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Discrete Morse theory for the moduli spaces of polygonal linkages, or solitaire on a circle

A. M. Zhukova and G. Yu. Panina

Abstract. We construct an exact discrete Morse function on the moduli space of a planar polygonal linkage. A cellular structure on the moduli space is used, and the number of cells is minimised by employing discrete Morse theory.

Bibliography: 12 entries.

Keywords: polygonal linkage, configuration space, cell complex, discrete vector field, exact Morse function.

§ 1. Introduction

A Morse function on a smooth manifold is *exact* if the number of its critical points equals the sum of the Betti numbers of the manifold. Similarly, a discrete Morse function on a cellular complex is *exact* if the number of its critical cells equals the sum of the Betti numbers of the complex[1]. An exact Morse function (smooth or discrete) is optimal in the following sense: all Morse inequalities turn to equalities, critical points (or critical cells in the discrete case) represent independent generators of the homology groups, and therefore the number of critical points (cells) is the least possible. Not every manifold admits an exact Morse function. Even if such a function exists, it can be difficult to find. In the discrete case this problem is NP-complete (see [1] and [2]).

In this paper we give an explicit construction of an exact Morse function on the moduli space of a polygonal linkage.

The starting point of our construction is a cellular decomposition of the moduli space, similar to that described in [3]. Initially, the number of cells in this decomposition is much larger than the sum of Betti numbers. Following Forman we construct a discrete Morse function on our cell complex and prove its exactness. This allows us to contract certain cells so that the number of remaining cells becomes the least possible. These manipulations with cells and the description of gradient paths resemble the card game of solitaire.

An exact Morse function is constructed in two stages. First, we define a natural pairing on the cell complex, which decreases the number of critical cells considerably,

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[1] In this paper we always consider homology with coefficients in $\mathbb{Z}$.

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although not to the minimal possible value. At the second stage we apply the technique of reversal of paths (again, following Forman), which changes the Morse function by decreasing the number of critical cells. This technique is a discrete analogue of the first cancellation theorem of Milnor and Smale; see [4]. Although in the original technique paths were reversed one at a time, we provide a modification in which several paths are reversed simultaneously, and use it to produce an exact Morse function. This idea of simultaneous reversal of paths is not new: it appeared in Hersh [5] in a different context.

Our approach can be used to calculate the homology groups of the moduli space of the linkage; the resulting method is independent of [6] by Farber and Schütz. However, such a calculation becomes quite involved.

There are no known examples of smooth exact Morse functions on the moduli space of a polygonal linkage. It is therefore natural to ask for a smooth analogue of the discrete Morse function constructed in this paper. Note that we are not interested in an existence theorem, but rather in a smooth Morse function itself, preferably expressed by a short explicit formula with a physical or geometric meaning.

Another interesting question would be to relate our construction to the works [5] and [7] by Babson and Hersh.

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§ 2. Definitions

2.1. Polygonal linkage: moduli space and cell complex. An *n-gonal linkage* is a sequence of positive numbers $L = (l_1, \dots, l_n)$. It should be interpreted as a collection of rigid bars linked to each other by revolute joints in a cyclic order. For such a cyclic chain of bars to exist, we impose the following inequalities on the lengths:

$$l_j < \frac{1}{2}\sum_{i=1}^{n} l_i \quad \forall\, j.$$

A *planar configuration* of the polygonal linkage L is a sequence of points (the vertices of the polygon)

$$P = (p_1, \dots, p_n), \qquad p_i \in \mathbb{R}^2,$$

satisfying $l_i = |p_i, p_{i+1}|$ and $l_n = |p_n, p_1|$. The edges of the polygon may intersect each other.

Definition 1. The *moduli space*, or the *configuration space*, $M(L)$ of a polygonal linkage L is the quotient space of all its configurations by the action of the group of orientation-preserving isometries of the plane.

In this paper we only consider generic polygonal linkages L, that is, those without configurations lying on a straight line. Under this assumption, the moduli space $M(L)$ is a closed smooth manifold of dimension $n-3$ (see [8]).

