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Topologically projective, injective and flat modules of harmonic analysis

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Abstract. We study homologically trivial modules of harmonic analysis on a locally compact group G. For $L_1(G)$- and $M(G)$-modules $C_0(G)$, $L_p(G)$ and $M(G)$ we give criteria for metric and topological projectivity, injectivity and flatness. In most cases, modules with these properties must be finite-dimensional.

Bibliography: 18 titles.

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§ 1. Introduction

Banach homology has a long history dating back to the 1950s. One of the main questions in this discipline is whether a given Banach module is homologically trivial, that is, projective, injective or flat. An example of a successful answer to this question is the work of Dales, Polyakov, Ramsden and Racher [1]–[3], where they gave criteria for homological triviality for classical modules of harmonic analysis. It is worth mentioning that all these studies were carried out for relative Banach homology. We answer the same questions but for two less-explored versions of Banach homology: topological and metric ones. Metric Banach homology was introduced by Graven in [4], where he applied modern (at that time) homological and Banach geometrical techniques to modules of harmonic analysis. The notion of topological Banach homology appeared in the work of White [5]. Seemingly, the latter theory looks much less restrictive then the metric one, but as we shall see this is not the case.

§ 2. Preliminaries on Banach homology

All Banach spaces under consideration are over the field of complex numbers. Let E be a Banach space. By B_E we denote the closed unit ball of E. If F is another Banach space, then a bounded linear operator $T\colon E \to F$ is called *isometric* (*c-topologically injective*) if $\|T(x)\| = \|x\|$ (respectively, $c\|T(x)\| \geqslant \|x\|$) for all $x \in E$. Similarly, T is *strictly coisometric* (*strictly c-topologically surjective*) if $T(B_E) = B_F$ (respectively, $cT(B_E) \supset B_F$). In some cases the constant c is omitted.

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We use the symbol $\bigoplus_p$ for an ℓ_p-sum of Banach spaces, and $\widehat{\otimes}$ for a projective tensor product of Banach spaces.

By A we denote an arbitrary Banach algebra. The symbol A_+ stands for the standard unitization of A. In what follows we consider Banach modules with contractive outer action only. A Banach A-module X is called essential (faithful, annihilator) if the linear span of $A \cdot X$ is dense in X ($a \cdot X = \{0\}$ implies that $a = 0$, $A \cdot X = \{0\}$, respectively). A bounded linear operator which is also a morphism of A-modules is called an A-morphism. The symbol A-**mod** stands for the category of left Banach A-modules with A-morphisms. By A-$\mathbf{mod}_1$ we denote the subcategory of A-**mod** with the same objects, but contractive A-morphisms only. The analogous categories of right modules are denoted by **mod**-A and $\mathbf{mod}_1$-A, respectively. We use the symbol $\cong$ to denote an isomorphism of two objects in a category. By $\widehat{\otimes}_A$ we denote the functor of projective module tensor product and by Hom, the usual morphism functor. Now we can give our main definitions.

Definition 1. A left Banach A-module P is *metrically* (*C-topologically*, *C-relatively*) *projective* if the morphism functor $\operatorname{Hom}_{A\text{-}\mathbf{mod}_1}(P, -)$ (respectively, $\operatorname{Hom}_{A\text{-}\mathbf{mod}}(P, -)$, $\operatorname{Hom}_{A\text{-}\mathbf{mod}}(P, -)$) maps all strictly coisometric morphisms (respectively, strictly c-topologically surjective morphisms, morphisms with right inverse operator of norm at most c) to strictly coisometric (respectively, strictly cC-topologically surjective, strictly cC-topologically surjective) operators.

Definition 2. A right Banach A-module J is *metrically* (*C-topologically*, *C-relatively*) *injective* if the morphism functor $\operatorname{Hom}_{\mathbf{mod}_1\text{-}A}(-, J)$ (respectively, $\operatorname{Hom}_{\mathbf{mod}\text{-}A}(-, J)$, $\operatorname{Hom}_{\mathbf{mod}\text{-}A}(-, J)$) maps all strictly isometric morphisms (respectively, c-topologically injective morphisms, morphisms with left inverse operator of norm at most c) to strictly coisometric (respectively, strictly cC-topologically surjective, strictly cC-topologically surjective) operators.

Definition 3. A left Banach A-module F is *metrically* (*C-topologically*, *C-relatively*) *flat* if the functor of module tensor product $-\widehat{\otimes}_A F$ maps all isometric morphisms (respectively, c-topologically injective morphisms, morphisms with left inverse operator of norm at most c) to isometric (respectively, cC-topologically injective, cC-topologically injective) operators.

We shall say that a Banach module is *topologically* (*relatively*) projective, injective or flat if it is C-topologically (respectively, C-relatively) projective, injective or flat for some $C > 0$.

These definitions were given in a slightly different form by Graven for metric theory [4], by White for topological theory [5] and by Helemskii for relative theory [6]. For topologically projective, injective and flat modules White used the terms strictly projective, injective and flat, respectively. It is worth mentioning that strictly injective and flat modules were originally introduced earlier by Helemskii in [7], Ch. VII, § 1. An overview of the basics of these theories is given in [8]. We shall heavily rely upon results in the latter paper.

