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Logarithmic nature of the long-time asymptotics of solutions of a Sobolev-type nonlinear equations with cubic nonlinearities

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Abstract. The Cauchy problem of the form

$$\begin{cases} i\,\partial_t(u-\partial_x^2 u)+\partial_x^2 u - a\,\partial_x^4 u = u^3, & t>0, \quad x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases}$$

is considered for a Sobolev-type nonlinear equation with cubic nonlinearity, where $a>1/5$, $a\neq 1$. It is shown that the asymptotic behaviour of the solution is characterized by an additional logarithmic decay in comparison with the corresponding linear case. To find the asymptotics of solutions of the Cauchy problem for a nonlinear Sobolev-type equation, factorization technique is developed. To obtain estimates for derivatives of the defect operators, $\mathbf{L}^2$-estimates of pseudodifferential operators are used.

Bibliography: 20 titles.

Keywords: nonlinear Sobolev-type equation, critical nonlinearity, factorization technique.

§ 1. Introduction

We consider a Cauchy problem for a Sobolev-type nonlinear equation with cubic nonlinearity in the one-dimensional case with respect to the space variable:

$$\begin{cases} i\,\partial_t(u-\partial_x^2 u)+\partial_x^2 u - a\,\partial_x^4 u = u^3, & t>0, \quad x\in\mathbb{R}, \\ u(0,x)=u_0(x), & x\in\mathbb{R}, \end{cases} \tag{1}$$

where $a>1/5$, $a\neq 1$. We exclude the case $a=1$, since then equation (1) is easily reducible to a nonlinear Schrödinger equation. Sobolev-type equations were first deduced in [1] in the description of small oscillations of a rotating fluid. Sobolev-type equations also arise in plasma theory and in modelling quasi-stationary processes in continuous electromagnetic media (see [2]). A discussion of the theory of nonlinear Sobolev-type equations can be found in [3]–[6].

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In the one-dimensional case with respect to the space variable cubic nonlinearities often behave themselves critically for large times. For example, the asymptotic behaviour of solutions of the nonlinear Schrödinger equation

$$i\,\partial_t u + \frac{1}{2}\,\partial_x^2 u = u^3$$

was considered in [7]–[9], where it was shown that the solution is characterized by an additional logarithmic decay in comparison with the linear Schrödinger equation. As far as we know, the large-time asymptotics of solutions of the Cauchy problem for the Sobolev-type nonlinear equation (1) has not been investigated yet. We fill this gap in our paper by developing the factorization technique proposed in [10]–[14]. We also use some known $\mathbf{L}^2$-estimates for pseudodifferential operators to estimate derivatives of the defect operators.

We introduce some notation. We let $\mathbf{L}^p$ denote the Lebesgue space with the norm

$$\|\phi\|_{\mathbf{L}^p} = \left(\int_{\mathbb{R}} |\phi(x)|^p\,dx\right)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$
$$\|\phi\|_{\mathbf{L}^\infty} = \operatorname*{ess\,sup}_{x\in\mathbb{R}} |\phi(x)| \quad \text{for } p = \infty.$$

We introduce the weighted Sobolev space

$$\mathbf{H}^{m,s} = \bigl\{\phi \in \mathbf{S}';\, \|\phi\|_{\mathbf{H}^{m,s}} = \|\langle x\rangle^s \langle i\,\partial_x\rangle^m \phi\|_{\mathbf{L}^2} < \infty\bigr\}$$

for $m, s \in \mathbb{R}$, where $\langle x\rangle = \sqrt{1+x^2}$, $\langle i\,\partial_x\rangle = \sqrt{1-\partial_x^2}$ and $\mathbf{S}'$ is the space of Schwartz distributions. We also set $\mathbf{H}^m = \mathbf{H}^{m,0}$. We let $\mathbf{C}(\mathbf{I};\mathbf{B})$ denote the space of continuous functions mapping the interval $\mathbf{I}$ to some Banach space $\mathbf{B}$. Similarly, $\mathbf{C}^1(\mathbf{I};\mathbf{B})$ denotes the space of continuously differentiable functions from $\mathbf{I}$ to $\mathbf{B}$.

We let $\mathcal{F}\phi$ or $\widehat{\phi}$ denote the Fourier transform

$$\widehat{\phi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} \phi(x)\,dx;$$

then the inverse Fourier transform $\mathcal{F}^{-1}$ has the form

$$\mathcal{F}^{-1}\phi = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ix\xi} \phi(\xi)\,d\xi.$$

Different positive constants are denoted by the same letter C.

We consider solutions of (1) in the space $\mathbf{C}([0,\infty);\mathbf{H}^5) \cap \mathbf{C}^1((0,\infty);\mathbf{H}^3)$; so equation (1) is understood in the classical sense. Multiplying (1) by the operator $(1-\partial_x^2)^{-1}$ we rewrite it in the pseudodifferential form as

$$\begin{cases} i\,\partial_t u - \mathbf{\Lambda} u = \langle i\,\partial_x\rangle^{-2} u^3, & t > 0, \quad x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{2}$$

where $\langle i\,\partial_x\rangle^{-2} = (1-\partial_x^2)^{-1}$ and the linear pseudodifferential operator $\mathbf{\Lambda} = (1-\partial_x^2)^{-1}(-\partial_x^2 + a\,\partial_x^4)$ is characterized by its symbol

$$\Lambda(\xi) = \frac{\xi^2 + a\xi^4}{1+\xi^2}.$$

Under the condition $a > 1/5$, we have $\Lambda''(\xi) > 0$ for all $\xi \in \mathbb{R}$, which guarantees the nonsingularity of the stationary point $\mu(x)$ defined as the root of the equation

$$\Lambda'(\xi) = \frac{2\xi(1 + 2a\xi^2 + a\xi^4)}{(1 + \xi^2)^2} = x$$

for each $x \in \mathbb{R}$. Thereafter, using the unperturbed evolution group $\mathcal{U}(t) = \mathcal{F}^{-1}e^{-it\Lambda(\xi)}\mathcal{F}$, we rewrite the Cauchy problem (2) as the integral equation

$$u(t) = \mathcal{U}(t)u_0 - \int_0^t \mathcal{U}(t-\tau)\langle i\,\partial_x\rangle^{-2}u^3(\tau)\,d\tau. \tag{3}$$

We introduce the extension operator $\mathcal{D}_t\phi = t^{-1/2}\phi(x/t)$, the scale transformation $(\mathcal{B}\phi)(x) = \phi(\mu(x))$ and the factor $M = e^{it\Theta(x)}$, where $\Theta(x) = -\Lambda(x) + x\Lambda'(x)$. We also introduce the notation $\widetilde{x} = x\sqrt{t}$.

The aim of this paper is to prove the following result.

Theorem 1.1. *There exist $\varepsilon_0 > 0$ and a positive constant C such that if the initial data $u_0 \in \mathbf{H}^5 \cap \mathbf{H}^{0,1}$ satisfy the inequalities*

$$\|u_0\|_{\mathbf{H}^5\cap\mathbf{H}^{0,1}} \leqslant C\varepsilon, \qquad \sup_{|\xi|\leqslant 1} |\arg \widehat{u_0}(\xi)| < \frac{\pi}{8} \quad \text{and} \quad \inf_{|\xi|\leqslant 1} |\widehat{u_0}(\xi)| \geqslant \varepsilon$$

for $\varepsilon \in (0, \varepsilon_0)$, then there exists a unique time-global solution

$$u \in \mathbf{C}([0,\infty); \mathbf{H}^5 \cap \mathbf{H}^{0,1}) \cap \mathbf{C}^1((0,\infty); \mathbf{H}^3)$$

of Cauchy problem (1). In addition,

$$u(t,x) = \mathcal{D}_t\mathcal{B}\frac{M|\widehat{u_0}|}{\sqrt{1 + \dfrac{|\widehat{u_0}|^2}{\sqrt{3}}\ln(t\langle x\rangle^2\langle\widetilde{x}\rangle^{-2})}} + O(t^{-1/2}(\ln(t\langle x\rangle^2\langle\widetilde{x}\rangle^{-2}))^{-3/4}) \tag{4}$$

as $t \to \infty$ uniformly with respect to $x \in \mathbb{R}$.

Remark 1.1. The asymptotic formula (4) describes an additional logarithmic decay in comparison with the corresponding linear case.

We briefly describe the rest of this paper. In § 2 we describe the factorization technique. Thereupon, we estimate the defect operators in the uniform metric. After that, applying some known estimates for pseudodifferential operators in the $\mathbf{L}^2$-norm, we obtain estimates for derivatives of the defect operators. In § 3 we prove *a priori* estimates for the solution in the norm $\|\mathcal{F}\mathcal{U}(-t)u(t)\|_{\mathbf{X}_T}$, where

$$\|\phi\|_{\mathbf{X}_T} = \sup_{t\in[1,T]} \bigl(\|\phi\|_{\mathbf{L}^\infty} + W^{1/2}\|\langle\widetilde{\xi}\rangle^{-\gamma}\phi\|_{\mathbf{L}^\infty} + t^{-\gamma}\|\langle\xi\rangle^5\phi\|_{\mathbf{L}^2} + K^{-1}\|\partial_\xi\phi\|_{\mathbf{L}^2}\bigr);$$

here $W(t) = 1+\varepsilon^2\ln(1+t)$, $K(t) = t^\gamma + \varepsilon^2 t^{1/4}W^{-3/2}(t)$ and $\gamma > 0$ is small. Finally, we prove Theorem 1.1 in § 4.

§ 2. Preliminary estimates

2.1. Factorization technique. We consider the following representation of the unperturbed evolution group:

$$\mathcal{U}(t) = \mathcal{F}^{-1} e^{-it\Lambda(\xi)} \mathcal{F}, \quad \text{where } \Lambda(\xi) = \frac{\xi^2 + a\xi^4}{1+\xi^2}.$$

For $a > 1/5$ we have

$$\Lambda''(\xi) = \frac{2(1+(6a-3)\xi^2 + 3a\xi^4 + a\xi^6)}{(1+\xi^2)^3} > 0 \quad \text{for all } \xi \in \mathbb{R}.$$

Therefore, there exists a unique nonsingular stationary point $\mu(x)$ defined as the root of the equation

$$\Lambda'(\xi) = \frac{2\xi(1+2a\xi^2 + a\xi^4)}{(1+\xi^2)^2} = x \quad \text{for each } x \in \mathbb{R}.$$

We write $\mathcal{U}(t)\mathcal{F}^{-1}\phi = \mathcal{D}_t \mathcal{B} M \mathcal{Q}\phi$, where $\mathcal{D}_t\phi = t^{-1/2}\phi(x/t)$ is the extension operator, $(\mathcal{B}\phi)(x) = \phi(\mu(x))$ is the scale transformation, $M = e^{it\Theta(\xi)}$ is the factor with phase

$$\Theta(\xi) = -\Lambda(\xi) + \xi\Lambda'(\xi) = \frac{\xi^2(1+\xi^2(3a-1)+a\xi^4)}{(1+\xi^2)^2},$$

and

$$\mathcal{Q}(t)\phi = \frac{t^{1/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-itS(\xi,\eta)}\phi(\xi)\,d\xi$$

is the defect operator with phase function $S(\xi,\eta) = \Lambda(\xi) - \Lambda(\eta) - \Lambda'(\eta)(\xi-\eta)$.

We also need the representation $\mathcal{F}\mathcal{U}(-t) = \mathcal{Q}^* \overline{M} \mathcal{B}^{-1} \mathcal{D}_t^{-1}$ for the inverse unperturbed evolution group, where $\mathcal{D}_t^{-1}\phi = t^{1/2}\phi(xt)$ is the inverse extension operator, $\mathcal{B}^{-1}\phi = \phi(\Lambda'(\eta))$ is the inverse scale transformation, and

$$\mathcal{Q}^*(t)\phi = \frac{t^{1/2}}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)\,d\eta$$

is the conjugate defect operator. Note that the defect operators can be written as $\mathcal{Q}(t) = \overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}\mathcal{F}^{-1}e^{-it\Lambda(\xi)}$ and $\mathcal{Q}^*(t) = M\mathcal{B}\mathcal{D}_t e^{it\Lambda(\xi)}\mathcal{F}$, which shows that they act from $\mathbf{L}^2$ to $\mathbf{L}^2$ and also that $\mathcal{Q}^*(t)$ is the conjugate operator of $\mathcal{Q}(t)$.