Definition 2. The *moduli space* $M(L)$ can also be defined as follows:

$$M(L) = \left\{(u_1, \dots, u_n) \in (S^1)^n \colon \sum_{i=1}^{n} l_i u_i = 0\right\} / \operatorname{SO}(2).$$

The equivalence of Definitions 1 and 2 implies that the diffeomorphism type of the manifold $M(L)$ does not depend on the order of the lengths $\{l_1, \dots, l_n\}$. Therefore, we may assume that

$$l_n \geqslant l_{n-1} \geqslant \dots \geqslant l_1.$$

Manifolds $M(L)$ arise naturally in topological robotics and are pretty well understood. The homology groups of $M(L)$ were calculated by Farber and Schütz in [6]. A subset I of the set $[n] = \{1, 2, \dots, n\}$ is called *short* if

$$\sum_{i \in I} l_i < \frac{1}{2} \sum_{i=1}^{n} l_i.$$

Theorem 1 (see [6]). *Let a_k be the number of short subsets of cardinality $k+1$ containing the element n (note that l_n is the maximal length of an edge). Then the homology group $H_k(M(L); Z)$ is free abelian of rank*

$$a_k + a_{n-3-k},$$

for any $k = 0, 1, \dots, n-3$.

According to Walker's conjecture, the edge lengths l_i are determined by the cohomology ring of $M(L)$; for a discussion, see [9]. The manifolds $M(L)$ were studied by smooth Morse-theoretic methods in [8] and [10], although the existence of an exact Morse function was not established there.

The regular cell decomposition of $M(L)$ from [3] is described next, after a couple more definitions.

A partition of the set $[n] = \{1, 2, \dots, n\}$ into subsets is called *admissible* if all its subsets are short.

In a partition of $[n] = \{1, 2, \dots, n\}$, the subset containing n is referred to as the *n-set*.

2.2. A remark on encoding cyclic partitions. We shall encode cyclicly ordered partitions of the set $[n]$ by (linearly ordered) sequences of subsets with the n-set at the last position. It is the order of subsets which matters, not the order of elements inside subsets. For instance,

$$(\{1\}\ \{3\}\ \{4,2,5,6\}) \neq (\{3\}\ \{1\}\ \{4,2,5,6\}) = (\{3\}\ \{1\}\ \{2,4,5,6\}).$$

We recall the definition of a regular CW-complex. The construction starts from the 0-skeleton, which is a finite set of points. Once the $(k-1)$-skeleton of the complex is defined, the k-skeleton is obtained by attaching several k-dimensional balls C_i by continuous maps φ_i from the boundaries ∂C_i to the $(k-1)$-skeleton. In a regular complex, each φ_i maps ∂C_i injectively to a subcomplex of the $(k-1)$-skeleton. A regular CW-complex is uniquely determined by its cell poset. Regularity also implies the existence of the barycentric subdivision and, in the case of manifolds, the existence of the dual complex.

Theorem 2. *The moduli space $M(L)$ has a regular* CW*-complex decomposition $\mathscr{K}(L)$. Its complete combinatorial description is as follows:*

1) *k-dimensional cells (k-cells) of $\mathscr{K}(L)$ are labelled by cyclicly ordered admissible partitions of $[n] = \{1, 2, \dots, n\}$ into $n-k$ nonempty subsets;*
2) *a cell C belongs to the boundary of another closed cell C' if and only if the label $\lambda(C)$ is a partition of the label $\lambda(C')$.*

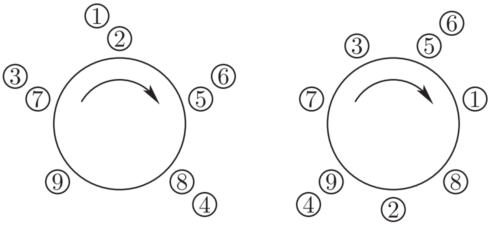

Figure 1. Examples of labels. The arrows denote the orientation of the circle.

An example of labels of a 4-cell and a 2-cell is shown in Figure 1. Following our convention, we write these labels in the following form: $\big(\{3,7\}\ \{1,2\}\ \{5,6\}\ \{4,8\}\ \{9\}\big)$ and $\big(\{7\}\ \{3\}\ \{5,6\}\ \{1\}\ \{8\}\ \{2\}\ \{4,9\}\big)$.

In what follows we identify cells with their labels.

Given a cell α, every facet of α corresponds to a partition of an entry of α into two nonempty ordered parts. For instance, each of the cells

$$\big(\{7\}\ \{3\}\ \{1,2\}\ \{5,6\}\ \{4,8\}\ \{9\}\big) \quad \text{and} \quad \big(\{3\}\ \{7\}\ \{1,2\}\ \{5,6\}\ \{4,8\}\ \{9\}\big)$$

is a facet of the cell $\big(\{3,7\}\ \{1,2\}\ \{5,6\}\ \{4,8\}\ \{9\}\big)$.