§ 3. Preliminaries on harmonic analysis

Let G be a locally compact group with unit e_G. The left Haar measure of G is denoted by m_G and the symbol Δ_G stands for the modular function of G. For an infinite and discrete (compact) group G we choose m_G to be the counting measure (respectively, the probability measure). In what follows, for $1 \leqslant p \leqslant +\infty$ we use the notation $L_p(G)$ to denote the Lebesgue space of functions that are p-integrable with respect to the Haar measure.

We regard $L_1(G)$ as a Banach algebra with convolution operator in the role of multiplication. This Banach algebra has a contractive two-sided approximate identity (see [9], Theorem 3.3.23). Clearly, $L_1(G)$ is unital if and only if G is discrete. In this case δ_{e_G}, the indicator function of e_G, is the identity of $L_1(G)$. Similarly, the space of complex finite Borel regular measures $M(G)$ endowed with convolution becomes a unital Banach algebra. The role of identity is played by the Dirac delta measure δ_{e_G} supported on e_G. Moreover, $M(G)$ is a coproduct, in the sense of category theory, in $L_1(G)$-$\mathbf{mod}_1$ (but not in $M(G)$-$\mathbf{mod}_1$) of the two-sided ideal $M_a(G)$ of measures which are absolutely continuous with respect to m_G and the subalgebra $M_s(G)$ of measures which are singular with respect to m_G. Note that $M_a(G) \cong L_1(G)$ in $M(G)$-$\mathbf{mod}_1$ and $M_s(G)$ is an annihilator $L_1(G)$-module. Finally, $M(G) = M_a(G)$ if and only if G is discrete.

Now we proceed to discuss the standard left and right modules over algebras $L_1(G)$ and $M(G)$. The Banach algebra $L_1(G)$ can be regarded as a two-sided ideal of $M(G)$ by means of the isometric left and right $M(G)$-morphism $i\colon L_1(G) \to M(G)$, $f \mapsto f m_G$. Therefore it is enough to define all module structures over $M(G)$. For any $1 \leqslant p < +\infty$, $f \in L_p(G)$ and $\mu \in M(G)$ we define

$$(\mu *_p f)(s) = \int_G f(t^{-1}s)\, d\mu(t) \quad \text{and} \quad (f *_p \mu)(s) = \int_G f(st^{-1})\Delta_G(t^{-1})^{1/p}\, d\mu(t).$$

These module actions turn all Banach spaces $L_p(G)$ for $1 \leqslant p < +\infty$ into left and right $M(G)$-modules. Note that for $p = 1$ and $\mu \in M_a(G)$ we get the usual definition of convolution. For $1 < p \leqslant +\infty$, $f \in L_p(G)$ and $\mu \in M(G)$ we define the module actions by

$$(\mu \cdot_p f)(s) = \int_G \Delta_G(t)^{1/p} f(st)\, d\mu(t) \quad \text{and} \quad (f \cdot_p \mu)(s) = \int_G f(ts)\, d\mu(t).$$

These module actions turn all Banach spaces $L_p(G)$ for $1 < p \leqslant +\infty$ into left and right $M(G)$-modules too. This special choice of module structure interacts nicely with duality. Indeed we have an isomorphism $(L_p(G), *_p)^* \cong (L_{p^*}(G), \cdot_{p^*})$ in $\mathbf{mod}_1$-$M(G)$ for all $1 \leqslant p < +\infty$. Here we set by definition $p^* = p/(p-1)$ for $1 < p < +\infty$ and $p^* = \infty$ for $p = 1$. Finally, the Banach space $C_0(G)$ also becomes a left and a right $M(G)$-module when endowed with $\cdot_\infty$ in the role of a module action. Moreover, $C_0(G)$ is a closed left and right $M(G)$-submodule of $L_\infty(G)$ such that $(C_0(G), \cdot_\infty)^* \cong (M(G), *)$ in $M(G)$-$\mathbf{mod}_1$.

By $\widehat{G}$ we denote the dual group of the group G. Any character $\gamma \in \widehat{G}$ gives rise to continuous characters

$$\varkappa_\gamma^L\colon L_1(G) \to \mathbb{C}, \quad f \mapsto \int_G f(s)\overline{\gamma(s)}\, dm_G(s),$$

and

$$\varkappa_\gamma^M\colon M(G) \to \mathbb{C}, \quad \mu \mapsto \int_G \overline{\gamma(s)}\, d\mu(s),$$

on $L_1(G)$ and $M(G)$, respectively. By $\mathbb{C}_\gamma$ we denote a left and right augmentation $L_1(G)$- or $M(G)$-module. Its module actions are defined by

$$f \cdot_\gamma z = z \cdot_\gamma f = \varkappa_\gamma^L(f)z \quad \text{and} \quad \mu \cdot_\gamma z = z \cdot_\gamma \mu = \varkappa_\gamma^M(\mu)z$$

for all $f \in L_1(G)$, $\mu \in M(G)$ and $z \in \mathbb{C}$.

