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Manifolds of isospectral arrow matrices

A. A. Ayzenberg and V. M. Buchstaber

Abstract. An arrow matrix is a matrix with zeros outside the main diagonal, the first row and the first column. We consider the space $M_{\mathrm{St}_n,\lambda}$ of Hermitian arrow $(n+1)\times(n+1)$-matrices with fixed simple spectrum λ. We prove that this space is a smooth $2n$-manifold with a locally standard torus action: we describe the topology and combinatorics of its orbit space. If $n\geqslant 3$, the orbit space $M_{\mathrm{St}_n,\lambda}/T^n$ is not a polytope, hence $M_{\mathrm{St}_n,\lambda}$ is not a quasitoric manifold. However, there is an action of a semidirect product $T^n\rtimes\Sigma_n$ on $M_{\mathrm{St}_n,\lambda}$, and the orbit space of this action is a certain simple polytope $\mathscr{B}^n$ obtained from the cube by cutting off codimension-2 faces. In the case $n=3$, the space $M_{\mathrm{St}_3,\lambda}/T^3$ is a solid torus with boundary subdivided into hexagons in a regular way. This description allows us to compute the cohomology ring and equivariant cohomology ring of the 6-dimensional manifold $M_{\mathrm{St}_3,\lambda}$ and another manifold, its twin.

Bibliography: 32 titles.

Keywords: sparse matrix, group action, moment map, fundamental domain, codimension-2 face cuts.

§ 1. Introduction

Spaces of isospectral Hermitian or symmetric matrices are at the interface between several areas of mathematics, including symplectic geometry, representation theory, toric topology and applied mathematics.

Let M_{n+1} be the space of all Hermitian matrices of size $n+1$ and let $M_\lambda\subset M_{n+1}$ denote the subspace of all Hermitian matrices with fixed simple spectrum $\lambda=\{\lambda_0,\lambda_1,\dots,\lambda_n\}$ (we assume that $\lambda_0<\lambda_1<\dots<\lambda_n$). The unitary group $U(n+1)$ acts on M_{n+1} by conjugation. Multiplying Hermitian matrices by $\sqrt{-1}$, we obtain skew Hermitian matrices, hence this action can be identified with the adjoint action of $U(n+1)$ on its tangent Lie algebra. For a simple spectrum λ, the subset M_λ is the principal orbit of this action. The manifold M_λ is diffeomorphic to the variety of full complex flags $\mathrm{Fl}_{n+1}=U(n+1)/T^{n+1}$. Here

$$T^{n+1}=\{D=\mathrm{diag}(t_0,\dots,t_n)\mid t_i\in\mathbb{C},\ |t_i|=1\}$$

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is the maximal compact torus of diagonal unitary matrices. The torus acts by matrix conjugation $A \mapsto DAD^{-1}$ on both matrix spaces M_{n+1} and M_λ. In the coordinate form we have

$$(a_{ij})_{i,j=0,\dots,n} \mapsto (t_i t_j^{-1} a_{ij})_{i,j=0,\dots,n}. \tag{1.1}$$

It is reasonable to look for subspaces in the flag manifold $M_\lambda \cong \mathrm{Fl}_{n+1}$ that are preserved by the torus action. Formula (1.1) implies that the torus action preserves zeros at the given positions. Hence we can study the spaces of Hermitian matrices with the given spectrum and zeros at prescribed positions.

The classical example is the space $M_{\mathbb{I}_n,\lambda}$ of isospectral tridiagonal Hermitian matrices. This space was investigated in [32], [10] and [16]. Tomei [32] introduced the real analogue of this space: he proved that this space is a smooth manifold, and its diffeomorphism type is independent of the spectrum. Bloch, Flaschka and Ratiu [10] studied the Hermitian case and demonstrated its connection with the toric variety of type A_n. The general theory developed in the seminal work of Davis and Januszkiewicz [16] allows one to describe the cohomology ring and the T^n-equivariant cohomology ring of $M_{\mathbb{I}_n,\lambda}$.

Instead of tridiagonal matrices one can consider staircase Hermitian matrices (also known as generalized Hessenberg matrices); see [25] and [17]. In such matrices nonzero elements are allowed only in the vicinity of the diagonal, which is encoded by the so-called Hessenberg function. The spaces of Hermitian staircase matrices can be studied similarly to the tridiagonal case: the properties of the generalized Toda flow can be used to prove the smoothness of these spaces. We collected the results on such 'matrix Hessenberg manifolds' in [7].

Note that the torus acting on matrix Hessenberg manifolds may have dimension less than half the dimension of the manifolds; in that case the theory of $(2n,k)$-manifolds is applicable; see [14].

Another way to generalize tridiagonal matrices is to allow two additional non-zero entries at the top-right and bottom-left corner of the matrix. Such matrices are called *periodic tridiagonal matrices*. They appear in the study of the discrete Schrödinger operator in mathematical physics (see [23] and [21]). The isospectral space of such matrices is investigated in the forthcoming paper [5].

In this paper we study the isospectral space $M_{\mathrm{St}_n,\lambda}$ of matrices that have zeros outside the diagonal, the first row and the first column. Matrices of this form will be called *arrow matrices*. We are indebted to Tadeusz Januszkiewicz[1] for telling us about this wonderful object.

In this paper we prove that $M_{\mathrm{St}_n,\lambda}$ is smooth and that its diffeomorphism type is independent of λ; see § 4. The action of the torus $T = T^n$ on $M_{\mathrm{St}_n,\lambda}$ is locally standard, so the orbit space $Q_n = M_{\mathrm{St}_n,\lambda}/T$ is a manifold with corners. In § 4 we describe the topology of the orbit space. We introduce a cubical complex Sq_n that is the union of the cubical faces of an n-dimensional permutohedron, and we prove that the orbit space Q_n is homotopy equivalent to Sq_{n-1}. It follows that for $n \geqslant 3$ the orbit space is not a simple polytope. The combinatorial face structure of Q_n is described in § 6 using the general notion of a cluster-permutohedron. The family of cluster-permutohedra contains two known examples: a permutohedron and a cyclopermutohedron of Panina; they provide interesting examples of

[1]Private communication.

partially ordered sets, which, as far as we know, have not yet been considered in combinatorial geometry.

In general, there is a natural permutation action of the symmetric group Σ_n on the manifold $M_{\mathrm{St}_n,\lambda}$ as well as on the orbit space Q_n. We show that the fundamental domain of the Σ_n-action on Q_n is diffeomorphic to a certain simple polytope, denoted by $\mathscr{B}_n$. In § 5 we describe the combinatorics of this polytope. We show that $\mathscr{B}_n$ can be obtained from a cube by cutting a sequence of codimension-2 faces, and hence $\mathscr{B}_n$ has a convex Delzant realization. We end § 5 with an observation relating our construction of the manifold $M_{\mathrm{St}_n,\lambda}$ to the general construction of symplectic implosion introduced in symplectic geometry (see [20]).

This description of the polytope $\mathscr{B}_n$ allows us to reconstruct the orbit space Q_n by stacking together $n!$ copies of this polytope. In the future we hope that this description of the orbit space Q_n will allow one to construct effective diagonalization algorithms for arrow-shaped matrices.

We are especially interested in arrow matrices of size 4×4, that is, in the case $n = 3$. In this case the orbit space $Q_3 = M_{\mathrm{St}_3,\lambda}$ is a solid torus whose boundary is subdivided into hexagons in a regular way. We have learned about this fact from Januszkiewicz. The cohomology and equivariant cohomology rings of the space $M_{\mathrm{St}_3,\lambda}$ itself can be computed using the theory developed by the first author in [1]–[4] and [9]. In general, this theory allows one to describe the cohomology and equivariant cohomology rings of manifolds with locally standard torus action whose orbit spaces have only acyclic proper faces. Since every facet of Q_3 is a hexagon, we are in a position to apply this theory to $M_{\mathrm{St}_3,\lambda}$. In § 7 we recall the notions of the Stanley-Reisner ring, h-, h'-, and h''-numbers of simplicial complexes, Novik-Swartz theory and related topological results in [16] and [22]. Theorem 7.1 is based on these results; it describes the homological structure of the manifold $M_{\mathrm{St}_3,\lambda}$. The polytope $\mathscr{B}_3$ is a cube with cut-off skew edges; this polytope is important in toric topology. Historically, it has been the starting point of the investigations of higher Massey products in the cohomology of moment-angle manifolds (a rapidly developing area of current research; see [12] and [19]).

In the last section, § 8, we introduce a manifold X_n, whose properties are similar to $M_{\mathrm{St}_n,\lambda}$. This manifold also carries a half-dimensional torus action and its orbit space is isomorphic to Q_n. However, X_n is more interesting from the topological point of view. For $n = 3$ we describe the cohomology ring and show that the first Pontryagin class of X_3 is nonzero.

§ 2. Spaces of sparse isospectral matrices

The action of T^{n+1} on the isospectral space M_λ is not effective, since the scalar matrices act trivially. Hence there is an effective action of $T = T^n \cong T^{n+1}/\Delta(T^1)$. Fixed points of the torus action on M_λ are diagonal matrices with the spectrum λ, that is, matrices of the form $\operatorname{diag}(\lambda_{\sigma(0)}, \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$ for all possible permutations $\sigma \in \Sigma_{n+1}$.

The action is Hamiltonian. Indeed, M_λ can be identified with the orbit of the (co)adjoint action of $U(n+1)$ on its (co)tangent algebra. This orbit possesses the Kostant-Kirillov symplectic form, and the action of $U(n+1)$ (hence of T^{n+1})

on this orbit is Hamiltonian. The moment map for the torus action is given by

$$\mu\colon M_\lambda \to \mathbb{R}^{n+1}, \qquad A = (a_{i,j}) \mapsto (a_{0,0}, a_{1,1}, \dots, a_{n,n})$$

(the image of this map lies in the hyperplane $\{\sum_{i=0}^n a_{i,i} = \sum \lambda_i = \text{const}\} \cong \mathbb{R}^n$). The Atiyah-Guillemin-Sternberg theorem tells us that the image of the moment map is the permutohedron

$$\mathrm{Pe}^n_\lambda = \mathrm{conv}\{(\lambda_{\sigma(0)}, \lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)}) \mid \sigma \in \Sigma_{n+1}\}.$$

We call it the *Schur-Horn permutohedron*, since the description of the diagonals of all Hermitian matrices with a given spectrum is the classical result due to Schur and Horn.

Construction 2.1. Let $\Gamma = (V, E)$ be a simple graph (that is, the graph with no loops and multiple edges) on the vertex set $V = \{0, 1, \dots, n\}$. Consider the vector subspace of Hermitian matrices:

$$M_\Gamma = \{A \in M_{n+1} \mid a_{ij} = 0 \text{ if } \{i, j\} \notin E\}.$$

As noted in § 1, the torus action preserves the set M_Γ for any Γ. Set

$$M_{\Gamma,\lambda} = M_\Gamma \cap M_\lambda.$$

The action of T^n can be restricted to $M_{\Gamma,\lambda}$. The space $M_{\Gamma,\lambda}$ is called the *space of isospectral sparse matrices of type* Γ. Then we have

$$\dim M_{\Gamma,\lambda} = 2|E|. \tag{2.1}$$

Example 2.1. If Γ is a complete graph on the set $\{0, \dots, n\}$, then $M_{\Gamma,\lambda} = M_\lambda \cong \mathrm{Fl}_{n+1}$.

Remark 2.1. Without loss of generality only connected graphs can be considered. Let $\Gamma_1, \dots, \Gamma_k$ be the connected components of a graph Γ with vertex sets $V_1, \dots, V_k \subset \{0, \dots, n\}$, respectively. Let Ω be the set of all possible partitions of the set $\{\lambda_0, \dots, \lambda_n\}$ into disjoint subsets S_i of cardinalities $|A_i|$, $i = 1, \dots, k$. Then $M_{\Gamma,\lambda} = \bigsqcup_\Omega \prod_{i=1}^k M_{\Gamma_i, S_i}$. We discuss the underlying combinatorial structures in detail in § 6.

Problem 2.1. Describe all graphs Γ such that the subspace $M_{\Gamma,\lambda}$ is a smooth manifold, whose diffeomorphism type is independent of the simple spectrum λ.

Remark 2.2. The Kostant-Kirillov form can be restricted to $M_{\Gamma,\lambda}$; however this restriction need not be symplectic even when $M_{\Gamma,\lambda}$ is smooth. Therefore, the Atiyah-Guillemin-Sternberg theorem is no longer applicable in the general case. Nevertheless, there is a map $\mu\colon M_{\Gamma,\lambda} \to \mathbb{R}^{n+1}$ taking a matrix to its diagonal. This map is constant on each torus orbit, hence there is an induced map $\widetilde{\mu}\colon M_{\Gamma,\lambda}/T^n \to \mathbb{R}^{n+1}$.

§ 3. Tree matrices

Let Γ be a tree on the vertex set $\{0,1,\dots,n\}$. In this case elements of M_Γ are called *tree matrices*. The $2n$-dimensional space $M_{\Gamma,\lambda}$ carries an effective action of a compact n-torus. This fact makes tree matrices an important object: they produce natural examples of half-dimensional torus actions. The moment map

$$\mu\colon M_{\Gamma,\lambda}\to H\subset\mathbb{R}^{n+1},\qquad H=\Bigl\{\sum a_{i,i}=\text{const}\Bigr\},$$

induces a map of the n-dimensional orbit space $M_{\Gamma,\lambda}/T$ to the Schur-Horn permutohedron $\mathrm{Pe}^n_\lambda\subset H\cong\mathbb{R}^n$. All vertices of Pe^n_λ belong to $\mu(M_{\Gamma,\lambda})$. In what follows it will be convenient to encode tree matrices in terms of labelled trees.