This cell structure arises from the following considerations. We assign labels to points in the configuration space. By Definition 2, each configuration corresponds to a set of unit vectors $\{u_i\}$. If all these vectors are distinct, then they define a cyclic ordering of the set $[n]$. If some vectors coincide, then we obtain a cyclicly ordered partition of $[n]$ into parts corresponding to the sets of coinciding vectors. The triangle inequality implies that all these sets are short. Therefore, all admissible partitions appear in this way.

We introduce an equivalence relation as follows: two points in $M(L)$ (that is, two configurations) are equivalent if they have the same labels. The resulting equivalence classes of $M(L)$ are open cells. The closure of an open cell is a closed cell; it is homeomorphic to a ball. For an open cell C, the label $\lambda(C)$ is defined as the label of its interior point. The set of open cells obtained in this way defines a regular CW-decomposition dual to the above complex $\mathscr{K}(L)$.

2.3. Discrete Morse functions on a regular cell complex. Here we give some basic definitions from discrete Morse theory; the details can be found in [11]. Assume given a regular cell complex. We denote its p-dimensional cells (p-cells) by α^p or β^p.

A *discrete vector field* on a cell complex is a collection of pairs of cells $\big(\alpha^p, \beta^{p+1}\big)$ satisfying the following conditions:

1) each cell belongs to at most one pair;
2) in each pair (α^p, β^{p+1}), the cell α^p is a facet of β^{p+1}.

A *gradient path* of a discrete vector field is a sequence of cells

$$\alpha_0^p,\ \beta_0^{p+1},\ \alpha_1^p,\ \beta_1^{p+1},\ \alpha_2^p,\ \beta_2^{p+1},\dots,\alpha_m^p,\ \beta_m^{p+1},\ \alpha_{m+1}^p$$

satisfying the conditions

1) all pairs $\left(\alpha_i^p,\ \beta_i^{p+1}\right)$ belong to the vector field;
2) α_i^p is a facet of β_{i-1}^{p+1} for any i;
3) $\alpha_i \neq \alpha_{i+1}$ for any i.

A gradient path is *closed* if $\alpha_{m+1}^p = \alpha_0^p$. A *discrete Morse function on a regular cell complex* is a discrete vector field without closed gradient paths.

Critical cells of a discrete Morse function are cells which are unpaired, that is, not contained in the pairs. Morse inequalities imply that critical cells always exist; our task is to minimise their number.

A *gradient path* of a discrete Morse function *from a critical cell β^{p+1} to a critical cell α^p* is a path

$$\beta^{p+1} = \beta_0^{p+1},\ \alpha_1^p,\ \beta_1^{p+1},\ \alpha_2^p,\ \beta_2^{p+1},\ \alpha_3^p,\ \beta_3^{p+1},\ \dots,\ \alpha_m^p,\ \beta_m^{p+1},\ \alpha_{m+1}^p = \alpha^p$$

satisfying the above conditions 1)–3).

A discrete Morse function is *exact* if the number of its k-dimensional critical cells equals the kth Betti number of the complex. This is equivalent to the condition that the total number of critical cells equals the sum of the Betti numbers.

§ 3. Pairing on $\mathscr{K}$

We introduce the following notation:

1) the symbol $\cdots$ denotes a (possibly empty) admissible sequence of nonintersecting subsets of $[n]$;
2) the symbol $*$ denotes any (possibly empty) subset of $[n]$;
3) a subset $I \subset [n]$ is called *k-pre-long* if I is short, but $I \cup \{k\}$ is long;
4) given a subset $I \subset [n]$ and $k \in [n]$, we write $k < I$ if $k < i$ for any $i \in I$;
5) given a subset $I \subset [n]$, we write $k = \mathrm{Min}(I)$ if k is the minimal element of I.

A discrete Morse function is defined by means of the following inductive procedure.

Step 1. We pair cells

$$\alpha = \left(\cdots\ \{1\}\ I\ \cdots\right) \quad \text{and} \quad \beta = \left(\cdots\ \{1\} \cup I\ \cdots\right)$$

which satisfy the following conditions:

1) $n \notin I$;
2) $\{1\} \cup I$ is a short set.

After performing all pairings at Step 1, cells which remain unpaired have one of the following forms:

$$\left(\cdots\ \{n, 1, *\}\right),$$
$$\left(\cdots\ \{1\}\ \{n, *\}\right),$$
$$\left(\cdots\ \{1\}\ \{1\text{-pre-long set}\}\ \cdots\right).$$

Step 2. We pair cells

$$\alpha = \big(\cdots\ \{2\}\ I\ \cdots\big) \quad \text{and} \quad \beta = \big(\cdots\ \{2\} \cup I\ \cdots\big)$$

satisfying the conditions

1) $n \notin I$, $1 \notin I$;
2) $\{2\} \cup I$ is a short set;
3) α and β were not paired at Step 1.

