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Existence of solutions of nonlinear elliptic equations with measure data in Musielak-Orlicz spaces

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Abstract. A second-order quasilinear elliptic equation with a measure of special form on the right-hand side is considered. Restrictions on the structure of the equation are imposed in terms of a generalized N-function such that the conjugate function obeys the Δ_2-condition and the corresponding Musielak-Orlicz space is not necessarily reflexive. In an arbitrary domain satisfying the segment property, the existence of an entropy solution of the Dirichlet problem is proved. It is established that this solution is renormalized.

Bibliography: 29 titles.

Keywords: quasilinear elliptic equation, entropy solution, renormalized solution, unbounded domain, diffuse measure, Musielak-Orlicz space.

§ 1. Introduction

This paper considers the problem of the existence of a solution of the Dirichlet problem

$$-\operatorname{div} \mathrm{a}(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) + b(\mathrm{x}, u, \nabla u) = \mu, \qquad \mathrm{x} \in \Omega, \tag{1.1}$$

$$u\big|_{\partial\Omega} = 0 \tag{1.2}$$

in an arbitrary unbounded domain $\Omega \subset \mathbb{R}^n = \{\mathrm{x} = (x_1, x_2, \dots, x_n)\}$, $n \geqslant 2$. Here the growth of the functions $\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) = (a_1(\mathrm{x}, s_0, \mathrm{s}), \dots, a_n(\mathrm{x}, s_0, \mathrm{s}))\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and $b(\mathrm{x}, s_0, \mathrm{s})\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is determined by a generalized N-function $M(\mathrm{x}, z)$, which does not necessarily satisfy the Δ_2-condition, and the bounded Radon measure μ has a special form.

The concept of renormalized solutions is the main step in the study of general degenerate elliptic equations with measure data. In [1] and [2], for an equation of the form

$$-\operatorname{div} \mathrm{a}(\mathrm{x}, \nabla u) = \mu, \qquad \mathrm{x} \in \Omega, \tag{1.3}$$

in Sobolev spaces, the stability and existence of a renormalized solution of the Dirichlet problem (1.3), (1.2) in a bounded domain Ω were proved.

In Musielak-Orlicz spaces, the existence of renormalized solutions with general measure data is a new problem even in the reflexive case. It was established by Chlebicka [3] under some regularity conditions on the Musielak-Orlicz function $M(\mathrm{x}, z)$ that each bounded Radon measure μ in a bounded domain $\Omega \subset \mathbb{R}^n$ is

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decomposable in the form $\mu = \mu_M + \mu_s$. Here, the measure μ_M is called diffuse with respect to the M-capacity Cap_M (an M-soft measure) and $\mu_M(E) = 0$ for any $E \subseteq \Omega$ such that $\mathrm{Cap}_M(E, \Omega) = 0$, whereas the measure μ_s is concentrated on a set of zero M-capacity and is called singular. It was established that μ_M is a diffuse measure with respect to the M-capacity if and only if $\mu_M \in L_1(\Omega) + W^{-1}_{\overline{M}}(\Omega)$ ($W^{-1}_{\overline{M}}(\Omega)$ is the conjugate space of $\mathring{W}^1_M(\Omega)$), that is, there exist functions $f \in L_1(\Omega)$ and $\mathrm{f} = (f_1, \dots, f_n) \in (L_{\overline{M}}(\Omega))^n$ such that

$$\mu_M = f - \operatorname{div} \mathrm{f}. \tag{1.4}$$

In [3] Chlebicka proved the existence of a renormalized solution of the Dirichlet problem (1.3), (1.2) and also its uniqueness for $\mu = \mu_M$.

In [4], for an anisotropic N-function $\Phi \in \Delta_2 \cap \nabla_2$, a similar measure decomposition with respect to the anisotropic Φ-capacity was established and the uniqueness of the approximation solution of the Dirichlet problem (1.3), (1.2) was proved in the case of a diffuse measure μ_Φ.

In [5], some class of second-order elliptic equations of the form

$$-\operatorname{div} \mathrm{a}(\mathrm{x}, \nabla u) + a_0(\mathrm{x}, u) = \mu, \qquad \mathrm{x} \in \Omega, \tag{1.5}$$

with variable nonlinearity exponents and right-hand side in the form of a general Radon measure with finite total variation was considered. The existence of a renormalized solution of (1.5), (1.2) was proved as a consequence of the stability with respect to the convergence of the right-hand side of the equation.

If the Musielak-Orlicz function M does not satisfy the Δ_2-condition, then the corresponding Musielak-Orlicz space is not reflexive, and even with a diffuse measure the problem under consideration becomes much more complicated. When no restriction is imposed on the growth of the generalized N-function $M(\mathrm{x}, z)$, it is commonly assumed that it satisfies the log-Hölder continuity condition with respect to the variable $\mathrm{x} \in \Omega$, which leads to good approximation properties of the Musielak-Orlicz space.

In [6] the existence of a renormalized solution of problem (1.3), (1.2) with $\mu \in L_1(\Omega)$ and an inhomogeneous anisotropic Musielak-Orlicz function was proved.

The authors of [7] and [8] established the existence of a renormalized and an entropy solution of the Dirichlet problem, respectively, for an equation of the form

$$-\operatorname{div}(\mathrm{a}(\mathrm{x}, u, \nabla u) + \mathrm{c}(u)) + a_0(\mathrm{x}, u, \nabla u) = f, \qquad f \in L_1(\Omega), \quad \mathrm{x} \in \Omega, \tag{1.6}$$

with $\mathrm{c} \in C_0(\mathbb{R}, \mathbb{R}^n)$. It was proved in [9], [10] ($a_0 \equiv 0$) and [11] that there exists an entropy solution of the Dirichlet problem for an equation of the form

$$-\operatorname{div}(\mathrm{a}(\mathrm{x}, u, \nabla u) + \mathrm{c}(\mathrm{x}, u)) + a_0(\mathrm{x}, u, \nabla u) = f, \qquad f \in L_1(\Omega), \quad \mathrm{x} \in \Omega, \tag{1.7}$$

with Carathéodory function $\mathrm{c}(\mathrm{x}, s_0) \colon \Omega \times \mathbb{R} \to \mathbb{R}^n$ subject to a growth condition with respect to s_0.

All the above results were deduced for entropy and renormalized solutions of elliptic problems in bounded domains. For elliptic equations with various types of nonlinearities and measure data (or L_1-data), existence and uniqueness results for entropy and renormalized solutions in arbitrary unbounded domains were derived in [12]–[20]. However, there are no results of this kind for equations with nonlinearities specified by Musielak-Orlicz functions.

The difficulty of generalizing to an unbounded domain is that the analogues of the Poincaré-Sobolev inequality and the compact embedding theorem for the Musielak-Orlicz-Sobolev space do not work in unbounded domains. These authors have managed to solve the problem by adding the term $M'(\mathrm{x},u)$ into (1.1) and requiring additionally that the function $M(\,\cdot\,,z)$ be integrable over Ω. In this paper we prove the existence of an entropy solution and establish that it is a renormalized solution of the problem (1.1), (1.2) with a diffuse-type measure μ in arbitrary (for instance, unbounded) domains Ω satisfying the segment property.

§ 2. Musielak-Orlicz-Sobolev spaces

In this section we provide some necessary information relating to the theory of generalized N-functions and Musielak-Orlicz spaces (see [21]–[23]).

Definition 2.1. Assume that a function $M(\mathrm{x},z)\colon \Omega\times\mathbb{R}\to\mathbb{R}_+$ satisfies the following conditions:

(1) $M(\mathrm{x},\cdot\,)$ is an N-function with respect to $z\in\mathbb{R}$, that is, it is downward convex, nondecreasing, even, continuous, $M(\mathrm{x},0)=0$ for a.a. $\mathrm{x}\in\Omega$, and also

$$\inf_{\mathrm{x}\in\Omega} M(\mathrm{x},z)>0 \quad \text{for all } z\neq 0,$$

$$\lim_{z\to 0}\sup_{\mathrm{x}\in\Omega}\frac{M(\mathrm{x},z)}{z}=0$$

and

$$\lim_{z\to\infty}\inf_{\mathrm{x}\in\Omega}\frac{M(\mathrm{x},z)}{z}=\infty;$$

(2) $M(\,\cdot\,,z)$ is a measurable function with respect to $\mathrm{x}\in\Omega$ for all $z\in\mathbb{R}$.

Such a function $M(\mathrm{x},z)$ is called a *Musielak-Orlicz function* or a *generalized N-function.*

The Young conjugate function $\overline{M}(\mathrm{x},\cdot\,)$ of the Musielak-Orlicz function $M(\mathrm{x},\cdot\,)$ is specified by

$$\overline{M}(\mathrm{x},z)=\sup_{y\geqslant 0}(yz-M(\mathrm{x},y))$$

for almost all $\mathrm{x}\in\Omega$ and any $z\geqslant 0$. This yields Young's inequality

$$|zy|\leqslant M(\mathrm{x},z)+\overline{M}(\mathrm{x},y),\qquad z,y\in\mathbb{R},\quad \mathrm{x}\in\Omega. \tag{2.1}$$

A Musielak-Orlicz function $M(\mathrm{x},z)$ has the integral representation

$$M(\mathrm{x},z)=\int_0^{|z|}M'(\mathrm{x},\theta)\,d\theta, \tag{2.2}$$

where $M'(\mathrm{x},\theta)\colon\Omega\times\mathbb{R}_+\to\mathbb{R}_+$, $M'(\mathrm{x},\cdot\,)$ is nondecreasing, continuous from the right, $M'(\mathrm{x},0)=0$ for almost all $\mathrm{x}\in\Omega$,

$$\inf_{\mathrm{x}\in\Omega}M'(\mathrm{x},\theta)>0 \quad \text{for a.a. } \theta>0$$

and

$$\lim_{\theta\to\infty}\inf_{\mathrm{x}\in\Omega} M'(\mathrm{x},\theta)=\infty. \tag{2.3}$$

For almost all $\mathrm{x}\in\Omega$ and $z\in\mathbb{R}$ it follows from (2.2) and (2.1) that

$$M(\mathrm{x},z)\leqslant M'(\mathrm{x},z)z, \tag{2.4}$$
$$M'(\mathrm{x},z)z\leqslant M(\mathrm{x},2z) \tag{2.5}$$

and

$$\overline{M}(\mathrm{x},M'(\mathrm{x},z))\leqslant M'(\mathrm{x},z)z. \tag{2.6}$$

Assume that $P(\mathrm{x},z)$ and $M(\mathrm{x},z)$ are Musielak-Orlicz functions. The fact that

$$\lim_{z\to\infty}\sup_{\mathrm{x}\in\Omega}\frac{P(\mathrm{x},lz)}{M(\mathrm{x},z)}=0 \tag{2.7}$$

for any positive constant l is denoted by $P\prec\prec M$; in this case P is said to grow slower than M at ∞.

A Musielak-Orlicz function M satisfies the Δ_2-condition if there exist constants $c>0$ and $z_0\geqslant 0$ and a function $H\in L_1(\Omega)$ such that

$$M(\mathrm{x},2z)\leqslant cM(\mathrm{x},z)+H(\mathrm{x})$$

for almost all $\mathrm{x}\in\Omega$ and any $|z|\geqslant z_0$. The Δ_2-condition is equivalent to the fulfillment of the inequality

$$M(\mathrm{x},lz)\leqslant c(l)M(\mathrm{x},z)+H_l(\mathrm{x}),\qquad H_l\in L_1(\Omega), \tag{2.8}$$

for almost all $\mathrm{x}\in\Omega$ and any $|z|\geqslant z_0$, where l is any number above one and $c(l)>0$.

We assume in this paper that the conjugate N-function $\overline{M}(\mathrm{x},z)$ satisfies the Δ_2-condition for all $z\in\mathbb{R}$ (that is, $z_0=0$). Thus,

$$\overline{M}(\mathrm{x},lz)\leqslant c(l)\overline{M}(\mathrm{x},z)+H_l(\mathrm{x}),\quad\text{where}\quad H_l\in L_1(\Omega),\quad z\in\mathbb{R}, \tag{2.9}$$

for any $l>0$ and almost all $\mathrm{x}\in\Omega$.

There are three Musielak-Orlicz classes.

$\mathcal{L}_M(\Omega)$ is the generalized Musielak-Orlicz class of measurable functions $v\colon\Omega\to\mathbb{R}$ such that

$$\varrho_{M,\Omega}(v)=\int_\Omega M(\mathrm{x},v(\mathrm{x}))\,d\mathrm{x}<\infty.$$

$L_M(\Omega)$ is the generalized Musielak-Orlicz space, which is the smallest linear space containing the class $\mathcal{L}_M(\Omega)$, with the Luxemburg norm

$$\|u\|_{M,\Omega}=\inf\biggl\{\lambda>0\;\bigg|\;\varrho_{M,\Omega}\Bigl(\frac{v}{\lambda}\Bigr)\leqslant 1\biggr\}.$$

$E_M(\Omega)$ is the closure of bounded measurable functions with compact support in $\overline{\Omega}$ with respect to the norm $\|u\|_{M,\Omega}$. The embeddings $E_M(\Omega)\subset\mathcal{L}_M(\Omega)\subset L_M(\Omega)$ are valid. Below we omit the index $Q=\Omega$ in the notation $\|\cdot\|_{M,Q}$, $\varrho_{M,Q}(\,\cdot\,)$.

In what follows we consider the following conditions on a Musielak-Orlicz function $M(\mathrm{x}, z)$.

(M1, loc) A function $M(\mathrm{x}, z)$ is locally integrable if

$$\varrho_{M,Q}(z) = \int_Q M(\mathrm{x}, z)\, d\mathrm{x} < \infty \quad \forall\, z \in \mathbb{R}$$

for any measurable set $Q \subset \Omega$ such that $\operatorname{meas} Q < \infty$.

(M1) A function $M(\mathrm{x}, z)$ is integrable if

$$\varrho_M(z) = \int_\Omega M(\mathrm{x}, z)\, d\mathrm{x} < \infty \quad \forall\, z \in \mathbb{R}.$$

(M2) A function $M(\mathrm{x}, z)$ satisfies the ϕ-regularity condition if there exists a function $\phi\colon [0, 1/2] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(\,\cdot\,, z)$ and $\phi(r, \cdot\,)$ are nondecreasing and for all $\mathrm{x}, \mathrm{y} \in \overline{\Omega}$, $|\mathrm{x} - \mathrm{y}| \leqslant 1/2$, $z \in \mathbb{R}^+$ and some constant $c > 0$ we have

$$M(\mathrm{x}, z) \leqslant \phi(|\mathrm{x} - \mathrm{y}|, z) M(\mathrm{y}, z) \quad \text{and} \quad \limsup_{\varepsilon \to 0^+} \phi(\varepsilon, c\varepsilon^{-n}) < \infty.$$

Assume that M and $\overline{M}$ obey condition (M1, loc). The space $E_M(\Omega)$ is separable and $(E_M(\Omega))^* = L_{\overline{M}}(\Omega)$. If M satisfies the Δ_2-condition, then $E_M(\Omega) = \mathcal{L}_M(\Omega) = L_M(\Omega)$ and $L_M(\Omega)$ is separable. The space $L_M(\Omega)$ is reflexive if and only if the Musielak-Orlicz functions M and $\overline{M}$ satisfy the Δ_2-condition.

For $v \in L_M(\Omega)$ it is true that

$$\|v\|_M \leqslant \varrho_M(v) + 1, \tag{2.10}$$

$$\varrho_M(v) \leqslant \|v\|_M \quad \text{if} \quad |v\|_M \leqslant 1 \tag{2.11}$$

and

$$\|v\|_M \leqslant \varrho_M(v) \quad \text{if} \quad \|v\|_M > 1. \tag{2.12}$$

A sequence of functions $\{v^j\}_{j \in \mathbb{N}} \in L_M(\Omega)$ *converges modularly* to $v \in L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{j \to \infty} \varrho_M\left(\frac{v^j - v}{\lambda}\right) = 0.$$

If M satisfies the Δ_2-condition, then the modular topology and the norm topology coincide.