Definition 3.1. A *labelled tree* Δ is a triple $\Delta=(\Gamma,a,b)$, where $\Gamma=(V,E)$ is a tree, $a\colon V\to\mathbb{R}$ and $b\colon E\to\mathbb{C}$.

In other words, a labelled tree is a tree with a real number a_i assigned to each vertex $i\in V$, and a complex number b_e assigned to each edge $e\in E$. A labelled tree determines a Hermitian matrix A_Δ as follows. The elements of A_Δ are:

$$(A_\Delta)_{i,j}=\begin{cases}a_i & \text{if } i=j,\\ b_e & \text{if } i<j \text{ and } e=\{i,j\}\in E,\\ \overline{b}_e & \text{if } i>j \text{ and } e=\{i,j\}\in E,\\ 0 & \text{if } i\neq j \text{ and } \{i,j\}\notin E.\end{cases}$$

Moreover, if a vertex k is fixed in a labelled tree, then we call it a *rooted labelled tree* with *root* k.

Example 3.1. Let $\mathbb{I}_n$ denote the path graph with edges $\{0,1\}$, $\{1,2\}$, …, $\{n-1,n\}$. Then $M_{\mathbb{I}_n,\lambda}$ is the space of tridiagonal isospectral Hermitian matrices. A classical result (see [32] and [10]) states that $M_{\mathbb{I}_n,\lambda}$ is a smooth manifold, and its diffeomorphism type is independent of λ. It follows from a result of Tomei that the orbit space $M_{\mathbb{I}_n,\lambda}/T^n$ is diffeomorphic to an n-dimensional permutohedron as a manifold with corners.

The moment map $\mu\colon \mathrm{Pe}^n\cong M_{\Gamma,\lambda}/T^n\to\mathrm{Pe}^n_\lambda$ is not the isomorphism of permotohedra. This map determines a bijection between the vertices of permutohedra, however this map is neither injective nor surjective in the interior of the permutohedron. For $n=2$ the image of the moment map is shown in Figure 1 (this figure is justified in Proposition 4.1 below).

Example 3.2. Let St_n denote the star graph with edges $\{0,1\}$, $\{0,2\}$, …, $\{0,n\}$. In this case matrices in M_{St_n} have the form

$$A_\Delta=\begin{pmatrix}a_0 & b_1 & \dots & b_n\\ \overline{b}_1 & a_1 & 0 & 0\\ \vdots & \vdots & \ddots & \vdots\\ \overline{b}_n & 0 & \dots & a_n\end{pmatrix}. \tag{3.1}$$

Such matrices are called *arrow matrices*.

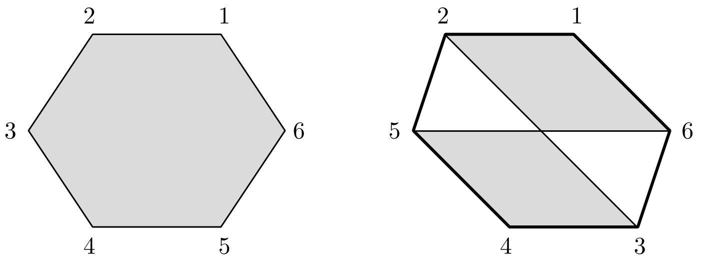


Figure 1. The image of the moment map for the tridiagonal (3×3)-matrices.

We formulate several technical statements about general tree matrices. First, there is a natural notion of a tree fraction, which generalizes continued fractions.

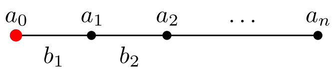
Definition 3.2. Let $\Delta = (\Gamma = (V, E), a, b)$ be a rooted labelled tree with root $k \in V$. Define the tree fraction $Q(\Delta, k)$ associated with (Δ, k) by recursion.

1. Let Δ be a labelled tree with root k and at least one more vertex. By deleting k from Δ the tree breaks down into s connected components, where each connected component is a rooted labelled tree Δ_i, whose root k_i is a descendant of k. Let $b_1, \dots, b_s \in \mathbb{C}$ be the labels on the edges of Δ connecting k with its descendants. Set

$$Q(\Delta, k) = a_k - \sum_{i=1}^{s} \frac{|b_i|^2}{Q(\Delta_i, k_i)}.$$

2. If Γ has a single vertex k labelled by $a_k \in \mathbb{R}$, then we set $Q(\Delta, k) = a_k$.

In particular, for a labelled path graph, rooted at the endpoint,



the tree fraction $Q(\Delta, k)$ is the continued fraction of the form

$$a_0 - \cfrac{|b_1|^2}{a_1 - \cfrac{|b_2|^2}{\ddots \, \cfrac{\vdots}{a_{n-1} - \cfrac{|b_n|^2}{a_n}}}}$$

Lemma 3.1. *Let Δ be a labelled tree and A_Δ be the corresponding Hermitian matrix. The diagonal elements of the inverse matrix are given by the tree fractions*:

$$(A_\Delta^{-1})_{k,k} = \frac{1}{Q(\Delta, k)}.$$

The proof is an exercise in linear algebra.

We say that S is a splitting of the tree Γ if S is a partition of the set of vertices into 1- and 2-element subsets, in which all 2-element subsets are edges of Γ. Let $S(\Gamma)$ be the set of all splittings of Γ. For $S \in S(\Gamma)$ set $\sigma(S) = (-1)^p$ where p is the number of edges in the splitting.

Lemma 3.2. *For a labelled tree* $\Delta = (\Gamma, a, b)$ *the following holds*:

$$\det A_\Delta = \sum_{S \in S(\Gamma)} \sigma(S) \prod_{i \text{ is a vertex of } S} a_i \prod_{e \text{ is an edge of } S} |b_e|^2.$$

Proof. Another exercise in linear algebra: expand the determinant with respect to the row corresponding to a leaf vertex of the tree.

Corollary 3.1. *For an arrow matrix* (*corresponding to the star graph*)

$$\det \begin{pmatrix} a_0 & b_1 & \dots & b_n \\ \bar{b}_1 & a_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{b}_n & 0 & \dots & a_n \end{pmatrix} = a_0 a_1 \cdots a_n - \sum_{i=1}^{n} |b_i|^2 a_1 \cdots \widehat{a_i} \cdots a_n.$$

The star graph is the only tree for which $\det(A_\Delta)$ *is at most quadratic in the variables* $|b_e|$.

§ 4. Space of isospectral arrow matrices

Let $\Delta = (\mathrm{St}_n, a, b)$ be a labelled star graph and A_Δ be the corresponding arrow matrix given by (3.1). Consider the isospectral space $M_{\mathrm{St}_n,\lambda} = \{A_\Delta \mid \operatorname{Spec} A_\Delta = \lambda\}$. We describe the image of the moment map of such matrices, that is, the set of all possible diagonals $(a_0, \dots, a_n) \in \mathbb{R}^{n+1}$. Since $a_0 = \sum_{i=0}^{n} \lambda_i - \sum_{i=1}^{n} a_i$, it is sufficient to describe all possible $(a_1, \dots, a_n) \in \mathbb{R}^n$.

Proposition 4.1. *Let* $I_j = [\lambda_{j-1}, \lambda_j] \subset \mathbb{R}$ *for* $j = 1, \dots, n$. *Then* $\mu(M_{\mathrm{St}_n,\lambda})$ *is the set*

$$\left\{ (a_0, a_1, \dots, a_n) \;\middle|\; (a_1, \dots, a_n) \in \mathscr{R}_n,\ a_0 = \sum_{i=0}^{n} \lambda_i - \sum_{i=1}^{n} a_i \right\},$$

where

$$\mathscr{R}_n = \bigcup_{\sigma \in \Sigma_n} I_{\sigma(1)} \times \dots \times I_{\sigma(n)} \subset \mathbb{R}^n$$

is the union of $n!$ *cubes of dimension* n.

Proof. First note that the permutation group Σ_n acts on the star graph Γ by permuting its rays. As a consequence, there is an action of Σ_n on the vector space M_{St_n}. The permutation action preserves the spectrum, hence there is an induced Σ_n-action on $M_{\mathrm{St}_n,\lambda}$. Therefore, we may assume that $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$.

Let us prove that under the conditions

$$\lambda_0 < a_1 < \lambda_1 < a_2 < \lambda_2 < \dots < a_n < \lambda_n \quad \text{and} \quad a_0 = \sum_{i=0}^{n} \lambda_i - \sum_{i=1}^{n} a_i$$

there exists an arrow matrix A_Δ with diagonal $(a_0, \dots, a_n)$ and eigenvalues $\lambda_0, \dots, \lambda_n$. This would imply that the interior of $\mathscr{R}_n$ lies in the image of the moment map. Consider the polynomials $P(\lambda) = \prod_{i=0}^{n} (\lambda - \lambda_i)$ and $Q(\lambda) = \prod_{i=1}^{n} (\lambda - a_i)$. Division by $Q(\lambda)$ yields

$$P(\lambda) = (\lambda - \alpha) Q(\lambda) + R(\lambda). \tag{4.1}$$

It is easy to check that $\alpha = \sum_{i=0}^{n} \lambda_i - \sum_{i=1}^{n} a_i = a_0$. Substituting all possible a_i into (4.1) we obtain

$$R(a_n) = P(a_n) < 0, \quad R(a_{n-1}) = P(a_{n-1}) > 0, \quad R(a_{n-2}) = P(a_{n-2}) < 0, \quad \dots .$$

For the rational function $P(\lambda)/Q(\lambda)$ we have the partial fraction expansion

$$\frac{P(\lambda)}{Q(\lambda)} = \lambda - a_0 + \frac{r_1}{\lambda - a_1} + \cdots + \frac{r_n}{\lambda - a_n},$$

where

$$r_i = \frac{R(a_i)}{\prod_{j \neq i}(a_i - a_j)} < 0 \quad \text{for all } i = 1, \dots, n.$$

Hence we can put $r_i = -|b_i|^2$ for some $b_i \in \mathbb{C}$. Consider the arrow matrix A_Δ of the form (3.1). According to Lemma 3.1, the top-left element of the matrix $(A_\Delta - \lambda E)^{-1}$ can be written as the tree fraction

$$(A_\Delta^{-1})_{0,0} = \frac{1}{a_0 - \lambda - \frac{|b_1|^2}{a_1 - \lambda} - \cdots - \frac{|b_n|^2}{a_n - \lambda}} = -\frac{1}{\frac{P}{Q}} = -\frac{\prod_{i=1}^{n}(\lambda - a_i)}{\prod_{i=0}^{n}(\lambda - \lambda_i)}.$$

This meromorphic function has poles at the points $\lambda_0, \dots, \lambda_n$. On the other hand the function $(A_\Delta - \lambda E)^{-1}$ is holomorphic outside the spectrum of A_Δ. Therefore, the λ_i are the eigenvalues of A_Δ.

We observe that a similar technique was applied by Moser [24] in the study of tridiagonal matrices: he attributed this technique to Stieltjes.

The converse reasoning shows that the eigenvalues and the diagonal elements $a_1, \dots, a_n$ alternate for any arrow matrix A_Δ. On the other hand, this fact also follows from Cauchy's interlace theorem (see, for example, [18]): *if A is a Hermitian matrix of size n and A' is its principal submatrix of size $n-1$, then the eigenvalues of A and A' interlace*. Application of this statement to the matrix A_Δ and its lower-right corner proves that points outside $\mathscr{R}_n$ do not lie in the image of the moment map.

Example 4.1. For $n = 2$ the image of μ consists of two squares, sitting inside a hexagon. Arrow matrices of size 3×3 coincide with tridiagonal matrices of this size up to permutation of rows and columns. This explains Example 3.1 and Figure 1.

Example 4.2. For $n = 3$ the image $\mu(M_{\mathrm{St}_n,\lambda})$ of the moment map is the union of six cubes shown in Figure 2. The contour shows the convex hull of these cubes, which is the Schur-Horn permutohedron Pe^3_λ.

Theorem 4.1. *The space $M_{\mathrm{St}_n,\lambda}$ is a smooth manifold of dimension $2n$. The space $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$ of isospectral real symmetric arrow matrices is a smooth manifold of dimension n. The action of T^n on $M_{\mathrm{St}_n,\lambda}$ and the action of $\mathbb{Z}_2^n$ on $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$ are locally standard.*

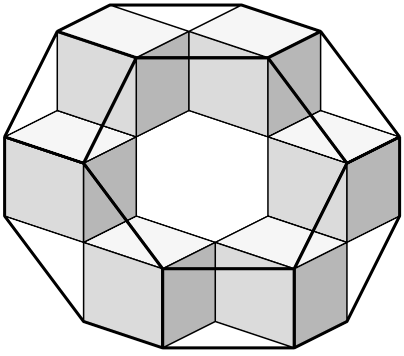


Figure 2. The moment map image for the star graph St_3.

We need a technical statement, well known in linear algebra.