We proceed similarly for all $k < n$:

Step k. We pair cells

$$\alpha = \big(\cdots\ \{k\}\ I\ \cdots\big) \quad \text{and} \quad \beta = \big(\cdots\ \{k\} \cup I\ \cdots\big)$$

satisfying the conditions

1) $n \notin I$, $1 \notin I$, $2 \notin I$, $\ldots$, $(k-1) \notin I$;
2) $\{k\} \cup I$ is a short cell;
3) α and β were not paired at the previous steps.

Examples of pairings are given in Figure 2. Let $n = 9$, and assume that all sets in the figure are short. The first pair is formed at Step 1, and the second at Step 2. The two bottom cells are unpaired: according to our rules no element merges to the n-set after pairing.

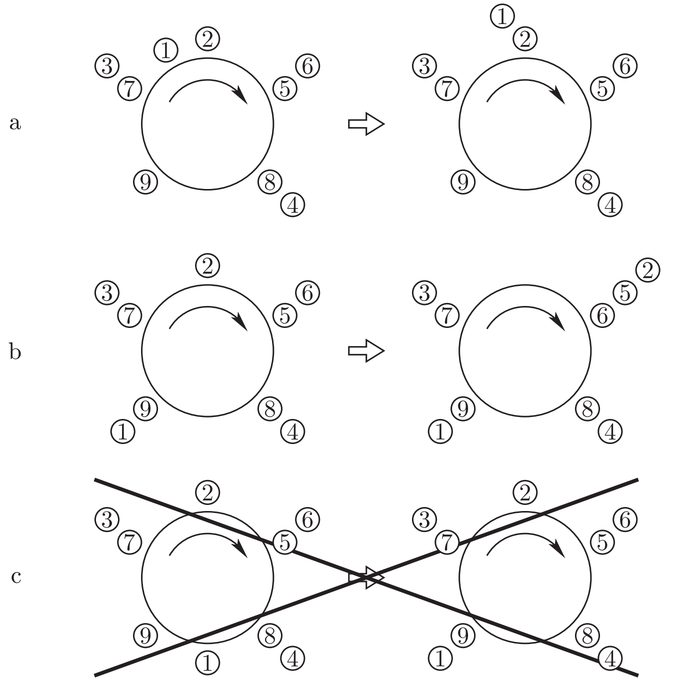

Figure 2. Two examples of paired cells (a and b), and two cells which are unpaired (c).

Search for a pair. Here we give a simple description of a cell paired with a given cell α (provided such a pairing exists). This description will be used to find gradient paths for the discrete Morse function being constructed.

Note that two cells in a pair differ by the position of just one element of $[n]$ and can be obtained from each other by separating this element from a set or (in the reverse direction) by adding this element to a set. Furthermore, paired cells have the same n-sets.

An element $k \neq n$ is called *movable forward* in a cell α if $\{k\}$ is a singleton in the partition α that is followed by a set I satisfying the conditions

1) $k < I$;
2) $n \notin I$;
3) the set $\{k\} \cup I$ is short.

An element k is called *movable backwards* in a cell α if it is contained in a set J, $|J| > 1$ and the following holds:

1) $n \notin J$;
2) $k = \mathrm{Min}(J)$;
3) one of the following conditions is satisfied:
   (a) the set preceding J is not a singleton,
   (b) J is preceded by a singleton $\{m\}$ with $m > k$,
   (c) J is preceded by the n-set.

Using this notation, the pairing is as follows. Let k be the minimal movable element in a cell α. Then α is paired at Step k with the cell obtained from α by moving k in an appropriate direction.

The description above implies the next proposition, which can informally be restated as follows: in the motion along a gradient path 'small' elements move forward, while 'large' elements move backwards.

Proposition 1. 1. *Assume given a gradient path for the vector field constructed above. Suppose that $m < k$ and a cell*

$$\alpha = \bigl(\cdots\ \{k,*\}\ \cdots\ \{m,*\}\ \cdots\bigr)$$

belongs to the path (that is, the elements k and m lie in distinct subsets, and k lies to the left of m). Then no cell following α in the path has k and m in the same subset or in subsets going in the reverse order.

2. *The discrete vector field constructed above is a discrete Morse function.*

Proof. Statement 1 follows from the description of the pairing.

2. In a closed path, at least two elements of $[n]$ are interchanged twice, which contradicts statement 1.

The proposition is proved.

§ 4. Critical cells of $\mathscr{K}$

Here we describe all the critical (that is, unpaired) cells in the complex. These are precisely the cells containing no movable elements.

