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Locally standard torus actions and sheaves over Buchsbaum posets

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Abstract. Manifolds with locally standard half-dimensional torus actions represent a large and important class of spaces. Cohomology rings of such manifolds are known in particular cases, but in general even Betti numbers are difficult to compute. Our approach to this problem is the following: we consider the orbit type filtration on a manifold with locally standard action and study the induced spectral sequence in homology. It collapses at the second page only in the case when the orbit space is homologically trivial. The cohomology ring in this case has already been computed. Nevertheless, we can completely describe the spectral sequence under more general assumptions, namely when all proper faces of the orbit space are acyclic. The theory of sheaves and cosheaves on finite partially ordered sets is used in the computation. We establish generalizations of the Poincaré duality and the Zeeman-McCrory spectral sequence for sheaves of ideals of exterior algebras.

Bibliography: 15 titles.

Keywords: locally standard action, manifold with corners, simplicial poset, sheaf over poset, Zeeman-McCrory spectral sequence.

§ 1. Introduction

An action of a compact torus T^n on a smooth compact manifold M of dimension $2n$ is called *locally standard* if it is locally modelled by the standard representation of T^n on \mathbb{C}^n . The orbit space of a local chart is isomorphic to the nonnegative cone $\mathbb{C}^n/T^n \cong \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0\}$, thus the orbit space $Q = M/T^n$ of the whole manifold has the natural structure of a manifold with corners. Points in the interior of k -dimensional faces of Q are k -dimensional orbits of the action. For any face G of Q consider the stabilizer subgroup $T_G \subset T^n$ of points in the interior of G . The mapping that associates the toric subgroup T_G to the face G is called the *characteristic map*.

For any manifold M with locally standard torus action there exists a principal T^n -bundle $Y \rightarrow Q$ over the orbit space $Q = M/T$ such that M is equivariantly homeomorphic to the identification space $X = Y/\sim$. Here \sim identifies points over

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a face $G \subset Q$ differing by the action of T_G (see [7] and [15]). Thus any manifold with locally standard action is uniquely determined by three objects: a manifold with corners Q , a principal torus bundle Y over Q (these bundles are encoded by their ‘Euler classes’ lying in $H^2(Q; \mathbb{Z}^n)$) and the characteristic maps.

For a manifold with corners Q consider the dual poset S_Q . The elements of S_Q are the faces of Q and the order is the reversed inclusion. If Q is the orbit space of a manifold with locally standard action, then S_Q is a simplicial poset.

The description of the topology of X in terms of the combinatorial data is difficult and, in general, far from being accomplished. The cohomology and equivariant cohomology rings are unknown, and even Betti numbers have not been explicitly calculated yet.

Nevertheless, there are several important particular cases which are known and well studied. If the orbit space Q is isomorphic to a simple polytope, the manifold X is called quasitoric. This particular case was introduced and studied in the seminal work of Davis and Januszkiewicz [7] and underlay the development of toric topology. Quasitoric manifolds are natural topological generalizations of smooth projective toric varieties. The reason why quasitoric manifolds are feasible from the topological viewpoint is that the orbit space has trivial topology (the convexity happens to be not so important).

This setting may be generalized to the case when all faces of Q are acyclic. This situation is very close to toric varieties or quasitoric manifolds, and the answer is also very similar [9]:

$$H_{T^n}^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q] \quad \text{and} \quad H^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q]/(\theta_1, \dots, \theta_n),$$

where $\mathbb{Z}[S_Q]$ is the face ring of the simplicial poset S_Q and $(\theta_1, \dots, \theta_n)$ is the regular sequence of degree 2 in $\mathbb{Z}[S_Q]$, determined by the characteristic map.

There are several papers where the calculation of topological invariants has been performed for more general examples of locally standard manifolds. In [1] we proved that whenever all proper faces of Q are acyclic and $Y \rightarrow Q$ is a trivial bundle, the equivariant cohomology ring is represented as a direct sum (of rings and of modules over $H^*(BT^n; \mathbb{Z})$ alike):

$$H_{T^n}^*(X; \mathbb{Z}) \cong \mathbb{Z}[S_Q] \oplus H^*(Q; \mathbb{Z}).$$

We also calculated the Betti numbers and partially described the ring structure of $H^*(X, \mathbb{Z})$ when X is an orientable toric origami-manifold with acyclic proper faces of the orbit space. This is a very restricted class of manifolds with locally standard actions, but even in this case many interesting phenomena spring up. The Betti numbers of 4-dimensional toric origami-manifolds without any restrictions on proper faces were calculated in [8]. The cohomology rings of 4-dimensional manifolds whose orbit spaces are polygons with polygonal holes were described in [13].

In [15] Yoshida introduced a cohomological spectral sequence converging to $H^*(X; \mathbb{Z})$ for any locally standard X , but in general this spectral sequence does not collapse at the second page, so it is difficult to extract any explicit information, such as Betti numbers, from it.

In this paper we study the homological structure of a manifold X using the filtration of X by orbit types

$$X_0 \subset X_1 \subset \dots \subset X_n = X. \tag{1.1}$$

Here X_i is the union of all T^n -orbits of dimension at most i , so $\dim X_i = 2i$. This filtration induces the spectral sequence $(E_X)_{p,q}^r \Rightarrow H_{p+q}(X)$, where $(E_X)_{p,q}^1 \cong H_{p+q}(X_p, X_{p-1})$. For reasons of dimension $(E_X)_{p,q}^r = 0$ for $p < q$ and $r \geq 1$.

There is a natural topological filtration of Y which covers the orbit type filtration of X , and the map $f: Y \rightarrow X$ induces the map of homological spectral sequences

$$f_*^r: (E_Y)_{p,q}^r \rightarrow (E_X)_{p,q}^r. \tag{1.2}$$

If the proper faces of Q are acyclic, we prove that the map f_*^2 is an isomorphism for $p > q$ (Theorem 3). Thus every entry of $(E_X)_{p,q}^r$ away from the diagonal is known, at least if the structure of $(E_Y)_{p,q}^2$ is known. The calculation of diagonal entries $(E_X)_{q,q}^r$ requires additional work. This is done in a different paper [2]. The diagonal terms play a special role, since they correspond to the equivariant cycles of X given by face submanifolds.

To prove the above-mentioned isomorphism (Theorem 3), we place the maps $f_*^2: (E_Y)_{p,q}^2 \rightarrow (E_X)_{p,q}^2$ into a long exact sequence and show that certain intermediate terms of this sequence vanish. These intermediate terms are the cohomology modules $H^*(S_Q; \mathcal{I})$ of a graded sheaf \mathcal{I} on S_Q , whose values are the ideals in the homology algebra $H_*(T^n)$ generated by the vector subspaces $H_1(T_G) \subset H_1(T^n)$. The vanishing of this sheaf cohomology in certain degrees is the most nontrivial and essential part of the work. It follows from the duality:

$$H^{n-1-i}(S_Q; \mathcal{I}) \cong H_i(S_Q; \widehat{\Pi}) \tag{1.3}$$

(Theorem 2) which holds for an arbitrary homology manifold S_Q and extends the Poincaré duality $H^{n-1-i}(S_Q; \mathbb{k}) \cong H_i(S_Q; \mathbb{k})$. Here $\widehat{\Pi}$ is an auxiliary cellular cosheaf on S_Q , which is defined in our paper.

We study this duality in a broader and quite natural setting. For a simplicial poset S there exists the Zeeman-McCrory spectral sequence $(E_{ZM})_{p,q}^r$. It converges to the homology of S , and its second page is the cohomology of local homology stacks \mathcal{U}_* on S . If S is a manifold, this sequence collapses at the second page and gives a standard proof of the Poincaré duality. Thus the Zeeman-McCrory spectral sequence can roughly be considered as a generalization of Poincaré duality to nonmanifolds.

We prove that there exists a spectral sequence, which starts with $H^*(S; \mathcal{U}_* \otimes \mathcal{I})$ and converges to $H_*(S; \widehat{\Pi})$ (Theorem 1). For homology manifolds it collapses and gives the isomorphism (1.3).

We mention one interesting connection of the topological task considered in this paper with tropical geometry. As we mentioned in the abstract, the spectral sequence $(E_X)_{p,q}^r$ collapses at the second page whenever the orbit space Q is homologically trivial (that is, all faces of Q are acyclic). In particular, this applies to the case when X is a complete nonsingular toric variety. The homology of X coincides with the cohomology of the sheaf \mathcal{L}/\mathcal{I} on S_Q . The value of \mathcal{L}/\mathcal{I} on

$I \in S_Q$ is the exterior algebra $H_*(T^n/T_I)$. One can construct the tropical toric variety $\text{trop}(X)$ corresponding to X (see [14], for instance) and compute its tropical homology groups using the recipe given in [3]. By definition these tropical homology groups coincide with $H^*(S_Q, \mathcal{L}/\mathcal{S})$ (up to reversing the index). Thus we get the statement that the homology of a complete nonsingular toric variety coincides with the tropical homology of the corresponding tropical toric variety, which is known. A similar statement holds for cohomology.

