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Eigenvalue asymptotics of long Kirchhoff plates with clamped edges

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Abstract. Asymptotic expansions are constructed for the eigenvalues and eigenfunctions of the Dirichlet problem for the biharmonic operator in thin domains (Kirchhoff plates with clamped edges). For a rectangular plate the leading terms are asymptotically determined from the Dirichlet problem for a second-order ordinary differential equation, while for a T-junction of plates they are determined from another limiting problem in an infinite waveguide formed by three half-strips in the shape of a letter T and describing a boundary-layer phenomenon. Open questions are stated for which the method developed gives no answer.

Bibliography: 33 titles.

Keywords: Kirchhoff plate, eigenvalues and eigenfunctions, asymptotic behaviour, dimension reduction, boundary layer.

§ 1. Introduction

1.1. Motivations. The Dirichlet and Neumann boundary-value problems (D) and (N) for the Laplace operator in thin domains, both static (st) and spectral (sp), are sufficiently well understood: full asymptotic expansions have been constructed for solutions of Poisson's equation, eigenvalues and eigenfunctions alike. It will be clear from what follows that, in order of increasing complexity of the asymptotic analysis, these problems can be put in the following order:

$$\mathsf{st}-\mathsf{D}\nearrow\mathsf{st}-\mathsf{N}\nearrow\mathsf{sp}-\mathsf{N}\nearrow\mathsf{sp}-\mathsf{D}. \tag{1.1}$$

A similar hierarchy holds for scalar problems on junctions of thin domains, that is, lattices which degenerate into graphs in the limit.

Vector problems in elasticity theory, which are extremely important in applications but require a more laborious asymptotic analysis, also follow the pattern (1.1): while there are many papers and books devoted to the Neumann problem (when the surface of the body is either loaded or free from external effects), there are essentially no publications devoted to the Dirichlet problem (when the surface of the elastic body is rigidly clamped). The reasons are twofold: the problem $\mathsf{st}-\mathsf{D}$ is too simple and, for all practical purposes, no different from the scalar one, while the problem $\mathsf{sp}-\mathsf{D}$ cannot be analyzed using the methods developed for scalar

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problems. In particular, we stress that many questions concerning the asymptotic analysis of junctions of elastic rods and beams (thin elastic lattices) so far have no answer in either the static or the spectral case.

It is impossible to give a more-or-less exhaustive list of references on this subject here, so we only mention several monographs [1]–[7] we know of.

Another range of problems, perhaps most important for engineering and construction industry practice, is presented by the theory of Kirchhoff plates, when the biharmonic equation in a plane domain is equipped with various boundary conditions, for instance, Dirichlet or Neumann, which are associated with the rigidly clamped or free edges of the plate, respectively. Note that Kirchhoff's theory itself is a result of asymptotic analysis, when the relative thickness of a three-dimensional plate is a small parameter and the statement of the problem considered in what follows, which is relevant to mechanics, concerns a long thin plate: after dimension reduction we make a scaling and the two-dimensional image of the plate becomes thin.

Results on the first three items in the list (1.1) were obtained in [8]–[10]; here we investigate the Dirichlet spectral problem for the biharmonic operator Δ^2, fourth in the list.

1.2. Fictitious paradoxes and boundary layers. In the technical theory of deformations of thin plates, going over from the spatial system of three second-order equations of elasticity theory to another system of three equations, one of order four and two of order two, so that four boundary conditions must be set at the edge of a two-dimensional plate, is viewed as a paradox, which is explained in the literature on a physical level of rigour. In this regard the asymptotic analysis of the static and spectral problems for the biharmonic operator provides its own pair of 'paradoxes'[1]; namely:

1) the Neumann problem for a fourth-order scalar equation gives rise to a system of (two) equations of orders four and two, while the number of boundary conditions increases by one (see [9] and [10]);

2) in the Dirichlet spectral problem for the biharmonic operator Δ^2 dimension reduction produces a second-order scalar equation, and the number of boundary conditions decreases by one (see § 2.3).

Of course, there is nothing paradoxical here whatsoever: it is well known (for instance, see the surveys [11] and [12] and the book [3], Ch. 16) that the size and orders of the limiting system obtained by dimension reduction are not determined by the analogous characteristics of the original system, but rather by the structure of the canonical system of Jordan chains of the operator pencil (see [13], Ch. 1) generated by the full boundary-value problem in a thin domain. For instance, in § 2.4 we verify that in the problem under consideration this pencil has the single real eigenvalue zero with Jordan chain of length two: this explains fact 2).

We will need the same Jordan chains in § 2.4, in the analysis of the boundary-layer phenomenon; the assumption that it decays exponentially gives rise to the boundary-value conditions in the limiting problem. For static problems that have the polynomial property (see [14] and [12]; all the equations we discuss have this property) the structure of the canonical system of Jordan chains of the

[1]We discuss another kind of 'paradox' in § 5.3.

pencil which corresponds to the eigenvalue zero can be determined by means of simple algebraic operations with polynomials on which the energy functional in the variational statement of the problem degenerates. In the static Dirichlet problem for Δ^2 there are no Jordan chains at all (see [12], Proposition 3.2 and Example 1.14), and the fact that the problem on a junction of several half-strips is solvable in the class of functions that decay exponentially at infinity is a simple consequence of Kondrat'ev's theorem (see [15] and also the book [16], Chs. 3 and 5, and the survey [12], § 3): unique solvability in the Sobolev class ensures that there exists a unique solution in the class of functions that decay exponentially at infinity.

Since the continuous spectrum of the Neumann problem for Δ^2 in a junction of half-strips is a half-axis $[\lambda_\dagger, +\infty)$ with cutoff point $\lambda_\dagger = 0$, we can again extract full information about the boundary layer by means of algebraic calculations based on the approach in [12], and the canonical system consists of two Jordan chains of length four and two, which explains fact 1) and gives rise to the boundary conditions (see [10]) or transmission conditions (see [9]) in one-dimensional models of long curved Kirchhoff plates or a straight cruciform junction of plates.

In the Dirichlet spectral problem for Δ^2 the cutoff point for the continuous spectrum is positive (see § 2.2). For this reason the approaches mentioned in the previous paragraph are useless and new techniques must be used. In § 2.4 we are only able to extract full information on the spectrum for the problem in a single half-strip, while in §§ 4.2 and 5.2 we verify that in problems in T- and X-junctions of strips with width one there must also be a discrete spectrum; however, its multiplicity and the existence or absence of threshold resonances are still open questions.

We stress again that the calculation of Jordan chains on the threshold of the continuous spectrum of the Dirichlet problem for the biharmonic operator, which we present in §§ 2.2 and 2.4, is the starting point for all the results that follow. On the other hand, the fact that the variables in this problem cannot be separated is a decisive difference between it and boundary-value problems for the Laplace operator, which have been well studied.

1.3. The contents of the paper. In § 2 we investigate the spectrum of the Dirichlet problem for the biharmonic operator Δ^2 in a thin rectangle (we stress that the variables in this problem cannot be separated). We investigate a model problem on a cross-section and the limiting problem on the lateral section, and also the boundary layer problem in a half-strip, one after the other. As a result, we propose formal asymptotic expressions for pairs {eigenvalue, eigenfunction}, which are justified in § 3. Verifying Theorem 3 on convergence proved to be the hardest part, whereas refining an estimate for the error of the one-dimensional model follows the standard approach, using the classical lemma on 'almost' eigenvalues and eigenfunctions. The final theorems, Theorems 1 and 4, give a full picture of the low-frequency part of the spectrum.

The results on the asymptotic behaviour of the spectrum of the same Dirichlet problem in a T-junction $\mathbb{T}^h$ of perpendicular plates with thickness $2h \ll 1$ presented in § 4 are much poorer. In Theorem 6 we only find the asymptotic behaviour of the first eigenvalue; we could not obtain information about the rest of the spectrum. This eigenvalue lies far below the total spectrum of the problem in a rectangle and has quite different origins from the eigenvalue in § 2: Theorem 5 shows that the

Dirichlet problem in an infinite T-shaped waveguide has a nonempty discrete spectrum, and in Theorem 6 we verify that it is precisely the bottom point of that spectrum which produces the first eigenvalue of the problem in the thin domain $\mathbb{T}^h$.

In § 5, the final section, we list some attendant question that we leave without answer. In particular, we explain why the incomplete investigation of the spectrum of the problem in $\mathbb{T}$ has prevented us from analysing the low-frequency part of the spectrum of the problem in $\mathbb{T}^h$. We discuss other shapes of junction, when new unexpected complications arise for asymptotic analysis, and also discuss boundary conditions for plates with simply supported edge, for which the accessible information is — unexpectedly — quite scanty.