Notation: in contrast to $\cdots$, the symbols ♠ and ♣ will denote a (possibly empty) sequence of singletons of $[n]$ in decreasing order. For instance, ♠ may denote the sequence $\bigl(\{7\}\ \{5\}\ \{3\}\bigr)$, but not $\bigl(\{7,5,3\}\bigr)$ or $\bigl(\{5\}\ \{3\}\ \{7\}\bigr)$.

We give some examples, and then formulate the theorem.

Examples of critical cells:

1) the cell $\big(\{7\}\ \{5\}\ \{3\}\ \{8,1,2,4,6\}\big)$ is critical;
2) the cell $\big(\{5\}\ \{3\}\ \{6,4\}\ \{1\}\ \{7,2\}\big)$ is critical if $\{6,4\}$ is a 3-pre-long set.

Noncritical cells:

1) the cell $\big(\{5,7\}\ \{3\}\ \{8,1,2,4,6\}\big)$ is not critical, as it is paired with $\big(\{5\}\ \{7\}\ \{3\}\ \{8,1,2,4,6\}\big)$; here 5 is a movable element;
2) the cell $\big(\{5\}\ \{6\}\ \{3\}\ \{2\}\ \{1\}\ \{8,4,7\}\big)$ is not critical, as it is paired with $\big(\{5,6\}\ \{3\}\ \{2\}\ \{1\}\ \{8,4,7\}\big)$; here the singletons are not in decreasing order, and 5 is a movable element;
3) the cell $\big(\{7\}\ \{5\}\ \{3\}\ \{2,6\}\ \{1\}\ \{8,4\}\big)$ is not critical, as it is paired with $\big(\{7\}\ \{5\}\ \{3\}\ \{2\}\ \{6\}\ \{1\}\ \{8,4\}\big)$.

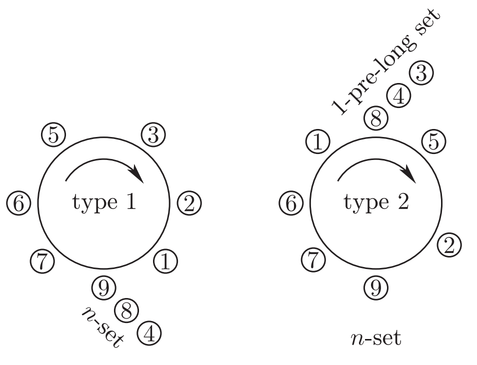

Figure 3. Critical cells for $n = 9$. We assume that $\{8,4,3\}$ is a 1-pre-long set.

Theorem 3. *There exist two types of critical cells for the above-constructed discrete Morse function* (*see Figure 3*):

1. *cells of the first type are of the form*

$$\big(\spadesuit\ \{n,*\}\big);$$

2. *cells of the second type are of the form*

$$\big(\spadesuit\ \{k\}\ I\ \clubsuit\ \{n,*\}\big),$$

where

(a) I *is a* k*-pre-long set*;
(b) $k < I$;
(c) $k < \spadesuit$.

Proof. Clearly, cells of the first and second type are critical. To prove that every critical cell is of one of the two types, we take a critical cell α and consider two cases.

1. All subsets in the label α except the n-set are singletons. Then these singletons must be arranged in decreasing order, as otherwise we would have a movable element. This corresponds to a critical cell of type 1.

2. The label α contains nonsingleton subsets distinct from the n-set. It is easy to see that each of these non-singleton subsets must be pre-long with respect to the element preceding it. Clearly, a label can contain at most one pre-long set. The other subsets in the label must be singletons arranged in decreasing order. This corresponds to a critical cell of type 2.

The theorem is proved.

Example 1. Let $L = (1, 1, \dots, 1, (n-1-\varepsilon))$ with small ε. It is known that the corresponding moduli space $M(L)$ is a sphere of dimension $n-3$ (see [8]). The discrete Morse function constructed above has exactly two critical cells:

$$\bigl(\{n-1\} \dots \{3\}\ \{2\}\ \{1\}\ \{n\}\bigr)$$

and

$$\bigl(\{1\}\ \{n-1, \dots, 3, 2\}\ \{n\}\bigr),$$

so the discrete Morse function is exact.

Example 2. Here is another example when the discrete Morse function constructed above is exact. Consider the polygonal linkage $L = \bigl(\varepsilon, \varepsilon, \varepsilon, \dots, \varepsilon, 1, 1, 1\bigr)$. The moduli space $M(L)$ is a disjoint union of two tori. Critical cells have labels

$$\bigl(\{n-1\}\ \{n-2\}\ \clubsuit\ \{n, *\}\bigr) \quad (\text{type } 1)$$

or

$$\bigl(\{n-2\}\ \{n-1\}\ \clubsuit\ \{n, *\}\bigr) \quad (\text{type } 2).$$

The number of critical k-cells equals the Betti number $b_k(M(L))$ for all k.