We introduce the new function $\widehat{\varphi} = \mathcal{F}\mathcal{U}(-t)u(t)$. Since $\mathcal{F}\mathcal{U}(-t)\mathcal{L} = i\,\partial_t \mathcal{F}\mathcal{U}(-t)$ for the operator $\mathcal{L} = i\,\partial_t - \mathbf{\Lambda}$, applying $\mathcal{F}\mathcal{U}(-t)$ to (2) we obtain

$$\begin{aligned} i\,\partial_t\widehat{\varphi} &= i\,\partial_t \mathcal{F}\mathcal{U}(-t)u(t) = \mathcal{F}\mathcal{U}(-t)\mathcal{L}u = \langle\xi\rangle^{-2}\mathcal{F}\mathcal{U}(-t)(\mathcal{U}(t)\mathcal{F}^{-1}\widehat{\varphi})^3 \\ &= \langle\xi\rangle^{-2}\mathcal{Q}^*\overline{M}\mathcal{B}^{-1}\mathcal{D}_t^{-1}(\mathcal{D}_t\mathcal{B}M\mathcal{Q}\widehat{\varphi})^3 = t^{-1}\langle\xi\rangle^{-2}\mathcal{Q}^*M^2v^3, \end{aligned}$$

where $v = \mathcal{Q}\widehat{\varphi}$. We have

$$S(\xi,\eta) + k\Theta(\eta) = \Omega_{k+1}(\xi) + (1+k)S\biggl(\frac{\xi}{1+k},\eta\biggr),$$

where

$$\Omega_{k+1} = \Lambda(\xi) - (k+1)\Lambda\left(\frac{\xi}{k+1}\right) \quad \text{for } k \neq -1.$$

By the definition of the conjugate defect operator $\mathcal{Q}^*(t)$,

$$\begin{aligned}
\mathcal{Q}^*(t)M^k\phi &= \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{it(S(\xi,\eta)+k\Theta(\eta))}\phi(\eta)\Lambda''(\eta)\,d\eta \\
&= e^{it\Omega_{k+1}}\mathcal{D}_{k+1}\sqrt{\frac{(1+k)t}{2\pi}}\int_{\mathbb{R}} e^{i(1+k)tS(\xi,\eta)}\phi(\eta)\Lambda''(\eta)\,d\eta \\
&= e^{it\Omega_{k+1}}\mathcal{D}_{k+1}\mathcal{Q}^*((k+1)t)\phi,
\end{aligned}$$

where $\mathcal{D}_{k+1}\phi = (k+1)^{-1/2}\phi(x/(k+1))$. Taking $k=2$ in this identity, we arrive at the main equation of the factorization method:

$$i\,\partial_t\widehat{\varphi} = t^{-1}\langle\xi\rangle^{-2}\mathcal{Q}^*M^2v^3 = t^{-1}\langle\xi\rangle^{-2}e^{it\Omega}\mathcal{D}_3\mathcal{Q}^*(3t)v^3, \tag{5}$$

where we have used the notation

$$\Omega = \Omega_3 = \Lambda(\xi) - 3\Lambda\left(\frac{\xi}{3}\right) = \frac{2\xi^2(9+(13a-3)\xi^2+a\xi^4)}{27(1+\xi^2)(1+\xi^2/9)} \geqslant 0.$$

We introduce the operators

$$\mathcal{A}_k\phi = \frac{1}{t\Lambda''(\eta)}\overline{M}^k\,\partial_\eta M^k\phi, \qquad k = 0, 1;$$

thus, $\mathcal{A}_1 = i\eta + \mathcal{A}_0$. The identities $i\xi\mathcal{Q}^* = \mathcal{Q}^*\mathcal{A}_1$ and $\mathcal{Q}i\xi = \mathcal{A}_1\mathcal{Q}$ hold.

2.2. Estimates for the defect operator $\mathcal{Q}$ in the uniform metric. We specify the kernel $A(t,\eta)$ as

$$\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\,d\xi.$$

The long-time asymptotics of $A(t,\eta)$ can be calculated using the method of stationary phase (see [15]). However, we need to estimate the remainder uniformly with respect to the parameter $\eta\in\mathbb{R}$. Therefore, for the convenience of the reader, we give a proof of the asymptotic formula.

Note that

$$S(\xi,\eta) = (\xi-\eta)^2P_1^2(\xi,\eta),$$

where

$$P_1^2(\xi,\eta) = \langle\xi\rangle^{-2}\langle\eta\rangle^{-4}P_2(\xi,\eta) \quad \text{and} \quad P_2(\xi,\eta) = (1-a)(\langle\xi\rangle^2-(\eta+\xi)^2)+a\langle\xi\rangle^2\langle\eta\rangle^4.$$

If $a \geqslant 1$, then

$$P_2(\xi,\eta) = (a-1)(\eta+\xi)^2 + \langle\xi\rangle^2(a\langle\eta\rangle^4-(a-1)) \geqslant C\langle\xi\rangle^2\langle\eta\rangle^4$$

for all $\xi, \eta \in \mathbb{R}$. In the case when $1/5 < a < 1$, using the inequality $\eta^2 \langle \eta \rangle^{-4} \leqslant 1/4$ we infer that

$$\begin{aligned} P_2(\xi, \eta) &\geqslant a\langle \eta \rangle^4 \left(\xi - \frac{1-a}{a} \frac{\eta}{\langle \eta \rangle^4} \right)^2 + a\left(\eta^2 + \frac{3a-1}{2a} \right)^2 + \frac{(1-a)(5a-1)}{2a} \\ &\geqslant C\langle \xi \rangle^2 \langle \eta \rangle^4 \end{aligned}$$

for all $\xi, \eta \in \mathbb{R}$. Hence $P_2(\xi, \eta) \geqslant C\langle \xi \rangle^2 \langle \eta \rangle^4$, and we can write

$$S(\xi, \eta) = z^2, \quad \text{where } z = (\xi - \eta)P_1(\xi, \eta) \text{ and } P_1(\xi, \eta) \geqslant C > 0,$$

for all $\xi, \eta \in \mathbb{R}$. On the other hand, since $P_2(\xi, \eta)$ is estimated from above by $C\langle \xi \rangle^2 \langle \eta \rangle^4$ for some positive constant C, the function $P_1(\xi, \eta)$ is bounded above and below by positive constants uniformly with respect to $\xi, \eta \in \mathbb{R}$.

Now we can replace the integration variable ξ by $z = (\xi - \eta)P_1$; thus, we have

$$A(t, \eta) = \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{-itz^2} f(\eta, z)\, dz,$$

where

$$f(\eta, z) = \frac{1}{\partial z(\xi, \eta)/\partial \xi} = \frac{2z}{\partial S(\xi, \eta)/\partial \xi}.$$

Note that $z = 0$ corresponds to $\xi = \eta$. We write

$$\frac{\partial S(\xi, \eta)}{\partial \xi} = 2(\xi - \eta)\langle \xi \rangle^{-4} \langle \eta \rangle^{-4} P_3(\xi, \eta),$$

where

$$\begin{aligned} P_3(\xi, \eta) = -(1-a)\eta\xi(2 + \xi^2 + \eta^2) + 1 + 2a\eta^2 + a\eta^4 + 2a\xi^2 + a\xi^4 \\ + (5a-1)\eta^2\xi^2 + 2a\eta^2\xi^2(\xi^2 + \eta^2) + a\eta^4\xi^4. \end{aligned}$$

When $a > 1$, we can easily see that $P_3(\xi, \eta) \geqslant C\langle \xi \rangle^4 \langle \eta \rangle^4$ for all $\xi, \eta \in \mathbb{R}$. To establish this estimate in the case when $1/5 < a < 1$ we use the inequalities

$$\xi\eta \leqslant \frac{1}{2} + \frac{1}{2}\eta^2\xi^2, \qquad \xi^2\eta^2 \leqslant \frac{1}{2}(\xi^4 + \eta^4) \quad \text{and} \quad \xi^2\eta^2 \leqslant \frac{1}{2} + \frac{1}{2}\eta^4\xi^4.$$

Then the first term in $P_3(\xi, \eta)$ satisfies

$$\begin{aligned} (1-a)\eta\xi(2 + \xi^2 + \eta^2) \leqslant \frac{5(1-a)}{4} + \frac{1-a}{4}(\xi^4 + \eta^4) + \frac{1-a}{4}\eta^4\xi^4 \\ + \frac{1-a}{2}(\xi^2 + \eta^2) + \frac{1-a}{2}\eta^2\xi^2(\xi^2 + \eta^2). \end{aligned}$$

It follows that

$$P_3(\xi, \eta) \geqslant \frac{5a-1}{4}\langle \xi \rangle^4 \langle \eta \rangle^4$$

for all $\xi, \eta \in \mathbb{R}$. In particular,

$$\frac{\partial S(\xi, \eta)/\partial \xi}{\xi - \eta} \geqslant C;$$

since $z/(\xi-\eta)$ is bounded, we have

$$0 < f(\eta,z) = \frac{2z/(\xi-\eta)}{S_\xi(\xi,\eta)/(\xi-\eta)} \leqslant C.$$

The boundedness of $1/f(\eta,z)$ can be established similarly.

Now we use the equalities

$$\sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itz^2}\,dz = \frac{1}{\sqrt{2i}}, \qquad P_1(\eta,\eta) = \sqrt{\frac{1}{2}\Lambda''(\eta)},$$
$$P_3(\eta,\eta) = \frac{1}{2}\langle\eta\rangle^4\Lambda''(\eta) \quad \text{and} \quad f(\eta,0) = \frac{\sqrt{2}}{\sqrt{\Lambda''(\eta)}};$$

then we have

$$A(t,\eta) = \frac{1}{\sqrt{i\Lambda''(\eta)}} + \sqrt{\frac{t}{2\pi}}R,$$

where the remainder R is $\int_{\mathbb{R}} e^{-itz^2}(f(\eta,z)-f(\eta,0))\,dz$. Integrating by parts in R and using the identity $e^{-itz^2} = (1-2itz^2)^{-1}\partial_z(ze^{-itz^2})$ we derive the equality

$$R = -\int_{\mathbb{R}} e^{-itz^2}\frac{z\,\partial_z f(\eta,z)}{1-2itz^2}\,dz - \int_{\mathbb{R}} e^{-itz^2}\frac{4itz^2(f(\eta,z)-f(\eta,0))}{(1-2itz^2)^2}\,dz.$$

Note that

$$\partial_z f(\eta,z) = \frac{1}{\partial z(\xi,\eta)/\partial\xi}\,\partial_\xi f(\eta,z) = \frac{1}{2}\,\partial_\xi f^2(\eta,z) = \frac{1}{2}\,\partial_\xi\frac{\langle\xi\rangle^6\langle\eta\rangle^4 P_2(\xi,\eta)}{P_3^2(\xi,\eta)}.$$

Therefore, in view of the estimates

$$|P_2(\xi,\eta)| \leqslant C\langle\xi\rangle^2\langle\eta\rangle^4, \qquad |\partial_\xi P_2(\xi,\eta)| \leqslant C\langle\xi\rangle\langle\eta\rangle^4, \qquad P_3(\xi,\eta) \geqslant C\langle\xi\rangle^4\langle\eta\rangle^4,$$
$$|\partial_\xi P_3(\xi,\eta)| \leqslant C\langle\xi\rangle^3\langle\eta\rangle^4 \quad \text{and} \quad |\partial_\xi\langle\xi\rangle^6| \leqslant C\langle\xi\rangle^5,$$

it is true that

$$\begin{aligned}|\partial_z f(\eta,z)| &= 2\left|\partial_\xi\frac{\langle\xi\rangle^6\langle\eta\rangle^4 P_2(\xi,\eta)}{P_3^2(\xi,\eta)}\right| \\ &\leqslant C\left|\frac{\langle\xi\rangle^6\langle\eta\rangle^4}{P_3^2(\xi,\eta)}\right||\partial_\xi P_2(\xi,\eta)| + C\left|\frac{\langle\xi\rangle^6\langle\eta\rangle^4 P_2(\xi,\eta)}{P_3^3(\xi,\eta)}\right||\partial_\xi P_3(\xi,\eta)| \\ &\quad + C\left|\frac{\langle\eta\rangle^4 P_2(\xi,\eta)}{P_3^2(\xi,\eta)}\right||\partial_\xi\langle\xi\rangle^6| \\ &\leqslant C\langle\xi\rangle^{-1}\end{aligned}$$

for all $\xi,\eta\in\mathbb{R}$. We also have

$$|f(\eta,z)-f(\eta,0)| \leqslant C|z| \quad \text{for } |z|\leqslant 1$$

and

$$|f(\eta,z)-f(\eta,0)| \leqslant f(\eta,z)+f(\eta,0) \leqslant C \quad \text{for } |z|\geqslant 1,$$

which implies the inequalities

$$|R| \leqslant C\int_0^1 \frac{z\,dz}{1+tz^2} + Ct^{-1}\int_1^\infty \langle\xi\rangle^{-1}z^{-1}\,dz + Ct^{-1}\int_1^\infty z^{-2}\,dz$$
$$\leqslant C\int_0^1 \frac{z\,dz}{1+tz^2} + Ct^{-1}\int_1^\infty \frac{dz}{z^2} + Ct^{-1}\int_{\mathbb{R}} \frac{d\xi}{\langle\xi\rangle^2} \leqslant Ct^{-1}\ln t.$$

Thus, we arrive at the asymptotic formula

$$A(t,\eta) = \frac{1}{\sqrt{i\Lambda''(\eta)}}(1+O(t^{-1/2}\ln t))$$

as $t\to\infty$, which holds uniformly with respect to $\eta\in\mathbb{R}$.

