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Approximating the trajectory attractor of the 3D Navier-Stokes system using various α -models of fluid dynamics

V. V. Chepyzhov

Abstract. We study the limit as $\alpha \rightarrow 0+$ of the long-time dynamics for various approximate α -models of a viscous incompressible fluid and their connection with the trajectory attractor of the exact 3D Navier-Stokes system. The α -models under consideration are divided into two classes depending on the orthogonality properties of the nonlinear terms of the equations generating every particular α -model. We show that the attractors of α -models of class I have stronger properties of attraction for their trajectories than the attractors of α -models of class II. We prove that for both classes the bounded families of trajectories of the α -models considered here converge in the corresponding weak topology to the trajectory attractor \mathfrak{A}_0 of the exact 3D Navier-Stokes system as time t tends to infinity. Furthermore, we establish that the trajectory attractor \mathfrak{A}_α of every α -model converges in the same topology to the attractor \mathfrak{A}_0 as $\alpha \rightarrow 0+$. We construct the minimal limits $\mathfrak{A}_{\min} \subseteq \mathfrak{A}_0$ of the trajectory attractors \mathfrak{A}_α for all α -models as $\alpha \rightarrow 0+$. We prove that every such set \mathfrak{A}_{\min} is a compact connected component of the trajectory attractor \mathfrak{A}_0 , and all the \mathfrak{A}_{\min} are strictly invariant under the action of the translation semigroup.

Bibliography: 39 titles.

Keywords: 3D Navier-Stokes system, α -models of fluid dynamics, trajectory attractor.

Introduction

Three-dimensional α -models of fluid dynamics are systems of differential equations that in a certain sense approximate and smooth the exact three-dimensional Navier-Stokes system, and the smoothing is effected by a certain filtration of the velocity vector which occurs in the argument of the nonlinear term of the original Navier-Stokes system. The small parameter α reflects the width of the scale of the spacial filtration of the model. Often the Green's function associated with the Helmholtz operator $I - \alpha^2 \Delta$ is considered as the kernel of the filtration. For $\alpha = 0$, the system of equations generating the α -model formally coincides with the ordinary 3D Navier-Stokes system.

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It seems that the first α -model described and studied in detail was the so-called Lagrange Averaged Navier-Stokes- α model (LANS- α model), also called the Camassa-Holm system with viscosity in the literature (see [1]–[6] and the bibliographies given in these papers). In a number of papers it has been shown, both analytically and numerically, that the LANS- α model gives a good approximation to turbulent flows (see [1]–[7]). In particular, it has been proved that explicit stationary solutions of the LANS- α system are very similar to empirical data and results in the numerical modelling of flows in channels and pipes for a fairly wide interval of values of the Reynolds numbers (see [1]–[3]).

After the LANS- α model, other α -models appeared which approximate the 3D Navier-Stokes system. For example, a model of turbulence was proposed in [8], which was called the Leray α -model. This name is related to the fact that Leray [9] considered a similar regularized system, which he used to prove the existence of weak solutions of the exact Navier-Stokes system. Note that Leray considered a more general smoothing kernel. The use of the Leray α -model in modelling turbulent flows was substantiated in [8]. There are also other approximating α -models that agree well with empirical data, for example, the Clark α -model [10], the modified Leray α -model [11], the simplified Bardina α -model [12], and others. Similar problems related to approximation and regularization of the three-dimensional Navier-Stokes system were also considered in [13] and [14].

For the α -models described in the papers listed above, the corresponding Cauchy problems were studied in detail, theorems on the existence and uniqueness of weak and strong solutions were proved, smoothing properties of these solutions were established, and global attractors were constructed for the infinite-dimensional dynamical semigroups generated by them. Furthermore, estimates of dimension were obtained for these attractors (that is, estimates of the number of degrees of freedom of the limit dynamics) depending on the physical parameters of the problem. In a number of papers (see [15]–[17]) other characteristics were also discussed related to the turbulence problem (energy spectrum, boundary layers, estimates of energy and enstrophy, etc.).