One of the numerous definitions of amenable group says that a locally compact group G is amenable if there exists an $L_1(G)$-morphism of right modules $M\colon L_\infty(G) \to \mathbb{C}_{e_{\widehat{G}}}$ such that $M(\chi_G) = 1$ (see [7], Ch. VII, § 2.5). We can even assume that M is contractive (see [7], Remark VII.1.54).

Most results in this section that are not supported with references are presented in full detail in [9], § 3.3.

§ 4. $L_1(G)$-modules

The metric homological properties of $L_1(G)$-modules of harmonic analysis were first studied in [4]. We generalise these ideas for the case of topological Banach homology. To clarify definitions we start from a general result on injectivity. It is instructive to prove it from first principles.

Proposition 1. *Let A be a Banach algebra with right contractive approximate identity; then the right A-module A^* is metrically injective.*

Proof. Let $\xi\colon Y \to X$ be an isometric A-morphism of right A-modules X and Y and an arbitrary contractive A-morphism $\varphi\colon X \to A^*$. By assumption A has a contractive approximate identity, say $(e_\nu)_{\nu\in N}$. For each $\nu \in N$ we define a bounded linear functional $f_\nu\colon Y \to \mathbb{C}$ by $y \mapsto \varphi(y)(e_\nu)$. By the Hahn-Banach theorem there exists a bounded linear functional $g_\nu\colon X \to \mathbb{C}$ such that $g_\nu\xi = f_\nu$ and $\|g_\nu\| = \|f_\nu\|$. It is routine to check that $\psi_\nu\colon X \to A^*$, $x \mapsto (a \mapsto g_\nu(x \cdot a))$, is an A-morphism of right modules such that $\|\psi_\nu\| \leqslant \|\varphi\|$ and $\psi_\nu(\xi(x))(a) = \varphi(x)(ae_\nu)$ for all $x \in X$ and $a \in A$. Since the net $(\psi_\nu)_{\nu\in N}$ is norm bounded, there exists a subnet $(\psi_\mu)_{\mu\in M}$ with the same norm bound that converges in the strong-to-weak* topology to some operator $\psi\colon X \to A^*$. Clearly, ψ is a morphism of right A-modules such that $\psi\xi = \varphi$ and $\|\psi\| \leqslant \|\varphi\|$. As φ is arbitrary, the map $\operatorname{Hom}_{\mathbf{mod}_1\text{-}A}(\xi, A^*)$ is strictly coisometric. Hence A^* is metrically injective. The proposition is proved.

Proposition 2. *Let G be a locally compact group. Then $L_\infty(G)$ is a metrically and topologically injective $L_1(G)$-module. As a result, the $L_1(G)$-module $L_1(G)$ is metrically and topologically flat.*

Proof. As $L_1(G)$ has a contractive approximate identity, by Proposition 1 the right $L_1(G)$-module $L_1(G)^*$ is metrically injective. As a consequence, it is topologically injective (see [8], Proposition 2.14). Therefore, it remains to recall that $L_\infty(G) \cong L_1(G)^*$ in $\mathbf{mod}_1$-$L_1(G)$. The result on the flatness of $L_1(G)$ follows from Proposition 2.21 in [8]. The proposition is proved.

Proposition 3. *Let G be a locally compact group, and $\gamma \in \widehat{G}$. Then the following are equivalent:*

1) *G is compact;*
2) *$\mathbb{C}_\gamma$ is a metrically projective $L_1(G)$-module;*
3) *$\mathbb{C}_\gamma$ is a topologically projective $L_1(G)$-module.*

The proof of 1) $\Rightarrow$ 2) and 3) $\Rightarrow$ 1) is similar to Theorem 4.2 in [4].
The implication 2) $\Rightarrow$ 3) follows from Proposition 2.4 in [8].

Proposition 4. *Let G be a locally compact group, and $\gamma \in \widehat{G}$. Then the following are equivalent*:

1) *G is amenable*;
2) *$\mathbb{C}_\gamma$ is a metrically injective $L_1(G)$-module*;
3) *$\mathbb{C}_\gamma$ is a topologically injective $L_1(G)$-module*;
4) *$\mathbb{C}_\gamma$ is a metrically flat $L_1(G)$-module*;
5) *$\mathbb{C}_\gamma$ is a topologically flat $L_1(G)$-module.*

Proof. 1) $\Rightarrow$ 2), 3) $\Rightarrow$ 1) The proof is similar to Theorem 4.5 in [4].

2) $\Rightarrow$ 3) This implication immediately follows from Proposition 2.14 in [8].

2) $\Rightarrow$ 4), 3) $\Rightarrow$ 5) Note that $\mathbb{C}_\gamma^* \cong \mathbb{C}_\gamma$ in $\mathbf{mod}_1$-$L_1(G)$, so all equivalences follow from the three previous paragraphs and the fact that flat modules are precisely the modules with injective dual (see [8], Proposition 2.21). The proposition is proved.

