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Asymptotic analysis of solutions of ordinary differential equations with distribution coefficients

A. M. Savchuk and A. A. Shkalikov

Abstract. Ordinary differential equations of the form

$$\tau(y) - \lambda^{2m}\varrho(x)y = 0, \qquad \tau(y) = \sum_{k,s=0}^{m} (\tau_{k,s}(x)y^{(m-k)}(x))^{(m-s)},$$

on the finite interval $x \in [0,1]$ are under consideration. Here the functions $\tau_{0,0}$ and ϱ are absolutely continuous and positive and the coefficients of the differential expression $\tau(y)$ are subject to the conditions

$$\tau_{k,s}^{(-l)} \in L_2[0,1], \qquad 0 \leqslant k, s \leqslant m, \quad l = \min\{k,s\},$$

where $f^{(-k)}$ denotes the kth antiderivative of the function f in the sense of distributions. Our purpose is to derive analogues of the classical asymptotic Birkhoff-type representations for the fundamental system of solutions of the above equation with respect to the spectral parameter as $\lambda \to \infty$ in certain sectors of the complex plane $\mathbb{C}$. We reduce this equation to a system of first-order equations of the form

$$\mathbf{y}' = \lambda\rho(x)\mathrm{B}\mathbf{y} + \mathrm{A}(x)\mathbf{y} + \mathrm{C}(x,\lambda)\mathbf{y},$$

where ρ is a positive function, B is a matrix with constant elements, the elements of the matrices $\mathrm{A}(x)$ and $\mathrm{C}(x,\lambda)$ are integrable functions, and $\|\mathrm{C}(x,\lambda)\|_{L_1} = o(1)$ as $\lambda \to \infty$. For systems of this kind, we obtain new results concerning the asymptotic representation of the fundamental solution matrix, which we use to make an asymptotic analysis of the above scalar equations of high order.

Bibliography: 44 titles.

Keywords: differential equations with distribution coefficients, asymptotics with respect to the spectral parameter, Birkhoff asymptotics, spectral asymptotics.

§ 1. Introduction

This study mainly aims to derive asymptotic formulae for solutions of ordinary differential equations of the form

$$\tau(y) - \lambda^{2m}\sigma(x)y = 0, \qquad \tau(y) = \sum_{k,s=0}^{m} (\tau_{k,s}(x)y^{(m-k)}(x))^{(m-s)}, \tag{1.1}$$

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under minimal assumptions on the smoothness of the coefficients $\tau_{k,s}$ and σ. Here λ is a complex parameter which plays the role of a spectral parameter when, in addition, boundary conditions are prescribed. We are going to obtain asymptotic formulae as $\lambda \to \infty$ in sectors of the complex plane that, in combination, cover the whole of the plane. In this work, we restrict ourselves to equations of even order $n = 2m$; the case of odd order requires a separate consideration.

This topic has more than a century of history, beginning with Birkhoff in his classical works (see [1] and [2]). The results currently available make it possible to establish Birkhoff asymptotics under the following conditions on the coefficients of the differential expression τ: $\tau_{0,0} \equiv 1$ and the $\tau_{k,s}$ have $m-s-1$ absolutely continuous derivatives, which is equivalent to the fact that the functions $\tau_{k,s}$ belong to the Sobolev space $W_1^{m-s}[0,1]$ (here $W_1^0[0,1] = L_1[0,1]$). These conditions are too restrictive. In this paper, we derive the required asymptotics not only in the case when the coefficients $\tau_{k,s}$ are classical (integrable) functions but also when they are generalized functions with a finite order of singularity. More precisely, we show that the necessary asymptotic formulae are only valid when the conditions

$$\frac{1}{\sqrt{|\tau_0|}}, \ \frac{1}{\sqrt{|\tau_0|}}\tau_{k,s}^{(-l)} \in L_2[0,1], \qquad 0 \leqslant k,s \leqslant m, \quad l = \min\{k,s\}, \tag{1.2}$$

hold, where $\tau_0 := \tau_{0,0}$ and $f^{(-k)}$ denotes the kth antiderivative of the function f, understood in the sense of the theory of distributions. In other words, under the additional condition that τ_0 is uniformly positive, it suffices to assume that the coefficients $\tau_{k,s}$ are the generalized derivatives of the lth order of square-integrable functions, where $l = \min\{k,s\}$. Of course, obtaining such a result requires not only serious technical work (which we have tried to simplify as much as possible) but also new methods. In particular, one of the most significant points in the implementation of our programme is the use of the *regularization method*. This method was initiated by the authors in [3] and [4] for the Sturm-Liouville equation. In subsequent studies (see [5]–[10]) it was extended to general second-order equations and some special classes of equations of fourth and higher orders. In this paper we use the result obtained recently by Mirzoev and Shkalikov in [11], where the general equations (1.1) were regularized.

Using the regularization method leads to the idea to derive asymptotic formulae by reducing (1.1) to a system of $2m$ equations of the first order. However, the available results for systems do not suit us completely, and we come to our second aim, which is to obtain asymptotic formulae for the fundamental solution matrix of a first-order system of the form

$$\mathbf{y}' = \lambda \mathrm{V}(x)\mathbf{y} + \mathrm{A}(x)\mathbf{y} + \mathrm{C}(x,\lambda)\mathbf{y}, \qquad x \in [0,1], \tag{1.3}$$

with a large complex parameter λ. Here

$$\mathbf{y} = \mathbf{y}(x) = (y_1(x), y_2(x), \dots, y_n(x))^\top$$

is a column vector of absolutely continuous functions $y_j(x)$ on $[0,1]$. As the set of values of the parameter λ, we consider a domain of the form $\{\lambda \in \mathbb{C} \colon |\lambda| > \lambda_0\}$, where $\lambda_0 > 0$ is an arbitrary fixed number.

We formulate the conditions under which we investigate this system:

(i) The matrix $\mathrm{V}(x)$ has the form $\mathrm{V}(x) = \rho(x)\mathrm{B}$, where $\mathrm{B} = \operatorname{diag}\{b_1, \dots, b_n\}$ is a constant diagonal matrix and the complex numbers b_j are nonzero. The function ρ is assumed to be integrable on $[0, 1]$ and positive almost everywhere. We denote its antiderivative by $p(x) := \int_0^x \rho(t)\,dt$.

(ii) All the coefficients of the system, $a_{jk}(x)$ and $c_{jk}(x, \lambda)$, (which are functions of the variable x for each fixed λ) are assumed to be Lebesgue integrable on the whole of the interval $[0, 1]$.

(iii) We assume that the relations $c_{jk}(x, \lambda) = o(1)$ hold in the L_1-norm as $|\lambda| \to \infty$, that is, $\int_0^1 |c_{jk}(x, \lambda)|\,dx \to 0$ for all $1 \leqslant j, k \leqslant n$.

In fact, systems with matrix $\mathrm{V}(x)$ of a more general form than the diagonal matrices involved in (i) can be reduced to the system (1.3). We shall discuss this in more detail at the beginning of § 2.

A *fundamental solution matrix* of (1.3) is a matrix $\mathrm{Y}(x, \lambda)$ of rank n defined for $x \in [0, 1]$ and $\lambda \in \mathbb{C}$, $|\lambda| > \lambda_0$, each column $\mathrm{Y}_k(x, \lambda)$ of which is a solution of (1.3). Thus, Y satisfies the matrix differential equation

$$\mathrm{Y}'(x, \lambda) = \bigl(\lambda\rho(x)\mathrm{B} + \mathrm{A}(x) + \mathrm{C}(x, \lambda)\bigr)\mathrm{Y}(x, \lambda) \tag{1.4}$$

for each fixed λ. It is well known (see [12], for example) that $\det \mathrm{Y}(x, \lambda)$ satisfies the equality

$$(\det \mathrm{Y}(x, \lambda))' = \operatorname{tr}\bigl(\lambda\rho(x)\mathrm{B} + \mathrm{A}(x) + \mathrm{C}(x, \lambda)\bigr) \det \mathrm{Y}(x, \lambda);$$

therefore, the condition that $\det \mathrm{Y}(\xi, \lambda) \neq 0$ for some $\xi \in [0, 1]$ implies that

$$\det \mathrm{Y}(x, \lambda) \neq 0 \quad \text{for all } x \in [0, 1].$$

As we already noted, our second aim is to derive asymptotic representations for the matrix $\mathrm{Y}(x, \lambda)$ as $|\lambda| \to \infty$. This topic is classical and has a long history. Following the works by Birkhoff [1] and [2] and Perron [13], Tamarkin published the monograph [14] and the paper [15] by Birkhoff and Langer also appeared. In these the asymptotics of the fundamental solution matrices for systems and spectral problems generated by systems supplemented by boundary conditions were studied. These works have not lost their relevance even now. The results deduced in [14] were later supplemented in [16]. It is well known that asymptotic results for systems are key to studying the spectral properties of ordinary differential operators of order $n \geqslant 2$ (see, for example, Naimark's monograph [17]). However, studying systems is undoubtedly of independent importance. In particular, one of the most important operators in mathematical physics, the Dirac operator, is generated by a second-order system (the relevant system in the case when $n = 2m$, $b_1 = \dots = b_m = \mathrm{i}$, and $b_{m+1} = \dots = b_n = -\mathrm{i}$ is also often called a *Dirac-type system* in the literature). The Sturm-Liouville equation $-y'' + qy = \lambda^2 \varrho y$ can be reduced to the Dirac system. We studied spectral problems for this equation in the singular case for potentials $q \in W_2^{-1}[0, 1]$ in [4] using asymptotic formulae for fundamental solutions, which were first established only in critical half-strips. Asymptotic formulae for general second-order equations with singular coefficients in half-planes of the complex plane were obtained later by Vladykina and Shkalikov [18]. In addition to asymptotic methods, the *operator transformation* method is efficient for Sturm-Liouville operators (see Marchenko's classical monograph [19] and the papers [20] and [21] by Albeverio, Hryniv, and Mykytyuk, where this method was developed in the singular case). For general first-order systems of the form (1.3) with coefficients

$$\mathrm{B} = \mathrm{diag}\{b_1 \mathrm{I}_{n_1}, \dots, b_k \mathrm{I}_{n_k}\} = B^*, \qquad \mathrm{A}(x) \in (L^\infty[0,1]; C^{n\times n}), \qquad \mathrm{C}(x,\lambda) \equiv 0,$$

transformation operators were constructed by Malamud [22]. For (2×2)-systems with matrices $\mathrm{B} = \mathrm{diag}\{b_1, b_2\} = \mathrm{B}^*$, $b_1 b_2 < 0$, transformation operators were constructed by Malamud and Lunyov [23] under the weaker condition $\mathrm{A}(x) \in (L^1[0,1]; C^{2\times 2})$, and they were applied to derive asymptotic formulae for eigenvalues and to investigate the basis property of the root functions of boundary value problems for the Dirac operator. Results on the unconditional basis property and asymptotics for the Dirac system with integrable coefficients were obtained previously by the authors in [24] and [25] using different methods.

In [14] and its supplement [16] Tamarkin assumes that the matrix V of the system (1.3) takes the following form after diagonalization:

$$\mathrm{B} = \mathrm{B}(x) = \mathrm{diag}\{\varphi_1(x), \dots, \varphi_n(x)\}$$

with additional conditions on values of the functions φ_j which, in view of Proposition 1 (see § 2.1), are equivalent to our condition (i). In this case, the elements of the matrices $\mathrm{V}(x)$, $\mathrm{A}(x)$ and $\mathrm{C}(x,\lambda)$ were assumed to be twice continuously differentiable, continuously differentiable, and continuous, respectively. The matrix elements of $\mathrm{C}(x,\lambda)$ were also assumed to decay at infinity with order $O(\lambda^{-1})$. Subsequently, the smoothness conditions were weakened repeatedly. In [26] Rapoport studied the asymptotic properties of solutions of systems of the form (1.3) with a real parameter λ under the assumption that the elements of the matrices $\mathrm{V}(x)$, $\mathrm{A}(x)$ and $\mathrm{C}(x,\lambda)$ belong to the spaces $W_1^2[0,1]$, $AC[0,1] = W_1^1[0,1]$ and $C[0,1]$, respectively, with the additional decay condition $\mathrm{C}(x,\lambda) = O(\lambda^{-1})$. In [27] Vagabov relaxed the smoothness conditions further: $\mathrm{V}(x) \in C^1[0,1]$ and $\mathrm{A}(x) \in C[0,1]$. It is likely that the currently most general case of systems of the form (1.3) was studied by Rykhlov in [28], where the elements of V are assumed to be absolutely continuous, while the elements of $\mathrm{A}(x)$ and $\mathrm{C}(x)$ are assumed to be Lebesgue integrable. Note that experts in the area did not pick up on [28]. In particular, results from it were proved again in [29] under stronger conditions on the coefficients namely, that $\mathrm{V} \in W_q^1[0,1]$ and $\mathrm{A}(x), \mathrm{C}(x) \in L_q[0,1]$, $q > 1$, whereas the remainder was estimated in the weaker norm $\|\cdot\|_{L_{q'}}$, $1/q + 1/q' = 1$, instead of $\|\cdot\|_{L_\infty}$.

Here we have limited ourselves to a short summary of the topic of our work. More details of results in the asymptotic theory for ordinary differential equations and systems and a more complete list of the literature on the topic can be found in the monographs by Rapoport [26], Wasov [30] and Naimark [17] and in the papers by Shkalikov [31], Rykhlov [28], and Lunyov and Malamud [32]. However, it is useful to note the main ways in which this paper is novel by comparison with the previous ones, and we do this below.

For an asymptotic analysis of solutions of (1.1) the situation is clear: such an analysis in the presence of distribution coefficients has only been carried out previously for second-order equations using other methods, which we have not managed to implement for higher orders. We shall now discuss the novelty in analyzing systems of the form (1.3) in more detail. It is well known that the asymptotic behaviour of the matrix $\mathrm{Y}(x,\lambda)$ as $|\lambda| \to +\infty$ depends fundamentally on a set Ω in the complex plane containing the parameter λ. Even a cursory glance at (1.3) makes it possible to conclude that the principal terms of the asymptotics must be functions of the form

$$\exp\{\lambda b_j p(x)\}, \quad \text{where } p(x) = \int \rho(t)\,dt,$$

whose asymptotic behaviour (growth, decrease or oscillation) is governed by the sign of the quantity $\operatorname{Re}(\lambda b_j p(x))$. Bearing in mind that the function p is nonnegative, we see that the asymptotics of $\mathrm{Y}(x,\lambda)$ must be sought in sectors $\arg\lambda \in (\alpha,\beta)$ in the complex plane. There must be either exponential growth or exponential decay of the solutions in such a sector. Oscillations occur only in half-strips in the complex plane containing the boundaries of the sectors. In what follows, we call these boundaries *critical rays*, while the arbitrary half-strips containing rays of this kind are called *critical half-strips*. However, problems concerning the asymptotic behaviour of eigenvalues of higher-order differential operators or operators generated directly by the differential expression (1.3) require precisely the knowledge of the asymptotics of $\mathrm{Y}(x,\lambda)$ in critical half-strips. Because of this, we cannot use the results in [32] and [33] due to Lunyov, Malamud, and Oridoroga, where systems of the form (1.3) with $\mathrm{C}(x,\lambda) = 0$ were studied but asymptotic representations were obtained in 'narrowed' sectors $\arg\lambda \in (\alpha+\varepsilon, \beta-\varepsilon)$, $\varepsilon > 0$. Sectors of this form do not cover the whole of the complex plane, and it is impossible to derive information about the behaviour of solutions in critical half-strips from [32] and [33].

The results obtained in [28], where the asymptotic behaviour of $\mathrm{Y}(x,\lambda)$ in closed sectors $\arg\lambda \in [\alpha,\beta]$ was found, are not sufficient for our needs either. There are three problems, one of which is substantial.

The first problem is that we need asymptotic formulae of this kind in wider domains. For definiteness, we fix a ray $\arg\lambda = \alpha$ and assume that it is the boundary of two adjacent sectors. Then the asymptotics obtained in [28] for the solution matrix $\mathrm{Y}_+(x,\lambda)$ are valid in the sector $\arg\lambda \in [\alpha,\beta]$, that is, in the half of the critical half-strip directed along the ray $\arg\lambda = \alpha$. Of course, the asymptotics of the solution in the other half of this half-strip are also given by the theorem in [28]; however, these are the asymptotics of another solution matrix, $\mathrm{Y}_-(x,\lambda)$. Thus, to obtain the asymptotics of the solution matrix inside the entire critical half-strip, we need to glue solutions, to study the asymptotics of the transition matrices, and so on. In this paper, we derive asymptotic formulae for $\mathrm{Y}(x,\lambda)$ in 'extended' sectors obtained by shifting the original sectors along the extension of the bisector. We admit a shift by an arbitrary distance, which makes it possible to cover critical half-strips of arbitrary width.

The second (substantial) problem lies in choosing a norm for the remainders. Roughly speaking, these terms have the form

$$\int_0^x e^{\mathrm{i}rt} a(t)\,dt, \quad \text{where } r \to +\infty,$$

the function $a(t)$ is integrable on the interval $[0,1]$, and the point $x \in [0,1]$ where we seek the value of the solution $\mathrm{Y}(x,\lambda)$ can be regarded as fixed. However, the specific character of the successive approximation method, which it is natural to apply when we seek a solution of the system (1.3), requires us to estimate quantities of the form

$$\int_0^\xi e^{\mathrm{i}rt} a(t)\,dt, \quad \text{where } \xi \in [0,x].$$

The point ξ cannot be regarded as fixed here; hence it is necessary to estimate some functional norm of the remainder, depending on ξ. How we choose this norm turns out to be very important. In the case when $a \in L_1[0,1]$, the norm $\|\cdot\|_{L_\infty}$ is natural for the remainders (the estimates in [28] use just this norm). It is straightforward to see that under additional smoothness of the function $a(x)$ (expressed, for example, by requirements on the integral modulus of continuity), estimates of the remainders in asymptotic formulae become sharper. However, if we require that the power of integrability of a be higher, that is, $a \in L_q[0,1]$, $q > 1$, the estimates for the remainders will not be sharper. This problem is of great importance to us, since elements of the matrix A turn out to be square-integrable when a differential equation of high order with distribution coefficients is reduced to a system. Using the norm $\|\cdot\|_{L_\infty}$ to estimate the remainders here would lead to considerably worse estimates in asymptotic formulae for the eigenvalues and eigenfunctions. For example, for the Sturm-Liouville operator, we would only obtain $s_n = o(1)$ instead of the asymptotics $\sqrt{\lambda_n} = n + s_n$, where $\{s_n\}_1^\infty \in l_2$, established in [3] and [4].