§ 2. Formal eigenvalue asymptotics in a rectangle

2.1. The problem in a rectangle. A long Kirchhoff plate

$$\Pi^h = (-1,1)\times(-h,h) = \bigl\{x=(y,z)\colon y\in(-1,1),\ z\in(-h,h)\bigr\}\subset\mathbb{R}^2$$

has the shape of a rectangle in which, after scaling, the lateral sides have a small half-length $h \ll 1$. The natural oscillations of a plate with rigidly clamped edge are described (for instance, see [17], § 30) by the Dirichlet spectral problem for the biharmonic operator

$$\Delta^2 u^h(x) = \Lambda^h u^h(x), \qquad x\in\Pi^h, \tag{2.1}$$

$$u^h(x)=0, \quad \partial_n u^h(x) = 0, \qquad x\in\partial\Pi^h, \tag{2.2}$$

where ∂_n is the outward normal derivative, which is defined on the boundary $\partial\Pi^h$ away from the corner points, and Λ^h is the spectral parameter. Problem (2.1), (2.2) has the variational form

$$\begin{aligned}\mathbf{a}(u^h,v;\Pi^h) &:= (\partial_y^2 u^h,\partial_y^2 v)_{\Pi^h} + 2(\partial_y\partial_z u^h,\partial_y\partial_z v)_{\Pi^h} + (\partial_z^2 u^h,\partial_z^2 v)_{\Pi^h}\\ &= \Lambda^h(u,v)_{\Pi^h} \quad \forall\, v\in H_0^2(\Pi^h),\end{aligned} \tag{2.3}$$

where $(\,\cdot\,,\cdot\,)_{\Pi^h}$ is the inner product in the Lebesgue space $L^2(\Pi^h)$, $\partial_y = \partial/\partial y$, $\partial_z = \partial/\partial z$, and $H_0^2(\Pi^h)$ is the Sobolev space of functions satisfying (2.2). The bilinear form $\mathbf{a}$ on the left-hand side of the integral identity (2.3) is positive definite and closed in $H_0^2(\Pi^h)$, so that by [18], Ch. 10, we can express (2.3) as the abstract equation

$$\mathbf{A}^h u^h = \Lambda^h u^h$$

for a certain unbounded positive definite selfadjoint operator $\mathbf{A}^h$ in $L^2(\Pi^h)$. This operator has discrete spectrum $\sigma(\mathbf{A}^h)$ because the embedding $H_0^2(\Pi^h)\subset L^2(\Pi^h)$ is compact. Counting multiplicities, the eigenvalues of $\mathbf{A}^h$ form a nondecreasing sequence

$$0<\Lambda_1^h\leqslant\Lambda_2^h\leqslant\cdots\leqslant\Lambda_n^h\leqslant\cdots\to+\infty, \tag{2.4}$$

and the corresponding eigenfunctions $u_1^h, u_2^h,\dots,u_n^h,\dots\in H_0^2(\Pi^h)$ can be taken to be orthogonal and normalized by

$$(u_n^h,u_m^h)_{\Pi^h} = \delta_{n,m}, \qquad n,m\in\mathbb{N}=\{1,2,3,\dots\},$$

where $\delta_{n,m}$ is the Kronecker delta. The asymptotic analysis of the eigenpairs $\{\Lambda_n^h, u_n^h\}$ as $h \to +0$ is the subject of our investigations in this section and the next.

2.2. A model problem on an interval. We start by dilating the transverse coordinate $z \mapsto \zeta = h^{-1}z$ and look at a model spectral problem on $(-1,1)$:

$$\partial_\zeta^4 U(\zeta) = MU(\zeta), \qquad \zeta \in (-1,1), \quad U(\pm 1) = \partial_\zeta U(\pm 1) = 0. \tag{2.5}$$

As direct calculations show, the eigenvalues

$$0 < M_1 < M_2 < \cdots < M_n < \cdots \to +\infty$$

of problem (2.5) can be found by solving the transcendental equation

$$\sinh^2(m_j)\cos^2(m_j) - \sin^2(m_j)\cosh^2(m_j) = 0,$$

in which $m_j = \sqrt[4]{M_j}$. Note that eigenfunctions with odd indices, which correspond to $m_1 \approx 2.365$, $m_3 \approx 5.497$, ..., are even functions of ζ,

$$U_k(\zeta) = \sinh(m_k)\cos(m_k\zeta) + \sin(m_k)\cosh(m_k\zeta), \tag{2.6}$$

while eigenfunctions with even indices, which correspond to $m_2 \approx 3.927$, $m_4 \approx 7.068$, ..., are odd functions of ζ,

$$U_k(\zeta) = \cosh(m_k)\sin(m_k\zeta) - \cos(m_k)\sinh(m_k\zeta).$$

2.3. Asymptotic ansätze. For an eigenvalue Λ^h and an eigenfunction u^h of problem (2.1), (2.2) we postulate the asymptotic ansätze

$$\Lambda^h = h^{-4}M_1 + h^{-2}\mu + \cdots \quad \text{and} \quad u^h(y,z) = w(y)U_1(\zeta) + h^2 u'(y,\zeta) + \cdots, \tag{2.7}$$

where the correction term μ and the functions w and u' are to be determined. Substituting (2.7) into (2.1) and collecting the terms of order h^{-2} we deduce the differential equation

$$\partial_\zeta^4 u'(y,\zeta) - M_1 u'(y,\zeta) = -2\,\partial_y^2 w(y)\,\partial_\zeta^2 U_1(\zeta) + \mu w(y)U_1(\zeta), \tag{2.8}$$

which can naturally be provided with the boundary conditions

$$u'(y,\pm 1) = 0 \quad \text{and} \quad \partial_\zeta u'(y,\pm 1) = 0, \tag{2.9}$$

which follow from (2.2). By Fredholm's theorem the boundary-value problem (2.8), (2.9) we have obtained is solvable if the right-hand side of (2.8) is orthogonal to U_1 in $L^2(-1,1)$. That is, the condition for solvability has the form of the differential equation with respect to w

$$-B\,\partial_y^2 w(y) - \mu w(y) = 0, \qquad y \in \Upsilon := (-1,1), \tag{2.10}$$

where

$$B = 2\frac{b}{a} \quad \text{and} \quad b = \int_{-1}^{1} |\partial_\zeta U_1(\zeta)|^2\,d\zeta > 0, \quad a = \int_{-1}^{1} |U_1(\zeta)|^2\,d\zeta > 0.$$

Equation (2.10) must be complemented with the boundary conditions

$$w(-1) = w(+1) = 0 \tag{2.11}$$

(see the explanations in § 2.4); then we have the countable system of eigenvalues

$$\mu_n = B\frac{\pi^2}{4}n^2, \qquad n \in \mathbb{N} = \{1, 2, 3, \dots\}, \tag{2.12}$$

and the corresponding eigenfunctions

$$w_n(x) = \sin\biggl(\frac{\pi}{2}n(x-1)\biggr). \tag{2.13}$$

In the rest of this section and in § 3 we prove the following statement on the eigenvalues (2.4); in addition, in Theorem 4 we present information concerning the corresponding eigenfunctions.

Theorem 1. *There exist positive quantities* C_n *and* h_n *such that for* $h \in (0, h_n)$ *the eigenvalues* (2.4) *of problem* (2.1), (2.2) *satisfy the estimate*

$$\biggl|\Lambda_n^h - h^{-4}M_1 + h^{-2}B\frac{\pi^2}{4}n^2\biggr| \leqslant C_n h^{-3/2}. \tag{2.14}$$

2.4. The boundary layer and the problem in a half-strip. Here we explain why we impose boundary conditions (2.11) for w. In principle, to the ansatz (2.7) for u^h we should add a boundary layer compensating for the residuals from regular terms in boundary conditions (2.2) at the edges $\{x\colon y = \pm 1, |z| < h\}$ (for instance, see [3], Ch. 16).

Using the partial Fourier transform $\mathscr{F}_{\eta\mapsto\vartheta}$, the Dirichlet problem for the biharmonic operator in the infinite strip $\mathbb{P} = \{(\eta, \zeta) \in \mathbb{R}^2\colon \eta \in \mathbb{R}, \zeta \in (-1, 1)\}$ generates the polynomial pencil

$$\begin{aligned} \mathfrak{u} \mapsto \mathfrak{A}(\vartheta)\mathfrak{u} = (\partial_\zeta^4\mathfrak{u} - 2\vartheta^2\,\partial_\zeta^2\mathfrak{u} + \vartheta^4\mathfrak{u} - M_1\mathfrak{u}, \mathfrak{u}(\pm 1), \partial_\zeta\mathfrak{u}(\pm 1))\colon \\ H_0^2(-1, 1) \to L^2(-1, 1) \times \mathbb{C} \times \mathbb{C} \end{aligned} \tag{2.15}$$

(see [13], Ch. 1, and also [16], § 1.2, for instance). Direct calculations show that the pencil (2.15) has the unique eigenvalue $\vartheta = 0$ with Jordan chain $\{\mathfrak{u}^0, \mathfrak{u}^1\}$, where $\mathfrak{u}^0 = U_1$ is the eigenfunction (2.6) with $k = 1$, and the associated vector $\mathfrak{u}^1$, which solves the equation

$$\mathfrak{A}(0)\mathfrak{u}^1 = \biggl(\frac{d\mathfrak{A}}{d\vartheta}(0)\mathfrak{u}^0, 0, 0\biggr) = (0, 0, 0),$$

can be taken to be zero. This chain cannot be extended further, because it is clear that the equation

$$\mathfrak{A}(0)\mathfrak{u}^2 = \biggl(\frac{1}{2}\,\frac{d^2\mathfrak{A}}{d\vartheta^2}(0)\mathfrak{u}^0, 0, 0\biggr) = (2\,\partial_\zeta^2 U_1, 0, 0)$$

for an associated vector of rank two has no solution. In addition, there exists $\beta_1 > 0$ such that the strip $\{\vartheta \in \mathbb{C}\colon |\operatorname{Im}\vartheta| \leqslant \beta_1\}$ contains no eigenvalues of the pencil, apart from the eigenvalue zero.

General asymptotic procedures (for instance, see [3], Ch. 16) show that Dirichlet conditions (2.11) close the resulting equation (2.10) if and only if the following problem has no bounded solutions in the half-strip $\mathbb{P}_+ = \{(\eta,\zeta) \in \mathbb{P}\colon \eta > 0\}$:

$$\begin{aligned} \Delta^2 Z(\eta,\zeta) - M_1 Z(\eta,\zeta) = 0, \qquad & (\eta,\zeta) \in \mathbb{P}_+, \\ Z(\eta,\pm 1) = 0, \quad \partial_\zeta Z(\eta,\pm 1) = 0, \qquad & \eta > 0, \\ Z(0,\zeta) = 0, \quad \partial_\eta Z(0,\zeta) = 0, \qquad & \zeta \in (-1,1). \end{aligned} \tag{2.16}$$

Throughout, we call domains $\mathbb{P}_+$ and other infinite domains waveguides.