The paper may be briefly outlined as follows. In §2 we review the basic notions for technical statements: simplicial posets, sheaves, cosheaves and the Zeeman-McCrory spectral sequence. The word ‘sheaf’ will always mean a sheaf over a finite poset. It is not used in its broadest topological sense, but rather replaces the term stack or local coefficient system. In §3 we introduce the notion of homological characteristic function, define two objects associated with this: the sheaf \mathcal{S} and the cosheaf $\widehat{\Pi}$, and formulate Theorems 1 and 2, proving the duality (1.3). Theorem 1 is proved in §4, and Theorem 2 follows as its particular case. Preliminaries on manifolds with locally standard actions are given in §5. We also introduce there topological filtrations on Q , X and Y and formulate Theorem 3, which states that the modules $(E_X)_{p,q}^r$ are isomorphic to $(E_Y)_{p,q}^2$ for $p > q$. Section 6 is devoted to the proof of Theorem 3; there we relate manifolds with locally standard torus actions to the sheaf-theoretical part of the work, which we develop in §§2–4.

§ 2. Sheaves and cosheaves over simplicial posets

2.1. Preliminaries on simplicial posets.

Definition 1. A finite partially ordered set (poset) S is called *simplicial* if there exists a minimal element $\widehat{0} \in S$ and, for any $I \in S$, the lower order ideal $\{J \in S \mid J \leq I\}$ is isomorphic to the boolean lattice $2^{[k]}$ (the poset of faces of a $(k - 1)$ -dimensional simplex) for some $k \geq 0$.

Elements of S are called *simplices*. The number k in the definition is denoted by $|I|$ and called the *rank of the simplex* I . Also set $\dim I = |I| - 1$. A simplex of rank 1 is called a *vertex*; the set of all vertices of S is denoted by $\text{Vert}(S)$. A subset $L \subset S$ closed under taking sub-simplices is called a *simplicial subposet*.

The notation $I \overset{i}{<} J$ is used whenever $I \leq J$ and $|J| - |I| = i$. If S is a simplicial poset, then for each $I \overset{2}{<} J \in S$ there exist exactly two simplices $J' \neq J''$ between I and J :

$$I \overset{1}{<} J', J'' \overset{1}{<} J. \tag{2.1}$$

For a simplicial poset S a *sign convention* can be chosen. It means that we can associate an incidence number $[J : I] = \pm 1$ with any pair $I \overset{1}{<} J \in S$ so that

$$[J : J'] \cdot [J' : I] + [J : J''] \cdot [J'' : I] = 0 \tag{2.2}$$

for any combination (2.1). Choosing a sign convention is the same as orienting each simplex in S . We fix an arbitrary sign convention and use it in the considerations that follow.

Notice that the set of simplices of any finite simplicial complex obviously forms a simplicial poset. Thus the notion of a simplicial poset is a straightforward generalization of an abstract simplicial complex.

For $I \in S$ consider the following subset of S :

$$\text{st}_S^\circ I = \{J \in S \mid J \geq I\},$$

which is called the *open star* of I . It is easily seen that $S \setminus \text{st}_S^\circ I$ is a simplicial subposet of S .

We also define the *link* of a simplex $I \in S$:

$$\text{lk}_S I = \{J \in S \mid J \geq I\}.$$

This set inherits the order relation from S , and $\text{lk}_S I$ is a simplicial poset with respect to this order, with minimal element I . The reason why we use two different notations for the same thing is that it is convenient to distinguish between $\text{st}_S^\circ I$, which is regarded as a subset (but not a subposet!) of S , and $\text{lk}_S I$, which is regarded as a simplicial poset on its own (and in general, is not included in S as a subposet in any meaningful way). Note that $\text{lk}_S \hat{0} = S$.

Let S' be the barycentric subdivision of S . By definition, S' is a simplicial complex on the set $S \setminus \hat{0}$ whose simplices are chains of elements of S . By definition, the geometric realization of S is the geometric realization of its barycentric subdivision $|S| \stackrel{\text{def}}{=} |S'|$. One can also think of $|S|$ as a CW-complex with simplicial cells. Such topological models of simplicial posets were called *simplicial cell complexes* and studied in [4].

A simplicial poset S is called *pure* if all its maximal elements have equal dimension. A poset S is pure whenever S' is pure.

In the following \mathbb{k} denotes the ground ring; it may be either a field or the ring of integers. The (co)homology of a simplicial poset S means the (co)homology of its geometrical realization $|S|$. If the coefficient ring in the notation of (co)homology is omitted, it is supposed to be \mathbb{k} .

Definition 2. A simplicial complex K of dimension $n - 1$ is called *Buchsbaum* (over \mathbb{k}) if $\tilde{H}_i(\text{lk}_K I; \mathbb{k}) = 0$ for all $\hat{0} \neq I \in K$ and $i \neq n - 1 - |I|$. If K is Buchsbaum and, moreover, $\tilde{H}_i(K; \mathbb{k}) = 0$ for $i \neq n - 1$ then K is called *Cohen-Macaulay*.

A simplicial poset S is called *Buchsbaum* (respectively Cohen-Macaulay) if S' is a Buchsbaum (respectively Cohen-Macaulay) simplicial complex.

Remark 1. By [12], § 6, S is Buchsbaum whenever $\tilde{H}_i(\text{lk}_S I; \mathbb{k}) = 0$ for all $\hat{0} \neq I \in S$ and $i \neq n - 1 - |I|$. Similarly, S is Cohen-Macaulay whenever $\tilde{H}_i(\text{lk}_S I; \mathbb{k}) = 0$ for all $I \in S$ and $i \neq n - 1 - |I|$.

Typical examples of Buchsbaum posets are triangulations (and, more generally, simplicial cell decompositions) of manifolds. Typical examples of Cohen-Macaulay posets are triangulations of spheres. A simplicial poset S is Buchsbaum whenever all its proper links are Cohen-Macaulay.

One can easily check that whenever S is Buchsbaum and connected, S is pure. In the following only pure simplicial posets are considered.

2.2. Cellular sheaves. Let $\text{MOD}_{\mathbb{k}}$ be the category of \mathbb{k} -modules. The notation $\dim V$ is used for the rank of a \mathbb{k} -module V .

Each simplicial poset S determines a small category $\text{CAT}(S)$ whose objects are elements of S and whose morphisms are inequalities $I \leq J$. A *cellular sheaf* [6] (or a *stack* [11], or a *local coefficient system* elsewhere) on S is a covariant functor $\mathcal{A} : \text{CAT}(S) \rightarrow \text{MOD}_{\mathbb{k}}$. We simply call \mathcal{A} a sheaf on S and hope this will not lead to confusion since other meanings of this word do not appear in the paper. The homomorphisms $\mathcal{A}(J_1 \leq J_2)$ are called *restriction maps*. The cochain complex $(C^*(S; \mathcal{A}), d)$ is defined as follows:

$$\begin{aligned}
 C^*(S; \mathcal{A}) &= \bigoplus_{i \geq -1} C^i(S; \mathcal{A}), & C^i(S; \mathcal{A}) &= \bigoplus_{\dim I=i} \mathcal{A}(I), \\
 d : C^i(S; \mathcal{A}) &\rightarrow C^{i+1}(S; \mathcal{A}), & d &= \bigoplus_{I \overset{1}{<} I', \dim I=i} [I' : I] \mathcal{A}(I \leq I').
 \end{aligned}
 \tag{2.3}$$

The sign convention (2.2) implies that $d^2 = 0$. Thus $(C^*(S; \mathcal{A}), d)$ is a differential complex. Define the cohomology of \mathcal{A} as the cohomology of this complex:

$$H^*(S; \mathcal{A}) \stackrel{\text{def}}{=} H^*(C^*(S; \mathcal{A}), d).
 \tag{2.4}$$

Remark 2. The cohomology of \mathcal{A} defined in this way coincides with other meaningful definitions of cohomology. For example, the derived functors of the functor of global sections give the same groups as (2.4) (see [6]).

A sheaf \mathcal{A} on S can be restricted to a simplicial subposet $L \subset S$. The complexes $(C^*(L, \mathcal{A}), d)$ and $(C^*(S; \mathcal{A})/C^*(L; \mathcal{A}), d)$ are defined as usual. The latter complex gives rise to a relative version of sheaf cohomology: $H^*(S, L; \mathcal{A})$.

Remark 3. It is standard in the topological literature to consider cellular sheaves which do not take values on $\widehat{0} \in S$, since in general this element does not have a geometrical meaning. However, this extra value $\mathcal{A}(\widehat{0})$ will be important in the considerations of §6. Therefore, the cohomology group may be nontrivial in degree $-1 = \dim \widehat{0}$. If a sheaf \mathcal{A} is defined on S , then we can consider its truncated version $\underline{\mathcal{A}}$ which coincides with \mathcal{A} on $S \setminus \{\widehat{0}\}$ and vanishes on $\widehat{0}$.