For two conjugate Musielak-Orlicz functions M and $\overline{M}$, functions $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$ satisfy the Hölder inequality

$$\left|\int_\Omega u(\mathrm{x}) v(\mathrm{x})\, d\mathrm{x}\right| \leqslant 2\|u\|_M \|v\|_{\overline{M}}. \tag{2.13}$$

We define the Musielak-Orlicz-Sobolev spaces

$$W^1 L_M(\Omega) = \big\{v \in L_M(\Omega) \mid |\nabla v| \in L_M(\Omega)\big\}$$

and

$$W^1 E_M(\Omega) = \big\{v \in E_M(\Omega) \mid |\nabla v| \in E_M(\Omega)\big\}$$

with the norm

$$\|v\|_M^1 = \|v\|_M + \||\nabla v|\|_M.$$

A sequence of functions $\{v^j\}_{j\in\mathbb{N}} \in W^1L_M(\Omega)$ *converges modularly* to $v \in W^1L_M(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{j\to\infty} \varrho_M\left(\frac{v^j - v}{\lambda}\right) = 0 \quad \text{and} \quad \lim_{j\to\infty} \varrho_M\left(\frac{|\nabla v^j - \nabla v|}{\lambda}\right) = 0.$$

For brevity we introduce the notation $(L_M(\Omega))^n = \mathrm{L}_M(\Omega)$, $(L_M(\Omega))^{n+1} = \mathbf{L}_M(\Omega)$, $(E_M(\Omega))^n = \mathrm{E}_M(\Omega)$ and $(E_M(\Omega))^{n+1} = \mathbf{E}_M(\Omega)$. The space $W^1L_M(\Omega)$ is identified with a subspace of the product $\mathbf{L}_M(\Omega)$; it is closed with respect to the $\sigma(\mathbf{L}_M, \mathbf{E}_{\overline{M}})$-topology. The space $\mathring{W}^1L_M(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the $\sigma(\mathbf{L}_M, \mathbf{E}_{\overline{M}})$-topology in $W^1L_M(\Omega)$. Finally, the space $\mathring{W}^1E_M(\Omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_M^1$ in $W^1L_M(\Omega)$.

The spaces $\mathring{W}^1L_M(\Omega)$ and $\mathring{W}^1E_M(\Omega)$ are Banach spaces (see [22], Theorem 10.2). We also define the Banach space

$$W^{-1}L_{\overline{M}}(\Omega) = \left\{F = f_0 - \operatorname{div}\mathrm{f} \;\middle|\; f_0 \in L_{\overline{M}}(\Omega),\, \mathrm{f} = (f_1, \dots, f_n) \in \mathrm{L}_{\overline{M}}(\Omega)\right\}.$$

The following embedding theorem is true (see [24], Theorem 4).

Lemma 2.1. *Assume that a Musielak-Orlicz function* $M(\mathrm{x}, z)$ *satisfies the following conditions*:

$$\int_1^\infty \frac{M^{-1}(\mathrm{x}, z)}{z^{(n+1)/n}}\,dz = \infty, \qquad \int_0^1 \frac{M^{-1}(\mathrm{x}, z)}{z^{(n+1)/n}}\,dz < \infty, \tag{2.14}$$

$$M_*^{-1}(\mathrm{x}, z) = \int_0^z \frac{M^{-1}(\mathrm{x}, \tau)}{\tau^{(n+1)/n}}\,d\tau, \qquad \mathrm{x} \in \Omega, \quad z \geqslant 0.$$

Then $M_*(\mathrm{x}, z)$ *is a generalized* N*-function and* $\mathring{W}^1L_M(\Omega) \hookrightarrow L_{M_*}(\Omega)$. *In addition, for any bounded subdomain* $Q \subset \Omega$ *the embedding* $\mathring{W}^1L_M(\Omega) \hookrightarrow L_P(Q)$ *holds and is compact for any Musielak-Orlicz function* $P \prec\prec M_*$ *such that* $P(\,\cdot\,, z)$ *is integrable over* Q.

Definition 2.2. A domain Ω *has the segment property* if there exist a finite open covering $\{\Theta_i\}_{i=1}^k$ of $\overline{\Omega}$ and nonzero vectors $\mathrm{z}_i \in \mathbb{R}^n$ such that $(\overline{\Omega} \cap \Theta_i) + t\mathrm{z}_i \subset \Omega$ for any $t \in (0, 1)$ and $i = 1, \dots, k$.

We state a theorem on the density of the smooth functions in the Musielak-Orlicz-Sobolev space (see [25], Theorem 3).

Lemma 2.2. *Assume that a domain* Ω *has the segment property, an* N*-function* M *satisfies conditions* (M1) *and* (M2), *and* $\overline{M}$ *satisfies condition* (M1). *Then for any* $u \in \mathring{W}^1L_M(\Omega)$ *there exists a sequence of functions* $u^j \in C_0^\infty(\Omega)$ *such that*

$$u^j \to u \quad \textit{modularly in } \mathring{W}^1L_M(\Omega), \qquad j \to \infty.$$

§ 3. Assumptions and statements of the results

We assume that the functions

$$\mathrm{a}(\mathrm{x}, s_0, \mathrm{s})\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \quad \text{and} \quad b(\mathrm{x}, s_0, \mathrm{s})\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

in (1.1) are measurable with respect to $\mathrm{x} \in \Omega$ for $\mathbf{s} = (s_0, \mathrm{s}) = (s_0, s_1, \dots, s_n) \in \mathbb{R}^{n+1}$ and continuous in $\mathbf{s} \in \mathbb{R}^{n+1}$ for almost all $\mathrm{x} \in \Omega$. We also assume that the following condition holds.

Condition M. There exist nonnegative functions $\Psi, \phi \in L_1(\Omega)$ and positive constants $\widehat{A}$, $\overline{a}$, $\overline{d}$ and $\widehat{d}$ such that for almost all $\mathrm{x} \in \Omega$ and any $s_0 \in \mathbb{R}$, $\mathrm{s}, \mathrm{t} \in \mathbb{R}^n$ and $\mathrm{s} \neq \mathrm{t}$,

$$\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) \cdot \mathrm{s} \geqslant \overline{a} M(\mathrm{x}, \overline{d}|\mathrm{s}|) - \phi(\mathrm{x}), \tag{3.1}$$

$$\overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, s_0, \mathrm{s})|) \leqslant \Psi(\mathrm{x}) + \widehat{A} P(\mathrm{x}, \widehat{d} s_0) + \widehat{A} M(\mathrm{x}, \widehat{d}|\mathrm{s}|) \tag{3.2}$$

and

$$\bigl(\mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) - \mathrm{a}(\mathrm{x}, s_0, \mathrm{t})\bigr) \cdot (\mathrm{s} - \mathrm{t}) > 0. \tag{3.3}$$

Here the Musielak-Orlicz functions $P(\mathrm{x}, z)$ and $M(\mathrm{x}, z)$ ($P \prec\prec M$) obey condition (M1), the continuously differentiable function $M(\mathrm{x}, z)$ obeys (M2), the conjugate function $\overline{M}(\mathrm{x}, z)$ of M obeys the Δ_2-condition and the condition (M1), $\mathrm{s} \cdot \mathrm{t} = \sum_{i=1}^n s_i t_i$ and $|\mathrm{s}| = (\sum_{i=1}^n s_i^2)^{1/2}$. Recall that $L_{\overline{M}}(\Omega) = E_{\overline{M}}(\Omega)$.

In addition, assume that there exist a nonnegative function $\Phi_0 \in L_1(\Omega)$ and a continuous nondecreasing function $\widehat{b}\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for almost all $\mathrm{x} \in \Omega$ and all $s_0 \in \mathbb{R}$ and $\mathrm{s} \in \mathbb{R}^n$ we have

$$|b(\mathrm{x}, s_0, \mathrm{s})| \leqslant \widehat{b}(|s_0|)\bigl(M(\mathrm{x}, \overline{d}|\mathrm{s}|) + \Phi_0(\mathrm{x})\bigr) \tag{3.4}$$

and

$$b(\mathrm{x}, s_0, \mathrm{s}) s_0 \geqslant 0. \tag{3.5}$$

Note that the fact that $M(\,\cdot\,, z) \in L_1(\Omega)$ implies that $M'(\,\cdot\,, z) \in L_1(\Omega)$ for any fixed $z \in \mathbb{R}$.

The condition M is satisfied, for example, by the functions

$$a_i(\mathrm{x}, \mathrm{s}) = M'(\mathrm{x}, |\mathrm{s}|) \frac{s_i}{|\mathrm{s}|} + f_i(\mathrm{x}), \quad f_i \in L_{\overline{M}}(\Omega), \qquad i = 1, \dots, n,$$

and

$$b(\mathrm{x}, s_0, \mathrm{s}) = b(s_0) \overline{R}^{-1}(M(\mathrm{x}, |\mathrm{s}|)) R^{-1}(\Phi_0)$$

for a continuous nondecreasing odd function $b\colon \mathbb{R} \to \mathbb{R}$, an arbitrary N-function $R(z)$ and a nonnegative function $\Phi_0 \in L_1(\Omega)$.

We assume that the measure μ has the form

$$\mu = f + f_0 - \operatorname{div} \mathrm{f}, \qquad f \in L_1(\Omega), \quad f_0 \in E_{\overline{M}}(\Omega), \quad \mathrm{f} \in (E_{\overline{M}}(\Omega))^n. \tag{3.6}$$

This choice is due to the representation (1.4), while the presence of the term f_0 is related to the unboundedness of Ω. However, we are not going to consider the problem of the diffuseness of the measure (3.6) in this paper.

Introducing the notation $\widetilde{\mathrm{a}}(\mathrm{x}, s_0, \mathrm{s}) = \mathrm{a}(\mathrm{x}, s_0, \mathrm{s}) - \mathrm{f}$, we derive from (1.1) that

$$-\operatorname{div}\widetilde{\mathrm{a}}(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) + b(\mathrm{x}, u, \nabla u) = f + f_0.$$

Applying (2.1), we can easily see that the function $\widetilde{\mathrm{a}}(\mathrm{x}, s_0, \mathrm{s})$ also obeys conditions of the forms (3.1)–(3.3). In addition, using (2.1), we arrive at

$$\int_\Omega |f_0|\, d\mathrm{x} \leqslant \int_\Omega \overline{M}(\mathrm{x}, f_0)\, d\mathrm{x} + \int_\Omega M(\mathrm{x}, 1)\, d\mathrm{x} < \infty.$$

Consequently, $f_0 \in L_1(\Omega)$. Therefore, we consider (1.1) for the measure

$$\mu = f, \qquad f \in L_1(\Omega). \tag{3.7}$$

We introduce the function $T_k(r) = \max(-k, \min(k, r))$ and let $\mathring{\mathcal{T}}_M^1(\Omega)$ denote the set of measurable functions $u\colon \Omega \to \mathbb{R}$ such that $T_k(u) \in \mathring{W}^1 L_M(\Omega)$ for any $k > 0$. For $u \in \mathring{\mathcal{T}}_M^1(\Omega)$ and any $k > 0$ we have

$$\nabla T_k(u) = \chi_{\{|u|<k\}} \nabla u \in \mathrm{L}_M(\Omega). \tag{3.8}$$

We introduce the notation $\langle u \rangle = \displaystyle\int_\Omega u\, d\mathrm{x}$.

Definition 3.1. An *entropy solution* of (1.1), (1.2), (3.7) is a function $u \in \mathring{\mathcal{T}}_M^1(\Omega)$ such that:

(1) $b(\mathrm{x}, u, \nabla u) \in L_1(\Omega)$;

(2) for all $k > 0$ and $\xi \in C_0^1(\Omega)$ we have

$$\langle (b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) - f) T_k(u - \xi) \rangle + \langle \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot \nabla T_k(u - \xi) \rangle \leqslant 0. \tag{3.9}$$

Definition 3.2. A *renormalized solution* of problem (1.1), (1.2), (3.7) is a function $u \in \mathring{\mathcal{T}}_M^1(\Omega)$ such that:

(1) $b(\mathrm{x}, u, \nabla u) \in L_1(\Omega)$;

(2) $\displaystyle\lim_{h\to\infty} \int_{\{\Omega\colon\, h \leqslant |u| < h+1\}} M(\mathrm{x}, \overline{d}|\nabla u|)\, d\mathrm{x} = 0$;

(3) for any smooth compactly supported function $S \in W_\infty^1(\mathbb{R})$ and any function $\xi \in C_0^1(\Omega)$, it is true that

$$\bigl\langle (b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) - f) S(u) \xi \rangle + \langle \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot (S'(u) \xi \nabla u + S(u) \nabla \xi) \bigr\rangle = 0. \tag{3.10}$$

The main results in this paper are the following theorems.

Theorem 3.1. *Assume that the domain Ω has the segment property and conditions* M *and* (2.14) *are satisfied. Then problem* (1.1), (1.2), (3.7) *has an entropy solution.*

Theorem 3.2. *Assume that the domain Ω has the segment property and conditions* M *and* (2.14) *are satisfied. Then the entropy solution constructed in Theorem* 3.1 *is a renormalized solution of problem* (1.1), (1.2), (3.7).

§ 4. Preparatory information

The integrals of the functions $M'(\mathrm{x},u)T_k(u-\xi)$ and $\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla T_k(u-\xi)$ in (3.9) are assumed to converge for any $k>0$. The convergence of the other integrals in (3.9) follows from (3.7) and condition (1) in Definition 3.1. The integral of the function $\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla(S(u)\xi)$ in (3.10) is assumed to converge, while the convergence of the other integrals follows from (3.7) and condition (1) in Definition 3.2.

In this section we establish some properties of the entropy solution of problem (1.1), (1.2), (3.7) and present auxiliary lemmas.

All constants occurring below in this paper are positive.

Lemma 4.1. *Let u be an entropy solution of problem* (1.1), (1.2), (3.7). *Then for any $k>0$*

$$\mathrm{a}(\mathrm{x},u,\nabla u)\chi_{\{\Omega:\,|u|<k\}}\in \mathrm{L}_{\overline{M}}(\Omega) \tag{4.1}$$

and

$$\int_{\{\Omega:\,|u|<k\}}\bigl(M(\mathrm{x},u)+M(\mathrm{x},\overline{d}|\nabla u|)\bigr)\,d\mathrm{x}+k\int_{\{\Omega:\,|u|\geqslant k\}}M'(\mathrm{x},|u|)\,d\mathrm{x}\leqslant C_1k+C_2. \tag{4.2}$$

Proof. For $\xi=0$ the inequality (3.9) takes the form

$$\begin{aligned}I&=\int_\Omega(M'(\mathrm{x},u)+b(\mathrm{x},u,\nabla u))T_k(u)\,d\mathrm{x}+\int_{\{\Omega:\,|u|<k\}}\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla u\,d\mathrm{x}\\&\leqslant\int_\Omega fT_k(u)\,d\mathrm{x}\leqslant C_3k.\end{aligned} \tag{4.3}$$

Applying (2.4) and (3.5) we infer the estimate

$$\begin{aligned}I\geqslant k\int_{\{\Omega:\,|u|\geqslant k\}}M'(\mathrm{x},|u|)\,d\mathrm{x}&+\int_{\{\Omega:\,|u|<k\}}M(\mathrm{x},u)\,d\mathrm{x}\\&+\int_{\{\Omega:\,|u|<k\}}\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla u\,d\mathrm{x}.\end{aligned} \tag{4.4}$$

Combining (4.4) and (4.3) we establish the inequality

$$\begin{aligned}k\int_{\{\Omega:\,|u|\geqslant k\}}M'(\mathrm{x},|u|)\,d\mathrm{x}&+\int_{\{\Omega:\,|u|<k\}}M(\mathrm{x},u)\,d\mathrm{x}\\&+\int_{\{\Omega:\,|u|<k\}}\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla u\,d\mathrm{x}\leqslant kC_3.\end{aligned} \tag{4.5}$$

Now, using (3.1) we obtain (4.2).