We define the antiderivative for $\xi\neq 0$ by

$$\partial_\xi^{-1} f = \begin{cases} -\displaystyle\int_\xi^\infty f(\zeta)\,d\zeta & \text{for } \xi>0,\\ \displaystyle\int_{-\infty}^\xi f(\zeta)\,d\zeta & \text{for } \xi<0.\end{cases}$$

We consider the kernel

$$G(t,\xi,\eta) = \sqrt{\frac{t}{2\pi}}\,\partial_\xi^{-1}(e^{-itS(\eta-\xi,\eta)}).$$

Integrating by parts in $\mathcal{Q}\phi$ yields

$$\mathcal{Q}\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\eta-\xi,\eta)}\phi(\eta-\xi)\,d\xi = A(t,\eta)\phi(\eta) + \int_{\mathbb{R}} G(t,\xi,\eta)\phi_\eta(\eta-\xi)\,d\xi,$$

since $-G(t,+0,\eta)+G(t,-0,\eta)=A(t,\eta)$.

The following lemma estimates the kernel $G(t,\xi,\eta)$. We introduce the automodel variable $\widetilde{\xi}=\xi\sqrt{t}$.

Lemma 2.1. *The following inequality holds*:

$$|G(t,\xi,\eta)| \leqslant C\langle\widetilde{\xi}\rangle^{-1} \quad \textit{for all } \xi,\eta\in\mathbb{R}, \quad t\geqslant 1.$$

Proof. To estimate the kernel $G(t,\xi,\eta)$ we integrate by parts on the basis of the identity

$$e^{-itS(\eta-\xi,\eta)} = H(t,\xi,\eta)\,\partial_\xi(\xi e^{-itS(\eta-\xi,\eta)}),$$

where $H(t,\xi,\eta) = (1-it\xi\,\partial_\xi S(\eta-\xi,\eta))^{-1}$. For $\xi>0$ we obtain

$$G(t,\xi,\eta) = -\sqrt{\frac{t}{2\pi}}\int_\xi^\infty e^{-itS(\eta-\zeta,\eta)}\,d\zeta$$
$$= \sqrt{\frac{t}{2\pi}}\,e^{-itS(\eta-\xi,\eta)}\xi H(t,\xi,\eta) + \sqrt{\frac{t}{2\pi}}\int_\xi^\infty e^{-itS(\eta-\zeta,\eta)}\zeta\,\partial_\zeta(H(t,\zeta,\eta))\,d\zeta.$$

Since $|\partial_\zeta S(\eta-\zeta,\eta)| \geqslant C|\zeta|$, we have

$$|H(t,\zeta,\eta)| + |\zeta\,\partial_\zeta(H(t,\zeta,\eta))| \leqslant C\langle\widetilde{\zeta}\rangle^{-2}, \quad \text{where } \widetilde{\zeta} = \zeta\sqrt{t}.$$

It follows for $\xi > 0$ that

$$|G(t,\xi,\eta)| \leqslant C|\widetilde{\xi}|\langle\widetilde{\xi}\rangle^{-2} + Ct^{1/2}\int_\xi^\infty \langle\widetilde{\zeta}\rangle^{-2}\,d\zeta \leqslant C\langle\widetilde{\xi}\rangle^{-1}.$$

The case $\xi < 0$ is considered similarly.

The lemma is proved.

The following lemma gives us the long-time asymptotics of the defect operator $\mathcal{Q}$.

Lemma 2.2. *The following inequality holds*:

$$\|\mathcal{Q}\phi - A\phi\|_{\mathbf{L}^\infty} \leqslant Ct^{-1/4}\|\phi_\xi\|_{\mathbf{L}^2} \quad \textit{for all } t \geqslant 1.$$

Proof. Using the estimate in Lemma 2.1 and the Cauchy-Bunyakovsky-Schwarz inequality we obtain

$$\begin{aligned}|\mathcal{Q}\phi - A\phi| &= \left|\int_{\mathbb{R}} G(t,\xi,\eta)\phi_\eta(\eta-\xi)\,d\xi\right| \leqslant C\|\phi_\xi\|_{\mathbf{L}^2}\left(\int_{\mathbb{R}}\langle\widetilde{\xi}\rangle^{-2}\,d\xi\right)^{1/2} \\ &\leqslant Ct^{-1/4}\|\phi_\xi\|_{\mathbf{L}^2}.\end{aligned}$$

The lemma is proved.

2.3. Estimates for the conjugate defect operator in the uniform metric. We define the conjugate defect operator $\mathcal{V}_h^*\phi$ by

$$\mathcal{V}_h^*\phi = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}h(t,\xi,\eta)\phi(\eta)\Lambda''(\eta)\,d\eta,$$

where the weight has the form

$$h(t,\xi,\eta) = \left(-\frac{1}{2} + itS_3(\xi,\eta)\right)^{-2} \quad \text{for } S_3(\xi,\eta) = S(\xi,\eta) + 2\Theta(\eta).$$

We consider the kernel

$$A_h^*(t,\xi) = \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{itS_3(\xi,\eta)}h(t,\xi,\eta)\Lambda''(\eta)\,d\eta.$$

Replacing the integration variable by $\widetilde{\eta} = t^{1/2}\eta$ we arrive at

$$\begin{aligned}A_h^*(t,0) &= \sqrt{\frac{1}{2\pi}}\int_{\mathbb{R}} e^{3i\widetilde{\eta}^2(1+O(t^{-1}\widetilde{\eta}^2))}\frac{\Lambda''(\widetilde{\eta}t^{-1/2})\,d\widetilde{\eta}}{(-1/2+3i\widetilde{\eta}^2(1+O(t^{-1}\widetilde{\eta}^2)))^2} \\ &= \sqrt{\frac{2}{\pi}}\int_{\mathbb{R}} e^{3i\widetilde{\eta}^2}\frac{d\widetilde{\eta}}{(-1/2+3i\widetilde{\eta}^2)^2} + O(t^{-1/2}) = \frac{\sqrt{8i}}{\sqrt{3}} + O(t^{-1/2}).\end{aligned}$$

We also obtain

$$A_h^*(t,\xi) = \frac{\sqrt{8i}}{\sqrt{3}} \langle \widetilde{\xi} \rangle^{-2} + O(t^{-1/2} \langle \widetilde{\xi} \rangle^{-2} + |\widetilde{\xi}| \langle \widetilde{\xi} \rangle^{-2}).$$

Denoting the kernel by

$$G_h^*(t,\xi,\eta) = \sqrt{\frac{t}{2\pi}}\, \partial_\eta^{-1} (e^{itS(\xi,\xi-\eta)} h(t,\xi,\xi-\eta) \Lambda''(\xi-\eta))$$

and integrating by parts in the integral $\mathcal{V}_h^*$, we obtain

$$\begin{aligned}\mathcal{V}_h^* \phi &= \sqrt{\frac{t}{2\pi}} \int_{\mathbb{R}} e^{itS_3(\xi,\xi-\eta)} h(t,\xi,\xi-\eta) \phi(\xi-\eta) \Lambda''(\xi-\eta)\, d\eta \\ &= A_h^*(t,\xi)\phi(\xi) + \int_{\mathbb{R}} G_h^*(t,\xi,\eta) \phi_\xi(\xi-\eta)\, d\eta,\end{aligned}$$

since $-G_h^*(t,\xi,+0) + G_h^*(t,\xi,-0) = A_h^*(t,\xi)$. The following lemma estimates the operator $\mathcal{V}_h^*$ in the uniform metric.

Lemma 2.3. *The following inequality holds*:

$$\|\mathcal{V}_h^* \phi - A_h^* \phi\|_{\mathbf{L}^\infty} \leqslant C t^{-1/4} \| \langle \widetilde{\eta} \rangle^{-2}\, \partial_\eta \phi \|_{\mathbf{L}^2} \quad \textit{for all } t \geqslant 1.$$

Proof. Since $h(t,\xi,\xi-\eta) \leqslant C \langle \widetilde{\xi} \rangle^{-1} \langle \widetilde{\eta} \rangle^{-3}$, we obtain the inequality

$$|G_h^*(t,\xi,\eta)| \leqslant C t^{1/2} \langle \widetilde{\xi} \rangle^{-1} \int_\eta^\infty \langle \widetilde{y} \rangle^{-3}\, dy \leqslant C \langle \widetilde{\xi} \rangle^{-1} \langle \widetilde{\eta} \rangle^{-2}.$$

From the Cauchy-Bunyakovsky-Schwarz inequality we infer that

$$\begin{aligned}|\mathcal{V}_h^* \phi - A_h^* \phi| &= \left| \int_{\mathbb{R}} G_h^*(t,\xi,\eta) \phi_\xi(\xi-\eta)\, d\eta \right| \\ &\leqslant C \| \langle \widetilde{\eta} \rangle^{-2}\, \partial_\eta \phi \|_{\mathbf{L}^2} \left(\int_{\mathbb{R}} \langle \widetilde{\xi} \rangle^{-2} \langle \widetilde{\eta} \rangle^{-4} \langle \widetilde{\xi} - \widetilde{\eta} \rangle^2\, d\eta \right)^{1/2} \\ &\leqslant C t^{-1/4} \| \langle \widetilde{\eta} \rangle^{-2}\, \partial_\eta \phi \|_{\mathbf{L}^2}.\end{aligned}$$

The lemma is proved.

2.4. Boundedness of pseudodifferential operators. There are many results concerning the $\mathbf{L}^2$-boundedness of pseudodifferential operators of the form

$$\mathbf{a}(t,x,\mathbf{D})\phi = \int_{\mathbb{R}} e^{ix\xi} \mathbf{a}(t,x,\xi) \widehat{\phi}(\xi)\, d\xi$$

(see [16]–[19]). Below we use the following result (see [18]).

Lemma 2.4. *Assume that the symbol* $\mathbf{a}(t,x,\xi)$ *satisfies the estimates*

$$\sup_{x,\xi \in \mathbb{R},\, t \geqslant 1} |\partial_x^k\, \partial_\xi^l \mathbf{a}(t,x,\xi)| \leqslant C$$

for $k,l = 0,1$. *Then*

$$\|\mathbf{a}(t,x,\mathbf{D})\phi\|_{\mathbf{L}_x^2} \leqslant C \|\phi\|_{\mathbf{L}^2}.$$

A similar result is true for the conjugate operator

$$\mathbf{a}^*(t,\xi,\mathbf{D})\phi = \int_{\mathbb{R}} e^{-ix\xi}\mathbf{a}^*(t,x,\xi)\overline{\widehat{\phi}(x)}\,dx.$$

Lemma 2.5. *Assume that the symbol* $\mathbf{a}^*(t,x,\xi)$ *satisfies the estimates*

$$\sup_{x,\xi\in\mathbb{R},\,t\geqslant 1} |\partial_x^k\,\partial_\xi^l \mathbf{a}^*(t,x,\xi)| \leqslant C$$

for $k,l=0,1$. *Then*

$$\|\mathbf{a}^*(t,\xi,\mathbf{D})\phi\|_{\mathbf{L}^2_\xi} \leqslant C\|\phi\|_{\mathbf{L}^2}.$$

2.5. Estimates for derivatives of the defect operator. We define the defect operator with weight $h(t,\xi,\eta)$ by

$$\mathcal{V}_h\phi = t^{1/2}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}h(t,\xi,\eta)\phi(\xi)\,d\xi.$$

The following lemma proves that the operator $\mathcal{V}_h$ is $\mathbf{L}^2$-bounded uniformly with respect to $t\geqslant 1$.