Remark 1. If Z is a solution of (2.16) and $e^{-\beta_1\eta} Z \in H^2_0(\mathbb{P}_+)$, then by Kondrat'ev's theorem on asymptotics (see [15] and also by Theorem 1.7 in [16]) and the above facts concerning the pencil (2.15) we have the representation

$$Z(\eta,\zeta) = K_0 U_1(\zeta) + K_1 \eta U_1(\zeta) + \widetilde{Z}(\eta,\zeta), \tag{2.17}$$

where K_0 and K_1 are constants and $e^{\beta_1\eta}\widetilde{Z} \in H^2_0(\mathbb{P}_+)$ is a remainder term which decays exponentially. Some solution of the form (2.17) must increase. If $K_0 = K_1 = 0$ in it, so that $Z \in H^2_0(\mathbb{P})$, then the asymptotic ansatz for the eigenvalue must be modified (cf. § 4.3). On the other hand, if the solution stabilizes as $\eta \to +\infty$, so that $K_1 = 0$ and $K_0 \neq 0$, then in place of the Dirichlet conditions (2.11) we must impose the Neumann condition $\partial_y w(\pm 1) = 0$.

In Lemma 1 and Theorem 2 we prove that if $K_1 = 0$ then the solution $Z = 0$ is trivial. Thus each nontrivial solution (2.17) of problem (2.16) has linear growth at infinity. We have already mentioned that if such a solution does exist, then by [3], Ch. 16, Dirichlet conditions (2.11) must be imposed.

Lemma 1. *Problem* (2.16) *has no nontrivial solutions in* $H^2_0(\mathbb{P}_+)$.

Proof. A solution $Z \in H^2_0(\mathbb{P}_+)$ satisfies

$$M_1 \|Z; L^2(\mathbb{P}_+)\|^2 = \|\partial^2_\zeta Z; L^2(\mathbb{P}_+)\|^2 + 2\|\partial_\eta\partial_\zeta Z; L^2(\mathbb{P}_+)\|^2 + \|\partial^2_\eta Z; L^2(\mathbb{P}_+)\|^2.$$

By the definition of the quantity M_1 we have $\|\partial^2_\zeta Z; L^2(\mathbb{P}_+)\|^2 \geqslant M_1 \|Z; L^2(\mathbb{P}_+)\|^2$. Furthermore, the one-dimensional Friedrichs and Hardy inequalities yield

$$\|\partial_\eta\partial_\zeta Z; L^2(\mathbb{P}_+)\|^2 \geqslant \frac{\pi^2}{4}\|\partial_\eta Z; L^2(\mathbb{P}_+)\|^2 \geqslant \frac{\pi^2}{16}\int_{\mathbb{P}_+} \eta^{-2}|Z(\eta,\zeta)|^2\,d\eta\,d\zeta.$$

This shows that $Z = 0$ indeed, which completes the proof.

Theorem 2. *Problem* (2.16) *has no nontrivial bounded solutions.*

Proof. It is known (see [19] and, for instance, [20]) that a solution $Z \in H^2_{\mathrm{loc}}(\overline{\mathbb{P}_+})$ of (2.16) belongs to $H^3_{\mathrm{loc}}(\overline{\mathbb{P}_+})$, so that if $K_1 = 0$ then the derivative $\partial_\eta Z$ belongs to $H^2(\overline{\mathbb{P}_+})$ and vanishes on $\partial\mathbb{P}_+$ together with $\partial_\eta\partial_\zeta Z$. Hence we see from Green's formula in $\mathbb{P}_+$ that

$$0 = -\int_{-1}^{1} |\partial^2_\eta Z(0,\zeta)|^2\,d\zeta.$$

Thus $\partial_\eta Z \in H^2_0(\mathbb{P}_+)$ solves (2.16), so that $\partial_\eta Z = 0$ by Lemma 1. The proof is complete.

§ 3. Justifying the asymptotics

3.1. A convergence theorem. The next observation will follow from the calculations in § 3.4 (see Remark 2).

Lemma 2. *For each $n \in \mathbb{N}$ there exist positive quantities $C_{\mu n}$ and h_n such that*

$$\mu^h_n := \Lambda^h_n h^2 - M_1 h^{-2} \leqslant C_{\mu n} \quad \text{for } h \in (0, h_n].$$

We will show that $\mu^h_n \to \mu_n$ as $h \to +0$. It this subsection we only establish a partial result (see Theorem 3).

By Lemma 2 the quantities μ^h_n are uniformly bounded for $h \in (0, h_n]$. We pick an infinitesimal sequence $\{h_p\}_{p\in\mathbb{N}}$ such that

$$\mu^h \to \mu^0 \tag{3.1}$$

along this sequence. Here and below we drop the subscripts n and p for brevity.

We can represent the eigenfunction u^h, normalized in $L^2(\Pi^h)$, in (2.1), (2.2) as

$$u^h(y, z) = w^h(y) U_1(h^{-1}z) + v^h(y, z),$$

where

$$\int_{-h}^{h} v^h(y, z) U_1(h^{-1}z)\, dz = 0 \ \text{ and } \ w^h(y) = \frac{1}{ah} \int_{-h}^{h} u^h(y, z) U_1(h^{-1}z)\, dz, \ y \in [-1, 1]. \tag{3.2}$$

Lemma 3. *There exist positive C_w, C_v and h_0 such that for $h < h_0$*

$$h^{3/2}\|\partial^2_y w^h; L^2(\Upsilon)\| + h^{1/2}\|\partial_y w^h; L^2(\Upsilon)\| \leqslant C_w, \tag{3.3}$$

$$\|v^h; H^2(\Pi^h)\| \leqslant C_v h^{-1} \quad \text{and} \quad \|v^h; L^2(\Pi^h)\| \leqslant C_v h. \tag{3.4}$$

Proof. Substituting the test function $v = u^h$ into the integral identity (2.3) gives

$$\Lambda^h = \Lambda^h \|u^h\|^2 = \|\partial^2_y u^h\|^2 + 2\|\partial_y \partial_z u^h\|^2 + \|\partial^2_z u^h\|^2 =: J_1 + 2J_2 + J_3. \tag{3.5}$$

In this subsection $\|\cdot\|$ will denote the norm in $L^2(\Pi^h)$. Setting $U^h_1(z) = U_1(h^{-1}z)$ we will bear in mind the orthogonality condition in (3.2) and the formula

$$\int_{-h}^{h} |U^h_1(z)|^2\, dz = h \int_{-1}^{1} |U_1(\zeta)|^2\, d\zeta = ah.$$

For the first term, from the right-hand side of (3.5) we deduce that

$$J_1 = \|\partial^2_y w^h\, U^h_1 + \partial^2_y v^h\|^2 = ah\|\partial^2_y w^h; L^2(\Upsilon)\|^2 + \|\partial^2_y v^h\|^2.$$

The relations

$$\int_{-h}^{h} |\partial^2_z U^h_1(z)|^2\, dz = M_1 h^{-4} ah = M_1 a h^{-3} \quad \text{and} \quad \int_{-h}^{h} \partial^2_z U^h_1(z)\, \partial^2_z v^h(z)\, dz = 0$$

follow from equation (2.5) for U_1 and the orthogonality condition in (3.2). They also take us to the formula

$$J_3 = M_1 a h^{-3} \|w^h; L^2(\Upsilon)\|^2 + \|\partial_z^2 v^h\|^2. \tag{3.6}$$

Integrating the one-dimensional Poincaré inequality for a function v^h which is orthogonal to U_1^h in $L^2(-h,h)$ with respect to $y \in (-1,1)$ we obtain

$$M_1 {M_2}^{-1} \|\partial_z^2 v^h\|^2 \geqslant M_1 h^{-4} \|v^h\|^2. \tag{3.7}$$

Taking (3.5), (3.6) and the equality

$$\|u^h\|^2 = ah\|w^h; L^2(\Upsilon)\|^2 + \|v^h\|^2 \tag{3.8}$$

into account we conclude that

$$Ch^{-2} \geqslant ah\|\partial_y^2 w^h; L^2(\Upsilon)\|^2 + \|\partial_y^2 v^h\|^2 + (1 - M_1 {M_2}^{-1})\|\partial_z^2 v^h\|^2. \tag{3.9}$$

From the relation

$$\int_{-h}^{h} |\partial_z U_1^h(z)|^2 \, dz = \frac{1}{h} \int_{-1}^{1} |\partial_\zeta U_1(\zeta)|^2 \, d\zeta = \frac{b}{h}$$

we obtain

$$\begin{aligned} J_2 &= \|\partial_y w^h \, \partial_z U_1^h + \partial_y \partial_z v^h\|^2 \\ &= bh^{-1} \|\partial_y w^h; L^2(\Upsilon)\|^2 + \|\partial_y \partial_z v^h\|^2 + 2(\partial_y w^h \, \partial_z U_1^h, \partial_y \partial_z v^h)_{\Pi^h}. \end{aligned}$$

In the last inner product we integrate by parts and treat the result using the Cauchy-Schwarz inequality and (3.9):

$$\begin{aligned} |2(\partial_y w^h \, \partial_z U_1^h, \partial_y \partial_z v^h)_{\Pi^h}| &= |2(\partial_y^2 w^h \, U_1^h, \partial_z^2 v^h)_{\Pi^h}| \\ &\leqslant 2(Ch^{-2})^{1/2}((1 - M_1/M_2)^{-1} Ch^{-2})^{1/2} \leqslant C' h^{-2}. \end{aligned}$$

Hence from (3.5) we obtain the first inequality in (3.4). The second follows from (3.9) and (3.7). Lemma 3 is proved.