As is well known, if we ask similar questions about the classical 3D Navier-Stokes system, then the answers remain unknown, since, in spite of many years of effort by the world mathematical community, the key uniqueness theorem has not been proved for global weak solutions of the inhomogeneous 3D Navier-Stokes system (which exist). The absence of a proof of the uniqueness theorem means we cannot apply the highly developed theory of global attractors of infinite-dimensional dynamical systems to this system directly. This theory is used successfully, for example, in the study of the 2D Navier-Stokes system and other equations of mathematical physics, as well as in the study of the aforementioned α -models (see the books [18]–[23] and the extensive bibliography in these books).

This substantial gap can be partially filled if we apply the theory of trajectory attractors of evolutionary partial differential equations, which was created with emphasis on the equations for which uniqueness theorems of the corresponding initial-boundary-value problems either do not hold, or have not yet been proved (see [23], [24]). The trajectory attractor was constructed for the 3D Navier-Stokes system (see [25], [22]), as well as for other important equations and systems of mathematical physics with no uniqueness theorem (see [26]–[30]).

The purpose of our paper is to study the connection between the long-time dynamics of solutions of various α -models and the trajectory attractor of the exact 3D Navier-Stokes system as $\alpha \rightarrow 0+$. In relation to concrete α -models, a similar problem was solved in [31] for the LANS- α model, and in [32] for the Leray α -model. Here, we generalize these, proving theorems which are applicable to many known α -models (as well as to new models which may appear in the future in papers on this topic). The α -models we consider are divided into two classes, depending on the orthogonality properties of the smoothed nonlinear term. For example, the Leray α -model belongs to class I, and the LANS- α model to class II. It turns out that the attraction of trajectories to the global attractor of the system for an α -model of class I is stronger than for an α -model of class II.

We now state the main results of the paper. We consider an arbitrary family of bounded (in the energy norm) solutions (trajectories) $B_\alpha = \{z_\alpha(x, t), t \geq 0\}$ of some α -model for $0 < \alpha \leq 1$. For $\alpha = 0$ we formally obtain the classical 3D Navier-Stokes system, for which we construct the trajectory attractor \mathfrak{A}_0 . This attractor describes the dynamics of the Leray-Hopf weak solutions $\{v(x, t), t \geq 0\}$ for the 3D Navier-Stokes system as $t \rightarrow +\infty$.

In the main theorem we prove that, as $\alpha \rightarrow 0+$ and as $h \rightarrow +\infty$, the time shifts

$$T(h)B_\alpha = \{z_\alpha(x, t + h), t \geq 0\}$$

of the trajectory set B_α of any α -model of class I or class II tend to the trajectory attractor \mathfrak{A}_0 of the exact 3D Navier-Stokes system in the corresponding local weak topology. We further prove that the trajectory attractors \mathfrak{A}_α of every α -model converge to the trajectory attractor \mathfrak{A}_0 of the Navier-Stokes system as $\alpha \rightarrow 0+$ in the same local weak topology.

The division of α -models into two classes proposed in the paper is characterized by the fact that the trajectory attractors \mathfrak{A}_α corresponding to an α -model of class I (for example, the Leray α -model) converge ‘more strongly’ as $\alpha \rightarrow 0+$ to the limit \mathfrak{A}_0 than the trajectory attractors of α -models of class II (for example, the LANS- α model). However, we note that there are α -models that do not fall into the proposed classification, for example, see [33]. This is related to the use of a different scheme of regularization of the exact 3D Navier-Stokes system in these models.

Finally, in the paper, following [31], we construct the minimal limits $\mathfrak{A}_{\min} \subseteq \mathfrak{A}_0$ of the trajectory attractors \mathfrak{A}_α as $\alpha \rightarrow 0+$ for every α -model of class I or II. We prove that every such set of trajectories \mathfrak{A}_{\min} is a compact connected component of the trajectory attractor \mathfrak{A}_0 of the Navier-Stokes system, and the sets \mathfrak{A}_{\min} are strictly invariant under the translation semigroup $\{T(h)\}$: $T(h)\mathfrak{A}_{\min} = \mathfrak{A}_{\min}$ for $h \geq 0$.

For simplicity, in our exposition we consider α -models with periodic boundary conditions, but a similar scheme can be used to construct α -models and analyze their attractors for other initial-boundary-value problems, for example in a bounded domain with the no-slip condition on the boundary.