Lemma 4.1. *Let d_i and ν_i, $i \in \{1, \dots, n\}$, be two sets of numbers such that all the $2n$ numbers are distinct. Then the square matrix*

$$B = \left(B_{i,j} = \frac{1}{d_i - \nu_j} \right)_{1 \leqslant i,j \leqslant n}$$

is invertible.

Proof. This follows from the uniqueness of the partial fraction expansion. Assume that $B \cdot (c_1, \dots, c_n)^\top = 0$ for a vector $(c_1, \dots, c_n) \in \mathbb{R}^n$. Then the rational function

$$R(d) = \frac{c_1}{d - \nu_1} + \dots + \frac{c_n}{d - \nu_n}$$

has roots $d_1, \dots, d_n$. Since the degree of the numerator of $R(d)$ is less than n, $R(d)$ is identically zero and $c_i = 0$ for all $i = 1, \dots, n$. Thus B is invertible. The lemma is proved.

Proof of Theorem 4.1. According to Corollary 3.1, the subspace $M_{\mathrm{St}_n,\lambda}$ is determined in the vector space M_{St_n} by the equations

$$P_j(\underline{a}, \underline{b}) = |b_1|^2 \prod_{i \neq 0,1} (a_i - \lambda_j) + \dots + |b_n|^2 \prod_{i \neq 0,n} (a_i - \lambda_j) - \prod_{i=0}^{n} (a_i - \lambda_j) = 0 \quad (4.2)$$

for $j \in [n] = \{1, \dots, n\}$, and

$$\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} \lambda_i = 0. \quad (4.3)$$

We use the vector notation

$$\frac{\partial P_j}{\partial b} = \left(\frac{\partial P_j}{\partial b_1}, \dots, \frac{\partial P_j}{\partial b_n} \right) \quad \text{and} \quad \frac{\partial P_j}{\partial a} = \left(\frac{\partial P_j}{\partial a_1}, \dots, \frac{\partial P_j}{\partial a_n} \right).$$

It is sufficient to show that the vectors $(\partial P_j/\partial b, \partial P_j/\partial a)$, $j \in [n]$, are linearly independent at all points in $M_{\mathrm{St}_n,\lambda}$. We have

$$\frac{\partial P_j}{\partial b_k} = \bar{b}_k \prod_{i\neq 0,k} (a_i - \lambda_j)$$

and

$$\begin{aligned}\frac{\partial P_j}{\partial b} &= \biggl(\bar{b}_1 \prod_{i\neq 0,1} (a_i - \lambda_j),\, \bar{b}_2 \prod_{i\neq 0,2} (a_i - \lambda_j),\, \dots,\, \bar{b}_n \prod_{i\neq 0,n} (a_i - \lambda_j)\biggr) \\ &= \prod_{i=1}^{n} (a_i - \lambda_j) \biggl(\frac{\bar{b}_1}{a_1 - \lambda_j}, \dots, \frac{\bar{b}_n}{a_n - \lambda_j}\biggr). \end{aligned} \tag{4.4}$$

First consider the general case: let all the a_i, $i \in [n]$, be distinct. Similarly to the proof of Proposition 4.1, we may assume that $a_1 < \dots < a_n$. Then

$$\lambda_0 < a_1 < \lambda_1 < a_2 < \dots < a_n < \lambda_n$$

and $b_i \neq 0$ for all $i \in [n]$.

It follows from (4.4) that the matrix formed by the vectors $\partial P_j/\partial b$, $j = 1, \dots, n$, has the form

$$\left(\frac{\partial P_j}{\partial b}\right) = \prod_{i=1}^{n} \bar{b}_i \prod_{i\neq j} (a_i - \lambda_j) \begin{pmatrix} \dfrac{1}{a_1 - \lambda_1} & \cdots & \dfrac{1}{a_n - \lambda_1} \\ \vdots & \ddots & \vdots \\ \dfrac{1}{a_1 - \lambda_n} & \cdots & \dfrac{1}{a_n - \lambda_n} \end{pmatrix}.$$

This matrix is nonsingular by Lemma 4.1. This proves the smoothness of $M_{\mathrm{St}_n,\lambda}$ at generic points.

Now we allow some of the points $\{a_i\}$ to collide. As noted in the proof of Proposition 4.1, only pairwise collisions can occur:

$$\dots < a_{j_1} = a_{j_1+1} < \dots < a_{j_2} = a_{j_2+1} < \dots < a_{j_s} = a_{j_s+1} < \cdots$$

and any pair of colliding diagonal entries determines an eigenvalue $\lambda_{j_l} = a_{j_l} = a_{j_l+1}$. All other eigenvalues still lie in the open intervals between a_i. We denote the set of all eigenvalues lying between the a_i by F, and denote the set of eigenvalues that come from colliding diagonal elements by D. We have $D = \{j_1, \dots, j_s\}$ and $F = \{0, \dots, n\} \setminus D$.

Let $A \in M_\Gamma$ be a matrix such that $a_{j_l} = a_{j_l+1} = \lambda_{j_l}$. A simple computation shows that $\frac{\partial P_{j_l}}{\partial b}(A) = 0$. Moreover, at a point A we have

$$\frac{\partial P_{j_l}}{\partial a_j}(A) = 0$$

if $j \neq j_l, j_l + 1$, and

$$\frac{\partial P_{j_l}}{\partial a_{j_l}}(A) = |b_{j_l+1}|^2 \prod_{i\neq j_l, j_l+1} (a_i - \lambda_{j_l}), \qquad \frac{\partial P_{j_l}}{\partial a_{j_l+1}}(A) = |b_{j_l}|^2 \prod_{i\neq j_l, j_l+1} (a_i - \lambda_{j_l}).$$

Reasoning similar to that used before shows that $-(|b_{j_l}|^2+|b_{j_l+1}|^2)$ is the coefficient of $\frac{1}{\lambda-a_{j_l}}$ in the partial fraction expansion of the (reducible) fraction $\frac{\prod_{i=0}^{n}(\lambda-\lambda_i)}{\prod_{i=1}^{n}(\lambda-a_i)}$. Hence $|b_{j_l}|^2+|b_{j_l+1}|^2\neq 0$, and one of the numbers $\frac{\partial P_{j_l}}{\partial a_{j_l}}(A)$, $\frac{\partial P_{j_l}}{\partial a_{j_l+1}}(A)$ is nonzero. Without loss of generality assume that $\frac{\partial P_{j_l}}{\partial a_{j_l}}(A)\neq 0$ for all $l=1,\dots,s$.

The rows of the rectangular matrix $\bigl(\frac{\partial P_j}{\partial b}(A)\bigr)$ that correspond to $j\in F$ (that is, $j\neq j_1,\dots,j_s$) are linearly independent by Lemma 4.1. In addition, the rows of the rectangular matrix $\bigl(\frac{\partial P_j}{\partial b}(A), \frac{\partial P_j}{\partial a}(A)\bigr)$ that correspond to $j\in D$ (that is, $j=j_l$ for some l) have zeros at all positions except for $\frac{\partial P_j}{\partial a_{j_l}}(A)$ and $\frac{\partial P_j}{\partial a_{j_l+1}}(A)$ and, moreover, $\frac{\partial P_j}{\partial a_{j_l}}(A)\neq 0$. It follows that the matrix $\bigl(\frac{\partial P_j}{\partial b}(A), \frac{\partial P_j}{\partial a}(A)\bigr)$ of the form

$$
\begin{array}{c}
\overbrace{}^{\partial P_j/\partial b}\;\overbrace{}^{\partial P_j/\partial a}\\
\begin{matrix} F\left\{\begin{matrix}\\ \\\end{matrix}\right. \\ D\left\{\begin{matrix}\\ \\ \\ \\\end{matrix}\right.\end{matrix}
\left(\begin{array}{ccc|cccccccccc}
* & \cdots & * & * & * & * & \cdots & * & * & \cdots & & & * \\
* & \cdots & * & * & * & * & \cdots & * & * & \cdots & * & * & * \\
\hline
0 & \cdots & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0
\end{array}\right)\\
\underbrace{}_{\{j_1,j_1+1\}}\quad\underbrace{}_{\{j_2,j_2+1\}}\quad\underbrace{}_{\{j_s,j_s+1\}}
\end{array}
$$

has the maximal rank. Therefore, $M_{\mathrm{St}_n,\lambda}$ is smooth at all points.

Since the action of T^n on $M_\Gamma=\mathbb{R}^{n+1}\times\mathbb{C}^n$ is locally standard and the smooth submanifold $M_{\mathrm{St}_n,\lambda}$ is preserved by this action, the induced action of T^n on $M_{\mathrm{St}_n,\lambda}$ is locally standard by the slice theorem.

Theorem 4.1 is proved.

For convenience we denote the orbit space $M_{\mathrm{St}_n,\lambda}/T^n$ by Q_n.

Proposition 4.2. *The map $\widetilde{\mu}\colon Q_n\to\mathscr{R}_n$ induced by μ is a homotopy equivalence.*

Proof. It is sufficient to prove that the preimage $\widetilde{\mu}^{-1}(a)$ is contractible for any point $a\in\mathscr{R}_n$. By Corollary 3.1 the condition on the spectrum yields the system of equations

$$
|b_1|^2\prod_{i\neq 0,1}(a_i-\lambda_j)+\dots+|b_1|^2\prod_{i\neq 0,n}(a_i-\lambda_j)=\prod_{i=0}^{n}(a_i-\lambda_j),\qquad j=0,1,\dots,n. \tag{4.5}
$$

Therefore, for fixed diagonal elements a_i and eigenvalues λ_i, the possible off-diagonal elements b_i lie on the intersection of Hermitian real quadrics of special type. By passing to the orbit space we simply forget the arguments of the numbers b_i. Setting $c_i=|b_i|^2$, we see that the parameters c_i satisfy a system of linear equations and conditions $c_i\geqslant 0$. Whenever this set is nonempty, it is a convex polytope, hence contractible. The proposition is proved.

Remark 4.1. For generic points $a \in \mathscr{R}_n$ the preimage $\widetilde{\mu}^{-1}(a)$ is a single point. In nongeneric points the preimage $\widetilde{\mu}^{-1}(a)$ is a cube. It can be seen that each pair of colliding values $a_{j_l} = a_{j_l+1}$ produces an interval in the preimage of $\widetilde{\mu}$. This interval is parametrized by the barycentric coordinates $|b_{j_l}|^2, |b_{j_l+1}|^2$ subject to the relation $|b_{j_l}|^2 + |b_{j_l+1}|^2 = \mathrm{const}$ (note that if $a_{j_l} = a_{j_l+1}$, then the expression $|b_{j_l}|^2 + |b_{j_l+1}|^2$ separates out in all the equations (4.5)).

The total preimage $\mu^{-1}(a)$ is therefore homeomorphic to the product of 3-spheres (a 3-sphere is the moment-angle manifold corresponding to the interval; see [13] for the general theory of moment-angle manifolds and complexes).

To determine the homotopy type of the orbit space $Q_n \simeq \mathscr{R}_n$, we need a description of the combinatorics of a permutohedron (details can be found in many sources, for instance, [32]). We fix a finite set $[n] = \{1, \dots, n\}$.

Construction 4.1. Let $S = (S_1, \dots, S_k)$ be an arbitrary linearly ordered partition of the set $[n] = \{1, \dots, n\}$ into nonempty subsets, that is $S_i \cap S_j = \varnothing$ for $i \neq j$, and $[n] = \bigcup_i S_i$. The set P of all such partitions is partially ordered: $S < S'$ if S is an order-preserving refinement of S'. It is known that P is isomorphic to the partially ordered set of faces of the permutohedron Pe^{n-1}. The polytope itself corresponds to the maximal partition $(S_1 = [n])$. Vertices correspond to ordered partitions of $[n]$ into one-element subsets $(\{s_1\}, \dots, \{s_n\})$, which are actually just the permutations $\tau \in \Sigma_n$, $\tau(i) = s_i$.

There is an edge between two vertices of a permutohedron if the corresponding permutations differ by a transposition of i and $i+1$. As a corollary, we obtain a standard fact that the 1-skeleton of the permutohedron is the Cayley graph of the group Σ_n with generators $(1,2), (2,3), \dots, (n-1,n)$.

Let F_S be the face of the permutohedron that corresponds to the ordered partition $S = (S_1, \dots, S_k)$. The polytope F_S is combinatorially isomorphic to the product of permutohedra $\mathrm{Pe}^{|S_1|-1} \times \dots \times \mathrm{Pe}^{|S_k|-1}$. If $|S_i| \leqslant 2$ for all i, then the corresponding face F_S is a product of intervals and points. We call such faces *cubical*. A cubical face of Pe^{n-1} has dimension at most $[n/2]$.

Remark 4.2. We make a remark on another important fact. Consider the standard convex realization of the permutohedron

$$\mathrm{Pe}^{n-1} = \mathrm{conv}\{(x_{\sigma(1)}, \dots, x_{\sigma(n)}) \mid \sigma \in \Sigma_n\},$$

where $x_1 < x_2 < \dots < x_n$. The vertex $(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ in this convex realization corresponds to the vertex $(\tau(1), \dots, \tau(n))$ in the combinatorial description, where $\tau = \sigma^{-1}$. For this reason, there is a duality in the notation: it is more convenient to encode vertices by permutations σ in geometric problems, and by $\tau = \sigma^{-1}$ in combinatorial ones.