Formula (3.3) ensures that the sequence $\{h^{1/2} w^h\}$ is bounded in $H^1(\Upsilon)$. Hence we can select a subsequence on which, besides (3.1), we also have the convergence

$$h^{1/2} w^h \to w^0 \quad \text{weakly in } H^1(\Upsilon) \text{ and strongly in } L^2(\Upsilon). \tag{3.10}$$

For the test function $\Phi(y,z) = \varphi(y) U_1^h(z)$ involving an arbitrary multiplier $\varphi \in C_0^\infty(\Upsilon)$ we write down an integral identity auxiliary to problem (2.3):

$$\begin{aligned} \Lambda^h ah(w^h, \varphi)_\Upsilon &= \Lambda^h (u^h, \Phi)_{\Pi^h} = (\Delta u^h, \Delta \Phi)_{\Pi^h} = (\partial_y^2 w^h \, U_1^h, \partial_y^2 \varphi \, U_1^h)_{\Pi^h} \\ &+ (\partial_y^2 w^h \, U_1^h, \varphi \, \partial_z^2 U_1^h)_{\Pi^h} + (w^h \, \partial_z^2 U_1^h, \partial_y^2 \varphi \, U_1^h)_{\Pi^h} + (w^h \, \partial_z^2 U_1^h, \varphi \, \partial_z^2 U_1^h)_{\Pi^h} \\ &+ (\Delta v^h, \partial_y^2 \varphi \, U_1^h)_{\Pi^h} + (\Delta v^h, \varphi \, \partial_z^2 U_1^h)_{\Pi^h} =: I_1 + \cdots + I_6. \end{aligned}$$

Easy transformations show that

$$I_1 = ha(\partial_y^2 w^h, \partial_y^2\varphi)_\Upsilon, \qquad I_2 = I_3 = bh^{-1}(\partial_y w^h, \partial_y\varphi)_\Upsilon,$$
$$I_4 = M_1 a h^{-3}(w^h, \varphi)_\Upsilon, \qquad I_5 = I_6 = (\partial_z^2 v^h, \partial_y^2\varphi\, U_1^h)_{\Pi^h}.$$

Thus,

$$B^{-1}\mu^h(w^h,\varphi)_\Upsilon - (\partial_y w^h, \partial_y\varphi)_\Upsilon = B^{-1}h^2(\partial_y^2 w^h, \partial_y^2\varphi)_\Upsilon + b^{-1}h(\partial_z^2 v^h, \partial_y^2\varphi\, U_1^h)_{\Pi^h}. \tag{3.11}$$

Multiplying both sides of (3.11) by $h^{1/2}$ we take the limit. The modified left-hand side converges to

$$B^{-1}\mu^0(w^0,\varphi)_\Upsilon - (\partial_y w^0, \partial_y\varphi)_\Upsilon,$$

while the right-hand side tends to zero. In fact, by (3.3) and (3.4)

$$h^{5/2}(\partial_y^2 w^h, \partial_y^2\varphi)_\Upsilon \leqslant h^{5/2}\|\partial_y^2 w^h; L^2(\Upsilon)\| \cdot \|\partial_y^2\varphi; L^2(\Upsilon)\| \to 0$$

and

$$h^{3/2}(\partial_z^2 v^h, \partial_y^2\varphi\, U_1^h)_{\Pi^h} \leqslant h^{3/2}(ah)^{1/2}\|\partial_z^2 v^h\| \cdot \|\partial_y^2\varphi; L^2(\Upsilon)\| \to 0.$$

It remains to observe that $\|w^0; L^2(\Upsilon)\| = 1$ because of the second estimate in (3.4) and relation (3.8).

The condition $w^0(\pm 1) = 0$ follows from the second equality in (3.2) and relation (3.10), since w^0 is smooth. We must stress that there is no convergence $h^{1/2}w^h \to w^0$ in H^2, so the condition for the derivative $\partial_y w^0$ is not preserved (cf. 'paradox' 2) in § 1.2).

Theorem 3. *The convergence* $h^2(\Lambda_n^h - h^{-4}M_1) \to \mu_{k(n)}$, *where* $k(n) \geqslant n$, *holds as* $h \to +0$.

Proof. The above arguments show that the limit exists and is equal to some eigenvalue $\mu_{k(n)}$ in the list (2.12). It is sufficient to verify that $k(n) \geqslant n$. To do this we select an infinitesimal sequence $h_p > 0$ such that the convergences (3.10) hold along this sequence for all w_j^h such that $1 \leqslant j \leqslant n$. Since

$$ah(w_j^h, w_l^h)_\Upsilon = (u_j^h, u_l^h)_{\Pi^h} - (v_j^h, v_l^h)_{\Pi^h} = \delta_{j,l} - O(h^2), \qquad 1 \leqslant j, l \leqslant n,$$

by (3.2) and (3.4) the limit functions $w_1^0, \dots, w_n^0$ form an orthogonal system of eigenfunctions in the list (2.13), and the corresponding limiting eigenvalues form an increasing sequence. This observation completes the proof.

3.2. The abstract equation. In the space $\mathscr{H}^h = H_0^2(\Pi^h)$ we consider the inner product

$$\langle u, v\rangle_h = \mathbf{a}(u,v) - M_1 h^{-4}(u,v)_{\Pi^h} + h^{-2}(u,v)_{\Pi^h} \tag{3.12}$$

and the operator $\mathscr{T}^h \colon \mathscr{H}^h \to \mathscr{H}^h$ defined by

$$\langle \mathscr{T}^h u, v\rangle_h = (u,v)_{\Pi^h} \quad \forall\, u, v \in \mathscr{H}^h. \tag{3.13}$$

It is obvious that $\mathscr{T}^h$ is compact, continuous, selfadjoint and positive, so that its essential spectrum consists of the unique point $\tau = 0$ and its discrete spectrum is a positive infinitesimal sequence $\{\tau_k^h\}_{k\in\mathbb{N}}$ (see [18], Theorems 10.1.5 and 10.2.2). Problem (2.1), (2.2) is equivalent to the abstract equation

$$\mathscr{T}^h u^h = \tau^h u^h.$$

Comparing the definitions (3.12) and (3.13) with the integral identity (2.3) we see that

$$\tau_k^h = (\Lambda_k^h - M_1 h^{-4} + h^{-2})^{-1}.$$

The corresponding eigenfunctions $\varphi_k^h = (\tau_k^h)^{1/2} u_k^h$ are orthonormal:

$$\begin{aligned}\langle \varphi_j^h, \varphi_k^h\rangle_h &= (\tau_j^h \tau_k^h)^{1/2}((\Delta u_j^h, \Delta u_k^h)_{\Pi^h} - M_1 h^{-4}(u_j^h, u_k^h)_{\Pi^h} + h^{-2}(u_j^h, u_k^h)_{\Pi^h}) \\ &= \delta_{j,k}\tau_k^h(\Lambda_k^h - M_1 h^{-4} + h^2) = \delta_{j,k}.\end{aligned}$$

The next result is known as the lemma on 'almost' eigenvalues and eigenvectors (see [21], for instance). We present a simplified version of it, which is sufficient for our purpose.

Lemma 4. *Let $\mathscr{T}$ be a compact selfadjoint operator in a Hilbert space $\mathscr{H}$. If there exist numbers $\beta > t > 0$ and a nontrivial vector $\psi \in \mathscr{H}$ such that*

$$\|\mathscr{T}\psi - \beta\psi; \mathscr{H}\| = t \quad \textit{and} \quad \|\psi; \mathscr{H}\| = 1,$$

then the closed interval $[\beta - t, \beta + t]$ contains at least one eigenvalue of $\mathscr{T}$. Furthermore, if for some $t_1 \in (t, \beta)$ the interval $[\beta - t_1, \beta + t_1]$ contains just one eigenvalue of $\mathscr{T}$, then the corresponding normalized eigenfunction Φ in $\mathscr{H}$ satisfies

$$\|\Phi - \psi; \mathscr{H}\| \leqslant 2t_1^{-1} t. \tag{3.14}$$

3.3. An approximate solution of the abstract equation. We construct approximations of eigenvectors of the operator $\mathscr{T}^h$ using the eigenfunctions (2.13) of the limiting problem (2.10), (2.11), and we use Lemma 4 to justify the asymptotic formula (2.14).

Let $\{w_k, \mu_k\}$ be an eigenpair in (2.12), (2.13). Setting $\beta_k^h = (\mu_k h^{-2} + h^{-2})^{-1}$, for an approximate eigenvector we take the product

$$\psi_k^h(y, z) = \chi^h(y) w_k(y) U_1^h(z),$$

where χ^h is a smooth cutoff function:

$$\begin{gathered}\chi^h = 1 \quad \text{for } y \in [-1 + 2h, 1 - 2h], \qquad \chi^h = 0 \quad \text{for } |y| \in [1 - h, 1], \quad 0 \leqslant \chi^h \leqslant 1, \\ \text{and} \quad |\partial_y^k \chi^h(y)| \leqslant c_\chi h^{-k} \quad \text{for } y \in [-1, 1], \quad k = 1, 2.\end{gathered} \tag{3.15}$$

We start by estimating the norm in $\mathscr{H}^h$ of the approximation we have constructed.

Lemma 5. *There exist positive $c_{\psi k}$ and h_0 such that for $h < h_0$*

$$\|\psi_k^h; \mathscr{H}^h\| \geqslant c_{\psi k} h^{-1/2}.$$

Proof. Direct calculations show that

$$\begin{aligned}&\|\Delta(w_kU_1^h)\|^2-M_1h^{-4}\|w_kU_1^h\|^2+h^{-2}\|w_kU_1^h\|^2\\&\qquad=\|\partial_y^2w_k;L^2(\Upsilon)\|^2ah+2\|\partial_yw_k;L^2(\Upsilon)\|^2bh^{-1}+\|w_k;L^2(\Upsilon)\|^2ah^{-1}\geqslant c_kh^{-1}.\end{aligned}$$

Taking account of the relation

$$\|((1-\chi^h)w_k)\,\partial_z^2U_1^h\|^2-M_1h^{-4}\|((1-\chi^h)w_k)U_1^h\|^2=0,$$

which follows from the integral identity for problem (2.5), we conclude that it suffices to have an upper bound for

$$\begin{gathered}I_1=\|\partial_y^2((1-\chi^h)w_k)\,U_1^h\|,\qquad I_2=\|\partial_y((1-\chi^h)w_k)\,\partial_zU_1^h\|\\ \text{and}\quad I_3=h^{-1}\|((1-\chi^h)w_k)U_1^h\|.\end{gathered}\tag{3.16}$$

The first expression is estimated by the sum

$$\|w_k\,\partial_y^2(1-\chi^h)\,U_1^h\|+2\|\partial_y(1-\chi^h)\,\partial_yw_k\,U_1^h\|+\|(1-\chi^h)\,\partial_y^2w_k\,U_1^h\|=:I_{11}+2I_{12}+I_{13},$$

whose terms satisfy the inequalities

$$\begin{aligned}I_{11}^2&=ah\int_{1-2h\leqslant|y|\leqslant1}|w_k(y)|^2\,|\partial_y^2\chi^h(y)|^2\,dy\leqslant ah4hC_kh^2(c_\chi h^{-2})^2\leqslant C_k',\\ I_{12}^2&=ah\int_{1-2h\leqslant|y|\leqslant1}|\partial_yw_k(y)|^2\,|\partial_y\chi^h(y)|^2\,dy\leqslant ah4hC_k(c_\chi h^{-1})^2\leqslant C_k'\end{aligned}$$

and

$$I_{13}^2=ah\int_{1-2h\leqslant|y|\leqslant1}|\partial_y^2w_k(y)|^2\,|1-\chi^h(y)|^2\,dy\leqslant ah4hC_k1\leqslant C_k'h^2.$$

In treating each of the three integrals we have used the following considerations: the interval of integration has length at most $4h$, the function w_k is bounded by C_kh on the interval of integration because $w_k(\pm1)=0$; finally, the derivatives of the cutoff function satisfy (3.15).