Notation

Let $\mathbb{T}^3 := [\mathbb{R} \bmod 2\pi]^3$ be the three-dimensional torus with the Euclidean metric and coordinates $x = (x_1, x_2, x_3) \in \mathbb{T}^3$. Let \mathcal{V} denote the space of trigonometric

vector-polynomials $y(x) = (y^1(x), y^2(x), y^3(x))$ with period 2π in every variable x_j , $j = 1, 2, 3$, which have zero divergence and zero mean over the torus \mathbb{T}^3 , that is,

$$\nabla \cdot y(x) := \partial_{x_1} y^1(x) + \partial_{x_2} y^2(x) + \partial_{x_3} y^3(x) = 0 \quad \forall x \in \mathbb{T}^3; \quad \int_{\mathbb{T}^3} y(x) dx = 0.$$

The closure of \mathcal{V} in the space $L_2(\mathbb{T}^3)^3$ is denoted by H . The Leray-Helmholtz orthogonal projector is denoted by $P: L_2(\mathbb{T}^3)^3 \rightarrow H$. The inner product and norm in H are denoted by $\langle u, v \rangle$ and $|u|$, respectively.

For $s \in \mathbb{R}$ we consider the scale of Hilbert spaces $H^s := \mathcal{D}(A^{s/2})$ equipped with the inner product and norm

$$\langle u, v \rangle_s := \langle A^{s/2} u, A^{s/2} v \rangle, \quad \|u\|_s = |A^{s/2} v|,$$

which corresponds to the (strictly) positive selfadjoint Stokes operator $A = -P\Delta$ acting in H with domain $D(A) := \{u \in H \mid \Delta u \in H\}$, which coincides with the space H^2 . Note that for periodic boundary conditions the Stokes operator is $A \equiv -\Delta$. Obviously, $H^0 = H$. We omit the index $s = 1$ in the notation for the norm of the space H^1 , $\|u\| := \|u\|_1$. The space H^{-s} is dual to H^s for any $s \in \mathbb{R}$.

Recall the standard Poincaré inequality

$$|u|^2 \leq \lambda_1^{-1} \|u\|^2 \quad \forall u \in H^1, \tag{0.1}$$

where λ_1 is the first eigenvalue of the Stokes operator A . For any $f \in H^{-1}$, let $\langle f, u \rangle$ also denote the action of the functional $f \in H^{-1}$ on a vector $u \in H^1$. The operator A effects an isomorphism between H^1 and H^{-1} , and $\langle u, v \rangle_1 = \langle Au, v \rangle$ for all $u, v \in H^1$.

In a Banach space X , the norm is denoted by $\|\cdot\|_X$. The Hausdorff distance in the space X from a set A to B is denoted by

$$\text{dist}_X(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\|_X. \tag{0.2}$$

§ 1. Systems of equations for 3D α -models

We consider a system of equations on the torus \mathbb{T}^3

$$\partial_t v = -\nu Av - PF(u, v) + g(x), \quad \nabla \cdot v = 0, \tag{1.1}$$

$$v = u + \alpha^2 Au, \quad \nabla \cdot u = 0, \tag{1.2}$$

in which $\nu > 0$ is the viscosity coefficient, A is the Stokes operator, P is the Leray-Helmholtz projector, and $F(u, v)$ is a certain bilinear differential operator of first order in u and v , which is described below. Here, α is a small positive parameter which corresponds to the given α -model. In the system (1.1), (1.2), the three-dimensional vector fields

$$v = (v^1(x, t), v^2(x, t), v^3(x, t)), \quad u = (u^1(x, t), u^2(x, t), u^3(x, t))$$

are unknown for $x \in \mathbb{T}^3$ and $t \geq 0$. By equation (1.2), the vector field $u(x, t)$ is uniquely determined by $v(x, t)$ and is a smoother function, that is, if $v \in H^s$,

then $u \in H^{s+2}$. The external force $g = (g^1(x), g^2(x), g^3(x))$, $x \in \mathbb{T}^3$, is assumed to be a known vector-function such that $\nabla \cdot g = 0$. It is assumed that all the vector-functions have zero mean over the torus \mathbb{T}^3 :

$$\int_{\mathbb{T}^3} u(x, t) \, dx = 0, \quad \int_{\mathbb{T}^3} v(x, t) \, dx = 0, \quad \int_{\mathbb{T}^3} g(x) \, dx = 0.$$