The notions of maps, (co)kernels, (co)images and tensor products of sheaves over S are defined in an obvious componentwise manner. For example, if \mathcal{A} and \mathcal{B} are two sheaves on S , then $\mathcal{A} \otimes \mathcal{B}$ is a sheaf on S with values $(\mathcal{A} \otimes \mathcal{B})(I) = \mathcal{A}(I) \otimes \mathcal{B}(I)$ and restriction maps $(\mathcal{A} \otimes \mathcal{B})(I \leq J) = \mathcal{A}(I \leq J) \otimes \mathcal{B}(I \leq J)$. In the realm of finite simplicial posets the distinction between sheaves and presheaves vanishes, which makes things simpler than they are in algebraic geometry.

Example 1. Let W be a \mathbb{k} -module. By abuse of notation let W denote the globally constant sheaf on S . It takes the constant value W on $I \neq \widehat{0}$ and vanishes on $\widehat{0}$. All nontrivial restriction maps are identity isomorphisms. If W is torsion-free, we have $H^*(S; W) \cong H^*(S; \mathbb{k}) \otimes W$ by the universal coefficients formula.

Example 2. A locally constant sheaf valued by $W \in \text{MOD}_{\mathbb{k}}$ is a sheaf \mathcal{W} that satisfies $\mathcal{W}(\widehat{0}) = 0$ and $\mathcal{W}(I) \cong W$ for $I \neq \widehat{0}$, and all nontrivial restriction maps are isomorphisms (but need not be identity isomorphisms).

Example 3. Following [11], define the i th local homology sheaf \mathcal{U}_i on S by setting $\mathcal{U}_i(\widehat{0}) = 0$ and

$$\mathcal{U}_i(J) = H_i(S, S \setminus \text{st}_S^\circ J; \mathbb{k}) \tag{2.5}$$

for $J \neq \widehat{0}$. The restriction maps $\mathcal{U}_i(J_1 < J_2)$ are induced by inclusions of subsets $\text{st}_S^\circ J_2 \hookrightarrow \text{st}_S^\circ J_1$. Standard topological arguments imply that a simplicial poset S is Buchsbaum if and only if $\mathcal{U}_i = 0$ for $i \neq n - 1$ (see also Remark 5 below).

Definition 3. A Buchsbaum simplicial poset S is called a *homology manifold* (orientable over \mathbb{k}) if its local homology sheaf \mathcal{U}_{n-1} is isomorphic to the constant sheaf \mathbb{k} .

A simplicial poset S is an orientable homology manifold if and only if its geometrical realization is an orientable homology manifold in the usual topological sense.

2.3. Cosheaves. A cellular cosheaf [6] is a contravariant functor $\widehat{\mathcal{A}} : \text{CAT}(S)^{\text{op}} \rightarrow \text{MOD}_{\mathbb{k}}$. The homology of a cosheaf is defined similarly to the cohomology of a sheaf:

$$\begin{aligned} C_*(S; \widehat{\mathcal{A}}) &= \bigoplus_{i \geq -1} C_i(S; \widehat{\mathcal{A}}), & C_i(S; \widehat{\mathcal{A}}) &= \bigoplus_{\dim I=i} \widehat{\mathcal{A}}(I), \\ d : C_i(S; \widehat{\mathcal{A}}) &\rightarrow C_{i-1}(S; \widehat{\mathcal{A}}), & d &= \bigoplus_{I \overset{\perp}{<} I', \dim I=i} [I : I'] \widehat{\mathcal{A}}(I \geq I'), \\ H_*(S; \widehat{\mathcal{A}}) &\stackrel{\text{def}}{=} H_*(C_*(S; \widehat{\mathcal{A}}), d). \end{aligned}$$

The relative homology groups $H_*(S, L; \widehat{\mathcal{A}})$ of a cosheaf $\widehat{\mathcal{A}}$ for $L \subset S$ are defined as the homology groups of the differential complex $C_*(S, L; \widehat{\mathcal{A}}) = C_*(S; \widehat{\mathcal{L}}) / C_*(L; \widehat{\mathcal{A}})$.

Example 4. Each locally constant sheaf \mathcal{W} on S determines the locally constant cosheaf $\widehat{\mathcal{W}}$ by inverting all maps, that is, $\widehat{\mathcal{W}}(I) \cong \mathcal{W}(I)$ and $\widehat{\mathcal{W}}(I > J) = (\mathcal{W}(J < I))^{-1}$.

Remark 4. Notice that the notation $H_*(S; \mathbb{k})$ can mean either the homology of the geometric realization $|S|$ or the homology of a globally constant cosheaf \mathbb{k} on S . Obviously, these two meanings are consistent, and the same holds for the cohomology of a constant sheaf.

2.4. Coskeleton filtration and dual faces. In the following we suppose that S is pure and $\dim S = n - 1$.

Construction 1. Let us recall the construction of the *coskeleton filtration* on $|S|$. Consider the barycentric subdivision S' of the pure simplicial poset S . By definition, S' is a simplicial complex on the set $S \setminus \widehat{0}$ and k -simplices of S' have the form $(I_0 < I_1 < \dots < I_k)$, where $I_i \in S \setminus \widehat{0}$. For each $I \in S \setminus \{\widehat{0}\}$ consider the subcomplex of the barycentric subdivision:

$$G_I = \{(I_0 < I_1 < \dots) \in S' \text{ such that } I_0 \geq I\} \subset S'$$

and the subsets

$$\partial G_I = \{(I_0 < I_1 < \dots) \in S' \text{ such that } I_0 > I\} \subset S', \quad G_I^\circ = G_I \setminus \partial G_I.$$

It is easily seen that $\dim G_I = n - 1 - \dim I$ since S is pure. We have $G_I \subset G_J$ whenever $J < I$. The complex G_I (or its geometrical realization $|G_I|$) is called the *face* or the *pseudocell* of $|S|$ dual to $I \in S$. The boundary ∂G_I of a face G_I is the union of some faces of smaller dimensions.

Let $S_i = \bigcup_{\dim G_I \leq i} G_I$ for $-1 \leq i \leq n - 1$. Thus S_i is a simplicial subcomplex of S' . The filtration

$$\emptyset = S_{-1} \subset S_0 \subset S_1 \subset \dots \subset S_{n-1} = S' \tag{2.6}$$

and the corresponding topological filtration

$$\emptyset = |S_{-1}| \subset |S_0| \subset |S_1| \subset \dots \subset |S_{n-1}| = |S| \tag{2.7}$$

are called the *coskeleton filtrations* of S' and $|S|$ respectively (see [11]).

For a pair $I \overset{1}{<} J \in S$ consider the map:

$$\begin{aligned} m_{I,J}^q &: H_{q+\dim G_I}(G_I, \partial G_I) \rightarrow H_{q+\dim G_{I-1}}(\partial G_I) \\ &\rightarrow H_{q+\dim G_{I-1}}(\partial G_I, \partial G_I \setminus G_J^o) \cong H_{q+\dim G_J}(G_J, \partial G_J), \end{aligned} \tag{2.8}$$

where the first map is the connecting homomorphism in the long exact sequence of homology for the pair $(G_I, \partial G_I)$ and the last isomorphism is due to excision. The homology spectral sequence associated with the filtration (2.7) runs

$$(E_S)_{p,q}^1 = H_{p+q}(S_p, S_{p-1}) \implies H_{p+q}(S).$$

The first differential $(d_S)^1$ is the sum of the maps $m_{I,J}^q$ over all pairs $I \overset{1}{<} J, I, J \in S$.

Construction 2. Given a sign convention on S , for each q consider the sheaf \mathcal{H}_q on S given by

$$\mathcal{H}_q(I) = H_{q+\dim G_I}(G_I, \partial G_I)$$

for $I \neq \widehat{0}$, and $\mathcal{H}_q(\widehat{0}) = 0$. For neighbouring simplices $I \overset{1}{<} J$ define the restriction map by $\mathcal{H}_q(I \overset{1}{<} J) = [J : I]m_{I,J}^q$. For general $I \overset{k}{<} J$ consider any saturated chain in S between I and J :

$$I \overset{1}{<} J_1 \overset{1}{<} \dots \overset{1}{<} J_{k-1} \overset{1}{<} J$$

and set

$$\mathcal{H}_q(I < J) \stackrel{\text{def}}{=} \mathcal{H}_q(J_{k-1} < J) \circ \dots \circ \mathcal{H}_q(I < J_1).$$

Lemma 1. *The map $\mathcal{H}_q(I < J)$ thus defined does not depend on the choice of a saturated chain between I and J .*

Proof. The differential $(d_S)^1$ satisfies $((d_S)^1)^2 = 0$, thus $m_{J',J}^q \circ m_{I,J'}^q + m_{J'',J}^q \circ m_{I,J''}^q = 0$. By combining this with (2.2) we see that $\mathcal{H}_q(I < J)$ is independent of the chain if $I \overset{2}{<} J$. In general, since $\{T \mid I \leq T \leq J\}$ is a boolean lattice, any two saturated chains between I and J are connected by a sequence of elementary flips $[J_k \overset{1}{<} T_1 \overset{1}{<} J_{k+2}] \rightsquigarrow [J_k \overset{1}{<} T_2 \overset{1}{<} J_{k+2}]$ and the statement follows.

Thus the sheaves \mathcal{H}_q are well defined. They will be called the *structure sheaves* of S . The next statement follows directly from the definition of a cochain complex.