Finally, the third problem is that only the principal term of the asymptotics was found in [28]. Of course, using the method of successive approximation, we can write down any required number of terms in the asymptotic representations (the second order of smallness in the remainders is attained in Theorem 2 in this paper). However, the problem lies in the fact that the terms of the asymptotic series become untractable in this case (to write down the second term of the asymptotics, we formally need $\sim n^3$ summands). In fact, not all of these summands have the same order. For example, we showed in [4], in the case of the Sturm-Liouville operator $-y'' + qy$ and $n = 2$, that the main contribution is made by only one asymptotic term, the next three are subordinate, and the rest of the terms are of a higher order of smallness. This, in particular, allowed us to prove a theorem on the Fréchet derivative of the nonlinear map $q \mapsto \{\lambda_n\}_1^\infty$ (here the λ_n are the eigenvalues) (see [34]), which is a key fact for solving the inverse problem. Thus, an important element of the novelty of our paper is that we derive the main component of the remainder in the asymptotic formulae explicitly and show that the remainder can be estimated efficiently when there is additional information about the matrix elements of the system.

Our assumptions on the function ρ are currently the most general used. In the classical theory, this function is taken to be absolutely continuous (we do not know of any works where weaker conditions are imposed). This assumption is natural for systems arising when high-order differential operators are being studied. The thing is that these operators can be reduced to systems of the form (1.3), whereas, as we can see from the proof of Proposition 1 (see § 2.1), the matrix $\mathrm{V}(x)$ can be diagonalized only provided that its elements are absolutely continuous. Nevertheless, the fact that we impose weaker requirements on the function $\rho(x)$ (only positivity and integrability) is essential in studying the systems (1.3) themselves (for example, in studying Dirac-type systems). We note here that our condition (i) on ρ cannot be weakened without considerably changing the final result. When the function ρ is zero on a set of positive measure, we lose the principal term on the right-hand side of (1.3). Changing the sign of $\rho(x)$ (sign change points are called turning points) is equivalent to changing the sign of the spectral parameter, that is, to a transition into another sector of the complex plane. Studying the asymptotics of solutions of second-order equations with alternating function $\rho(x)$ requires other methods and approaches; for these see [35]–[38], where additional references are given.

We briefly comment on conditions (i)–(iii). We deliberately restrict ourselves to the form $\rho(x)\mathrm{B}$ of the leading term (with respect to the order) on the right-hand side of (1.3), although our main result concerning the asymptotics of the fundamental matrix $\mathrm{Y}(x,\lambda)$ can be extended to the more general case when

$$\mathrm{B} = \mathrm{B}(x) = \operatorname{diag}\{\varphi_1(x), \dots, \varphi_n(x)\}$$

under some additional conditions on φ_j. In this case, the asymptotics are not valid in sectors, only in more complicated domains $\Omega_j \subset \mathbb{C}$, which need not cover the whole of the complex plane. Nevertheless, this statement of the problem is quite meaningful. For example, it arises when we study polynomial operator pencils, more precisely, high-order differential equations whose coefficients are polynomials in the spectral parameter λ. To solve some problems (for example, to prove the completeness of the eigenfunctions and associated functions of pencils) it suffices to know the asymptotics of the solution $\mathrm{Y}(x,\lambda)$ on certain rays in $\mathbb{C}$ and the general estimate for the growth of $\mathrm{Y}(x,\lambda)$ as $|\lambda| \to \infty$; then the Phragmén-Lindelöf theorem applies (see [31] and [39]). However, we do not consider these problems in our paper, for this would involve us in technical complications.

Furthermore, we admit equalities among the numbers b_j, that is, we do not assume that the b_j are pairwise distinct. In fact, this means that it is convenient to consider system (1.3) in blocks, combining the equations where the b_j coincide into one group. This generalization does not affect the exponents in the asymptotic representation of the solution $\mathrm{Y}(x,\lambda)$ but changes factors in front of the corresponding exponentials, which are functions depending only on x. These functions can be written down explicitly in the case when the numbers b_j are pairwise distinct, but they are defined as solutions of systems of differential equations independent of λ in the general case. If there are Jordan cells in (the normal form of) the matrix B, then we arrive at another structure of the leading asymptotic term, and the above factors become polynomials in the spectral parameter. This case is technically complicated, and it hardly makes sense to study it in the general case rather than for a particular model problem. The coefficients of the system (1.3) being integrable is a quite natural condition, though we know of the result due to Amirov and Guseinov [40], where increasing the smoothness of the function $a_{12}(x)$ in the Dirac system makes it possible to consider the case when $a_{21}(x)$ is a distribution. Evidently, the requirement that the functions $c_{jk}(x,\lambda)$ vanish as $|\lambda| \to \infty$ is maximally general (when systems arising in studying operators with distribution coefficients or polynomial pencils are under consideration, we have $c_{jk}(x,\lambda) = O(|\lambda|^{-1})$). Of course, strengthening the conditions on the smoothness of the functions $a_{ij}(x)$, which opens the way to better estimates for the remainders in the asymptotics of the matrix $\mathrm{Y}(x,\lambda)$, must be accompanied by strengthening the conditions on the decrease of the functions $c_{jk}(x,\lambda)$.

This paper is structured as follows. In § 2 we define the sectors we deal with to obtain asymptotics and we carry out some preparatory work for the proof of the first basic theorem on the fundamental solution matrix of (1.3) in § 3. In § 4, using the matrix of the Mirzoev-Shkalikov regularization, we reduce equation (1.1) to the system (1.3) and obtain the second basic result on the asymptotics of the fundamental system of solutions of (1.1) with distribution coefficients subject to conditions (1.2). At the end of the paper, we write down our general results in the most important case when $n=2$.

Note that the results in this work form the basis for studying the spectral properties of higher-order differential operators with distribution coefficients (by spectral properties here we understand the asymptotics of eigenvalues and eigenfunctions, estimates for the resolvents of the operators obtained by prescribing boundary conditions, theorems on the unconditional basis property of eigenfunctions and associated functions, and so on). The authors plan to carry out this research in subsequent works. Note that the first version of this paper was presented by the authors in the preprint [41].

§ 2. Auxiliary results

2.1. We first give a useful result on the possibility of diagonalizing the matrix $\mathrm{V}(x)$ in (1.3). (Recall that conditions (i)–(iii), which we impose in what follows on the coefficients of the system under study, were formulated in § 1 after equation (1.3).)

Proposition 1. *Assume that the matrix* $\mathrm{V}(x)$ *of the system*[1]

$$\mathbf{u}' = \lambda \mathrm{V}(x)\mathbf{u} + \mathrm{A}_0(x)\mathbf{u} + \mathrm{C}_0(x,\lambda)\mathbf{u} \tag{2.1}$$

admits a representation $\mathrm{V}(x) = \mathrm{W}(x)(\rho(x)\mathrm{B})\mathrm{W}^{-1}(x)$, *where the matrices* $\mathrm{W}(x)$ *and* $\mathrm{W}^{-1}(x)$ *are absolutely continuous and the matrix* B *and the function* $\rho(x)$ *are subject to condition* (i). *Furthermore, assume that* $\mathrm{A}_0(x)$ *and* $\mathrm{C}_0(x,\lambda)$ *are subject to conditions* (ii) *and* (iii). *Then system* (2.1) *can be reduced to the form* (1.3) *by means of the change of variables* $\mathbf{u}(x) = \mathrm{W}(x)\mathbf{y}(x)$.

Proof. Making the change of variables $\mathbf{u}(x) = \mathrm{W}(x)\mathbf{y}(x)$ we obtain

$$\begin{aligned}
&\mathrm{W}'(x)\mathbf{y}(x) + \mathrm{W}(x)\mathbf{y}'(x)\\
&\qquad = \lambda\mathrm{W}(x)(\rho(x)\mathrm{B})\mathbf{y}(x) + \mathrm{A}_0(x)\mathrm{W}(x)\mathbf{y}(x) + \mathrm{C}_0(x,\lambda)\mathrm{W}(x)\mathbf{y}(x)\\
&\iff \quad \mathbf{y}' = \lambda\rho(x)\mathrm{B}\mathbf{y}\\
&\qquad\qquad + \bigl(\mathrm{W}^{-1}(x)\mathrm{A}_0(x)\mathrm{W}(x) - \mathrm{W}^{-1}(x)\mathrm{W}'(x)\bigr)\mathbf{y} + \mathrm{W}^{-1}(x)\mathrm{C}_0(x,\lambda)\mathrm{W}(x)\mathbf{y}.
\end{aligned}$$

We set

$$\mathrm{A}(x) = \mathrm{W}^{-1}(x)\mathrm{A}_0(x)\mathrm{W}(x) - \mathrm{W}^{-1}(x)\mathrm{W}'(x), \qquad \mathrm{C}(x,\lambda) = \mathrm{W}^{-1}(x)\mathrm{C}_0(x,\lambda)\mathrm{W}(x).$$

It is evident that the conditions (ii)–(iii) hold for these matrices.

[1] Thus, for each $x\in[0,1]$ the matrix $\mathrm{V}(x)$ has n eigenvalues $\rho(x)b_j$, $1\leqslant j\leqslant n$. We admit multiple eigenvalues among these but do not admit Jordan cells, that is, we assume that the geometric and algebraic multiplicity of each eigenvalue of $\mathrm{V}(x)$ coincide.

2.2. In what follows we need an existence and uniqueness theorem for the $(n \times n)$-system $\mathbf{y}' = \mathrm{T}(x)\mathbf{y} + \mathbf{f}(x)$ with initial condition $\mathbf{y}(\xi) = \mathbf{y}^0$, where the vector function $\mathbf{f}(x)$ and the matrix function $\mathrm{T}(x)$ are integrable. This theorem is certainly well known (see, for example, [12]). However, first we need estimates for the norm of the solution and for the norm of the difference of the solutions when ξ, $\mathbf{f}$ and T vary.

We set

$$\begin{aligned} \|\mathbf{y}\|_{AC} &:= \max_{1\leqslant j\leqslant n} \int_0^1 |y_j(x)| + |y_j'(x)|\,dx, \\ \|\mathbf{y}\|_{\xi} &:= \max_{1\leqslant j\leqslant n} \biggl(|y_j(\xi)| + \int_0^1 |y_j'(x)|\,dx\biggr), \end{aligned} \tag{2.2}$$

$$\|\mathbf{y}\|_{\mathrm{C}} = \|\mathbf{y}\|_{L_\infty} := \max_{\substack{1\leqslant j\leqslant n\\ x\in[0,1]}} |y_j(x)| \quad \text{and} \quad \|\mathbf{y}\|_{L_p} = \biggl(\sum_{j=1}^n \int_0^1 |y_j(x)|^p\,dx\biggr)^{1/p} \tag{2.3}$$

and note that $\|\mathbf{y}\|_{AC} \leqslant 2\|\mathbf{y}\|_\xi$ for any $\xi \in [0,1]$ and $\|\mathbf{y}\|_C \leqslant \|\mathbf{y}\|_{AC}$. For an arbitrary vector $\mathbf{y}$ we set $|\mathbf{y}| := \max_{1\leqslant j\leqslant n} |y_j|$. Similarly, for a vector function $\mathbf{f}(x)$ and a matrix $\mathrm{T}(x)$ we introduce the notation

$$|\mathbf{f}(x)| := \max_{1\leqslant j\leqslant n} |f_j(x)|, \qquad |\mathrm{T}(x)| = \max_{1\leqslant j\leqslant n} \sum_{k=1}^n |t_{jk}(x)| \tag{2.4}$$

and we write $\mathbf{f} \in L_1[0,1]$ or $\mathrm{T} \in L_1[0,1]$ for short, if all the coordinate functions $f_j(x)$ belong to $L_1[0,1]$ or all the matrix elements $t_{jk}(x)$ belong to $L_1[0,1]$. We also define the matrix norms

$$\|\mathrm{T}(x)\|_{L_p} = \sum_{j,k=1}^n \biggl(\sum_{k=1}^n \int_0^1 |t_{jk}(x)|^p\,dx\biggr)^{1/p}, \qquad \|\mathrm{T}(x)\|_{\mathrm{C}} = \max_{1\leqslant j,k\leqslant n} |t_{jk}(x)|$$
$$\text{and} \quad \|\mathrm{T}(x)\|_{AC} = \|\mathrm{T}(x)\|_{L_1} + \|\mathrm{T}'(x)\|_{L_1}.$$

Proposition 2. *Assume that the elements of an $(n \times n)$-matrix $\mathrm{T}(x)$ and all n coordinates of the vector function $\mathbf{f}(x)$ belong to the space $L_1[0,1]$. Then for any fixed $\xi \in [0,1]$ the system*

$$\mathbf{z}' = \mathrm{T}(x)\mathbf{z} + \mathbf{f}(x), \qquad x \in [0,1], \quad \mathbf{z}(\xi) = \mathbf{z}^0, \tag{2.5}$$

has a unique solution $\mathbf{z}(x) \in AC[0,1]$, which satisfies the estimates

$$\|\mathbf{z}\|_{AC} \leqslant (1 + 2\tau e^{\tau})\|\mathbf{g}\|_{AC} \quad \text{and} \quad |\mathbf{z}(x)| \leqslant e^{\tau(x)}\|\mathbf{g}\|_C, \tag{2.6}$$

where

$$\tau(x) = \biggl|\int_\xi^x |\mathrm{T}(t)|\,dt\biggr|, \qquad \tau = \int_0^1 |\mathrm{T}(t)|\,dt, \qquad \mathbf{g}(x) = \mathbf{z}^0 + \int_\xi^x \mathbf{f}(t)\,dt.$$

Proof. We write the system (2.5) in the integral form

$$\mathbf{z}(x) = \mathbf{g}(x) + (\mathscr{T}\mathbf{z})(x), \quad \text{where } (\mathscr{T}\mathbf{z})(x) = \int_\xi^x \mathrm{T}(t)\mathbf{z}(t)\,dt,$$

and first we prove the estimate

$$|(\mathscr{T}^l\mathbf{z})(x)| \leqslant \frac{\tau^l(x)}{l!}\|\mathbf{z}\|_C, \qquad l \geqslant 1. \tag{2.7}$$

It follows from the definitions (2.2)–(2.4) that

$$|(\mathscr{T}\mathbf{z})(x)| \leqslant \tau(x)\|\mathbf{z}\|_C.$$

The rest of proof is carried out by induction. Assume that the estimate is valid for $l-1$; then we have

$$\begin{aligned} |(\mathscr{T}^l\mathbf{z})(x)| &\leqslant \left|\int_\xi^x |\mathscr{T}(t)|\cdot|(\mathrm{T}^{l-1}\mathbf{z})(t)|\,dt\right| \leqslant \|\mathbf{z}\|_C\left|\int_\xi^x |\mathrm{T}(t)|\frac{\tau^{l-1}(t)}{(l-1)!}\,dt\right| \\ &= \|\mathbf{z}\|_C\left|\int_\xi^x \frac{\tau^{l-1}(t)}{(l-1)!}\,d\tau(t)\right| = \frac{\tau^l(x)}{l!}\|\mathbf{z}\|_C. \end{aligned} \tag{2.8}$$

Thus, (2.7) is proved. This implies another estimate

$$\begin{aligned} \|\mathscr{T}^l\mathbf{z}\|_{AC} &\leqslant 2\|\mathscr{T}^l\mathbf{z}\|_\xi \leqslant 2|(\mathscr{T}^l\mathbf{z})(\xi)| + 2\int_0^1 |\mathrm{T}(x)|\cdot|(\mathscr{T}^{l-1}\mathbf{z})(x)|\,dx \\ &\leqslant 2\|\mathbf{z}\|_C\int_0^1 |\mathrm{T}(x)|\frac{\tau^{l-1}(x)}{(l-1)!}\,dx \leqslant \frac{2\tau^l}{(l-1)!}\|\mathbf{z}\|_C \leqslant \frac{2\tau^l}{(l-1)!}\|\mathbf{z}\|_{AC}. \end{aligned}$$

In particular, we have

$$\|\mathscr{T}^l\|_{AC} \leqslant \frac{2\tau^l}{(l-1)!}. \tag{2.9}$$

Now we can write the solution of (2.5) as the series

$$\mathbf{z}(x) = \sum_{l=0}^{\infty}(\mathscr{T}^l\mathbf{g})(x), \tag{2.10}$$

which, due to (2.7) and (2.9), converges both in the norm of the space $C[0,1]$ and in the norm of $AC[0,1]$. We have

$$\begin{aligned} |\mathbf{z}(x)| &\leqslant \sum_{l=0}^{\infty}|(\mathscr{T}^l\mathbf{g})(x)| \leqslant \|\mathbf{g}\|_C\sum_{l=0}^{\infty}\frac{\tau^l(x)}{l!} \leqslant e^{\tau(x)}\|\mathbf{g}\|_C, \\ \|\mathbf{z}\|_{AC} &\leqslant \sum_{l=0}^{\infty}\|\mathscr{T}^l\mathbf{g}\|_{AC} \leqslant \|\mathbf{g}\|_{AC}\left(1 + 2\sum_{l=1}^{\infty}\frac{\tau^l}{(l-1)!}\right) = (1+2\tau e^\tau)\|\mathbf{g}\|_{AC}. \end{aligned}$$

The uniqueness of the solution can be proved in the standard way: if $\mathbf{z}$ and $\widetilde{\mathbf{z}}$ are two solutions of the system (2.5), then

$$\mathbf{z}-\widetilde{\mathbf{z}} = \mathscr{T}(\mathbf{z}-\widetilde{\mathbf{z}}) = \dots = \mathscr{T}^l(\mathbf{z}-\widetilde{\mathbf{z}}), \qquad l = 2, 3, \dots,$$

which yields $\mathbf{z}-\widetilde{\mathbf{z}} = 0$, since the operator $\mathscr{T}^l$ is a contraction in the space $AC[0,1]$ for sufficiently large l.

Proposition 2 is proved.