Every α -model (1.1), (1.2) is characterized by its first-order vector differential operator

$$F(u, v) = (F^1(u, v), F^2(u, v), F^3(u, v)),$$

in which the components $F^i(u, v)$ are linear combinations of all possible operator monomials of the form $u^k \partial_{x_j} v^n$, $v^k \partial_{x_j} u^n$, and $u^k \partial_{x_j} u^n$:

$$F^i(u, v) = \sum_{k,j,n=1}^3 C_{kjn}^i u^k \partial_{x_j} v^n + D_{kjn}^i v^k \partial_{x_j} u^n + E_{kjn}^i u^k \partial_{x_j} u^n, \quad (1.3)$$

where C_{kjn}^i , D_{kjn}^i , and E_{kjn}^i are some real coefficients. Note that monomials of the form $v^k \partial_{x_j} v^n$ are not used in the representation (1.3), since they do not contain components of the ‘smoothed’ vector field u . Otherwise, we might obtain a system with ‘bad’ properties, as with the exact Navier-Stokes system, which involves the quadratic term $(v \cdot \nabla)v$. (For $F(u, v) = (v \cdot \nabla)v$, equation (1.1) is obviously independent of α and coincides with the 3D Navier-Stokes system.)

We consider various operators $F(u, v)$ of the form (1.3) corresponding to various α -models which are characterized by the following two basic properties.

Property 1. It is assumed that for $u = v \in \mathcal{V}$ the operator

$$PF(v, v) = P \sum_{j=1}^3 v^j \partial_{x_j} v = P(v \cdot \nabla)v \quad (1.4)$$

coincides with the quadratic operator in the classical 3D Navier-Stokes system.

Consequently, for $\alpha = 0$ the system (1.1), (1.2) takes the following form:

$$\partial_t v = -\nu Av - P(v \cdot \nabla)v + g(x), \quad \nabla \cdot v = 0, \quad (1.5)$$

that is, this system coincides formally with the classical 3D Navier-Stokes system for the unknown vector field $v = v(x, t)$, in which the pressure function is excluded in the standard fashion by applying the operator P to both sides of the system (see [13], [34]–[36]).

Property 2. It is assumed that one of two properties of *orthogonality* of the function $F(u, v)$ to u or v holds in the space H , which divides the α -models under consideration into two classes:

$$\langle F(u, v), v \rangle = 0 \quad \forall u, v \in \mathcal{V} \quad (\text{class I}), \quad (1.6)$$

$$\langle F(u, v), u \rangle = 0 \quad \forall u, v \in \mathcal{V} \quad (\text{class II}). \quad (1.7)$$

We look at some examples of α -models that belong to these classes.

Example 1.1 (Leray α -model, class I; see [8]). The nonlinear operator is

$$F(u, v) = (u \cdot \nabla)v.$$

The system (1.1), (1.2) has the form

$$\begin{aligned} \partial_t v &= -\nu Av - P(u \cdot \nabla)v + g(x), & \nabla \cdot v &= 0, \\ v &= u + \alpha^2 Au, & \nabla \cdot u &= 0. \end{aligned}$$

For $\alpha = 0$, $u = v$ we obviously obtain the exact 3D Navier-Stokes system (1.5). As is well known, the bilinear term in the Navier-Stokes system satisfies the identity

$$\langle (u \cdot \nabla)v, v \rangle = 0 \quad \forall u, v \in \mathcal{V}.$$

Consequently, $\langle F(u, v), v \rangle = 0$ for any $u, v \in \mathcal{V}$.