Examples 1 and 2 are rather exceptional: in other cases the constructed Morse function is far from being exact, and even a rough calculation shows that the number of critical cells exceeds the sum of the Betti numbers greatly.

§ 5. Gradient paths between critical points

A gradient path between two critical cells is an alternating sequence of merger and splitting steps. A merger occurs between a cell α_i^p and its paired cell β_i^{p+1}; here a minimal movable element in the label of the cell is moved to the next subset. A splitting occurs between a cell β_i^{p+1} and its facet α_{i+1}^p; here a subset in the label of a cell splits into two parts. A gradient path always begins and ends with a splitting.

Note that not all gradient paths end at a critical cell. This resembles the card game of solitaire: it is not always possible to complete, sometimes the player 'gets stuck'. When constructing a gradient path, there is some freedom in choosing each splitting step, but in many cases this freedom is illusory: if we want to arrive at a critical cell, then in most cases all steps are uniquely determined. If after such a step the minimal movable element is movable backwards, then there can be no merger step, as the resulting cell is paired with a cell whose dimension is not larger but smaller.

Proposition 2. *Assume there is a gradient path from a critical cell*

$$\beta = \big(\spadesuit_1\ \{j_1\}\ I_1\ \clubsuit_1\ \{n, *_1\}\big)$$

to a critical cell

$$\alpha = \big(\spadesuit_2\ \{j_2\}\ I_2\ \clubsuit_2\ \{n, *_2\}\big).$$

If $I_1 \neq I_2$, *then* $j_1 \neq j_2$ *and* $j_2 \in *_1$.

Proof. Consider the last merger step creating the set I_2 (so that this set will be preserved until the end of the path). At this step the element $k = \mathrm{Min}(I_2)$ is merged with the set $I_2 \setminus \{k\}$. At this time k is the minimal movable element, so for an element $j_2 < k$ we have two possibilities: (1) j_2 is in the n-set; (2) j_2 is to the right of I_2.

Case (2) is impossible, as no element can travel through the n-set.

The proposition is proved.

One can use Proposition 2 in combination with case by case analysis to describe all possible gradient paths for the Morse function constructed. We do not include such a full classification since we do not need all gradient paths. In § 6 we reduce the number of critical cells by inverting some gradient paths and obtain an exact Morse function.

Proposition 3. *There exists no gradient path from a critical cell of type* 1 *to a critical cell of type* 2.

Proof. Assume the converse, that is, let there exist a gradient path from a cell

$$\beta = \big(\spadesuit_1\ \{n, *_1\}\big)$$

to a cell

$$\alpha = \big(\spadesuit_2\ \{k\}\ I\ \clubsuit\ \{n, *_2\}\big).$$

By Proposition 1, at most one singleton j of $\spadesuit_1$ is contained in I. Furthermore, since all other elements of I fall into this set at some point, the pairing algorithm implies that $j = \mathrm{Max}(I)$. All the other elements of I, as well as k, come from $*_1$. Therefore,

$$\big(I \setminus \{\mathrm{Max}(I)\}\big) \cup \{k\} \subseteq *_1.$$

The set $I \cup \{k\}$ is long, so $\mathrm{Max}(I) \cup \{*_1\}$ is also long. This implies that $\{n, *_1\}$ is long too, a contradiction. The proposition is proved.

Proposition 4. *Let*

$$\beta = \big(\spadesuit\ \{k\}\, I\ \clubsuit\ \{n, *, j\}\big) \quad \textit{and} \quad \alpha = \big(\spadesuit\ \{k\}\, I\ \clubsuit \cup \{j\}\ \{n, *\}\big)$$

be critical cells of type 2. *If* I *is a* j*-pre-long set, then there exists exactly one gradient path between* α *and* β. *Along this path, the element* j *splits from the* n*-set and merges with* $\clubsuit$ (*see Figure* 4).

Proof. Suppose there exists a path from β to α. By Proposition 2 the set I is preserved along this path. Therefore, the first step in the path is a splitting of the n-set, when the element j is split to the left. The proposition is proved.