Corollary 1. *The cochain complex of structure sheaves coincides with $(E_S)^1_{*,*}$ up to change of indices:*

$$((E_S)^1_{*,q}, (d_S)^1) \cong (C^{n-1-*}(\mathcal{H}_q), d).$$

Here d is the standard differential in the cochain complex of a sheaf as defined in (2.3).

Remark 5. There exists an isomorphism of sheaves

$$\mathcal{H}_q \cong \mathcal{U}_{q+n-1}, \tag{2.9}$$

where the \mathcal{U}_* are the sheaves of local homology defined in Example 3. Indeed, it can be shown that $H_i(S, S \setminus \text{st}_S^\circ I) \cong H_{i-\dim I}(G_I, \partial G_I)$ and these isomorphisms can be chosen compatible with restriction maps. For simplicial complexes this fact was proved in [11], §6.1; the case of simplicial posets is similar. Note that the definition of \mathcal{H}_* depends on the sign convention while \mathcal{U}_* does not. This makes no contradiction since the isomorphism (2.9) itself depends on the choice of orientations.

The isomorphism (2.9) implies that S is Buchsbaum if and only if $\mathcal{H}_q = 0$ for $q \neq 0$. A simplicial poset S is an orientable manifold if it is Buchsbaum and, moreover, $\mathcal{H}_0 \cong \mathbb{k}$.

2.5. Zeeman-McCrory spectral sequence. The considerations in the previous subsection imply the following

Proposition (McCrory, [11]). *There exists a spectral sequence, located in the fourth quadrant,*

$$(E_{ZM})^r_{p,q}, \quad d^r : (E_{ZM})^r_{p,q} \rightarrow (E_{ZM})^r_{p-r, q+r-1}, \tag{2.10}$$

$$(E_{ZM})^2_{p,q} \cong H^{n-1-p}(S; \mathcal{U}_{n-1+q}) \Rightarrow H_{p+q}(S; \mathbb{k}). \tag{2.11}$$

It is isomorphic to the homological spectral sequence associated with the coskeleton filtration of $|S|$.

For us, however, it will be more convenient to work with the structure sheaves \mathcal{H}_* rather than the local homology sheaves \mathcal{U}_* . For a Buchsbaum simplicial poset the sheaf \mathcal{H}_i vanishes for $i \neq 0$. Thus $(E_{ZM})^2_{p,q} = 0$ for $q \neq 0$ and the spectral sequence collapses at the second page, inducing the isomorphism

$$H^{n-1-p}(S; \mathcal{H}_0) \cong H_p(S; \mathbb{k}).$$

When S is an orientable homology manifold, this gives the Poincaré duality isomorphism

$$H^{n-1-p}(S; \mathbb{k}) \cong H_p(S; \mathbb{k}).$$

2.6. Corefinements of sheaves. In this section we develop a technical notion which will be used further in the proofs. Let \mathcal{A} be a sheaf on S . Define a cosheaf $\widehat{\mathcal{A}}$ on the barycentric subdivision S' by

$$\widehat{\mathcal{A}}(I_1 < \dots < I_k) = \mathcal{A}(I_1),$$

with corestriction maps determined naturally by restriction maps of \mathcal{A} :

$$\widehat{\mathcal{A}}((I_1 < \dots < I_k) \supset (J_1 < \dots < J_s)) = \mathcal{A}(I_1 \leq J_1).$$

We call $\widehat{\mathcal{A}}$ a *corefinement of a sheaf* \mathcal{A} . Faces G_I and their boundaries ∂G_I are simplicial subcomplexes of S' , so we can restrict $\widehat{\mathcal{A}}$ to them. The next lemma easily follows from the definitions.

Lemma 2.

$$H_q(S_p, S_{p-1}, \widehat{\mathcal{A}}) \cong \bigoplus_{I, \dim G_I=p} H_q(G_I, \partial G_I; \widehat{\mathcal{A}}).$$

Similar to (2.8) there is a map

$$\begin{aligned} m_{I,J}^{q,\mathcal{A}} : H_{q+\dim G_I}(G_I, \partial G_I; \widehat{\mathcal{A}}) &\rightarrow H_{q+\dim G_{I-1}}(\partial G_I; \widehat{\mathcal{A}}) \\ &\rightarrow H_{q+\dim G_{I-1}}(\partial G_I, \partial G_I \setminus G_J^\circ; \widehat{\mathcal{A}}) \cong H_{q+\dim G_J}(G_J, \partial G_J; \widehat{\mathcal{A}}). \end{aligned} \quad (2.12)$$

These maps allow us to define new sheaves $\overline{\mathcal{A}}_q$ on S by setting $\overline{\mathcal{A}}_q(I) = H_{q+\dim G_I}(G_I, \partial G_I; \widehat{\mathcal{A}})$ with restriction maps defined similarly to Construction 2.

Lemma 3. *If $\mathcal{A}(I)$ is torsion-free for all $I \in S$, then there exist natural isomorphisms*

$$H_r(G_I, \partial G_I; \widehat{\mathcal{A}}) \cong H_r(G_I, \partial G_I; \mathbb{k}) \otimes \mathcal{A}(I).$$

The maps $m_{I,J}^{q,\mathcal{A}}$ coincide with $m_{I,J}^q \otimes \mathcal{A}(I < J)$ up to these isomorphisms. Thus the sheaf $\overline{\mathcal{A}}_q$ is isomorphic to $\mathcal{H}_q \otimes \mathcal{A}$.

Proof. By the definition of $\widehat{\mathcal{A}}$ we have

$$H_r(G_I, \partial G_I; \widehat{\mathcal{A}}) \cong H_r(G_I, \partial G_I; \mathcal{A}(I)),$$

since the value of $\widehat{\mathcal{A}}$ on all simplices of G_I° is exactly $\mathcal{A}(I)$. The rest follows from the universal coefficients formula.

§ 3. Exterior algebras and characteristic functions

Let V be a free \mathbb{k} -module of dimension N . Let $\Lambda[V]$ denote the free exterior algebra generated by V , that is, the quotient of a free tensor algebra $T[V]$ by the relations $v \otimes v = 0$ for all $v \in V$. The algebra $\Lambda[V]$ is graded by degrees of exterior forms.

Definition 4. Fix a simplicial poset S and a locally constant sheaf \mathcal{V} on S . A collection of vectors $\{\omega_i \in \mathcal{V}(i) \mid i \in \text{Vert}(S)\}$ is called a *homological \mathbb{k} -characteristic function* if it satisfies the following $(*_\mathbb{k})$ -condition:

for each simplex $I \in S \setminus \widehat{0}$ whose vertices are i_1, \dots, i_k , the vectors

$$\mathcal{V}(i_1 \leq I)(\omega_{i_1}), \dots, \mathcal{V}(i_k \leq I)(\omega_{i_k}) \in \mathcal{V}(I)$$

are linearly independent over \mathbb{k} and span a direct summand in $\mathcal{V}(I)$.

For a locally constant sheaf \mathcal{V} on S , valued by the vector space V , consider the sheaf $\mathcal{L} = \Lambda[\mathcal{V}]$ of graded exterior algebras generated by \mathcal{V} . This means that $\mathcal{L}(I) = \Lambda[\mathcal{V}(I)]$, and $\mathcal{L}(I \leq J)$ is an isomorphism of graded exterior algebras generated by the isomorphism $\mathcal{V}(I \leq J): \mathcal{V}(I) \rightarrow \mathcal{V}(J)$. Let $\widehat{\mathcal{L}}$ denote the locally constant cosheaf of exterior algebras corresponding to the sheaf \mathcal{L} (see Example 4).

Let $\{\omega_i \in \mathcal{V}(i) \mid i \in \text{Vert}(S)\}$ be a homological characteristic function. If i is a vertex of a simplex I , then the restriction map $\mathcal{V}(i \leq I)$ sends the vector $\omega_i \in \mathcal{V}(i)$ to some vector in $\mathcal{V}(I)$. By abuse of notation we denote the target vector by the same letter ω_i . So far the definition of homological characteristic function implies that the set $\{\omega_{i_1}, \dots, \omega_{i_k}\}$ spans freely a direct summand of $\mathcal{V}(I)$ whenever i_1, \dots, i_k are vertices of I . Note that $\mathcal{L}(I)$ is the exterior algebra generated by $\mathcal{V}(I)$, so the vectors ω_i can be considered as elements of degree 1 in $\mathcal{L}(I)$.

Construction 3. Consider the subsheaf $\mathcal{I} \subset \mathcal{L}$ defined as follows. For a simplex I with vertices i_1, \dots, i_k we set the value of \mathcal{I} on I to be the ideal of $\mathcal{L}(I)$ generated by the linear forms $\omega_{i_1}, \dots, \omega_{i_k}$:

$$\mathcal{I}(I) = (\omega_{i_1}, \dots, \omega_{i_k}) \subset \mathcal{L}(I).$$

It is easily seen that whenever $I \leq J$, the restriction map $\mathcal{L}(I \leq J)$ sends the ideal $\mathcal{I}(I)$ generated by the smaller set of elements into the ideal $\mathcal{I}(J)$ generated by the larger set of elements. Thus the restriction maps of the sheaf \mathcal{I} are induced from those of \mathcal{L} and are well defined.