Remark 1. For fixed ξ, $\mathbf{z}^0$ and $\mathbf{f} \in L_1[0,1]$ the solution of equation (2.5) depends continuously on the matrix $\mathrm{T} \in L_1[0,1]$ in the following sense. If $\widetilde{\mathbf{z}}$ is a solution of the initial value problem (2.5) with the same ξ, $\mathbf{z}^0$ and $\mathbf{f}$ but with another matrix $\widetilde{\mathrm{T}}$, then

$$\|\mathbf{z}-\widetilde{\mathbf{z}}\|_{AC} \leqslant 2\|\mathrm{T}-\widetilde{\mathrm{T}}\|_{L_1}(1+2\tau e^{\tau})e^{\widetilde{\tau}}\|\mathbf{g}\|_{AC}. \tag{2.11}$$

In fact, let $\mathbf{u}(x)=\mathbf{z}(x)-\widetilde{\mathbf{z}}(x)$; this function is absolutely continuous, vanishes at ξ and satisfies the equation

$$\mathbf{u}'=\mathrm{T}(x)\mathbf{u}+(\mathrm{T}(x)-\widetilde{\mathrm{T}}(x))\widetilde{\mathbf{z}}(x).$$

According to (2.6), we obtain

$$\begin{aligned}
\|\mathbf{u}\|_{AC} &\leqslant (1+2\tau e^{\tau})\biggl\|\int_{\xi}^{x}(\mathrm{T}(t)-\widetilde{\mathrm{T}}(t))\widetilde{\mathbf{z}}(t)\,dt\biggr\|_{AC} \\
&\leqslant 2(1+2\tau e^{\tau})\biggl\|\int_{\xi}^{x}(\mathrm{T}(t)-\widetilde{\mathrm{T}}(t))\widetilde{\mathbf{z}}(t)\,dt\biggr\|_{\xi} \\
&= 2(1+2\tau e^{\tau})\int_0^1\bigl|(\mathrm{T}(x)-\widetilde{\mathrm{T}}(x))\widetilde{\mathbf{z}}(x)\bigr|\,dx \\
&\leqslant 2(1+2\tau e^{\tau})\|\mathrm{T}-\widetilde{\mathrm{T}}\|_{L_1}\|\widetilde{\mathbf{z}}\|_{C} \leqslant 2\|\mathrm{T}-\widetilde{\mathrm{T}}\|_{L_1}(1+2\tau e^{\tau})e^{\widetilde{\tau}}\|\mathbf{g}\|_{AC}.
\end{aligned}$$

Remark 2. Assume that the initial value $\mathbf{z}^0$, the function $\mathbf{f}$ and the matrix T depend on a parameter λ varying in a domain $D\subset\mathbb{C}$. Assume that the functions $z_j^0(\lambda)$ and the mappings $\lambda\mapsto f_j(\,\cdot\,,\lambda)$ and $\lambda\mapsto t_{jk}(\,\cdot\,,\lambda)$ from D to $L_1[0,1]$, $1\leqslant j,k\leqslant n$, are holomorphic with respect to λ. Then the solution $\mathbf{z}$ of (2.5) is holomorphic as the map $\lambda\mapsto\mathbf{z}(\,\cdot\,,\lambda)$ from D to $(AC[0,1])^n$.

The above result, which we need, is a variant of known results on the holomorphic dependence of solutions of equations on a parameter, provided that the coefficients and the initial conditions are holomorphic in this parameter. It can be proved in the standard way. The function $\mathbf{g}(\,\cdot\,,\lambda)$ in $(AC[0,1])^n$ is holomorphic in view of the representation $\mathbf{g}(\,\cdot\,,\lambda)=\mathbf{z}^0(\lambda)+\displaystyle\int\mathbf{f}(\,\cdot\,,\lambda)$. The function $(\mathscr{T}\mathbf{g})(\,\cdot\,,\lambda)$ is holomorphic in view of the definition of the operator $\mathscr{T}$. We establish that the functions $(\mathscr{T}^m\mathbf{g})(\,\cdot\,,\lambda)$ are holomorphic by induction. Since the function $\tau(\lambda)$ is continuous in $\lambda\in D$, it is bounded on each compact set. Then the estimates (2.9) guarantee that the series (2.10) converges uniformly on this compact set. Now it follows from Weierstrass's theorem that the function $\mathbf{z}(\,\cdot\,,\lambda)$ is holomorphic.

Corollary 1. *For each fixed $\lambda\in\mathbb{C}$, $|\lambda|>\lambda_0$, the system (1.3), supplemented by the initial condition $\mathbf{y}(\xi,\lambda)=\mathbf{y}^0(\lambda)$, has a unique solution in the class of absolutely continuous functions on $[0,1]$. The AC-norm of this solution admits the estimate*

$$\|\mathbf{y}\|_{AC}\leqslant(1+2\tau e^{\tau})|\mathbf{y}^0(\lambda)|,\qquad \tau=|\lambda|\|\rho\|_{L_1}\max_{1\leqslant j\leqslant n}|b_j|+a+c,$$
$$|\mathbf{y}^0(\lambda)|=\max_{1\leqslant j\leqslant n}|y_j^0(\lambda)|,$$

where

$$a=\int_0^1|\mathrm{T}(t)|\,dt\quad\textit{and}\quad c=\sup_{|\lambda|>\lambda_0}\gamma(\lambda),\quad\textit{where }\gamma(\lambda)=\int_0^1|\mathrm{C}(t,\lambda)|\,dt. \tag{2.12}$$

If all the functions $y_j^0 = y_j^0(\lambda)$ *and the map* $\lambda \mapsto c_{jk}(\,\cdot\,,\lambda)$, $1 \leqslant j,k \leqslant n$, *from the domain* $\lambda \in \{|\lambda| > \lambda_0\}$ *to* $L_1[0,1]$ *are holomorphic, then the solution* $\mathbf{y}$ *is also holomorphic as the map* $\lambda \mapsto \mathbf{y}(\,\cdot\,,\lambda)$ *from* $\{|\lambda| > \lambda_0\}$ *to* $(AC[0,1])^n$.

2.3. We define a family of sectors

$$\Gamma_\varkappa = \{\lambda\colon\ \arg\lambda \in (\alpha_{\varkappa-1}, \alpha_\varkappa)\}, \qquad \varkappa = 1,\dots,J,$$

in which we seek solutions with the necessary asymptotic behaviour (it is well known that in general, due to the Stokes phenomenon, solutions of this kind do not preserve their asymptotics in the whole of the complex plane). If all the b_j coincide, then our family consists of exactly one sector $\Gamma_1 = \mathbb{C}$. Otherwise we fix two arbitrary indices $1 \leqslant k < l \leqslant n$ such that $b_k \neq b_l$ and consider the equation

$$\operatorname{Re}(b_k\lambda) = \operatorname{Re}(b_l\lambda) \quad\Longleftrightarrow\quad \operatorname{Re}((b_k - b_l)\lambda) = 0. \tag{2.13}$$

It is easy to see that a solution of this equation is some straight line passing through the origin. The total number of equations of the form (2.13) is $n(n-1)/2$; hence we finally obtain a partition of the complex plane into $1 \leqslant J \leqslant (n^2 - n)$ sectors. For $J > 1$, each sector is bounded by two polar rays, which we number in the ascending order of their arguments:

$$\alpha_0 \leqslant 0 < \alpha_1 < \alpha_2 < \dots < \alpha_{J-1} < \alpha_J = \alpha_0 + 2\pi.$$

We fix some index $\varkappa \leqslant J$, consider the sector $\Gamma_\varkappa$, and seek solutions of equation (1.4) on it that have special asymptotic representations. In addition, we show that these solutions not only preserve the required asymptotics in $\Gamma_\varkappa$ but also in a wider sector $\widetilde{\Gamma}_\varkappa$. We define the sector $\widetilde{\Gamma}_\varkappa$ to be a parallel shift of the sector $\Gamma_\varkappa$ along its bisector (more precisely, along the extension of this bisector beyond $\Gamma_\varkappa$):

$$\widetilde{\Gamma}_\varkappa = \widetilde{\Gamma}_\varkappa(r) := \{\lambda \in \mathbb{C}\colon \lambda + re^{(\alpha_{\varkappa-1}+\alpha_\varkappa)/2} \in \Gamma_\varkappa\},$$

where $r > 0$ is fixed. In the case when $J = 1$ and $\Gamma_1 = \mathbb{C}$, we set $\widetilde{\Gamma}_1 = \mathbb{C}$. Of course, $\widetilde{\Gamma}_\varkappa$ depends on the choice of r; however, to keep the notation simple, we do not explicitly indicate this dependence and assume that r is fixed.

Note that for any pair of indices $1 \leqslant k,l \leqslant n$ such that $b_k \neq b_l$ the following property holds in the sector $\Gamma_\varkappa$: either $\operatorname{Re}(b_k\lambda) > \operatorname{Re}(b_l\lambda)$ for all $\lambda \in \Gamma_\varkappa$ or $\operatorname{Re}(b_k\lambda) < \operatorname{Re}(b_l\lambda)$ for all $\lambda \in \Gamma_\varkappa$. Let ν be the number of distinct b_j. We renumber the equations in (1.3) so that

$$\begin{aligned}\operatorname{Re}(b_1\lambda) = \operatorname{Re}(b_2\lambda) = \dots &= \operatorname{Re}(b_{n_1}\lambda) > \operatorname{Re}(b_{n_1+1}\lambda)\\ &= \dots = \operatorname{Re}(b_{n_1+n_2}\lambda) > \dots > \operatorname{Re}(b_{n_1+\dots+n_{\nu-1}+1}\lambda)\\ &= \dots = \operatorname{Re}(b_n\lambda) \quad \forall\,\lambda \in \Gamma_\varkappa, \qquad n = n_1 + n_2 + \dots + n_\nu.\end{aligned} \tag{2.14}$$

It is convenient to introduce the notation $b_1 = \dots = b_{n_1} =: \beta_1$, $b_{n_1+1} = \dots = b_{n_1+n_2} =: \beta_2$, and so on, thus switching to the set $\beta_1,\dots,\beta_\nu$ of pairwise distinct numbers. In $\widetilde{\Gamma}_\varkappa$ inequalities (2.14) obviously fail (except in the case when $\widetilde{\Gamma}_1 = \Gamma_1 = \mathbb{C}$).

Lemma 1. *There are numbers $h > 0$ and $\lambda_0 > 0$ (which depend on the value of the displacement r) such that the inequalities*

$$\begin{aligned} &\operatorname{Re}((b_k - b_l)\lambda) > -h, \qquad 1 \leqslant k < l \leqslant n, \\ &\operatorname{Re}((\beta_k - \beta_l)\lambda) > -h, \qquad 1 \leqslant k < l \leqslant \nu, \end{aligned} \tag{2.15}$$

hold in the domain $\mathfrak{D}_{\kappa,\lambda_0} := \{\lambda \in \widetilde{\Gamma}_\kappa,\ |\lambda| > \lambda_0\}$.

Proof. Evidently, it suffices to prove the inequalities for b_j. First note that the sector $\widetilde{\Gamma}_\kappa$ is bounded by a ray ℓ_1 of the form

$$\lambda = -re^{\frac{\mathrm{i}}{2}(\alpha_{\kappa-1}+\alpha_\kappa)} + te^{\mathrm{i}\alpha_{\kappa-1}}, \qquad t > 0,$$

which is parallel to the ray $\arg\lambda = \alpha_{\kappa-1}$, and by a ray ℓ_2 of the form

$$\lambda = -re^{\frac{\mathrm{i}}{2}(\alpha_{\kappa-1}+\alpha_\kappa)} + te^{\mathrm{i}\alpha_\kappa}, \qquad t > 0,$$

which is parallel to the ray $\arg\lambda = \alpha_\kappa$. Clearly, a ray ℓ of the form $\lambda = a + te^{\mathrm{i}\varphi}$, $t > 0$, beginning with some t, does not lie in the sector $\Gamma = \{\lambda\colon \arg\lambda \in (\varphi_0, \varphi_1)\}$ if $\varphi \notin [\varphi_0, \varphi_1]$. Hence beginning with some $t = t_1$ the ray ℓ_1 either lies in Γ_κ or in the adjacent sector $\Gamma_{\kappa-1}$ (we assume that $\Gamma_0 := \Gamma_J$ and $\Gamma_{J+1} := \Gamma_1$). The first case is impossible since $\widetilde{\Gamma}_\kappa \supset \Gamma_\kappa$ by definition. Similarly, beginning with some $t = t_2$ the ray ℓ_2 lies, in the sector $\Gamma_{\kappa+1}$. Choosing λ_0 equal to the maximum of $|\ell_1(t_1)|$ and $|\ell_2(t_2)|$, we find that the domain $\mathfrak{D}_{\kappa,\lambda_0}$ is embedded in the union of the sectors $\Gamma_{\kappa-1} \cup \overline{\Gamma_\kappa} \cup \Gamma_{\kappa+1}$.

We now fix an arbitrary pair of indices $1 \leqslant k < l \leqslant n$ and find a number h_{kl} such that the inequality $\operatorname{Re}((b_k - b_l)\lambda) > -h_{kl}$ holds everywhere in $\mathfrak{D}_{\kappa,\lambda_0}$. With this aim in view, we consider the linear function $w_{kl}(\lambda) = \operatorname{Re}((b_k - b_l)\lambda)$ in the λ-plane. We assume that $b_k \neq b_l$; otherwise we set $h_{kl} = 0$. By virtue of (2.14), this function is positive for $\lambda \in \Gamma_\kappa$. Four cases are possible. The first is that this function is positive in both $\Gamma_{\kappa-1}$ and $\Gamma_{\kappa+1}$. In this case we set $h_{kl} = 0$ and arrive at the inequality $w_{kl}(\lambda) > -h_{kl}$ for $\lambda \in \mathfrak{D}_{\kappa,\lambda_0}$. The function w_{kl} can be positive in both $\Gamma_{\kappa-1}$ and Γ_κ but equal to zero on the ray $\arg\lambda = \alpha_\kappa$. Then w_{kl} is negative in $\Gamma_{\kappa+1}$; since ℓ_2 is parallel to the ray $\arg\lambda = \alpha_\kappa$, w_{kl} is constant on ℓ_2. In this case we set $h_{kl} = -w_{kl}|_{\ell_2}$. Then on each ray parallel to ℓ_2 and lying in the strip between the rays ℓ_2 and $\arg\lambda = \alpha_\kappa$, the function w_{kl} is also constant and assumes values in the interval $(-h_{kl}, 0)$. This implies the inequality $w_{kl}(\lambda) > -h_{kl}$ in $\mathfrak{D}_{\kappa,\lambda_0}$. The function w_{kl} can be positive in Γ_κ and $\Gamma_{\kappa+1}$ but negative in $\Gamma_{\kappa-1}$. This case is similar to the previous one: we set $h_{kl} = -w_{kl}|_{\ell_1}$ and again arrive at the inequality $w_{kl}(\lambda) > -h_{kl}$ everywhere in $\mathfrak{D}_{\kappa,\lambda_0}$. Finally, it is possible that w_{kl} is positive only in the sector Γ_κ and negative in $\Gamma_{\kappa-1}$ and $\Gamma_{\kappa+1}$. Then we have $w_{kl} = 0$ on both the rays $\arg\lambda = \alpha_\kappa$ and $\arg\lambda = \alpha_{\kappa-1}$. As w_{kl} is linear, this is possible only when these rays are parallel. Since $\alpha_\kappa \neq \alpha_{\kappa-1}$, we have $\alpha_\kappa = \alpha_{\kappa-1} + \pi$. In this case, the sector Γ_κ turns into a half-plane; hence, the rays ℓ_1 and ℓ_2 form a straight line. We set $h_{kl} = -w_{kl}|_{\ell_1} = -w_{kl}|_{\ell_2}$ and again infer that $w_{kl}(\lambda) > -h_{kl}$ in $\mathfrak{D}_{\kappa,\lambda_0}$. We have thus deduced the assertion of the lemma for $h = \max_{1\leqslant k<l\leqslant n} h_{kl}$.

Lemma 1 is proved.

2.4. We fix some sector Γ_κ and renumber the equations in (1.3) so that inequalities (2.14) hold in Γ_κ. Now we assume that $\lambda \in \mathfrak{D}_{\kappa,\lambda_0} \subset \widetilde{\Gamma}_\kappa$. The numbers κ and λ_0 are fixed; therefore, for brevity we denote this domain by $\mathfrak{D}$. We let h denote the number defined in Lemma 1. It is important in what follows that inequalities (2.15) hold in the domain $\mathfrak{D} \subset \widetilde{\Gamma}_\kappa$.

We start by seeking the leading term in the asymptotic representation for $\mathrm{Y}(x, \lambda)$ as $\mathfrak{D} \ni \lambda \to \infty$. For this purpose, we discard the term $\mathrm{C}(x, \lambda)\mathrm{Y}(x, \lambda)$ on the right-hand side of (1.4). We represent the matrix $\mathrm{A}(x)$ as

$$\mathrm{A}(x) = \mathrm{D}(x) + (\mathrm{A}(x) - \mathrm{D}(x)), \quad \text{where } \mathrm{D}(x) = \begin{cases} a_{jk}(x) & \text{if } b_j = b_k, \\ 0 & \text{if } b_j \neq b_k. \end{cases}$$

We also discard the term $(\mathrm{A}(x) - \mathrm{D}(x))\mathrm{Y}(x, \lambda)$ on the right-hand side of (1.4). As a result, we arrive at the matrix differential equation

$$\mathrm{Y}^0(x, \lambda)' = \lambda\rho(x)\mathrm{B}\mathrm{Y}^0(x, \lambda) + \mathrm{D}(x)\mathrm{Y}^0(x, \lambda), \tag{2.16}$$

$$\mathrm{B} = \begin{pmatrix} b_1 & 0 & 0 & \dots & 0 \\ 0 & b_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_n \end{pmatrix}, \qquad \mathrm{D}(x) = \begin{pmatrix} \mathrm{D}_1(x) & 0 & 0 & \dots & 0 \\ 0 & \mathrm{D}_2(x) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{D}_\nu(x) \end{pmatrix}$$

(the matrices $\mathrm{D}_j(x)$ have size $n_j \times n_j$); we call this the *model equation* for the basic equation (1.4). From now on, we let I denote the identity matrix.

Proposition 3. *The solution* $\mathrm{Y}^0(x, \lambda)$ *of equation* (2.16) *with the initial condition* $\mathrm{Y}^0(0, \lambda) = \mathrm{I}$ *has the form*

$$\mathrm{Y}^0(x, \lambda) = \mathrm{E}(x, \lambda) \cdot \mathrm{M}(x), \qquad \mathrm{E}(x, \lambda) = \operatorname{diag}\{e^{b_1\lambda p(x)}, \dots, e^{b_n\lambda p(x)}\}, \tag{2.17}$$

where $p(x) = \int_0^x \rho(t)\,dt$, *each matrix* $\mathrm{M}_j(x)$, $1 \leqslant j \leqslant \nu$, *has size* $n_j \times n_j$, *and*

$$\mathrm{M}(x) = \begin{pmatrix} \mathrm{M}_1(x) & 0 & 0 & \dots & 0 \\ 0 & \mathrm{M}_2(x) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{M}_\nu(x) \end{pmatrix}.$$

Each matrix $\mathrm{M}_j(x)$ *is a solution of the equation* $\mathrm{M}_j'(x) = \mathrm{D}_j(x)\mathrm{M}_j(x)$ *with the initial condition* $\mathrm{M}_j(0) = \mathrm{I}$.