Lemma 2.6. *Assume that the kernel* h *satisfies the estimates*

$$|\partial_\eta^m\,\partial_\xi^n h(t,\xi,\eta)| \leqslant Ct^{(m+n)/2}$$

for all $\xi,\eta\in\mathbb{R}$, $t\geqslant 1$, *where* $m,n=0,1$. *Then*

$$\|\sqrt{\Lambda''}\mathcal{V}_h\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{L}^2} \quad \textit{for all } t\geqslant 1.$$

Proof. We make the change of variable $\eta=\mu(x)$; then

$$\mathcal{V}_h\phi = t^{1/2}\overline{M}\mathcal{B}^{-1}\int_{\mathbb{R}} e^{itx\xi}h(t,\xi,\mu(x))e^{-it\Lambda(\xi)}\phi(\xi)\,d\xi.$$

Next, we change the variable of integration in accordance with the relation $\xi=t^{-1/2}\xi'$ (and omit primes in what follows):

$$\begin{aligned}\mathcal{V}_h\phi &= \overline{M}\mathcal{B}^{-1}\mathcal{D}_{t^{1/2}}^{-1}\int_{\mathbb{R}} e^{ix\xi}h\biggl(t,\frac{\xi}{\sqrt{t}},\mu\biggl(\frac{x}{\sqrt{t}}\biggr)\biggr)\mathcal{D}_{t^{1/2}}e^{-it\Lambda(\xi)}\phi(\xi)\,d\xi\\ &= \overline{M}\mathcal{B}^{-1}\mathcal{D}_{t^{1/2}}^{-1}\mathbf{a}(t,x,\mathbf{D})\mathcal{F}^{-1}\mathcal{D}_{t^{1/2}}e^{-it\Lambda}\phi,\end{aligned}$$

where the symbol $\mathbf{a}(t,x,\xi)$ is $h(t,\xi/\sqrt{t},\mu(x/\sqrt{t}))$. Since $\mu(x)=O(x)$ and $\mu'(x)=1/\Lambda''(\mu(x))=O(1)$, we have

$$|\partial_x^m\,\partial_\xi^n\mathbf{a}(t,x,\xi)| = O\biggl(\frac{t^{-m/2}}{(\Lambda''(\eta))^m}\,\partial_\eta^m\,\partial_\xi^n h(t,\xi t^{-1/2},\eta)\bigg|_{\eta=\mu(xt^{-1/2})}\biggr) \leqslant C$$

for all $x,\xi\in\mathbb{R}$, $t\geqslant 1$ and $m,n=0,1$. An application of Lemma 2.4 yields

$$\|\mathbf{a}(t,x,\mathbf{D})\phi\|_{\mathbf{L}^2_x} \leqslant C\|\phi\|_{\mathbf{L}^2}.$$

Therefore, using the equalities

$$\|\sqrt{\Lambda''}\mathcal{B}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}, \qquad \|\mathcal{D}_{t^{1/2}}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2},$$
$$\|\mathcal{F}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2} \quad \text{and} \quad \|\mathcal{D}_{t^{1/2}}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2},$$

we deduce the inequality

$$\|\sqrt{\Lambda''}\mathcal{V}_h\phi\|_{\mathbf{L}^2} \leqslant C\|\sqrt{\Lambda''}\mathcal{B}^{-1}\mathcal{D}_{t^{1/2}}^{-1}\mathbf{a}(t,x,\mathbf{D})\mathcal{F}^{-1}\mathcal{D}_{t^{1/2}}e^{-it\Lambda}\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{L}^2}.$$

The lemma is proved.

Now we estimate the derivative $\partial_\eta\mathcal{Q}$.

Lemma 2.7. *The following inequality holds*:

$$\|\partial_\eta\mathcal{Q}\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{H}^1} \quad \textit{for all } t \geqslant 1.$$

Proof. Integration by parts yields

$$\partial_\eta\mathcal{Q}\phi = C\mathcal{V}_{q_1}\,\partial_\xi\phi + C\mathcal{V}_{q_2}\phi,$$

where

$$q_1(\xi,\eta) = \frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)} = \frac{\Lambda''(\eta)}{\int_0^1 \Lambda''(\eta+(\xi-\eta)z)\,dz} = O(1)$$

and

$$q_2(\xi,\eta) = \partial_\xi\biggl(\frac{\partial_\eta S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\biggr) = \partial_\xi\biggl(\frac{\Lambda''(\eta)}{\int_0^1 \Lambda''(\eta+(\xi-\eta)z)\,dz}\biggr) = O(1),$$

which implies the estimates $\sup_{\eta,\xi\in\mathbb{R}}|\partial_\eta^k\,\partial_\xi^l q_j(\eta,\xi)| \leqslant C$ for $k,l=0,1,\ j=1,2$. Therefore, using Lemma 2.6 we obtain

$$\|\partial_\eta\mathcal{Q}\phi\|_{\mathbf{L}^2} \leqslant C\|\sqrt{\Lambda''}\,\partial_\eta\mathcal{Q}\phi\|_{\mathbf{L}^2_x} \leqslant C\|\partial_\xi\phi\|_{\mathbf{L}^2} + C\|\phi\|_{\mathbf{L}^2}.$$

The lemma is proved.

The next lemma estimates the derivative $\mathcal{Q}_t$.

Lemma 2.8. *The following identity holds*:

$$t\mathcal{Q}_t\phi = i\mathcal{A}_1\mathcal{V}_{h_1}\,\partial_\xi\phi + \eta\mathcal{V}_{h_1}\,\partial_\xi\phi + \frac{1}{t}\mathcal{V}_{h_2}\,\partial_\xi\phi + \frac{1}{t}\mathcal{V}_{h_3}\phi,$$

where the weight functions h_j are defined below. In addition,

$$\|\mathcal{V}_{h_j}\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{L}^2} \quad \textit{for all } t \geqslant 1, \quad j=1,2,3.$$

Proof. Integrating by parts yields

$$\begin{aligned} t\mathcal{Q}_t\phi &= \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\biggl(\frac{1}{2} - itS(\xi,\eta)\biggr)\phi(\xi)\,d\xi \\ &= \frac{1}{2}\mathcal{Q}\phi - \sqrt{\frac{t}{2\pi}}\int_{\mathbb{R}} e^{-itS(\xi,\eta)}\,\partial_\xi\biggl(\frac{S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\phi(\xi)\biggr)\,d\xi \\ &= -\mathcal{V}_{h_1}\xi\,\partial_\xi\phi + \eta\mathcal{V}_{h_1}\,\partial_\xi\phi + \mathcal{V}_{q_3}\phi, \end{aligned}$$

where $h_1(\xi,\eta) = \dfrac{S(\xi,\eta)}{\sqrt{2\pi}(\xi-\eta)\,\partial_\xi S(\xi,\eta)}$ and

$$q_3(\xi,\eta) = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2} - \partial_\xi \frac{S(\xi,\eta)}{\partial_\xi S(\xi,\eta)}\right) = \frac{1}{\sqrt{2\pi}}\left(\frac{\Lambda''(\xi) f_1(\xi,\eta)}{f_2^2(\xi,\eta)} - \frac{1}{2}\right)$$

for

$$f_1(\xi,\eta) = \int_0^1 \Lambda''(\eta + (\xi-\eta)z)(1-z)\,dz \quad \text{and} \quad f_2(\xi,\eta) = \int_0^1 \Lambda''(\eta+(\xi-\eta)z)\,dz.$$

Note that $q_3(\eta,\eta) = 0$. Thus, integrating by parts once again, we obtain

$$V_{q_3}\phi = \frac{1}{t} V_{q_4}\,\partial_\xi \phi + \frac{1}{t} V_{q_5}\phi,$$

where

$$q_4(\xi,\eta) = \frac{q_3(\xi,\eta)}{i\,\partial_\xi S(\xi,\eta)} \quad \text{and} \quad q_5(\xi,\eta) = \partial_\xi \frac{q_3(\xi,\eta)}{i\,\partial_\xi S(\xi,\eta)}.$$

We introduce the notation $f_3(\xi,\eta) = f_1(\xi,\eta) - \frac{1}{2} f_2^2(\xi,\eta)$. Since $f_3(\eta,\eta) = 0$, we can write

$$f_3(\xi,\eta) = (\xi-\eta)(f_4(\xi,\eta) + f_5(\xi,\eta) + f_6(\xi,\eta)),$$

where

$$f_4(\xi,\eta) = \int_0^1 \Lambda'''(\eta+(\xi-\eta)z_1)\int_0^1 \Lambda''(\eta+(\xi-\eta)z_1 z)(1-z)\,dz\,dz_1,$$
$$f_5(\xi,\eta) = \int_0^1 \Lambda''(\eta+(\xi-\eta)z_1)\int_0^1 \Lambda'''(\eta+(\xi-\eta)z_1 z)z(1-z)\,dz\,dz_1$$

and

$$f_6(\xi,\eta) = -\int_0^1 \left(\int_0^1 \Lambda''(\eta+(\xi-\eta)z_1 z)\,dz\right)\left(\int_0^1 \Lambda'''(\eta+(\xi-\eta)z_1 z)z\,dz\right) dz_1.$$

Hence we have

$$q_4(\xi,\eta) = \frac{f_4(\xi,\eta) + f_5(\xi,\eta) + f_6(\xi,\eta)}{i\sqrt{2\pi} f_2^3(\xi,\eta)}$$

and

$$q_5(\xi,\eta) = \partial_\xi \frac{f_4(\xi,\eta) + f_5(\xi,\eta) + f_6(\xi,\eta)}{i\sqrt{2\pi} f_2^3(\xi,\eta)}.$$

In view of the fact that $S(\xi,\eta) = (\xi-\eta)^2 f_1(\xi,\eta)$ we deduce the relation

$$h_1(\xi,\eta) = \frac{f_1(\xi,\eta)}{\sqrt{2\pi} f_2(\xi,\eta)} = O(1).$$

Therefore, using the operator $\mathcal{A}_1 = \dfrac{1}{t\Lambda''(\eta)}\overline{M}\,\partial_\eta M$ we can write

$$\mathcal{V}_{h_1}\xi\,\partial_\xi\phi = -i\mathcal{A}_1\mathcal{V}_{h_1}\,\partial_\xi\phi + t^{-1}\mathcal{V}_{q_6}\,\partial_\xi\phi,$$

where $q_6(\xi,\eta) = it\mathcal{A}_0 h_1(\xi,\eta)$. Thus, we arrive at the representation

$$t\mathcal{Q}_t\phi = i\mathcal{A}_1\mathcal{V}_{h_1}\,\partial_\xi\phi + \eta\mathcal{V}_{h_1}\,\partial_\xi\phi + t^{-1}\mathcal{V}_{h_2}\,\partial_\xi\phi + \frac{1}{t}\mathcal{V}_{h_3}\phi,$$

where $h_2 = q_4 - q_6$ and $h_3 = q_5$. Since $\sup_{\eta,\xi\in\mathbb{R}} |\partial_\eta^k\,\partial_\xi^l h_j(\eta,\xi)| \leqslant C$ for $k,l = 0,1$, $j = 1,2,3$, using Lemma 2.6 we infer the estimates

$$\|\mathcal{V}_{h_j}\phi\|_{\mathbf{L}^2} \leqslant C\|\sqrt{\Lambda''}\mathcal{V}_{h_j}\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{L}^2}.$$

Lemma 2.8 is proved.

2.6. Estimates for derivatives of the conjugate defect operator $\mathcal{Q}^*$. Here we establish the $\mathbf{L}^2$-boundedness of the weighted conjugate defect operator

$$\mathcal{V}_h^*\phi = t^{1/2}\int_{\mathbb{R}} e^{itS(\xi,\eta)}h(t,\xi,\eta)\phi(\eta)\Lambda''(\eta)\,d\eta.$$

Lemma 2.9. *Assume that the weight function h satisfies*

$$|\partial_\eta^m\,\partial_\xi^n h(t,\xi,\eta)| \leqslant Ct^{(m+n)/2}$$

for all $\xi,\eta\in\mathbb{R}$, $t\geqslant 1$ and $m,n = 0,1$. Then

$$\|\mathcal{V}_h^*\phi\|_{\mathbf{L}^2} \leqslant C\|\sqrt{\Lambda''}\phi\|_{\mathbf{L}^2} \quad \textit{for all } t\geqslant 1.$$

Proof. We make the change of the variable of integration $\eta = \mu(x)$; then

$$\mathcal{V}_h^*\phi = e^{it\Lambda(\xi)}t^{1/2}\int_{\mathbb{R}} e^{-itx\xi}h(t,\xi,\mu(x))\mathcal{B}M\phi\,dx.$$

Next, we make the change $x = t^{-1/2}x'$ (primes are omitted below); then we have

$$\mathcal{V}_h^*\phi = e^{it\Lambda(\xi)}\mathcal{D}_{t^{1/2}}^{-1}\int_{\mathbb{R}} e^{-ix\xi}h(t,\xi t^{-1/2},\mu(xt^{-1/2}))\mathcal{D}_{t^{-1/2}}\mathcal{B}M\phi\,dx.$$

We define the pseudodifferential operator by

$$\mathbf{a}^*(t,\xi,\mathbf{D})\phi = \int_{\mathbb{R}} e^{-ix\xi}\mathbf{a}^*(t,\xi,x)\overline{\widehat{\phi}(x)}\,dx,$$

where the symbol is $\mathbf{a}^*(t,\xi,x) = h(t,\xi t^{-1/2},\mu(xt^{-1/2}))$. Then we can write

$$\mathcal{V}_h^*\phi = e^{it\Lambda(\xi)}\mathcal{D}_{t^{1/2}}^{-1}\mathbf{a}^*(t,\xi,\mathbf{D})\mathcal{F}^{-1}\overline{\mathcal{D}_{t^{1/2}}\mathcal{B}M\phi}.$$

