partial r} = \frac{1}{l(r)} + \frac{|x| - r}{l^2(r)} l'(r) \geq \frac{1}{l(r)}, \qquad x \in \omega(r). \tag{50}$$

We introduce the notation

$$\Theta(t) = \exp\left(-\int_0^t M(\tau)\, d\tau\right)$$

and

$$H(t, r) = \Theta(t)\left\{\int_\Omega \eta|\nabla\mathbf{u}(t, x)|^2\, dx + \int_0^t \int_\Omega \eta\mathbf{u}_t^2\, dx\, d\tau\right\}. \tag{51}$$

We note the inequalities

$$H(t, r) \leq b_{17}, \qquad \Theta(t) \geq b_{18} > 0, \qquad t > 0, \quad r \geq R_0, \tag{52}$$

which follow from (21) and (30).

Lemma. *Suppose that Ω has boundary uniformly of class C^3, condition* B *holds, and the initial function satisfies the relations* (4) *and* (9). *Then there is a number β such that for all $t > 0$ and $r \geq R_0/q^2$*

$$H(t, r) \leq -\beta l(r)\left(H_r(t, r) + \int_0^t g(\tau, r) H_r(\tau, r)\, d\tau\right). \tag{53}$$

Here the index r denotes the derivative, and the constant β does not depend on the initial function φ.

Proof. Suppose that the set F consists of the values of $t > 0$ for which $\mathbf{u}_t(t, x) \in \overset{\circ}{\mathbf{J}}(\Omega)$. By the property (18), the measure of the complement $(0, \infty)\backslash F$ is equal to 0. We fix one of the values $t \in F$.

Let us multiply the Navier-Stokes equations by the function $\eta\mathbf{u}_t$ and integrate over Ω. After uncomplicated transformations we arrive at the equality

$$\begin{aligned} &\int_\Omega \mathbf{u}_t^2 \eta\, dx + \int_\Omega \eta\nabla\mathbf{u} \colon \nabla\mathbf{u}_t\, dx \\ &= -\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\eta\mathbf{u}_t\, dx - \int_{\omega(r)} \left(\sum_{i,j=1}^3 (u_i)_t \frac{\partial u_i}{\partial x_j}\frac{\partial\eta}{\partial x_j} - p(\mathbf{u}_t, \nabla\eta)\right) dx. \end{aligned} \tag{54}$$

We obtain upper estimates for the terms on the right-hand side:

$$\left|\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\eta\mathbf{u}_t\,dx\right| \le \int_\Omega \eta(\mathbf{u}^2|\nabla\mathbf{u}|^2+\mathbf{u}_t^2)/2\,dx$$

$$\le \int_\Omega \eta(M(t)|\nabla\mathbf{u}|^2+\mathbf{u}_t^2)/2\,dx.$$

Further, in view of (49),

$$\left|\int_{\omega(r)}\sum_{i,j=1}^3 (u_i)_t\frac{\partial u_i}{\partial x_j}\frac{\partial\eta}{\partial x_j}\,dx\right| \le \int_{\omega(r)}|\nabla\mathbf{u}|\,|\mathbf{u}_t|/l(r)\,dx$$

$$\le \int_{\omega(r)}(|\nabla\mathbf{u}|^2/l^2(r)+\mathbf{u}_t^2/2)\,dx.$$

To estimate the remaining integral we prove that $\int_{\omega^i}(\mathbf{u}_t,\nabla\eta)\,dx = 0$ for each connected component ω^i of the set $\omega(r)$, $r>R_0$. Since $\mathbf{u}_t(t,x)\in\overset{\circ}{\mathbf{J}}(\Omega)$, it suffices to establish the equality for vectors $\mathbf{v}\in\overset{\bullet}{\mathbf{J}}(\Omega)$. The latter is obvious, because the flow of a solenoidal vector with compact support in Ω across the section $S^i(r)=\{x\in\omega^i : |x|=r\}$ of the ith "sleeve" is equal to zero:

$$0=\int_{S^i(r)}(\mathbf{v},x)/|x|\,dS = l(r)\int_{S^i(r)}(\mathbf{v},\nabla\eta)\,dS.$$

Thus, by condition B there is a vector $\mathbf{w}\in\overset{\circ}{\mathbf{W}}{}_2^1(\omega(r))$ such that $\operatorname{div}\mathbf{w}=(\mathbf{u}_t,\nabla\eta)$ and

$$\|\nabla\mathbf{w}\|_{\omega(r)}\le d_1\|(\mathbf{u}_t,\nabla\eta)\|_{\omega(r)}\le\frac{d_1}{l(r)}\|\mathbf{u}_t\|_{\omega(r)}.$$