Proof. By definition, $\mathrm{D}(x)$ is the diagonal part of the matrix $\mathrm{A}(x)$ consisting of ν square blocks, each of size $n_j \times n_j$, $1 \leqslant j \leqslant \nu$. In (2.16), each block of the matrix D is associated with the same number b_j lying on the diagonal of B. Therefore, the system (2.16) splits into ν separate systems

$$\mathrm{Y}_j'(x, \lambda) = \lambda\beta_j\rho(x)\mathrm{Y}_j(x, \lambda) + \mathrm{D}_j(x)\mathrm{Y}_j(x, \lambda), \quad \mathrm{Y}_j(0, \lambda) = \mathrm{I}, \qquad j = 1, \dots, \nu.$$

Using a direct substitution, we verify that $\mathrm{Y}_j(x, \lambda) = \exp\{\lambda\beta_j p(x)\}\mathrm{M}_j(x)$.

The proposition is proved.

Remark 3. In the general case, the matrix $\mathrm{M}(x)$ cannot be expressed explicitly in terms of elements of $\mathrm{A}(x)$. However, if $\mathrm{D}_j(x)$ is a diagonal matrix, that is,

$$\mathrm{D}_j(x) = \operatorname{diag}\{a_{k_j,k_j}(x), \dots, a_{k_j+n_j,k_j+n_j}(x)\}, \quad \text{where } k_j = n_1 + \dots + n_{j-1} + 1,$$

then $\mathrm{M}_j(x)$ is also diagonal and has the form

$$\mathrm{M}_j(x) = \operatorname{diag}\left\{\exp\left\{\int_0^x a_{k_j,k_j}(t)\,dt\right\}, \dots, \exp\left\{\int_0^x a_{k_j+n_j,k_j+n_j}(t)\,dt\right\}\right\}.$$

In particular, this is the case when the b_j, $1 \leqslant j \leqslant n$, are pairwise distinct.

Remark 4. The matrix $\mathrm{M}(x)$ commutes with B and $\mathrm{E}(x,\lambda)$.

Lemma 2. *The matrix* $\mathrm{M}(x)$ *admits the estimate* $\|\mathrm{M}(x)\|_C \leqslant e^a$, *where the number* a *is defined in* (2.12) *with* T *replaced by* D (*or* A). *It is invertible for any* $x \in [0,1]$; *the inverse matrix satisfies the equation* $(\mathrm{M}^{-1}(x))' = -\mathrm{M}^{-1}(x)\mathrm{D}(x)$ *with the initial condition* $\mathrm{M}(0) = \mathrm{I}$ *and also admits the estimate* $\|\mathrm{M}^{-1}(x)\|_C \leqslant e^a$.

Proof. The estimate for M follows from (2.6). That $\mathrm{M}(x)$ is nonsingular can be proved in a usual way: its determinant satisfies

$$(\det \mathrm{M}(x))' = \operatorname{tr} \mathrm{D}(x) \cdot \det \mathrm{M}(x)$$

(for example, see [12]) and the initial condition $\det \mathrm{M}(0) = 1$; thus, it is nonzero on the whole of $[0,1]$. Differentiating the identity $\mathrm{I} = \mathrm{M}^{-1}(x)\mathrm{M}(x)$, taking account of the equality $\mathrm{M}'(x) = \mathrm{D}(x)\mathrm{M}(x)$, and multiplying on the right by $\mathrm{M}^{-1}(x)$, we obtain $(\mathrm{M}^{-1}(x))' = -\mathrm{M}^{-1}(x)\mathrm{D}(x)$. Passing to the conjugate matrices, according to (2.6), we infer that $\|(\mathrm{M}^{-1})^*\|_C \leqslant e^a$, which is equivalent to $\|\mathrm{M}^{-1}(x)\|_C \leqslant e^a$.

2.5. We define the matrices

$$\mathrm{Q}(x) = \mathrm{M}^{-1}(x)(\mathrm{A}(x) - \mathrm{D}(x))\mathrm{M}(x) \quad \text{and} \quad \mathrm{R}(x,\lambda) = \mathrm{M}^{-1}(x)\mathrm{C}(x,\lambda)\mathrm{M}(x),$$

whose elements are denoted by $q_{jk}(x)$ and $r_{jk}(x,\lambda)$. Note that

$$\int_0^1 |\mathrm{Q}(x)|\,dx \leqslant ae^{2a}, \qquad \int_0^1 |\mathrm{R}(x,\lambda)|\,dx \leqslant \gamma(\lambda)e^{2a}, \tag{2.18}$$

where a and γ are defined in (2.12) with T replaced by A (or $\mathrm{A} - \mathrm{D}$).

Lemma 3. *If* $b_k = b_l$ *for any pair of indices*, *then we have*

$$q_{kl}(x) = q_{lk}(x) = 0.$$

Proof. We first note that the matrix $\mathrm{M}^{-1}(x)$ has the same structure as $\mathrm{M}(x)$, with blocks $\mathrm{M}_j^{-1}(x)$ with the same size and location. Decomposing the matrix $\mathrm{A}(x)$ into the corresponding blocks, we obtain

$$\mathrm{Q}(x) = \mathrm{M}^{-1}(x)(\mathrm{A}(x) - \mathrm{D}(x))\mathrm{M}(x) = \begin{pmatrix} \mathrm{M}_1^{-1}(x) & 0 & 0 & \dots & 0 \\ 0 & \mathrm{M}_2^{-1}(x) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{M}_\nu^{-1}(x) \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & \mathrm{A}_{12}(x) & \mathrm{A}_{13}(x) & \dots & \mathrm{A}_{1\nu}(x) \\ \mathrm{A}_{21}(x) & 0 & \mathrm{A}_{23}(x) & \dots & \mathrm{A}_{2\nu}(x) \\ \dots & \dots & \dots & \dots & \dots \\ \mathrm{A}_{\nu 1}(x) & 0 & \mathrm{A}_{\nu,\nu-1}(x) & \dots & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} \mathrm{M}_1(x) & 0 & 0 & \dots & 0 \\ 0 & \mathrm{M}_2(x) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{M}_\nu(x) \end{pmatrix}.$$

We can easily see that the product of these three matrices has a zero block diagonal. This implies the assertion of the lemma.

We set $\mathrm{v}_{jl}(x,\lambda) = q_{jl}(x) + r_{jl}(x,\lambda)$. We let the symbol $(\pm)_{jk}$ denote the quantity equal to -1 for $j < k$ and 1 for $j \geqslant k$. To write down the remainders, we define functions υ_{jkl} and ϱ_{jkl} in the following way:

$$\upsilon_{jkl}(s,x,\lambda) = (\pm)_{jk}(\pm)_{lk} \int q_{jl}(t) e^{(b_l - b_k)\lambda(p(t)-p(s)) + (b_j - b_k)\lambda(p(x)-p(t))}\,dt,$$
$$\varrho_{jkl}(s,x,\lambda) = (\pm)_{jk}(\pm)_{lk} \int r_{jl}(t,\lambda) e^{(b_l - b_k)\lambda(p(t)-p(s)) + (b_j - b_k)\lambda(p(x)-p(t))}\,dt, \tag{2.19}$$

where the limits of integration are as follows (we assume that the integral is zero if the lower limit is larger that the upper limit):

$$\begin{cases} \text{from } x \text{ to } s & \text{for } j,l < k, \\ \text{from } \max\{x,s\} \text{ to } 1 & \text{for } j < k \leqslant l, \\ \text{from } 0 \text{ to } \min\{x,s\} & \text{for } l < k \leqslant j, \\ \text{from } s \text{ to } x & \text{for } k \leqslant j,l. \end{cases} \tag{2.20}$$

To estimate the remainders in the representation (3.1) we use the functions

$$\Upsilon(\lambda) = \Upsilon_\infty(\lambda) = \max_{j,k,l,s,x} |\upsilon_{jkl}(s,x,\lambda)|, \qquad \Upsilon(x,\lambda) = \max_{j,k} |\upsilon_{jkk}(0,x,\lambda)|,$$
$$\Upsilon_\mu(\lambda) = \max_{j,k,l}\biggl(\int_0^1\int_0^1 |\upsilon_{jkl}(s,x,\lambda)|^\mu\,ds\,dx\biggr)^{1/\mu} + \max_{j,k}\biggl(\int_0^1 |\upsilon_{jkk}(0,x,\lambda)|^\mu\,dx\biggr)^{1/\mu}, \tag{2.21}$$

where $\mu \in [1,\infty)$. It is easy to see that $\Upsilon_\mu(\lambda) \leqslant \Upsilon(\lambda)$ and $\Upsilon(x,\lambda) \leqslant \Upsilon(\lambda)$ for any $x \in [0,1]$. We need the following analogue of the Riemann-Lebesgue lemma. (Recall that $\mathfrak{D}$ is a domain in the complex plane that is the intersection of the exterior of a circle of large radius and the extended sector $\widetilde{\Gamma}_\kappa$, in which the inequalities (2.15) hold.)

Lemma 4. *Let $\lambda \in \mathfrak{D} \subset \widetilde{\Gamma}_\kappa$. Then as $\lambda \to \infty$, $\Upsilon(\lambda) \to 0$. In addition,*

$$\max_{j,k,l,s,x} |\varrho_{jkl}(s,x,\lambda)| \leqslant e^{2hp}\gamma(\lambda), \quad \text{where } p := \int_0^1 \rho(t)\,dt, \tag{2.22}$$

in $\mathfrak{D}$, where the functions ϱ_{jkl} and γ are defined in (2.19) and (2.12), respectively.

Proof. We fix arbitrary indices j, k and l and points s and $x \in [0,1]$. We first note that for $l < k$, the limits of integration in (2.19) are such that $t \leqslant s$; therefore, it follows from (2.15) that the estimates

$$\operatorname{Re}(b_l - b_k)\lambda(p(t) - p(s)) < h(p(s) - p(t)), \qquad p(x) = \int_0^x \rho(\tau)\,d\tau,$$

hold for $\lambda \in \mathfrak{D} \subset \widetilde{\Gamma}$. For $l \geqslant k$ we have $t \geqslant s$ and

$$\operatorname{Re}(b_l - b_k)\lambda(p(t) - p(s)) < h(p(t) - p(s)).$$

Consequently, in either case we obtain

$$\operatorname{Re}(b_l - b_k)\lambda(p(t) - p(s)) < h|p(s) - p(t)| \leqslant hp.$$

Similarly, we derive the estimates

$$\operatorname{Re}(b_j - b_k)\lambda(p(x) - p(t)) < h|p(x) - p(t)| \leqslant hp.$$

In view of (2.18), the above estimates immediately yield (2.22).

We now prove that $\Upsilon(\lambda) \to 0$ as $\widetilde{\Gamma}_\kappa \ni \lambda \to \infty$. Due to Lemma 3, we have $\upsilon_{jkl} \equiv 0$ if $b_j = b_l$. If $b_j \neq b_l$, then

$$|\upsilon_{jkl}(s,x,\lambda)| \leqslant \left| \int q_{jl}(t) \exp\{(b_l - b_k)\lambda(p(t) - p(s)) + (b_j - b_k)\lambda(p(x) - p(t))\}\,dt \right|,$$

where the limits of integration in the last integral are arranged according to (2.20). Making the substitution $\xi = p(t)$, $\xi \in [0,p]$, the integral takes the form

$$\left| \int f_{jl}(\xi) e^{(b_l - b_k)\lambda(\xi - p(s)) + (b_j - b_k)\lambda(p(x) - \xi)}\,d\xi \right|, \qquad f_{jl}(\xi) = \frac{q_{jl}(t(\xi))}{\rho(t(\xi))}. \tag{2.23}$$

The limits of integration are arranged here according to the rule (2.20) with allowance for the fact that the inequality $a \leqslant t \leqslant b$ is equivalent to $p(a) \leqslant \xi \leqslant p(b)$. Note that the function f_{jl} is integrable on the interval $[0,p]$, since

$$\int_0^p \frac{|q_{jl}(t(\xi))|}{\rho(t(\xi))}\,d\xi = \int_0^1 |q_{jl}(t)|\,dt.$$

The exponential in the integral (2.23) is bounded as $\lambda \in \widetilde{\Gamma}_\kappa \ni \lambda \to \infty$; for $\xi \neq p(s)$ and $\xi \neq p(x)$ it decays or oscillates in this sector. Therefore, the proof is completed in the same way as in the Riemann-Lebesgue lemma. For a small fixed $\varepsilon > 0$ we choose a continuously differentiable function $\widetilde{q}_{jl}(\xi)$ such that

$$\int_0^p \left| \frac{q_{jl}(t(\xi))}{\rho(t(\xi))} - \widetilde{q}_{jl}(\xi) \right| d\xi < \varepsilon.$$

Since, as before, the real part of the expression in the exponent of the exponential in the integral (2.23) does not exceed $2hp$, the integral in (2.23) (which we denote by J) admits the estimate

$$J \leqslant \varepsilon e^{2hp} + \biggl|\int \widetilde{q}_{jl}(\xi) e^{(b_l-b_k)\lambda(\xi-p(s))+(b_j-b_k)\lambda(p(x)-\xi)}\,d\xi\biggr|. \tag{2.24}$$

Integrating by parts in the integral, we arrive at the estimate

$$J \leqslant \varepsilon e^{2hp} + \frac{e^{2hp}}{|b_l-b_j|\cdot|\lambda|}\biggl(2\max_{0\leqslant\xi\leqslant p}|\widetilde{q}_{jl}(\xi)| + \int_0^p |\widetilde{q}'_{jl}(\xi)|\,d\xi\biggr) \leqslant 2\varepsilon e^{2hp}$$

if $|\lambda|$ is sufficiently large.

Lemma 4 is proved.

§ 3. The main result for systems

We proceed to derive asymptotic formulae for the fundamental matrix of the system (1.3).

Theorem 1. *Assume that the matrices in* (1.3) *are subject to conditions* (i)–(iii). *Fix one of the sectors* $\widetilde{\Gamma}_\kappa$ *defined in* § 2.3. *Then there exists a fundamental solution matrix* $\mathrm{Y}(x,\lambda)$ *for equation* (1.3) *that has the following asymptotic representation in the sector* $\widetilde{\Gamma}_\kappa$ *as* $\lambda\to\infty$:

$$\mathrm{Y}(x,\lambda) = \mathrm{Y}^0(x,\lambda) + \mathrm{S}(x,\lambda)\mathrm{E}(x,\lambda) \tag{3.1}$$

where

$$\mathrm{S}(x,\lambda) = (s_{jk}(x,\lambda))_{j,k=1}^n, \qquad \max_{j,k,x}|s_{jk}(x,\lambda)| \leqslant C(\Upsilon(\lambda)+\gamma(\lambda)),$$

and the functions $\mathrm{Y}^0(x,\lambda)$ *and* $\mathrm{E}(x,\lambda)$ *are defined in* (2.17).

If $\mathrm{D}(x)$ *is the matrix of the model equation* (2.16) *and elements of the matrix* $\mathrm{A}(x)-\mathrm{D}(x)$ *are integrable to the power* $\mu'\in(1,\infty)$, *then the estimate of the remainder can be sharpened. More precisely, then*

$$\max_{j,k}|s_{jk}(x,\lambda)| \leqslant \Upsilon(x,\lambda) + C(\Upsilon_\mu(\lambda)+\gamma(\lambda)) \tag{3.2}$$

and

$$\max_{j,k}\biggl(\int_0^1 |s_{jk}(x,\lambda)|^\mu\,dx\biggr)^{1/\mu} \leqslant C(\Upsilon_\mu(\lambda)+\gamma(\lambda)), \tag{3.3}$$

where μ *is defined by the condition* $1/\mu+1/\mu'=1$. *The majorant functions* Υ *and* γ *are defined by equalities* (2.21) *and* (2.12).

Remark 5. In view of Lemma 4, Theorem 1 yields the asymptotic representation

$$\mathrm{Y}(x,\lambda) = \bigl(\mathrm{M}(x,\lambda)+o(1)\bigr)\mathrm{E}(x,\lambda), \qquad \widetilde{\Gamma}_\kappa \ni \lambda\to\infty,$$

which is uniform with respect to $x\in[0,1]$.

Proof of Theorem 1. *Step* 1. *Passing to the system of integral equations.* First we represent the matrix equation (1.4) in a more convenient form and then reduce it to a system of integral equations.

Recall that the equations in (1.3) are numbered so that the inequalities (2.15) hold in $\mathfrak{D}$. We seek a solution of (1.4) in the form $\mathrm{Y}(x,\lambda)=\mathrm{M}(x)\mathrm{Z}(x,\lambda)\mathrm{E}(x,\lambda)$. Substituting this expression into (1.4) we obtain

$$\begin{gathered}\mathrm{M}'\mathrm{ZE}+\mathrm{MZ}'\mathrm{E}+\mathrm{MZE}'=\lambda\rho\mathrm{BMZE}+\mathrm{DMZE}+(\mathrm{A}-\mathrm{D}+\mathrm{C})\mathrm{MZE}\\ \Longleftrightarrow\quad \mathrm{MZ}'\mathrm{E}+\lambda\rho\mathrm{MZBE}=\lambda\rho\mathrm{BMZE}+(\mathrm{A}-\mathrm{D}+\mathrm{C})\mathrm{MZE}.\end{gathered}$$

We multiply this equality by M^{-1} on the right and by E^{-1} on the left and use the equality $\mathrm{M}^{-1}\mathrm{B}=\mathrm{BM}^{-1}$. As a result, our equation takes the form

$$\mathrm{Z}'=\lambda\rho(\mathrm{BZ}-\mathrm{ZB})+(\mathrm{Q}+\mathrm{R})\mathrm{Z}$$

(note that $\mathrm{Q}(x):=\mathrm{M}^{-1}(x)(\mathrm{A}(x)-\mathrm{D}(x))\mathrm{M}(x)$ and $\mathrm{R}(x,\lambda):=\mathrm{M}^{-1}(x)\mathrm{C}(x,\lambda)\mathrm{M}(x)$). In coordinate form this equation is as follows:

$$z_{jk}'(x,\lambda)=\lambda(b_j-b_k)\rho(x)z_{jk}(x,\lambda)+\sum_{l=1}^{n}\mathrm{v}_{jl}(x,\lambda)z_{lk}(x,\lambda), \tag{3.4}$$

where the v_{jl} are elements of the matrix $\mathrm{Q}+\mathrm{R}$.