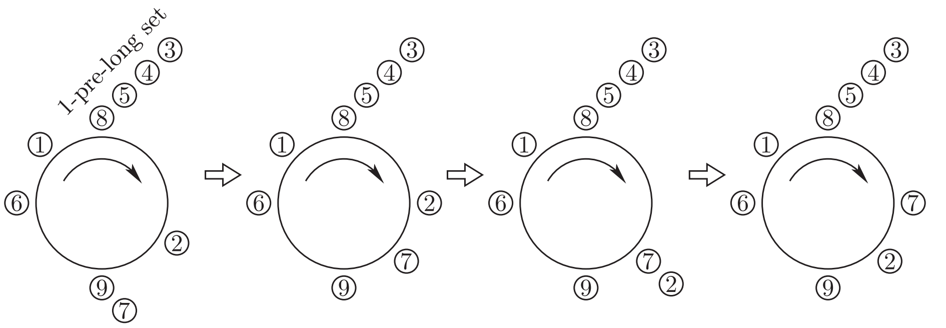

Figure 4. An example of a reversible path.

§ 6. Reversal of paths: a new Morse function

The following theorem will allow us to reduce the number of critical cells.

Theorem 4 (see [12]). *Assume given a discrete Morse function on a cell complex with critical cells α and β. Assume further that there exists only one gradient path between these cells. Then reversing this path defines a new discrete Morse function for which the cells α and β are no longer critical.*

Note that paths must be reversed one by one, because after reversing a path some new paths may appear between other critical cells. With this in mind, we do not reverse all paths described in Proposition 4, but impose some additional conditions.

Namely, we reverse a path between two critical cells

$$\beta = \bigl(\spadesuit\ \{k\}\ I\ \clubsuit\ \{n,*,j\}\bigr) \quad \text{and} \quad \alpha = \bigl(\spadesuit\ \{k\}\ I\ \clubsuit\ \cup \{j\}\ \{n,*\}\bigr)$$

if and only if the following conditions are satisfied:

1) $j > *$;
2) $j > \clubsuit$;
3) $j > k$.

We list the cells that remain critical after the reversal of paths (Figure 5):

1) all cells of type 1;
2) all cells $\bigl(\spadesuit\ \{k\}\ I\ \clubsuit\ \{n,*\}\bigr)$ of type 2 satisfying

$$k > * \quad \text{and} \quad k > \clubsuit.$$

Proposition 5. *After reversing paths as described above, the vector field remains well-defined.*

Proof. Condition 2) from the definition of a discrete vector field is obvious. To see that condition 1) holds note that each cell is contained in at most one reversed path. The proposition is proved.

Proposition 6. *The vector field constructed above is a discrete Morse function.*

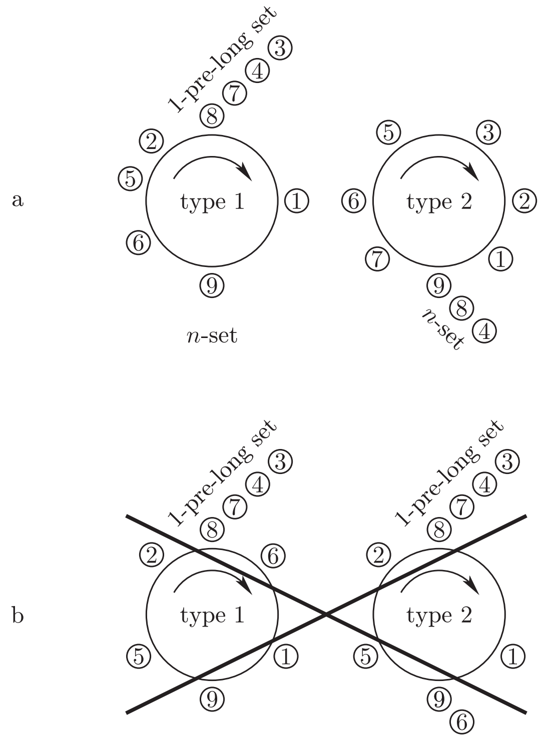

Figure 5. Cells that remain critical after the reversal of paths (a), and cells which are no longer critical (b).

Proof. Suppose the converse: there exists a closed path Γ. It can be written as a sequence of reversed and unreversed paths between former critical cells. As there exist no paths from cells of type 1 to cells of type 2, all the former critical cells appearing in Γ have type 2. We consider two cases:

1. All former critical cells contain the same element preceding the pre-long set. Then Proposition 2 implies that this pre-long set is preserved along the path. Therefore, no element larger than k can travel through the pre-long set (as otherwise the set would become long). Also, no element less than k can travel through the n-set. It follows that no element can make a full turn inside the label.

A closed path Γ necessarily contains at least one reversed path. This means that some elements larger than k leave the sequence of singletons to the right of the pre-long set and fall into the n-set. Let i be the smallest of these elements.