The Friedrichs inequality enables us to get the estimate

$$\|\mathbf{w}\|_{\omega(r)}\le c\|\mathbf{u}_t\|_{\omega(r)}.$$

We can now write the following chain of inequalities for the remaining integral on the right-hand side of (54):

$$\left|\int_{\omega(r)}p(\mathbf{u}_t,\nabla\eta)\,dx\right| = \left|\int_{\omega(r)}p\operatorname{div}\mathbf{w}\,dx\right| = \left|\int_{\omega(r)}\nabla p\cdot\mathbf{w}\,dx\right|$$

$$=\left|\int_{\omega(r)}(\mathbf{u}_t-\Delta\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{w}\,dx\right|$$

$$\le\int_{\omega(r)}[(\mathbf{u}_t^2+\mathbf{u}^2|\nabla\mathbf{u}|^2)/2+\mathbf{w}^2+|\nabla\mathbf{u}:\nabla\mathbf{w}|]\,dx$$

$$\le\int_{\omega(r)}[(\mathbf{u}_t^2+M(t)|\nabla\mathbf{u}|^2)/2+c^2\mathbf{u}_t^2+(|\nabla\mathbf{u}|^2/l^2(r)+d_1^2\mathbf{u}_t^2)/2]\,dx.$$

Let us substitute the estimates just obtained in (54). After simple transformations we have

$$\int_\Omega\mathbf{u}_t^2\eta\,dx+\frac{d}{dt}\int_\Omega\eta|\nabla\mathbf{u}|^2\,dx$$

$$\le\int_\Omega M(t)\eta|\nabla\mathbf{u}|^2\,dx+\int_{\omega(r)}[(2/l^2(r)+M(t))|\nabla\mathbf{u}|^2+\beta\mathbf{u}_t^2]\,dx.$$

Integration with respect to t over (s, T) yields the inequality

$$\int_s^T \int_\Omega \eta \mathbf{u}_t^2 \, dx \, dt + \int_\Omega \eta |\nabla \mathbf{u}(T, x)|^2 \, dx \leq \int_s^T \int_\Omega \eta M(t) |\nabla \mathbf{u}|^2 \, dx \, dt + \int_\Omega \eta |\nabla \mathbf{u}(s, x)|^2 \, dx + h(T),$$

where

$$h(T) \equiv h_1(T) + h_2(T) = \int_0^T \int_{\omega(r)} g(t, r) |\nabla \mathbf{u}|^2 \, dx \, dt + \beta \int_0^T \int_{\omega(r)} \mathbf{u}_t^2 \, dx \, dt.$$

The properties of the solution of (1), (2) formulated in Theorem 2 enable us to pass to the limit as $s \to 0$. In view of the identity $\eta\varphi \equiv 0$ for $r \geq R_0$, one of the integrals on the right-hand side disappears:

$$\int_0^T \int_\Omega \eta \mathbf{u}_t^2 \, dx \, dt + \int_\Omega \eta |\nabla \mathbf{u}(T, x)|^2 \, dx \leq \int_0^T \int_\Omega \eta M(t) |\nabla \mathbf{u}|^2 \, dx \, dt + h(T). \tag{55}$$

From (55), we get the differential inequality $z' \leq M(T)(z + h(T))$ for the function

$$z(T) = \int_0^T \int_\Omega \eta M(t) |\nabla \mathbf{u}|^2 \, dx \, dt.$$

Consequently,

$$zT \leq \int_0^T \exp\left(\int_t^T M(\tau) \, d\tau\right) M(t) h(t) \, dt = -\int_0^T h(t) \Theta'(t)/\Theta(T) \, dt. \tag{56}$$

By integrating by parts we easily establish that

$$-\int_0^T h_1(t) \Theta'(t)/\Theta(T) \, dt = -h_1(T) + \int_0^T h_1'(t) \Theta(t)/\Theta(T) \, dt.$$

Combining this with (56) and substituting in (55), we have

$$\int_0^T \int_\Omega \eta \mathbf{u}_t^2 \, dx \, dt + \int_\Omega \eta |\nabla \mathbf{u}(T, x)|^2 \, dx \leq h_2(T) + \int_0^T (h_1'(t) + M(t) h_2(t)) \Theta(t)/\Theta(T) \, dt.$$