We fix an index k and integrate in (3.4), choosing the initial conditions $z_{jk}(1,\lambda)=0$ for $j<k$, $z_{jk}(0,\lambda)=0$ for $j>k$ and $z_{kk}(0,\lambda)=1$. We finally arrive at

$$\begin{aligned} z_{jk}(x,\lambda)&=-\sum_l\int_x^1\mathrm{v}_{jl}(t,\lambda)e^{(b_j-b_k)\lambda(p(x)-p(t))}z_{lk}(t,\lambda)\,dt, && j<k,\\ z_{kk}(x,\lambda)-1&=\sum_l\int_0^x\mathrm{v}_{kl}(t,\lambda)z_{lk}(t,\lambda)\,dt, \\ z_{jk}(x,\lambda)&=\sum_l\int_0^x\mathrm{v}_{jl}(t,\lambda)e^{(b_j-b_k)\lambda(p(x)-p(t))}z_{lk}(t,\lambda)\,dt, && j>k.\end{aligned} \tag{3.5}$$

We let $\mathrm{V}_k(\lambda)$ denote the integral operator defined by the right-hand side of this system, and we let $\mathbf{z}_k$ denote the kth column of the matrix Z. Then the system assumes the form $\mathbf{z}_k=\mathbf{z}_k^0+\mathrm{V}_k\mathbf{z}_k$, where $\mathbf{z}_k^0=(\delta_j^k)_{j=1}^n$. We also introduce the operators $\mathrm{Q}_k(\lambda)$ and $\mathrm{R}_k(\lambda)$, defined by the right-hand side of (3.5) with v_{jk} replaced by q_{jk} and r_{jk}, respectively. Since $\mathrm{v}_{jk}(x,\lambda)=q_{jk}(x)+r_{jk}(x,\lambda)$, we have $\mathrm{V}_k=\mathrm{Q}_k+\mathrm{R}_k$.

Step 2. *The operator* $(V_k(\lambda))^2$ *is a contraction.* We seek a solution of the system (3.5) as a series $\mathbf{z}_k=\sum_{\nu=0}(\mathrm{V}_k(\lambda))^\nu\mathbf{z}_k^0$ converging in the space $(L_\infty[0,1])^n$. To prove that the series converges and estimate the remainders in (3.1), we need to estimate the norms of the operators $\mathrm{Q}_k(\lambda)$ and $\mathrm{R}_k(\lambda)$. Note that the estimates for $\|\mathrm{Q}_k(\lambda)\|$ and $\|\mathrm{R}_k(\lambda)\|$ have different natures. The first are related to the Riemann-Lebesgue lemma, whereas the second reflect the fact that the norm

$\|\mathrm{C}(\,\cdot\,,\lambda)\|_{L_1[0,1]} = \gamma(\lambda)$ decreases as $\lambda \to \infty$. We start with the estimates

$$\|\mathrm{R}_k(\lambda)\|_{L_\infty \to L_\infty} \leqslant e^{hp+2a}\gamma(\lambda), \qquad \|\mathrm{V}_k(\lambda)\|_{L_\infty \to L_\infty} \leqslant e^{hp+2a}(a+c),$$
$$\|\mathrm{Q}_k(\lambda)\|_{L_\mu \to L_\infty} \leqslant e^{hp+2a}\|\mathrm{A}(x)-\mathrm{D}(x)\|_{L_{\mu'}}, \tag{3.6}$$
$$a = \int_0^1 |\mathrm{A}(t)|\,dt, \qquad p = \int_0^1 \rho(t)\,dt.$$

Note that the limits of integration in (3.5) are arranged so that

$$\mathrm{Re}(b_j - b_k)\lambda(p(x)-p(t)) < h|p(x)-p(t)| \leqslant hp$$

for $\lambda \in \widetilde{\Gamma}_\kappa$. We assume that $\mathbf{f} \in L_\infty[0,1]$ and set $\mathbf{g}_k = \mathrm{R}_k(\lambda)\mathbf{f} \in AC[0,1]$. It follows from (2.18) and (3.5) that

$$\|\mathbf{g}_k(x,\lambda)\|_{L_\infty} \leqslant e^{hp}\sum_{j=1}^n\sum_{l=1}^n \int_0^1 |r_{jl}(t,\lambda)||f_l(t)|\,dt \leqslant e^{hp+2a}\gamma(\lambda)\|\mathbf{f}\|_{L_\infty};$$

thus, the first inequality in (3.6) is proved. The proof of the second inequality is completely similar. It remains to estimate the norm of $\mathrm{Q}_k(\lambda)$ as an operator from L_μ to L_∞. We assume that $\mathbf{f} \in L_\mu[0,1]$ and set $\mathbf{g}_k = \mathrm{Q}_k(\lambda)\mathbf{f} \in AC[0,1]$. Using (2.18) and (3.5) we obtain

$$\|\mathbf{g}_k(x,\lambda)\|_{L_\infty} \leqslant e^{hp}\sum_{j=1}^n\sum_{l=1}^n \int_0^1 |q_{jl}(t)|\,|f_l(t)|\,dt \leqslant e^{hp}\|\mathrm{Q}(x)\|_{L_{\mu'}}\|\mathbf{f}\|_{L_\mu},$$

which implies (3.6).

We show that for sufficiently large $|\lambda|$, $\lambda \in \mathfrak{D}$, the operator $(\mathrm{V}_k(\lambda))^2$ is a contraction in $L_\infty[0,1]$. If $\mathbf{f} \in L_\infty$ and $\mathbf{g}_k = \mathrm{V}_k^2(\lambda)\mathbf{f}$, then we have

$$g_{jk}(x,\lambda) = \sum_{l,m=1}^n (\pm)_{jk}(\pm)_{lk}$$
$$\times \int\biggl(\mathrm{v}_{jl}(t,\lambda)e^{(b_j-b_k)\lambda(p(x)-p(t))}\int \mathrm{v}_{lm}(s,\lambda)e^{(b_l-b_k)\lambda(p(t)-p(s))}f_m(s)\,ds\biggr)dt.$$

The outer integral is taken over the interval $[x,1]$ for $j<k$ and over $[0,x]$ for $j \geqslant k$. The inner integrals are taken over the interval $[t,1]$ for $l<k$ and over $[0,t]$ for $l \geqslant k$. We interchange the integrals; after this, in each of the four cases for the arrangement of the indices ($j,l<k$; $j<k\leqslant l$; $l<k\leqslant j$; $k\leqslant j,k$), we arrive at

$$g_{jk}(x,\lambda) = \sum_{m=1}^n \int_0^1 \biggl(\sum_{l=1}^n \mathrm{v}_{lm}(s,\lambda)(\upsilon_{jkl}(s,x,\lambda)+\varrho_{jkl}(s,x,\lambda))\biggr)f_m(s)\,ds. \tag{3.7}$$

We show this, using the first case, when $j, l < k$, as an example. Here $x \leqslant t \leqslant 1$ and $t \leqslant s \leqslant 1$; thus, we have

$$
\begin{aligned}
&(\pm)_{jk}(\pm)_{lk} \int \left(\mathrm{v}_{jl}(t,\lambda) e^{(b_j - b_k)\lambda(p(x)-p(t))} \int \mathrm{v}_{lm}(s,\lambda) e^{(b_l - b_k)\lambda(p(t)-p(s))} f_m(s)\, ds \right) dt \\
&\quad = (-1)^2 \int_x^1 \mathrm{v}_{lm}(s,\lambda) f_m(s) \int_x^s \mathrm{v}_{jl}(t,\lambda) e^{(b_j - b_k)\lambda(p(x)-p(t)) + (b_l - b_k)\lambda(p(t)-p(s))}\, dt\, ds \\
&\quad = \int_0^1 \left(\sum_{l=1}^n \mathrm{v}_{lm}(s,\lambda)(\upsilon_{jkl}(s,x,\lambda) + \varrho_{jkl}(s,x,\lambda)) \right) f_m(s)\, ds,
\end{aligned}
$$

since we have agreed in the definition of the functions υ_{jkl} and ϱ_{jkl} that the integrals $\int_x^s$ for $s < x$ are zero. The three other possible arrangements of the indices j, k, and l can be treated similarly. According to (2.18) and Lemma 4, we have

$$
\begin{aligned}
\sup_{0 \leqslant x \leqslant 1} \int_0^1 \bigl| \mathrm{v}_{lm}(s,\lambda)(\upsilon_{jkl}(s,x,\lambda) + \varrho_{jkl}(s,x,\lambda)) \bigr|\, ds& \\
\leqslant (a + \gamma(\lambda)) e^{2a} (\Upsilon(\lambda) + \gamma(\lambda)) \to 0&
\end{aligned}
$$

as $\mathfrak{D} \ni \lambda \to \infty$. Then the relation

$$
\|(\mathrm{V}_k(\lambda))^2 \mathbf{f}\|_{L_\infty} = \max_{j,x} |g_{jk}(x,\lambda)| \leqslant n^2 (a + \gamma(\lambda)) e^{2a} (\Upsilon(\lambda) + \gamma(\lambda)) \max_{m,s} |f_m(s)|
$$

holds. Introducing the notation $C_0 = n^2(a + c)e^{2a}$, we derive the estimate

$$
\|(\mathrm{V}_k(\lambda))^2\|_{L_\infty} \leqslant C_0(\Upsilon(\lambda) + \gamma(\lambda)) \to 0 \quad \text{as } |\lambda| \to \infty, \tag{3.8}
$$

that is, the operator $(\mathrm{V}_k(\lambda))^2$ in L_∞ is a contraction for large $|\lambda|$.

Step 3. *Final estimates.* We return to (3.5) and represent its solution as the (formal, for now) series

$$
\mathbf{z}_k = \mathbf{z}_k^0 + \mathrm{V}_k(\lambda) \sum_{\nu=0}^{\infty} (\mathrm{V}_k(\lambda))^\nu \mathbf{z}_k^0.
$$

Note that the operator $V_k(\lambda)$ acting from L_∞ to L_∞ is bounded (this was proved in (3.6)). We rewrite this series in the form

$$
\mathbf{z}_k = \mathbf{z}_k^0 + \mathrm{V}_k \mathbf{z}_k^0 + \sum_{\nu=0}^{\infty} (\mathrm{V}_k(\lambda))^{2\nu} \bigl((\mathrm{V}_k(\lambda))^2 \mathbf{z}_k^0 + (\mathrm{V}_k(\lambda))^3 \mathbf{z}_k^0 \bigr).
$$

Taking account of (3.8), we increase the number λ_0 so that $\|(\mathrm{V}_k(\lambda))^2\|_{L_\infty} < 1/2$ for all $\lambda \in \mathfrak{D}$. This guarantees that the series converges in the L_∞-norm and yields the estimate

$$
\|\mathbf{z}_k - \mathbf{z}_k^0 - \mathrm{V}_k(\lambda)\mathbf{z}_k^0\|_{L_\infty} \leqslant 2 \bigl\| (\mathrm{V}_k(\lambda))^2 \mathbf{z}_k^0 + (\mathrm{V}_k(\lambda))^3 \mathbf{z}_k^0 \bigr\|_{L_\infty}. \tag{3.9}
$$

We calculate the function $\widetilde{\mathbf{z}}_k^1 = (\widetilde{z}_{jk}^1(x,\lambda))_{j=1}^n = \mathrm{Q}_k(\lambda)\mathbf{z}_k^0$ as follows:

$$\begin{aligned} \widetilde{z}_{jk}^1(x,\lambda) &= -\int_x^1 q_{jk}(t,\lambda)e^{(b_j-b_k)\lambda(p(x)-p(t))}\,dt, \qquad j<k, \\ \widetilde{z}_{jk}^1(x,\lambda) &= \int_0^x q_{jk}(t,\lambda)e^{(b_j-b_k)\lambda(p(x)-p(t))}\,dt, \qquad j\geqslant k. \end{aligned} \tag{3.10}$$

Hence we have $\widetilde{z}_{jk}^1(x,\lambda) = \upsilon_{jkk}(0,x,\lambda)$ and

$$\max_{1\leqslant j,k\leqslant n} |\widetilde{z}_{jk}^1(x,\lambda)| \leqslant \Upsilon(x,\lambda). \tag{3.11}$$

This implies the estimates $\|\widetilde{\mathbf{z}}_k^1\|_{L_\mu} \leqslant \Upsilon_\mu(\lambda)$ and

$$\|\mathrm{V}_k\mathbf{z}_k^0\|_{L_\mu} \leqslant \|\widetilde{\mathbf{z}}_k^1\|_{L_\mu} + \|\mathrm{R}_k(\lambda)\mathbf{z}_k^0\|_{L_\infty} \leqslant C_1(\Upsilon_\mu(\lambda)+\gamma(\lambda)), \qquad C = e^{hp+2a}. \tag{3.12}$$

In what follows we use the notation $\mathbf{z}_k^\nu := (\mathrm{V}_k)^\nu\mathbf{z}_k^0$. Note that

$$\begin{aligned} \|\mathbf{z}_k^2\|_{L_\infty} &\leqslant \|\mathrm{Q}_k\|_{L_\mu\to L_\infty}\|\mathbf{z}_k^1\|_{L_\mu} + \|\mathrm{R}_k\|_{L_\infty\to L_\infty}\|\mathbf{z}_k^1\|_{L_\infty} \\ &\leqslant C_1\|\mathrm{Q}_k\|_{L_\mu\to L_\infty}(\Upsilon_\mu(\lambda)+\gamma(\lambda)) + e^{hp+2a}\gamma(\lambda)\|\mathrm{V}_k\|_{L_\infty} \\ &\leqslant C_2(\Upsilon_\mu(\lambda)+\gamma(\lambda)), \end{aligned} \tag{3.13}$$

where $C_2 = 4e^{hp+4a}(a+c+m)$, $m = \|\mathrm{A}-\mathrm{D}\|_{L_{\mu'}}$. Finally, we arrive at

$$\|\mathbf{z}_k^3\|_{L_\infty} \leqslant \|\mathrm{V}_k(\lambda)\|_{L_\infty} C_2(\Upsilon_\mu(\lambda)+\gamma(\lambda)) \leqslant C_3(\Upsilon_\mu(\lambda)+\gamma(\lambda)), \tag{3.14}$$

where $C_3 = 4e^{3hp+6a}(a+c)(a+c+m)$. Substituting the above estimates into (3.9) we obtain

$$\|\mathbf{z}_k - \mathbf{z}_k^0 - \mathbf{z}_k^1\|_{L_\infty} \leqslant C_4(\Upsilon_\mu(\lambda)+\gamma(\lambda)), \qquad C_4 = 2C_2 + 2C_3. \tag{3.15}$$

In view of (3.12), we have

$$\|\mathbf{z}_k - \mathbf{z}_k^0\|_{L_\mu} \leqslant C_5(\Upsilon_\mu(\lambda)+\gamma(\lambda)), \qquad C_5 = C_4 + 2e^{hp+2a}. \tag{3.16}$$

Multiplying the matrix Z by $\mathrm{M}(x) \in AC[0,1]$ on the left, we derive (3.3) (the constant in (3.3) is $C = C_5e^a$). Estimates (3.1) are a particular case of (3.3) for $\mu = \infty$.

To prove (3.2) we note that

$$|\mathbf{z}_k^1(x,\lambda)| \leqslant |\widetilde{\mathbf{z}}_k^1(x,\lambda)| + \|\mathrm{R}_k(\lambda)\mathbf{z}_k^0\|_{L_\infty} \leqslant \Upsilon(x,\lambda) + 2e^{hp+2a}\gamma(\lambda).$$

The estimate (3.2) follows from these inequalities and (3.15).

Theorem 1 is proved.

Remark 6. If elements $c_{jk}(x,\lambda)$ of the matrix $\mathrm{C}(x,\lambda)$ are holomorphic with respect to the parameter λ in $\mathfrak{D}$, then the function $\mathrm{Y}(x,\lambda)$ is also holomorphic in this domain (as an element of $(AC[0,1])^{n^2}$).

This follows from Remark 2.

Remark 7. The determinant of the matrix $\mathrm{Y}(x,\lambda)$ has the form

$$\det \mathrm{Y}(x,\lambda) = \det \mathrm{Y}^0(x,\lambda)(\det \mathrm{M}(x) + o(1)) \quad \text{as } \lambda \to \infty.$$

In particular, this determinant is nonzero and $Y(x,\lambda)$ is indeed a fundamental solution matrix for the system (1.3) in the domain $\mathfrak{D}$.

In fact, in Theorem 1 we only found the leading term in the asymptotic representation of the matrix $\mathrm{Y}(x,\lambda)$. Below we give useful refinements of the asymptotic formulae (3.1)–(3.3).

Theorem 2. *Let $\mathrm{Z}^\nu(x,\lambda)$ be the matrices formed of the columns $\mathbf{z}_k^\nu(x,\lambda) = \mathrm{V}_k^\nu \mathbf{z}_k^0$, where $\mathbf{z}_k^0 = \{\delta_j^k\}_{j=1}^n$. If the assumptions of Theorem 1 hold, then the following asymptotic representations are valid in the domain $\mathfrak{D}$:*

$$\begin{gathered}\mathrm{Y}(x,\lambda) = \mathrm{M}(x,\lambda)\mathrm{Z}(x,\lambda)\mathrm{E}(x,\lambda), \quad \textit{where } \mathrm{Z}(x,\lambda) = \mathrm{I} + \mathrm{Z}^1(x,\lambda) + \mathrm{Z}^2(x,\lambda) + \mathrm{S}(x,\lambda),\\ \mathrm{S}(x,\lambda) = (s_{jk}(x,\lambda))_{j,k=1}^n, \quad \textit{where } \max_{j,k,x} |s_{jk}(x,\lambda)| \leqslant C(\Upsilon(\lambda) + \gamma(\lambda))^2,\end{gathered} \tag{3.17}$$

$$\mathrm{Z}(x,\lambda) = \mathrm{I} + \mathrm{Z}^1(x,\lambda) + \mathrm{S}(x,\lambda), \quad \textit{where } \max_{j,k} \|s_{jk}(x,\lambda)\|_{L_\infty} \leqslant C(\Upsilon(\lambda) + \gamma(\lambda)). \tag{3.18}$$

In addition, if the elements of the matrix $\mathrm{A}(x) - \mathrm{D}(x)$ are integrable to the power $\mu' \in (1,\infty)$, then

$$\mathrm{Z}(x,\lambda) = \mathrm{I} + \mathrm{Z}^1(x,\lambda) + \mathrm{S}(x,\lambda), \quad \textit{where } \max_{j,k,x} |s_{jk}(x,\lambda)| \leqslant C(\Upsilon_\mu(\lambda) + \gamma(\lambda)), \tag{3.19}$$

$$\begin{gathered}\mathrm{Z}(x,\lambda) = \mathrm{I} + \mathrm{Z}^1(x,\lambda) + \mathrm{Z}^2(x,\lambda) + \mathrm{Z}^3(x,\lambda) + \mathrm{S}(x,\lambda),\\ \textit{where } \max_{j,k} \|s_{jk}(x,\lambda)\|_{L_\mu} \leqslant C(\Upsilon_\mu(\lambda) + \gamma(\lambda))^2,\end{gathered} \tag{3.20}$$

$$\begin{gathered}\mathrm{Z}(x,\lambda) = \mathrm{I} + \mathrm{Z}^1(x,\lambda) + \mathrm{Z}^2(x,\lambda) + \mathrm{S}(x,\lambda),\\ \textit{where } \max_{j,k} \|s_{jk}(x,\lambda)\|_{L_\infty} \leqslant C(\Upsilon_\mu(\lambda) + \gamma(\lambda))(\Upsilon(\lambda) + \gamma(\lambda)).\end{gathered} \tag{3.21}$$

Proof. The representations (3.17)–(3.19) follow from the estimates that we obtained in the proof of Theorem 1. To deduce (3.17), it suffices to use inequalities (3.12) and (3.8). Estimate (3.19) is derived from (3.15). Writing down the equality

$$\mathbf{z}_k - \mathbf{z}_k^0 - \mathrm{V}_k \mathbf{z}_k^0 = \mathrm{V}_k(\mathbf{z}_k - \mathbf{z}_k^0)$$

and using (3.6) and (3.16) for $\mu = \infty$ we arrive at (3.18).