Construction 4. Let us define another type of ideals associated with a characteristic function. Let $J = \{i_1, \dots, i_k\}$ be a nonempty subset of vertices of a simplex $I \in S$. Consider the element $\pi_J \in \mathcal{L}(I) = \widehat{\mathcal{L}}(I)$, $\pi_J = \bigwedge_{i \in J} \omega_i$. By the definition of the characteristic function, the elements $\{\omega_i \mid i \in J\}$ are linearly independent, thus π_J is a nonzero form of degree $|J|$. Let $\Pi_J \subset \mathcal{L}(I)$ be the principal ideal generated by π_J . The restriction maps $\mathcal{L}(I < I')$ (and corestriction maps $\widehat{\mathcal{L}}(I' > I) = \mathcal{L}(I < I')^{-1}$) identify $\Pi_J \subset \mathcal{L}(I)$ with $\Pi_J \subset \mathcal{L}(I')$.

Let us define a subcosheaf $\widehat{\Pi}$ of ideals in $\widehat{\mathcal{L}}$. If J is the whole set of vertices of a simplex $I \neq \widehat{0}$ we set $\widehat{\Pi}(I) \stackrel{\text{def}}{=} \Pi_J \subset \widehat{\mathcal{L}}(I)$. If $I' < I$, the corestriction map $\widehat{\mathcal{L}}(I' > I)$ injects $\widehat{\Pi}(I')$ into $\widehat{\Pi}(I)$ since the form $\pi_{I'}$ is divisible by π_I . Thus $\widehat{\Pi}$ is a well-defined graded sub-cosheaf of $\widehat{\mathcal{L}}$. We formally set $\widehat{\Pi}(\widehat{0}) = 0$.

Now we can formulate our main homological results.

Theorem 1. *Let S be a pure simplicial poset of dimension $n - 1$, and \mathcal{I} and $\widehat{\Pi}$ the sheaf and cosheaf over S determined by some homological \mathbb{k} -characteristic function. Then there exists a spectral sequence*

$$E_{s,k}^2 \cong H^{n-1-s}(S; \mathcal{H}_k \otimes \mathcal{I}) \Rightarrow H_{s+k}(S; \widehat{\Pi}),$$

$$d^r : E_{s,k}^r \rightarrow E_{s-r, k+r-1}^r,$$

which respects the inner gradings of \mathcal{I} and $\widehat{\Pi}$.

If S is Buchsbaum, the spectral sequence of Theorem 1 collapses at the second page and implies the following.

Theorem 2. *For a Buchsbaum simplicial poset S of dimension $n - 1$ there exists an isomorphism $H^k(S; \mathcal{H}_0 \otimes \mathcal{S}) \cong H_{n-1-k}(S; \widehat{\Pi})$ which respects the inner gradings of \mathcal{S} and $\widehat{\Pi}$.*

Corollary 2. *If S is a homology $(n - 1)$ -manifold, then there is an isomorphism $H^k(S; \mathcal{S}) \cong H_{n-1-k}(S; \widehat{\Pi})$ respecting the inner gradings.*

Let $\mathcal{S}^{(q)}$ and $\widehat{\Pi}^{(q)}$ denote the homogeneous parts of inner degree q of the corresponding sheaves \mathcal{S} and $\widehat{\Pi}$, respectively.

Corollary 3 (key corollary). *If S is a Buchsbaum simplicial poset, then $H^j(S; \mathcal{H}_0 \otimes \mathcal{S}^{(q)}) = 0$ for $j \leq n - 1 - q$.*

Proof. By Theorem 2, it is sufficient to prove that $H_j(S; \widehat{\Pi}^{(q)}) = 0$ for $j \geq q$. The ideal $\widehat{\Pi}(I) = \Pi_I$ is generated by the element π_I of degree $|I| = \dim I + 1$. Thus $\Pi_I^{(q)} = 0$ for $q \leq \dim I$. Hence the corresponding part of the chain complex vanishes, and the homology in these degrees vanishes as well.

Remark 6. The exterior forms of the top power, $\Lambda[V]^{(N)} \cong \mathbb{k}$, lie in every ideal $\mathcal{S}(I)$ and $\widehat{\Pi}(I)$. Thus the isomorphism in Theorem 2, when restricted to the top degree, gives the Poincaré duality:

$$H^k(S; \mathcal{H}_0) = H^k(S; \mathcal{H}_0 \otimes \mathcal{S}^{(N)}) \cong H_{n-1-k}(S; \widehat{\Pi}^{(N)}) = H_{n-1-k}(S; \mathbb{k}).$$

The restriction of the spectral sequence in Theorem 1 to the top degree gives the Zeeman-McCroy spectral sequence in a similar way.

§ 4. Proof of Theorem 1

The idea of the proof is the following. We construct a filtered double differential complex $\mathcal{X}_{k,l}$ and then play with various spectral sequences converging to its total homology.

Before we proceed we need a small technical lemma. Let $J \in S$ be a simplex. If i is a vertex of J , we have a map $\eta_i: \Pi_i \hookrightarrow \mathcal{S}(J)$, which embeds the ideal Π_i generated by the linear form ω_i in the ideal $\mathcal{S}(J)$ generated by a larger set of linear forms. Consider the sequence of maps

$$0 \leftarrow \mathcal{S}(J) \xleftarrow{\eta} \bigoplus_{\substack{I, \dim I=0 \\ I \leq J}} \Pi_I \xleftarrow{\xi} \bigoplus_{\substack{I, \dim I=1 \\ I \leq J}} \Pi_I \xleftarrow{\xi} \bigoplus_{\substack{I, \dim I=2 \\ I \leq J}} \Pi_I \xleftarrow{\xi} \dots, \tag{4.1}$$

where η is the direct sum of the maps η_i over $i \in \text{Vert}(S)$, $i \leq J$; and ξ is the direct sum of the inclusion maps $\Pi_I \hookrightarrow \Pi_{I'}$, each rectified by the incidence sign $[I : I']$. The sign convention obviously implies that (4.1) is a differential complex. But what is more important,

Lemma 4. *The sequence (4.1) is exact.*

Proof. This is very similar to the Taylor resolution of the monomial ideal in a commutative polynomial ring (or the Koszul resolution), but our situation is a bit different, since the Π_I are not free modules over Λ . Nevertheless, the proof is similar to the commutative case: the exactness of (4.1) follows from the inclusion-exclusion principle. To make things precise (and also to tackle the case $\mathbb{k} = \mathbb{Z}$) we proceed as follows.

By the $(*_\mathbb{k})$ -condition, the subspace $\langle \omega_j \mid j \in J \rangle$ is a direct summand in $V \cong \mathbb{k}^N$. Let $\{\nu_1, \dots, \nu_N\}$ be a basis of V such that its first $|J|$ vectors are exactly ω_j , $j \in J$. We simply identify J with the subset $\{1, \dots, |J|\} \subseteq [N]$ by abuse of notation. The module $\Lambda[V]$ splits into multidegree components: $\Lambda = \bigoplus_{A \subseteq [N]} \Lambda_A$, where Λ_A is the one-dimensional \mathbb{k} -module generated by $\bigwedge_{i \in A} \nu_i$. All the modules and maps in (4.1) respect this splitting. Thus (4.1) can be written as

$$0 \longleftarrow \bigoplus_{A \cap J \neq \emptyset} \Lambda_A \longleftarrow \bigoplus_{I \subseteq J, |I|=1} \bigoplus_{A \supseteq I} \Lambda_A \longleftarrow \bigoplus_{I \subseteq J, |I|=2} \bigoplus_{A \supseteq I} \Lambda_A \longleftarrow \dots,$$

$$\bigoplus_{A, A \cap J \neq \emptyset} \left(0 \longleftarrow \Lambda_A \longleftarrow \bigoplus_{I \subseteq A \cap J, |I|=1} \Lambda_A \longleftarrow \bigoplus_{I \subseteq A \cap J, |I|=2} \Lambda_A \longleftarrow \dots \right).$$

For each A , the homology of the complex in brackets coincides with $\widetilde{H}_*(\Delta_{A \cap J}; \Lambda_A) \cong \widetilde{H}_*(\Delta_{A \cap J}; \mathbb{k})$, the reduced simplicial homology of the simplex on the set $A \cap J \neq \emptyset$. Thus the homology vanishes.

Let us define a cosheaf $\widehat{\mathcal{N}}$ on S taking values in graded differential complexes. We set $\widehat{\mathcal{N}}(I) = C_*(G_I; \Pi_I)$, the simplicial chains of the simplicial complex G_I . The corestriction maps $\widehat{\mathcal{N}}(I > J)$ are naturally induced by the inclusions of faces $G_I \hookrightarrow G_J$ and the inclusions of coefficient modules $\widehat{\Pi}(I > J): \Pi_I \hookrightarrow \Pi_J$.