Let $\mathrm{w}\in \mathrm{E}_M(\Omega)$ be arbitrary. Then condition (3.3) yields

$$(\mathrm{a}(\mathrm{x},u,\nabla u)-\mathrm{a}(\mathrm{x},u,\mathrm{w}))\cdot(\nabla u-\mathrm{w})>0.$$

It follows that

$$\begin{aligned}\int_{\{\Omega:\,|u|<k\}}\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\mathrm{w}\,d\mathrm{x}&\leqslant\int_{\{\Omega:\,|u|<k\}}\mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla u\,d\mathrm{x}\\&-\int_{\{\Omega:\,|u|<k\}}\mathrm{a}(\mathrm{x},u,\mathrm{w})\cdot(\nabla u-\mathrm{w})\,d\mathrm{x}.\end{aligned} \tag{4.6}$$

Using (3.2) we then deduce that

$$\int_{\{\Omega:\ |u|<k\}} \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, u, \mathrm{w})|)\, dx \leqslant \|\Psi\|_1 + \widehat{A}\int_\Omega P(\mathrm{x}, \widehat{d}k)\, d\mathrm{x} + \widehat{A}\int_\Omega M(\mathrm{x}, \widehat{d}|\mathrm{w}|))\, d\mathrm{x} \leqslant C_4. \tag{4.7}$$

Combining (4.5), (4.6), (4.7) and (3.8), we derive the estimate

$$\int_{\{\Omega:\ |u|<k\}} \mathrm{a}(\mathrm{x}, u, \nabla u)\cdot \mathrm{w}\, d\mathrm{x} \leqslant C_5 \quad \forall\, \mathrm{w} \in \mathrm{E}_M(\Omega).$$

Replacing w by −w we obtain

$$-\int_{\{\Omega:\ |u|<k\}} \mathrm{a}(\mathrm{x}, u, \nabla u)\cdot \mathrm{w}\, d\mathrm{x} \leqslant C_5 \quad \forall\, \mathrm{w} \in \mathrm{E}_M(\Omega).$$

Therefore, it is true that

$$\left|\int_{\{\Omega:\ |u|<k\}} \mathrm{a}(\mathrm{x}, u, \nabla u)\cdot \mathrm{w}\, d\mathrm{x}\right| \leqslant C_5 \quad \forall\, \mathrm{w} \in \mathrm{E}_M(\Omega),$$

which implies (4.1). The lemma is proved.

The following lemma holds.

Lemma 4.2. *Let $v\colon \Omega \to \mathbb{R}$ be a measurable function and let*

$$\int_{\{\Omega:\ |v|\geqslant k\}} M'(\mathrm{x}, |v|)\, d\mathrm{x} \leqslant C_1 + \frac{C_2}{k} \tag{4.8}$$

for all $k > 0$. Then

$$\operatorname{meas}\{\Omega\colon |v| \geqslant k\} \to 0, \qquad k \to \infty, \tag{4.9}$$

and

$$M'(\mathrm{x}, |v|) \in L_1(\Omega). \tag{4.10}$$

Proof. Relation (4.8) yields

$$\operatorname{meas}\{\Omega\colon |v| \geqslant k\} \inf_{\mathrm{x}\in\Omega} M'(\mathrm{x}, k) \leqslant C_1 + \frac{C_2}{k}.$$

Applying (2.3), we derive (4.9) from this inequality.

Furthermore, due to (4.8) and the fact that $M'(\mathrm{x}, k) \in L_1(\Omega)$, we have

$$\int_\Omega M'(\mathrm{x}, |v|)\, d\mathrm{x} = \int_{\{\Omega:\ |v|\geqslant k\}} M'(\mathrm{x}, |v|)\, d\mathrm{x} + \int_{\{\Omega:\ |v|<k\}} M'(\mathrm{x}, |v|)\, d\mathrm{x} \leqslant C_1 + \frac{C_2}{k} + \int_\Omega M'(\mathrm{x}, k)\, d\mathrm{x} = C_6(k).$$

The lemma is proved.

Remark 1. If u is an entropy solution of problem (1.1), (1.2), (3.7), then it follows from Lemmas 4.1 and 4.2 that

$$\operatorname{meas}\{\Omega\colon |u| \geqslant k\} \to 0, \quad k \to \infty, \tag{4.11}$$

and

$$M'(\mathrm{x}, |u|) \in L_1(\Omega). \tag{4.12}$$

In addition,

$$\forall\, k > 0 \quad (M(\mathrm{x}, |u|) + M(\mathrm{x}, \overline{d}|\nabla u|))\chi_{\{\Omega\colon |u|<k\}} \in L_1(\Omega).$$

Lemma 4.3 (see [24], Lemma 2). *Let $\{v^j\}_{j\in\mathbb{N}}$ and v be functions in $L_M(\Omega)$ such that*

$$\|v^j\|_M \leqslant C, \qquad j \in \mathbb{N},$$
$$v^j \to v \quad \text{a.e. in } \Omega, \qquad j \to \infty.$$

Then $v^j \rightharpoonup v$ as $j \to \infty$ in the $\sigma(L_M, E_{\overline{M}})$-topology on the space $L_M(\Omega)$.

Lemma 4.4. *Let g^j, $j \in \mathbb{N}$, and g be functions in $L_1(\Omega)$ such that $g^j \geqslant 0$ a.e. in Ω and*

$$g^j \to g \quad \text{strongly in } L_1(\Omega), \qquad j \to \infty.$$

Let v^j, $j \in \mathbb{N}$, and v be measurable functions in Ω such that

$$v^j \to v \quad \text{a.e. in } \Omega, \qquad j \to \infty,$$

and

$$|v^j| \leqslant g^j, \quad j \in \mathbb{N}, \quad \text{a.e. in } \Omega.$$

Then

$$\int_\Omega v^j \, d\mathrm{x} \to \int_\Omega v \, d\mathrm{x}, \qquad j \to \infty.$$

Lemma 4.5. *Let w^j, $j \in \mathbb{N}$, and w be functions in $L_1(\Omega)$ such that $w^j \geqslant 0$ a.e. in Ω,*

$$w^j \to w \quad \text{a.e. in } \Omega, \qquad j \to \infty,$$

and

$$\int_\Omega w^j \, d\mathrm{x} \to \int_\Omega w \, d\mathrm{x}, \qquad j \to \infty.$$

Then

$$w^j \to w \quad \text{strongly in } L_1(\Omega), \qquad j \to \infty.$$

Lemma 4.6. *If the domain Ω has the segment property and u is an entropy solution of problem (1.1), (1.2), (3.7), then inequality (3.9) is valid for any $\xi \in \mathring{W}^1 L_M(\Omega) \cap L_\infty(\Omega)$.*

Proof. Due to Lemma 2.2, for any $\xi \in \mathring{W}^1 L_M(\Omega) \cap L_\infty(\Omega)$ there exists a sequence $\{\xi^j\}_{j\in\mathbb{N}} \in C_0^\infty(\Omega)$ such that

$$\nabla\xi^j \to \nabla\xi, \quad \xi^j \to \xi \quad \text{modularly in } L_M(\Omega), \qquad j \to \infty. \tag{4.13}$$

It follows that

$$\xi^j \to \xi \quad \text{and} \quad \nabla\xi^j \to \nabla\xi \quad \text{a.e. in } \Omega \quad \text{as } j \to \infty. \tag{4.14}$$

Then for any $k > 0$

$$T_k(u-\xi^j) \to T_k(u-\xi) \quad \text{and} \quad \nabla T_k(u-\xi^j) \to \nabla T_k(u-\xi) \quad \text{a.e. in } \Omega, \qquad j \to \infty. \tag{4.15}$$

Note that we can choose $\{\xi^j\}_{j\in\mathbb{N}}$ so that this sequence is bounded in $L_\infty(\Omega)$. Let $K = \sup_{j\in\mathbb{N}} \|\xi^j\|_\infty$ and let $\widehat{k} = k + K$; then we have

$$|\nabla T_k(u-\xi^j)| \leqslant |\nabla T_{\widehat{k}}(u)| + |\nabla\xi^j|, \qquad \mathrm{x} \in \Omega, \quad j \in \mathbb{N}.$$

Since the modularly convergent sequence $\nabla\xi^j$ is bounded in $\mathrm{L}_M(\Omega)$ and in accordance with (3.8), the norms $\|\nabla T_k(u-\xi^j)\|_M$, $j \in \mathbb{N}$, are bounded. Using (4.15) and Lemma 4.3, for any $k > 0$ we obtain

$$\nabla T_k(u-\xi^j) \rightharpoonup \nabla T_k(u-\xi) \text{ in the } \sigma(\mathrm{L}_M, \mathrm{E}_{\overline{M}})\text{-topology on } \mathrm{L}_M(\Omega), \qquad j \to \infty. \tag{4.16}$$

Now we pass to the limit as $j \to \infty$ in the inequality

$$\int_\Omega (b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) - f) T_k(u-\xi^j)\, d\mathrm{x} + \int_\Omega \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot \nabla T_k(u-\xi^j)\, d\mathrm{x} \leqslant 0. \tag{4.17}$$

Since $b(\mathrm{x}, u, \nabla u)$, $M'(\mathrm{x}, u)$, $f \in L_1(\Omega)$ (see Definition 3.1, (1) and (4.12)), using (4.15) and Lebesgue's theorem, we can take the limit as $j \to \infty$ in the first term.

In view of the fact that $\mathrm{a}(\mathrm{x}, u, \nabla u)\chi_{\{\Omega:\ |u|<\widehat{k}\}} \in \mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}$ (see (4.1)), applying (4.16) we establish that the second term in the last inequality also converges as $j \to \infty$. Thus, taking the limit in (4.17) we obtain (3.9). The lemma is proved.

Lemma 4.7. *Let u be an entropy solution of problem* (1.1), (1.2), (3.7). *Then for all $k > 0$*

$$\lim_{h\to\infty} \int_{\{h \leqslant |u| < k+h\}} M(\mathrm{x}, \overline{d}|\nabla u|)\, d\mathrm{x} = 0. \tag{4.18}$$

Proof. Setting $\xi = T_h(u)$ in (3.9) we arrive at

$$\int_{\{h \leqslant |u| < k+h\}} \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot \nabla u\, d\mathrm{x} + \int_{\{h \leqslant |u|\}} \big(b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u)\big) T_k(u - T_h(u))\, d\mathrm{x} \leqslant k \int_{\{h \leqslant |u|\}} |f|\, d\mathrm{x}.$$

Now, using (3.5) we infer the inequality

$$\int_{\{h\leqslant|u|<k+h\}} \mathrm{a}(\mathrm{x},u,\nabla u)\cdot\nabla u\,d\mathrm{x} + k\int_{\{|u|\geqslant k+h\}} \big(|b(\mathrm{x},u,\nabla u)| + M'(\mathrm{x},|u|)\big)\,d\mathrm{x} + \int_{\{h\leqslant|u|<k+h\}} \big(b(\mathrm{x},u,\nabla u)+M'(\mathrm{x},u)\big)(u-h\operatorname{sign}u)\,d\mathrm{x} \leqslant k\int_{\{h\leqslant|u|\}} |f|\,d\mathrm{x}.$$

In view of (3.5) we have

$$(b(\mathrm{x},u,\nabla u)+M'(\mathrm{x},u))(u-h\operatorname{sign}u)\geqslant 0$$

for $h\leqslant|u|$. Combining the two last inequalities and using (3.1) for any $k>0$ we derive that

$$\overline{a}\int_{\{h\leqslant|u|<k+h\}} M(\mathrm{x},\overline{d}|\nabla u|)\,d\mathrm{x} \leqslant k\int_{\{h\leqslant|u|\}} (|f|+|\phi|)\,d\mathrm{x}.$$

As $f,\phi\in L_1(\Omega)$, taking (4.11) into account and passing to the limit as $h\to\infty$, we deduce (4.18). The lemma is proved.

Lemma 4.8. *Let v^j, $j\in\mathbb{N}$, and v be functions in $\mathcal{L}_M(\Omega)$ such that*

$$v^j\to v \quad \text{a.e. in } \Omega, \qquad j\to\infty,$$

and

$$M(\mathrm{x},v^j)\leqslant g\in L_1(\Omega), \qquad j\in\mathbb{N}.$$

Then

$$v^j\to v \quad \text{modularly in } L_M(\Omega), \qquad j\to\infty.$$

Lemma 4.8 follows from Lebesgue's theorem.

Lemma 4.9. *Let Q be a measurable set, assume that the Carathéodory function $\mathrm{a}(\mathrm{x},s_0,\mathrm{s})\colon Q\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ satisfies (3.3), and let $v^j\colon Q\to\mathbb{R}$ and $\mathrm{w}^j\colon Q\to\mathbb{R}^n$ be sequences of measurable functions such that $v^j\to v$ a.e. in Q, $|v|\leqslant k$, and*

$$\int_Q (\mathrm{a}(\mathrm{x},v^j,\mathrm{w}^j)-a(\mathrm{x},v^j,\mathrm{w}))\cdot(\mathrm{w}^j-\mathrm{w})\,d\mathrm{x}\to 0, \qquad j\to\infty,$$

where $|\mathrm{w}|\leqslant r$. Then

$$\mathrm{w}^j\to\mathrm{w} \quad \text{a.e. in } Q, \qquad j\to\infty.$$

This lemma can be proved based on Lemma 2.4 in [26], similarly to the proof of Lemma 6.2 in [27] by Vorob'ev and Mukminov.

Lemma 4.10. *Assume that conditions (3.1)–(3.3) are satisfied, and for some fixed $k>0$ assume that the sequence $(T_k(u^j),\nabla T_k(u^j))\in\mathbf{L}_M(\Omega)$, $j\in\mathbb{N}$, satisfies*

$$\nabla T_k(u^j)\rightharpoonup\nabla T_k(u) \quad \text{in the } \sigma(\mathrm{L}_M,\mathrm{E}_{\overline{M}})\text{-topology on } \mathrm{L}_M(\Omega), \qquad j\to\infty, \tag{4.19}$$

$$T_k(u^j)\to T_k(u) \quad \text{a.e. in } \Omega, \qquad j\to\infty, \tag{4.20}$$

$$\mathrm{a}(\mathrm{x},T_k(u^j),\nabla T_k(u^j)), \quad j\in\mathbb{N}, \quad \text{is bounded in } \mathrm{L}_{\overline{M}}(\Omega); \tag{4.21}$$

$$\lim_{s\to\infty}\lim_{j\to\infty}\int_\Omega q_s^j(\mathrm{x})\,d\mathrm{x}=0, \tag{4.22}$$

and

$$q_s^j(\mathrm{x}) = (\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) - \mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s)) \cdot (\nabla T_k(u^j) - \nabla T_k(u)\chi_s), \tag{4.23}$$

where χ_s *is the characteristic function of the set* $\Omega_s = \{\mathrm{x} \in \Omega \mid |\nabla T_k(u)| \leqslant s\}$. *Then it is true for some subsequence that*

$$\nabla T_k(u^j) \to \nabla T_k(u) \quad a.e.\ in\ \Omega, \qquad j \to \infty, \tag{4.24}$$

$$\nabla T_k(u^j) \to \nabla T_k(u) \quad modularly\ in\ \mathrm{L}_M(\Omega), \qquad j \to \infty, \tag{4.25}$$

and

$$\begin{aligned} &\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) \cdot \nabla T_k(u^j) \\ &\qquad \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) \quad in\ L_1(\Omega), \qquad j \to \infty. \end{aligned} \tag{4.26}$$

Proof. We fix $r > 0$. Let $s > r$. Applying (3.3) we can infer the relations

$$\begin{aligned} 0 &\leqslant \int_{\Omega_r} \big(\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) - \mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u))\big) \cdot \nabla(T_k(u^j) - T_k(u))\, d\mathrm{x} \\ &= \int_{\Omega_r} (\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) - \mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s)) \cdot (\nabla T_k(u^j) - \nabla T_k(u)\chi_s)\, d\mathrm{x} \\ &\leqslant \int_{\Omega} (\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) - \mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s)) \cdot (\nabla T_k(u^j) - \nabla T_k(u)\chi_s)\, d\mathrm{x}. \end{aligned}$$

Then from (4.22) we obtain

$$\lim_{j\to\infty} \int_{\Omega_r} \big(\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) - \mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u))\big) \cdot \nabla(T_k(u^j) - T_k(u))\, d\mathrm{x} = 0. \tag{4.27}$$

Using Lemma 4.9 for $Q = \Omega_r$, $v^j = T_k(u^j)$, $v = T_k(u)$, $\mathrm{w}^j = \nabla T_k(u^j)$ and $\mathrm{w} = \nabla T_k(u)$, we establish the convergence

$$\nabla T_k(u^j) \to \nabla T_k(u) \quad \text{a.e. in } \Omega_r, \qquad j \to \infty;$$

thereupon we deduce the convergence (4.24) using the diagonal process.