We need additional arguments to prove (3.20) and (3.21). We show that for $\mu' > 1$ the operator $(\mathrm{Q}_k(\lambda))^2$, regarded as an operator from L_∞ to L_μ, is also a contraction. We take an arbitrary function $\mathbf{f} \in L_\infty$ and set $\mathbf{g}_k = (\mathrm{Q}_k(\lambda))^2\mathbf{f}$.

Repeating the arguments used to derive (3.7) above, we arrive at the same equality (3.7), but with the functions $\upsilon_{lm}(s,\lambda)$ replaced by the $q_{lm}(s)$. Then we have

$$\begin{aligned}
&\left(\int_0^1\left|\int_0^1 q_{lm}(s)\upsilon_{jkl}(s,x,\lambda)\,ds\right|^\mu dx\right)^{1/\mu}\\
&\qquad\leqslant\left(\int_0^1\left(\int_0^1 |q_{lm}(s)|^{\mu'}\,ds\right)^{\mu/\mu'}\int_0^1|\upsilon_{jkl}(s,x,\lambda)|^\mu\,ds\right)^{1/\mu}\leqslant\|q_{lm}\|_{L_{\mu'}}\Upsilon_\mu(\lambda),\\
&\qquad\int_0^1|g_{jk}(x,\lambda)|^\mu\,dx\leqslant n^{2/\mu'}\|\mathbf{f}\|^\mu_{L_\infty}\sum_{l,m=1}^n\int_0^1\left|\int_0^1 q_{lm}(s)\upsilon_{jkl}(s,x,\lambda)\,ds\right|^\mu dx\\
&\qquad\qquad\leqslant n^{2/\mu'}\|\mathbf{f}\|^\mu_{L_\infty}\bigl(\|\mathrm{Q}(x)\|_{L_{\mu'}}\Upsilon_\mu(\lambda)\bigr)^\mu.
\end{aligned}$$

It follows from these inequalities that $\|\mathbf{g}_k\|_{L_\mu}\leqslant n^2\|\mathrm{Q}(x)\|_{L_{\mu'}}\Upsilon_\mu(\lambda)\|\mathbf{f}\|_{L_\infty}$, that is,

$$\|(\mathrm{Q}_k(\lambda))^2\|_{L_\infty\to L_\mu}\leqslant n^2e^{2a}\|\mathrm{A}(x)-\mathrm{D}(x)\|_{L_{\mu'}}\Upsilon_\mu(\lambda). \tag{3.22}$$

Setting $C_6=n^2e^{2a}\max\{a+c,\|\mathrm{A}(x)-\mathrm{D}(x)\|_{L_{\mu'}}\}$ we obtain $\|(\mathrm{Q}_k(\lambda))^2\|_{L_\infty\to L_\mu}\leqslant C_6\Upsilon_\mu(\lambda)$. Then

$$\begin{aligned}
\|(\mathrm{V}_k(\lambda))^2\|_{L_\infty\to L_\mu}&\leqslant\|(\mathrm{Q}_k(\lambda))^2\|_{L_\infty\to L_\mu}+\|\mathrm{Q}_k(\lambda)\mathrm{R}_k(\lambda)\|_{L_\infty\to L_\mu}\\
&\qquad+\|\mathrm{R}_k(\lambda)\mathrm{Q}_k(\lambda)\|_{L_\infty\to L_\mu}+\|(\mathrm{R}_k(\lambda))^2\|_{L_\infty\to L_\mu}\\
&\leqslant C_6\Upsilon_\mu(\lambda)+8e^{2hp+2a}\|\mathrm{A}(x)-\mathrm{D}(x)\|_{L'_\mu}\gamma(\lambda)+4e^{2hp}\gamma^2(\lambda).
\end{aligned}$$

Therefore, for sufficiently large $|\lambda|>\lambda_0$, when $\gamma(\lambda)<1$, the estimate

$$\|(\mathrm{V}_k(\lambda))^2\|_{L_\infty\to L_\mu}\leqslant C_7(\Upsilon_\mu(\lambda)+\gamma(\lambda)) \tag{3.23}$$

is valid. We bounded the norms $\|\mathbf{z}_k^2\|_{L_\infty}$ and $\|\mathbf{z}_k^3\|_{L_\infty}$ in (3.13) and (3.14) by the quantity $C(\Upsilon_\mu(\lambda)+\gamma(\lambda))$. It remains to note that $\|(\mathrm{V}_k(\lambda)^2\|_{L_\infty}\leqslant 1/2$; thus, we have

$$\left\|\sum_{\nu=0}^\infty(\mathrm{V}_k(\lambda))^{2\nu}(\mathbf{z}_k^2+\mathbf{z}_k^3)\right\|_{L_\infty}\leqslant 2\|\mathbf{z}_k^2+\mathbf{z}_k^3\|_{L_\infty}\leqslant C_8(\Upsilon_\mu(\lambda)+\gamma(\lambda)).$$

Applying the operator $(\mathrm{V}_k(\lambda))^2$ to the function whose norm is estimated in this inequality and taking (3.23) into account we infer the inequality

$$\left\|\sum_{\nu=1}^\infty(\mathrm{V}_k(\lambda))^{2\nu}(\mathbf{z}_k^2+\mathbf{z}_k^3)\right\|_{L_\mu}\leqslant C_9(\Upsilon_\mu(\lambda)+\gamma(\lambda))^2.$$

This proves (3.20).

To prove (3.21) we use estimates (3.23) and (3.11), which yield

$$\|\mathbf{z}_k^3\|_{L_\mu}\leqslant\|V_k(\lambda)^2\|_{L_\infty\to L_\mu}\|\mathbf{z}_k^1\|_{L_\infty}\leqslant C_7(\Upsilon_\mu(\lambda)+\gamma(\lambda))(\Upsilon(\lambda)+\gamma(\lambda)).$$

Theorem 2 is proved.

Theorems 1 and 2 give estimates for the remainders in the asymptotic representation of the matrix $\mathrm{Y}(x,\lambda)$ in terms of the functions $\Upsilon(\lambda)$, $\Upsilon_\mu(\lambda)$, $\Upsilon(x,\lambda)$ and $\gamma(\lambda)$. In the general situation, when conditions (i)–(iii) on the coefficients of the system (1.3) hold, these functions tend to zero as $\mathfrak{D}\ni\lambda\to\infty$ (see Lemma 4). The rate at which the function $\gamma(\lambda)$ tends to zero is completely governed by the rate of decrease of the functions $c_{ij}(\,\cdot\,,\lambda)$. It can be seen from Theorem 3 given below that $\gamma(\lambda)=O(|\lambda|^{-1})$ in the case of high-order operators. The way the functions $\Upsilon(\lambda)$, $\Upsilon_\mu(\lambda)$, and $\Upsilon(x,\lambda)$ approach zero is governed by the 'regularity' of $\rho(x)$ and the elements of the matrix $\mathrm{A}(x)-\mathrm{D}(x)$. When the integrability exponent $\mu'\in(1,2]$ of these functions increases, we can estimate the norms of $\Upsilon_\mu(\lambda)$ and $\Upsilon(x,\lambda)$ in the Hardy spaces H^μ. When we increase the smoothness expressed in terms of estimates for the modulus of continuity or the norm in the Besov spaces of the function $\rho(x)$ and elements of the matrix $\mathrm{A}(x)-\mathrm{D}(x)$, estimates for the rate of decrease of the functions $\Upsilon(\lambda)$, $\Upsilon_\mu(\lambda)$, and $\Upsilon(x,\lambda)$ can be obtained. The relevant theorems will be given in the next paper by these authors.

§ 4. Asymptotics for ordinary differential equations with distribution coefficients

4.1. In this section we obtain the required asymptotics for the fundamental system of solutions of equation (1.1). In this case the result in [11] concerning the regularization of (1.1) and the results we deduced in § 3 play a key role. We write out the most general form of the (nonselfadjoint) differential expression of even order which we are dealing with:

$$\tau(y)=\sum_{k,s=0}^{m}(\tau_{k,s}(x)y^{(m-k)}(x))^{(m-s)} \tag{4.1}$$

and assume that the complex-valued functions $\tau_{k,s}$ are subject to conditions (1.2), where $\tau_0:=\tau_{0,0}$. Note that the choice of the constants of integration in (1.2) does not affect the fulfillment of these conditions.

We say that a differential expression *is reduced to normal form* if it as follows:

$$\begin{aligned}l(y)=\sum_{k=0}^{m}&(-1)^{m-k}(\tau_k(x)y^{(m-k)}(x))^{(m-k)}\\&+\mathrm{i}\sum_{k=0}^{m-1}(-1)^{m-k-1}\bigl[(\sigma_k(x)y^{(m-k-1)})^{(m-k)}+(\sigma_k(x)y^{(m-k)})^{(m-k-1)}\bigr].\end{aligned} \tag{4.2}$$

This expression is a particular case of (4.1) when $\tau_{k,s}\equiv 0$ for $|k-s|\geqslant 2$ and $\tau_{k,k+1}\equiv\tau_{k+1,k}$. For differential expressions of the form (4.2) conditions (1.2) take the form

$$\frac{1}{\sqrt{|\tau_0|}},\ \frac{1}{\sqrt{|\tau_0|}}\tau_k^{(-k)},\ \frac{1}{\sqrt{|\tau_0|}}\sigma_k^{(-k)}\in L^2[0,1]. \tag{4.3}$$

Note that in [11] conditions (1.2) and (4.3) are assumed to hold only locally on the interval $(0,1)$. This makes it possible to define differential expressions consistently and to construct minimal and maximal operators. However, for our purposes we need the functions in (4.3) to be square summable on the whole of the interval $[0,1]$.

The following assertion repeats Theorem 1 in [11]. We give a detailed proof of it.

Proposition 4. *For any differential expression $\tau(y)$ of the form (4.1) there is an expression $l(y)$ of the form (4.2) coinciding with it. The equality $\tau(y)=l(y)$ on the interval $(0,1)$ is understood in the sense of the theory of distributions.*

Proof. We call the integer $k-s$ the index of the term $(\tau_{k,s}y^{(m-k)})^{(m-s)}$. We first consider the (only) term $(\tau_{m,0}y)^{(m)}$ of index m. Due to (1.2), we have $|\tau_0|^{-1/2}\tau_{m,0}\in L_2[0,1]$. We replace this term in the differential expression $\tau(y)$ by the sum

$$(\tau_{m,0}y)^{(m)}=(\tau'_{m,0}y)^{(m-1)}+(\tau_{m,0}y')^{(m-1)}$$

of two terms with indices $m-1$ and $m-2$. In this case conditions (1.2) also hold for the transformed differential expression, since

$$l=1,\qquad |\tau_0|^{-1/2}(\tau'_{m,0})^{(-1)}\in L_2[0,1],\quad |\tau_0|^{-1/2}(\tau_{m,0})^{(-1)}\in L_2[0,1]$$

(here the power -1 means taking the antiderivative). As a result, we arrive at a differential expression with no terms of index m for which conditions (1.2) are satisfied. Using the transformation

$$(\tau_{k,s}y^{(m-k)})^{(m-s)}=(\tau'_{k,s}y^{(m-k)})^{(m-s-1)}+(\tau_{k,s}y^{(m-k+1)})^{(m-s-1)},$$

we do exactly the same with the two terms with index $m-1$, and so on, until all expressions with indices $k-s\geqslant 2$ are eliminated. It is straightforward to see that conditions (4.3) are preserved in this case. Thereafter, using the identity

$$(\tau_{k,s}y^{(m-k)})^{(m-s)}=(\tau_{k,s}y^{(m-k-1)})^{(m-s+1)}-(\tau'_{k,s}y^{(m-k-1)})^{(m-s)}$$

(the index of the expression increases under this transformation), we eliminate all terms with indices $k-s\leqslant -2$. As a result, only expressions with indices -1, 0 and 1 remain. Finally, the identity

$$\begin{aligned}&(\tau_{k+1,k}y^{(m-k-1)})^{(m-k)}+(\tau_{k,k+1}y^{(m-k)})^{(m-k-1)}\\&\quad=\left(\frac12(\tau_{k+1,k}+\tau_{k,k+1})y^{(m-k-1)}\right)^{(m-k)}+\left(\frac12(\tau_{k+1,k}+\tau_{k,k+1})y^{(m-k)}\right)^{(m-k-1)}\\&\qquad-\left(\frac12(\tau_{k+1,k}+\tau_{k,k+1})'y^{(m-k-1)}\right)^{(m-k-1)},\end{aligned}$$

where $k=0,\dots,m-1$, reduces the differential expression to the form (4.2).

The proposition is proved.

Remark 8. We can easily see that the above transformations do not change the leading term $(\tau_0(x)y^{(m)})^{(m)}$.

Thus, in what follows, we deal with expressions of the form (4.2). We show how we can go over from the equation $l(y)=\lambda^n\varrho y$ to a system of the form (2.1).

To perform this transition, it suffices to assume that the weight function satisfies $\varrho \in L_1[0,1]$). However, to derive asymptotic representations we will assume later that $\varrho \in AC[0,1]$. We introduce some notation

$$\mathscr{T}_k = \tau_k^{(-k)}, \qquad \mathscr{S}_k = \sigma_k^{(-k)}, \qquad \varphi_k = \mathscr{T}_k + \mathrm{i}\mathscr{S}_{k-1}, \qquad \psi_k = \mathscr{T}_k - \mathrm{i}\mathscr{S}_{k-1}. \tag{4.4}$$

Now we introduce the matrix $\mathrm{F}(x) = (f_{j,k}(x))_{j,k=1}^n$ into consideration. First of all, we set

$$f_{j,k} = 0$$

if one of the following conditions hold ($m = n/2$):

$$\begin{cases} 1 \leqslant j \leqslant m-1, \\ k \neq j+1, \end{cases} \qquad \begin{cases} j = m, \\ k \geqslant m+1, \end{cases} \qquad \begin{cases} m+1 \leqslant j \leqslant n, \\ k \geqslant m+2, \\ k \neq j+1. \end{cases}$$

We define the functions above the main diagonal by the equality

$$f_{j,j+1} = \begin{cases} 1 & \text{for } j \neq m, \\ \tau_0^{-1} & \text{for } j = m; \end{cases}$$

hence, the matrix F assumes the form

$$\mathrm{F} = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ f_{m,1} & f_{m,2} & \dots & \dots & f_{m,m} & \tau_0^{-1} & 0 & \dots & 0 & 0 \\ f_{m+1,1} & f_{m+1,2} & \dots & \dots & \dots & f_{m+1,m+1} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ f_{n-1,1} & f_{n-1,2} & \dots & \dots & \dots & f_{n-1,m+1} & 0 & \dots & 0 & 1 \\ f_{n,1} & f_{n,2} & f_{n,3} & \dots & \dots & f_{n,m+1} & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{4.5}$$

We define the first m elements of the mth row of F by

$$f_{m,k} = (-1)^{m-k}\varphi_{m+1-k}\tau_0^{-1}, \qquad 1 \leqslant k \leqslant m,$$

while the last m elements of the $(m+1)$th column are defined by

$$f_{j,m+1} = -\psi_{j-m}\tau_0^{-1}, \qquad m+1 \leqslant j \leqslant n.$$

We define the remaining elements of F by

$$\begin{aligned} f_{m+k,m-j} = {} & (-1)^{j+1}\varphi_{j+1}\psi_k\tau_0^{-1} \\ & + \chi_{j+k<m}(-1)^j \binom{j+k+1}{k}\left[\mathscr{T}_{j+k+1} + \mathrm{i}\frac{j-k+1}{j+k+1}\mathscr{S}_{j+k}\right], \end{aligned} \tag{4.6}$$

where $1 \leqslant k \leqslant m$, $0 \leqslant j \leqslant m-1$, $\binom{j+k+1}{k}$ is the binomial coefficient, and the number $\chi_{j+k<m}$ is 1 for $j+k<m$ and 0 otherwise. We note the following important fact, which follows directly by (4.3): *the functions $f_{j,k}(x)$ are integrable on $[0,1]$.*

Now we introduce the quasi-derivatives[2] of a function y, which we denote by $y^{[j]}(x)$, in the following way:

$$y^{[0]} = y, \qquad y^{[k]} = \frac{1}{f_{k,k+1}(x)}\biggl[(y^{[k-1]})' - \sum_{j=1}^{k} f_{k,j}(x)y^{[j-1]}\biggr], \quad 1 \leqslant k \leqslant n-1. \tag{4.7}$$

We state Theorem 2 from [11] (its proof needs serious technical work and is not given here).

Proposition 5. *If $l(y)$ is a differential expression of the form (4.2), then it takes the following form in the new notation*:

$$l(y) = (y^{[n-1]})' - \sum_{j=1}^{n} f_{n,j}(x)y^{[j-1]}, \tag{4.8}$$

where the functions $f_{n,j}$ are elements of the matrix F defined by (4.5) and (4.6).

4.2. Recall that our purpose is to study the equation $l(y) = \lambda^n\varrho(x)y$, where $\lambda^n \in \mathbb{C}$ is a spectral parameter. Note that equalities (4.7) and (4.8) make it possible to write the equation $l(y) = \lambda^n\varrho(x)y$ as a system of n first-order differential equations with respect to the vector function formed of the quasi-derivatives $y^{[j]}(x)$, $0 \leqslant j \leqslant n-1$. The matrix of this system is evidently $\mathrm{F}(x)$, with the only change that the term $(-1)^m\lambda^n\varrho(x)$ is added to $f_{n,1}(x)$. However, it is more convenient to go over to a system by setting $\mathbf{u}(x) = (u_j(x))_{j=1}^n$, where $u_j(x) := \lambda^{1-j}y^{[j-1]}(x)$.