In a similar way, for I_2 in (3.16) we obtain

$$I_2\leqslant\|w_k\,\partial_y(1-\chi^h)\,\partial_zU_1^h\|+\|(1-\chi^h)\,\partial_yw_k\,\partial_zU_1^h\|=:I_{21}+I_{22},$$

where

$$I_{21}^2=bh^{-1}\int_{1-2h\leqslant|y|\leqslant1}|w_k(y)|^2\,|\partial_y\chi^h(y)|^2\,dy\leqslant bh^{-1}4hC_kh^2(c_\chi h^{-1})^2\leqslant C_k'$$

and

$$I_{22}^2=bh^{-1}\int_{1-2h\leqslant|y|\leqslant1}|\partial_yw_k(y)|^2\,|1-\chi^h(y)|^2\,dy\leqslant bh^{-1}4hC_k1\leqslant C_k'.$$

It remains to observe that

$$I_3^2=h^{-2}ah\int_{1-2h\leqslant|y|\leqslant1}|w_k(y)|^2\,|1-\chi^h(y)|^2\,dy\leqslant h^{-2}ah4hC_kh^21\leqslant C_k'h^2.$$

Lemma 5 is proved.

Lemma 6. *There exist positive quantities $C_{\psi k}$ and h_0 such that*

$$\|\mathscr{T}^h\psi_k^h - \beta_k^h\psi_k^h; \mathscr{H}^h\| \leqslant C_{\psi k}h^2$$

for $h \in (0, h_0)$.

Proof. We have the equality

$$\|\mathscr{T}^h\psi_k^h - \beta_k^h\psi_k^h; \mathscr{H}^h\| = \sup_{\varphi\in\mathbb{B}}\langle\mathscr{T}^h\psi_k^h - \beta_k^h\psi_k^h, \varphi\rangle_h,$$

where the supremum is calculated over the unit ball $\mathbb{B}$ in $\mathscr{H}^h$. Taking the definitions (3.12) and (3.13) into account we transform the inner product as follows:

$$\begin{aligned}\langle\mathscr{T}^h\psi_k^h - \beta_k^h, \varphi\rangle_h &= \beta_k^h\big((\beta_k^h)^{-1}\langle\mathscr{T}^h\psi_k^h, \varphi\rangle_h - \langle\psi_k^h, \varphi\rangle_h\big)\\ &= -\beta_k^h\big((\Delta\psi_k^h, \Delta\varphi)_{\Pi^h} - M_1h^{-4}(\psi_k^h, \varphi)_{\Pi^h} - \mu_kh^{-2}(\psi_k^h, \varphi)_{\Pi^h}\big) =: \beta_k^h\gamma.\end{aligned}$$

As $|\beta_k^h| \leqslant h^2$, it is sufficient to verify that γ is bounded.

The equalities

$$(\partial_z^2\psi_k^h, \partial_z^2\varphi)_{\Pi^h} - M_1h^{-4}(\psi_k^h, \varphi)_{\Pi^h} = 0$$

and

$$2(\partial_y w_k\,\partial_z U_1^h, \partial_y\partial_z(\chi^h\varphi))_{\Pi^h} - \mu_kh^{-2}(\psi_k^h, \varphi)_{\Pi^h} = 0$$

follow from the integral identities for problems (2.5) and (2.10), (2.11), respectively. Thus,

$$|\gamma| \leqslant 2|K_1| + 2|K_2| + |K_3|,$$

and we also have

$$K_1 = (w_k\,\partial_z U_1^h\,\partial_y\chi^h, \partial_y\partial_z\varphi)_{\Pi^h}, \qquad K_2 = (\partial_y w_k\,\partial_z^2 U_1^h, \varphi\,\partial_y\chi^h)_{\Pi^h}$$
$$\text{and}\quad K_3 = (\partial_y^2(w_k\chi^h)\,U_1^h, \partial_y^2\varphi)_{\Pi^h}.$$

Note that by the Cauchy-Schwarz inequality we have

$$K_1 \leqslant (bh^{-1})^{1/2}\|w_k\,\partial_y\chi^h; L^2(\Upsilon)\|\,\|\partial_y\partial_z\varphi\| \leqslant C_k,$$

because, first, Poincaré's inequality yields

$$\|\partial_y^2\varphi\|^2 + \|\partial_y\partial_z\varphi\|^2 + h^{-2}\|\varphi\|^2 \leqslant 1, \tag{3.17}$$

and, second, the norm $\|w_k\,\partial_y\chi^h; L^2(\Upsilon)\|$ is no greater than $c_kh^{1/2}$ for similar reasons to the ones presented in the proof of Lemma 5. The inner product K_3 is estimated similarly: namely,

$$|K_3| \leqslant (ah)^{1/2}\|\partial_y^2(w_k\chi^h); L^2(\Upsilon)\|\,\|\partial_y^2\varphi\| \leqslant C_k.$$

Here, we have again used (3.17) and the estimates for I_{11}, I_{12} and I_{13} in the proof of Lemma 5.

Finally, we turn to K_2. From the Cauchy-Schwarz inequality we can conclude that

$$|K_2| \leqslant (M_1h^{-4}ah)^{1/2}\|\partial_y w_k\,\partial_y\chi^h; L^2(\Upsilon)\|\,\|\varphi; L^2(\Pi^h \cap \operatorname{supp}|\partial_y\chi^h|)\|.$$

It is clear that $\|w_k\,\partial_y\chi^h; L^2(\Upsilon)\| \leqslant c_kh^{-1/2}$, so it is sufficient to verify that

$$\|\varphi; L^2(\Pi^h \cap \operatorname{supp}|\partial_y\chi^h|)\| \leqslant ch^2.$$

This is a simple consequence of Friedrichs' inequality and (3.8). Lemma 6 is proved.

3.4. The proof of Theorem 1. Lemmas 4–6 ensure that for $h < h_0$ the operator $\mathscr{T}^h$ has an eigenvalue $\tau^h_{j(n)}$ such that

$$|\tau^h_{j(n)} - (\mu_n h^{-2} + h^{-2})^{-1}| \leqslant C_{\psi n} c^{-1}_{\psi n} h^{5/2}.$$

In other words

$$\begin{aligned}|\Lambda^h_{j(n)} - M_1 h^{-4} - \mu_n h^{-2}| &\leqslant C_{\psi n} c^{-1}_{\psi n} h^{5/2}(\mu_n + 1)h^{-2}(\Lambda^h_{j(n)} - M_1 h^{-4} + h^{-2}) \\ &\leqslant C_n h^{-3/2}.\end{aligned} \tag{3.18}$$

It is also obvious that $j(n) \geqslant n$. Combining this with the statement of Theorem 3 we derive Theorem 1.

Remark 2. The estimate (3.18) shows that all the μ^h_k satisfy $|\mu^h_k - \mu_n| \leqslant C_k h^{-1/2}$ for some n, which proves Lemma 2.

3.5. Eigenfunction asymptotics. In particular, Theorem 1 associates with each $N \in \mathbb{N}$ a number $h_N > 0$ such that for $h \in (0, h_N]$ all the eigenvalues $\Lambda^h_1, \dots, \Lambda^h_N$ are simple (as the eigenvalues (2.12) are simple and (2.14) holds). As a result, our calculations in § 3.3 and the second part of Lemma 4 which is related to formula (3.14) lead to a statement concerning the eigenfunctions of (2.3).

Theorem 4. *There exist positive values C_n and h_n such that for $h \in (0, h_n)$ the eigenfunctions of* (2.1), (2.2) *satisfy*

$$\|u^h_n - \alpha_n \chi^h w_n U^h_1; \mathscr{H}^h\| \leqslant C_n, \tag{3.19}$$

where α_n is the coefficient normalizing $\psi^h_n = \chi^h w_n U^h_1$ in the space $\mathscr{H}^h$.

The asymptotic construction of an eigenfunction can be made simpler by eliminating the cutoff function χ^h or, more precisely, by adding terms of boundary-layer type. In our paper we leave out these standard procedures and content ourselves with the estimate (3.19) which is easy to deduce.

§ 4. Asymptotic behaviour of the spectrum of a T-junction of plates

4.1. Spectral problems. Consider a problem similar to (2.1), (2.2):

$$\Delta^2 u^h_\top(x) = \Lambda^h_\top u^h_\top(x), \qquad x \in \mathbb{T}^h, \tag{4.1}$$

$$u^h_\top(x) = 0, \quad \partial_n u^h_\top(x) = 0, \qquad x \in \partial\mathbb{T}^h, \tag{4.2}$$

which relates to a T-junction of Kirchhoff plates with small ($h \ll 1$) thickness (Figure 1, a)

$$\mathbb{T}^h = \Pi^h \cup \{(y, z) \colon z \in (-\ell, 0),\ y \in (-h, h)\}, \qquad \ell > 0. \tag{4.3}$$

All the conclusions made in § 2.1 also hold for the spectrum of the operator $\mathbf{A}^h_\top$ of problem (4.1), (4.2). We keep our notation for the attributes of this problem, adding the subscript $\top$ to the symbols. By dilating the coordinates

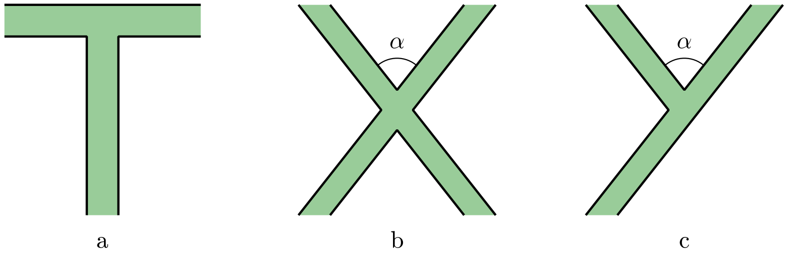

Figure 1. T-, X- and Y-junctions.