In the following statement we study specific ideals of the Banach algebra $L_1(G)$, namely the ideals of the form $L_1(G) * \mu$ for some idempotent measure μ. In fact, for the case of commutative compact groups this class of ideals coincides with those left ideals of $L_1(G)$ that admit a right bounded approximate identity.

Theorem 1. *Let G be a locally compact group and $\mu \in M(G)$ be an idempotent measure, that is $\mu * \mu = \mu$. Assume that the left ideal $I = L_1(G) * \mu$ of the Banach algebra $L_1(G)$ is a topologically projective $L_1(G)$-module. Then $\mu = pm_G$ for some $p \in I$.*

Proof. Let $\varphi\colon I \to L_1(G)$ be an arbitrary morphism of left $L_1(G)$-modules. Consider the $L_1(G)$-morphism $\varphi'\colon L_1(G) \to L_1(G)$, $x \mapsto \varphi(x*\mu)$. By Wendel's theorem (see [10], Theorem 1) there exists a measure $\nu \in M(G)$ such that $\varphi'(x) = x * \nu$ for all $x \in L_1(G)$. In particular, $\varphi(x) = \varphi(x * \mu) = \varphi'(x) = x * \nu$ for all $x \in I$. It is clear now that $\psi\colon I \to I$, $x \mapsto \nu * x$, is a morphism of right I-modules satisfying $\varphi(x)y = x\psi(y)$ for all $x, y \in I$. By Lemma 2, (ii), in [11] the ideal I has a right identity, say $e \in I$. Then $x * \mu = x * \mu * e$ for all $x \in L_1(G)$. Two measures are equal if their convolutions with all functions of $L_1(G)$ coincide (see [9], Corollary 3.3.24), so $\mu = \mu * em_G$. Since $e \in I \subset L_1(G)$, we have $\mu = \mu * em_G \in M_a(G)$. Set $p = \mu * e \in I$, then $\mu = pm_G$. The theorem is proved.

We conjecture that a left ideal of the form $L_1(G)*\mu$ for an idempotent measure μ is a metrically projective $L_1(G)$-module if and only if $\mu = pm_G$ for $p \in I$ with $\|p\| = 1$. In [4] Graven gave a criterion of metric projectivity of the $L_1(G)$-module $L_1(G)$. Now we can prove this fact as a mere corollary.

Theorem 2. *Let G be a locally compact group. Then the following are equivalent*:

1) *G is discrete*;
2) *$L_1(G)$ is a metrically projective $L_1(G)$-module*;
3) *$L_1(G)$ is a topologically projective $L_1(G)$-module.*

Proof. 1) $\Rightarrow$ 2) If G is discrete, then $L_1(G)$ is unital with unit of norm 1. From [11], Proposition 7, we conclude that $L_1(G)$ is metrically projective as an $L_1(G)$-module.

2) $\Rightarrow$ 3) This implication is a direct corollary of Proposition 2.4 in [8].

3) $\Rightarrow$ 1) Clearly, δ_{e_G} is an idempotent measure. Since $L_1(G) = L_1(G) * \delta_{e_G}$ is topologically projective, then by Theorem 1 we have $\delta_{e_G} = f m_G$ for some $f \in L_1(G)$. This is possible only if G is discrete. The theorem is proved.

Note that the $L_1(G)$-module $L_1(G)$ is relatively projective for any locally compact group G (see [7], Exercise VII.1.17).

Proposition 5. *Let G be a locally compact group. Then the following are equivalent*:

1) *G is discrete*;
2) *$M(G)$ is a metrically projective $L_1(G)$-module*;
3) *$M(G)$ is a topologically projective $L_1(G)$-module*;
4) *$M(G)$ is a metrically flat $L_1(G)$-module.*

Proof. 1) $\Rightarrow$ 2) We have $M(G) \cong L_1(G)$ in $L_1(G)$-$\mathbf{mod}_1$ for discrete G, so the result follows from Theorem 2.

2) $\Rightarrow$ 3) See [8], Proposition 2.4.

2) $\Rightarrow$ 4) The implication follows from Proposition 2.26 in [8].

3) $\Rightarrow$ 1) Recall that $M(G) \cong L_1(G) \bigoplus_1 M_s(G)$ in $L_1(G)$-$\mathbf{mod}_1$, so $M_s(G)$ is topologically projective as a retract of a topologically projective module (see [8], Proposition 2.2). Note that $M_s(G)$ is also an annihilator $L_1(G)$-module, and therefore the algebra $L_1(G)$ has a right identity (see [8], Proposition 3.3). Recall that $L_1(G)$ also has a two-sided bounded approximate identity, so $L_1(G)$ is unital. The latter is equivalent to G being discrete.