Multiplying by $\Theta(T)$ and using the notation (51), we get

$$\begin{aligned} H(T, r) \leq \beta \Theta(T) \int_0^T \int_{\omega(r)} \mathbf{u}_t^2 \, dx \, dt \\ + \int_0^T g(t, r) \Theta(t) \left[\int_{\omega(r)} |\nabla \mathbf{u}|^2 \, dx \, dt + \beta \int_0^t \int_{\omega(r)} \mathbf{u}_t^2 \, dx \, d\tau\right] dt. \end{aligned} \tag{57}$$

From (50),

$$-l(r) \frac{\partial H(t, r)}{\partial r} \geq \Theta(t) \left[\int_{\omega(r)} |\nabla \mathbf{u}|^2 \, dx \, dt + \int_0^t \int_{\omega(r)} \mathbf{u}_t^2 \, dx \, d\tau\right],$$

hence the assertion of the lemma follows from (57).

Theorem 3. *Assume the conditions of the lemma. Then there exist positive numbers* Γ, γ, *and* A_1 *such that for all* $R \geq 1$ *and* $T > 0$ *the solution of the problem* (1), (2) *satisfies the inequality*

$$\int_0^T \int_{\Omega_R^\infty} \mathbf{u}_t^2(t, x)\, dx\, dt \leq A_1 \exp\left(\frac{\Gamma T}{l^2(R)} - \frac{\gamma R}{l(R)}\right). \tag{58}$$

The constants Γ *and* γ *do not depend on the initial function* φ.

Proof. Fix a number $r_0 \geq R_0$. Let the function $r(s)$ be the solution of the Cauchy problem $r' = \beta l(r)$, $r(0) = r_0$, and let $h(t, s) = H(t, r(s))$. The function $g(t, r)$ is monotonically nonincreasing with respect to r. Therefore, fixing the argument $r = r_0$ of this function, we can rewrite the inequality (53) in the form

$$h(t, s) \leq -h_s(t, s) - \int_0^t g(\tau, r_0) h_s(\tau, s)\, d\tau, \qquad s \geq 0.$$

Integrating the last inequality with respect to s, we establish that

$$\int_s^\infty h(t, \rho)\, d\rho \leq h(t, s) + \int_0^t g(\tau, r_0) h(\tau, s)\, d\tau, \qquad s \geq 0.$$

Continuing the integration, we get by induction on n the inequality

$$\int_0^\infty \frac{s^{n-1}}{(n-1)!} h(t, s)\, ds < \sum_{i=0}^n \binom{h}{i} (G^i h)(t, 0), \tag{59}$$

where G denotes the integral operator

$$(Gh)(t, s) = \int_0^t g(\tau, r_0) h(\tau, s)\, d\tau.$$

Using the nonnegativity of the function h and the fact that it is monotonically nonincreasing in the second argument, we can write

$$\int_{s/2}^s \rho^{n-1} h(t, \rho)\, d\rho \geq \left(\frac{s}{2}\right)^n h(t, s).$$

The binomial coefficients do not exceed 2^n, and if we estimate the function h from above by the number b_{17} in (52), then the right-hand side of (59) does not exceed the number

$$2^n b_{17} \exp \int_0^t g(\tau, r_0)\, d\tau.$$

If we now use the relation (48) and Stirling's formula, we get the following consequence of (59):

$$h(t, s) \leq c \left(\frac{4}{s}\right)^n \frac{n!}{n} \exp \frac{2t}{l^2(r_0)} \leq c_1 \left(\frac{4n}{se}\right)^n \exp \frac{2t}{l^2(r_0)}.$$

Here we substitute $n = [s/4] \geq s/4 - 1$. This yields

$$h(t, s) \leq c_1 \exp\left(\frac{2t}{l^2(r_0)} + 1 - \frac{s}{4}\right), \qquad s \geq 0.$$

Returning to the variable r, we have

$$H(t, r) \leq c_1 \exp\left(\frac{2t}{l^2(r_0)} - \int_{r_0}^r \frac{d\rho}{4\beta l(\rho)}\right), \qquad r \geq r_0. \tag{60}$$

Let $R = r_0/q^2$ and $r = qR$. Then (44) and the fact that $l(r)$ is monotonically nondecreasing give us the inequalities

$$\int_{r_0}^{r} \frac{d\rho}{4\beta l(\rho)} \geq c_3 R/l(R), \qquad l^2(r_0) \geq c_3 l^2(R); \qquad r_0 \geq R_0,$$

where the constant c_3 is independent of r_0. Using (45), we conclude that $\eta(x, r) = 1$ for $|x| > R$. Therefore, from (60), the notation (51), and the boundedness of $\mathbf{\Theta}(t)$ from below (see (52)) we get the estimate (58) for $R \geq R_0/q^2$. It is valid also for the remaining values of $R \geq 1$ in view of the boundedness of the left-hand side of the inequality and the continuity of the right-hand side with respect to R.