According to the Banach-Steinhaus theorem, the convergence (4.19) yields that the sequence $\{\nabla T_k(u^j)\}_{j\in\mathbb{N}}$ is bounded, that is,

$$\|\nabla T_k(u^j)\|_M \leqslant C_7, \qquad j \in \mathbb{N}. \tag{4.28}$$

We conclude from (4.20), (4.24), since $\mathrm{a}(\mathrm{x}, s_0, \mathrm{s})$ is continuous in $\mathbf{s} = (s_0, \mathrm{s})$, that

$$\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)) \quad \text{a.e. in } \Omega, \qquad j \to \infty.$$

By virtue of Lemma 4.3, from this and (4.21) we derive that

$$\begin{aligned} \mathrm{a}\big(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)\big) \rightharpoonup \mathrm{a}\big(\mathrm{x}, T_k(u), \nabla T_k(u)\big) \quad &\text{in the } \sigma(\mathrm{L}_{\overline{M}}, \mathrm{E}_M)\text{-topology} \\ &\text{on } \mathrm{L}_{\overline{M}}(\Omega), \qquad j \to \infty. \end{aligned} \tag{4.29}$$

It follows from (4.20) that

$$\mathrm{a}\big(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s\big) \to \mathrm{a}\big(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s\big) \quad \text{a.e. in } \Omega, \qquad j \to \infty,$$

while (3.2) implies the estimate

$$\overline{M}\big(\mathrm{x}, |\mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s)|\big) \leqslant \Psi(\mathrm{x}) + \widehat{A}P(\mathrm{x}, \widehat{d}k) + \widehat{A}M(\mathrm{x}, \widehat{d}s) \in L_1(\Omega), \qquad j \in \mathbb{N}.$$

By Lemma 4.8 we obtain the convergence

$$\mathrm{a}\big(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s\big) \to \mathrm{a}\big(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s\big) \quad \text{modularly in } \mathrm{L}_{\overline{M}}(\Omega), \qquad j \to \infty.$$

Since the function $\overline{M}$ satisfies the Δ_2-condition, we have

$$\mathrm{a}\big(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s\big) \to \mathrm{a}\big(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s\big) \quad \text{strongly in } \mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}(\Omega), \qquad j \to \infty. \tag{4.30}$$

We set $y^j = \mathrm{a}(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)) \cdot \nabla T_k(u^j)$ and $y = \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u)$ and write

$$\int_\Omega y^j \, d\mathrm{x} = \int_\Omega q_s^j(\mathrm{x}) \, d\mathrm{x} + \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^j), \nabla T_k(u)\chi_s\big) \cdot \big(\nabla T_k(u^j) - \nabla T_k(u)\chi_s\big) \, d\mathrm{x} + \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^j), \nabla T_k(u^j)\big) \cdot \nabla T_k(u)\chi_s \, d\mathrm{x} = I_{s1}^j + I_{s2}^j + I_{s3}^j. \tag{4.31}$$

Applying (4.19) and (4.30) we establish that

$$\lim_{j \to \infty} I_{s2}^j = \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s\big) \cdot \nabla T_k(u)(1 - \chi_s) \, d\mathrm{x} = \int_{\{\Omega\colon |\nabla T_k(u)| > s\}} \mathrm{a}(\mathrm{x}, T_k(u), 0) \cdot \nabla T_k(u) \, d\mathrm{x}.$$

As $\mathrm{a}(\mathrm{x}, T_k(u), 0) \cdot \nabla T_k(u) \in L_1(\Omega)$ and $\operatorname{meas}\{\Omega\colon |\nabla T_k(u)| > s\} \to 0$ as $s \to \infty$, we obtain

$$\lim_{s,j \to \infty} I_{s2}^j = 0. \tag{4.32}$$

Due to (4.29),

$$\lim_{s,j \to \infty} I_{s3}^j = \int_\Omega y \, d\mathrm{x}. \tag{4.33}$$

Combining (4.22) and (4.31)–(4.33) we establish the convergence

$$\int_\Omega y^j \, d\mathrm{x} \to \int_\Omega y \, d\mathrm{x}, \qquad j \to \infty. \tag{4.34}$$

Now, using (4.20) and (4.24) we conclude that

$$y^j \to y \quad \text{a.e. in } \Omega, \qquad j \to \infty. \tag{4.35}$$

According to (3.1), the functions $w^j = y^j + \phi$ and $w = y + \phi$ are nonnegative; therefore, from Lemma 4.5 we deduce the convergence (4.26). Then, in view of (3.1) and (4.26) and using Lemma 4.4 for $v^j = M(\mathrm{x}, \overline{d}|\nabla(T_k(u^j) - T_k(u))|/2)$ and $g^j = (y^j + y + 2\phi)(\overline{a}2)^{-1}$ we derive (4.25). The lemma is proved.

Lemma 4.11. *Let v^j, $j \in \mathbb{N}$, and $v \in L_\infty(\Omega)$ be functions such that the sequence $\{v^j\}_{j\in\mathbb{N}}$ is bounded in $L_\infty(\Omega)$ and*

$$v^j \to v \quad a.e. \ in\ \Omega, \qquad j \to \infty.$$

Then $v^j \rightharpoonup v$ as $j \to \infty$ in the $\sigma(L_\infty, L_1)$-topology on $L_\infty(\Omega)$.

In addition, if g is in $L_M(\Omega)(E_M(\Omega))$, then

$$v^j g \to vg \quad modularly\ (strongly)\ in\ L_M(\Omega)(E_M(\Omega)), \qquad j \to \infty.$$

The proof of Lemma 4.11 follows from Lebesgue's theorem.

Below we use Vitali's theorem in the following form (see [28], Ch. III, § 6, Theorem 15).

Lemma 4.12. *Let v^j, $j \in \mathbb{N}$, and v be measurable functions in a bounded domain Q such that*

$$v^j \to v \quad a.e.\ in\ Q, \qquad j \to \infty,$$

and the integrals

$$\int_Q |v^j(\mathrm{x})|\, d\mathrm{x}, \qquad j \in \mathbb{N},$$

are uniformly absolutely continuous. Then

$$v^j \to v \quad strongly\ in\ L_1(Q), \qquad j \to \infty.$$

Lemma 4.13. *Let v^j, $j \in \mathbb{N}$, and $v \in L_M(\Omega)$ and let*

$$v^j \to v \quad modularly\ in\ L_M(\Omega), \qquad j \to \infty.$$

Then $v^j \rightharpoonup v$ in the $\sigma(L_M, L_{\overline{M}})$-topology on $L_M(\Omega)$.

See Lemma 2 in [25].

§ 5. Proofs of Theorems 3.1 and 3.2

Proof of Theorem 3.1. Step 1. We set

$$f^m(\mathrm{x}) = T_m f(\mathrm{x})\chi_{\Omega(m)}, \quad \text{where } \Omega(m) = \{\mathrm{x} \in \Omega \colon |\mathrm{x}| < m\}, \quad m \in \mathbb{N}.$$

Throughout, χ_Q is the characteristic function of the set Q. It is straightforward to show that

$$f^m \to f \quad \text{in } L_1(\Omega), \qquad m \to \infty, \tag{5.1}$$

and

$$|f^m(\mathrm{x})| \leqslant |f(\mathrm{x})|, \quad |f^m(\mathrm{x})| \leqslant m\chi_{\Omega(m)}, \qquad \mathrm{x} \in \Omega, \quad m \in \mathbb{N}. \tag{5.2}$$

We consider the equations

$$-\operatorname{div} \mathrm{a}^m(\mathrm{x}, u, \nabla u) + a_0^m(\mathrm{x}, u, \nabla u) = f^m(\mathrm{x}), \qquad \mathrm{x} \in \Omega, \quad m \in \mathbb{N}, \tag{5.3}$$

where

$$\mathrm{a}^m(\mathrm{x}, s_0, \mathrm{s}) = \mathrm{a}(\mathrm{x}, T_m(s_0), \mathrm{s}) \quad \text{and} \quad a_0^m(\mathrm{x}, s_0, \mathrm{s}) = b^m(\mathrm{x}, s_0, \mathrm{s}) + M'(\mathrm{x}, s_0).$$

Here

$$\mathrm{a}^m(\mathrm{x}, s_0, \mathrm{s}) = (a_1^m(\mathrm{x}, s_0, \mathrm{s}), \dots, a_n^m(\mathrm{x}, s_0, \mathrm{s})) \quad \text{and} \quad b^m(\mathrm{x}, s_0, \mathrm{s}) = T_m b(\mathrm{x}, s_0, \mathrm{s})\chi_{\Omega(m)}.$$

It is evident that

$$|b^m(\mathrm{x}, s_0, \mathrm{s})| \leqslant |b(\mathrm{x}, s_0, \mathrm{s})| \quad \text{and} \quad |b^m(\mathrm{x}, s_0, \mathrm{s})| \leqslant m\chi_{\Omega(m)}, \quad \mathrm{x} \in \Omega, \quad (s_0, \mathrm{s}) \in \mathbb{R}^{n+1}. \tag{5.4}$$

In addition, using (3.5) we establish the inequality

$$b^m(\mathrm{x}, s_0, \mathrm{s})s_0 \geqslant 0, \qquad \mathrm{x} \in \Omega, \quad (s_0, \mathrm{s}) \in \mathbb{R}^{n+1}. \tag{5.5}$$

We define an operator $\mathbf{A}^m \colon \mathring{W}^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ with domain

$$D(\mathbf{A}^m) = \{u \in \mathring{W}^1 L_M(\Omega) \mid a_i^m(\mathrm{x}, u, \nabla u) \in L_{\overline{M}}(\Omega),\ i = 0, \dots, n\}$$

by setting

$$\langle \mathbf{A}^m(u), v\rangle = \big\langle \mathrm{a}^m(\mathrm{x}, u, \nabla u) \cdot \nabla v\big\rangle + \big\langle a_0^m(\mathrm{x}, u, \nabla u) v\big\rangle \tag{5.6}$$

for $v \in \mathring{W}^1 L_M(\Omega)$. A generalized solution of problem (5.3), (1.2) is a function u in $\mathring{W}^1 L_M(\Omega)$ satisfying the integral identity

$$\langle \mathbf{A}^m(u), v\rangle = \langle f^m v\rangle \tag{5.7}$$

for $v \in \mathring{W}^1 L_M(\Omega)$.

The following theorem holds.

Theorem 5.1. *If conditions* M *and* (2.14) *are satisfied, then the problem* (5.3), (1.2) *has a generalized solution.*

Proving Theorem 5.1 reduces to verifying the assumptions in the theorem in [29].

Due to Theorem 5.1, for each $m \in \mathbb{N}$ problem (5.3), (1.2) has a generalized solution u in $\mathring{W}^1 L_M(\Omega)$. Thus, the integral identity

$$\begin{aligned} &\big\langle (b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m) - f^m(\mathrm{x}))v\big\rangle \\ &\qquad + \langle \mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot \nabla v\rangle = 0, \qquad m \in \mathbb{N}, \end{aligned} \tag{5.8}$$

holds for all $v \in \mathring{W}^1 L_M(\Omega)$.

Step 2. At this step we establish a priori estimates for the sequence $\{u^m\}_{m\in\mathbb{N}}$.

Setting $v = T_{k,h}(u^m) = T_k(u^m - T_h(u^m))$ in (5.8), $h > k > 0$, we obtain

$$\begin{aligned} &\int_{\{h\leqslant |u^m| < k+h\}} \mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot \nabla u^m \, d\mathrm{x} \\ &\quad + \int_{\{h\leqslant |u^m|\}} \big(b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m)\big) T_{k,h}(u^m)\, d\mathrm{x} \leqslant k \int_{\{|u^m|\geqslant h\}} |f^m|\, d\mathrm{x}. \end{aligned} \tag{5.9}$$

By virtue of (5.5), on the set $\{\Omega \colon h \leqslant |u^m|\}$ we have

$$\big(b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m)\big) T_{k,h}(u^m) \geqslant 0.$$

In view of this fact, applying (5.2) we derive from (5.9) that

$$\int_{\{h\leqslant|u^m|<k+h\}} \mathrm{a}(\mathrm{x},T_m(u^m),\nabla u^m)\cdot\nabla u^m\,d\mathrm{x} + \int_{\{h\leqslant|u^m|<k+h\}} \big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)|u^m-h\operatorname{sign}u^m|\,d\mathrm{x} + k\int_{\{|u^m|\geqslant k+h\}} \big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)\,d\mathrm{x} \leqslant k\int_{\{|u^m|\geqslant h\}} |f|\,d\mathrm{x}.$$

This yields the inequality

$$\int_{\{h\leqslant|u^m|<k+h\}} \big(\mathrm{a}(\mathrm{x},T_m(u^m),\nabla u^m)\cdot\nabla u^m+\phi\big)\,d\mathrm{x} + k\int_{\{|u^m|\geqslant k+h\}} \big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)\,d\mathrm{x} \leqslant \int_{\{|u^m|\geqslant h\}} (k|f|+\phi)\,d\mathrm{x} \leqslant C_3k+C_4, \qquad m\in\mathbb{N}. \tag{5.10}$$

Now we take $T_k(u^m)$ as a test function in (5.8); making similar transformations and taking account of (2.4) we obtain the inequality

$$\int_{\{|u^m|<k\}} \mathrm{a}(\mathrm{x},T_k(u^m),\nabla T_k(u^m))\cdot\nabla u^m\,d\mathrm{x} + \int_{\{|u^m|<k\}} M(\mathrm{x},u^m)\,d\mathrm{x} + k\int_{\{|u^m|\geqslant k\}} \big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)\,d\mathrm{x} \leqslant C_3k, \qquad m\geqslant k. \tag{5.11}$$

Using (3.1), we infer that

$$\overline{a}\int_{\{|u^m|<k\}} M(\mathrm{x},\overline{d}|\nabla u^m|)\,d\mathrm{x} + k\int_{\{|u^m|\geqslant k\}} \big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)\,d\mathrm{x} + \int_{\{|u^m|<k\}} M(\mathrm{x},u^m)\,d\mathrm{x} \leqslant C_3k+C_4, \qquad m\geqslant k. \tag{5.12}$$

If follows from (5.12) that

$$\int_\Omega M(\mathrm{x},T_k(u^m))\,d\mathrm{x} = \int_{\{|u^m|<k\}} M(\mathrm{x},u^m)\,d\mathrm{x} + \int_{\{|u^m|\geqslant k\}} M(\mathrm{x},k)\,d\mathrm{x} \leqslant \int_{\{|u^m|<k\}} M(\mathrm{x},u^m)\,d\mathrm{x} + k\int_{\{|u^m|\geqslant k\}} M'(\mathrm{x},|u^m|)\,d\mathrm{x} \leqslant C_8(k), \qquad m\geqslant k, \tag{5.13}$$

and

$$\int_\Omega |M'(\mathrm{x},u^m)|\,d\mathrm{x} \leqslant \int_{\{|u^m|<k\}} M'(\mathrm{x},k)\,d\mathrm{x} + \int_{\{|u^m|\geqslant k\}} M'(\mathrm{x},|u^m|)\,d\mathrm{x} \leqslant C_9(k), \qquad m\geqslant k. \tag{5.14}$$

In addition, inequality (5.12) implies the estimate

$$\int_{\{|u^m|<k\}} M(\mathrm{x},\overline{d}|\nabla u^m|)\,d\mathrm{x} = \int_\Omega M(\mathrm{x},\overline{d}|\nabla T_k(u^m)|)\,d\mathrm{x} \leqslant C_{10}(k), \qquad m \geqslant k. \tag{5.15}$$

Combining (5.4), (3.4) and (5.15) we deduce the inequality

$$\int_{\{|u^m|<k\}} |b^m(\mathrm{x},u^m,\nabla u^m)|\,d\mathrm{x} \leqslant \widehat{b}(k)\int_{\{|u^m|<k\}} \big(M(\mathrm{x},\overline{d}|\nabla u^m|)+\Phi_0(\mathrm{x})\big)\,d\mathrm{x} \leqslant C_{11}(k) \tag{5.16}$$

for $m \geqslant k$. It follows from (5.16) and (5.12) that

$$\|b^m(\mathrm{x},u^m,\nabla u^m)\|_1 \leqslant C_{12}(k), \qquad m \geqslant k. \tag{5.17}$$

In addition, from (5.13) and (5.15) we derive that

$$\|T_k(u^m)\|_M + \|\nabla T_k(u^m)\|_M \leqslant C_{13}(k), \qquad m \geqslant k. \tag{5.18}$$

Step 3. According to Lemma 4.2, estimate (5.12) implies that

$$\operatorname{meas}\{|u^m| \geqslant \rho\} \to 0 \quad \text{uniformly with respect to } m \in \mathbb{N}, \qquad \rho \to \infty. \tag{5.19}$$

Now we establish the convergence

$$u^m \to u \quad \text{a.e. in } \Omega, \qquad m \to \infty, \tag{5.20}$$

over a subsequence. It follows from (5.18) that the set $\{T_\rho(u^m)\}_{m\in\mathbb{N}}$ is bounded in $\mathring{W}^1L_M(\Omega)$. Owing to (2.14), by Lemma 2.1 the space $\mathring{W}^1L_M(\Omega)$ is compactly embedded in $L_P(\Omega(R))$ for any Musielak-Orlicz function $P \in L_{1,\mathrm{loc}}(\overline{\Omega})$, $P \prec\prec M_*$. Here $L_{1,\mathrm{loc}}(\overline{\Omega})$ is the space consisting of functions $v\colon \Omega \to \mathbb{R}$ such that $v \in L_1(Q)$ for any bounded set $Q \subset \Omega$.