Example 1.2 (LANS- α model, class II; see [5]). The nonlinear operator is

$$F(u, v) = -u \times (\nabla \times v).$$

Here, $a \times b$ denotes the vector product in the space \mathbb{R}^3 . The system has the form

$$\begin{aligned} \partial_t v &= -\nu Av + P(u \times (\nabla \times v)) + g(x), & \nabla \cdot v &= 0, \\ v &= u + \alpha^2 Au, & \nabla \cdot u &= 0. \end{aligned}$$

Note that the nonlinear term $u \times (\nabla \times v)$ can be rewritten in the form

$$u \times (\nabla \times v) = -(u \cdot \nabla)v - \sum_{j=1}^3 u^j \nabla v^j;$$

therefore for $v = u$,

$$u \times (\nabla \times u) = -(u \cdot \nabla)u - \frac{1}{2} \nabla(u \cdot u)$$

and, consequently, $P(u \times (\nabla \times u)) = -P(u \cdot \nabla)u$, since the operator P projects any gradient vector-function to zero. Therefore for $\alpha = 0$ we again obtain the 3D Navier-Stokes system (1.5). Using standard formulae from vector analysis we can establish that

$$\langle F(u, v), u \rangle = \langle u \times (\nabla \times v), u \rangle = 0 \quad \forall u, v \in \mathcal{V}.$$

Two more α -models can be constructed if in the preceding example we interchange the variables u and v . Indeed, it is obvious that after such an interchange any α -model changes its class, that is, if $F(u, v)$ generates an α -model of class I, then for the operator $\bar{F}(u, v) := F(v, u)$ the corresponding α -model belongs to class II, and vice versa.

Class II—the modified Leray α -model: $F(u, v) = (v \cdot \nabla)u$ (see [11]).

Class I—the modified LANS- α model: $F(u, v) = -v \times (\nabla \times u)$ (the author has not come across this model in the literature).

In all the examples considered above, α -models were constructed by smoothing just one argument in the bilinear term of the Navier-Stokes system. When both arguments are smoothed, we obtain the so-called simplified Bardina α -model which is in class II.

Class II – simplified Bardina α -model: $F(u, v) = -(u \cdot \nabla)u$ (see [12]):

$$\begin{aligned} \partial_t v &= -\nu Av - P(u \cdot \nabla)u + g(x), & \nabla \cdot v &= 0, \\ v &= u + \alpha^2 Au, & \nabla \cdot u &= 0. \end{aligned}$$

To conclude this section we note that there are infinitely many α -models of both classes. Indeed, if we have two different α -models of the same class I or II with operators $F_0(u, v)$ and $F_1(u, v)$, then the linear interpolation

$$F_\theta(u, v) = (1 - \theta)F_0(u, v) + \theta F_1(u, v) \quad \forall \theta \in (0, 1),$$

generates an α -model of the same class as the operators $F_0(u, v)$ and $F_1(u, v)$.

§ 2. The trajectory attractor of the 3D Navier-Stokes system

Suppose that the external force satisfies $g \in H^{-1}$. We consider the exact Navier-Stokes system:

$$\partial_t v = -\nu Av - B(v, v) + g(x), \quad \nabla \cdot v = 0, \quad x \in \mathbb{T}^3, \quad t \geq 0. \tag{2.1}$$

From now on, $B(u, v)$ denotes the bilinear operator $B(u, v) = P(u \cdot \nabla)v$.

Note that for any $u, v \in H^1$ and any $w \in H^2$ we have the inequalities

$$|\langle B(u, v), w \rangle| \leq c_0 |u| \cdot \|v\| \cdot \|w\|_{L_\infty} \leq c |u| \cdot \|v\| \cdot \|w\|_2, \tag{2.2}$$

since $H^2 \subset L_\infty(\mathbb{T}^3)$ by the Sobolev embedding theorem. Recall that H^{-2} is the dual space of H^2 . Consequently, $B(u, v) \in H^{-2}$ and

$$\|B(u, v)\|_{-2} \leq c |u| \cdot \|v\| \quad \forall u, v \in H^1. \tag{2.3}$$