Corollary. *Assume the conditions of the lemma. Then the solution of the problem* (1), (2) *for all* $t > 0$ *and* $R \geq 1$ *satisfies the estimate*

$$\int_{\Omega_R^\infty} \mathbf{u}^2(t, x)\,dx < tA_1 \exp\left(\frac{\Gamma t}{l^2(2R)} - \frac{\gamma R}{l(2R)}\right). \tag{61}$$

The constants A_1, Γ, *and* γ *are the same as in Theorem* 3.

The proof of the corollary is not complicated (see, for example, the proof of the analogous assertion in [29]), and thus is not given here.

4. Proof of Theorem 1

We first prove an assertion analogous to Theorem 1 for the class of domains satisfying condition B. Then we derive Theorem 1 from it as a simple corollary.

We define the function $\rho(t)$, $t > 0$, as the inverse of the monotonically increasing continuous function $l(\rho)$, $\rho > 0$.

Theorem 4. *Suppose that the domain* Ω *satisfies the conditions of the lemma in* §3. *Then there exist positive numbers* κ *and* A_2 *such that the solution of the problem* (1), (2) *satisfies for all* $t \geq 2$ *the estimates*

$$\|\mathbf{u}(t)\| \leq A_2 t^{1/2} \exp(-\kappa \rho^2(t)/t), \tag{62}$$

$$\|\nabla \mathbf{u}(t)\| \leq A_2 \exp(-\kappa \rho^2(t)/t), \tag{63}$$

$$\|D^2 \mathbf{u}(t)\| \leq A_2 \exp(-\kappa \rho^2(t)/t), \tag{64}$$

$$\sup_{x \in \Omega} |\mathbf{u}(t, x)| \leq A_2 \exp(-\kappa \rho^2(t)/t), \tag{65}$$

$$\|\nabla p\| \leq A_2 \exp(-\kappa \rho^2(t)/t). \tag{66}$$

The constant κ *does not depend on* $\boldsymbol{\varphi}$.

Proof. The Γ and γ be the numbers in Theorem 3. Since $\rho(t)$ tends to infinity as $t \to \infty$, there is a number $T > 1$ such that $\rho(2\Gamma t/\gamma) \geq 2$ for all $t \geq T$.

Fix $t \geq T$ and let $R = \rho(2\Gamma t/\gamma)/2$. We have equalities analogous to (8):

$$\frac{2\Gamma t}{l^2(R)} = \frac{\gamma R}{l(R)} = \frac{\gamma^2 R^2}{2\Gamma t}. \tag{67}$$

Consequently, the estimate (61) can be represented in the form

$$\int_{\Omega_R^\infty} \mathbf{u}^2(\tau, x)\,dx \leq \delta \equiv tA_2 \exp\left(-\frac{\gamma^2 R^2}{4\Gamma t}\right) \tag{68}$$

for all $\tau \in [0, t]$.

We write the inequality (46) for the vector $\mathbf{u}$ when $\tau \geq 0$:

$$\frac{1}{al^2(R)} \int_{\Omega_R} \mathbf{u}^2(\tau, x)\,dx \leq \int_{\Omega_R} |\nabla \mathbf{u}(\tau, x)|^2\,dx \leq \|\nabla \mathbf{u}(\tau, x)\|^2.$$

Using (68) and (15), we derive a differential inequality for the absolutely continuous function $E(\tau) = \|\mathbf{u}(\tau)\|^2$:

$$\frac{1}{al^2(R)}(E(\tau) - \delta) \leq \frac{1}{al^2(R)} \int_{\Omega_R} \mathbf{u}^2(\tau, x)\,dx$$
$$\leq \|\nabla \mathbf{u}(\tau)\|^2 = -\frac{1}{2}\frac{d}{d\tau}E(\tau), \qquad \tau \in [0, t].$$

Solving it for the monotonically nonincreasing function $E(\tau)$, we get the estimate

$$E(t) \leq \delta + E(0)\exp(-2t/al^2(R)).$$

Replacement of δ by its value and use of the equalities (67) gives us

$$E(t) \leq t(E(0) + A_2)\exp(-\kappa\rho^2(2\Gamma t/\gamma)/t),$$

where $\kappa = \min\{\gamma^2/4\Gamma, \gamma^2/2a\Gamma^2\}$. It can be assumed without loss of generality that $2\Gamma/\gamma \geq 1$. In view of the monotone increase of $\rho(t)$, the inequality (62) is thereby proved for $t \geq T$. It remains valid also for $t \in [1, T]$ by the boundedness of the left-hand side and the continuity of the right-hand side with respect to t.