Definition 4.1. Let Sq_{n-1} be the cell complex consisting of all cubical faces of an $(n-1)$-dimensional permutohedron.

The complex Sq_{n-1} is connected: every edge of Pe^{n-1} is a cubical face, hence lies in Sq_{n-1}.

Proposition 4.3. *The orbit space* $Q_n = M_{\Gamma,\lambda}/T \simeq \mathscr{R}_n$ *is homotopy equivalent to* Sq_{n-1}.

Proof. We construct an intermediate simplicial complex N which is homotopy equivalent to both Sq_{n-1} and $\mathscr{R}_n$.

Let N be the simplicial complex obtained from Sq_{n-1} by substituting each cubical face with vertices $\{v_1, \dots, v_{2^k}\}$ by a simplex on the same vertex set. The natural map $N \to \mathrm{Sq}_{n-1}$, which is identical on vertices and linear on each simplex, is a homotopy equivalence (see [6], where objects of this kind were studied in the framework of the theory of nerve complexes).

Proposition 4.1 implies that $\mathscr{R}_n = \mu(M_{\Gamma,\lambda})$ is the union of cubes $\bigcup_{\tau\in\Sigma} I^n_\tau$, where $I^n_\tau = I_{\tau(1)} \times \dots \times I_{\tau(n)}$, $I_j = [\lambda_{j-1}, \lambda_j]$. All cubes I^n_τ are convex. Hence the nerve N' of the covering $\bigcup_{\tau\in\Sigma} I^n_\tau = \mathscr{R}_n$ is homotopy equivalent to $\mathscr{R}_n$ by Alexandroff's nerve theorem.

Inspecting all coordinates, we see that the cubes $I^n_{\tau_1}$ and $I^n_{\tau_2}$ intersect if and only if, for every $i \in [n]$, we have $|\tau_1(i) - \tau_2(i)| \leqslant 1$. This means that

$$\tau_1\tau_2^{-1} = (i_1, i_1+1)(i_2, i_2+1)\cdots(i_s, i_s+1), \qquad i_{l+q} > i_l + 1, \tag{4.6}$$

is a product of independent transpositions interchanging neighbouring elements. Therefore, the vertices F_{τ_1} and F_{τ_2} of the permutohedron Pe^{n-1} lie in a cubical face F_S corresponding to the partition

$$S = \bigl(\{\tau(1)\}, \{\tau(2)\}, \dots, \{\tau(i_1), \tau(i_1+1)\}, \dots, \{\tau(i_s), \tau(i_s+1)\}, \dots, \{\tau(n)\}\bigr),$$

where $\tau = \tau_1$ or $\tau = \tau_2$. More generally, let a family of cubes $I^n_{\tau_i}$, $i = 1, \dots, l$, intersect jointly. Then each pair $i < j$ determines its own product of transpositions of the form (4.6). All permutations $s_{1,j} = \tau_j\tau_1^{-1}$ have order 2 and commute (since $s_{1,j}s_{1,i}^{-1} = \tau_j\tau_i^{-1}$ is of order 2 as well). Therefore, there exists a common partition into 1- and 2-element subsets, which governs all these permutations. Again, all the vertices F_{τ_i} lie in the same cubical face of the permutohedron. Thus we have $\mathrm{Sq}_{n-1} \simeq N = N' \simeq \mathscr{R}_n$.

The proposition is proved.

Example 4.3. For $n = 3$, the orbit space Q_3 and the image of the moment map are homotopy equivalent to Sq_2. This complex is just the union of the cubical faces of a hexagon, which is in fact its boundary: $Q \simeq S^1$. This can be seen from Figure 2.

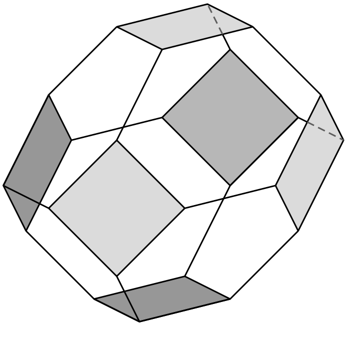


Figure 3. The cubical complex $\mathrm{Sq}_3 \simeq \bigvee_7 S^1$.

For $n = 4$, the orbit space Q_4 is homotopy equivalent to Sq_3. This complex is shown in Figure 3. The complex Sq_3 is homotopy equivalent to a wedge of seven circles.

§ 5. Permutation action and the fundamental polytope

Definition 5.1. Let P be a simple polytope, $\dim P = n$, and $G_1, \dots, G_s$ be a collection of its codimension-2 faces. Let $\mathscr{F}_i'$ and $\mathscr{F}_i''$ be the facets of P such that $\mathscr{F}_i' \cap \mathscr{F}_i'' = G_i$. If the sets $\{\mathscr{F}_i', \mathscr{F}_i''\}$, $i = 1, \dots, s$, are pairwise disjoint, we call $\{G_i\}$ a collection *in general position.*

Lemma 5.1. *If $\{G_1, \dots, G_s\}$ is a collection in general position, then a simple combinatorial polytope obtained by cutting off these faces successively does not depend on the order of cut-offs.*

Proof. The statement easily follows from the consideration of the dual simplicial sphere. Cutting off a face of codimension 2 corresponds to a stellar subdivision of an edge in a simplicial sphere. The general position implies that the subdivided edges do not intersect. The independence of the result follows from the definition of a stellar subdivision. The lemma is proved.

Let $I^n = I_1 \times \dots \times I_n$ be a cube, $I_j = [-1, 1]$. The facets are indexed by the set $W = \{\pm 1, \pm 2, \dots, \pm n\}$: the element δk, $\delta = \pm 1$, encodes the facet

$$\mathscr{F}_{\delta k} = I_1 \times \dots \times \overset{k}{\{\delta\}} \times \dots \times I_n.$$

The group $\mathbb{Z}_2^n$ acts on the set of facets: for an element $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{Z}_2^n$, $\varepsilon_i = \pm 1$, we have $\varepsilon \delta k = \varepsilon_k \delta k$.

Given a transposition of neighbouring elements $\sigma_i = (i, i+1) \in \Sigma_n$, $i = 1, \dots, n-1$, we denote by F_{σ_i} the codimension-2 face of I^n of the form

$$F_{\sigma_i} = I_1 \times \dots \times \overset{i}{\{1\}} \times \overset{i+1}{\{-1\}} \times \dots \times I_n = \mathscr{F}_i \cap \mathscr{F}_{-(i+1)}.$$

Definition 5.2. Let $\mathscr{B}^n$ denote a simple polytope obtained from I^n by cutting off all faces F_{σ_i}, $i = 1, \dots, n-1$.

Note that the faces F_{σ_i} may intersect, however the result of the cutting off is well defined according to Lemma 5.1.

Example 5.1. The polytope $\mathscr{B}^3$ is obtained from I^3 by cutting off two skew edges (see Figure 4). This polytope plays an important role in toric topology (see [13], § 4.9), although it emerged under different circumstances.

The facet of $\mathscr{B}^n$ obtained by cutting off F_{σ_i} will be denoted by $\mathscr{F}_{\sigma_i}$. The original facets of the cube remain facets of $\mathscr{B}^n$ and will be denoted by the same letters $\mathscr{F}_{\delta k}$. In total, the polytope $\mathscr{B}^n$ has $3n - 1$ facets.

Whenever a codimension-2 face is cut off a polytope P^n, the resulting facet F^{n-1}_{cut} has the combinatorial type of the product $I^1 \times G^{n-2}$, for some polytope G^{n-2}. This can be seen by considering the dual triangulation of a sphere: whenever an edge e is subdivided by a vertex v, the link of v is the suspension over the link of e in the original triangulation.

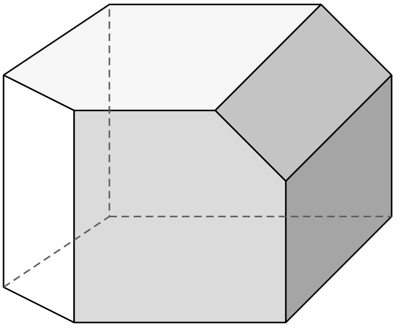


Figure 4. The polytope $\mathscr{B}^3$.

Every facet F_{σ_i} of $\mathscr{B}^n$ has the form $I^1 \times G_i$. Let $\alpha_i\colon F_{\sigma_i} \to F_{\sigma_i}$ be the antipodal map, that is, the map that is constant on G_i and antipodal on the interval I^1.

Construction 5.1. Note that the permutation group Σ_n acts on the star graph St_n. This action induces the action of Σ_n on the space of arrow matrices which preserves the spectrum. Therefore, there is an action of Σ_n on $M_{\mathrm{St}_n,\lambda}$, which commutes with the torus action up to a natural action of Σ_n on T^n. Let $\mathscr{N}$ denote the semidirect product $\mathscr{N}_n = T^n \rtimes \Sigma_n$, where Σ_n acts on T^n by permuting the coordinates. The group $\mathscr{N}_n$ arises naturally as the normalizer of the maximal torus T^n in the Lie group $U(n)$.

We have an action of $\mathscr{N}_n$ on $M_{\mathrm{St}_n,\lambda}$. The orbit space $Q^n = M_{\mathrm{St}_n,\lambda}/T^n$ carries the remaining action of $\mathscr{N}/T^n \cong \Sigma_n$. The image $\mathscr{R}_n = \bigcup_{\sigma\in\Sigma_n} I^n_\sigma$ of the moment map carries a natural action of Σ_n, which permutes the coordinates (in particular, this action permutes the cubes in the union). The map $\widetilde{\mu}\colon Q_n \to \mathscr{R}_n$ is Σ_n-equivariant, as can easily be seen from its definition.

Similar considerations hold true in the real case. The finite group $\mathscr{N}_n^{\mathbb{R}} = \mathbb{Z}_2^n \rtimes \Sigma_n$ acts on $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$. Note that $\mathscr{N}_n^{\mathbb{R}}$ coincides with the Weyl group of type B.

Proposition 5.1. *The preimage of a single cube I^n_σ in the set $\mathscr{R}_n = \bigcup_{\sigma\in\Sigma_n} I^n_\sigma$ under the map $\widetilde{\mu}\colon Q_n \to \mathscr{R}_n$ is diffeomorphic to the polytope $\mathscr{B}^n$. The map $\widetilde{\mu}\colon \widetilde{\mu}^{-1}(I^n_\sigma) \to I^n_\sigma$ is the map $\mathscr{B}^n \to I^n$ which blows down the cut-off faces.*

The polytope $\mathscr{B}^n$ is the fundamental domain of the Σ_n-action on Q_n, $\mathscr{N}_n$-action on $M_{\mathrm{St}_n,\lambda}$, and $\mathscr{N}_n^{\mathbb{R}}$-action on $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$.

Proof. Without loss of generality consider the single cube $I^n_{\mathrm{id}} = I_1 \times \cdots \times I_n$, corresponding to the trivial permutation $\mathrm{id} \in \Sigma$. For points in this cube we have $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n$. As follows from § 4 (see Remark 4.1), the preimage $\widetilde{\mu}^{-1}(x)$ consists of a single point for generic $x \in I^n_{\mathrm{id}}$. If a collision $a_j = a_{j+1}$ occurs for a point x, then x lies on a codimension-2 face F_{σ_i} of the cube. It was noted in Remark 4.1 that in this case the preimage $\widetilde{\mu}^{-1}(x)$ is a cube of dimension equal to the number of pairwise collisions. Therefore, the preimage $\widetilde{\mu}^{-1}(I^n_{\mathrm{id}})$ is given by blowing up the cube at the faces $F_{\sigma_1}, \dots, F_{\sigma_{n-1}}$. The proposition is proved.

The orbit space Q_n can be represented as the union of $n!$ copies of the polytope $\mathscr{B}^n$ attached along the cut faces. More precisely, we have

$$Q_n = \bigcup_{\sigma\in\Sigma_n} \mathscr{B}^n_\sigma/\sim, \qquad \mathscr{B}^n_\sigma \cong \mathscr{B}^n.$$

The relation $\sim$ identifies the point $x \in \mathscr{F}_{\sigma_i}$ of the polytope $\mathscr{B}^n_\sigma$ with the point $\alpha_i(x) \in \mathscr{F}_{\sigma_i}$ of the polytope $\mathscr{B}^n_\tau$ whenever $\sigma = \tau\sigma_i$. Recall that α_i is the antipodal involution of the facet $\mathscr{F}_{\sigma_i}$. One should not forget about this involution: in going over from $\mathscr{B}^n_\sigma$ to $\mathscr{B}^n_\tau$ the barycentric coordinates $|b_i|^2$ and $|b^2_{i+1}|$ on the blown up face interchange.

Example 5.2. For $n = 3$ the space Q_3 is obtained by stacking six copies of the polytope $\mathscr{B}^3$ shown in Figure 4. The result of this stacking is shown in Figure 5. It can be seen that Q_3 is a solid torus and its boundary is subdivided into hexagons in a regular way. Note that stacking does not produce additional faces, so the picture should be smoothened at stack points. This is similar to the construction of origami templates in the theory of toric origami manifolds (see [15] and [9]).