$x \mapsto \xi = (\eta, \zeta) = (h^{-1}y, h^{-1}z)$ and setting $h = 0$ formally, we transform the thin domain into the infinite T-shaped waveguide

$$\mathbb{T} = \{(\eta, \zeta) \in \mathbb{R}^2 \colon |\zeta| < 1\} \cup \{(\eta, \zeta) \in \mathbb{R}^2 \colon |\eta| < 1,\ \zeta < 0\}, \tag{4.4}$$

in which we pose the Dirichlet spectral problem for the biharmonic operator

$$\Delta_\xi^2 W(\xi) = \nu W(\xi), \quad \xi \in \mathbb{T}, \quad \text{and} \quad W(\xi) = 0, \quad \partial_n W(\xi) = 0, \quad \xi \in \partial\mathbb{T}. \tag{4.5}$$

4.2. Existence of a discrete spectrum in a T-shaped waveguide. The spectral problem (4.5) has the variational formulation

$$(\Delta W, \Delta V)_{\mathbb{T}} = \nu (W, V)_{\mathbb{T}} \quad \forall V \in H_0^2(\mathbb{T}). \tag{4.6}$$

The bilinear form on the left-hand side of (4.6) is positive and closed in $H_0^2(\mathbb{T})$, and we can re-write (4.6) as the abstract equation

$$\mathbb{A}_{\top} W = \nu W$$

for a certain unbounded positive selfadjoint operator $\mathbb{A}_{\top}$ in $L^2(\mathbb{T})$, which has a continuous spectrum $\sigma_c(\mathbb{A}_{\top}) = [\nu_\dagger, +\infty)$ with cutoff point $\nu_\dagger = M_1$ (the first eigenvalue of problem (2.5)). In what follows we let ν_1 denote the infimum of $\sigma(\mathbb{A}_{\top})$.

Theorem 5. *The discrete spectrum $\sigma_{\mathrm{di}}(\mathbb{A}_{\top})$ contains at least one eigenvalue $\nu_1 \in (0, \nu_\dagger)$.*

Proof. Assume that $\sigma_{\mathrm{di}}(\mathbb{A}_{\top}) = \varnothing$. Then by [18], § 10.2,

$$\|\Delta W; L^2(\mathbb{T})\|^2 \geqslant M_1 \|W; L^2(\mathbb{T})\|^2 \tag{4.7}$$

for all $W \in H_0^2(\mathbb{T})$. We construct a test function W^δ for which (4.7) fails. It depends on a small parameter $\delta > 0$ and is given by

$$W^\delta(\eta, \zeta) = \begin{cases} e^{-\delta(|\eta|-1)^2} U_1(\zeta), & (\eta, \zeta) \in \mathbb{T}_1^{\pm} := \{\pm\eta \geqslant 1,\ |\zeta| < 1\}, \\ U_1(\zeta) + \delta\Psi(\eta, \zeta), & (\eta, \zeta) \in \mathbb{T}_0 := \{|\eta| < 1,\ |\zeta| < 1\}, \\ \delta\Psi(\eta, \zeta), & (\eta, \zeta) \in \mathbb{T}_2^{-} := \{|\eta| < 1,\ \zeta < -1\}, \end{cases} \tag{4.8}$$

where Ψ is a smooth function with support $\operatorname{supp}\Psi \subset \{(\eta, \zeta) \colon |\eta| < 1,\ \zeta \in (-2, 0)\}$.

The function W^δ decays exponentially at infinity and is piecewise smooth. It is easy to see that the definitions (4.8) are compatible on the straight-line separators $\Upsilon_1^\pm = \{\eta = \pm 1, |\zeta| < 1\}$ and $\Upsilon_2^- = \{|\eta| < 1, \zeta = -1\}$, so that W^δ belongs to $H_0^2(\mathbb{T})$.

We substitute (4.8) into (4.7) and calculate the norms. Note that

$$\|\Delta W^\delta; L^2(\mathbb{T}_j^\pm)\|^2 - M_1\|W^\delta; L^2(\mathbb{T}_j^\pm)\| = O(\delta^2), \qquad j = 1, 2,$$

and

$$\|\Delta W^\delta; L^2(\mathbb{T}_0)\|^2 - M_1\|W^\delta; L^2(\mathbb{T}_0)\|^2 = 2\delta Y(\Psi) + O(\delta^2),$$

where the expression

$$Y(\Psi) = (\Delta U_1, \Delta\Psi)_{\mathbb{T}_0} - M_1(U_1, \Psi)_{\mathbb{T}_0} = -(\partial_\zeta^2 U_1, \partial_\zeta \Psi)_{\Upsilon_2^-} + (\partial_\zeta^3 U_1, \Psi)_{\Upsilon_2^-}$$

can be made negative by selecting Ψ appropriately. That is, (4.7) does indeed fail for small $\delta > 0$. The proof is complete.

Proposition 1. *The eigenfunctions W of problem (4.5) that are associated with eigenvalue ν_1 satisfy the estimates*

$$|\nabla_\xi^p W(\xi)| \leqslant c_p e^{-\beta_1|\xi|}, \qquad p \in \mathbb{N}_0, \quad |\xi| > 2, \tag{4.9}$$

for some positive parameter β_1.

Proof. The required result is established using Kondrat'ev's theory (see [15] and also [16], §§ 3.2 and 5.2) and our calculations in §§ 2.2 and 2.4; note that we introduced the last condition in (4.9) to exclude corner points, at which derivatives of the eigenfunction are singular by the same theory, from consideration. The proof is complete.

Here we give an elementary proof of a weaker result, which is all we need for the computations that follow.

Lemma 7. *The eigenfunctions W in Proposition 1 satisfy the estimates*

$$\|e^{\gamma_1|\xi|}\nabla_\xi^p W; L^2(\mathbb{T} \cap \{|\xi| > 2\})\| \leqslant c_p, \qquad p = 0, 1, 2,$$

for some positive parameter γ_1.

Proof. We look at $W_X(\xi) = X(\eta)W(\xi)$, where X is a smooth cutoff function that is equal to one for $\eta > 2$ and to zero for $\eta < 3/2$. The function $W_X \in H_0^2(\mathbb{P}_+)$ solves the problem

$$\Delta_\xi^2 W_X(\xi) - \nu_1 W_X(\xi) = f(\xi), \qquad \xi \in \mathbb{P}_+, \tag{4.10}$$

where f is a function with compact support. Problem (4.10) is uniquely solvable thanks to Friedrichs' inequality on a cross-section.

The auxiliary function $\mathbf{W}(\xi) = e^{\gamma\eta} W_X(\xi)$ solves the problem

$$\Delta_\xi^2 \mathbf{W}(\xi) - \nu_1 \mathbf{W}(\xi) + e^{\gamma\eta}[\Delta_\xi^2, e^{-\gamma\eta}]\mathbf{W}(\xi) = e^{\gamma\eta} f(\xi), \qquad \xi \in \mathbb{P}_+,$$
$$\mathbf{W}(\xi) = 0, \qquad \partial_\nu \mathbf{W}(\xi) = 0, \qquad \xi \in \partial\mathbb{P}_+,$$

where the perturbation operator $e^{\gamma\eta}[\Delta_\xi^2, e^{-\gamma\eta}]$, which involves the commutator of the bi-Laplacian Δ_ξ^2 and the function $e^{-\gamma\eta}$, satisfies

$$(e^{\gamma\eta}[\Delta_\xi^2, e^{-\gamma\eta}]w, w)_{\mathbb{P}_+} \leqslant C\gamma\|w; H_0^2(\mathbb{P}_+)\|^2, \qquad w \in H_0^2(\mathbb{P}_+).$$

Hence there exists $\gamma_1 > 0$ such that $\mathbf{W} \in H_0^2(\mathbb{P}_+)$, and so $e^{\gamma_1\eta}W_X \in H_0^2(\mathbb{P}_+)$ too. In the other 'arms' of the waveguide (4.4) the proof is similar. The lemma is proved.

4.3. A new asymptotic ansatz. Our next statement describes the asymptotic behaviour of the first eigenvalue of problem (4.1), (4.2). We stress that, in contrast to the asymptotic construction in § 3.5, the first eigenfunction of the problem in $\mathbb{T}^h$ has the properties of a boundary layer: it is localized quite closely to the middle part $\theta^h = \{(y,z)\colon -h < y, z < h\}$ of the junction (4.3) and decays at an exponential rate as the point moves away from θ^h; see (4.12) and (4.9).

Theorem 6. *For each $h > 0$ the first eigenvalue $\Lambda_{1,\top}^h$ of problem (4.1), (4.2) satisfies*

$$\nu_1 h^{-4} \leqslant \Lambda_{1,\top}^h \leqslant \nu_1 h^{-4} + Ch^{-4}e^{-\beta_1/h} \tag{4.11}$$

for some positive C_1.

Proof. We apply the minimum principle (for instance, see [18], Theorem 10.2.1) to the operator of problem (4.5):

$$\nu_1 = \inf_{w\in H_0^2(\mathbb{T})\setminus\{0\}} \frac{\|\Delta w; L^2(\mathbb{T})\|^2}{\|w; L^2(\mathbb{T})\|^2}.$$

Substituting the function $\xi \mapsto u_{1,\top}^h(h^{-1}x)$ which is extended by zero outside the set $\{\xi \in \mathbb{T}\colon |\eta| < h^{-1},\ \zeta > -h^{-1}\}$ into the Rayleigh quotient yields the lower bound in (4.11).