The relations (63)–(66) follows from (62) and Theorem 2.

Proof of Theorem 1. As noted in §3, all the requirements of the lemma hold under the conditions of Theorem 1, and $l(r) = f(r)$. Consequently, Theorem 4 is applicable. Since $\rho(t) \equiv r(t)$, Theorem 1 is a consequence of Theorem 4.

BIBLIOGRAPHY

1. H. Beirão da Veiga, *Existence and asymptotic behavior for strong solutions of Navier-Stokes equations in the whole space*, Indiana Univ. Math. J. **36** (1987), 149–166.
2. M. E. Bogovskiĭ, *Solution of some problems of vector analysis connected with the operators* div *and* grad, Theory of Cubature Formulas and the Application of Functional Analysis to Problems of Mathematical Physics (Proc. Sem. S. L. Sobolev, Part 1), Inst. Mat. Sibirsk Otdel. Akad Nauk SSSR, Novosibirsk, 1980, pp. 5–40. (Russian)
3. M. E. Bogovskiĭ and V. N. Maslennikova, *On Sobolev systems with three space variables*, Partial Differential Equations (Proc. Sem. S. L. Sobolev, Part 2), Inst. Mat. Sibirsk. Otdel. Akad. Nauk SSSR, Novosibirsk, 1976, pp. 49–69. (Russian)
4. Wolfgang Borchers and Tetsuro Miyakawa, *L^2 decay for the Navier-Stokes flow in halfspaces*, Math. Ann. **282** (1988), 139–155.
5. ______, *Algebraic L^2 decay for Navier-Stokes flows in exterior domains*, Acta Math. **165** (1990), 189–227.
6. Wolfgang Borchers and Hermann Sohr, *On the equations not* $\mathbf{v} = \mathbf{g}$ *and* $\operatorname{div} \mathbf{u} = f$ *with zero boundary conditions*, Hokkaido Math. J. **19** (1990), 67–87.
7. Giovanni P. Galdi and Paolo Maremonti, *Monotonic decreasing and asymptotic behavior of the kinetic energy for weak solutions of the Navier-Stokes equations in exterior domains*, Arch. Rational Mech. Anal. **94** (1986), 253–266.
8. A. V. Glushko, *Time asymptotics of the solution of the Cauchy problem for the linearized system of Navier-Stokes equations with zero right-hand side*, Theory of Cubature Formulas and the Application of Functional Analysis to Problems of Mathematical Physics (Proc. Sem. S. L. Sobolev, Part 1), Inst. Mat. Sibirsk Otdel. Akad Nauk SSSR, Novosibirsk, 1981, pp. 5–33. (Russian)
9. A. V. Glushko and V. N. Maslennikova, *Localization theorems of Tauberian type and the rate of decay of a solution of the system of the hydrodynamics of a viscous compressible fluid*, Trudy Mat. Inst. Steklov **181** (1988), 156–187; English transl. in Proc. Steklov Inst. Math. **1989**, no. 4 (181).

10. A. K. Gushchin, *On uniform stabilization of solutions of the second mixed problem for a parabolic equation*, Mat. Sb. **119** **(161)** (1982), 451–508; English transl. in Math. USSR Sb. **47** (1984).

11. Eberhard Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nachr. **4** (1951), 213–231.

12. John G. Heywood, *On uniqueness questions in the theory of viscous flow*, Acta Math. **136** (1976), 61–102.

13. ——, *The Navier-Stokes equations: on the existence, regularity and decay of solutions*, Indiana Univ. Math. J. **29** (1980), 639–681.

14. Ryuji Kajikiya and Tetsuro Miyakawa, *On L^2 decay of weak solutions of the Navier-Stokes equations in $\mathbf{R}^n$*, Math. Z. **192** (1986), 135–148.

15. Tosio Kato, *Strong L^p-solutions of the Navier-Stokes equation in $\mathbf{R}^m$, with applications to weak solutions*, Math. Z. **187** (1984), 471–480.