The chain complex

$$\mathcal{X}_{*,*} = (C_*(S; \widehat{\mathcal{N}}_*); d_H), \quad \mathcal{X}_{k,l} = \bigoplus_{I, \dim I=k} C_l(G_I; \Pi_I)$$

is a double complex. It has the horizontal homological differential $d_H: \mathcal{X}_{k,l} \rightarrow \mathcal{X}_{k-1,l}$ (sheaf-differential) and the vertical differential $d_V: C_l(G_I; \Pi_I) \rightarrow C_{l-1}(G_I; \Pi_I)$ (inner differential). These differentials commute: $d_H d_V = d_V d_H$, so we can form the totalized differential complex

$$\mathcal{X}_j = \bigoplus_{k+l=j} \mathcal{X}_{k,l}, \quad d_{\text{Tot}} = d_H + (-1)^k d_V: \mathcal{X}_j \rightarrow \mathcal{X}_{j-1}.$$

Lemma 5. $H_k(\mathcal{X}, d_{\text{Tot}}) \cong H_k(S; \widehat{\Pi})$.

Proof. Consider the vertical spectral sequence [10] converging to $H_k(\mathcal{X}, d_{\text{Tot}})$:

$$(E_V)_{*,*}^r, \quad (d_V)_r: (E_V)_{k,l}^r \rightarrow (E_V)_{k-r,l+r-1}^r,$$

which first computes the vertical homology and then the horizontal. We have

$$(E_V)_{k,l}^1 = \bigoplus_{I, \dim I=k} H_l(G_I; \Pi_I).$$

Since G_I is contractible, $H_l(G_I; \Pi_I) = 0$ for $l \neq 0$ and $H_0(G_I; \Pi_I) = \Pi_I$. Thus

$$(E_V)_{k,l}^1 = \begin{cases} \bigoplus_{\dim I=k} \Pi_I = C_k(S; \widehat{\Pi}) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0, \end{cases}$$

$$(E_V)_{k,l}^2 = \begin{cases} H_k(S; \widehat{\Pi}) & \text{if } l = 0, \\ 0 & \text{if } l \neq 0. \end{cases}$$

The spectral sequence collapses at the second page, thus

$$H_k(\mathcal{X}, d_{\text{Tot}}) \cong H_k(S; \widehat{\Pi}).$$

Our next goal is to compute the homology of totalization by first computing the horizontal homology and then the vertical. Recall that G_I is a simplicial subcomplex of S' , so the module $C_*(G_I; \Pi_I)$ is regarded as the chain complex of the constant cosheaf Π_I . Let the cosheaf $\widehat{\mathcal{F}}^I$ be the corefinement of the sheaf \mathcal{F} defined in § 2.6.

Lemma 6. *The sequence*

$$0 \longleftarrow C_*(S'; \widehat{\mathcal{F}}^I) \longleftarrow \bigoplus_{I, \dim I=0} C_*(G_I; \Pi_I) \longleftarrow \bigoplus_{I, \dim I=1} C_*(G_I; \Pi_I) \longleftarrow \dots \quad (4.2)$$

is exact.

Proof. Since all the maps $C_*(G_I; \Pi_I) \rightarrow C_*(G_I; \Pi_I)$ are induced by inclusions of simplicial subcomplexes, the sequence (4.2) decomposes as the direct sum over all simplices $\Delta = (I_1 < \dots < I_k) \in S'$:

$$\bigoplus_{\Delta \in S'} \left(0 \longleftarrow \widehat{\mathcal{F}}^I(\Delta) \longleftarrow \bigoplus_{\substack{I, \dim I=0 \\ \Delta \in G_I}} \Pi_I \longleftarrow \bigoplus_{\substack{I, \dim I=1 \\ \Delta \in G_I}} \Pi_I \longleftarrow \dots \right).$$

Since the condition $\Delta \in G_I$ is equivalent to $I_1 \geq I$, by the definition of corefinement $\widehat{\mathcal{F}}^I$ the expression in brackets is equal to

$$0 \longleftarrow \mathcal{F}(I_1) \longleftarrow \bigoplus_{\substack{I, \dim I=0 \\ I \leq I_1}} \Pi_I \longleftarrow \bigoplus_{\substack{I, \dim I=1 \\ I \leq I_1}} \Pi_I \longleftarrow \dots.$$

This sequence is exact by Lemma 4.

We return to the double complex \mathcal{X} . Consider the horizontal spectral sequence

$$(E_H)_{*,*}^r \Rightarrow H_*(\mathcal{X}, d_{\text{Tot}}), \quad (d_H)_r: (E_H)_{k,l}^r \rightarrow (E_H)_{k+r-1,l-r}^r,$$

which computes the horizontal homology first and then the vertical.

Lemma 7. $H_l(\mathcal{X}, d_{\text{Tot}}) \cong H_l(S'; \widehat{\mathcal{F}}^I)$.

Proof. By Lemma 6 the horizontal homology of \mathcal{X} vanishes everywhere except in degree $k = 0$, where it is isomorphic to $C_*(S'; \widehat{\mathcal{F}}')$. Thus

$$(E_H)_{k,l}^2 \cong \begin{cases} H_l(S'; \widehat{\mathcal{F}}') & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases}$$

The spectral sequence collapses and the statement follows.

Finally, we make use of the coskeleton filtration on S' .

Lemma 8. *There exists a spectral sequence*

$$E_{s,k}^r \Rightarrow H_{s+k}(S'; \widehat{\mathcal{F}}'), \quad d^r : E_{s,k}^r \rightarrow E_{s-r,k+r-1}^r, \quad E_{s,k}^2 \cong H^{n-1-s}(S; \mathcal{H}_k \otimes \mathcal{I}).$$

This spectral sequence respects the inner gradings on \mathcal{I} and $\widehat{\mathcal{F}}'$.

Proof. Consider the spectral sequence associated with the coskeleton filtration of S' for the coefficient system $\widehat{\mathcal{F}}'$:

$$E_{s,k}^r \Rightarrow H_{s+k}(S'; \widehat{\mathcal{F}}'), \quad d^r : E_{s,k}^r \rightarrow E_{s-r,k+r-1}^r, \\ E_{s,k}^1 \cong H_{s+k}(S_s, S_{s-1}; \widehat{\mathcal{F}}').$$

We have

$$E_{s,k}^1 \cong H_{s+k}(S_s, S_{s-1}; \widehat{\mathcal{F}}') = \bigoplus_{I, \dim G_I = s} H_{s+k}(G_I, \partial G_I; \widehat{\mathcal{F}}') = \bigoplus_{I, \dim G_I = s} \overline{\mathcal{F}}_k(I)$$

by Lemma 2. Since the values of \mathcal{I} are torsion-free, Lemma 3 implies

$$\bigoplus_{I, \dim G_I = s} \overline{\mathcal{F}}_k(I) \cong \bigoplus_{I, \dim G_I = s} (\mathcal{I} \otimes \mathcal{H}_k)(I) = C^{n-1-s}(S; \mathcal{I} \otimes \mathcal{H}_k).$$

Therefore, $E_{s,k}^2 \cong H^{n-1-s}(S; \mathcal{I} \otimes \mathcal{H}_k)$, which proves the statement.

The combination of Lemmas 5, 7 and 8 proves Theorem 1.

§ 5. Manifolds with locally standard torus actions

5.1. Orbit spaces. Let T^n be a compact n -dimensional torus. The *standard representation* of T^n is a representation of T^n on \mathbb{C}^n by coordinate-wise rotations, that is,

$$(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n),$$

for $z_i, t_i \in \mathbb{C}$, $|t_i| = 1$. An action of T^n on a (compact connected smooth) manifold M^{2n} is called *locally standard* if M has an atlas of standard charts, each isomorphic to a subset of the standard representation. More precisely, a *standard chart* on M is a triple (U, f, ψ) , where $U \subset M$ is a T^n -invariant open subset, ψ is an automorphism of T^n , and f is a ψ -equivariant homeomorphism $f : U \rightarrow W$ onto a T^n -invariant open subset $W \subset \mathbb{C}^n$ (that is, $f(t \cdot y) = \psi(t) \cdot f(y)$ for all $t \in T^n$, $y \in U$).

The orbit space \mathbb{C}^n/T^n of the standard representation is the nonnegative cone $\mathbb{R}_{\geq}^n = \{x \in \mathbb{R}^n \mid x_i \geq 0\}$. Therefore, the orbit space of a locally standard action

obtains the structure of a compact connected n -dimensional manifold with corners. Recall that a manifold with corners is a topological space locally modeled by open subsets of \mathbb{R}_{\geq}^n with the combinatorial stratification induced from the face structure of \mathbb{R}_{\geq}^n (the reader is referred to [5] or [15] for details relevant to the study of torus actions).

5.2. Characteristic functions. Let $Q = M/T^n$ be the orbit space of a locally standard action. Let $\text{Fac}(Q)$ denote the set of facets (that is, faces of codimension 1). Every face F of codimension k lies in exactly k distinct facets of Q (such manifolds with corners are called *nice* in [9] or *manifolds with faces* elsewhere). Consider the set S_Q of all faces of Q , including Q itself, and define the order on S_Q by reversed inclusion. Since Q is nice, S_Q is a simplicial poset. The minimal element of S_Q is the maximal face, that is, the space Q itself. The facets of Q correspond to the vertices of S_Q . For convenience we denote abstract elements of S_Q by I, J and so on, and the corresponding faces of Q will be denoted by F_I, F_J and so on.