For any fixed $\rho, R > 0$, this yields the convergence $T_\rho(u^m) \to v_\rho$ in $L_P(\Omega(R))$ and also the convergence $T_\rho(u^m) \to v_\rho$ over a subsequence almost everywhere in $\Omega(R)$. Next, (5.20) is established in the same way as in [19], § 5.3. It follows from (5.20) that

$$T_k(u^m) \to T_k(u) \quad \text{a.e. in } \Omega, \qquad m \to \infty, \tag{5.21}$$

for any $k > 0$.

Due to (5.20) and (5.21), we deduce from (5.12)–(5.14) that

$$\operatorname{meas}\{|u| \geqslant \rho\} \to 0, \qquad \rho \to \infty, \tag{5.22}$$

$$M'(\mathrm{x},|u|) \in L_1(\Omega), \tag{5.23}$$

$$\forall\, k > 0 \quad M(\mathrm{x},T_k(u)) \in L_1(\Omega). \tag{5.24}$$

Now we prove that

$$M'(\mathrm{x},u^m) \to M'(\mathrm{x},u) \quad \text{in } L_{1,\mathrm{loc}}(\overline{\Omega}), \qquad m \to \infty. \tag{5.25}$$

In view of the convergence (5.20) we have

$$M'(\mathrm{x},u^m)\to M'(\mathrm{x},u)\quad \text{a.e. in } \Omega,\qquad m\to\infty. \tag{5.26}$$

From (5.10) for $k=1$ and any $h>0$ we derive that

$$\begin{aligned}\int_{\{\Omega:\ h\leqslant|u^m|<1+h\}}&\big(\mathrm{a}(\mathrm{x},T_m(u^m),\nabla u^m)\cdot\nabla u^m+\phi(\mathrm{x})\big)\,d\mathrm{x}\\&+\int_{\{\Omega:\ |u^m|\geqslant h+1\}}\big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)\,d\mathrm{x}\\&\leqslant\int_{\{\Omega:\ |u^m|\geqslant h\}}(|f|+\phi)\,d\mathrm{x},\qquad m\in\mathbb{N}.\end{aligned}$$

In view of the fact that $f,\phi\in L_1(\Omega)$ and the absolute continuity of the integral on the right-hand side of the last inequality, taking account of (5.19), for any $\varepsilon>0$, we can choose sufficiently large $h(\varepsilon)>1$ such that

$$\begin{aligned}\int_{\{\Omega:\ h-1\leqslant|u^m|<h\}}&\big(\mathrm{a}(\mathrm{x},T_m(u^m),\nabla u^m)\cdot\nabla u^m+\phi(\mathrm{x})\big)\,d\mathrm{x}\\&+\int_{\{\Omega:\ |u^m|\geqslant h\}}\big(|b^m(\mathrm{x},u^m,\nabla u^m)|+M'(\mathrm{x},|u^m|)\big)\,d\mathrm{x}<\frac{\varepsilon}{2},\qquad m\in\mathbb{N}.\end{aligned} \tag{5.27}$$

Let Q be an arbitrary bounded subset of Ω; then for any measurable set $E\subset Q$ we have

$$\int_E M'(\mathrm{x},|u^m|)d\mathrm{x}\leqslant\int_{\{E:\ |u^m|<h\}}M'(\mathrm{x},|u^m|)\,d\mathrm{x}+\int_{\{\Omega:\ |u^m|\geqslant h\}}M'(\mathrm{x},|u^m|)\,d\mathrm{x}. \tag{5.28}$$

The fact that $M'(\mathrm{x},z)\in L_1(\Omega)$ for any fixed $z\in\mathbb{R}$ implies the inequality

$$\int_{\{E:\ |u^m|<h\}}M'(\mathrm{x},|u^m|)\,d\mathrm{x}\leqslant\int_E M'(\mathrm{x},h)\,d\mathrm{x}<\frac{\varepsilon}{2} \tag{5.29}$$

for any E such that $\operatorname{meas}E<\alpha(\varepsilon)$.

Combining (5.27)–(5.29) we infer the estimate

$$\int_E M'(\mathrm{x},|u^m|)\,d\mathrm{x}<\varepsilon\quad\forall\,E \text{ such that } \operatorname{meas}E<\alpha(\varepsilon),\qquad m\in\mathbb{N}.$$

It follows that the integrals $\int_Q M'(\mathrm{x},|u^m|)\,d\mathrm{x}$, $m\in\mathbb{N}$, are uniformly absolutely continuous; by Lemma 4.12 we have the convergence

$$M'(\mathrm{x},|u^m|)\to M'(\mathrm{x},|u|)\quad\text{in } L_1(Q),\qquad m\to\infty.$$

As $Q\subset\Omega$ is arbitrary, the convergence (5.25) is proved.

Step 4. We show that $T_k(u)\in\mathring{W}^1L_M(\Omega)$ for any $k>0$. Since the set $\{T_k(u^m)\}_{m\in\mathbb{N}}$ is bounded in $\mathring{W}^1L_M(\Omega)$, we can extract a weakly convergent subsequence such that

$T_k(u^m) \rightharpoonup v_k$ as $m \to \infty$ in the $\sigma(\mathbf{L}_M, \mathbf{E}_{\overline{M}})$-topology on $\mathring{W}^1 L_M(\Omega)$, and moreover $v_k \in \mathring{W}^1 L_M(\Omega)$. The continuity of the natural map $\mathring{W}^1 L_M(\Omega) \to \mathbf{L}_M(\Omega)$ implies the weak convergences

$$\nabla T_k(u^m) \rightharpoonup \nabla v_k \quad \text{in the } \sigma(\mathrm{L}_M, \mathrm{E}_{\overline{M}})\text{-topology in } \mathrm{L}_M(\Omega), \qquad m \to \infty,$$

and

$$T_k(u^m) \rightharpoonup v_k \quad \text{in the } \sigma(L_M, E_{\overline{M}})\text{-topology in } \mathrm{L}_M(\Omega), \qquad m \to \infty.$$

Using (5.21) and Lemma 4.3, we obtain the weak convergence

$$T_k(u^m) \rightharpoonup T_k(u) \quad \text{in the } \sigma(L_M, E_{\overline{M}})\text{-topology in } L_M(\Omega), \quad m \to \infty.$$

It follows that $v_k = T_k(u) \in \mathring{W}^1 L_M(\Omega)$; therefore,

$$\nabla T_k(u^m) \rightharpoonup \nabla T_k(u) \quad \text{in the } \sigma(\mathrm{L}_M, \mathrm{E}_{\overline{M}})\text{-topology in } \mathrm{L}_M(\Omega), \qquad m \to \infty. \tag{5.30}$$

Step 5. We prove the convergences

$$\nabla T_k(u^m) \to \nabla T_k(u) \quad \text{modularly in } \mathrm{L}_M(\Omega), \qquad m \to \infty, \tag{5.31}$$

and

$$\begin{aligned} &\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot \nabla T_k(u^m) \\ &\qquad \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)) \cdot \nabla T_k(u) \quad \text{in } L_1(\Omega), \qquad m \to \infty. \end{aligned} \tag{5.32}$$

Let $\mathrm{w} \in \mathrm{E}_M(\Omega)$ be arbitrary; then condition (3.3) yields the inequality

$$\big(\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) - \mathrm{a}(\mathrm{x}, T_k(u^m), \mathrm{w})\big) \cdot (\nabla T_k(u^m) - \mathrm{w}) \geqslant 0. \tag{5.33}$$

From this we derive that

$$\begin{aligned} \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot \mathrm{w}\, d\mathrm{x} &\leqslant \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot \nabla T_k(u^m)\, d\mathrm{x} \\ &\quad - \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \mathrm{w}) \cdot (\nabla T_k(u^m) - \mathrm{w})\, d\mathrm{x}. \end{aligned} \tag{5.34}$$

Furthermore, applying (3.2) we infer that

$$\begin{aligned} &\int_\Omega \overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, T_k(u^m), \mathrm{w})|)\, d\mathrm{x} \\ &\qquad \leqslant \|\Psi\|_1 + \widehat{A} \int_\Omega P(\mathrm{x}, \widehat{d}k)\, d\mathrm{x} + \widehat{A} \int_\Omega M(\mathrm{x}, \widehat{d}|\mathrm{w}|))\, d\mathrm{x} = C_{14}. \end{aligned}$$

Using (2.10), we establish the estimate

$$\|\mathrm{a}(\mathrm{x}, T_k(u^m), \mathrm{w})\|_{\overline{M}} \leqslant C_{15}. \tag{5.35}$$

Combining (5.34), (5.11), (5.18), and (5.35) we arrive at the inequality

$$\int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot \mathrm{w}\, d\mathrm{x} \leqslant C_{16} \quad \forall\, \mathrm{w} \in \mathrm{E}_M(\Omega).$$

Using the principle of uniform boundedness, for any $k > 0$ we have

$$\|\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m))\|_{\overline{M}} \leqslant C_{17}(k), \qquad m \geqslant k. \tag{5.36}$$

Estimate (5.36) implies that, over a subsequence,

$$\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \rightharpoonup \widetilde{\mathrm{a}}_k$$
$$\text{in the } \sigma(\mathrm{L}_{\overline{M}}, \mathrm{E}_M)\text{-topology on } \mathrm{L}_{\overline{M}}(\Omega), \qquad m \to \infty. \tag{5.37}$$

For positive real numbers m, j, δ, ε, and s we let $\omega(m, j, \delta, \varepsilon, s)$ denote an arbitrary quantity such that

$$\lim_{s\to+\infty} \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \lim_{j\to+\infty} \lim_{m\to+\infty} \omega(m, j, \delta, \varepsilon, s) = 0.$$

Let h, k, $h - 1 > k > 0$.

Due to Lemma 2.2, there exists a sequence $v^j \in C_0^\infty(\Omega)$ such that

$$v^j \to T_k(u) \quad \text{modularly in } \mathring{W}^1 L_M(\Omega), \qquad j \to \infty.$$

Then

$$T_k(v^j) \to T_k(u) \quad \text{modularly in } \mathring{W}^1 L_M(\Omega), \qquad j \to \infty,$$

and

$$T_k(v^j) \to T_k(u), \quad \nabla T_k(v^j) \to \nabla T_k(u) \quad \text{a.e. in } \Omega, \qquad j \to \infty. \tag{5.38}$$

In addition, by Lemma 4.13 we have

$$\nabla T_k(v^j) \rightharpoonup \nabla T_k(u) \quad \text{in the } \sigma(\mathrm{L}_M, \mathrm{L}_{\overline{M}})\text{-topology}, \qquad j \to \infty. \tag{5.39}$$

We set

$$z^{mj} = T_k(u^m) - T_k(v^j), \quad z^j = T_k(u) - T_k(v^j), \qquad m, j \in \mathbb{N},$$

and $\varphi_k(\rho) = \rho \exp(\gamma^2\rho^2)$, where $\gamma = \widehat{b}(k)/\overline{a}$. It is evident that

$$\psi_k(\rho) = \varphi_k'(\rho) - \gamma|\varphi_k(\rho)| \geqslant \frac{7}{8}, \qquad \rho \in \mathbb{R}.$$

It follows that

$$\frac{7}{8} \leqslant \psi_k(z^{mj}) \leqslant \max_{[-2k,2k]} \psi_k(\rho) = C_{18}(k), \quad m, j \in \mathbb{N}. \tag{5.40}$$

In view of (5.21) and (5.38) we have

$$\varphi_k(z^{mj}) \to \varphi_k(z^j), \quad \varphi_k'(z^{mj}) \to \varphi_k'(z^j), \quad \psi_k(z^{mj}) \to \psi_k(z^j)$$
$$\text{a.e. in } \Omega, \qquad m \to \infty, \tag{5.41}$$
$$\varphi_k(z^j) \to \varphi_k(0) = 0, \quad \varphi_k'(z^j) \to \varphi_k'(0) = 1, \quad \psi_k(z^j) \to \psi_k(0) = 1$$
$$\text{a.e. in } \Omega, \qquad j \to \infty, \tag{5.42}$$

and also

$$|\varphi_k(z^{mj})| \leqslant \varphi_k(2k), \quad 1 \leqslant \varphi_k'(z^{mj}) \leqslant \varphi_k'(2k), \qquad m, j \in \mathbb{N}, \tag{5.43}$$

and

$$|\varphi_k(z^{j})| \leqslant \varphi_k(2k), \quad 1 \leqslant \varphi_k'(z^{j}) \leqslant \varphi_k'(2k), \qquad j \in \mathbb{N}. \tag{5.44}$$

Using (5.41), (5.43), (5.42), and (5.44), due to Lemma 4.11 we establish the convergences

$$\varphi_k(z^{mj}) \rightharpoonup \varphi_k(z^j) \quad \text{in the } \sigma(L_\infty, L_1)\text{-topology in } L_\infty(\Omega), \qquad m \to \infty, \tag{5.45}$$

and

$$\varphi_k(z^{j}) \rightharpoonup 0 \quad \text{in the } \sigma(L_\infty, L_1)\text{-topology in } L_\infty(\Omega), \qquad j \to \infty. \tag{5.46}$$

We set $\zeta(r) = \min(1, \max(0, r))$, $\eta_h(r) = \zeta(h - r + 1)$, $\eta_{s,\varepsilon}(r) = \zeta((s - r)/\varepsilon + 1)$ and $\nu_{k,\delta}(r) = \zeta((r - k)/\delta + 1)$, $r \in \mathbb{R}$. It is obvious that

$$\eta_{s,\varepsilon}(|r|) \to \chi(\{|r| \leqslant s\}) \quad \text{in } \mathbb{R}, \qquad \varepsilon \to 0,$$

and

$$\nu_{k,\delta}(|r|) \to \chi(\{|r| \geqslant k\}) \quad \text{in } \mathbb{R}, \qquad \delta \to 0.$$

For brevity, we use the notation $\eta^m_{h-1}(\mathrm{x}) = \eta_{h-1}(|u^m|)$, $\widetilde{\eta}_{h-1}(\mathrm{x}) = \eta_{h-1}(|u|)$, $\eta^j_{s,\varepsilon}(\mathrm{x}) = \eta_{s,\varepsilon}(|\nabla T_k(v^j)|)$, $\widetilde{\eta}_{s,\varepsilon}(\mathrm{x}) = \eta_{s,\varepsilon}(|\nabla T_k(u)|)$, $\nu^m_{k,\delta}(\mathrm{x}) = \nu_{k,\delta}(|u^m|)$, and $\widetilde{\nu}_{k,\delta}(\mathrm{x}) = \nu_{k,\delta}(|u|)$.