Consider a function $v(\cdot) \in L_2(0, M; H^1) \cap L_\infty(0, M; H)$ for some $M > 0$. Then $Av \in L_2(0, M; H^{-1})$ and by (2.3),

$$B(v(\cdot), v(\cdot)) \in L_2(0, M; H^{-2}). \tag{2.4}$$

A function $v(\cdot) \in L_2(0, M; H^1) \cap L_\infty(0, M; H)$ is called a *weak solution* of equation (2.1) if it satisfies this equation in the space of distributions $\mathcal{D}'(0, M; H^{-2})$ (see, for example, [13]). Then it follows from property (2.4) that for any weak solution $v(\cdot)$ the derivative with respect to time satisfies $\partial_t v(\cdot) \in L_2(0, M; H^{-2})$ and, consequently, $v(\cdot) \in C([0, M]; H^{-2})$. Recall that $v(\cdot) \in L_\infty(0, M; H)$. Therefore, by the well-known Lions-Magenes lemma [37] (see also [35]), the function $v(\cdot)$ belongs to the space $C_w([0, M]; H)$ of weakly continuous functions with values in H . Thus, for equation (2.1) the initial condition

$$v|_{t=0} = v_0(x) \in H \tag{2.5}$$

makes sense in the class of weak solutions in $L_2(0, M; H^1) \cap L_\infty(0, M; H)$.

The classical theorem on the existence of weak solutions of the Cauchy problem for the three-dimensional Navier-Stokes system asserts that for any $v_0 \in H$ there exists a weak solution $v(t)$ of equation (2.1) in the space $L_2(0, M; H^1) \cap L_\infty(0, M; H)$ such that $v(0) = v_0$ and $v(t)$ satisfies the energy inequality

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2 + \nu \|v(t)\|^2 \leq \langle g, v(t) \rangle, \quad t \in [0, M]. \tag{2.6}$$

Inequality (2.6) means that

$$-\frac{1}{2} \int_0^M |v(t)|^2 \psi'(t) dt + \nu \int_0^M \|v(t)\|^2 \psi(t) dt \leq \int_0^M \langle g, v(t) \rangle \psi(t) dt \tag{2.7}$$

for any $\psi(\cdot) \in C_0^\infty(]0, M[)$, $\psi(t) \geq 0$.

Such solutions are customarily called Leray-Hopf weak solutions (see, for example, [13], [23], [34]–[36]).

Remark 2.1. As is well known, the question of uniqueness of a Leray-Hopf weak solution of problem (2.1), (2.5) remains open. It is also not known whether any weak solution satisfies the energy inequality (2.6). Nevertheless it is known that every weak solution obtained by the Faedo-Galerkin approximation method satisfies the energy inequality.

In this section we construct the trajectory attractor for the system (2.1). (See more details in [23], [25].)

We consider the family $\mathcal{X}_0^+ = \{v(x, t), t \geq 0\}$ of all possible Leray-Hopf weak solutions of the system (2.1), that is, $v(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+; H^1) \cap L_\infty^{\text{loc}}(\mathbb{R}_+; H)$ and the function $v(t) := v(\cdot, t)$ satisfies (2.6) for any $M > 0$. It follows from the existence theorem stated above that arbitrary solutions of problem (2.1), (2.5) with an arbitrary initial condition $v(0) \in H$ that are constructed by the Galerkin approximation method belong to \mathcal{X}_0^+ .

We call the family \mathcal{X}_0^+ the *trajectory space* of the 3D Navier-Stokes system. This family is not empty and is fairly large. It follows from equation (2.1) that $\partial_t v(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+; H^{-2})$ for any trajectory $v(\cdot) \in \mathcal{X}_0^+$.

We consider the following Banach space \mathcal{F}_+^b , which we will use to describe sets of bounded trajectories in \mathcal{X}_0^+ :

$$\mathcal{F}_+^b = \{z(\cdot) \mid z(\cdot) \in L_2^b(\mathbb{R}_+; H^1) \cap L_\infty(\mathbb{R}_+; H), \partial_t z(\cdot) \in L_2^b(\mathbb{R}_+; H^{-2})\} \tag{2.8}$$

with the norm

$$\|z\|_{\mathcal{F}_+^b} = \|z\|_{L_2^b(\mathbb{R}_+; H^1)} + \|z\|_{L_\infty(\mathbb{R}_+; H)} + \|\partial_t z\|_{L_2^b(\mathbb{R}_+; H^{-2})},$$

where

$$\begin{aligned} \|z\|_{L_2^b(\mathbb{R}_+; H^1)}^2 &= \sup_{t \geq 0} \int_t^{t+1} \|z(s)\|^2 ds, & \|z\|_{L_\infty(\mathbb{R}_+; H)} &= \text{ess sup}_{t \geq 0} |z(t)|, \\ \|\partial_t z\|_{L_2^b(\mathbb{R}_+; H^{-2})}^2 &= \sup_{t \geq 0} \int_t^{t+1} \|\partial_t z(s)\|_{-2}^2 ds. \end{aligned}$$