Proposition 6. *If $l(y)$ is a differential expression of the form (4.2), then the equation $l(y) = \lambda^n\varrho(x)y$ is equivalent in our notation to the system of equations*

$$\mathbf{u}' = \mathrm{F}(x,\lambda)\mathbf{u}, \quad \textit{where } \mathrm{F}(x,\lambda) = \bigl(f_{j,k}(x)\lambda^{k-j} + (-1)^m\lambda\varrho(x)\delta_j^n\delta_k^1\bigr)_{j,k=1}^n. \tag{4.9}$$

Proof. We set $\mathbf{v}(x) = (y^{[j-1]}(x))_{j=1}^n$. Then $\mathbf{v}' = (\mathrm{F}(x) + (-1)^m\lambda^n\mathrm{E}_{n,1})\mathbf{v}$, where $\mathrm{E}_{\alpha,\beta}$ denotes the matrix $\mathrm{E}_{\alpha,\beta} = \bigl(\delta_j^\alpha\delta_k^\beta\bigr)_{j,k=1}^n$ and δ_j^k is the Kronecker symbol. According to our notation, we have $\mathbf{v} = \operatorname{diag}\{1, \lambda, \dots, \lambda^{n-1}\}\mathbf{u}$; hence the relation

$$\mathbf{u}' = \operatorname{diag}\{1, \lambda^{-1}, \dots, \lambda^{1-n}\} \cdot (\mathrm{F}(x) + (-1)^m\lambda^n\mathrm{E}_{n,1}) \cdot \operatorname{diag}\{1, \lambda, \dots, \lambda^{n-1}\}\mathbf{u}$$

holds. Multiplying these three matrices, we arrive at (4.9).

The proposition is proved.

We now note that the matrix $\mathrm{F}(x,\lambda)$ in (4.9) can be represented as

$$\mathrm{F}(x,\lambda) = \lambda\mathrm{F}_1(x) + \mathrm{F}_0(x) + \lambda^{-1}\mathrm{F}_{-1}(x) + \dots + \lambda^{1-n}\mathrm{F}_{1-n}(x), \tag{4.10}$$

[2] In different publications (for example, see [42] and [43]), including these authors' papers [3] and [4], different definitions of quasi-derivatives are used. For example, in the case of the Sturm-Liouville operator, when $n = 2$ and only the first quasi-derivative must be defined, it is convenient to discard the factor $1/f_{1,2}$ in the definition. Distinctions of this kind must be taken into account when results on the asymptotic behaviour of solutions being compared.

where

$$\mathrm{F}_1(x) = \sum_{j=1}^{m-1} \mathrm{E}_{j,j+1} + \tau_0^{-1}(x)\mathrm{E}_{m,m+1} + \sum_{j=m+1}^{n-1} \mathrm{E}_{j,j+1} + (-1)^m \varrho(x)\mathrm{E}_{n,1},$$
$$\mathrm{F}_0(x) = f_{m,m}(x)\mathrm{E}_{m,m} + f_{m+1,m+1}(x)\mathrm{E}_{m+1,m+1},$$
$$\mathrm{F}_{-k}(x) = \sum_{j=k+1}^{n} f_{j,j-k}\mathrm{E}_{j,j-k}, \quad \mathrm{E}_{\alpha,\beta} = \big(\delta_j^\alpha \delta_k^\beta\big)_{j,k=1}^n, \qquad 1 \leqslant k \leqslant n-1.$$

Thus, the system (4.9) has the form (2.1). Now we show that the assumptions of Proposition 1 are satisfied for $\mathrm{F}_1(x)$, that is, this matrix admits diagonalization and can be reduced to a system of the form (1.3).

Theorem 3. *Assume that a differential expression τ has the form (4.1), its coefficients are subject to conditions (4.3), and functions τ_0 and ϱ are absolutely continuous and positive on the interval $[0,1]$. Then the equation $\tau(y) = \lambda^n \varrho(x) y$ is reducible to a system of the form (1.3) with matrix $\mathrm{B} = \operatorname{diag}\{\omega_0, \omega_1, \dots, \omega_{n-1}\}$, where the ω_k are the nth roots of $(-1)^m$, $m = n/2$, representable as*

$$\omega_k = \begin{cases} \epsilon^{(2k+1)/2} & \text{if } m \text{ is odd,} \\ \epsilon^k & \text{if } m \text{ is even,} \end{cases} \qquad \epsilon = e^{i\pi/m}, \quad k = 1, \dots, n.$$

The function ρ has the form

$$\rho(x) = \varrho^{1/n}(x)\tau_0^{-1/n}(x) \tag{4.11}$$

and the elements $(a_{j,k}(x))_{j,k=1}^n$ of the matrix $\mathrm{A}(x)$ are given by the formulae

$$a_{j,k}(x) = \frac{1}{n}\begin{cases} \dfrac{1}{1-\epsilon^{k-j}}\left(\dfrac{\varrho'(x)}{\varrho(x)} - \epsilon^{m(k-j)}\dfrac{\tau_0'(x)}{\tau_0(x)}\right) \\ \qquad + \dfrac{\epsilon^{m(k-j)}}{\tau_0(x)}\big(\varphi_1(x)\epsilon^{j-k} - \psi_1(x)\big), & j \neq k, \\ \dfrac{1-n}{2}(\ln \varrho(x))' - \dfrac{1}{2}(\ln \tau_0(x))' + \dfrac{2\mathrm{i}\sigma_0(x)}{\tau_0(x)}, & j = k. \end{cases} \tag{4.12}$$

All elements of the matrices $\mathrm{A}(x)$ and $\mathrm{C}(x,\lambda)$ are integrable on $[0,1]$; furthermore, $\|\mathrm{C}(\,\cdot\,,\lambda)\|_{L_1} = O(|\lambda|^{-1})$ as $\lambda \to \infty$.

Proof. Most of the work has already been done: we have shown that the equation $\tau(y) = \lambda^n \varrho(x) y$ can be reduced to the system (4.9). It remains to diagonalize the matrix $\mathrm{F}_1(x)$ in (4.10). It is straightforward to see that the characteristic polynomial of this matrix is $\chi(s) = s^n + (-1)^{m+1}\varrho(x)\tau^{-1}(x)$; thus, its roots are $s_k = \omega_k \rho(x)$. The function ρ is positive; therefore, for each $x \in [0,1]$ the roots s_k of the characteristic polynomial are pairwise distinct. This means that there is a transition matrix $\mathrm{W}(x)$ such that $\mathrm{W}^{-1}(x)\mathrm{F}_1(x)\mathrm{W}(x) = \rho(x)\operatorname{diag}\{\omega_0, \omega_1, \dots, \omega_{n-1}\}$. We find an explicit form for $\mathrm{W}(x)$:

$$\mathrm{W}(x)=(w_{jk}(x))_{j,k=1}^{n}, \qquad w_{jk}(x)=(\omega_{k-1}\rho(x))^{j-1}\cdot\begin{cases}1, & j\leqslant m,\\ \tau_0, & j>m,\end{cases} \tag{4.13}$$

$$\mathrm{W}^{-1}(x)=(\widetilde{w}_{jk}(x))_{j,k=1}^{n}, \qquad \widetilde{w}_{jk}(x)=\frac{1}{n}(\omega_{j-1}\rho(x))^{1-k}\cdot\begin{cases}1, & k\leqslant m,\\ \tau_0^{-1}, & k>m.\end{cases}$$

We show that this matrix is the one required. First we note that

$$\sum_{l=1}^{n} w_{jl}\widetilde{w}_{lk}=\frac{1}{n}\sum_{l=1}^{n}\omega_{l-1}^{j-k}\cdot\left\{\begin{matrix}\rho^{j-k} & \text{if } j,k\leqslant m\\ \rho^{j-k}\tau_0 & \text{if } k\leqslant m<j\\ \rho^{j-k}\tau_0^{-1} & \text{if } j\leqslant m<k\\ \rho^{j-k} & \text{if } m<j,k\end{matrix}\right\}=\delta_j^k,$$

since all the sums $\sum_{l=1}^{n}\omega_{l-1}^{p}$ with $p=-n+1,-n+2,\dots,-1,1,\dots,n-2,n-1$ are zero. Thus, the matrices $\mathrm{W}(x)$ and $\mathrm{W}^{-1}(x)$ really are inverse to each other.

It is somewhat more difficult to verify the equality

$$\mathrm{W}^{-1}(x)\mathrm{F}_1(x)\mathrm{W}(x)=\rho(x)\,\mathrm{diag}\{\omega_0,\omega_1,\dots,\omega_{n-1}\}.$$

Multiplying $\mathrm{F}_1(x)$ and $\mathrm{W}(x)$, we see that

$$(\mathrm{F}_1(x)\mathrm{W}(x))_{jk}=(\omega_{k-1}\rho(x))^{j}\cdot\begin{cases}1, & j\leqslant m,\\ \tau_0, & j>m.\end{cases}$$

Then the element in the ith row and kth column of the matrix $\mathrm{W}^{-1}\mathrm{F}_1\mathrm{W}$ is as follows:

$$\frac{1}{n}\sum_{j=1}^{n}(\omega_{i-1}\rho)^{1-j}(\omega_{k-1}\rho)^{j}=\frac{\omega_{i-1}\rho(x)}{n}\sum_{j=1}^{n}\epsilon^{j(k-i)}=\begin{cases}0, & k-i\neq 0,\\ \omega_{i-1}\rho(x), & k-i=0.\end{cases}$$

So the substitution $\mathbf{y}=\mathrm{W}^{-1}\mathbf{u}$ reduces the system (4.9) to the form

$$\mathbf{y}'=\lambda\rho(x)\mathrm{B}\mathbf{y}+\mathrm{A}(x)\mathbf{y}+\mathrm{C}(x,\lambda)\mathbf{y}, \quad \text{where } \mathrm{B}=\mathrm{diag}\{\omega_0,\omega_1,\dots,\omega_{n-1}\}. \tag{4.14}$$

In this case we have

$$\mathrm{A}(x)=-\mathrm{W}^{-1}(x)\mathrm{W}'(x)+\mathrm{W}^{-1}(x)\mathrm{F}_0(x)\mathrm{W}(x)$$

and

$$\mathrm{C}(x,\lambda)=\mathrm{W}^{-1}(x)\sum_{k=1}^{n-1}\lambda^{-k}\mathrm{F}_{-k}(x)\mathrm{W}(x).$$

Substituting in the explicit expressions for the matrices W, W^{-1} and F_0 we obtain formulae (4.12) for the elements of A. Since the elements of the matrices $\mathrm{W}(x)$ and $\mathrm{W}^{-1}(x)$ are absolutely continuous and the elements of the matrices $\mathrm{F}_k(x)$ are integrable, hence the elements of the matrices $\mathrm{A}(x)$ and $\mathrm{C}(x,\lambda)$ are also integrable. In addition, the last equality yields the estimate $\|\mathrm{C}(\,\cdot\,,\lambda)\|_{L_1}=O(|\lambda|^{-1})$ as $\lambda\to\infty$.

Theorem 3 is proved.

We now obtain asymptotic formulae for the solutions of the differential equation $l(y) = \lambda^n \varrho(x) y$ of order $n = 2m$ with the spectral parameter on the right-hand side. We assume that the differential expression $l(y)$ has been already reduced to the normal form (4.2). In what follows we use the notation introduced above without mentioning this. Due to Theorem 3, the matrix B of the resulting system is $\mathrm{B} = \operatorname{diag}\{\omega_0, \dots, \omega_{n-1}\}$. First we describe the system of sectors Γ_k whose boundaries are given by the equations

$$\operatorname{Re}(\omega_j \lambda) = \operatorname{Re}(\omega_k \lambda), \qquad 0 \leqslant j \neq k \leqslant n-1.$$

In view of the explicit form of the numbers ω_j, these equations are easy to solve (we omit solving the trigonometric equations). The final result is as follows: for $n = 2$, the plane is divided into two sectors which are the upper and lower half-planes; for $n > 2$, the plane is divided into $2n$ sectors of the form

$$\Gamma_k = \left\{\lambda \colon \frac{\pi(k-1)}{n} \leqslant \arg \lambda \leqslant \frac{\pi k}{n}\right\}, \qquad k = 1, \dots, 2n.$$

As before, we extend each sector Γ_k by shifting it by r along the bisector. We denote the sectors obtained by $\widetilde{\Gamma}_k$.

Theorem 4. *Assume that $l(y)$ is a differential expression of the form (4.2) whose coefficients satisfy conditions (4.3), and that $\tau_0(x)$ and $\varrho(x)$ are absolutely continuous and positive functions on $[0,1]$. Let $\widetilde{\Gamma}_\kappa$ be one of the sectors defined above. Then the equation $l(y) = \lambda^n \varrho(x) y$ has a fundamental system of solutions $y_k(x, \lambda)$, $k = 1, \dots, n$, in this sectors, which admit the asymptotic representations*

$$y_k(x,\lambda) = e^{\omega_{k-1}\lambda p(x)} \left[\varrho^{(1-n)/(2n)}(x) \tau_0^{-1/(2n)}(x) \exp\left\{\frac{2\mathrm{i}}{n} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\, dt\right\} + \zeta_{1k}(x,\lambda)\right] \tag{4.15}$$

in $\widetilde{\Gamma}_\kappa$, where $\tau_0 := \tau_{0,0}$, $p(x) = \int_0^x \rho(t)\, dt$, and the function ρ is defined by (4.11). The quasi-derivatives of orders $j = 1, \dots, m-1$ of these functions have the asymptotic representations

$$y_k^{[j]}(x) = \lambda^j e^{\omega_{k-1}\lambda p(x)} \biggl[(\omega_{k-1})^j \varrho^{(2j-n+1)/(2n)}(x) \tau_0^{-(1+2j)/(2n)}(x) \times \exp\left\{\frac{2\mathrm{i}}{n} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\, dt\right\} + \zeta_{jk}(x,\lambda)\biggr], \tag{4.16}$$

while the quasi-derivatives of orders $j = m, \dots, n-1$ have the representations

$$y_k^{[j]}(x) = \lambda^j e^{\omega_{k-1}\lambda p(x)} \biggl[(\omega_{k-1})^j \varrho^{(2j-n+1)/(2n)}(x) \tau_0^{1-(1+2j)/(2n)}(x) \times \exp\left\{\frac{2\mathrm{i}}{n} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\, dt\right\} + \zeta_{jk}(x,\lambda)\biggr] \tag{4.17}$$

(here the quasi-derivatives are defined by (4.7)). As $\widetilde{\Gamma}_\kappa \ni \lambda \to \infty$, the remainders in these representations admit the estimate

$$\max_{j,k} |\zeta_{j,k}(x,\lambda)| \leqslant C(\Upsilon(x,\lambda) + \Upsilon_\mu(\lambda) + |\lambda|^{-1}). \tag{4.18}$$

Here the functions $\Upsilon_\mu(\lambda)$ *and* $\Upsilon(x,\lambda)$ *are defined by* (2.21) *and* (2.19), *where* $q_{jl}(x) = a_{jl}(x)$, *and the matrix* $\mathrm{A}(x)$ *is defined by* (4.12).

In the case when the functions $\varrho'(x)$, $\tau_0'(x)$, $\tau_1^{(-1)}(x)$, *and* $\sigma_0(x)$ *are integrable to power* μ', *the remainders in the asymptotic representations admit the estimates*

$$\max_{j,k} \|\zeta_{j,k}(x,\lambda)\|_{L_\mu} \leqslant C(\Upsilon_\mu(\lambda) + |\lambda|^{-1}), \qquad \mu = \left(1 - \frac{1}{\mu'}\right)^{-1}. \tag{4.19}$$

Proof. We set

$$\mathbf{u}(x) = (u_j(x))_{j=1}^n, \quad u_j(x) = \lambda^{1-j} y^{[j-1]}(x) \quad \text{and} \quad \mathbf{y}(x) = \mathrm{W}^{-1}(x)\mathbf{u}(x),$$

where the matrices $\mathrm{W}(x)$ and $\mathrm{W}^{-1}(x)$ are defined in (4.13). Then the vector $\mathbf{y}(x)$ satisfies (4.14) with matrix $\mathrm{A}(x)$ defined in (4.12). We apply Theorem 1 to seek an asymptotic representation for the fundamental matrix $\mathrm{Y}(x,\lambda)$ of this system. First we find the principal term of the expansion, that is, the matrix $\mathrm{Y}^0(x,\lambda)$. Note that all numbers $b_j = \omega_{j-1}$, $j = 1,\dots,n$, are pairwise distinct; hence (see Remark 3) $\mathrm{M}(x)$ is a diagonal matrix. In addition, we can see from (4.12) that all the elements a_{jj}, $1 \leqslant j \leqslant n$, are equal and

$$\int_0^x a_{jj}(t)\,dt = \frac{1-n}{2n} \ln \varrho(x) - \frac{1}{2n} \ln \tau_0(x) + \frac{2\mathrm{i}}{n} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\,dt.$$

Then we have

$$\mathrm{Y}^0(x,\lambda) = \varrho^{(1-n)/(2n)}(x)\tau_0^{-1/(2n)}(x) \exp\left(\frac{2\mathrm{i}}{n} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\,dt\right)\mathrm{E}(x,\lambda),$$
$$\mathrm{E}(x,\lambda) = \operatorname{diag}\{e^{\omega_0\lambda p(x)}, \dots, e^{\omega_{n-1}\lambda p(x)}\}.$$

We now use Theorem 1 and find the solution of the system $\mathrm{Y}(x,\lambda) = \mathrm{Y}^0(x,\lambda) + \mathrm{S}(x,\lambda)\mathrm{E}(x,\lambda)$, where the elements of the matrix S satisfy (3.2) and (3.3). It remains to make the inverse change by setting $\mathrm{U}(x,\lambda) = \mathrm{W}(x)\mathrm{Y}(x,\lambda)$. Multiplying the matrices, we arrive at

$$\mathrm{U}(x,\lambda) = \mathrm{U}^0(x,\lambda) + \mathrm{W}(x)\mathrm{S}(x,\lambda),$$

where

$$u_{jk}^0(x,\lambda) = \varrho^{(1-n)/(2n)}(x)\tau_0^{-1/(2n)}(x) \exp\left(\omega_{k-1}\lambda p(x) + \frac{2\mathrm{i}}{n} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\,dt\right)$$
$$\times (\omega_{k-1}\rho(x))^{j-1} \cdot \begin{cases} 1, & j \leqslant m, \\ \tau_0, & j > m. \end{cases}$$

Substituting $j = 1$ yields (4.15); for $2 \leqslant j \leqslant n$, we obtain equalities (4.16) and (4.17) since $y_k^{[j-1]}(x) = \lambda^{j-1} u_{jk}(x)$. Inequalities (3.2) and (3.3), which are valid for elements of the matrix $\mathrm{S}(x,\lambda)$, give us the estimates (4.18) and (4.19) for the functions $\zeta_{jk}(x,\lambda)$, which are elements of the matrix $\mathrm{W}(x)\mathrm{S}(x,\lambda)$, since $\mathrm{W}(x)$ is absolutely continuous and independent of λ.