4) $\Rightarrow$ 1) Note that $M(G) \cong L_1(G) \bigoplus_1 M_s(G)$ in $L_1(G)$-$\mathbf{mod}_1$, so $M_s(G)$ is metrically flat as a retract of a metrically flat module (see [8], Proposition 2.27). Recall also that $M_s(G)$ is an annihilator module over a nonzero algebra $L_1(G)$, therefore $M_s(G)$ must be a zero module (see [8], Proposition 3.6). The latter is equivalent to G being discrete. The proposition is proved.

Proposition 6. *Let G be a locally compact group. Then $M(G)$ is a topologically flat $L_1(G)$-module.*

Proof. Since $M(G)$ is an L_1-space it is *a fortiori* an $\mathscr{L}_1^g$-space (see [12], § 3.13, Exercise 4.7, (b)). Since $M_s(G)$ is complemented in $M(G)$, $M_s(G)$ is an $\mathscr{L}_1^g$-space too (see [12], Corollary 23.2.1, (2)). Moreover, since $M_s(G)$ is an annihilator $L_1(G)$-module, it is a topologically flat $L_1(G)$-module (see [8], Proposition 3.6). The $L_1(G)$-module $L_1(G)$ is also topologically flat by Proposition 2. Note that $M(G) \cong L_1(G) \bigoplus_1 M_s(G)$ in $L_1(G)$-$\mathbf{mod}_1$, so the $L_1(G)$-module $M(G)$ is topologically flat as a sum of topologically flat modules (see [8], Proposition 2.27). The proposition is proved.

§ 5. *M(G)*-modules

We turn to the study of the standard $M(G)$-modules of harmonic analysis. As we shall see, most of the results can be derived from previous theorems and propositions on $L_1(G)$-modules.

Proposition 7. *Let G be a locally compact group, and X be an essential (faithful) $L_1(G)$-module. Then*

1) *X is a metrically projective or metrically flat (respectively, injective) $M(G)$-module if and only if it is a metrically projective or metrically flat (respectively, injective) $L_1(G)$-module;*
2) *X is a topologically projective or topologically flat (respectively, injective) $M(G)$-module if and only if it is a topologically projective or topologically flat (respectively, injective) $L_1(G)$-module.*

Recall that $L_1(G)$ is a two-sided contractively complemented ideal of $M(G)$. Now 1) and 2) follow from Proposition 2.6 in [8] or Proposition 2.24 in [8] (respectively, Proposition 2.16 in [8]).

It is worth mentioning here that $L_1(G)$-modules $C_0(G)$, $L_p(G)$ for $1 \leqslant p < \infty$ and $\mathbb{C}_\gamma$ for $\gamma \in \widehat{G}$ are essential and $L_1(G)$-modules $C_0(G)$, $M(G)$, $L_p(G)$ for $1 \leqslant p \leqslant \infty$ and $\mathbb{C}_\gamma$ for $\gamma \in \widehat{G}$ are faithful.

Proposition 8. *Let G be a locally compact group. Then $M(G)$ is a metrically and topologically projective $M(G)$-module. As a consequence, it is a metrically and topologically flat $M(G)$-module.*

Proof. As $M(G)$ is a unital algebra, the metric (topological) projectivity of $M(G)$ follows from Proposition 7 in [11] since one may regard $M(G)$ as a unital ideal of $M(G)$. It remains to recall that any metrically (topologically) projective module is metrically (topologically) flat (see [8], Proposition 2.26). The proposition is proved.

§ 6. Banach geometric restrictions

In this section we show that many modules of harmonic analysis fail to be metrically or topologically projective, injective or flat for purely Banach geometric reasons. In metric theory for infinite-dimensional $L_1(G)$-modules $L_p(G)$, $M(G)$ and $C_0(G)$ this was done in [4], Theorems 4.12–4.14.

Proposition 9. *Let G be an infinite locally compact group. Then*

1) *$L_1(G)$, $C_0(G)$, $M(G)$ and $L_\infty(G)^*$ are not topologically injective Banach spaces;*
2) *$C_0(G)$ and $L_\infty(G)$ are not complemented in any L_1-space.*

Proof. Since G is infinite all modules in question are infinite dimensional.

1) If an infinite-dimensional Banach space is topologically injective, it contains a copy of $\ell_\infty(\mathbb{N})$ (see [13], Corollary 1.1.4) and consequently, a copy of $c_0(\mathbb{N})$. The Banach space $L_1(G)$ is weakly sequentially complete (see [14], Part III.C, Corollary 14), so by Corollary 5.2.11 in [15] it cannot contain a copy of $c_0(\mathbb{N})$. Therefore, $L_1(G)$ is not a topologically injective Banach space. Assume that $M(G)$ is topologically injective, then so is its complemented subspace $M_a(G)$, which is isometrically isomorphic to $L_1(G)$. By the previous argument this is impossible, a contradiction. By Corollary 3 in [16] the Banach space $C_0(G)$ is not complemented in $L_\infty(G)$, hence it cannot be topologically injective. Note that $L_1(G)$ is complemented in $L_\infty(G)^*$ which is isometrically isomorphic to $L_1(G)^{**}$ (see [12], Proposition B10). Therefore, if $L_\infty(G)^*$ is topologically injective as a Banach space,

then so is its retract $L_1(G)$. By the previous argument this is impossible, a contradiction.