We introduce the semigroup of translations $\{T(h)\} := \{T(h), h \geq 0\}$ acting on a function $\{z(t), t \geq 0\}$ by the formula $T(h)z(t) = z(t + h), t \geq 0$. Clearly, the semigroup $\{T(h)\}$ is well defined on \mathcal{F}_+^b . We consider the action of the semigroup $\{T(h)\}$ in \mathcal{X}_0^+ , the trajectory space of equation (2.1). From the definition of \mathcal{X}_0^+ we conclude that if $v(\cdot) \in \mathcal{X}_0^+$, then $T(h)v(\cdot) = v(\cdot + h) \in \mathcal{X}_0^+$ for any $h \geq 0$. Therefore,

$$T(h)\mathcal{X}_0^+ \subseteq \mathcal{X}_0^+ \quad \forall h \geq 0. \tag{2.9}$$

Proposition 2.1 (see [23]). *Let $g \in H^{-1}$. Then $\mathcal{X}_0^+ \subset \mathcal{F}_+^b$ and for any trajectory $v(\cdot) \in \mathcal{X}_0^+$ the inequality*

$$\|T(h)v(\cdot)\|_{\mathcal{F}_+^b} \leq C_0 \|v(\cdot)\|_{L_\infty(0,1;H)}^2 \exp(-\nu\lambda_1 t) + R_0 \tag{2.10}$$

holds, where the constants C_0, R_0 depend on ν, λ_1 , and $\|g\|_{-1}$.

In order to define a topology on the trajectory space \mathcal{X}_0^+ , we introduce the space

$$\mathcal{F}_+^{\text{loc}} = \{z(\cdot) \mid z(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+; H^1) \cap L_\infty^{\text{loc}}(\mathbb{R}_+; H), \partial_t z(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+; H^{-2})\},$$

in which we define the structure of a topological space Θ_+^{loc} with the topology of local weak convergence generated by the following convergence: by definition a sequence $\{z_n(\cdot)\} \subset \mathcal{F}_+^{\text{loc}}$ converges to $z(\cdot) \in \mathcal{F}_+^{\text{loc}}$ in Θ_+^{loc} as $n \rightarrow +\infty$ if for any $M > 0$ as $n \rightarrow \infty$ we have

$$\begin{aligned} z_n(\cdot) &\rightharpoonup z(\cdot) \quad \text{weakly in } L_2(0, M; H^1), \\ z_n(\cdot) &\rightharpoonup z(\cdot) \quad \text{weak-* in } L_\infty(0, M; H), \\ \partial_t z_n(\cdot) &\rightharpoonup \partial_t z(\cdot) \quad \text{weakly in } L_2(0, M; H^{-2}). \end{aligned} \tag{2.11}$$

Note that the topology of Θ_+^{loc} can be described using a suitable system of neighbourhoods. The space Θ_+^{loc} is a Hausdorff topological space. Obviously $\mathcal{F}_+^b \subseteq \Theta_+^{\text{loc}}$. It is known that any ball $B_R = \{z \in \mathcal{F}_+^b \mid \|u\|_{\mathcal{F}_+^b} \leq R\}$ is compact in Θ_+^{loc} . Thus, the ball B_R with the topology induced from Θ_+^{loc} is a metrizable space, and the corresponding metric space is complete (but the entire space Θ_+^{loc} is not metrizable, see [23], [25] for the details). The fact that the topology is metrizable on the balls B_R simplifies the construction of the trajectory attractor of the semigroup $\{T(h)\}$ acting on \mathcal{X}_0^+ . The definition of the topology of Θ_+^{loc} immediately implies that the translation semigroup $\{T(h)\}$ is continuous in Θ_+^{loc} .