Consider the splitting step following the merger of i with the n-set:

(a) if an element j splits from the n-set to the right, then it cannot return to the n-set, since this will require a full turn;

(b) if an element j splits from the n-set to the left, then it cannot return to the n-set , since it is smaller than i.

2. In some former critical cells different elements precede the pre-long set. Let j be the smallest of these elements. At some step along the path, j leaves the

position in front of the pre-long set. The element j is smaller than the next element that takes the position in front of the pre-long set, so in the next former critical cell j can appear neither in the pre-long set, nor in the sequence of singletons in front of it. Therefore, j falls into the sequence of singletons following the pre-long set. To get back to the position in front of the pre-long set j must get back into the n-set at some point, which is possible only through some reversed path. By condition 3) for this reversed path, it must contain an element smaller than j in front of the pre-long set, which is impossible. The proposition is proved.

Theorem 5. *For the Morse function constructed above,*[2] *the number of critical cells equals the sum of Betti numbers of the manifold $M(L)$. In other words, the Morse function constructed is exact.*

Proof. By Theorem 1, each short subset J of $[n]$ containing the element n contributes 2 to the sum of Betti numbers. Therefore, it is sufficient to construct a bijection between short subsets of $[n]$ and some pairs of critical cells. More precisely, with each short set of cardinality $k+1$ containing n we associate one k-cell of type 1 and one $(n-3-k)$-cell of type 2.

1. *The cell of type* 1. We take J as the n-set for a cell of type 1. Such a cell is uniquely determined. Conversely, every critical cell of type 1 corresponds to a short set containing n, namely, to its n-set.

2. *The cell of type* 2.
   (a) *Compose I.* The set $\overline{J} := [n] \setminus J$ is long. We compose the set I from the elements of $\overline{J}$, starting with the largest one. We add these elements in decreasing order, as long as I remains short. The process stops once the set I becomes pre-long with respect to the appropriate element of J (one step before becoming long).
   (b) *Choose the element preceding I.* Let j be the largest element of $\overline{J} \setminus I$. We make j into a singleton and place it in front of the set I.
   (c) *Compose the n-set.* The n-set is defined as $\big(\overline{J} \setminus (I \cup \{j\})\big) \cup \{n\}$. This is a short set, as its complement is long. Furthermore, each element of the n-set (except n itself) is less than j.
   (d) *The positions of the remaining elements are determined uniquely.* We make all remaining elements into singletons and place them either in front of $\{j\}$ (if they are larger than j), or after I (if they are smaller than j), in decreasing order.

Now we compute the number of subsets in the resulting partition. All elements of J, except n, form singletons. We also have a singleton j and two nonsingleton subsets. There are $k+3$ subsets in total, so the dimension of the constructed cell is $n-3-k$.

Conversely, every critical cell of type 2 for the new Morse function defines a short set in the following way. Take all singletons except the one preceding the pre-long set, merge them together and add n. Then we obtain a short set containing n. The theorem is proved.

Let $L = (1, 1, 1, 1, 1, 1, 1)$ be an equilateral heptagonal linkage.

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[2]Recall that this Morse function is constructed via pairing and reversal of paths.

Example 3. For the short set $J=\{7\}$,

(a) the corresponding cell of type 1 is

$$\bigl(\{6\}\ \{5\}\ \{4\}\ \{3\}\ \{2\}\ \{1\}\ \{7\}\bigr);$$

(b) $\overline{J}=\{1,2,3,4,5,6\}$, $I=\{4,5,6\}$, $j=3$, so the corresponding cell of type 2 is

$$\bigl(\{3\}\ \{4,5,6\}\ \{7,1,2\}\bigr).$$

Example 4. For the short set $J=\{5,6,7\}$,

(a) the corresponding cell of type 1 is

$$\bigl(\{4\}\ \{3\}\ \{2\}\ \{1\}\ \{7,5,6\}\bigr);$$

(b) $\overline{J}=\{1,2,3,4\}$, $I=\{2,3,4\}$, $j=1$, so the corresponding cell of type 2 is

$$\bigl(\{6\}\ \{5\}\ \{1\}\ \{2,3,4\}\ \{7\}\bigr).$$

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Alena M. Zhukova
Faculty of Liberal Arts and Sciences,
St Petersburg State University
E-mail: a.zhukova@spbu.ru

Gaiane Yu. Panina
St Petersburg Institute
for Informatics and Automation
of the Russian Academy of Sciences;
St Petersburg Department
of V. A. Steklov Institute of Mathematics
of the Russian Academy of Sciences
E-mail: gaiane-panina@rambler.ru

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