To establish the $\mathbf{L}^2$-boundedness of the pseudodifferential operator $\mathbf{a}^*(t,\xi,\mathbf{D})$, we estimate the symbol $\mathbf{a}^*(t,\xi,x) = h(t,\xi t^{-1/2},\eta)|_{\eta=\mu(xt^{-1/2})}$. Since $\mu'(x) = 1/\Lambda''(\mu(x)) = O(1)$, we have

$$|\partial_x^m\,\partial_\xi^n\mathbf{a}^*(t,\xi,x)| = O\biggl(\frac{t^{-m/2}}{(\Lambda''(\eta))^m}\,\partial_\eta^m\,\partial_\xi^n h(t,\xi t^{-1/2},\eta)\biggr|_{\eta=\mu(xt^{-1/2})}\biggr) \leqslant C$$

for all $x, \xi \in \mathbb{R}$, $t \geqslant 1$ and $m, n = 0, 1$. Therefore, from Lemma 2.5 we derive the estimate $\|\mathbf{a}^*(t,\xi,\mathbf{D})\phi\|_{\mathbf{L}^2_\xi} \leqslant C\|\phi\|_{\mathbf{L}^2}$; in view of the equalities

$$\|\mathcal{D}^{-1}_{t^{1/2}}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2}, \qquad \|\mathcal{F}^{-1}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2},$$
$$\|\mathcal{D}_{t^{1/2}}\phi\|_{\mathbf{L}^2} = \|\phi\|_{\mathbf{L}^2} \quad \text{and} \quad \|\mathcal{B}\phi\|_{\mathbf{L}^2} = \|\sqrt{\Lambda''}\phi\|_{\mathbf{L}^2},$$

it follows that

$$\|\mathcal{V}^*_h\phi\|_{\mathbf{L}^2} = \|\mathbf{a}^*(t,\xi,\mathbf{D})\mathcal{F}^{-1}\overline{\mathcal{D}_{t^{1/2}}\mathcal{B}M\phi}\|_{\mathbf{L}^2_\xi} \leqslant C\|\sqrt{\Lambda''}\phi\|_{\mathbf{L}^2}.$$

The lemma is proved.

Lemma 2.10. *The following inequality holds*:

$$\|\partial_\xi\mathcal{Q}^*\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{H}^1} \quad \text{for all } t \geqslant 1.$$

Proof. Integration by parts yields

$$\partial_\xi\mathcal{Q}^*\phi = C\mathcal{V}^*_{q_7}\,\partial_\eta\phi + C\mathcal{V}^*_{q_8}\phi,$$

where

$$q_7(\xi,\eta) = \frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)} = \frac{1}{\Lambda''(\eta)}\int_0^1 \Lambda''(\eta + (\xi-\eta)z)\,dz = O(1)$$

and

$$q_8(\xi,\eta) = \frac{1}{\Lambda''(\eta)}\,\partial_\eta\biggl(\frac{\partial_\xi S(\xi,\eta)}{\partial_\eta S(\xi,\eta)}\Lambda''(\eta)\biggr)$$
$$= \frac{1}{\Lambda''(\eta)}\,\partial_\eta\int_0^1 \Lambda''(\eta + (\xi-\eta)z)\,dz = O(1).$$

Taking account of the estimates $\sup_{\eta,\xi\in\mathbb{R}}|\partial_\eta^k\,\partial_\xi^l q_j(\xi,\eta)| \leqslant C$ for $k, l = 0, 1$ and $j = 7, 8$ and using Lemma 2.9 we obtain

$$\|\partial_\xi\mathcal{Q}^*\phi\|_{\mathbf{L}^2_x} \leqslant C\|\sqrt{\Lambda''}\,\partial_\eta\phi\|_{\mathbf{L}^2} + C\|\sqrt{\Lambda''}\phi\|_{\mathbf{L}^2} \leqslant C\|\phi\|_{\mathbf{H}^1}.$$

The lemma is proved.

The next lemma estimates the commutator $[\langle\xi\rangle^{-2}, \mathcal{Q}^*]$.

Lemma 2.11. *The following inequality holds*:

$$\|[\langle\xi\rangle^{-2}, \mathcal{Q}^*]\phi\|_{\mathbf{L}^2} \leqslant Ct^{-1}\|\phi\|_{\mathbf{H}^1} \quad \text{for all } t \geqslant 1.$$

Proof. Integrating by parts we obtain

$$t[\langle\xi\rangle^{-2}, \mathcal{Q}^*]\phi = CV^*_{q_9}\,\partial_\eta\phi + CV^*_{q_{10}}\phi,$$

where

$$q_9(\xi,\eta) = \frac{\langle\xi\rangle^{-2} - \langle\eta\rangle^{-2}}{\partial_\eta S(\xi,\eta)} = \frac{\xi+\eta}{\langle\xi\rangle^2\langle\eta\rangle^2\Lambda''(\eta)} = O(1)$$

and

$$q_{10}(\xi,\eta)=\frac{1}{\Lambda''(\eta)}\,\partial_\eta\biggl(\frac{\langle\xi\rangle^{-2}-\langle\eta\rangle^{-2}}{\partial_\eta S(\xi,\eta)}\Lambda''(\eta)\biggr)=\frac{1}{\Lambda''(\eta)}\,\partial_\eta\biggl(\frac{\xi+\eta}{\langle\xi\rangle^2\langle\eta\rangle^2}\biggr)=O(1).$$

Taking account of the estimates $\sup_{\eta,\xi\in\mathbb{R}}|\partial_\eta^k\,\partial_\xi^l q_j(\xi,\eta)|\leqslant C$ for $k,l=0,1$ and $j=9,10$ and using Lemma 2.9 we obtain

$$t\|[\langle\xi\rangle^{-2},\mathcal{Q}^*]\phi\|_{\mathbf{L}^2_x}\leqslant C\|\phi\|_{\mathbf{H}^1}.$$

The lemma is proved.

§ 3. *A priori* estimates for the solution

First we state a result on the time-local existence of a solution of the Cauchy problem (2) in the function space $\mathbf{C}([0,\infty);\mathbf{H}^5\cap\mathbf{H}^{0,1})\cap\mathbf{C}^1((0,\infty);\mathbf{H}^3)$ (see [20] for a proof).

Theorem 3.1. *Assume that the initial data satisfy $u_0\in\mathbf{H}^5\cap\mathbf{H}^{0,1}$. Then for some $T>0$ Cauchy problem (2) has a unique solution*

$$u\in\mathbf{C}([0,\infty);\mathbf{H}^5\cap\mathbf{H}^{0,1})\cap\mathbf{C}^1((0,\infty);\mathbf{H}^3)$$

such that $\|u\|_{\mathbf{X}_T}<C$. If the norm $\|u_0\|_{\mathbf{H}^5\cap\mathbf{H}^{0,1}}$ is small, then the existence time T is greater than 1.

To establish the time-global existence of solutions we need to establish *a priori* estimates for solutions in the norm $\|\widehat{\varphi}\|_{\mathbf{X}_T}$ that are uniform with respect to $T\geqslant 1$. Here

$$\|\phi\|_{\mathbf{X}_T}=\sup_{t\in[1,T]}(\|\phi(t)\|_{\mathbf{L}^\infty}+W^{1/2}(t)\|\langle\widetilde{\xi}\rangle^{-\gamma}\phi(t)\|_{\mathbf{L}^\infty}+t^{-\gamma}\|\langle\xi\rangle^5\phi\|_{\mathbf{L}^2}+K^{-1}(t)\|\partial_\xi\phi(t)\|_{\mathbf{L}^2}),$$

where $W(t)=1+\varepsilon^2\ln(1+t)$, $K(t)=t^\gamma+\varepsilon^2t^{1/4}W^{-3/2}(t)$, $\widetilde{\xi}=\xi\sqrt{t}$, and $\gamma>0$ is small.

3.1. An estimate for the derivative. The next lemma estimates the function

$$\Phi=\partial_\xi\widehat{\varphi}-\langle\xi\rangle^{-2}\Omega'(\xi)\int_1^t\mathcal{Q}^*(\tau)M^2(\tau)v^3(\tau)\,d\tau.$$

Lemma 3.1. *Assume that $\|\widehat{\varphi}\|_{\mathbf{X}_T}\leqslant C\varepsilon$. Then*

$$\|\Phi(t)\|_{\mathbf{L}^2}\leqslant C\varepsilon K(t)\quad \textit{for all } t\in[1,T].$$

Proof. Differentiating (5) we arrive at the equality

$$i\,\partial_t\widehat{\varphi}_\xi=i\langle\xi\rangle^{-2}\Omega'(\xi)e^{it\Omega(\xi)}\mathcal{D}_3\mathcal{Q}^*(3t)v^3+R=i\langle\xi\rangle^{-2}\Omega'(\xi)\mathcal{Q}^*(t)M^2(t)v^3(t)+R,$$

where $R=t^{-1}e^{it\Omega}\,\partial_\xi(\langle\xi\rangle^{-2}\mathcal{D}_3\mathcal{Q}^*(3t)v^3)$, which yields $i\,\partial_t\Phi=R$. To estimate the remainder R we use Lemma 2.10; then $\|R\|_{\mathbf{L}^2}\leqslant Ct^{-1}\|v^3\|_{\mathbf{H}^1}$. From Lemma 2.7 we also deduce that

$$\|\partial_\eta v\|_{\mathbf{L}^2}\leqslant C\|\widehat{\varphi}\|_{\mathbf{H}^1}\leqslant C\varepsilon K(t).$$

Using Lemma 2.2 yields the inequality

$$|v| \leqslant C|\widehat{\varphi}| + Ct^{-1/4}\|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} \leqslant C\varepsilon,$$

which implies that

$$\|R\|_{\mathbf{L}^2} \leqslant Ct^{-1}\|v\|^2_{\mathbf{L}^\infty}\|v\|_{\mathbf{H}^1} \leqslant C\varepsilon^3 t^{-1}K(t).$$

Thus, we have

$$\frac{d}{dt}\|\Phi(t)\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 t^{-1}K(t).$$

Integration with respect to time leads to the inequality

$$\|\Phi(t)\|_{\mathbf{L}^2} \leqslant C\varepsilon K(t) \quad \text{for } t \in [1, T].$$

The lemma is proved.

Now we estimate the derivative $\partial_\xi \widehat{\varphi}$.

Lemma 3.2. *Assume that* $\|\widehat{\varphi}\|_{\mathbf{X}_T} \leqslant C\varepsilon$. *Then*

$$\|\partial_\xi \widehat{\varphi}\|_{\mathbf{L}^2} \leqslant C\varepsilon K(t) \quad \textit{for all } t \in [1, T].$$

Proof. In view of Lemma 3.1 we need to estimate the integral

$$I = \langle \xi \rangle^{-2}\Omega'(\xi) \int_1^t \mathcal{Q}^*(\tau)M^2(\tau)v^3(\tau)\, d\tau.$$

Using the identity $t^{1/2}e^{itS_3(\xi,\eta)} = H_1\, \partial_t(t^{3/2}e^{itS_3(\xi,\eta)})$, where $H_1(t,\xi,\eta) = (3/2 + itS_3(\xi,\eta))^{-1}$ and $S_3(\xi,\eta) = S(\xi,\eta) + 2\Theta(\eta)$, and integrating by parts, we infer that

$$\begin{aligned}\int_1^t \mathcal{Q}^* M^2 \phi\, d\tau &= t\mathcal{Q}^*(t)M^2(t)H_1(t)\phi(t) - \mathcal{Q}^*(1)M^2(1)H_1(1)\phi(1)\\ &\quad - \int_1^t \mathcal{Q}^*(\tau)M^2(\tau)H_2(\tau)\phi(\tau)\, d\tau - \int_1^t \mathcal{Q}^*(\tau)M^2(\tau)H_1(\tau)\tau\, \partial_\tau\phi(\tau)\, d\tau,\end{aligned}$$

where $H_2 = \frac{3}{2}H_1^2 - H_1$. Note that $S(\xi,\eta) + 2\Theta(\eta) = \Omega(\xi) + 3S(\xi/3,\eta)$, which yields the estimate

$$S_3(\xi,\eta) = \frac{3}{4}\Omega(\xi) + \frac{1}{2}\Theta(\eta) + \frac{3}{4}\left(S(\xi,\eta) + S\left(\frac{\xi}{3},\eta\right)\right) \geqslant \frac{1}{8}(\xi^2 + \eta^2),$$

since

$$S(\xi,\eta) = \frac{1}{2}\int_\eta^\xi \Lambda''(z)(\xi - z)\, dz \geqslant C(\xi - \eta)^2.$$