To verify the upper bound we look at the smooth cutoff function

$$\chi^h(\xi) = \begin{cases} 1, & \xi \in \{|\eta| \leqslant h^{-1} - 2,\ z \in (-1,1)\} \\ & \qquad \cup\{\zeta \geqslant -\ell h^{-1} + 2,\ \eta \in (-1,1)\}, \\ 0, & \xi \in \{|\eta| \geqslant h^{-1} - 1\} \cup \{|\zeta| \geqslant \ell h^{-1} - 1\}, \\ \chi_0(h^{-1} - |\eta|), & |\eta| \in [h^{-1} - 2, h^{-1} - 1], \\ \chi_0(\ell h^{-1} - |\zeta|), & |\zeta| \in [\ell h^{-1} - 2, \ell h^{-1} - 1], \end{cases}$$

where χ_0 is a function on the interval $[1,2]$. The definition is rather complicated because we have to ensure that the error has the optimal rate of decay as $h \to +0$ (cf. (4.11) and (4.9)). If we wished to have a majorant $\nu_1 h^{-4} + ce^{-\delta/h}$ for some $\delta > 0$, we could take a cutoff function equal to one for $|x| < d/2$ and to zero for $|x| > d$, where $d = \min\{1, \ell\}$ in accordance with the definition (4.3).

Applying the minimum principle to (4.1), (4.2), we substitute the function

$$u_0(x) = W_1(h^{-1}x)\chi^h(h^{-1}x), \tag{4.12}$$

where W_1 is an eigenfunction of $\mathbb{A}_\top$ which is normalized in $L^2(\mathbb{T})$ and corresponds to the eigenvalue ν_1, into the Rayleigh quotient

$$\Lambda_{1,\top}^h = \inf_{u\in H_0^2(\mathbb{T}^h)\setminus\{0\}} \frac{\|\Delta u; L^2(\mathbb{T}^h)\|^2}{\|u; L^2(\mathbb{T}^h)\|^2}.$$

By (4.9) and the choice of χ^h we have the chains of inequalities

$$\|\Delta(W_1\chi^h);L^2(\mathbb{T})\|^2 \leqslant (\|\Delta W_1;L^2(\mathbb{T})\| + \|\Delta(W_1(1-\chi^h));L^2(\mathbb{T})\|)^2 \leqslant \nu_1 + ce^{-\beta_1/h}$$

and

$$\|W_1\chi^h;L^2(\mathbb{T})\|^2 \geqslant (\|W_1;L^2(\mathbb{T})\| - \|W_1(1-\chi^h);L^2(\mathbb{T})\|)^2 \geqslant 1 - ce^{-\beta_1/h}.$$

This completes the proof.

§ 5. Open questions

5.1. A T-junction of plates. Of course, we cannot say that our asymptotic analysis of problem (4.1), (4.2) in § 4 is complete: the results in § 4.2 do not cover the whole of the spectrum of problem (4.5) in the infinite waveguide (4.4). For instance, we do not know the total multiplicity of the discrete spectrum $\sigma_{\mathrm{di}}(\mathbb{A}_\top)$ of the operator $\mathbb{A}_\top$ of the problem. And we could neither prove nor disprove the existence of a threshold resonance. Note that none of the diverse approaches developed for the Dirichlet problem for the Laplace operator (see [22]–[25] and many other papers) is fit to treat a fourth-order equation.

It is not difficult to predict (cf. [26], [27] and § 4.3) that each eigenvalue of the operator $\mathbb{A}_\top$ that lies in the half-open interval $(0,\nu_\dagger]$ (which includes the threshold) produces an eigenvalue similar to that in Theorem 1. On the other hand, we could not even show that the first eigenvalue ν_1 in Theorem 5 is simple.

By [28] (see also [24]–[26] and other papers) a threshold resonance occurs when, for the threshold value $\nu = \nu_\dagger$ of the spectral parameter, problem (4.5) has a bounded solution, which either decays at infinity and is a trapped wave or stabilizes at infinity and is an almost standing wave. We mentioned the trapped-wave case in the preceding paragraph. Almost standing waves affect another series of eigenvalues, which are generated by the triple of ordinary differential equations with respect to $y \in (-1,1)$ and $z \in (-\ell,0)$ (see the components of the junction (4.4)) with Dirichlet conditions for $y = \pm 1$ and $z = -\ell$, and with certain transmission conditions at the common central point with coordinates $y = \pm 0$ and $z = -0$. The same arguments as in [26] and [27] suggest that when there are no almost standing waves, the said transmission conditions turn into Dirichlet conditions, that is, the junction splits into three disjoint line segments in the limit as $h \to +0$. When one or several standing waves exist, some or other sophisticated transmission conditions bind the said equations together into a coherent problem[2]. Thus we have left the question of finding an eigenvalue of problem (4.1), (4.2) satisfying $\Lambda^h > M_1 h^{-4}$ completely unanswered.

5.2. Other shapes of junction. The method in § 4.2 can easily be applied to the junctions shown in Figure 1, b and c. In fact, to verify a result similar to Theorem 5 it is sufficient to distinguish a whole strip in the junction: as before, the required test function is constructed using (4.8). Neither the V- or Z-shaped nor the symmetric Y-shaped infinite Kirchhoff waveguide shown in Figure 2 contains such a strip, and so far we have no information about the discrete spectrum of the Dirichlet problem for Δ^2 for these.

[2]For Neumann boundary conditions a threshold resonance necessarily occurs (see [9]).

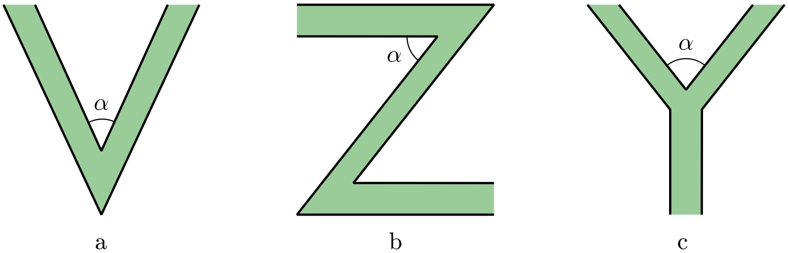

Figure 2. V-, Z- and Y-shaped junctions.

The X-shaped waveguide in Figure 1, b, contains two strips; if they have distinct widths, then we take the wider strip in constructing a test function (see the proof of Theorem 5), and this will determine the lower bound of the continuous spectrum. According to the maximum-minimum principle ([18], Theorem 10.2.2) a nonempty discrete spectrum for the waveguides in Figure 2, b and c, follows from a similar property for the V-shaped waveguide in Figure 2, a. Using asymptotic analysis from [29] we can find an isolated eigenvalue for angles α close to π. Furthermore, relying on [30] we can conjecture that the total multiplicity of the discrete spectrum of the waveguides in Figure 1, b and c, and Figure 2, a–c, increases without limit as $\alpha \to +0$.

Now we return to a T-shaped waveguide, for which we assume that the infinite crossbeam has width two while the half-infinite leg has width $H \in (0, +\infty)$. As decreasing the latter ($H < 2$) does not affect the cutoff point $\nu_\dagger(H) = M_1$ of the continuous spectrum, the discrete spectrum is still nonempty; however, as this width increases ($H > 2$) the cutoff point $\nu_\dagger(H) = (2/H)^4 M_1$ goes down, and we cannot use the approach in § 4.2. The next simple statement estimates the value H_* for which (and above which) there is no discrete spectrum.

Lemma 8. *For $H \geqslant 8/\pi(M_1)^{1/4}$ problem (4.5) in the waveguide $\mathbb{T}(H)$ has no discrete spectrum.*

Proof. For an arbitrary function $U \in H_0^2(\mathbb{T}(H))$, where $H > 2$, we write down the obvious inequalities

$$\|\Delta U; L^2(Q_{j\pm})\|^2 \geqslant \nu_\dagger(H)\|U; L^2(\mathbb{T}(Q_{j\pm}))\|^2$$

in the arms $Q_{1\pm}\{\xi \in \mathbb{T}(H)\colon \pm 2\eta > H\}$ and the leg $Q_{2-} = \{\xi \in \mathbb{T}(H)\colon \zeta < -1\}$ of the waveguide. In the remaining rectangle $\Theta = \{\xi\colon 2|\eta| < H,\ |\zeta| < 1\}$ we use Friedrichs' inequality twice, which yields

$$\|\partial_\zeta^2 U; L^2(\Theta)\|^2 \geqslant \left(\frac{\pi}{4}\right)^2 \|\partial_\zeta U; L^2(\Theta)\|^2 \geqslant \left(\frac{\pi}{4}\right)^4 \|U; L^2(\Theta)\|^2. \tag{5.1}$$

Hence, for $(\pi/4)^2 \geqslant (2/H)^4 M_1$ we have

$$\|\Delta U; L^2(\mathbb{T}(H))\|^2 \geqslant \nu_\dagger(H)\|U; L^2(\mathbb{T}(H))\|^2,$$

so there is indeed no discrete spectrum. The proof is complete.

Remark 3. By Sobolev's embedding theorem, $H^2 \subset C$ in the plane, and we have two extra constraints $U(\pm H/2,-1)=0$ in the rectangle Θ in the proof of Lemma 8, which are induced by the Dirichlet conditions on $\partial\mathbb{T}(H)$. There is no way to take them into account in the classical Friedrichs inequalities used twice in (5.1) above and adapted to boundary-value problems for the Laplace operator. So there is no way that the estimate for the critical width H_* in Lemma 8 will be sharp in any sense.