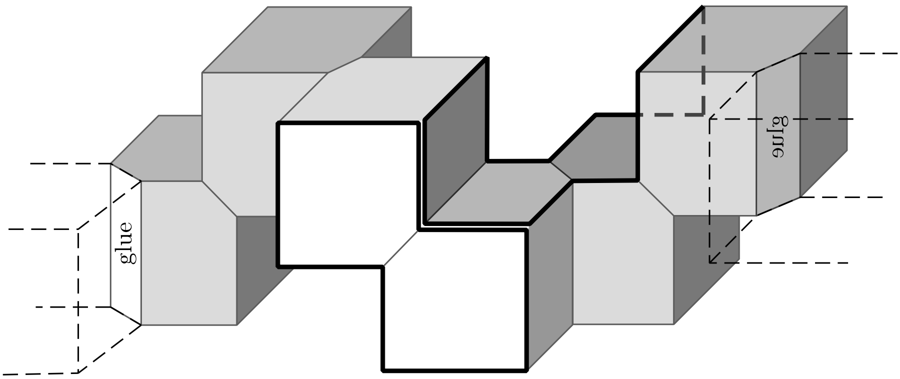


Figure 5. Reconstructing Q_3 by stacking six copies of $\mathscr{B}^3$.

The manifold $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$ is a small cover over $Q_n = M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}/\mathbb{Z}^n_2$ in which all stabilizers of the action $\mathbb{Z}^n_2 \circlearrowright M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$ are coordinate subgroups of $\mathbb{Z}^n_2$. The constructed cellular subdivision on Q_n can be lifted to $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$, and we have the following statement.

Theorem 5.1. *The manifold $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$ is canonically subdivided into $2^n n!$ cells which are combinatorially isomorphic to $\mathscr{B}^n$. More precisely,*

$$M^{\mathbb{R}}_{\mathrm{St}_n,\lambda} = \bigcup_{\sigma\in\Sigma_n,\varepsilon\in\mathbb{Z}^n_2} \mathscr{B}^n_{\sigma,\varepsilon}/\sim, \qquad \mathscr{B}^n_{\sigma,\varepsilon} \cong \mathscr{B}^n,$$

where the equivalence relation $\sim$ is generated by the following relations:

- *a point $x \in \mathscr{F}_{\sigma_i} \subset \mathscr{B}^n_{\sigma,\varepsilon}$ is identified with the point $\alpha_i(x) \in \mathscr{F}_{\sigma_i} \subset \mathscr{B}^n_{\tau,\varepsilon}$ if $\sigma\tau^{-1} = \sigma_i$;*

- *a point $x \in \mathscr{F}_{\delta k} \subset \mathscr{B}^n_{\sigma,\varepsilon_1}$ is identified with the point $x \in \mathscr{F}_{\delta k} \subset \mathscr{B}^n_{\sigma,\varepsilon_2}$ if*

$$\varepsilon_1\varepsilon_2^{-1} = (+1,\dots,+1,\overset{k}{-1},+1,\dots,+1).$$

Each codimension-s cell of the cellular structure on $M^{\mathbb{R}}_{\mathrm{St}_n,\lambda}$ lies in exactly 2^{n-s} top-dimensional cells.

Remark 5.1. We make a remark about the general construction, called *symplectic implosion*, which is related to the space $M_{\mathrm{St}_n,\lambda}$ and the cubes appearing in our work. Details on symplectic implosion can be found in [20].

Assume that there is a Hamiltonian action of a compact Lie group K on a symplectic manifold M. There is a moment map $\Phi\colon M \to \mathfrak{k}^*$, where $\mathfrak{k}$ is the Lie algebra of K. Let T be a maximal torus of K, $\mathfrak{t}$ be its Lie algebra, and $\mathfrak{t}^*_+$ be a closed Weyl chamber in the dual space $\mathfrak{t}$. Using the Killing form we can identify Lie algebras with their duals and assume that the vector space $\mathfrak{t}^*$ is embedded into $\mathfrak{k}^*$ as a subspace. With all the choices fixed, we can consider the open symplectic submanifold $\Phi^{-1}((\mathfrak{t}^*_+)^\circ)$, the preimage of the open Weyl chamber under the moment map. A certain compactification of this submanifold is considered. More precisely, one should take the closed preimage $\Phi^{-1}(\mathfrak{t}^*_+)$ and identify two points m_1 and m_2 in the preimage whenever $m_2 = gm_1$ for some element g of the commutator subgroup $[K_{\Phi(m_1)}, K_{\Phi(m_1)}]$. Here $K_{\Phi(m_1)}$ is the stabilizer of $\Phi(m_1) \in \mathfrak{t}^*_+$ under the (co)adjoint K-action on $\mathfrak{t}$. The space $M_{\mathrm{impl}} = \Phi^{-1}(\mathfrak{t}^*_+)/{\sim}$ defined in this way is called the *imploded cross-section*.

In our case, M is the manifold M_λ of Hermitian matrices of size $n+1$ having the given simple spectrum λ. As we mentioned in § 1, the conjugation action of $U(n+1)$ on M_λ is Hamiltonian, and the moment map is just the identity map. Now consider the subgroup $U(n) \subset U(n+1)$ embedded as the lower-right corner; it will play the role of K from the general construction. The induced action of $K = U(n)$ on M_λ is again Hamiltonian, and its moment map Φ assigns to a matrix $A \in M_\lambda$ its lower-right $(n \times n)$-corner. The maximal torus $T \subset K = U(n)$ consists of the diagonal unitary matrices of size n. The embedding $\mathfrak{t}^* \subset \mathfrak{k}^*$ is given by the diagonal matrices, and therefore the space $\Phi^{-1}(\mathfrak{t}^*)$ is exactly the space of all arrow matrices with spectrum λ. The Weyl chamber $\mathfrak{t}^*_+$ consists of the n-tuples $a = (a_1,\dots,a_n)$ with $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$ (while the Weyl group is the symmetric group Σ_n). Hence M_{impl} is the quotient space of $\mu^{-1}(I^n_{\mathrm{id}})$ by a certain relation (recall that I^n_σ is the cube in the union $\mathscr{R}_n$ that corresponds to the permutation σ).

The preimage $\Phi^{-1}(a)$ is nonempty if and only if the only collisions of a_i are pairwise. In this case it can be easily shown that the subgroup $[K_a, K_a]$ is the product of $\mathrm{SU}(2)$; each $\mathrm{SU}(2)$ corresponds to a pair of collided values. By recalling Remark 4.1 we see that the space $\Phi^{-1}(\mathfrak{t}^*_+)/{\sim}$ is (at least topologically) just the toric manifold over a cube. Therefore, the symplectic implosion for the $U(n)$-action on M_λ, the coadjoint orbit of $U(n+1)$, is the toric manifold $(\mathbb{C}P^1)^n$.

These observations give a hint of how to generalize the manifold $M_{\mathrm{St}_n,\lambda}$ and its properties to other Lie groups.

Remark 5.2. Note that the polytope $\mathscr{B}^n$ has a convex Delzant realization, since it is obtained from a cube by a sequence of codimension-2 face cutoffs. Therefore, there exists a symplectic toric manifold over $\mathscr{B}^n$. Note, however, that this symplectic manifold does not coincide with the imploded cross-section described in Remark 5.1.

§ 6. Combinatorics of strata

In this section Γ is an arbitrary graph on the set V, $|V| = n$, $\Delta = (\Gamma, a, b)$ is a labelled graph, A_Δ is its corresponding Hermitian matrix and $M_{\Gamma,\lambda}$ is the space of Γ-shaped matrices with simple spectrum λ.

The orbit space $M_{\Gamma,\lambda}/T^n$ is stratified by the dimensions of torus orbits. Suppose an orbit $[A] \in M_{\Gamma,\lambda}/T^n$ is represented by a matrix A such that $b_e = 0$, $e \in W$, for some set W of edges. Removing the edges of W from the graph Γ, we obtain a new graph $\widetilde{\Gamma}$ on the same vertex set V. Let k be the number of connected components of $\widetilde{\Gamma}$. We can see that the dimension of the orbit T^nA is $n-k$. The set of matrices having zeros at the positions $e \in W$ can in general be disconnected, since the spectrum λ can be distributed in different ways between blocks of a matrix. These considerations motivate the following definition.

Definition 6.1. Let Γ be a graph on a vertex set V, $|V| = n$. Let $\widetilde{\Gamma} \subset \Gamma$ be a subgraph on V, and $V_1, \dots, V_k$ be the vertex sets of the connected components of $\widetilde{\Gamma}$. Consider $\mathscr{V} = \{V_1, \dots, V_k\}$, the unordered partition of V. We say that two bijections $p_1, p_2 \colon V \to [n]$ are *equivalent with respect to* $\mathscr{V}$ (or simply $\mathscr{V}$*-equivalent*) if they differ by permutations within each subset V_i, that is,

$$p_1 = p_2 \cdot \sigma, \qquad \sigma \in \Sigma_{V_1} \times \dots \times \Sigma_{V_k} \subseteq \Sigma_V.$$

The class of $\mathscr{V}$-equivalent bijections is called a *cluster* subject to the partition $\mathscr{V}$. Let $\mathscr{P}_\Gamma$ be the set of all clusters subject to partitions of subgraphs $\widetilde{\Gamma} \subset \Gamma$ into connected components. Now we define a partial order on $\mathscr{P}_\Gamma$. Note that the inclusion of subgraphs $\widetilde{\Gamma}' \subset \widetilde{\Gamma}$ implies that the partition $\widetilde{\mathscr{V}}'$ is a refinement of $\widetilde{\mathscr{V}}$. We say that a cluster $[p']$ subject to $\widetilde{\mathscr{V}}'$ is less than a cluster $[p]$ subject to $\widetilde{\mathscr{V}}$ if p and p' are $\mathscr{V}$-equivalent. In other words, $p' < p$ if p' is a refinement of p. The poset $\mathscr{P}_\Gamma$ is called the *cluster-permutohedron* corresponding to Γ.

Remark 6.1. A cluster-permutohedron is a graded poset: the rank of a cluster subject to a subgraph $\widetilde{\Gamma}$ is equal to the rank of this subgraph, that is, the number of edges in its spanning forest. Therefore, the rank is equal to the number of vertices of $\widetilde{\Gamma}$ minus the number of its connected components.

Note that the poset $\mathscr{P}_\Gamma$ has a unique maximal element of rank $n-1$, which is represented by the cluster subject to the whole graph Γ. The elements in this cluster can be permuted arbitrarily. A smallest element of the poset $\mathscr{P}_\Gamma$ corresponds to a partition of the vertex set into n singletons: such clusters are encoded by all possible permutations $p \colon V \to [n]$. This means $\mathscr{P}_\Gamma$ has exactly $n!$ atoms for any Γ.

Example 6.1. Let Γ be a path graph with edges $(1,2), (2,3), \dots, (n-1,n)$. Its subgraph is represented by a sequence of path graphs, and the corresponding partition of the vertex set has the form

$$\mathscr{V} = \{\{1, \dots, s_1\}, \{s_1+1, \dots, s_2\}, \dots, \{s_k+1, \dots, n\}\},$$

where $1 \leqslant s_1 < s_2 < \dots < s_k < n$. A cluster subject to this partition has the form

$$(\{\sigma(1), \dots, \sigma(s_1)\}, \{\sigma(s_1+1), \dots, \sigma(s_2)\}, \dots, \{\sigma(s_k+1), \dots, \sigma(n)\}),$$

which is a linearly ordered partition of $V = [n]$. Therefore, the cluster-permutohedron in this case coincides with the poset of faces of a permutohedron; see Construction 4.1.

Example 6.2. Let Γ be a simple cycle on $[n]$, that is, the graph with edges $(1,2)$, $(2,3)$, ..., $(n-1,n)$, $(n,1)$. In this case, clusters subject to partitions into connected components are cyclically ordered partitions of $[n]$. The poset of such cyclically ordered partitions is called a *cyclopermutohedron*; it was introduced and studied by Panina; see [28].

We formulate several basic properties of general cluster-permutohedra.

Proposition 6.1. *Let $p \in \mathscr{P}_\Gamma$ be an element of the cluster-permutohedron, subject to a partition $\mathscr{V} = \{V_1, \dots, V_k\}$ of the vertex set of Γ, and let Γ_i be the induced subgraph of Γ on the set V_i. Then the lower order ideal*

$$(\mathscr{P}_\Gamma)_{\leqslant p} = \{q \in \mathscr{P}_\Gamma \mid q < p\}$$

is isomorphic to the direct product of posets

$$\mathscr{P}_{\Gamma_1} \times \cdots \times \mathscr{P}_{\Gamma_k}.$$

The proof is straightforward from the construction of a partial order on the cluster-permutohedron. Recall a basic definition from the theory of posets.

Definition 6.2. A poset S is called *simplicial* if it has a unique minimal element $\widehat{0} \subset S$, and for each $I \in S$ the lower order ideal $S_{\leqslant I} = \{J \in S \mid J \leqslant I\}$ is isomorphic to a Boolean lattice. Elements of a simplicial poset are called *simplices*. If every two simplices $I, J \in S$ have a unique common lower bound, then S is called a *simplicial complex*.

A poset S is called *simple* (or *dually simplicial*) if S^* is a simplicial poset. Here S^* is the set S with the reversed order. By the definition of a simplicial complex, each simplex is uniquely determined by its set of vertices. A simplicial complex S is called *flag* if, whenever a collection σ of vertices is pairwise connected by edges, σ is a simplex of K.