2) Suppose $C_0(G)$ is a retract of an L_1-space; then $M(G)$, which is isometrically isomorphic to $C_0(G)^*$, is a retract of L_∞-space. Therefore, $M(G)$ must be a topologically injective Banach space. This contradicts part 1). Note that $\ell_\infty(\mathbb{N})$ embeds in $L_\infty(G)$, hence so does $c_0(\mathbb{N})$. If $L_\infty(G)$ is a retract of L_1-space, then there exists an L_1-space containing a copy of $c_0(\mathbb{N})$. This is impossible as already shown in part 1). The proposition is proved.

From now on, by A we denote either $L_1(G)$ or $M(G)$. Recall that $L_1(G)$ and $M(G)$ are both L_1-spaces.

Proposition 10. *Let G be an infinite locally compact group. Then*

1) *$C_0(G)$ and $L_\infty(G)$ are neither topologically nor metrically projective A-modules*;
2) *$L_1(G)$, $C_0(G)$, $M(G)$ and $L_\infty(G)^*$ are neither topologically nor metrically injective A-modules*;
3) *$L_\infty(G)$ and $C_0(G)$ are neither topologically nor metrically flat A-modules*;
4) *$L_p(G)$ for $1 < p < \infty$ are neither topologically nor metrically projective, injective or flat A-modules.*

Proof. 1) Every metrically or topologically projective A-module is complemented in some L_1-space (see [8], Proposition 3.8). Now the result follows from Proposition 9, 2).

2) Every metrically or topologically injective A-module is topologically injective as a Banach space (see [8], Proposition 3.8). It remains to apply Proposition 9, 1).

3) Note that $C_0(G)^* \cong M(G)$ in $\mathbf{mod}_1$-A. Now the result follows from part 1) and the fact that the dual module of a flat module is injective (see [8], Proposition 2.21).

4) Since $L_p(G)$ is reflexive for $1 < p < \infty$, the result follows from Corollary 3.14 in [8]. The proposition is proved.

Now we consider the metric and topological homological properties of A-modules when G is finite.

Proposition 11. *Let G be a nontrivial finite group and let $1 \leqslant p \leqslant \infty$. Then the A-module $L_p(G)$ is metrically projective* (*injective*) *if and only if $p = 1$* (*respectively*, *$p = \infty$*).

Proof. Assume that $L_p(G)$ is metrically projective (injective) as an A-module. As $L_p(G)$ is finite dimensional, there exists an isometric isomorphism $L_p(G) \cong \ell_1(\mathbb{N}_n)$ (respectively, $L_p(G) \cong \ell_\infty(\mathbb{N}_n)$), where $n = \mathrm{Card}(G) > 1$; see [8], Proposition 3.8, (i) and (ii). Now we use the result of Theorem 1 in [17] for Banach spaces over the field $\mathbb{C}$: if for $2 \leqslant m \leqslant k$ and $1 \leqslant r, s \leqslant \infty$ there exists an isometric embedding of $\ell_r(\mathbb{N}_m)$ into $\ell_s(\mathbb{N}_k)$, then either $r = 2$, $s \in 2\mathbb{N}$ or $r = s$. Therefore, $p = 1$ (respectively, $p = \infty$). The converse easily follows from Theorem 2 (respectively, Proposition 2). The proposition is proved.

Proposition 12. *Let G be a finite group. Then*

1) *$C_0(G)$ and $L_\infty(G)$ are metrically injective A-modules*;

2) $C_0(G)$ *and* $L_p(G)$ *for* $1 < p \leqslant \infty$ *are metrically projective* A*-modules if and only if* G *is trivial*;
3) $M(G)$ *and* $L_p(G)$ *for* $1 \leqslant p < \infty$ *are metrically injective* A*-modules if and only if* G *is trivial*;
4) $C_0(G)$ *and* $L_p(G)$ *for* $1 < p \leqslant \infty$ *are metrically flat* A*-modules if and only if* G *is trivial.*

Proof. 1) Since G is finite, we have $C_0(G) = L_\infty(G)$. The result follows from Proposition 2.

2) If G is trivial, that is $G = \{e_G\}$, then $L_p(G) = C_0(G) = L_1(G)$ and the result follows from part 1). If G is nontrivial, then we recall that $C_0(G) = L_\infty(G)$ and use Proposition 11.

3) If $G = \{e_G\}$, then $M(G) = L_p(G) = L_\infty(G)$ and the result follows from part 1). If G is nontrivial, then we note that $M(G) = L_1(G)$ and use Proposition 11.

4) From part 3) it follows that $L_p(G)$ for $1 \leqslant p < \infty$ is a metrically injective A-module if and only if G is trivial. Recall that a Banach module is flat if and only if its dual is injective (see [8], Proposition 2.21). Now the result for $L_p(G)$ follows from the identifications $L_p(G)^* \cong L_{p^*}(G)$ in $\mathbf{mod}_1$-$L_1(G)$ for $1 \leqslant p^* < \infty$. Similarly, using the above characterisation of flat modules and isomorphisms $C_0(G)^* \cong M(G) \cong L_1(G)$ in $\mathbf{mod}_1$-$L_1(G)$ we get a criterion of injectivity of $M(G)$. The proposition is proved.