Theorem 4 is proved.

4.3. We write out the assertion of Theorem 4 for $n = 2$. We can do this directly by taking $n = 2$ in the assertion of Theorem 4; however, for clarity, we briefly repeat the main points of its proof in this case. In addition, the explicit form of the matrix $\mathrm{A}(x)$ makes it possible to weaken the assumptions of Theorem 4 slightly. So, for $n = 2$, the equation under study has the form

$$l(y) = -(\tau_0(x)y')' + \mathrm{i}(\sigma_0(x)y)' + \mathrm{i}\sigma_0(x)y' + \tau_1(x)y = \lambda^2\varrho(x)y, \tag{4.20}$$

where

$$\begin{gathered} \tau_0(x), \varrho(x) \in W_1^1[0,1], \qquad \tau_0(x) > 0, \quad \varrho(x) > 0, \\ \tau_1(x) = \mathscr{T}_1'(x), \qquad \mathscr{T}_1(x), \sigma_0(x) \in L_2[0,1]. \end{gathered} \tag{4.21}$$

We pass to the system by means of the substitution

$$u_1(x) = y(x), \qquad u_2(x) = \lambda^{-1}y^{[1]}(x) = \lambda^{-1}\tau_0(x)(y'(x) - f_{1,1}(x)y),$$

where the elements of $\mathrm{F}(x)$ are as follows:

$$f_{1,1}(x) = \frac{\mathscr{T}_1(x) + \mathrm{i}\sigma_0(x)}{\tau_0(x)}, \qquad f_{1,2}(x) = \frac{1}{\tau_0(x)},$$
$$f_{2,1}(x) = -\frac{\mathscr{T}_1^2(x) + \sigma_0^2(x)}{\tau_0(x)}, \qquad f_{2,2}(x) = \frac{-\mathscr{T}_1(x) + \mathrm{i}\sigma_0(x)}{\tau_0(x)}.$$

Thus, equation (4.20) reduces to the system

$$\mathbf{u}'(x) = \begin{pmatrix} f_{1,1}(x) & \dfrac{\lambda}{\tau_0(x)} \\ -\lambda\varrho(x) + \dfrac{f_{2,2}(x)}{\lambda} & f_{2,1}(x) \end{pmatrix}\mathbf{u}(x), \qquad \mathbf{u}(x) = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}.$$

Making the substitution

$$\mathbf{y} = \frac{1}{2}\begin{pmatrix} 1 & -\dfrac{\mathrm{i}}{\sqrt{\varrho\tau_0}} \\ 1 & \dfrac{\mathrm{i}}{\sqrt{\varrho\tau_0}} \end{pmatrix}\mathbf{u}, \qquad \mathbf{u} = \begin{pmatrix} 1 & 1 \\ \mathrm{i}\sqrt{\varrho\tau_0} & -\mathrm{i}\sqrt{\varrho\tau_0} \end{pmatrix}\mathbf{y},$$

this system takes the form

$$\mathbf{y}' = \lambda\sqrt{\frac{\varrho(x)}{\tau_0(x)}}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\mathbf{y} + \begin{pmatrix} d(x) & e(x) \\ e(x) & d(x) \end{pmatrix}\mathbf{y} + \frac{1}{\lambda}\cdot\frac{\mathrm{i}(\mathscr{T}_1^2 + \sigma_0^2)}{\tau_0^{3/2}\varrho^{1/2}}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}\mathbf{y} \tag{4.22}$$

(here $\rho(x) = \varrho(x)/\tau_0(x)$), where

$$d(x) = -\frac{(\varrho\tau_0)'}{4\varrho\tau_0} + \frac{\mathrm{i}\sigma_0}{\tau_0} \quad \text{and} \quad e(x) = \frac{(\varrho\tau_0)'}{4\varrho\tau_0} + \frac{\mathscr{T}_1}{\tau_0}.$$

Corollary 2. *Let $\widetilde{\Gamma}_1 = \{\lambda\colon \operatorname{Im}\lambda > -h\}$ and $\widetilde{\Gamma}_2 = \{\lambda\colon \operatorname{Im}\lambda < h\}$, where $h > 0$ is arbitrary. If condition (4.21) holds, then equation (4.20) has a pair of linearly independent solutions $y_\pm(x,\lambda)$ in $\widetilde{\Gamma}_1$ such that the following asymptotic representations are valid as $\widetilde{\Gamma}_1 \ni \lambda \to \infty$:*

$$y_\pm(x,\lambda) = e^{\pm\mathrm{i}\lambda p(x)}\left[\varrho^{-1/4}(x)\tau_0^{-1/4}(x)\exp\left(\mathrm{i}\int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\,dt\right) + \zeta_\pm(x,\lambda)\right]$$

and

$$y_{\pm}^{[1]}(x,\lambda) = \pm \mathrm{i}\lambda e^{\pm \mathrm{i}\lambda p(x)} \left[\varrho^{1/4}(x)\tau_0^{1/4}(x) \exp\left(\mathrm{i} \int_0^x \frac{\sigma_0(t)}{\tau_0(t)}\, dt \right) + \zeta_{\pm,1}(x,\lambda) \right],$$

where $p(x) = \displaystyle\int_0^x \varrho^{1/2}(t)\tau_0^{-1/2}(t)\, dt$ *and the remainders* $\zeta_\pm$ *and* $\zeta_{\pm,1}$ *satisfy the estimates*

$$\|\zeta(\,\cdot\,,\lambda)\|_{L_\infty} \leqslant C\big(\Upsilon(\lambda) + |\lambda|^{-1}\big).$$

If $(\varrho(x)\tau_0(x))'$ *and* $\mathscr{T}_1(x)$ *are integrable to the power* $\mu' \in [1,\infty]$, *then*[3]

$$\|\zeta(\,\cdot\,,\lambda)\|_{L_\mu} \leqslant C\big(\Upsilon_\mu(\lambda) + |\lambda|^{-1}\big) \quad \text{and} \quad |\zeta(x,\lambda)| \leqslant \Upsilon(x,\lambda) + C\big(\Upsilon_\mu(\lambda) + |\lambda|^{-1}\big),$$

where μ *is the Hölder conjugate number of* μ'. *Exactly the same assertion (with another pair of functions* $y_\pm(x,\lambda)$*) is valid for* $\widetilde{\Gamma}_2$.

Remark 9. The explicit form of the matrix in (4.22) makes it possible to relax somewhat the conditions on the coefficients of the differential expression (4.20). Instead of the assumption that σ_0 and $\mathscr{T}_1 \in L_2[0,1]$, it suffices that the conditions $\sigma_0, \mathscr{T}_1 \in L_1[0,1]$ and $\sigma_0^2 + \mathscr{T}_1^2 \in L_1[0,1]$ hold.

Note that the asymptotic representations given in Corollary 2 were derived in [18] and [44] using other methods. Essentially, the conditions on the coefficients in these works are the same as in (4.21); however, the estimate for the remainder is cruder and only in terms of $o(1)$.

We also note that we used yet another method, namely, the Prüfer angle method, to obtain asymptotic representations in [4] (in the case when $l(y) = -y'' + \tau_1 y$) and in [25] (for the Dirac system). The results deduced in these works and concerning asymptotic representations for the fundamental systems of solutions coincide with (3.2) and (3.3); however, we only proved their validity in strips in the complex plane containing the real axis. In the case of the Sturm-Liouville operator, the parameter μ was taken equal to 2; for the Dirac system, the range $\mu \in [2,\infty)$ was considered. In [4], however, in addition to the 'short' asymptotics for eigenvalues and eigenfunctions, we also derived (in the case of Dirichlet boundary conditions) 'long' asymptotics based on the asymptotic representation (3.21). The final result does not contain terms generated by the functions Z^3 and Z^4, while only one term due to Z^2 is involved. The reasons for this phenomenon are hidden quite deeply and consist in the fact that, although the functions $\|\mathrm{Z}^j(\,\cdot\,,\lambda)\|_{L_2}$, $j = 3,4$, are not estimated themselves in terms of $\Upsilon_2^2(\lambda)$, the norms of these functions in the Hardy space $H^1(\widetilde{\Gamma})$ admit estimates of the same order as $\|\Upsilon_2^2(\lambda)\|$. A detailed analysis of these estimates will be done in another work by these authors.

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[3] Recall that $\mathscr{T}_1(x) \in L_2[0,1]$ in any case; hence the condition $\mathscr{T}_1 \in L_{\mu'}[0,1]$ is meaningful only for $\mu' > 2$.

Bibliography

[1] G. D. Birkhoff, "On the asymptotic character of the solutions of certain linear differential equations containing a parameter", *Trans. Amer. Math. Soc.* **9**:2 (1908), 219–231.

[2] G. D. Birkhoff, "Boundary value and expansion problems of ordinary linear differential equations", *Trans. Amer. Math. Soc.* **9**:4 (1908), 373–395.

[3] A. M. Savchuk and A. A. Shkalikov, "Sturm-Liouville operators with singular potentials", *Mat. Zametki* **66**:6 (1999), 897–912; English transl. in *Math. Notes* **66**:6 (1999), 741–753.

[4] A. M. Savchuk and A. A. Shkalikov, "Sturm-Liouville operators with distribution potentials", Tr. Mosk. Mat. Obshch., vol. 64, URSS, Moscow 2003, pp. 159–219; English transl. in *Trans. Moscow Math. Soc.* **64** (2003), 143–192.

[5] A. A. Vladimirov, "On the convergence of sequences of ordinary differential equations", *Mat. Zametki* **75**:6 (2004), 941–943; English transl. in *Math. Notes* **75**:6 (2004), 877–880.

[6] I. M. Guseinov and R. T. Pashaev, "A class of differential operators with singular coefficients", *Dokl. Nats. Akad. Nauk Azerbaijan* **62**:1 (2006), 7–10. (Russian)

[7] K. A. Mirzoev and A. A. Shkalikov, "Two-term differential operators with singular coefficient", *International conference "Differential equations and related questions" dedicated to the* 110*th anniversary of birth of I. G. Petrovskii*: *Abstracts of talks*, XXIII joint session of the Moscow Mathematical Society and the I. G. Petrovskii Seminar (Moscow 2011), Moscow State University Press, Moscow; INTUIT.RU Ltd, Moscow 2011, pp. 274–275. (Russian)

[8] J. Eckhardt, F. Gesztesy, R. Nichols and G. Teschl, "Weyl-Titchmarsh theory for Sturm-Liouville operators with distributional potentials", *Opuscula Math.* **33**:3 (2013), 467–563.

[9] A. Goriunov, V. Mikhailets and K. Pankrashkin, "Formally self-adjoint quasi-differential operators and boundary-value problems", *Electron. J. Differential Equations* **2013** (2013), 101, 16 pp.

[10] K. A. Mirzoev, "Sturm-Liouville operators", Tr. Mosk. Mat. Obshch., vol. 75, no. 2, Moscow Center for Continuous Mathematical Education, Moscow 2014, pp. 335–359; English transl. in *Trans. Moscow Math. Soc.* **75** (2014), 281–299.

[11] K. A. Mirzoev and A. A. Shkalikov, "Differential operators of even order with distribution coefficients", *Mat. Zametki* **99**:5 (2016), 788–793; English transl. in *Math. Notes* **99**:5 (2016), 779–784.

[12] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York–Toronto–London 1955, xii+429 pp.

[13] O. Perron, "Über lineare Differentialgleichungen, bei denen die unabhängig Variable reel ist. I", *J. Reine Angew. Math.* **142** (1913), 254–270; II, **143**, 25–49.

[14] J. D. Tamarkin, *Some general problems in the theory of ordinary differential equations and expanding arbitrary functions in series*, M. P. Frolova Prints, Petrograd 1917, xiv+308 pp. (Russian)

[15] G. D. Birkhoff and R. E. Langer, "The boundary problems and developments associated with a system of ordinary differential equations of the first order", *Proc. Amer. Acad. Arts Sci.* **58**:2 (1923), 51–128.

[16] J. Tamarkin, "Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions", *Math. Z.* **27**:1 (1928), 1–54.

[17] M. A. Naimark, *Linear differential operators*, 2nd ed., Nauka, Moscow 1969, 526 pp.; English transl. of 1st ed., vols. I, II, Frederick Ungar Publishing Co., New York 1967, 1968, xiii+144 pp., xv+352 pp.

[18] A. A. Shkalikov and V. E. Vladykina, "Asymptotics of the solutions of the Sturm-Liouville equation with singular coefficients", *Mat. Zametki* **98**:6 (2015), 832–841; English transl. in *Math. Notes* **98**:6 (2015), 891–899.

[19] V. A. Marchenko, *Sturm-Liouville operators and applications*, Naukova Dumka, Kiev 1977, 331 pp.; English transl., Oper. Theory Adv. Appl., vol. 22, Birkhäuser Verlag, Basel 1986, xii+367 pp.

[20] R. O. Hryniv and Ya. V. Mykytyuk, "Transformation operators for Sturm-Liouville operators with singular potentials", *Math. Phys. Anal. Geom.* **7**:2 (2004), 119–149.

[21] S. Albeverio, R. O. Hryniv and Ya. V. Mykytyuk, "Inverse spectral problems for Dirac operators with summable potentials", *Russ. J. Math. Phys.* **12**:4 (2005), 406–423.

[22] M. M. Malamud, "Uniqueness questions in inverse problems for systems of differential equations on a finite interval", Tr. Mosk Mat. Obshch., vol. 60, URSS, Moscow 1999, pp. 199–258; English transl. in *Trans. Moscow Math. Soc.*, 1999, 173–224.

[23] A. A. Lunyov and M. M. Malamud, "On the Riesz basis property of root vectors system for 2×2 Dirac type operators", *J. Math. Anal. Appl.* **441**:1 (2016), 57–103.

[24] A. M. Savchuk and A. A. Shkalikov, "Spectral properties of Dirac operators on $(0, 1)$ with summable potentials", *The Sixth International Conference on Differential and Functional Differential Equations. Abstracts* (Moscow 2011), 2011, pp. 63.

[25] A. M. Savchuk and A. A. Shkalikov, "Dirac operator with complex-valued summable potential", arXiv: 1412.6757; English transl. in *Math. Notes* **96**:5 (2014), 777–810.

[26] I. M. Rapoport, *On some asymptotic methods in the theory of differential equations*, Publishing house of the Academy of Sciences of Ukr. SSR, Kiev 1954, 292 pp. (Russian)

[27] A. I. Vagabov, "On sharpening an asymptotic theorem of Tamarkin", *Differ. Uravn.* **29**:1 (1993), 41–49; English transl. in *Differ. Equ.* **29**:1 (1993), 33–41.

[28] V. S. Rykhlov, "Asymptotical formulas for solutions of linear differential systems of the first order", *Results Math.* **36**:3–4 (1999), 342–353.

[29] A. I. Vagabov, "On the asymptotics with respect to a parameter of solutions of differential systems with coefficients in the class L_q", *Differ. Uravn.* **46**:1 (2010), 16–22; English transl. in *Differ. Equ.* **46**:1 (2010), 17–23.

[30] W. Wasow, *Asymptotic expansions for ordinary differential equations*, Pure Appl. Math., vol. 14, Intersci. Publ. John Wiley & Sons, Inc., New York–London–Sydney 1965, ix+362 pp.

[31] A. A. Shkalikov, "Boundary problems for ordinary differential equations with parameter in the boundary conditions", Tr. Semin. im. I. G. Petrovskogo, vol. 9, 1983, pp. 190–229; English transl. in *J. Soviet Math.* **33** (1986), 1311–1342.

[32] A. A. Lunyov and M. M. Malamud, "On the completeness and Riesz basis property of root subspaces of boundary value problems for first order systems and applications", *J. Spectr. Theory* **5**:1 (2015), 17–70.

[33] M. M. Malamud and L. L. Oridoroga, "On the completeness of root subspaces of boundary value problems for first order systems of ordinary differential equations", *J. Funct. Anal.* **263**:7 (2012), 1939–1980.

[34] A. M. Savchuk and A. A. Shkalikov, "Inverse problems for Sturm-Liouville operators with potentials in Sobolev spaces: uniform stability", *Funktsional. Anal. i Prilozhen.* **44**:4 (2010), 34–53; English transl. in *Funct. Anal. Appl.* **44**:4 (2010), 270–285.

[35] C. Bennewits, "Spectral asymptotics for Sturm-Liouville equations", *Proc. London Math. Soc.* (3) **59**:2 (1989), 294–338.

[36] A. Fleige, *Spectral theory of indefinite Krein-Feller differential operators*, Math. Res., vol. 98, Akademie Verlag, Berlin 1996, 133 pp.

[37] A. Zettl, *Sturm-Liouville theory*, Math. Surveys Monogr., vol. 121, Amer. Math. Soc., Providence, RI 2005, xii+328 pp.

[38] J. Behrndt, F. Philipp and C. Trunk, "Bounds on the non-real spectrum of differential operators with indefinite weights", *Math. Ann.* **357**:1 (2013), 185–213.

[39] A. A. Shkalikov and V. T. Pliev, "Compact perturbations of strongly damped operator pencils", *Mat. Zametki* **45**:2 (1989), 118–128; English transl. in *Math. Notes* **45**:2 (1989), 167–174.

[40] R. Kh. Amirov and I. M. Guseinov, "Some classes of Dirac operators with singular potentials", *Differ. Uravn.* **40**:7 (2004), 999–1001; English transl. in *Differ. Equ.* **40**:7 (2004), 1066–1068.

[41] A. M. Savchuk and A. A. Shkalikov, *Asymptotic formulae for fundamental system of solutions of high order ordinary differential equations with coefficients which are distributions*, arXiv: 1704.02736V1. (Russian)

[42] F. V. Atkinson, W. N. Everitt and A. Zettl, "Regularization of a Sturm-Liouville problem with an interior singularity using quasi-derivatives", *Differential Integral Equations* **1**:2 (1988), 213–221.

[43] W. N. Everitt, "A note on linear ordinary quasi-differential equations", *Proc. Roy. Soc. Edinburgh Sect. A* **101**:1–2 (1985), 1–14.

[44] V. E. Vladykina, "Asymptotics of fundamental solutions to Sturm-Liouville problem with respect to spectral parameter", *Vestn. Moskov. Univ. Ser. 1. Mat. Mekh.*, 2019, no. 1, 57–61; English transl. in *Moscow Univ. Math. Bull.* **74**:1 (2019), 38–41.

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