16. A. A. Kiselev and O. A. Ladyzhenskaya, *On existence and uniqueness of a solution of the time-dependent problem for a viscous incompressible fluid*, Izv. Akad. Nauk SSSR Ser. Mat. **21** (1957), 655–680; English transl. in Amer. Math. Soc. Transl. (2) **24** (1963).

17. O. A. Ladyzhenskaya, *Mathematical questions in the dynamics of a viscous incompressible fluid*, 2nd rev. aug. ed., "Nauka", Moscow, 1970; English transl. of 1st ed., *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York, 1963; revised 1969.

18. O. A. Ladyzhenskaya and V. A. Solonnikov, *On an initial-boundary-value problem for linearized Navier-Stokes equations in domains with noncompact boundaries*, Trudy Mat. Inst. Steklov **159** (1983), 37–40; English transl. in Proc. Steklov Inst. Math. **1984**, no. 2 (159).

19. ——, *On some problems of vector analysis and on generalized statements of boundary value problems for Navier-Stokes equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **59** (1976), 81–116; English transl. in J. Soviet Math. **10** (1978), no. 2.

20. ——, *On the determination of solutions of boundary value problems for steady-state Stokes and Navier-Stokes equations having an unbounded Dirichlet integral*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **96** (1980), 117–160; English transl. in J. Soviet Math. **21** (1983), no. 5.

21. Jean Leray, *Sur le mouvement d'un liquide visqueux emplissant l'espace*, Acta Math. **63** (1934), 193–248.

22. Paolo Maremonti, *On the asymptotic behaviour of the L^2-norm of suitable weak solutions to the Navier-Stokes equations in three-dimensional exterior domains*, Comm. Math. Phys. **118** (1988), 385–400.

23. Kyûya Masuda, *On the stability of viscous incompressible fluid motions past objects*, J. Math. Soc. Japan **27** (1975), 294–327.

24. ——, *Weak solutions of Navier-Stokes equations*, Tôkoku Math. J. (2) **36** (1984), 623–646.

25. V. N. Maslennikova, *On the rate of decay of a vortex in a viscous fluid*, Trudy Mat. Inst. Steklov **126** (1973), 46–72; English transl. in Proc. Steklov. Inst. Math. **126** (1973).

26. ——, *On the large-time rate of decay of a solution of the Sobolev system with viscosity taken into account*, Mat. Sb. **92** **(134)** (1973), 589–610; English transl. in Math. USSR Sb. **21** (1973).

27. Tetsuro Miyakawa and Hermann Sohr, *On energy inequality, smoothness and large time behavior in L^2 for weak solutions of the Navier-Stokes equations in exterior domains*, Math. Z. **199** (1988), 455–478.

28. F. Kh. Mukminov, *Stabilization of solutions of the first mixed problem for a second-order parabolic equation*, Mat. Sb. **111** **(153)** (1980), 503–521; English transl. in Math. USSR Sb. **39** (1981).

29. ——, *On the decay of a solution of the first mixed problem for the linearized system of Navier-Stokes equations in a domain with noncompact boundary*, Mat. Sb. **183** (1992), 123–144; English transl. in Math. USSR Sb. **77** (1994).

30. I. M. Petunin, *On an asymptotic estimate of the solution of the first boundary value problem in a half-space for the equations of motion of a viscous rotating fluid*, Differential Equations and Functional Analysis (V. N. Maslennikova, ed.), Univ. Druzhby Narodov, Moscow, 1983, pp. 64–85. (Russian)

31. Maria Elena Schonbek, *L^2 decay for weak solutions of the Navier-Stokes equations*, Arch. Rational Mech. Anal. **88** (1985), 209–222.

32. ——, *Large time behaviour of solutions to the Navier-Stokes equations*, Comm. Partial Differential Equations **11** (1986), 733–763.

33. James Serrin, *The initial value problem for the Navier-Stokes equations*, Nonlinear Problems (Proc. Sympos., Madison, WI, 1962; R. E. Langer, editor), Univ. of Wisconsin Press, Madison, WI, 1963, pp. 69–98.

34. V. A. Solonnikov and V. E. Shchadilov, *On a certain boundary value problem for the steady-state system of Navier-Stokes equations*, Trudy Mat. Inst. Steklov **125** (1973), 196–210; English transl. in Proc. Steklov Inst. Math. **125** (1973).

35. Michael Wiegner, *Decay results for weak solutions of the Navier-Stokes equations on* $\mathbf{R}^n$, J. London Math. Soc. (2) **35** (1987), 303–313.

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Received 16/APR/92

Translated by H. H. McFADEN