It follows from (5.20) and (5.38) that

$$\eta^m_{h-1} \to \widetilde{\eta}_{h-1} \quad \text{a.e. in } \Omega, \qquad m \to \infty, \tag{5.47}$$

$$\eta^j_{s,\varepsilon} \to \widetilde{\eta}_{s,\varepsilon} \quad \text{a.e. in } \Omega, \qquad j \to \infty, \tag{5.48}$$

$$\nu^m_{k,\delta} \to \widetilde{\nu}_{k,\delta} \quad \text{a.e. in } \Omega, \qquad m \to \infty. \tag{5.49}$$

Taking $\varphi_k(z^{mj})\eta^m_{h-1}$ as a test function in (5.8), we obtain

$$\begin{aligned} &\int_\Omega \mathrm{a}(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla(\varphi_k(z^{mj})\eta^m_{h-1})\, d\mathrm{x} \\ &\qquad + \int_\Omega b^m(\mathrm{x}, u^m, \nabla u^m)\varphi_k(z^{mj})\eta^m_{h-1}\, d\mathrm{x} \\ &\qquad + \int_\Omega M'(\mathrm{x}, u^m)\varphi_k(z^{mj})\eta^m_{h-1}\, d\mathrm{x} \\ &\qquad - \int_\Omega f^m\varphi_k(z^{mj})\eta^m_{h-1}\, d\mathrm{x} = I_1 + I_2 + I_3 + I_4 = 0, \qquad m \geqslant h. \end{aligned} \tag{5.50}$$

Estimates for the integrals I_2–I_4. In view of the inequality $|M'(\mathrm{x}, u^m)|\eta^m_{h-1} \leqslant M'(\mathrm{x}, h) \in L_1(\Omega)$ and the convergences (5.45) and (5.46), we infer that

$$|I_3| \leqslant \int_\Omega M'(\mathrm{x}, h)|\varphi_k(z^{mj})|\, d\mathrm{x} = \omega(m) + \int_\Omega M'(\mathrm{x}, h)|\varphi_k(z^{j})|\, d\mathrm{x} = \omega_h(m, j). \tag{5.51}$$

Similarly, owing to (5.2), in view of the fact that $f \in L_1(\Omega)$ we obtain

$$|I_4| \leqslant \int_\Omega |f|\,|\varphi_k(z^{mj})|\,d\mathrm{x} = \omega(m) + \int_\Omega |f|\,|\varphi_k(z^{j})|\,d\mathrm{x} = \omega(m,j). \tag{5.52}$$

It is obvious that $z^{mj}u^m \geqslant 0$ for $|u^m| \geqslant k$; therefore, in view of (5.5), we arrive at the estimate

$$b^m(\mathrm{x}, u^m, \nabla u^m)\varphi_k(z^{mj}) \geqslant 0 \quad \text{for } |u^m| \geqslant k.$$

Taking it into account and applying (5.4) and (3.4) we estimate the integrals I_2:

$$\begin{aligned} -I_2 &\leqslant \int_{\{\Omega:\ |u^m|<k\}} |b^m(\mathrm{x}, u^m, \nabla u^m)|\,|\varphi_k(z^{mj})|\,d\mathrm{x} \\ &\leqslant \widehat{b}(k)\int_\Omega \big(M(\mathrm{x}, \overline{d}|\nabla T_k(u^m)|) + \Phi_0(\mathrm{x})\big)|\varphi_k(z^{mj})|\,d\mathrm{x}, \qquad m \in \mathbb{N}. \end{aligned}$$

Using (3.1), we infer the estimate

$$\begin{aligned} -I_2 \leqslant \frac{\widehat{b}(k)}{\overline{a}} &\int_\Omega \big(\overline{a}\Phi_0(\mathrm{x}) + \phi(\mathrm{x})\big)|\varphi_k(z^{mj})|\,d\mathrm{x} \\ &+ \frac{\widehat{b}(k)}{\overline{a}}\int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot \nabla T_k(u^m)|\varphi_k(z^{mj})|\,d\mathrm{x} = I_{21} + I_{22}. \end{aligned} \tag{5.53}$$

In view of (5.45) and (5.46) we have

$$I_{21} = \frac{\widehat{b}(k)}{\overline{a}}\int_\Omega \big(\overline{a}\Phi_0(\mathrm{x}) + \phi(\mathrm{x})\big)|\varphi_k(z^{mj})|\,d\mathrm{x} = \omega(m,j). \tag{5.54}$$

We set $I_1 = I_{11} - I_{12}$, where

$$I_{12} = \int_{\{\Omega:\ h-1\leqslant|u^m|<h\}} \mathrm{a}(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla u^m|\varphi_k(z^{mj})|\,d\mathrm{x}.$$

Now, using (5.51)–(5.54) we derive from (5.50) that

$$\begin{aligned} I_5 = I_{11} - I_{22} &= (I_1 + I_2) + I_{12} - I_{22} - I_2 = -(I_3 + I_4) + I_{12} + \omega(m,j) \\ &= \omega_h(m,j) + I_{12}, \qquad m \geqslant h. \end{aligned} \tag{5.55}$$

Using (5.43), we obtain the estimate

$$\begin{aligned} |I_{12}| \leqslant \varphi_k(2k) &\int_{\{\Omega:\ h-1\leqslant|u^m|<h\}} \big(\mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot \nabla u^m + \phi\big)\,d\mathrm{x} \\ &+ \varphi_k(2k)\int_{\{\Omega:\ h-1\leqslant|u^m|<h\}} \phi\,d\mathrm{x}, \qquad m \geqslant h. \end{aligned}$$

Due to (5.27), we have

$$I_{12} \leqslant \omega(h), \qquad m \geqslant h. \tag{5.56}$$

Combining (5.55) and (5.56), we establish the inequalities

$$I_5 \leqslant \omega(h) + \omega_h(m,j), \qquad m \geqslant h. \tag{5.57}$$

A representation for I_5. Making elementary transformations, we obtain

$$\begin{aligned}
I_5 &= \int_\Omega \mathrm{a}\big(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)\big) \cdot \nabla T_k(u^m) \varphi_k'(z^{mj})\, d\mathrm{x} \\
&\quad - \int_\Omega \mathrm{a}\big(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)\big) \cdot \nabla T_k(v^j) \varphi_k'(z^{mj}) \eta_{h-1}^m \, d\mathrm{x} \\
&\quad - \frac{\widehat{b}(k)}{\overline{a}} \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)\big) \cdot \nabla T_k(u^m) |\varphi_k(z^{mj})| \, d\mathrm{x} \\
&= \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)\big) \cdot \nabla T_k(u^m) \psi_k(z^{mj})\, d\mathrm{x} \\
&\quad - \int_\Omega \mathrm{a}\big(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)\big) \cdot \nabla T_k(v^j) \varphi_k'(z^{mj}) \eta_{h-1}^m \, d\mathrm{x} \\
&= \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)\big) \cdot \big(\nabla T_k(u^m) - \nabla T_k(v^j)\eta_{s,\varepsilon}^j\big) \psi_k(z^{mj})\, d\mathrm{x} \\
&\quad + \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)\big) \cdot \nabla T_k(v^j)\eta_{s,\varepsilon}^j \psi_k(z^{mj})\, d\mathrm{x} \\
&\quad - \int_\Omega \mathrm{a}\big(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)\big) \cdot \nabla T_k(v^j) \varphi_k'(z^{mj}) \eta_{s,\varepsilon}^j \eta_{h-1}^m \, d\mathrm{x} \\
&\quad + \int_\Omega \mathrm{a}\big(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)\big) \cdot \nabla T_k(v^j) (\eta_{s,\varepsilon}^j - 1) \varphi_k'(z^{mj}) \eta_{h-1}^m \, d\mathrm{x}.
\end{aligned}$$

It is obvious that

$$\begin{aligned}
I_5 &= \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)\big) \cdot \big(\nabla T_k(u^m) - \nabla T_k(v^j)\eta_{s,\varepsilon}^j\big) \psi_k(z^{mj})\, d\mathrm{x} \\
&\quad - \frac{\widehat{b}(k)}{\overline{a}} \int_\Omega \mathrm{a}\big(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)\big) \cdot \nabla T_k(v^j)\eta_{s,\varepsilon}^j |\varphi_k(z^{mj})| \, d\mathrm{x} \\
&\quad + \int_\Omega \big(\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) - \eta_{h-1}^m \mathrm{a}(\mathrm{x}, T_h(u^m), \nabla T_h(u^m))\big) \\
&\qquad \times \nabla T_k(v^j) \nu_{k,\delta}^m \eta_{s,\varepsilon}^j \varphi_k'(z^{mj})\, d\mathrm{x} \\
&\quad + \int_\Omega \mathrm{a}\big(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)\big) \cdot \nabla T_k(v^j) (\eta_{s,\varepsilon}^j - 1) \varphi_k'(z^{mj}) \eta_{h-1}^m \, d\mathrm{x} \\
&= I_{51} + I_{52} + I_{53} + I_{54}, \qquad m \geqslant h.
\end{aligned} \tag{5.58}$$

Estimates for the integrals I_{52}–I_{54}.

Applying (5.41), (5.43), and Lemma 4.11 for $g = \nabla T_k(v^j)\eta_{s,\varepsilon}^j \in \mathrm{E}_M(\Omega)$, we deduce that

$$\nabla T_k(v^j)\eta_{s,\varepsilon}^j |\varphi_k(z^{mj})| \to \nabla T_k(v^j)\eta_{s,\varepsilon}^j |\varphi_k(z^j)| \quad \text{strongly in } \mathrm{E}_M(\Omega), \qquad m \to \infty.$$

In view of the convergence (5.37), we establish the equality

$$I_{52} = -\frac{\widehat{b}(k)}{\overline{a}} \int_\Omega \widetilde{\mathrm{a}}_k \cdot \nabla T_k(v^j)\eta_{s,\varepsilon}^j |\varphi_k(z^j)| \, d\mathrm{x} + \omega(m).$$

Using (5.42), (5.44), (5.48) and Lemma 4.11 for $g = \widetilde{\mathrm{a}}_k \in \mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}(\Omega)$, we obtain that

$$\widetilde{\mathrm{a}}_k |\varphi_k(z^j)| \eta^j_{s,\varepsilon} \to \widetilde{\mathrm{a}}_k |\varphi_k(0)| \widetilde{\eta}_{s,\varepsilon} = 0 \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad j \to \infty.$$

Taking account of the convergence (5.39), we have

$$I_{52} = \omega(m,j). \tag{5.59}$$

Applying (5.41), (5.43), (5.49) and Lemma 4.11 for $g = \nabla T_k(v^j)\eta^j_{s,\varepsilon} \in \mathrm{E}_M(\Omega)$, we derive that

$$\nabla T_k(v^j)\eta^j_{s,\varepsilon}\varphi'_k(z^{mj})\nu^m_{k,\delta} \to \nabla T_k(v^j)\eta^j_{s,\varepsilon}\varphi'_k(z^j)\widetilde{\nu}_{k,\delta}$$
$$\text{strongly in } \mathrm{E}_M(\Omega), \qquad m \to \infty.$$

In view of the convergence (5.37) it follows that

$$I_{53} = \int_\Omega (\widetilde{\mathrm{a}}_k - \widetilde{\eta}_{h-1}\widetilde{\mathrm{a}}_h) \cdot \nabla T_k(v^j)\widetilde{\nu}_{k,\delta}\eta^j_{s,\varepsilon}\varphi'_k(z^j)\, d\mathrm{x} + \omega(m).$$

Furthermore, applying (5.42), (5.44), (5.48) and Lemma 4.11 for $g = (\widetilde{\mathrm{a}}_k - \widetilde{\eta}_{h-1}\widetilde{\mathrm{a}}_h) \in \mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}(\Omega)$, we obtain

$$(\widetilde{\mathrm{a}}_k - \widetilde{\eta}_{h-1}\widetilde{\mathrm{a}}_h)\varphi'_k(z^j)\widetilde{\nu}_{k,\delta}\eta^j_{s,\varepsilon} \to (\widetilde{\mathrm{a}}_k - \widetilde{\eta}_{h-1}\widetilde{\mathrm{a}}_h)\widetilde{\nu}_{k,\delta}\widetilde{\eta}_{s,\varepsilon}$$
$$\text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad j \to \infty.$$

In view of the convergence (5.39) we have

$$I_{53} = \int_\Omega (\widetilde{\mathrm{a}}_k - \eta_{h-1}\widetilde{\mathrm{a}}_h) \cdot \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon}\widetilde{\nu}_{k,\delta}\, d\mathrm{x} + \omega(m,j).$$

Taking account of the fact that $(\widetilde{\mathrm{a}}_k - \eta_{h-1}\widetilde{\mathrm{a}}_h) \cdot \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon} \in L_1(\Omega)$, we pass to the limit as $\delta \to 0$ and arrive at the relation

$$I_{53} = \int_\Omega (\widetilde{\mathrm{a}}_k - \eta_{h-1}\widetilde{\mathrm{a}}_h) \cdot \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon}\chi(\{|u| \geqslant k\})\, d\mathrm{x} + \omega(m,j,\delta) = \omega(m,j,\delta). \tag{5.60}$$

Then using (5.41), (5.43), (5.47) and Lemma 4.11 for $g = \nabla T_k(v^j)(\eta^j_{s,\varepsilon} - 1) \in \mathrm{E}_M(\Omega)$ we infer that

$$\nabla T_k(v^j)(\eta^j_{s,\varepsilon} - 1)\varphi'_k(z^{mj})\eta^m_{h-1} \to \nabla T_k(v^j)(\eta^j_{s,\varepsilon} - 1)\varphi'_k(z^j)\widetilde{\eta}_{h-1}$$
$$\text{strongly in } \mathrm{E}_M(\Omega), \qquad m \to \infty.$$

In view of the convergence (5.37), we establish the equality

$$I_{54} = \int_\Omega \widetilde{\mathrm{a}}_h \cdot \nabla T_k(v^j)(\eta^j_{s,\varepsilon} - 1)\varphi'_k(z^j)\widetilde{\eta}_{h-1}\, d\mathrm{x} + \omega(m).$$

Using (5.42), (5.44), (5.48) and Lemma 4.11 for $g = \widetilde{\mathrm{a}}_h \in \mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}(\Omega)$, we obtain

$$\widetilde{\mathrm{a}}_h\varphi'_k(z^j)(\eta^j_{s,\varepsilon} - 1)\widetilde{\eta}_{h-1} \to \widetilde{\mathrm{a}}_h(\widetilde{\eta}_{s,\varepsilon} - 1)\widetilde{\eta}_{h-1} \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad j \to \infty.$$

In view of the convergence (5.39) we have

$$I_{54} = \int_\Omega \widetilde{\mathrm{a}}_h \cdot \nabla T_k(u)(\widetilde{\eta}_{s,\varepsilon} - 1)\, d\mathrm{x} + \omega(m,j).$$

Due to the fact that $\widetilde{\mathrm{a}}_h \cdot \nabla T_k(u) \in L_1(\Omega)$, we obtain the equality

$$I_{54} = \int_\Omega \widetilde{\mathrm{a}}_h \cdot \nabla T_k(u)(\chi_s - 1)\, d\mathrm{x} + \omega(m,j,\varepsilon) = \omega(m,j,\varepsilon,s). \tag{5.61}$$