Proposition 6.2. *If Γ is a tree, then the poset $\mathscr{P}_\Gamma$ is simple. Its dual $\mathscr{P}_\Gamma^*$ is a flag simplicial complex.*

Proof. Let an element $p \in \mathscr{P}_\Gamma$ be subject to a partition $\mathscr{V} = \{V_1, \dots, V_k\}$. This partition defines a forest $\widetilde{\Gamma} \subset \Gamma$ uniquely. Let $\{e_1, \dots, e_{k-1}\}$ be the set of edges of Γ which do not lie in $\widetilde{\Gamma}$. The upper order ideal $\{q \in \mathscr{P}_\Gamma \mid q \geqslant p\}$ is isomorphic to the Boolean lattice of subsets of the set $\{e_1, \dots, e_{k-1}\}$. The condition on a simplicial complex and the flag property hold for similar reasons. The proposition is proved.

Example 6.3. Let $\Gamma = \mathrm{St}_n$ be a star graph with edges $(0,1), (0,2), \dots, (0,n)$. Its cluster-permutohedron has the following property: each of its lower order ideals is again a cluster-permutohedron of a star graph. Indeed, each subgraph of St_n is a disjoint union of a discrete set of vertices $\{i_1, \dots, i_s\} \subset \{1, \dots, n\}$ and a star graph on the remaining vertices.

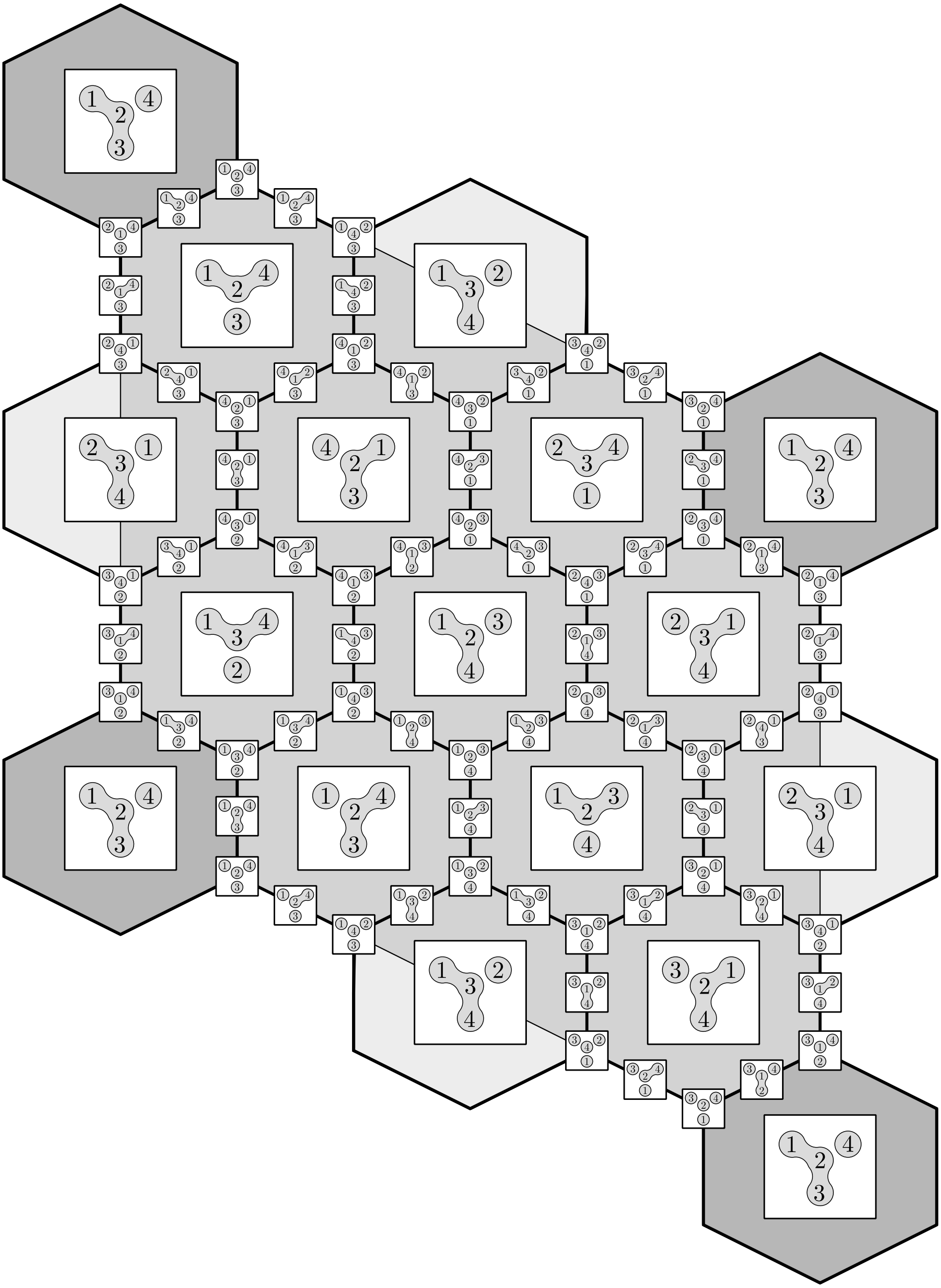


Figure 6. The combinatorics of the cluster-permutohedron for the star graph with three rays.

This example is particularly interesting in connection with the results of § 4. The space $M_{\mathrm{St}_n,\lambda}$ is a smooth manifold with half-dimensional torus action. Therefore, the orbit space $Q_n = M_{\mathrm{St}_n,\lambda}/T$ is a manifold with corners. The poset of faces of this manifold with corners is isomorphic to the cluster-permutohedron $\mathscr{P}_{\mathrm{St}_n}$ according to the arguments from the beginning of this section.

We consider the case $n = 3$, the star with three rays, in detail. First, note that the star graph St_2 is just a simple path on three vertices, hence its cluster-permutohedron $\mathscr{P}_\Gamma$ is the poset of faces of a hexagon, according to Example 3.1. All 2-dimensional faces of Q_3 are hexagons. Indeed, each of these faces is the orbit space of the manifold of tridiagonal (3×3)-matrices. This orbit space is a permutohedron of dimension 2, that is, a hexagon.

As shown in § 5, the space Q_3 is a solid torus, whose boundary is subdivided into hexagons in a regular simple fashion: each vertex is contained in exactly three hexagons. The combinatorics of this hexagonal subdivision can be described in terms of the cluster-permutohedron (see Figure 6).

We observe that the cell complex ∂Q_3 is a nanotube with chiral vectors $(2, 2)$ and $(4, -2)$, using the terminology adopted in discrete geometry and chemistry (see [11]).

§ 7. Arrow matrices 4 × 4

We apply the results in [2], [3] and [9] to describe the cohomology and equivariant cohomology rings of the manifold $M_{\mathrm{St}_3,\lambda}$.

Recall the necessary definitions. Let Q be a manifold with corners, $\dim Q = n$. Assume that every vertex of Q lies in exactly n facets (such manifolds with corners were called *nice* in [22] or otherwise *manifolds with faces*). We call Q an *almost homological polytope* (or simply an *almost polytope*) if all of its proper faces are acyclic. If, moreover, Q itself is acyclic, Q is called a *homology polytope*.

Let $\mathscr{P}_Q$ denote a poset of faces of Q. The poset $K_Q = \mathscr{P}_Q^*$ with reversed order is simplicial for a nice manifold with corners. If Q is an almost polytope, then the simplicial poset K_Q is a homology manifold (see the definition below).

We recall the basic definitions needed for our considerations. The definitions are given for simplicial complexes, since general simplicial posets do not appear in our examples.

Definition 7.1. Let K be a simplicial complex on a set $[m]$ and $I \in K$ be a simplex. The *link* of I is a simplicial complex $\mathrm{link}_K I = \{J \subseteq [m] \setminus I \mid I \cap J \in K\}$. In particular, $\mathrm{link}_K \varnothing = K$. The complex K is called *pure* if all of its maximal-by-inclusion simplices have equal dimensions. A pure simplicial complex K of dimension $n-1$ is called a *homology manifold* whenever, for any nonempty simplex $I \in K$, $I \neq \varnothing$, the complex $\mathrm{link}_K I$ has the same homology as a sphere of the corresponding dimension:

$$\widetilde{H}_j(\mathrm{link}_K I; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } j = n - 1 - |I|, \\ 0 & \text{otherwise}. \end{cases} \tag{7.1}$$

A manifold K is called *orientable* if its geometric realization possesses an orienting cycle. The complex K is called a *homology sphere* if condition (7.1) also holds for $I = \varnothing$ and $\mathrm{link}_K \varnothing = K$.

Remark 7.1. In all the definitions the coefficient ring should be specified. If the coefficient ring is omitted, it is assumed that the coefficients are in $\mathbb{Z}$.

Definition 7.2. Let R be either a field or the ring $\mathbb{Z}$, and $R[m] = R[v_1, \dots, v_m]$ be the polynomial ring on m generators, $\deg v_i = 2$. The *Stanley-Reisner ring* of K is the quotient ring

$$R[K] = R[m]/(v_{i_1} v_{i_2} \cdots v_{i_k} \mid \{i_1, \dots, i_k\} \notin K).$$

The ring $R[K]$ has a natural structure of an $R[m]$-module.

Definition 7.3. Let K be a pure simplicial complex of dimension $n-1$. A function $\lambda\colon [m] \to R^n$ is called a *characteristic function* for K if for each maximal simplex $I = \{i_1, \dots, i_n\}$ the collection $\{\lambda(i_1), \dots, \lambda(i_n)\}$ is a basis of the free module R^n.

Each function $\lambda\colon [m] \to R^n$ determines a sequence of degree-2 elements in the ring $R[K]$ as described below (abusing the terminology we call such elements linear, since all rings are generated in degree 2 and there are no odd components at all). Let $\lambda(i) = (\lambda_{i,1}, \dots, \lambda_{i,n}) \in R^n$, $i \in [m]$. Consider $\theta_j = \lambda_{1,j} v_1 + \dots + \lambda_{m,j} v_m \in R[K]$, $j = 1, \dots, n$. Let Θ be the ideal in $R[K]$ generated by $\theta_1, \dots, \theta_n$.

Lemma 7.1 (see, for example, [13]). *Let R be a field. Then the set of linear elements $\theta_1, \dots, \theta_n$ is a linear system of parameters in $R[K]$ if and only if $\lambda\colon [m] \to R^n$ is a characteristic function.*

In what follows $\lambda\colon [m] \to R^n$ always denotes a characteristic function.

Example 7.1. Let $c\colon [m] \to [n]$ be a proper colouring of the vertices of K, that is, a map taking different values at the endpoints of any edge of K. For such a colouring there is an associated characteristic function $\lambda_c\colon [m] \to R^n$, $\lambda_c(i) = e_{c(i)}$, where $e_1, \dots, e_n$ is the fixed basis of R^n. For the characteristic function λ_c, the elements of the linear system of parameters have the form

$$\theta_j = \sum_{i, c(i) = j} v_i \in R[K]_2.$$

Characteristic functions of this kind will be called *chromatic*.

Proposition 7.1 (Reisner [29], Stanley [31] and Schenzel [30]). *If K is a homology sphere, then $R[K]$ is a Cohen-Macaulay ring. In this case $\theta_1, \dots, \theta_n$ is a regular sequence in $R[K]$, which means that $R[K]$ is a free module over the subring $R[\theta_1, \dots, \theta_n]$. If K is a homology manifold, then $R[K]$ is a Buchsbaum ring* (*which means that $\theta_1, \dots, \theta_n$ is a weak regular sequence*).

Recall the basic combinatorial characteristics of a simplicial complex. Let f_j denote the number of j-dimensional simplices of K for $-1 \leqslant j \leqslant n-1$; in particular, $f_{-1} = 1$ (the empty simplex has formal dimension -1). The h-numbers of K are defined using the relation

$$\sum_{j=0}^{n} h_j t^{n-j} = \sum_{j=0}^{n} f_{j-1} (t-1)^{n-j}, \tag{7.2}$$

where t is a formal variable. Let $\widetilde{\beta}_j(K) = \dim \widetilde{H}_j(K)$ be the reduced Betti number of K. The h'- *and* h''*-numbers* of K are defined by the relations

$$h'_j = h_j + \binom{n}{j} \left(\sum_{s=1}^{j-1} (-1)^{j-s-1} \widetilde{\beta}_{s-1}(K) \right) \quad \text{for } 0 \leqslant j \leqslant n, \tag{7.3}$$

and

$$h''_j = h'_j - \binom{n}{j}\widetilde{\beta}_{j-1}(K) = h_j + \binom{n}{j}\Bigl(\sum_{s=1}^{j}(-1)^{j-s-1}\widetilde{\beta}_{s-1}(K)\Bigr) \tag{7.4}$$

for $0 \leqslant j \leqslant n-1$ and $h''_n = h'_n$. Summation over an empty set is assumed to produce zero.