If $F \in \text{Fac}(Q)$ and x is a point in the interior of F , then the stabilizer of x , denoted by $\lambda(F)$, is a one-dimensional toric subgroup in T^n . If F_I is a codimension- k face of Q contained in facets $F_1, \dots, F_k \in \text{Fac}(Q)$, then the stabilizer of an orbit $x \in F_I^\circ$ is the k -dimensional torus $T_I = \lambda(F_1) \times \dots \times \lambda(F_k) \subset T^n$, where the product is free inside T^n . This puts a specific restriction on the subgroups $\lambda(F)$, $F \in \text{Fac}(Q)$. In general, the map

$$\lambda: \text{Fac}(Q) \rightarrow \{\text{one-dimensional toric subgroups of } T^n\} \tag{5.1}$$

is called a *characteristic function* if, whenever the facets F_1, \dots, F_k have nonempty intersection, the map

$$\lambda(F_1) \times \dots \times \lambda(F_k) \rightarrow T^n,$$

induced by inclusions $\lambda(F_i) \hookrightarrow T^n$, is injective and splits. This is called the *(*)-condition*. Notice that F_1, \dots, F_k have nonempty intersection whenever the corresponding vertices of S_Q are the vertices of some simplex.

From the *(*)-condition* it follows that the map

$$H_1(\lambda(F_1) \times \dots \times \lambda(F_k); \mathbb{k}) \rightarrow H_1(T^n; \mathbb{k}) \tag{5.2}$$

is also injective and splits for any ground ring \mathbb{k} . Thus the homology classes $\omega_1, \dots, \omega_k$ of the subgroups $\lambda(F_1), \dots, \lambda(F_k)$ span freely a direct summand in $H_1(T^n; \mathbb{k})$. This motivates the definition of homological characteristic function given in §3. Surely, the exterior algebra $\Lambda[V]$ generated by a \mathbb{k} -module V has a clear meaning as the whole homology algebra of a torus: $\Lambda[H_1(T^n; \mathbb{k})] \cong H_*(T^n; \mathbb{k})$.

If the function (5.1) satisfies (5.2) for some particular ground ring \mathbb{k} , we say that λ *satisfies the $(*_\mathbb{k})$ -condition*. It is easy to see that the topological *(*)-condition* is equivalent to the *(*_ \mathbb{Z})-condition*, and that the *(*_ \mathbb{Z})-condition* implies the *(*_ \mathbb{k})-condition* for any \mathbb{k} .

5.3. Model spaces. Let M be a manifold with a locally standard action and $\mu: M \rightarrow Q$ be the projection onto the orbit space. The free part of the action has the form $\mu|_{Q^\circ}: \mu^{-1}(Q^\circ) \rightarrow Q^\circ$, where $Q^\circ = Q \setminus \partial Q$ is the interior of the manifold with corners. The free part is a principal torus bundle over Q° . It can be uniquely extended over Q and defines a principal T^n -bundle $\rho: Y \rightarrow Q$.

Therefore any manifold with a locally standard action determines three objects: the nice manifold with corners Q , the principal torus bundle $\rho: Y \rightarrow Q$, and the characteristic function λ . One can recover the manifold M from these data using the following standard construction.

Construction 5 (model space). Let $\rho: Y \rightarrow Q$ be a principal T^n -bundle over a nice manifold with corners Q and λ be a characteristic function on $\text{Fac}(Q)$. Consider the space $X \stackrel{\text{def}}{=} Y/\sim$, where $y_1 \sim y_2$ if and only if $\rho(y_1) = \rho(y_2) \in F_I^\circ$ for some face F_I of Q and y_1 and y_2 lie in the same T_I -orbit of the action. Then there exists a natural T^n -equivariant map $f: Y \rightarrow X$.

Every manifold with a locally standard torus action is equivariantly homeomorphic to its model space (see [15], Corollary 2), so in the following we will work with X instead of M .

5.4. Filtrations. Since Q is a manifold with corners, there is a natural filtration on Q :

$$\emptyset = Q_{-1} \subset Q_0 \subset Q_1 \subset \cdots \subset Q_{n-1} = \partial Q \subset Q = Q_n, \tag{5.3}$$

where Q_i is the union of all faces of dimension $\leq i$. It lifts to the T^n -invariant filtration on Y :

$$\emptyset = Y_{-1} \subset Y_0 \subset Y_1 \subset \cdots \subset Y_{n-1} \subset Y_n = Y, \tag{5.4}$$

where $Y_i = \rho^{-1}(Q_i)$. In turn, this descends to the filtration on X :

$$\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X, \tag{5.5}$$

$X_i = f(Y_i)$. It is easily seen that (5.5) is the filtration of X by orbit types, that is, X_i is the union of all orbits of dimension at most i . We have $\dim X_i = 2i$. The maps $\mu: X \rightarrow Q$, $\rho: Y \rightarrow Q$ and $f: Y \rightarrow X$ preserve the filtrations.

The filtrations give rise to homological spectral sequences:

$$\begin{aligned} (E_Q)^1_{p,q} &= H_{p+q}(Q_p, Q_{p-1}) \Rightarrow H_{p+q}(Q), & (d_Q)^r &: (E_Q)^r_{*,*} \rightarrow (E_Q)^r_{*-r, *+r-1}, \\ (E_Y)^1_{p,q} &\cong H_{p+q}(Y_p, Y_{p-1}) \Rightarrow H_{p+q}(Y), & (d_Y)^r &: (E_Y)^r_{*,*} \rightarrow (E_Y)^r_{*-r, *+r-1}, \\ (E_X)^1_{p,q} &\cong H_{p+q}(X_p, X_{p-1}) \Rightarrow H_{p+q}(X), & (d_X)^r &: (E_X)^r_{*,*} \rightarrow (E_X)^r_{*-r, *+r-1}. \end{aligned}$$

In the following we also need the spectral sequence associated to the filtration of Q truncated at $Q_{n-1} = \partial Q$:

$$\begin{aligned} \emptyset &= Q_{-1} \subset Q_0 \subset Q_1 \subset \cdots \subset Q_{n-1} = \partial Q, & (5.6) \\ (E_{\partial Q})^1_{p,q} &= \begin{cases} H_{p+q}(Q_p, Q_{p-1}) & \text{for } p < n, \\ 0 & \text{for } p = n \end{cases} \Rightarrow H_{p+q}(\partial Q). \end{aligned}$$

Note that $(E_X)^1_{p,q} = 0$ for $q > p$ for reasons of dimension. The map $f: Y \rightarrow X$ induces the map of spectral sequences

$$f_*^r: (E_Y)^r_{p,q} \rightarrow (E_X)^r_{p,q}.$$

The main topological result in this paper is the following.

Theorem 3. *If Q is orientable and all proper faces of Q are acyclic over \mathbb{k} , then the map $f_*^2: (E_Y)^2_{p,q} \rightarrow (E_X)^2_{p,q}$ is an isomorphism for $q < p$ or $q = p = n$ and is injective for $q = p < n$.*

§ 6. Proof of Theorem 3

First we prove a technical lemma, which is extremely useful when from the topology of Q we go over to the topology of its underlying simplicial poset S_Q .

For a poset S consider the space $P = \text{Cone}|S|$. The coskeleton filtration of S extends to the coskeleton filtration of P :

$$|S_0| \subset \cdots \subset |S_{n-1}| = |S| \subset P$$

and the corresponding homological spectral sequence is denoted by $(E_P)_{*,*}^*$.

For convenience we introduce the following definition.

Definition 5. An oriented manifold with corners Q is called *Buchsbaum* if all its proper faces are acyclic over \mathbb{k} . If Q is Buchsbaum and Q itself is acyclic over \mathbb{k} , then Q is called *Cohen-Macaulay*.

Each face G of a Buchsbaum manifold with corners Q is an orientable manifold with corners. The acyclicity of G implies that $H_j(G, \partial G) = 0$ for $j \neq \dim G$ and $H_{\dim G}(G, \partial G) \cong \mathbb{k}$ by Poincaré-Lefschetz duality.

Lemma 9. (1) _{n} Let Q be a Buchsbaum manifold with corners, $\dim Q = n$, S_Q be its underlying poset, and let $P = \text{Cone}(|S_Q|)$. Then there exists a face-preserving map $\varphi: Q \rightarrow P$ which induces the identity isomorphism on the posets of faces and an isomorphism of spectral sequences $\varphi_*: (E_{\partial Q})_{*,*}^r \xrightarrow{\cong} (E_S)_{*,*}^r$ for $r \geq 1$.

(2) _{n} If Q is Cohen-Macaulay of dimension n , then φ induces an isomorphism of spectral sequences $\varphi_*: (E_Q)_{*,*}^r \xrightarrow{\cong} (E_P)_{*,*}^r$ for $r \geq 1$.