5.3. A plate with simply supported edge. Apart from the Dirichlet and Neumann conditions we have discussed in our paper, in the theory of Kirchhoff plates authors also consider the mixed boundary conditions

$$u^h(x)=0 \quad \text{and} \quad \Delta u^h(x)=0, \quad x\in\partial\Pi^h, \tag{5.2}$$

which mean that the deflection and the bending moment vanish on the edge of the plate, so that this edge is 'simply supported' (see [17], § 30). It is easy to see that the eigenvalues of problem (2.1), (5.2) are the squares of eigenvalues of the Dirichlet problem for the Laplace operator

$$-\Delta v^h(x)=\beta^h v^h(x) \quad \text{for } x\in\Pi^h, \qquad v^h(x)=0 \quad \text{for } x\in\partial\Pi^h. \tag{5.3}$$

It appears ar first glance that the same must also hold if we impose the conditions of simple support (5.2) on equation (4.1) in a T-junction (4.3) of plates; however, this is wrong. In fact, iterating the solution of problem (5.3) we obtain 'eigenfunctions' in the Sobolev class H^1, which belong to the energy class H^2 only when the polygon has no concave angles (with opening greater than π). This observation, called the Sapondzhyan paradox in mechanics (see [31]), was rigorously justified in the note [32] (see also [3], Ch. 18, and [33]), where, in particular, the reader can find a correct algorithm for reducing (4.1), (5.2) to the iterated problem (5.3). Unfortunately, this algorithm does not establish any relation between the eigenvalues of the above problems. Moreover, the conditions of simple support do not allow the trick in § 4.2, so that we have no information on the spectra of problem (4.1), (5.2) in the junction $\mathbb{T}^h$ or of the problem

$$\Delta^2_\xi W(\xi)=\nu W(\xi), \quad \xi\in\mathbb{T}, \qquad W(\xi)=0, \quad \Delta_\xi W(\xi)=0, \quad \xi\in\partial\mathbb{T}, \tag{5.4}$$

in the infinite waveguide (4.4). This also relates to the other junctions in Figures 1 and 2.

Finally, note that there is no sense in using the minimum principle and calculating the Rayleigh quotient for the eigenfunction $W_1\in H^2_0(\mathbb{T})$ of problem (4.6) provided by Theorem 5. As the cutoff point $\pi^4/64$ of the continuous spectrum of problem (5.4) lies below $\nu_\dagger$, we cannot prove the existence of an isolated eigenvalue in problem (5.4) using this method.

Bibliography

[1] P. G. Ciarlet, *Plates and junctions in elastic multi-structures. An asymptotic analysis*, Rech. Math. Appl., vol. 14, Masson, Paris; Springer-Verlag, Berlin 1990, viii+215 pp.

[2] J. Sanchez-Hubert and É. Sanchez-Palencia, *Coques élastiques minces. Propriétés asymptotiques*, Rech. Math. Appl., Masson, Paris 1997, xix+376 pp.

[3] V. Maz'ya, S. Nazarov and B. Plamenevskii, *Asymptotic theory of elliptic boundary value problems in singularly perturbed domains*, vols. 1, 2, Oper. Theory Adv. Appl., vol. 111, 112, Birkhäuser Verlag, Basel 2000, xxiv+435 pp., xxiv+323 pp.

[4] S. A. Nazarov, *Asymptotic theory of thin plates and rods. Dimension reduction and integral bounds*, Nauchnaya Kniga, Novosibirsk 2002, 406 pp. (Russian)

[5] G. Panasenko, *Multi-scale modelling for structures and composites*, Springer, Dordrecht 2005, xiv+398 pp.

[6] O. Post, *Spectral analysis on graph-like spaces*, Lecture Notes in Math., vol. 2039, Springer, Berlin–Heidelberg 2012, xvi+431 pp.

[7] P. Exner and H. Kovařík, *Quantum waveguides*, Theoret. Math. Phys., vol. 22, Springer, Cham 2015, xxii+382 pp.

[8] A. Gaudiello, G. Panasenko and A. Piatnitski, "Asymptotic analysis and domain decomposition for a biharmonic problem in a thin multi-structure", *Commun. Contemp. Math.* **18**:5 (2016), 1550057, 27 pp.

[9] S. A. Nazarov, "Asymptotics of the deflection of a cruciform junction of two narrow Kirchhoff plates", *Zh. Vychisl. Mat. Mat. Fiz.* **58**:7 (2018), 1197–1218; English transl. in *Comput. Math. Math. Phys.* **58**:7 (2018), 1150–1171.

[10] S. A. Nazarov, "The asymptotics of natural oscillations of a long two-dimensional Kirchhoff plate with variable cross-section", *Mat. Sb.* **209**:9 (2018), 35–86; English transl. in *Sb. Math.* **209**:9 (2018), 1287–1336.

[11] S. A. Nazarov, "General averaging procedure for selfadjoint elliptic systems in many-dimensional domains, including thin ones", *Algebra i Analiz* **7**:5 (1995), 1–92; English transl. in *St. Petersburg Math. J.* **7**:5 (1996), 681–748.

[12] S. A. Nazarov, "The polynomial property of self-adjoint elliptic boundary-value problems and an algebraic description of their attributes", *Uspekhi Mat. Nauk* **54**:5(329) (1999), 77–142; English transl. in *Russian Math. Surveys* **54**:5 (1999), 947–1014.

[13] I. C. Gohberg and M. G. Krein, *Introduction to the theory of linear nonselfadjoint operators in Hilbert space*, Nauka, Moscow 1965, 448 pp.; English transl., Transl. Math. Monogr., vol. 18, Amer. Math. Soc., Providence, RI 1969, xv+378 pp.

[14] S. A. Nazarov, "Self-adjoint elliptic boundary-value problems. The polynomial property and formally positive operators", Probl. Mat. Anal., vol. 16, Publishing house of St Petersburh University, St Petersburh 1997, pp. 167–192; English transl., *J. Math. Sci. (N. Y.)* **92**:6 (1998), 4338–4353.

[15] V. A. Kondrat'ev, "Boundary problems for elliptic equations in domains with conical or angular points", Tr. Mosk. Mat. Obshch., vol. 16, Publishing house of Moscow University, Moscow 1967, pp. 209–292; English transl. in Trans. Moscow Math. Soc., vol. 16, Amer. Math. Soc., Providence, RI 1967, pp. 227–313.

[16] S. A. Nazarov and B. A. Plamenevsky, *Elliptic problems in domains with piecewise smooth boundaries*, Nauka, Moscow 1991, 336 pp.; English transl., De Gruyter Exp. Math., vol. 13, Walter de Gruyter & Co., Berlin 1994, viii+525 pp.

[17] S. G. Mikhlin, *Variational methods in mathematical physics*, 2nd revised and augmented ed., Nauka, Moscow 1970, 512 pp.; English transl. of 1st ed. in Monographs Pure Appl. Math., vol. 50, Pergamon Press Book, Macmillan Co., New York 1964, xxxii+582 pp.

[18] M. Sh. Birman and M. Z. Solomyak, *Spectral theory of self-adjoint operators in Hilbert space*, Publishing house of Leningrad University, Leningrad 1980, 264 pp.; English transl., Math. Appl. (Soviet Ser.), vol. 5, D. Reidel Publishing Co., Dordrecht 1987, xv+301 pp.

[19] M. L. Williams, "Stress singularities resulting from various boundary conditions in angular corners of plates in extension", *J. Appl. Mech.* **19**:2 (1952), 526–528, https://authors.library.caltech.edu/47672/.

[20] C. De Coster, S. Nicaise and G. Sweers, "Solving the biharmonic Dirichlet problem on domains with corners", *Math. Nach.* **288**:8–9 (2015), 854–871.

[21] M. I. Višik (Vishik) and L. A. Ljusternik, "Regular degeneration and boundary layer for linear differential equations with small parameter", *Uspekhi Mat. Nauk* **12**:5(77) (1957), 3–122; English transl. in *Amer. Math. Soc. Transl.* (2) **20** (1962), 239–364.

[22] S. A. Nazarov, "Bounded solutions in a T-shaped waveguide and the spectral properties of the Dirichlet ladder", *Zh. Vychisl. Mat. Mat. Fiz.* **54**:8 (2014), 1299–1318; English transl. in *Comput. Math. Math. Phys.* **54**:8 (2014), 1261–1279.

[23] F. L. Bakharev, S. G. Matveenko and S. A. Nazarov, "The discrete spectrum of cross-shaped waveguides", *Algebra i Analiz* **28**:2 (2016), 58–71; English transl. in *St. Petersburg Math. J.* **28**:2 (2017), 171–180.

[24] F. L. Bakharev and S. A. Nazarov, *Criteria for the absence and existence of bounded solutions at the threshold frequency in a junction of quantum waveguides*, arXiv: 1705.10481.

[25] K. Pankrashkin, "Eigenvalue inequalities and absence of threshold resonances for waveguide junctions", *J. Math. Anal. Appl.* **449**:1 (2017), 907–925.

[26] D. Grieser, "Spectra of graph neighborhoods and scattering", *Proc. Lond. Math. Soc.* (3) **97**:3 (2008), 718–752.

[27] S. A. Nazarov, "The spectra of rectangular lattices of quantum waveguides", *Izv. Ross. Akad. Nauk Ser. Mat.* **81**:1 (2017), 31–92; English transl. in *Izv. Math.* **81**:1 (2017), 29–90.

[28] S. Molchanov and B. Vainberg, "Scattering solutions in networks of thin fibers: small diameter asymptotics", *Comm. Math. Phys.* **273**:2 (2007), 533–559.

[29] S. A. Nazarov, "Discrete spectrum of cranked quantum and elastic waveguides", *Zh. Vychisl. Mat. Mat. Fiz.* **56**:5 (2016), 879–895; English transl. in *Comput. Math. Math. Phys.* **56**:5 (2016), 864–880.

[30] F. L. Bakharev, S. G. Matveenko and S. A. Nazarov, "Examples of plentiful discrete spectra in infinite spatial cruciform quantum waveguides", *Z. Anal. Anwend.* **36**:3 (2017), 329–341.

[31] O. N. Sapondzhyan, "Bending of a simply supported polygonal plate", *Izv. Akad. Nauk Arm. SSR Ser. Fiz.-Mat. Estestv. Tekhn. Nauk* **5**:2 (1952), 29–46. (Russian)

[32] V. G. Maz'ya and S. A. Nazarov, "On the Sapondzhan-Babushka paradox in problems of the theory of thin plates", *Dokl. Akad. Nauk Arm. SSR* **78**:3 (1984), 127–130. (Russian)

[33] S. A. Nazarov and G. Sweers, "A hinged plate equation and iterated Dirichlet Laplace operator on domains with concave corners", *J. Differential Equations* **233**:1 (2007), 151–180.

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