Proposition 7.2 (Reisner [29], Stanley [31] and Schenzel [30]). *For each pure simplicial complex K of dimension $n-1$*

$$\operatorname{Hilb}(R[K];t) = \frac{h_0 + h_1 t^2 + \dots + h_n t^n}{(1-t^2)^n}.$$

For a homology sphere K, $\operatorname{Hilb}(R[K]/\Theta;t) = \sum_i h_i t^{2i}$. For a homology manifold K, $\operatorname{Hilb}(R[K]/\Theta;t) = \sum_i h'_i t^{2i}$.

Proposition 7.3 (Novik and Swartz [26], [27]). *Let K be a connected orientable homology manifold of dimension $n-1$. The $2j$th graded component of the module $R[K]/\Theta$ contains a vector subspace $(I_{\mathrm{NS}})_{2j} \cong \binom{n}{j}\widetilde{H}^{j-1}(K;R)$ which is a trivial $R[m]$-submodule (that is, $R[m]_+(I_{\mathrm{NS}})_{2j} = 0$). Let $I_{\mathrm{NS}} = \bigoplus_{j=0}^{n-1}(I_{\mathrm{NS}})_{2j}$ be the sum of all submodules except for the top one. Then the quotient module $R[K]/\Theta/I_{\mathrm{NS}}$ is a Poincaré duality algebra, and $\operatorname{Hilb}(R[K]/\Theta/I_{\mathrm{NS}};t) = \sum_i h''_i t^{2i}$.*

This result implies in particular the generalized Dehn-Sommerville relations for manifolds: $h''_j = h''_{n-j}$.

Let $\Lambda^* R^n$ denote the exterior algebra over R^n. For a characteristic function $\lambda\colon [m] \to R^n$ and an oriented simplex $I = \{i_1, \dots, i_s\} \in K$ consider the nonzero skew-form

$$\lambda_I = \lambda(i_1) \wedge \dots \wedge \lambda(i_s) \in \Lambda^s R^n.$$

Proposition 7.4 (see [2] and [4]). *Let K be a connected orientable homology manifold, $\dim K = n-1$, and $\theta_1, \dots, \theta_n$ be a linear system of parameters corresponding to a characteristic function λ. Let R be either $\mathbb{Q}$ or $\mathbb{Z}$. Then the $2j$th graded component of the algebra $R[K]/\Theta$ is additively generated by the elements*

$$v_I = v_{i_1} \cdots v_{i_j}, \qquad I = \{i_1, \dots, i_j\} \in K,$$

and we have the following.

(1) *All additive relations on the elements v_I in the module $R[K]/\Theta$ have the form*

$$\sum_{I \in K,\, |I| = j} \langle \omega, \lambda_I \rangle \sigma(I) v_I, \tag{7.5}$$

where μ runs over $(\Lambda^k R^n)^*$ and σ runs over the vector space of simplicial $(j-1)$-coboundaries of the complex K: $\sigma \in \mathscr{C}^{j-1}(K;R)$, $\sigma = d\tau$.

(2) *All additive relations on the elements v_I in the module $R[K]/\Theta/I_{\mathrm{NS}}$ for $j < n$ have the form (7.5), where σ runs over the space of simplicial cocycles: $\sigma \in \mathscr{C}^{j-1}(K;R)$, $d\sigma = 0$.*

In particular, this statement gives an explicit formula for the generators of the Novik-Swartz ideal $I_{\mathrm{NS}} \subset R[K]/\Theta$. In [8] we called (7.5) *Minkowski-type relations* by analogy with the terminology adopted in toric geometry. It was shown that these relations have a simple geometrical explanation, coming from the theory of multi-polytopes.

The theory sketched above can be used to describe the cohomological structure of manifolds with half-dimensional torus action. Let X be a $2n$-manifold, and let a compact torus T^n act on X in a locally standard way. In this case the orbit space $Q = X/T$ is a nice manifold with corners. Let K_Q be the simplicial poset dual to the simple poset of faces of Q. Let $[m]$ be the vertex set of K_Q, and therefore $\{\mathscr{F}_1, \dots, \mathscr{F}_m\}$ be the set of facets of Q.

The preimage of $\mathscr{F}_i \subset Q$ under the map $p\colon X \to Q$ is a submanifold $X_i \subset X$ of codimension 2, which is called a *characteristic submanifold*. The cohomology class dual to $[X_i] \in H_{2n-2}(X)$ is denoted by $v_i \in H^2(X;\mathbb{Z})$ (one should orient X_i somehow to make things well-defined). For a point x in the interior of $\mathscr{F}_i$ the stabilizer of the action is a one-dimensional subgroup. It has the form $\lambda(i)(S^1)$, where $\lambda(i) \in \operatorname{Hom}(S^1, T^n) \cong \mathbb{Z}^n$. Since the action is locally standard, the map λ is a characteristic function on K_Q.

If Q is a homology polytope, then K_Q is a homology sphere. If Q is an almost polytope, then K_Q is a homology manifold. The proper faces of an almost polytope Q determine the homological cell subdivision of the boundary ∂Q, which is dual to K_Q; for details, see [1].

Proposition 7.5 (Masuda and Panov [22]). *If Q is a homology polytope, then*

$$H^*_T(X;\mathbb{Z}) \cong \mathbb{Z}[K_Q], \qquad H^*(X;\mathbb{Z}) \cong \mathbb{Z}[K_Q]/\Theta,$$
$$H^{2i+1}(X;\mathbb{Z}) = 0, \qquad H^{2i}(X;\mathbb{Z}) \cong \mathbb{Z}^{h_i(K_Q)}.$$

Proposition 7.6 (see [9]). *If Q is an orientable connected almost polytope and the projection map $p\colon X \to Q$ admits a section, then*

$$H^*_T(X;R) \cong R[K_Q] \oplus H^*(Q;R)$$

(*the units are identified in the direct sum of the rings*).

Proposition 7.7 (see [3]). *Assume Q is an orientable connected almost polytope and the projection map $p\colon X \to Q$ admits a section.*

(1) *Let $A^*(X;R)$ be the subring in $H^*(X;R)$ generated by the classes v_i of characteristic submanifolds. Then there exists a sequence of epimorphisms*

$$R[K_Q]/\Theta \twoheadrightarrow A^*(X;R) \twoheadrightarrow R[K_Q]/\Theta/I_{\mathrm{NS}}.$$

The component $A^{2j}(X;R)$ is additively generated by classes v_I, $I \in K_Q$, $|I| = j$. The relations on these classes in $A^{2j}(X;R)$ for $j < n$ have the form (7.5), where $\sigma \in \mathscr{C}^{j-1}(K;R) \cong \mathscr{C}_{n-j}(\partial Q;R)$ runs over all cellular chains vanishing in $H_{n-j}(Q;R)$.

(2) *The submodule $A^+ = \bigoplus_{j>0} A^{2j}(X;R)$ is an ideal in $H^*(X;R)$. Then there is an isomorphism of graded rings*

$$H^*(X)/A^+ \cong \biggl(\bigoplus_{i<j} H^i(Q,\partial Q) \otimes H^j(T^n)\biggr) \oplus \biggl(\bigoplus_{i\geqslant j} H^i(Q) \otimes H^j(T^n)\biggr).$$

All nontrivial products on the right-hand side are given by cup products in cohomology and relative cohomology.

Now we apply this technique to the 6-dimensional manifold $M_{\mathrm{St}_3,\lambda}$ of isospectral arrow (4×4)-matrices. According to the results of §§ 4 and 5, the orbit space $Q_3 = M_{\mathrm{St}_3,\lambda}/T^3$ is a manifold with corners, homeomorphic to $D^2\times S^1$, and its boundary is subdivided into hexagons as shown in Figure 7, a. Therefore, Q^3_{St} is an almost polytope. Note that the map $p\colon M_{\mathrm{St}_3,\lambda}\to Q_3$ admits a section. Indeed, Q_3 may be identified with the space of arrow matrices with nonnegative off-diagonal elements. This is a natural subset of $M_{\mathrm{St}_3,\lambda}$.

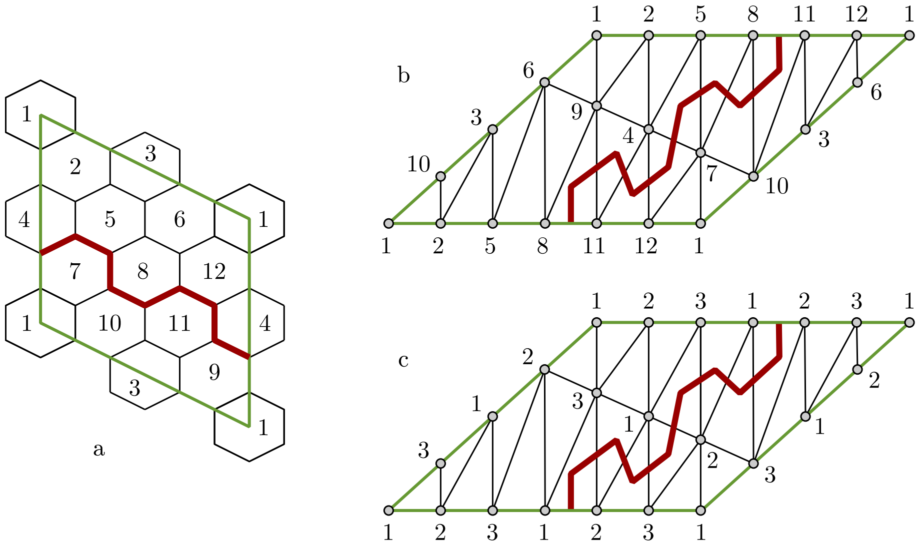


Figure 7. The combinatorics of the boundary ∂Q_3 (a) and its dual simplicial complex (b). The bold line denotes the cycle in $\partial Q_3\cong T^2$ which is contractible within the solid torus Q_3. The proper colouring (c) of the vertices of $\mathscr{P}^*_{\mathrm{St}_3}$ (that is, 2-faces of Q_3).

The simplicial complex $\mathscr{P}^*_{\mathrm{St}_3}$ dual to Q_3 is the triangulation of a 2-torus with twelve vertices that is shown in Figure 7, b. It can be seen that its f-vector is $(f_{\{-1\}}, f_0, f_1, f_2) = (1, 12, 36, 24)$, the h-vector is $(1, 9, 15, -1)$, and the h'-vector is $(1, 9, 15, 1)$.

All stabilizers of the T^3-action on $M_{\mathrm{St}_3,\lambda}$ are coordinate subtori in T^3. Therefore, the characteristic function λ of the action is chromatic. It comes from the proper colouring of vertices of $\mathscr{P}^*_{\mathrm{St}_3}$ indicated in Figure 7, c. Propositions 7.6 and 7.7 applied to $M_{\mathrm{St}_3,\lambda}$ give the following result.

Theorem 7.1. *The relation* $H^*_T(M_{\mathrm{St}_3,\lambda};R)\cong R[\mathscr{P}^*_{\mathrm{St}_3}]\oplus H^*(S^1;R)$ *holds. The subring* $A^*(M_{\mathrm{St}_3,\lambda};R)\subset H^*(M_{\mathrm{St}_3,\lambda};R)$ *generated by the classes* v_i *of the characteristic submanifolds has the form*

$$A^* = A^*(M_{\mathrm{St}_3,\lambda};R) = R[\mathscr{P}^*_{\mathrm{St}_3}]/\Theta/\mathscr{I},$$

where:

- *the ideal* Θ *of the Stanley-Reisner ring* $R[\mathscr{P}^*_{\mathrm{St}_3}]$ *is generated by the linear forms*

$$\theta_1 = v_1 + v_3 + v_4 + v_8, \quad \theta_2 = v_5 + v_9 + v_{10} + v_{12}, \quad \theta_3 = v_2 + v_6 + v_7 + v_{11}$$

(*the numeration of vertices is as shown in Figure* 7);

- *the ideal* $\mathscr{I}$ *is additively generated by the elements*

$$v_8v_{11} + v_7v_8 + v_4v_7 + v_4v_{11}, \quad v_8v_{10} + v_4v_{12}, \quad v_5v_7 + v_9v_{11}$$

(*the choice of these elements is noncanonical*).

The graded components of the subring A^* *have dimensions* $(1, 0, 9, 0, 12, 0, 1)$.

The quotient ring $H^*(M_{\mathrm{St}_3,\lambda})/A^+$ *has the following nonempty components*: R *in degree* 0, R *in degree* 1, R^3 *in degree* 2 *and* R *in degree* 5. *The product in the quotient ring* $H^*(M_{\mathrm{St}_3,\lambda})/A^+$ *is trivial.*

The integral cohomology of X^6_{St} *is torsion free*, *the Betti numbers are* $(1, 1, 12, 0, 12, 1, 1)$.

Proof. Two statements require an explanation: the form of the generators of the ideal $\mathscr{I}$, and the torsion freeness of the cohomology. According to Proposition 7.7, the relations on the classes $v_I = v_{i_1} \cdots v_{i_k} \in H^*(X^6_{\mathrm{St}})$ are given by all possible skew forms and all possible cellular cycles in $\partial Q^3_{\mathrm{St}}$, which vanish in the homology of Q^3_{St}. Each such pair gives a relation $\sum_I \sigma(I)\langle\omega, \lambda_I\rangle v_I$. There is a unique basis cycle in $\partial Q^3_{\mathrm{St}}$, which is homologous to zero in Q^3_{St}. This cycle can be recognized by analyzing the image of the moment map (see Figure 2). This vanishing cycle is shown in Figure 7.