Hence we arrive at the inequality

$$|H_1(t,\xi,\eta)| \leqslant \frac{C}{1 + t(\xi^2 + \eta^2)}.$$

Since $\tau\,\partial_\tau v(\tau) = \mathcal{Q}\tau\widehat{\varphi}_\tau + \tau\mathcal{Q}_\tau\widehat{\varphi}$, we have the representation

$$I = \langle\xi\rangle^{-2}\frac{\Omega'(\xi)}{\xi}\sum_{j=1}^{4} I_j,$$

where

$$I_1 = t\xi\mathcal{Q}^*(t)M^2(t)H_1(t)v^3(t) - \xi\mathcal{Q}^*(1)M^2(1)H_1(1)v^3(1),$$
$$I_2 = -\xi\int_1^t \mathcal{Q}^*(\tau)M^2(\tau)H_2(\tau)v^3(\tau)\,d\tau,$$
$$I_3 = -3\xi\int_1^t \mathcal{Q}^*(\tau)M^2(\tau)H_1(\tau)v^2\mathcal{Q}\tau\widehat{\varphi}_\tau\,d\tau$$

and

$$I_4 = -3\xi\int_1^t \mathcal{Q}^*(\tau)M^2(\tau)H_1(\tau)v^2\tau\mathcal{Q}_\tau\widehat{\varphi}\,d\tau.$$

By virtue of Lemma 2.2 we have

$$\langle\widetilde{\xi}\rangle^{-\gamma}|v| \leqslant C\langle\widetilde{\xi}\rangle^{-\gamma}|\widehat{\varphi}| + Ct^{-1/4}\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \leqslant C\varepsilon W^{-1/2}(t)$$

and also

$$\|\langle\widetilde{\xi}\rangle^{-1/2-2\gamma}v\|_{\mathbf{L}^2} \leqslant C\varepsilon W^{-1/2}(t)\|\langle\widetilde{\xi}\rangle^{-1/2-\gamma}\|_{\mathbf{L}^2} \leqslant C\varepsilon t^{-1/4}W^{-1/2}(t).$$

Now, using Lemma 2.9 for $h(t,\xi,\eta) = \widetilde{\xi}\langle\widetilde{\eta}\rangle H_1$ we derive the estimates

$$\|t\xi\mathcal{Q}^*(t)M^2H_1v^3(t)\|_{\mathbf{L}^2} \leqslant Ct^{1/2}\|\langle\widetilde{\xi}\rangle^{-1}v^3\|_{\mathbf{L}^2} \leqslant C\varepsilon^3t^{1/4}W^{-3/2}(t) \leqslant C\varepsilon K(t),$$
$$\|\xi\mathcal{Q}^*(1)M^2(1)H_1(1)v^3(1)\|_{\mathbf{L}^2} \leqslant C\|\langle\xi\rangle^{-1}v^3(1)\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 \leqslant C\varepsilon K(t);$$

so that $\|I_1\|_{\mathbf{L}^2} \leqslant C\varepsilon K(t)$. In a similar way we obtain the estimates

$$\|I_2\|_{\mathbf{L}^2} \leqslant \left\|\xi\int_1^t \mathcal{Q}^*(\tau)M^2H_2v^3(\tau)\,d\tau\right\|_{\mathbf{L}^2} \leqslant C\varepsilon^3\int_1^t \tau^{-3/4}W^{-3/2}(\tau)\,d\tau \leqslant C\varepsilon K(t).$$

Furthermore, it follows from (5) that

$$\mathcal{Q}t\widehat{\varphi}_t = -i\mathcal{Q}\langle\xi\rangle^{-2}\mathcal{Q}^*M^2v^3 = -i\mathcal{Q}\langle\xi\rangle^{-2}e^{i\tau\Omega}\mathcal{D}_3\mathcal{Q}^*(3\tau)v^3,$$

which yields the relations

$$\begin{aligned}\mathcal{Q}^*M^2H_1v^2\mathcal{Q}\tau\widehat{\varphi}_\tau &= -i\mathcal{Q}^*M^2H_1v^2\mathcal{Q}e^{i\tau\Omega}\mathcal{D}_3\langle 3\xi\rangle^{-2}\mathcal{Q}^*(3\tau)v^3\\ &= -i\mathcal{Q}^*M^4H_1\langle 3\eta\rangle^{-2}v^5 - i\mathcal{Q}^*M^2H_1v^2\mathcal{Q}e^{i\tau\Omega}\mathcal{D}_3[\langle 3\xi\rangle^{-2},\mathcal{Q}^*(3\tau)]v^3.\end{aligned}$$

Thus we have $I_3 = I_5 + I_6$, where

$$I_5 = 3i\xi\int_1^t \mathcal{Q}^*M^4H_1\langle 3\eta\rangle^{-2}v^5\,d\tau$$

and

$$I_6 = 3i\xi \int_1^t \mathcal{Q}^* M^2 H_1 v^2 \mathcal{Q} e^{i\tau\Omega} \mathcal{D}_3[\langle 3\xi \rangle^{-2}, \mathcal{Q}^*(3\tau)] v^3 \, d\tau.$$

As above, owing to Lemma 2.9 for $h = \widetilde{\xi} \langle \widetilde{\eta} \rangle H_1$ we obtain

$$\|I_5\|_{\mathbf{L}^2} \leqslant C \int_1^t \tau^{-1/2} \|\langle \widetilde{\eta} \rangle^{-1} v^5(\tau)\|_{\mathbf{L}^2} \, d\tau \leqslant C\varepsilon^3 \int_1^t \tau^{-3/4} W^{-5/2}(\tau) \, d\tau \leqslant C\varepsilon K(t).$$

Using Lemma 2.11 we also deduce that

$$\begin{aligned} \|I_6\|_{\mathbf{L}^2} &\leqslant C\varepsilon^2 \int_1^t \tau^{-1/2} \|[\langle 3\xi \rangle^{-2}, \mathcal{Q}^*(3\tau)] v^3\|_{\mathbf{L}^2} \, d\tau \\ &\leqslant C\varepsilon^2 \int_1^t \tau^{-3/2} \|v^3\|_{\mathbf{H}^1} \, d\tau \leqslant C\varepsilon^4 \int_1^t \tau^{-3/2} K(\tau) \, d\tau \leqslant C\varepsilon K(t). \end{aligned}$$

Finally, we estimate the term I_4. It follows from Lemma 2.8 that

$$t \mathcal{Q}_t \phi = i \mathcal{A}_1 \mathcal{V}_{h_1} \partial_\xi \phi + \eta \mathcal{V}_{h_1} \partial_\xi \phi + \frac{1}{t} \mathcal{V}_{h_2} \partial_\xi \phi + \frac{1}{t} \mathcal{V}_{h_3} \phi.$$

Therefore, we have the representation $I_4 = \sum_{j=7}^{10} I_j$, where

$$I_7 = -3\xi \int_1^t \mathcal{Q}^*(\tau) M^2 H_1 v^2 \mathcal{V}_{h_3} \widehat{\varphi} \, \frac{d\tau}{\tau}, \qquad I_8 = -3\xi \int_1^t \mathcal{Q}^*(\tau) M^2 H_1 v^2 \mathcal{V}_{h_2} \partial_\xi \widehat{\varphi} \, \frac{d\tau}{\tau},$$

$$I_9 = -3\xi \int_1^t \mathcal{Q}^*(\tau) M^2 H_1 \eta v^2 \mathcal{V}_{h_1} \partial_\xi \widehat{\varphi} \, d\tau,$$

$$I_{10} = -3i\xi \int_1^t \mathcal{Q}^*(\tau) M^2 H_1 v^2 \mathcal{A}_1 \mathcal{V}_{h_1} \partial_\xi \widehat{\varphi} \, d\tau.$$

Using the equalities $\mathcal{Q}^*(t) \mathcal{A}_1(t) = i\xi \mathcal{Q}^*(t)$ and $v_1 = \mathcal{A}_1 v = \mathcal{Q} i\xi \widehat{\varphi} = i\eta v + \mathcal{A}_0 v$ we conclude from this that

$$\begin{aligned} \mathcal{Q}^* M^2 H_1 v^2 \mathcal{A}_1 \phi &= \mathcal{Q}^* \frac{1}{t\Lambda''(\eta)} \overline{M} H_1 (Mv)^2 \, \partial_\eta M \phi \\ &= \mathcal{Q}^* \mathcal{A}_1 (M^2 H_1 v^2 \phi) - 2\mathcal{Q}^* M^2 H_1 v v_1 \phi - \mathcal{Q}^*(t) M^2 (\mathcal{A}_0 H_1) v^2 \phi \\ &= i\xi \mathcal{Q}^* M^2 H_1 v^2 \phi - 2i \mathcal{Q}^* M^2 H_1 \eta v^2 \phi - 2\mathcal{Q}^* M^2 H_1 (\mathcal{A}_0 v) v \phi - \mathcal{Q}^* M^2 (\mathcal{A}_0 H_1) v^2 \phi. \end{aligned}$$

Hence $I_{10} = \sum_{j=11}^{14} I_j$, where

$$I_{11} = 3\xi^2 \int_1^t \mathcal{Q}^* M^2 H_1 v^2 \mathcal{V}_{h_1} \partial_\xi \widehat{\varphi} \, d\tau, \qquad I_{12} = -6\xi \int_1^t \mathcal{Q}^* M^2 H_1 \eta v^2 \mathcal{V}_{h_1} \partial_\xi \widehat{\varphi} \, d\tau,$$

$$I_{13} = 3i\xi \int_1^t \mathcal{Q}^* M^2 (\mathcal{A}_0 H_1) v^2 \mathcal{V}_{h_1} \partial_\xi \widehat{\varphi} \, d\tau$$

and

$$I_{14} = 6i\xi \int_1^t \mathcal{Q}^* M^2 H_1 (\mathcal{A}_0 v) v \mathcal{V}_{h_1} \partial_\xi \widehat{\varphi} \, d\tau.$$

Lemmas 2.8 and 2.9 for $h = \widetilde{\xi}\langle\widetilde{\eta}\rangle H_1$ yield the estimate

$$\|I_7\|_{\mathbf{L}^2} \leqslant C\left\|\xi\int_1^t \mathcal{Q}^*(\tau)M^2H_1v^2\mathcal{V}_{h_3}\widehat{\varphi}\,\frac{d\tau}{\tau}\right\|_{\mathbf{L}^2}$$
$$\leqslant C\int_1^t \tau^{-3/2}\|\langle\widetilde{\eta}\rangle^{-1}v^2\mathcal{V}_{h_3}\widehat{\varphi}\|_{\mathbf{L}^2}\,d\tau \leqslant C\varepsilon^3\int_1^t \tau^{\gamma-3/2}\,d\tau \leqslant C\varepsilon^3K(t).$$

In a similar way we have

$$\|I_8\|_{\mathbf{L}^2}+\|I_9\|_{\mathbf{L}^2} \leqslant C\int_1^t \tau^{-3/2}\|\langle\widetilde{\eta}\rangle^{-1}v^2\mathcal{V}_{h_2}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}\,d\tau + C\int_1^t \tau^{-1}\|v^2\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}\,d\tau$$
$$\leqslant C\varepsilon^3\int_1^t \tau^{-3/2}K(\tau)\,d\tau + C\varepsilon^3\int_1^t \tau^{-1}K(\tau)\,d\tau \leqslant C\varepsilon^3K(t).$$

Using Lemmas 2.8 and 2.9 we obtain

$$\|I_{11}\|_{\mathbf{L}^2}+\|I_{12}\|_{\mathbf{L}^2}+\|I_{13}\|_{\mathbf{L}^2} \leqslant C\int_1^t \tau^{-1}\|v^2\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2}\,d\tau$$
$$\leqslant C\varepsilon^3\int_1^t \tau^{-1}K(\tau)\,d\tau \leqslant C\varepsilon^3K(t).$$

Finally, in view of inequality $\|\mathcal{Q}^*\phi\|_{\mathbf{L}^\infty} \leqslant Ct^{1/2}\|\phi\|_{\mathbf{L}^1}$ we arrive at the inequality

$$\|I_{14}\|_{\mathbf{L}^2} \leqslant C\int_1^t \|\langle\widetilde{\xi}\rangle^{-1}\|_{\mathbf{L}^2}\|\mathcal{Q}^*(\tau)M^2\widetilde{\xi}\langle\widetilde{\xi}\rangle H_1(\mathcal{A}_0v)v\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^\infty}\,d\tau$$
$$\leqslant C\varepsilon\int_1^t \tau^{-5/4}\|(\tau\mathcal{A}_0v)\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^1}\,d\tau$$
$$\leqslant C\varepsilon^3\int_1^t \tau^{-5/4}K^2(\tau)\,d\tau \leqslant C\varepsilon^3\int_1^t \tau^{-1}K(\tau)\,d\tau \leqslant C\varepsilon^3K(t),$$

which implies the assertion of the lemma for all $t \in [1,T]$.

Lemma 3.2 is proved.

3.2. Estimates in the uniform metric. We introduce the notation

$$y = \widehat{\varphi} + i\langle\xi\rangle^{-2}\mathcal{Q}^*M^2H_3v^3,$$

where $H_3(t,\xi,\eta) = (-1/2 + itS_3(\xi,\eta))^{-1}$ and $S_3(\xi,\eta) = S(\xi,\eta) + 2\Theta(\eta)$.