Since I_{51} is independent of h, it follows from (5.57)–(5.61) that

$$I_{51} \leqslant \omega_h(m,j) + \omega(m,j,\delta,\varepsilon,s) + \omega(h). \tag{5.62}$$

An estimate for the integral

$$\begin{aligned} I_6 &= \int_\Omega \big(\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) - \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon})\big) \\ &\qquad \times (\nabla T_k(u^m) - \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^{mj})\, d\mathrm{x} \\ &= \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot (\nabla T_k(u^m) - \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^{mj})\, d\mathrm{x} \\ &\qquad - \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \cdot (\nabla T_k(u^m) - \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^{mj})\, d\mathrm{x} \\ &= I_{51} - I_{61}. \end{aligned} \tag{5.63}$$

The convergence (5.21) implies that

$$\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \quad \text{a.e. in } \Omega, \qquad m \to \infty,$$

while (3.2) yields the estimates

$$\begin{aligned} &\overline{M}(\mathrm{x}, |\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon})|) \\ &\qquad \leqslant \widehat{A}M(\mathrm{x}, \widehat{d}(s+1)) + \widehat{A}P(\mathrm{x}, \widehat{d}k) + \Psi(\mathrm{x}) \in L_1(\Omega), \qquad \varepsilon < 1, \quad m,j \in \mathbb{N}. \end{aligned}$$

By virtue of Lemma 4.8,

$$\begin{aligned} &\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \\ &\qquad \text{modularly in } \mathrm{L}_{\overline{M}}(\Omega), \qquad m \to \infty. \end{aligned}$$

Since the function $\overline{M}$ satisfies the Δ_2-condition, it is true that $\mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}(\Omega)$ and

$$\begin{aligned} &\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \\ &\qquad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad m \to \infty. \end{aligned} \tag{5.64}$$

In a similar way we establish that

$$\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u)\chi_s) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \quad m \to \infty, \tag{5.65}$$

$$\mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon}) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \quad j \to \infty, \tag{5.66}$$

$$\mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon}) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \quad \varepsilon \to 0, \tag{5.67}$$

and

$$\mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s) \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \quad s \to \infty. \tag{5.68}$$

Applying (5.40)–(5.42), (5.48), and (5.49), from (5.64), (5.66)–(5.68) we derive that

$$\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^{mj}) \\ \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^j) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad m \to \infty, \tag{5.69}$$

$$\mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^j) \\ \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon}) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad j \to \infty, \tag{5.70}$$

$$\mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon})(1 - \widetilde{\eta}_{s,\varepsilon}) \\ \to \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s)(1 - \chi_s) \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad \varepsilon \to 0, \tag{5.71}$$

and

$$\mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s)(1 - \chi_s) \to 0 \quad \text{strongly in } \mathrm{E}_{\overline{M}}(\Omega), \qquad s \to \infty. \tag{5.72}$$

Using (5.69) and (5.30) we infer the relation

$$I_{61} = \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \cdot (\nabla T_k(u) - \nabla T_k(v^j)\eta^j_{s,\varepsilon})\psi_k(z^j)\, d\mathrm{x} + \omega(m), \\ j \in \mathbb{N}.$$

Due to Lemma 4.8, it follows from (5.38) and (5.48) that

$$\nabla T_k(v^j)\eta^j_{s,\varepsilon} \to \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon} \quad \text{modularly in } \mathrm{L}_M(\Omega), \qquad j \to \infty.$$

In accordance with Lemma 4.13, we derive from this that

$$\nabla T_k(v^j)\eta^j_{s,\varepsilon} \rightharpoonup \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon} \quad \text{in the } \sigma(\mathrm{L}_M, \mathrm{L}_{\overline{M}})\text{-topology}, \qquad j \to \infty. \tag{5.73}$$

Applying (5.70), (5.73) and (5.71) we obtain the equality

$$I_{61} = \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\widetilde{\eta}_{s,\varepsilon}) \cdot \nabla T_k(u)(1 - \widetilde{\eta}_{s,\varepsilon})\, d\mathrm{x} + \omega(m, j) \\ = \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s) \cdot \nabla T_k(u)(1 - \chi_s)\, d\mathrm{x} + \omega(m, j, \varepsilon).$$

Finally, owing to (5.72) we arrive at

$$I_{61} = \omega(m, j, \varepsilon, s). \tag{5.74}$$

Combining (5.63), (5.74), and (5.62) and using (5.40) we deduce the estimate

$$I_7 = \int_\Omega \big(\mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) - \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon})\big) \\ \times (\nabla T_k(u^m) - \nabla T_k(v^j)\eta^j_{s,\varepsilon})\, d\mathrm{x} \leqslant \frac{8}{7} I_6 \\ \leqslant \omega_h(m, j) + \omega(m, j, \delta, \varepsilon, s) + \omega(h). \tag{5.75}$$

Using the notation (4.23) we have

$$
\begin{aligned}
0 &\leqslant \int_\Omega q_s^m(\mathrm{x})\,d\mathrm{x} \\
&= I_7 + \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \cdot (\nabla T_k(v^j)\eta^j_{s,\varepsilon} - \nabla T_k(u)\chi_s)\,d\mathrm{x} \\
&\qquad - \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u)\chi_s) \cdot (\nabla T_k(u^m) - \nabla T_k(u)\chi_s)\,d\mathrm{x} \\
&\qquad + \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(v^j)\eta^j_{s,\varepsilon}) \cdot (\nabla T_k(u^m) - \nabla T_k(v^j)\eta^j_{s,\varepsilon})\,d\mathrm{x} \\
&= I_7 + I_{71} + I_{72} + I_{73}.
\end{aligned} \tag{5.76}
$$

Estimates for the integrals I_{71}–I_{73}. In view of the convergences (5.37) and (5.73), we have

$$
\begin{aligned}
I_{71} &= \int_\Omega \widetilde{\mathrm{a}}_k \cdot (\nabla T_k(v^j)\eta^j_{s,\varepsilon} - \nabla T_k(u)\chi_s)\,d\mathrm{x} + \omega(m) \\
&= \int_\Omega \widetilde{\mathrm{a}}_k \cdot \nabla T_k(u)(\widetilde{\eta}_{s,\varepsilon} - \chi_s)\,d\mathrm{x} + \omega(m,j) = \omega(m,j,\varepsilon).
\end{aligned} \tag{5.77}
$$

Applying (5.30) and (5.65) we obtain the equality

$$
I_{72} = \int_\Omega \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)\chi_s) \cdot \nabla T_k(u)(\chi_s - 1)\,d\mathrm{x} + \omega(m).
$$

In view of (5.72), it follows that

$$
I_{72} = \omega(m,s). \tag{5.78}
$$

The integral I_{73} is estimated in the same way as I_{61}: more precisely, we have

$$
I_{73} = \omega(m,j,\varepsilon,s). \tag{5.79}
$$

Combining (5.75)–(5.79) we obtain

$$
\int_\Omega q_s^m(\mathrm{x})\,d\mathrm{x} \leqslant \omega_h(m,j) + \omega(m,j,\delta,\varepsilon,s) + \omega(h), \qquad m \geqslant h.
$$

As the left-hand side of the last inequality is independent of j, δ, ε and h, taking the limits as $m \to \infty$, $j \to \infty$, $\delta \to 0$, $\varepsilon \to 0$ and $s \to \infty$ in succession we establish the relation

$$
\lim_{s\to\infty} \lim_{m\to\infty} \int_\Omega q_s^m(\mathrm{x})\,d\mathrm{x} \leqslant \omega(h).
$$

Passing to the limit as $h \to \infty$, we infer that

$$
\lim_{s\to\infty} \lim_{m\to\infty} \int_\Omega q_s^m(\mathrm{x})\,d\mathrm{x} = 0.
$$

By Lemma 4.10, we have the convergences (5.31), (5.32) and

$$
\nabla T_k(u^m) \to \nabla T_k(u) \quad \text{a.e. in } \Omega, \qquad m \to \infty. \tag{5.80}
$$

Furthermore, like in [19], § 5.5, we establish the convergence

$$\nabla u^m \to \nabla u \quad \text{a.e. in } \Omega, \qquad m \to \infty, \tag{5.81}$$

over a subsequence.
Step 6. Using (5.36) and the convergences (5.21) and (5.80), due to Lemma 4.3 we establish the weak convergence

$$\begin{gathered} \mathrm{a}(\mathrm{x}, T_k(u^m), \nabla T_k(u^m)) \rightharpoonup \mathrm{a}(\mathrm{x}, T_k(u), \nabla T_k(u)) \\ \text{in the } \sigma(\mathrm{L}_{\overline{M}}, \mathrm{E}_M)\text{-topology in the space } \mathrm{L}_{\overline{M}}(\Omega), \qquad m \to \infty. \end{gathered} \tag{5.82}$$

The continuity of $b(\mathrm{x}, s_0, \mathrm{s})$ in (s_0, s) and the convergences (5.20) and (5.81) imply that

$$b^m(\mathrm{x}, u^m, \nabla u^m) \to b(\mathrm{x}, u, \nabla u) \quad \text{a.e. in } \Omega, \qquad m \to \infty. \tag{5.83}$$

In view of (5.83), according to Fatou's lemma we can derive from (5.17) that

$$b(\mathrm{x}, u, \nabla u) \in L_1(\Omega). \tag{5.84}$$

Thus, condition (1) in Definition 3.1 is satisfied.

Now we establish the convergence

$$b^m(\mathrm{x}, u^m, \nabla u^m) \to b(\mathrm{x}, u, \nabla u) \quad \text{in } L_{1,\mathrm{loc}}(\overline{\Omega}), \qquad m \to \infty. \tag{5.85}$$

Let Q be an arbitrary bounded subset in Ω. For any measurable set $E \subset Q$ we have

$$\begin{aligned} &\int_E |b^m(\mathrm{x}, u^m, \nabla u^m)|\, d\mathrm{x} \\ &\quad \leqslant \int_{\{E:\ |u^m| < h\}} |b^m(\mathrm{x}, u^m, \nabla u^m)|\, d\mathrm{x} + \int_{\{\Omega:\ |u^m| \geqslant h\}} |b^m(\mathrm{x}, u^m, \nabla u^m)|\, d\mathrm{x}. \end{aligned} \tag{5.86}$$

From (5.4), (3.4) and (3.1) we deduce the estimate

$$\begin{aligned} \int_{\{E:\ |u^m| < h\}} |b^m(\mathrm{x}, u^m, \nabla u^m)|\, d\mathrm{x} &\leqslant \widehat{b}(h) \int_{\{E:\ |u^m| < h\}} \big(M(\mathrm{x}, \overline{d}|\nabla u^m|) + \Phi_0(\mathrm{x})\big)\, d\mathrm{x} \\ &\leqslant \frac{\widehat{b}(h)}{\overline{a}} \int_E \big(\mathrm{a}(\mathrm{x}, T_h(u^m), \nabla T_h(u^m)) \cdot \nabla T_h(u^m) + \phi\big)\, d\mathrm{x} + \widehat{b}(h) \int_E \Phi_0(\mathrm{x})\, d\mathrm{x}. \end{aligned}$$

In view of the fact that $\Phi_0, \phi \in L_1(E)$, the convergence (5.32), and the absolute continuity of the integrals on the right-hand side of the last inequality, for any $\varepsilon > 0$ there is an $\alpha(\varepsilon)$ such that

$$\int_{\{E:\ |u^m| < h\}} |b^m(\mathrm{x}, u^m, \nabla u^m)|\, d\mathrm{x} < \frac{\varepsilon}{2}, \qquad m \in \mathbb{N}, \tag{5.87}$$

for any E with $\operatorname{meas} E < \alpha(\varepsilon)$.

Combining (5.27), (5.86) and (5.87), we establish the estimate

$$\int_E |b^m(\mathrm{x}, u^m, \nabla u^m)|\, dx < \varepsilon \quad \forall\, E \text{ such that } \operatorname{meas} E < \alpha(\varepsilon), \qquad m \in \mathbb{N}.$$

It follows that the sequence $\{b^m(\mathrm{x}, u^m, \nabla u^m)\}_{m\in\mathbb{N}}$ has uniformly absolutely continuous integrals over the set Q. By Lemma 4.12, we obtain

$$b^m(\mathrm{x}, u^m, \nabla u^m) \to b(\mathrm{x}, u, \nabla u) \quad \text{in } L_1(Q), \qquad m \to \infty,$$

for any bounded set $Q \subset \Omega$. The convergence (5.85) is proved.
Step 7. To prove (3.9) we take a test function $v = T_k(u^m - \xi)$ in (5.8), where $\xi \in C_0^1(\Omega)$, and arrive at the relation

$$\int_\Omega \mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot \nabla T_k(u^m - \xi)\, d\mathrm{x} \\ + \int_\Omega \big(b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m) - f^m\big) T_k(u^m - \xi)\, d\mathrm{x} = I^m + J^m. \tag{5.88}$$

We set $\widehat{k} = k + \|\xi\|_\infty$. If $|u^m| \geqslant \widehat{k}$, then $|u^m - \xi| \geqslant |u^m| - \|\xi\|_\infty \geqslant k$; therefore, $\{\Omega\colon |u^m - \xi| < k\} \subseteq \{\Omega\colon |u^m| < \widehat{k}\}$. Consequently,

$$\begin{aligned} I^m &= \int_\Omega \mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot \nabla T_k(u^m - \xi)\, d\mathrm{x} \\ &= \int_{\{\Omega\colon |u^m - \xi| < k\}} \mathrm{a}(\mathrm{x}, u^m, \nabla u^m) \cdot \nabla (u^m - \xi)\, d\mathrm{x} \\ &= \int_{\{\Omega\colon |u^m - \xi| < k\}} \big(\mathrm{a}(\mathrm{x}, u^m, \nabla u^m) - \mathrm{a}(\mathrm{x}, u^m, \nabla \xi)\big) \cdot \nabla (u^m - \xi)\, d\mathrm{x} \\ &\quad + \int_{\{\Omega\colon |u^m - \xi| < k\}} \mathrm{a}(\mathrm{x}, u^m, \nabla \xi) \cdot \nabla (u^m - \xi)\, d\mathrm{x} \\ &\geqslant \int_\Omega \eta_{k-\varepsilon,\varepsilon}(|u^m - \xi|)\big(\mathrm{a}(\mathrm{x}, u^m, \nabla u^m) - \mathrm{a}(\mathrm{x}, u^m, \nabla \xi)\big) \cdot \nabla (u^m - \xi)\, d\mathrm{x} \\ &\quad + \int_\Omega \mathrm{a}(\mathrm{x}, T_{\widehat{k}}(u^m), \nabla \xi) \cdot \nabla T_k(u^m - \xi)\, d\mathrm{x} = I_1^{m\varepsilon} + I_2^m, \qquad m \geqslant \widehat{k}. \end{aligned} \tag{5.89}$$

Furthermore, as $\mathrm{a}(\mathrm{x}, s_0, \mathrm{s})$ is continuous in the variables (s_0, s), due to Fatou's lemma, we derive from (5.20), (5.81) and (3.3) that

$$\liminf_{m\to\infty} I_1^{m\varepsilon} \geqslant \int_\Omega \eta_{k-\varepsilon,\varepsilon}(|u - \xi|)\big(\mathrm{a}(\mathrm{x}, u, \nabla u) - \mathrm{a}(\mathrm{x}, u, \nabla \xi)\big) \cdot \nabla (u - \xi)\, d\mathrm{x}.$$

Since $\eta_{k-\varepsilon,\varepsilon}(|u - \xi|) \to \chi(\{|u - \xi| < k\})$ in Ω as $\varepsilon \to 0$, passing to the limit yields the inequality

$$\lim_{\varepsilon\to 0} \liminf_{m\to\infty} I_1^{m\varepsilon} \geqslant \int_\Omega \big(\mathrm{a}(\mathrm{x}, T_{\widehat{k}}(u), \nabla T_{\widehat{k}}(u)) - \mathrm{a}(\mathrm{x}, T_{\widehat{k}}(u), \nabla \xi)\big) \cdot \nabla T_k(u - \xi)\, d\mathrm{x}. \tag{5.90}$$

From Lemma 4.8 we obtain

$$\mathrm{a}(\mathrm{x}, T_{\widehat{k}}(u^m), \nabla\xi) \to \mathrm{a}(\mathrm{x}, T_{\widehat{k}}(u), \nabla\xi) \quad \text{strongly in } \mathrm{L}_{\overline{M}}(\Omega) = \mathrm{E}_{\overline{M}}(\Omega), \qquad m \to \infty. \tag{5.91}$$

Let $v^m = u^m - \xi$ and $v = u - \xi$. Since $|v| = k$ in a set where $|v^m| \to k$ as $m \to \infty$, we have $\nabla v = 0$. It follows that

$$\begin{aligned} \nabla T_k(v^m) - \nabla T_k(v) &= \chi_{\{\Omega:\, |v^m|<k\}}(\nabla v^m - \nabla v) \\ &\quad + \big(\chi_{\{\Omega:\, |v^m|<k\}} - \chi_{\{\Omega:\, |v|<k\}}\big)\nabla v \to 0 \quad \text{a.e. in } \Omega, \qquad m \to \infty. \end{aligned} \tag{5.92}$$

Evidently,

$$|\nabla T_k(u^m - \xi)| \leqslant |\nabla T_{\widehat{k}}(u^m)| + |\nabla\xi|, \qquad \mathrm{x} \in \Omega, \quad m \in \mathbb{N}.$$

Then (5.18) implies that the sequence $\{\nabla T_k(u^m - \xi)\}_{m\in\mathbb{N}}$ is bounded in $\mathrm{L}_M(\Omega)$. Using (5.92) and Lemma 4.3, we derive from this that

$$\begin{gathered} \nabla T_k(u^m - \xi) \rightharpoonup \nabla T_k(u - \xi) \\ \text{in the } \sigma(\mathrm{L}_M, \mathrm{E}_{\overline{M}})\text{-topology in } \mathrm{L}_M(\Omega), \qquad m \to \infty, \end{gathered} \tag{5.93}$$

for any $k > 0$.