Torsion freeness of $\mathbb{Z}[K]/\Theta/\mathscr{I}$ can be checked by direct computation based on part (1) of Proposition 7.4.

The theorem is proved.

§ 8. The twin manifold of $M_{\mathrm{St}_n,\lambda}$

In [7] we introduced the notion of a twin manifold in the variety of complete complex flags. Given a smooth T-invariant submanifold $X \subset M_\lambda \cong \mathrm{Fl}_n$, we construct another smooth T-invariant submanifold $\widetilde{X} = p_{\mathrm{l}}p_{\mathrm{r}}^{-1}(X)$, called the *twin* of X, where $p_{\mathrm{l}}\colon U(n) \to \mathrm{Fl}_n$ ($p_{\mathrm{r}}\colon U(n) \to \mathrm{Fl}_n$) is the quotient map defined by the left free action of a torus on $U(n)$ (by the right action, respectively):

$$\mathrm{Fl}_n \cong T^n \setminus U(n) \xleftarrow{p_{\mathrm{l}}} U(n) \xrightarrow{p_{\mathrm{r}}} U(n)/T^n \cong \mathrm{Fl}_n. \tag{8.1}$$

It can be seen that $\widetilde{X}/T \cong X/T$ since both spaces are homeomorphic to the double quotient $T \setminus p_{\mathrm{l}}p_{\mathrm{r}}^{-1}(X)/T$. However, the characteristic data of the T-manifolds X and $\widetilde{X}$ are different in general. The twins may be nondiffeomorphic.

Construction 8.1. Let us describe the twin of the manifold $M_{\mathrm{St}_n,\lambda}$. A complex flag in $\mathbb{C}^{n+1}$ can be naturally identified with the sequence of 1-dimensional linear subspaces $L_0, L_1, \dots, L_n \subset \mathbb{C}^{n+1}$, which are pairwise orthogonal: $L_i \perp L_j$, $i \neq j$. Consider a diagonalizable operator $S\colon \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ with distinct real eigenvalues and define a subset $X_n \subset \mathrm{Fl}_{n+1}$:

$$X_n = \{\{L_i\} \in \mathrm{Fl}_{n+1} \mid S(L_i) \subset L_0 \oplus L_i \text{ for } i \neq 0\}.$$

The action of T^{n+1} on $\mathbb{C}^{n+1}$ induces an effective action of $T^n = T^{n+1}/\Delta(T^1)$ on the space X_n. We can assume that $S = \Lambda = \mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_n)$.

Note that there is also an action of Σ_n on X_n which permutes the straight lines $L_1, \dots, L_n$. This action commutes with the T^n-action, hence there is a combined action of the direct product $T^n \times \Sigma_n$ on X_n.

Proposition 8.1. *The space X_n is the twin of $M_{\mathrm{St}_n,\lambda}$. In particular, X_n is a smooth manifold. Its orbit space by the action of T^n is isomorphic to Q_n as a manifold with corners and it is homotopy equivalent to Sq_{n-1}. The orbit space of the $(T^n \times \Sigma_n)$-action on X_n is diffeomorphic to the polytope $\mathscr{B}^n$.*

Proof. We recall the construction in [7]. For a Hermitian matrix A consider its spectral decomposition $A = U^{-1}\Lambda U$. Here the unitary operator U is defined up to left multiplication by diagonal matrices. Given a subspace $X \subset M_\lambda$ of matrices with the given spectrum such that X is preserved by the torus action, we consider the twin space

$$\widetilde{X} = \{A \in M_\lambda \mid A = U\Lambda U^{-1}, \text{ where } U^{-1}\Lambda U \in X\}.$$

Let $e_0, e_1, \dots, e_n$ be the standard basis of $\mathbb{C}^{n+1}$. Then $\widetilde{X}$ is identified with the collection of flags

$$U\langle e_0\rangle \subset U\langle e_0, e_1\rangle \subset \dots \subset U\langle e_0, e_1, \dots, e_n\rangle$$

for all possible unitary matrices $U \in U(n)$ with the property $U^{-1}\Lambda U \in X$.

Let $L_i = U\langle e_i\rangle$. The condition $U^{-1}\Lambda U \in M_{\mathrm{St}_n,\lambda}$ is equivalent to

$$U^{-1}\Lambda U(e_i) \subset \langle e_0, e_i\rangle \quad \text{for } i \neq 0,$$

which is the same condition as

$$\Lambda L_i \subset L_0 \oplus L_i.$$

Hence the twin of $M_{\mathrm{St}_n,\lambda}$ is exactly X_n. Since $M_{\mathrm{St}_n,\lambda}$ is smooth, so is X_n. The orbit spaces of a manifold and its twin coincide (see [7] for details).

The proposition is proved.

Remark 8.1. In [7] we noticed that Hessenberg varieties are the twins of manifolds of staircase isospectral matrices. Note that unlike in this case, the twin X_n of $M_{\mathrm{St}_n,\lambda}$ is not a subvariety of Fl_{n+1}.

Remark 8.2. The manifold $M_{\mathrm{St}_n,\lambda}$ is a submanifold of $M_{\mathrm{St}_n} \cong \mathbb{R}^{3n+1}$ defined by a system of smooth functions with nondegenerate intersections of level hypersurfaces. Hence $M_{\mathrm{St}_n,\lambda}$ has a trivial normal bundle, and all of its Pontryagin classes and numbers vanish. However, this may not be the case for its twin X_n. This makes the twin X_n a more interesting object from a topological point of view.

Construction 8.2. Let us describe the characteristic function on X_n. Recall that the facets of the orbit space $X_n/T \cong Q_n$ are encoded by clusters subject to the partitions $\{\{j\}, \{0, 1, \dots, \widehat{j}, \dots, n\}\}$, $j \neq 0$; see § 6. A cluster is given by

$$\{\{p(j)\}, \{p(0), p(1), \dots, \widehat{p(j)}, \dots, p(n)\}\}$$

for a bijection $p\colon \{0,1,\dots,n\}\to\{0,1,\dots,n\}$. The facet $F_{[p]}$ corresponding to this cluster consists of flags $\{L_i\}\in X_n$ such that $L_j=\langle e_{p(j)}\rangle$. These flags are stabilized by the circle subgroup $T_{p(j)}\subset T^n$, which is the image of the $p(j)$th coordinate circle of T^{n+1} in the quotient $T^n=T^{n+1}/\Delta(T^1)$. In particular, the characteristic function of X_n takes $n+1$ values. This function is not chromatic.

Example 8.1. The values of the characteristic function on X_3 are shown in Figure 8. The characteristic function takes four values in $\mathbb{Z}^3$ which sum to zero, hence we can assume that its values are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and $(-1,-1,-1)$. These values come from a proper 4-colouring of facets of the hexagonal subdivision of a torus. The colouring of a facet is determined by the number standing on a separate vertex of the cluster encoding the facet.

Note that the quotient map $X_3\to X_3/T\cong Q_3$ admits a section. Indeed, the space $Q_3\cong S^1$ has no second cohomology, hence every principal T-bundle over the interior of Q_3 is trivial. The following proposition is completely similar to Theorem 7.1.

Proposition 8.2. *The relation* $H_T^*(X_3;R)\cong R[\mathscr{P}^*_{\mathrm{St}_3}]\oplus H^*(S^1;R)$ *holds. The subring* $A^*(X_3;R)\subset H^*(X_3;R)$ *generated by the classes* v_i *of the characteristic submanifolds has the form*

$$A^*=A^*(X_3;R)=R[\mathscr{P}^*_{\mathrm{St}_3}]/\Theta/\mathscr{I},$$

where:

- *the ideal* Θ *of the Stanley-Reisner ring* $R[\mathscr{P}^*_{\mathrm{St}_3}]$ *is generated by the linear forms*

$$\theta_1=v_4+v_6+v_{10}-v_1-v_5-v_{11},\quad \theta_2=v_3+v_7+v_{12}-v_1-v_5-v_{11},$$
$$\theta_3=v_2+v_8+v_9-v_1-v_5-v_{11}$$

under the numeration of vertices shown in Figure 7;

- *the ideal* $\mathscr{I}$ *is additively generated by the elements*

$$v_5v_7-v_7v_8+v_8v_{11}-v_{11}v_{12},\quad v_8v_{10}-v_8v_{11}+v_4v_{11}-v_4v_9,$$
$$v_4v_7-v_5v_7+v_{11}v_{12}-v_4v_{11}$$

(*the choice of these elements is noncanonical*).

The graded components of the subring A^* *have dimensions* $(1,0,9,0,12,0,1)$. *The integral cohomology of* X^6_{St} *is torsion free, the Betti numbers are* $(1,1,12,0,12,1,1)$.

The second Pontryagin class of X_3 is given by the class $p_1=\sum_{i=1}^{12}v_i^2\in A^*\subset H^*(X_3)$. It can be shown by direct calculation that this class is nontrivial. Actually, the integral of this class over any characteristic submanifold equals ±8 (one needs to introduce an omniorientation to specify the sign). This calculation can be done by simplifying the expression v_ip_1, with the use of the relations in the cohomology ring given by Proposition 8.2.

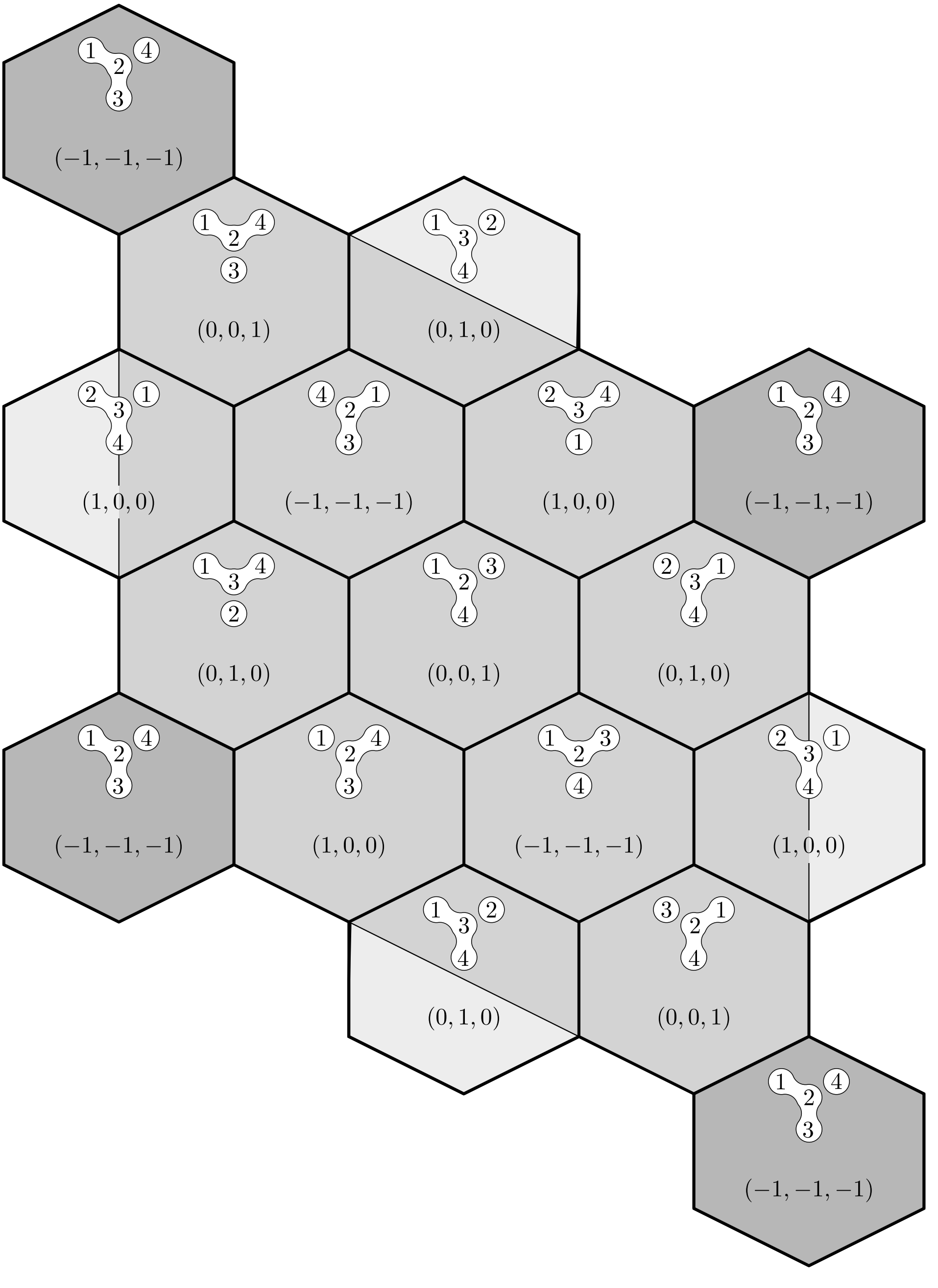


Figure 8. The characteristic function for the twin of $M_{\mathrm{St}_3,\lambda}$.

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Anton A. Ayzenberg
Faculty of Computer Science,
National Research University
Higher School of Economics,
Moscow, Russia
E-mail: ayzenberga@gmail.com

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Victor M. Buchstaber
Steklov Mathematical Institute
of Russian Academy of Sciences,
Moscow, Russia
E-mail: buchstab@mi-ras.ru