Lemma 3.3. *Assume that $\|\widehat{\varphi}\|_{\mathbf{X}_T} \leqslant C\varepsilon$. Then $y = \widehat{\varphi} + O(\varepsilon^3W^{-3/2})$. In addition, the function $y(t,\xi)$ satisfies the equality*

$$\partial_t y(t,\xi) = -\frac{1}{2t\sqrt{3}\langle\widetilde{\xi}\rangle^2}y^3(t,\xi) + g(t,\xi) \tag{6}$$

for all $t \geqslant 1$ and $x \in \mathbb{R}$, where

$$g(t,\xi) = O(g_1(t,\xi))$$

and

$$g_1(t,\xi) = \varepsilon^5t^{-1}W^{-5/2}(t) + \varepsilon^3t^{\gamma-5/4} + \varepsilon^3t^{-1}|\widetilde{\xi}|\langle\widetilde{\xi}\rangle^{-2}W^{-3/2}.$$

Proof. Using the identity $e^{itS_3(\xi,\eta)} = H_3 t^{3/2}\,\partial_t(t^{-1/2}e^{itS_3(\xi,\eta)})$ and integrating by parts we obtain

$$t^{-1}\mathcal{Q}^* M^2 v^3 = \partial_t(\mathcal{Q}^* M^2 H_3 v^3) - t^{-1}\mathcal{Q}^* M^2 H_4 v^3 - 3\mathcal{Q}^* M^2 H_3 v^2\,\partial_t v,$$

where $H_4 = -\frac{1}{2}H_3^2$. Therefore, it follows from (5) that

$$i\,\partial_t y = -iR_0 - 3\langle\xi\rangle^{-2}R_1, \tag{7}$$

where $R_0 = -it^{-1}\langle\xi\rangle^{-2}\mathcal{Q}^* M^2 H_4 v^3$ and $R_1 = \mathcal{Q}^* M^2 H_3 v^2\,\partial_t v$. We have $\partial_t v = \mathcal{Q}\widehat{\varphi}_t + (\mathcal{Q})_t\widehat{\varphi}$; hence we can set $R_1 = R_2 + R_3$, where

$$R_2 = \mathcal{Q}^* M^2 H_3 v^2 \mathcal{Q}\widehat{\varphi}_t \quad \text{and} \quad R_3 = \mathcal{Q}^* M^2 H_3 v^2 (\mathcal{Q})_t\widehat{\varphi}.$$

We derive from (5) that

$$\mathcal{Q}\widehat{\varphi}_t = -it^{-1}M^2\langle 3\eta\rangle^{-2}v^3 - it^{-1}\mathcal{Q}e^{it\Omega}\mathcal{D}_3[\langle 3\xi\rangle^{-2}, \mathcal{Q}^*(3t)]v^3;$$

thus, we can write $R_2 = R_4 + R_5$, where

$$R_4 = -it^{-1}\mathcal{Q}^* M^4 H_3 v^5\langle 3\eta\rangle^{-2}$$

and

$$R_5 = -it^{-1}\mathcal{Q}^* M^2 H_3 v^2 \mathcal{Q}e^{it\Omega}\mathcal{D}_3[\langle 3\xi\rangle^{-2}, \mathcal{Q}^*(3t)]v^3.$$

From Lemma 2.2 we infer the inequality $\|\langle\widetilde{\eta}\rangle^{-\gamma}v\|_{\mathbf{L}^\infty} \leqslant C\varepsilon W^{-1/2}$, which, in view of the estimate $\|\mathcal{V}_h^*\phi\|_{\mathbf{L}^\infty} \leqslant Ct^{1/2}\|h\phi\|_{\mathbf{L}^1}$, implies that

$$\|R_4\|_{\mathbf{L}^\infty} \leqslant Ct^{-1/2}\|\langle\widetilde{\eta}\rangle^{-\gamma}v\|_{\mathbf{L}^\infty}^5\|\langle\widetilde{\eta}\rangle^{5\gamma-2}\|_{\mathbf{L}^1} \leqslant C\varepsilon^5 t^{-1}W^{-5/2} \leqslant C|g_1|;$$

from Lemma 2.11 we obtain

$$\begin{aligned}\|R_5\|_{\mathbf{L}^\infty} &\leqslant Ct^{-1/2}\|\langle\widetilde{\eta}\rangle^{-\gamma}v\|_{\mathbf{L}^\infty}^2\|\langle\widetilde{\eta}\rangle^{2\gamma-2}\|_{\mathbf{L}^2}\|[\langle 3\xi\rangle^{-2}, \mathcal{Q}^*(3t)]v^3\|_{\mathbf{L}^2}\\ &\leqslant C\varepsilon^2 t^{-7/4}\|v^3\|_{\mathbf{H}^1} \leqslant C|g_1|.\end{aligned}$$

Furthermore, owing to Lemma 2.8,

$$t(\mathcal{Q})_t\phi = t^{-1}\eta\mathcal{V}_{h_1}\,\partial_\xi\phi + t^{-2}\mathcal{V}_{h_2}\,\partial_\xi\phi + t^{-2}\mathcal{V}_{h_3}\phi + it^{-1}\mathcal{A}_1\mathcal{V}_{h_1}\,\partial_\xi\phi.$$

Therefore, $R_3 = \sum_{j=6}^{9} R_j$, where

$$R_6 = t^{-1}\mathcal{Q}^* M^2 H_3\eta v^2\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}, \qquad R_7 = t^{-2}\mathcal{Q}^* M^2 H_3 v^2\mathcal{V}_{h_2}\,\partial_\xi\widehat{\varphi},$$
$$R_8 = t^{-2}\mathcal{Q}^* M^2 H_3 v^2\mathcal{V}_{h_3}\widehat{\varphi} \quad \text{and} \quad R_9 = it^{-1}\mathcal{Q}^* M^2 H_3 v^2\mathcal{A}_1\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}.$$

Using the inequality $\|\mathcal{Q}^*\phi\|_{\mathbf{L}^\infty} \leqslant Ct^{1/2}\|\phi\|_{\mathbf{L}^1}$ and Lemma 2.8 we deduce the relations

$$\begin{aligned}&\|R_6\|_{\mathbf{L}^\infty} + \|R_7\|_{\mathbf{L}^\infty} + \|R_8\|_{\mathbf{L}^\infty}\\ &\quad\leqslant Ct^{-1/2}\|\langle\widetilde{\eta}\rangle^{-2}v^2\eta\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^1}\\ &\qquad + Ct^{-3/2}\|\langle\widetilde{\eta}\rangle^{-2}v^2\mathcal{V}_{h_2}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^1} + Ct^{-3/2}\|\langle\widetilde{\eta}\rangle^{-2}v^2\mathcal{V}_{h_3}\widehat{\varphi}\|_{\mathbf{L}^1}\\ &\quad\leqslant Ct^{-1}\|\langle\widetilde{\eta}\rangle^{-\gamma}v\|_{\mathbf{L}^\infty}^2\|\langle\widetilde{\eta}\rangle^{2\gamma-1}\|_{\mathbf{L}^2}(\|\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + \|\mathcal{V}_{h_2}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + \|\mathcal{V}_{h_3}\widehat{\varphi}\|_{\mathbf{L}^2})\\ &\quad\leqslant C\varepsilon^2 t^{-5/4}W^{-1}(\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} + \|\widehat{\varphi}\|_{\mathbf{L}^2})\\ &\quad\leqslant C\varepsilon^3 t^{-5/4}W^{-1}K(t) \leqslant C|g_1|.\end{aligned}$$

Now from the identities $\mathcal{Q}^*(t)\mathcal{A}_1(t) = i\xi\mathcal{Q}^*(t)$ and $v_1 = \mathcal{A}_1 v = \mathcal{Q}i\xi\widehat{\varphi} = i\eta v + \mathcal{A}_0 v$ we obtain

$$\mathcal{Q}^* M^2 H_3 v^2 \mathcal{A}_1 \phi = i\xi \mathcal{Q}^* M^2 H_3 v^2 \phi - 2i\mathcal{Q}^* M^2 H_3 \eta v^2 \phi \\ - 2\mathcal{Q}^* M^2 H_3 (\mathcal{A}_0 v) v\phi - \mathcal{Q}^* M^2 (\mathcal{A}_0 H_3) v^2 \phi.$$

Consequently, we can write $R_9 = \sum_{j=10}^{13} R_j$, where

$$R_{10} = -t^{-1}\xi\mathcal{Q}^* M^2 H_3 v^2 \mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}, \qquad R_{11} = 2t^{-1}\mathcal{Q}^* M^2 H_3 \eta v^2 \mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi},$$
$$R_{12} = -it^{-1}\mathcal{Q}^* M^2 (\mathcal{A}_0 H_3) v^2 \mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi} \quad \text{and} \quad R_{13} = -2it^{-1}\mathcal{Q}^* M^2 H_3 (\mathcal{A}_0 v) v \mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}.$$

Note that $(\mathcal{A}_0 H_3) = O(t^{-1/2}\langle\widetilde{\eta}\rangle^{-2})$; hence, using the inequality $\|\mathcal{Q}^*\phi\|_{\mathbf{L}^\infty} \leqslant Ct^{1/2}\|\phi\|_{\mathbf{L}^1}$ and Lemma 2.8, we deduce the relations

$$\begin{aligned}\|R_{10}\|_{\mathbf{L}^\infty} + \|R_{11}\|_{\mathbf{L}^\infty} + \|R_{12}\|_{\mathbf{L}^\infty} &\leqslant Ct^{-1}\|\langle\widetilde{\eta}\rangle^{-1} v^2 \mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^1} \\ &\leqslant Ct^{-1}\|\langle\widetilde{\eta}\rangle^{-\gamma} v\|^2_{\mathbf{L}^\infty}\|\langle\widetilde{\eta}\rangle^{2\gamma-1}\|_{\mathbf{L}^2}\|\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \leqslant C\varepsilon^2 t^{-5/4} W^{-1}\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \\ &\leqslant C\varepsilon^3 t^{-5/4} W^{-1} K(t) \leqslant C|g_1|.\end{aligned}$$

We estimate the term R_{13} using the inequality $\|\mathcal{Q}^*\phi\|_{\mathbf{L}^\infty} \leqslant Ct^{1/2}\|\phi\|_{\mathbf{L}^1}$, Lemma 2.8, and Lemma 2.7 as follows:

$$\begin{aligned}\|R_{13}\|_{\mathbf{L}^\infty} &\leqslant Ct^{-1/2}\|\langle\widetilde{\eta}\rangle^{-2} v(\mathcal{A}_0 v)\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^1} \\ &\leqslant Ct^{-1/2}\|\langle\widetilde{\eta}\rangle^{-2} v\|_{\mathbf{L}^\infty}\|\mathcal{A}_0 v\|_{\mathbf{L}^2}\|\mathcal{V}_{h_1}\,\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \\ &\leqslant C\varepsilon^3 t^{-3/2} W^{-1/2} K^2(t) \leqslant C\varepsilon^3 t^{-5/4} W^{-1} K(t) \leqslant C|g_1|.\end{aligned}$$

So we arrive at the estimate $\|R_1\|_{\mathbf{L}^\infty} \leqslant C|g_1|$.

Now consider the asymptotic behaviour of the first term R_0 on the right-hand side of (7). As above, in view of Lemma 2.7 we have

$$\|\partial_\eta v^3\|_{\mathbf{L}^2} \leqslant C\|\langle\widetilde{\eta}\rangle^{-2} v^2\|_{\mathbf{L}^\infty}\|\partial_\eta v\|_{\mathbf{L}^2} \leqslant C\varepsilon^3 W^{-1} K(t).$$

Therefore, an application of Lemma 2.3 yields

$$\mathcal{V}_h^* v^3 = \frac{\sqrt{8i}}{\sqrt{3}}\langle\widetilde{\xi}\rangle^{-2} v^3 + O(\varepsilon^3|\widetilde{\xi}|\langle\widetilde{\xi}\rangle^{-2}) + O(\varepsilon^3 t^{-1/4} W^{-1} K(t)).$$

Thus, it is true that

$$R_0 = \frac{i}{2t\langle\xi\rangle^2}\mathcal{V}_h^* v^3 = \frac{i\sqrt{2i}}{t\sqrt{3}\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^2} v^3(t,\xi) + O(\varepsilon^3(|\widetilde{\xi}|\langle\widetilde{\xi}\rangle^{-2} + t^{-1/4} W^{-1} K(t))).$$

Lemma 2.2 implies that $v(t,\xi) = \dfrac{1}{\sqrt{2i}}\,\widehat{\varphi}(t,\xi) + O(\varepsilon t^{-1/4} K(t))$. Hence

$$R_0 = \frac{1}{2t\sqrt{3}\langle\widetilde{\xi}\rangle^2}\,\widehat{\varphi}^3(t,\xi) + O(\varepsilon^3(|\widetilde{\xi}|\langle\widetilde{\xi}\rangle^{-2} + t^{-1/4} W^{-1} K(t))).$$

We also have the estimate

$$|y - \widehat{\varphi}| \leqslant \|\mathcal{Q}^* M^2 H_3 v^3\|_{\mathbf{L}^\infty} \leqslant Ct^{-1/2}\|\langle\widetilde{\eta}\rangle^{3\gamma-2}\|_{\mathbf{L}^1}\|\langle\widetilde{\eta}\rangle^{-\gamma} v\|^3_{\mathbf{L}^\infty} \leqslant C\varepsilon^3 W^{-3/2},$$

which yields (6).