It is worth mentioning here that if we consider all Banach spaces over the field of real numbers, then $L_\infty(G)$ and $L_1(G)$ will be metrically projective and injective, respectively, for the group G consisting of two elements. The reason is that $L_\infty(\mathbb{Z}_2) \cong \mathbb{R}_{\gamma_0} \bigoplus_1 \mathbb{R}_{\gamma_1}$ in $L_1(\mathbb{Z}_2)$-$\mathbf{mod}_1$ and $L_1(\mathbb{Z}_2) \cong \mathbb{R}_{\gamma_0} \bigoplus_\infty \mathbb{R}_{\gamma_1}$ in $\mathbf{mod}_1$-$L_1(\mathbb{Z}_2)$. Here, $\mathbb{Z}_2$ denotes the unique group of two elements and $\gamma_0, \gamma_1 \in \widehat{\mathbb{Z}_2}$ are the characters defined by $\gamma_0(0) = \gamma_0(1) = \gamma_1(0) = -\gamma_1(1) = 1$.

Proposition 13. *Let* G *be a finite group. Then for* $1 \leqslant p \leqslant \infty$ *the* A*-modules* $C_0(G)$, $M(G)$ *and* $L_p(G)$ *are topologically projective*, *injective and flat.*

Proof. For a finite group G we have $M(G) = L_1(G)$ and $C_0(G) = L_\infty(G)$, so the modules $C_0(G)$ and $M(G)$ do not require special considerations. Since $M(G) = L_1(G)$, we can restrict our considerations to the case $A = L_1(G)$. The identity map $i\colon L_1(G) \to L_p(G)$, $f \mapsto f$, is a topological isomorphism of Banach spaces, because $L_1(G)$ and $L_p(G)$ for $1 \leqslant p < +\infty$ are of equal finite dimension. Since G is finite, it is unimodular. Therefore, the module actions in $(L_1(G), *)$ and $(L_p(G), *_p)$ coincide for $1 \leqslant p < +\infty$. Hence i is an isomorphism in $L_1(G)$-$\mathbf{mod}$ and $\mathbf{mod}$-$L_1(G)$. One can show similarly that $(L_\infty(G), \cdot_\infty)$ and $(L_p(G), \cdot_p)$, where $1 < p \leqslant +\infty$, are isomorphic in $L_1(G)$-$\mathbf{mod}$ and $\mathbf{mod}$-$L_1(G)$. Finally, one can easily check that $(L_1(G), *)$ and $(L_\infty(G), \cdot_\infty)$ are isomorphic in $L_1(G)$-$\mathbf{mod}$ and $\mathbf{mod}$-$L_1(G)$ via the map $j\colon L_1(G) \to L_\infty(G)$, $f \mapsto (s \mapsto f(s^{-1}))$. Thus all the modules in question are pairwise isomorphic. It remains to recall that $L_1(G)$ is topologically projective and flat by Theorem 2 and Proposition 2, while $L_\infty(G)$ is topologically injective by Proposition 2. The proposition is proved.

We summarise the results on the homological properties of modules of harmonic analysis into Tables 1 and 2. Each cell in each table contains a condition under which the corresponding module has the corresponding property and references to the proofs. The arrow ⇒ indicates that only a necessary condition is known. We should mention that results for modules $L_p(G)$, where $1 < p < \infty$, are valid for both module actions $*_p$ and $\cdot_p$. Characterisations and proofs for the homologically trivial modules $\mathbb{C}_\gamma$ in the case of relative theory are the same as in Propositions 3 and 4, but these results are already well known. For example, the projectivity of $\mathbb{C}_\gamma$ was characterized in [7], Theorem IV.5.13, and the criterion of injectivity was given in [18], Theorem 2.5. For algebras $L_1(G)$ and $M(G)$ the notions of projectivity (injectivity or flatness) coincide for all the three theories when one deals with the modules $M(G)$ and $\mathbb{C}_\gamma$ (respectively, $L_\infty(G)$, $C_0(G)$ and $\mathbb{C}_\gamma$ or $L_1(G)$ and $\mathbb{C}_\gamma$). Finally, the $M(G)$-modules $M(G)$ also have the same characterization of flatness in metric, topological and relative theory.