Proof. The map φ is constructed inductively. The 0-skeleta of Q and P are naturally identified since both correspond to the set of maximal simplices of S . There always exists an extension of φ to higher-dimensional faces since all faces of P are cones. The statement is proved using the following scheme of induction: (2) _{$\leq n-1$} \Rightarrow (1) _{n} \Rightarrow (2) _{n} . The case $n = 0$ is clear.

We prove the implication (1) _{n} \Rightarrow (2) _{n} . The map φ induces the homomorphism of the long exact sequences:

$$\begin{array}{ccccccccc} \tilde{H}_*(\partial Q) & \longrightarrow & \tilde{H}_*(Q) & \longrightarrow & H_*(Q, \partial Q) & \longrightarrow & \tilde{H}_{*-1}(\partial Q) & \longrightarrow & \tilde{H}_{*-1}(Q) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \tilde{H}_*(\partial P) & \longrightarrow & \tilde{H}_*(P) & \longrightarrow & H_*(P, \partial P) & \longrightarrow & \tilde{H}_{*-1}(\partial P) & \longrightarrow & \tilde{H}_{*-1}(P) \end{array}$$

The maps $\tilde{H}_*(Q) \rightarrow \tilde{H}_*(P)$ are isomorphisms since both groups are trivial. The maps $\tilde{H}_*(\partial Q) \rightarrow \tilde{H}_*(\partial P)$ are isomorphisms since $(E_{\partial Q}) \xrightarrow{\cong} H_*(\partial Q)$, $(E_{\partial P}) \xrightarrow{\cong} H_*(\partial P)$ and the spectral sequences are isomorphic by (1) _{n} . The Five Lemma shows that $\varphi_*: (E_Q)_{n,*}^1 \rightarrow (E_P)_{n,*}^1$ is an isomorphism as well. This proves (2) _{n} .

Now we prove the implication (2) _{$\leq n-1$} \Rightarrow (1) _{n} . Let F_I be faces of Q and G_I be faces of P . All proper faces of Q are Cohen-Macaulay of dimension $\leq n - 1$. Thus (2) _{$\leq n-1$} implies the isomorphisms $H_*(F_I, \partial F_I) \rightarrow H_*(G_I, \partial G_I)$ which sum together to the isomorphism $\varphi_*: (E_{\partial Q})_{*,*}^1 \xrightarrow{\cong} (E_{\partial P})_{*,*}^1$.

Corollary 4. *If Q is a Buchsbaum manifold then S_Q is Buchsbaum. Moreover, in this case S_Q is a homology manifold. If Q is Cohen-Macaulay, then S_Q is a homology sphere.*

From now on we suppose that Q is Buchsbaum, as stated in the hypothesis of Theorem 3. Thus S_Q is also Buchsbaum.

We return to the spaces Y and X over Q . As before, let F_I be the face of Q corresponding to $I \in S_Q$. Let $Y_I = \rho^{-1}(F_I)$ and $X_I = f(Y_I)$ be the corresponding subsets of Y and X , respectively. In fact, $X_I \subset X$ is a closed submanifold of dimension $2 \dim F_I$, called a face submanifold. We set $\partial Y_I = \rho^{-1}(\partial F_I)$ and $\partial X_I = f(\partial Y_I)$ (the set ∂X_I does not have the meaning of a boundary in the topological sense, this is just conventional notation). Note that $Y_{\hat{0}} = Y$ and $X_{\hat{0}} = X$. We have

$$(E_Y)_{p,q}^1 \cong H_{p+q}(Y_p, Y_{p-1}) \cong \bigoplus_{|I|=n-p} H_{p+q}(Y_I, \partial Y_I),$$

$$(E_X)_{p,q}^1 \cong H_{p+q}(X_p, X_{p-1}) \cong \bigoplus_{|I|=n-p} H_{p+q}(X_I, \partial X_I).$$

Remark 7. The map $f_*^1: (E_Y)_{n,q}^1 \rightarrow (E_X)_{n,q}^1$, which coincides with $f_*: H_*(Y, \partial Y) \rightarrow H_*(X, \partial X)$, is an isomorphism since the identification \sim in Construction 5 touches only the boundary ∂Y , thus $Y/\partial Y \cong X/\partial X$.

The space Y_I is a principal T^n -bundle over Q_I . For each $I \in S \setminus \hat{0}$ the face Q_I is acyclic. Thus there exists a trivialization $Y_I \cong Q_I \times T^n$ and we have

$$H_{p+q}(Y_I, \partial Y_I) \cong \bigoplus_{i+j=p+q} H_i(F_I, \partial F_I) \otimes H_j(T^n) \cong H_q(T^n) \tag{6.1}$$

(the groups $H_i(F_I, \partial F_I)$ vanish for $i \neq p$, and $H_p(F_I, \partial F_I) \cong \mathbb{k}$). Similarly, for X we have the identification

$$H_*(X_I, \partial X_I) \cong H_*(F_I \times T^n/T_I, \partial F_I \times T^n/T_I),$$

thus

$$H_{p+q}(X_I, \partial X_I) \cong H_q(T^n/T_I). \tag{6.2}$$

Consider the graded sheaf \mathcal{H}_q^Y on S_Q that takes the value $H_{p+q}(Y_I, \partial Y_I)$ on each $I \in S_Q$ (including $I = \hat{0}$), with the restriction maps extracted from the differential $(d_Y)^1$ similarly to Construction 2. By (6.1), the truncated part $\mathcal{H}_*^Y = \bigoplus_q \mathcal{H}_q^Y$ (see Remark 3) is the locally constant sheaf \mathcal{L} valued by exterior algebras.

Similarly, we can define a graded sheaf \mathcal{H}_q^X on S_Q that takes the value $H_{p+q}(X_I, \partial X_I)$ on $I \in S$. Its truncated part $\mathcal{H}_*^X = \bigoplus_q \mathcal{H}_q^X$ is the sheaf of quotient algebras \mathcal{L}/\mathcal{I} according to (6.2). Indeed, it is easily seen that the homology algebra $H_*(T^n/T_I)$ is naturally isomorphic to the quotient of $H_*(T^n)/\mathcal{I}(I)$, where $\mathcal{I}(I)$ is the ideal generated by the subspace $H_1(T_I) \subset H_1(T^n)$.

The map $f_*^1: (E_Y)_{*,*}^1 \rightarrow (E_X)_{*,*}^1$ is equal to $f_*: C^*(S; \mathcal{H}_*^Y) \rightarrow C^*(S; \mathcal{H}_*^X)$. This last map coincides with $f_*: C^*(S; \mathcal{L}) \rightarrow C^*(S; \mathcal{L}/\mathcal{I})$ away from $\hat{0}$.

Lemma 10. *There exists a short exact sequence of graded sheaves*

$$0 \rightarrow \mathcal{I} \rightarrow \mathcal{H}^Y \rightarrow \mathcal{H}^X \rightarrow 0.$$

Proof. This follows from the diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{I} \hookrightarrow & \mathcal{H}^Y & \longrightarrow & \mathcal{H}^X & \longrightarrow 0 \\
 & & \parallel & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \mathcal{I} \hookrightarrow & \mathcal{H}^Y & \longrightarrow & \mathcal{H}^X & \longrightarrow 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & \mathcal{H}^Y / \mathcal{H}^Y & \xrightarrow{\cong} & \mathcal{H}^X / \mathcal{H}^X & \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 &
 \end{array}$$

in which all the vertical and two horizontal lines are exact. The lower sheaves are concentrated in $\widehat{0} \in S_Q$ and the graded isomorphism between them is due to Remark 7.

Finally, the short exact sequence of Lemma 10 induces the long exact sequence in sheaf cohomology:

$$\begin{aligned}
 \dots \rightarrow H^{i-1}(S_Q; \mathcal{I}^{(q)}) &\rightarrow H^{i-1}(S_Q; \mathcal{H}_q^Y) \\
 \xrightarrow{f_*^2} H^{i-1}(S_Q; \mathcal{H}_q^X) &\longrightarrow H^i(S_Q; \mathcal{I}^{(q)}) \rightarrow \dots
 \end{aligned} \tag{6.3}$$

The poset S_Q is a homology manifold. Thus its structure sheaf is constant: $\mathcal{H}_0 \cong \mathbb{k}$. Corollary 3 implies that the groups $H^i(S_Q; \mathcal{I}^{(q)})$ vanish for $i \leq n - 1 - q$. From the long exact sequence (6.3) we can see that the map

$$f_*: H^{i-1}(S_Q; \mathcal{H}_q^Y) \rightarrow H^{i-1}(S_Q; \mathcal{H}_q^X)$$

is an isomorphism for $i \leq n - 1 - q$ and is injective for $i = n - q$. This map coincides with

$$f_*^2: (E_Y)_{n-i,q}^2 \rightarrow (E_X)_{n-i,q}^2.$$

The change of indices $p = n - i$ completes the proof of Theorem 3.

Remark 8. Note that a similar argument proves that the map $f_*: (E_{\partial Y})_{p,q}^2 \rightarrow (E_{\partial X})_{p,q}^2$ is an isomorphism for $p > q$ and injective for $p = q$.

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