Lemma 3.3 is proved.

Now consider a Cauchy problem for an ordinary differential equation depending on a parameter $\xi \in \mathbb{R}$:

$$\begin{cases} \partial_t y(t,\xi) = -\dfrac{1}{2t\sqrt{3}\langle\widetilde{\xi}\rangle^2}\, y^3(t,\xi) + g(t,\xi), \quad t \geqslant 1, \\ y(1,\xi) = y_1(\xi), \end{cases} \tag{8}$$

where $g(t,\xi) = O(\varepsilon^5 t^{-1} W^{-5/2}(t) + \varepsilon^3 t^{\gamma-5/4} + \varepsilon^3 t^{-1} |\widetilde{\xi}| \langle\widetilde{\xi}\rangle^{-2} W^{-3/2})$.

Lemma 3.4. *Assume that the initial perturbation y_1 satisfies the conditions*

$$\varepsilon \leqslant |y_1(\xi)| \leqslant C\varepsilon \quad \textit{and} \quad |\arg y_1(\xi)| < \frac{\pi}{8} \quad \textit{for } |\xi| \leqslant 1,$$

where $\varepsilon > 0$ is sufficiently small. Then the solution of Cauchy problem (8) has the estimates

$$|y(t)| \leqslant C\varepsilon\Psi^{-1/2} \quad \textit{and} \quad |\arg y(t)| \leqslant C\Psi^{1/2}$$

for all $t \in [1,T]$ and $\xi \in \mathbb{R}$, where $\Psi = 1 + \varepsilon^2 \ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2})$.

Proof. In the case $|\xi| > 1$ we have $\langle\widetilde{\xi}\rangle^{-2} \leqslant t^{-1}$; thus, we derive from (8) that $y_t = O(\varepsilon^3 t^{-2}) + O(\varepsilon^5 t^{-1} W^{-5/2})$. Integrating with respect to time, we infer from this relation that $|y(t)| \leqslant \varepsilon + \varepsilon^2 \leqslant 2\varepsilon\Psi^{-1/2}$. Now we consider the case $|\xi| \leqslant 1$. We make the substitution $y = re^{i\omega}$, where $r > 0$ and ω is a real function. Taking the real and imaginary parts, from (8) we deduce that

$$r_t = -\frac{1}{2t\sqrt{3}\langle\widetilde{\xi}\rangle^2}\, r^3 \cos 2\omega + \operatorname{Re}(ge^{-i\omega}) \tag{9}$$

and

$$\omega_t = -\frac{1}{2t\sqrt{3}\langle\widetilde{\xi}\rangle^2}\, r^2 \sin 2\omega + \operatorname{Im}(gr^{-1}e^{-i\omega}) \tag{10}$$

with the initial conditions $r(1,\xi) = |y_1(\xi)|$ and $\omega(1,\xi) = \arg y_1(\xi)$.

Now we prove the inequalities

$$\frac{1}{2}\Psi < \frac{|y_1(\xi)|^2}{r^2(t)} < 2\Psi \quad \text{and} \quad |\omega(t,\xi)| < \frac{\pi}{8} \tag{11}$$

for all $t \in [1,T]$ and $|\xi| \leqslant 1$. Reasoning by contradiction we assume that there is maximum time $\widetilde{T} \in (1,T]$ such that

$$\frac{1}{2}\Psi \leqslant \frac{|y_1(\xi)|^2}{r^2(t)} \leqslant 2\Psi \quad \text{and} \quad |\omega(t,\xi)| \leqslant \frac{\pi}{8} \tag{12}$$

for all $t \in [1,\widetilde{T}]$ and $|\xi| \leqslant 1$. Dividing (9) by r^3 we obtain

$$\partial_t r^{-2} = \frac{\cos 2\omega}{t\sqrt{3}\,\langle\widetilde{\xi}\rangle^2} - 2\operatorname{Re}(ge^{-i\omega}),$$

which implies the inequalities

$$\frac{1}{t\sqrt{6}\,\langle\widetilde{\xi}\rangle^2} - 2r^{-3}|g| \leqslant \partial_t r^{-2} \leqslant \frac{1}{t\sqrt{3}\,\langle\widetilde{\xi}\rangle^2} + 2r^{-3}|g|.$$

Integrating with respect to time we find that

$$1+\frac{\varepsilon^2}{\sqrt{6}}\ln\frac{t\langle\xi\rangle^2}{\langle\widetilde{\xi}\rangle^2}-2\varepsilon^2\int_1^t r^{-3}|g|\,d\tau \leqslant \frac{|y_1(\xi)|^2}{r^2(t)} \leqslant 1+\frac{\varepsilon^2}{\sqrt{3}}\ln\frac{t\langle\xi\rangle^2}{\langle\widetilde{\xi}\rangle^2}+2\varepsilon^2\int_1^t r^{-3}|g|\,d\tau,$$

since

$$\int_1^t \langle\sqrt{\tau}\,\xi\rangle^{-2}\,\frac{d\tau}{\tau}=\int_{\xi^2}^{\widetilde{\xi}^2}\frac{dz}{(1+z)z}=\ln\frac{t\langle\xi\rangle^2}{\langle\widetilde{\xi}\rangle^2}.$$

Using (12) and the assumptions concerning g we infer that

$$\begin{aligned}\int_1^t r^{-3}|g|\,d\tau &\leqslant C\int_1^t (1+\varepsilon^2\ln(\tau\langle\widetilde{\xi}\rangle^{-2}))^{3/2}(\varepsilon^5\tau^{-1}W^{-5/2}(\tau)\\ &\qquad + \varepsilon^3\tau^{\gamma-5/4}+\varepsilon^3\tau^{-1}|\widetilde{\xi}|\langle\widetilde{\xi}\rangle^{-2}W^{-3/2}(\tau))\,d\tau\\ &\leqslant C\Psi^{-1/2}\end{aligned}$$

for all $t\in[1,\widetilde{T}]$ and $|\xi|\leqslant 1$. Thus, we have

$$\int_1^t r^{-3}|g|\,d\tau \leqslant C\Psi^{1/2},$$

which yields the estimate $|y_1(\xi)|^2/r^2(t) < 2\Psi$. In a similar way we find the lower estimate

$$\frac{|y_1(\xi)|^2}{r^2(t)} \geqslant 1+\frac{\varepsilon^2}{\sqrt{6}}\ln\frac{t\langle\xi\rangle^2}{\langle\widetilde{\xi}\rangle^2}-2\varepsilon^2\int_1^t r^{-3}|g|\,d\tau > \frac{1}{3}\Psi$$

for all $t\in[1,\widetilde{T}]$ and $|\xi|\leqslant 1$. Hence (11) holds for all $t\in[1,\widetilde{T}]$ and $|\xi|\leqslant 1$. Multiplying both sides of (10) by ω we arrive at the equality

$$\partial_t\omega^2=-\frac{1}{t\sqrt{3}\,\langle\widetilde{\xi}\rangle^2}r^2\omega\sin 2\omega+2\omega r^{-1}\operatorname{Im}(ge^{-i\omega})\leqslant \omega^2\,\partial_t\ln r^2+Cr^{-1}|g|,$$

since $2\omega\sin 2\omega\geqslant\omega^2$ for $|\omega|\leqslant\pi/8$. Integrating with respect to time, we derive from this that

$$\omega^2(t)\leqslant r^2\left(\varepsilon^{-2}\omega^2(0)+C\int_1^t r^{-3}|g|\,d\tau\right)\leqslant\Psi^{-1}(\omega^2(0)+C\varepsilon^2\Psi^{1/2})$$

for all $t\in[1,\widetilde{T}]$, $|\xi|\leqslant 1$. Consequently, $|\omega(t,\xi)|<\pi/8$ for all $t\in[1,\widetilde{T}]$ and $|\xi|\leqslant 1$. This contradiction proves the estimates in the lemma for all $t\in[1,T]$.

Lemma 3.4 is proved.

The next lemma establishes *a priori* estimates for solutions.

Lemma 3.5. *Assume that* $\|\widehat{\varphi}\|_{\mathbf{X}_T}\leqslant C\varepsilon$. *Then*

$$\|\widehat{\varphi}\|_{\mathbf{X}_T}<C\varepsilon.$$

Proof. It follows from (5) that

$$\frac{d}{dt}\|\langle\xi\rangle^5\widehat{\varphi}(t)\|_{\mathbf{L}^2} \leqslant C\varepsilon^2 t^{-1}\|\langle\xi\rangle^3\widehat{\varphi}(t)\|_{\mathbf{L}^2},$$

which implies the estimate $\|\langle\xi\rangle^5\widehat{\varphi}(t)\|_{\mathbf{L}^2} < C\varepsilon t^{\gamma}$ by means of integration with respect to time. Using Lemma 3.4 and Lemma 3.3 we obtain

$$\begin{aligned}|\widehat{\varphi}(t,\xi)| &\leqslant |y(t,\xi)| + C\varepsilon^3 W^{-3/2}\\ &\leqslant C\varepsilon(1+\varepsilon^2\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2}))^{-1/2} + C\varepsilon^3 W^{-3/2} < C\varepsilon\end{aligned}$$

and

$$\langle\widetilde{\xi}\rangle^{-\gamma}|\widehat{\varphi}(t,\xi)| \leqslant C\varepsilon\langle\widetilde{\xi}\rangle^{-\gamma}(1+\varepsilon^2\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2}))^{-1/2} + C\varepsilon^3 W^{-3/2} < C\varepsilon W^{-1/2}$$

for all $t\in[1,T]$. Using Lemma 3.2 we also infer the estimate $\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2} \leqslant C\varepsilon K(t)$ for the derivative. Thus, we have $\|\widehat{\varphi}\|_{\mathbf{X}_T} < C\varepsilon$.

The lemma is proved.

§ 4. Proof of Theorem 1.1

The time-global existence of a solution

$$u \in \mathbf{C}([0,\infty);\mathbf{H}^5\cap\mathbf{H}^{0,1})\cap\mathbf{C}^1((0,\infty);\mathbf{H}^3)$$

of the Cauchy problem (1) satisfying the estimate $\|\widehat{\varphi}\|_{\mathbf{X}_T} < C\varepsilon$ is a consequence of Lemma 3.5 and the local existence guaranteed by Theorem 3.1. Hence it only remains to prove the asymptotic formula (4). From the formulae of the factorization method, Lemma 2.2, and Lemma 3.3 we see that

$$\begin{aligned}u(t) = \mathcal{D}_t\mathcal{B}M\mathcal{Q}\widehat{\varphi} &= \mathcal{D}_t\mathcal{B}M\frac{\widehat{\varphi}}{\sqrt{i\Lambda''}} + O(t^{-3/4}\|\partial_\xi\widehat{\varphi}\|_{\mathbf{L}^2})\\ &= \mathcal{D}_t\mathcal{B}M\frac{re^{i\omega}}{\sqrt{i\Lambda''}} + O(t^{-1/2}(\ln t)^{-3/2})\end{aligned}$$

as $t\to\infty$. Like in the proof of Lemma 3.4, we obtain

$$r^{-2}(t,\xi) = |\widehat{u_0}(\xi)|^{-2} + \frac{1}{\sqrt{3}}\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2}) + O(\Psi^{1/2})$$

and $|\omega(t,\xi)| \leqslant C\Psi^{-1/4}$, where $\Psi = 1+\varepsilon^2\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2})$. Consequently,

$$r(t,\xi) = |\widehat{u_0}(\xi)|\biggl(1+\frac{|\widehat{u_0}(\xi)|^2}{\sqrt{3}}\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2})\biggr)^{-1/2} + O(\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2})^{-3/4}).$$

Thus, we have

$$\begin{aligned}u(t) = \mathcal{D}_t\mathcal{B}M|\widehat{u_0}(\xi)|&\biggl(1+\frac{|\widehat{u_0}(\xi)|^2}{\sqrt{3}}\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2})\biggr)^{-1/2}\\ &+ O(t^{-1/2}(\ln(t\langle\xi\rangle^2\langle\widetilde{\xi}\rangle^{-2}))^{-3/4}),\end{aligned}$$

which yields the asymptotics (4).

Theorem 1.1 is proved.

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