Table 1. Homologically trivial modules of harmonic analysis

	$L_1(G)$-modules		
	Projectivity	Injectivity	Flatness
Metric theory			
$L_1(G)$	G is discrete Theorem 2	$G = \{e_G\}$ Propositions 10, 12	G is arbitrary Proposition 2
$L_p(G)$	$G = \{e_G\}$ Propositions 10, 11	$G = \{e_G\}$ Propositions 10, 11	$G = \{e_G\}$ Propositions 10, 12
$L_\infty(G)$	$G = \{e_G\}$ Propositions 10, 11	G is arbitrary Proposition 2	$G = \{e_G\}$ Propositions 10, 12
$M(G)$	G is discrete Proposition 5	$G = \{e_G\}$ Propositions 10, 12	G is discrete Proposition 6
$C_0(G)$	$G = \{e_G\}$ Propositions 10, 12	G is finite Propositions 10, 12	$G = \{e_G\}$ Propositions 10, 12
$\mathbb{C}_\gamma$	G is compact Proposition 3	G is amenable Proposition 4	G is amenable Proposition 4
Topological theory			
$L_1(G)$	G is discrete Theorem 2	G is finite Propositions 10, 13	G is arbitrary Proposition 2
$L_p(G)$	G is finite Propositions 10, 13	G is finite Propositions 10, 13	G is finite Propositions 10, 13
$L_\infty(G)$	G is finite Propositions 10, 13	G is arbitrary Proposition 2	G is finite Propositions 10, 13
$M(G)$	G is discrete Proposition 5	G is finite Propositions 10, 13	G is arbitrary Proposition 6
$C_0(G)$	G is finite Propositions 10, 13	G is finite Propositions 10, 13	G is finite Propositions 10, 13
$\mathbb{C}_\gamma$	G is compact Proposition 3	G is amenable Proposition 4	G is amenable Proposition 4

Table 1 (continuation)

Relative theory

$L_1(G)$	G is arbitrary [1], § 6	G is amenable and discrete [1], § 6	G is arbitrary [1], § 6
$L_p(G)$	G is compact [1], § 6	G is amenable [3]	G is amenable [3]
$L_\infty(G)$	G is finite [1], § 6	G is arbitrary [1], § 6	G is amenable [1], § 6
$M(G)$	G is discrete [1], § 6	G is amenable [1], § 6	G is arbitrary [2], § 3.5
$C_0(G)$	G is compact [1], § 6	G is finite [1], § 6	G is amenable [1], § 6
$\mathbb{C}_\gamma$	G is compact Proposition 3	G is amenable Proposition 4	G is amenable Proposition 4

Table 2. Homologically trivial modules of harmonic analysis

	$M(G)$-modules		
	Projectivity	Injectivity	Flatness
	Metric theory		
$L_1(G)$	G is discrete Theorem 2, Proposition 7	$G = \{e_G\}$ Propositions 10, 12	G is arbitrary Propositions 2, 7
$L_p(G)$	$G = \{e_G\}$ Propositions 10, 11	$G = \{e_G\}$ Propositions 10, 11	$G = \{e_G\}$ Propositions 10, 12
$L_\infty(G)$	$G = \{e_G\}$ Propositions 10, 11	G is arbitrary Propositions 2, 7	$G = \{e_G\}$ Propositions 10, 12
$M(G)$	G is arbitrary Proposition 8	$G = \{e_G\}$ Propositions 10, 12	G is arbitrary Proposition 8
$C_0(G)$	$G = \{e_G\}$ Propositions 10, 12	G is finite Propositions 10, 12	$G = \{e_G\}$ Propositions 10, 12
$\mathbb{C}_\gamma$	G is compact Propositions 3, 7	G is amenable Propositions 4, 7	G is amenable Propositions 4, 7
	Topological theory		
$L_1(G)$	G is discrete Theorem 2, Proposition 7	G is finite Propositions 10, 13	G is arbitrary Propositions 2, 7
$L_p(G)$	G is finite Propositions 10, 13	G is finite Propositions 10, 13	G is finite Propositions 10, 13
$L_\infty(G)$	G is finite Propositions 10, 13	G is arbitrary Propositions 2, 7	G is finite Propositions 10, 13

TABLE 2 (continuation)

$M(G)$	G is arbitrary Proposition 8	G is finite Propositions 10, 13	G is arbitrary Proposition 8
$C_0(G)$	G is finite Propositions 10, 13	G is finite Propositions 10, 13	G is finite Propositions 10, 13
$\mathbb{C}_\gamma$	G is compact Propositions 3, 7	G is amenable Propositions 4, 7	G is amenable Propositions 4, 7
Relative theory			
$L_1(G)$	G is arbitrary [2], § 3.5	G is amenable and discrete [2], § 3.5	G is arbitrary [2], § 3.5
$L_p(G)$	G is compact [2], § 3.5	G is amenable [2], § 3.5, [3]	G is amenable [2], § 3.5
$L_\infty(G)$	G is finite [2], § 3.5	G is arbitrary [2], § 3.5	G is amenable ($\Rightarrow$) [2], § 3.5
$M(G)$	G is arbitrary [2], § 3.5	G is amenable [2], § 3.5	G is arbitrary [2], § 3.5
$C_0(G)$	G is compact [2], § 3.5	G is finite [2], § 3.5	G is amenable [2], § 3.5
$\mathbb{C}_\gamma$	G is compact Propositions 3, 7	G is amenable Propositions 4, 7	G is amenable Propositions 4, 7

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