Combining (5.89)–(5.93) we conclude that

$$\begin{aligned} \liminf_{m\to\infty} I^m &\geqslant \int_\Omega \mathrm{a}(\mathrm{x}, T_{\widehat{k}}(u), \nabla T_{\widehat{k}}(u)) \cdot \nabla T_k(u - \xi)\, d\mathrm{x} \\ &= \int_\Omega \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot \nabla T_k(u - \xi)\, d\mathrm{x}. \end{aligned} \tag{5.94}$$

According to Lemma 4.11, we derive from (5.20) that

$$\begin{gathered} T_k(u^m - \xi) \rightharpoonup T_k(u - \xi) \\ \text{in the } \sigma(L_\infty, L_1)\text{-topology on } L_\infty(\Omega), \qquad m \to \infty. \end{gathered} \tag{5.95}$$

We split the integral J^m into two terms. The first integral

$$J_1^m = \int_\Omega \big(b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m)\big) T_k(u^m - \xi)\, d\mathrm{x}$$

is estimated as follows. Let $\operatorname{supp}\xi \subset \Omega(l)$, $l \geqslant l_0$, and let $c^m(\mathrm{x}, u^m, \nabla u^m) = b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m)$ and $c(\mathrm{x}, u, \nabla u) = b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u)$. Then in view of (5.5) we have

$$\begin{aligned} &\int_{\Omega\setminus\Omega(l)} c^m(\mathrm{x}, u^m, \nabla u^m) T_k(u^m)\, d\mathrm{x} + \int_{\Omega(l)} c^m(\mathrm{x}, u^m, \nabla u^m) T_k(u^m - \xi)\, d\mathrm{x} \\ &\qquad \geqslant \int_{\Omega(l)} c^m(\mathrm{x}, u^m, \nabla u^m) T_k(u^m - \xi)\, d\mathrm{x} = \overline{J}_1^{\,lm} \end{aligned}$$

for $l \geqslant l_0$. Applying (5.25), (5.85) and (5.95) we pass to the limit as $m \to \infty$. Next, taking (5.23) and (5.84) into account, we take the limit as $l \to \infty$ and arrive at

$$\int_\Omega (b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u)) T_k(u - \xi)\, d\mathrm{x} = \lim_{l\to\infty} \lim_{m\to\infty} \overline{J}_1^{\,lm} \leqslant \liminf_{m\to\infty} J_1^m. \tag{5.96}$$

Using (5.1) and (5.95) and passing to the limit as $m \to \infty$ in the second integral, we establish the relation

$$\lim_{m\to\infty} J_2^m = \lim_{m\to\infty} \int_\Omega f^m T_k(u^m - \xi)\,d\mathrm{x} = \int_\Omega f T_k(u - \xi)\,d\mathrm{x}. \tag{5.97}$$

Combining (5.88), (5.94), (5.96) and (5.97) we obtain (3.9). Theorem 3.1 is proved.

Proof of Theorem 3.2. We prove that the entropy solution constructed in Theorem 3.1 has all the properties of a renormalized solution. Condition (1) holds since it coincides with condition (1) in Definition 3.1. Then condition (2) also holds (see (4.18)).

We prove (3.10). Let $\{u^m\}_{m\in\mathbb{N}}$ be a sequence of weak solutions of problem (5.3), (1.2) and let $S \in W^1_\infty(\mathbb{R})$ be a function such that $\operatorname{supp} S \subset [-M, M]$ for $M > 0$. For any function $\xi \in C_0^1(\Omega)$ we take $S(u^m)\xi \in \mathring{W}^1 L_M(\Omega)$ as a test function in (5.8) and infer that

$$\begin{aligned}
&\langle \mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot (S'(u^m)\xi \nabla u^m + S(u^m)\nabla\xi)\rangle \\
&\qquad + \langle (b^m(\mathrm{x}, u^m, \nabla u^m) + M'(\mathrm{x}, u^m) - f^m(\mathrm{x}))S(u^m)\xi\rangle \\
&= I^m + J^m = 0, \qquad m \in \mathbb{N}.
\end{aligned} \tag{5.98}$$

Obviously, we have

$$\begin{aligned}
I^m &= \int_\Omega \mathrm{a}(\mathrm{x}, T_m(u^m), \nabla u^m) \cdot (S'(u^m)\xi\nabla u^m + S(u^m)\nabla\xi)\,d\mathrm{x} \\
&= \int_\Omega \mathrm{a}(\mathrm{x}, T_M(u^m), \nabla T_M(u^m)) \cdot \nabla T_M(u^m) S'(u^m)\xi\,d\mathrm{x} \\
&\quad + \int_\Omega \mathrm{a}(\mathrm{x}, T_M(u^m), \nabla T_M(u^m)) \cdot \nabla\xi S(u^m)\,d\mathrm{x} = I_1^m + I_2^m, \qquad m \geqslant M.
\end{aligned} \tag{5.99}$$

In view of the convergences (5.20), (5.32), and (5.80), applying Lemma 4.4 we establish the relation

$$I_1^m = \int_\Omega \mathrm{a}(\mathrm{x}, T_M(u), \nabla T_M(u)) \cdot \nabla T_M(u) S'(u)\xi\,d\mathrm{x} + \omega(m), \qquad m \to \infty. \tag{5.100}$$

By Lemma 4.11, the convergence (5.20) yields that

$$S(u^m)\nabla\xi \to S(u)\nabla\xi \quad \text{strongly in } \mathrm{E}_M(\Omega), \qquad m \to \infty.$$

In view of (5.82), we obtain

$$I_2^m = \int_\Omega \mathrm{a}(\mathrm{x}, T_M(u), \nabla T_M(u)) \cdot \nabla\xi S(u)\,d\mathrm{x} + \omega(m), \qquad m \to \infty. \tag{5.101}$$

Combining (5.99)–(5.101) yields

$$\begin{aligned}
\lim_{m\to\infty} I^m &= \int_\Omega \mathrm{a}(\mathrm{x}, T_M(u), \nabla T_M(u)) \cdot (S'(u)\xi\nabla T_M(u) + S(u)\nabla\xi)\,d\mathrm{x} \\
&= \int_\Omega \mathrm{a}(\mathrm{x}, u, \nabla u) \cdot (S'(u)\xi\nabla u + S(u)\nabla\xi)\,d\mathrm{x}.
\end{aligned} \tag{5.102}$$

Due to Lemma 4.11 we have

$$S(u^m)\xi \rightharpoonup S(u)\xi \quad \text{in the } \sigma(L_\infty, L_1)\text{-topology}, \qquad m \to \infty.$$

Then taking account of the convergences (5.1), (5.25), and (5.85), we establish the equality

$$\lim_{m\to\infty} J^m = \int_\Omega (b(\mathrm{x}, u, \nabla u) + M'(\mathrm{x}, u) - f) S(u)\xi \, d\mathrm{x}. \tag{5.103}$$

Combining (5.98), (5.102) and (5.103) we arrive at (3.10). Thus, we conclude that u is a renormalized solution of problem (1.1), (1.2), (3.7). Theorem 3.2 is proved.

Bibliography

[1] G. Dal Maso, F. Murat, L. Orsina and A. Prignet, "Renormalized solutions of elliptic equations with general measure data", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **28**:4 (1999), 741–808.

[2] A. Malusa, "A new proof of the stability of renormalized solutions to elliptic equations with measure data", *Asymptot. Anal.* **43**:1–2 (2005), 111–129.

[3] I. Chlebicka, *Measure data elliptic problems with generalized Orlicz growth*, 2020, arXiv: 2008.02495.

[4] I. Chlebicka and P. Nayar, "Essentially fully anisotropic Orlicz functions and uniqueness to measure data problem", *Math. Methods Appl. Sci.*, 2021, 1–25, Publ. online.

[5] L. M. Kozhevnikova, "Renormalized solutions of elliptic equations with variable exponents and general measure data", *Mat. Sb.* **211**:12 (2020), 83–122; English transl. in *Sb. Math.* **211**:12 (2020), 1737–1776.

[6] P. Gwiazda, I. Skrzypczak and A. Zatorska-Goldstein, "Existence of renormalized solutions to elliptic equation in Musielak-Orlicz space", *J. Differential Equations* **264**:1 (2018), 341–377.

[7] M. Ait Khellou and A. Benkirane, "Renormalized solution for nonlinear elliptic problems with lower order terms and L^1 data in Musielak-Orlicz spaces", *An. Univ. Craiova Ser. Mat. Inform.* **43**:2 (2016), 164–187.

[8] M. S. B. Elemine Vall, T. Ahmedatt, A. Touzani and A. Benkirane, "Existence of entropy solutions for nonlinear elliptic equations in Musielak framework with L^1 data", *Bol. Soc. Parana. Mat.* (3) **36**:1 (2018), 125–150.

[9] R. Elarabi, M. Rhoudaf and H. Sabiki, "Entropy solution for a nonlinear elliptic problem with lower order term in Musielak-Orlicz spaces", *Ric. Mat.* **67**:2 (2018), 549–579.

[10] M. Ait Khellou, S. M. Douiri and Y. El Hadfi, "Existence of solutions for some nonlinear elliptic equations in musielak spaces with only the log-Hölder continuity condition", *Mediterr. J. Math.* **17**:1 (2020), 33, 18 pp.

[11] A. Talha and A. Benkirane, "Strongly nonlinear elliptic boundary value problems in Musielak-Orlicz spaces", *Monatsh. Math.* **186**:4 (2018), 745–776.

[12] Ph. Benilan, L. Boccardo, Th. Gallouët, R. Gariepy, M. Pierre and J. L. Vazquez, "An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **22**:2 (1995), 241–273.

[13] A. Malusa and M. M. Porzio, "Renormalized solutions to elliptic equations with measure data in unbounded domains", *Nonlinear Anal.* **67**:8 (2007), 2370–2389.

[14] F. Kh. Mukminov, "Uniqueness of the renormalized solution of an elliptic-parabolic problem in anisotropic Sobolev-Orlicz spaces", *Mat. Sb.* **208**:8 (2017), 106–125; English transl. in *Sb. Math.* **208**:8 (2017), 1187–1206.

[15] L. M. Kozhevnikova, "On the entropy solution to an elliptic problem in anisotropic Sobolev-Orlicz spaces", *Zh. Vychisl. Mat. Mat. Fiz.* **57**:3 (2017), 429–447; English transl. in *Comput. Math. Math. Phys.* **57**:3 (2017), 434–452.

[16] L. M. Kozhevnikova, "Existence of entropic solutions of an elliptic problem in anisotropic Sobolev-Orlicz spaces", *Differential equations. Mathematical physics.*, Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., vol. 139, VINITI, Moscow 2017, pp. 15–38; English transl. in *J. Math. Sci.* (*N.Y.*) **241**:3 (2019), 258–284.

[17] L. M. Kozhevnikova, "On entropy solutions of anisotropic elliptic equations with variable nonlinearity indices in unbounded domains", *Differential and functional differential equations*, Sovr. Mat. Fundam. Napravl., vol. 63, no. 3, RUDN University, Moscow 2017, pp. 475–493; English transl. in *J. Math. Sci.* (*N.Y.*) **253**:5 (2021), 692–709.

[18] L. M. Kozhevnikova, "On solutions of anisotropic elliptic equations with variable exponent and measure data", *Complex Var. Elliptic Equ.* **65**:3 (2020), 333–367.

[19] L. M. Kozhevnikova, "Entropy and renormalized solutions of anisotropic elliptic equations with variable nonlinearity exponents", *Mat. Sb.* **210**:3 (2019), 131–161; English transl. in *Sb. Math.* **210**:3 (2019), 417–446.

[20] L. M. Kozhevnikova, "Equivalence of entropy and renormalized solutions of the anisotropic elliptic problem in unbounded domains with measure data", *Izv. Vyssh. Uchebn. Zaved. Math.*, 2020, no. 1, 30–45; English transl. in *Russian Math.* (*Iz. VUZ*) **64**:1 (2020), 25–39.

[21] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex functions and Orlicz spaces*, Fizmatgiz, Moscow 1958, 271 pp.; English transl., P. Noordhoff Ltd., Groningen 1961, xi+249 pp.

[22] J. Musielak, *Orlicz spaces and modular spaces*, Lecture Notes in Math., vol. 1034, Springer-Verlag, Berlin 1983, iii+222 pp.

[23] I. Chlebicka, "A pocket guide to nonlinear differential equations in Musielak-Orlicz spaces", *Nonlinear Anal.* **175** (2018), 1–27.

[24] A. Benkirane and M. Sidi El Vally, "An existence result for nonlinear elliptic equations in Musielak-Orlicz-Sobolev spaces", *Bull. Belg. Math. Soc. Simon Stevin* **20**:1 (2013), 57–75.

[25] Y. Ahmida, I. Chlebicka, P. Gwiazda and A. Youssfi, "Gossez's approximation theorems in Musielak-Orlicz-Sobolev spaces", *J. Funct. Anal.* **275**:9 (2018), 2538–2571.

[26] G. I. Laptev, "Weak solutions of second-order quasilinear parabolic equations with double non-linearity", *Mat. Sb.* **188**:9 (1997), 83–112; English transl. in *Sb. Math.* **188**:9 (1997), 1343–1370.

[27] N. A. Vorob'yov and F. Kh. Mukminov, "Existence of a renormalized solution of a parabolic problem in anisotropic Sobolev-Orlicz spaces", *Differential equations*, Itogi Nauki Tekh. Ser. Sovrem. Mat. Prilozh. Temat. Obz., vol. 163, VINITI, Moscow 2019, pp. 39–64; English transl. in *J. Math. Sci.* (*N.Y.*) **258**:1 (2021), 37–64.

[28] N. Dunford and J. T. Schwartz, *Linear operators*, vol. I: *General theory*, Pure Appl. Math., vol. 7, Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London 1958, xiv+858 pp.

[29] L. M. Kozhevnikova and A. P. Kashnikova, "Existence of solutions to quasilinear elliptic equations in Musielak-Orlicz-Sobolev spaces for unbounded domains", *Vestn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki* **24**:4 (2020), 621–643. (Russian)

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