Proposition 2.2 (see [23]). *The trajectory space \mathcal{X}_0^+ is sequentially closed in the topology of Θ_+^{loc} .*

Recall that a set $P \subset \mathcal{F}_+^{\text{loc}}$ is said to be *attracting* for the semigroup $\{T(h)\}|_{\mathcal{X}_0^+}$ in the topology of Θ_+^{loc} if for any set $B \subset \mathcal{X}_0^+$ which is bounded (in \mathcal{F}_+^b) we have

$$T(h)B \rightarrow P \quad \text{as } h \rightarrow +\infty \tag{2.12}$$

in the topology of Θ_+^{loc} , that is, for any neighbourhood $\mathcal{O}(P)$ (in Θ_+^{loc}) there exists a number $h_1 = h_1(B, \mathcal{O}) > 0$ such that $T(h)B \subseteq \mathcal{O}(P)$ for all $h \geq h_1$.

Definition 2.1. A set $\mathcal{A}_0 \subset \mathcal{X}_0^+$ is called the *trajectory attractor* of the semigroup $\{T(h)\}$ in the topology of Θ_+^{loc} if

- a) the set \mathfrak{A}_0 is bounded in \mathcal{F}_+^b and compact in Θ_+^{loc} ;
- b) \mathfrak{A}_0 is an attracting set of the semigroup $\{T(h)\}$ in the topology of Θ_+^{loc} ;
- c) \mathfrak{A}_0 is strictly invariant with respect to the translation semigroup: $T(h)\mathfrak{A}_0 = \mathfrak{A}_0$ for all $h \geq 0$.

It follows from inequality (2.10) that the ball $B_{2R_0} = \{v(\cdot) \in \mathcal{F}_+^b \mid \|v\|_{\mathcal{F}_+^b} \leq 2R_0\}$ in the space \mathcal{F}_+^b with radius $2R_0$ is an *absorbing set* of the semigroup $\{T(h)\}_{\mathcal{K}_0^+}$, that is, $T(h)B \subseteq B_{2R_0}$ for any set $B \subset \mathcal{K}_0^+$ bounded in \mathcal{F}_+^b and for $h \geq h_2$ if $h_2 = h_2(B)$ is sufficiently large. It follows from Proposition 2.2 that the set $P = B_{2R_0} \cap \mathcal{K}_0^+$ is also a compact absorbing set.

As already mentioned, the ball B_{2R_0} is a compact metric subspace of Θ_+^{loc} on which the continuous semigroup $\{T(h)\}$ acts. Therefore it follows from the classical theorem on attractors of semigroups that the semigroup $\{T(h)\}$ on \mathcal{K}_0^+ has a compact (in Θ_+^{loc}) global attractor $\mathfrak{A} \subset \mathcal{K}_0^+ \cap B_{2R_0}$:

$$\mathfrak{A}_0 = \bigcap_{\tau > 0} \left[\bigcup_{h \geq \tau} T(h)P \right]_{\Theta_+^{\text{loc}}},$$

where $[\cdot]_{\Theta_+^{\text{loc}}}$ denotes closure in Θ_+^{loc} (see [18], [19] or [23]). This attractor is obviously a trajectory attractor of the 3D Navier-Stokes system (2.1).

Note that the following inclusions are continuous:

$$\Theta_+^{\text{loc}} \subset L_2^{\text{loc}}(\mathbb{R}_+; H^{1-\delta}), \quad \Theta_+^{\text{loc}} \subset C^{\text{loc}}(\mathbb{R}_+; H^{-\delta}), \quad 0 < \delta \leq 1 \tag{2.13}$$

(see [13] and [23]). Therefore the trajectory attractor \mathfrak{A}_0 satisfies the following strong attraction properties: for any set $B \subset \mathcal{K}_0^+$, bounded in \mathcal{F}_+^b , and for any $M > 0$,

$$\text{dist}_{L_2(0,M;H^{1-\delta})}(T(h)B, \mathfrak{A}_0) \rightarrow 0, \quad \text{dist}_{C([0,M];H^{-\delta})}(T(h)B, \mathfrak{A}_0) \rightarrow 0, \quad h \rightarrow +\infty. \tag{2.14}$$

In order to describe the structure of the trajectory attractor \mathfrak{A}_0 , we recall the definition of the *kernel* \mathcal{K}_0 of the system (2.1). The kernel \mathcal{K}_0 consists of all the complete weak solutions $\{v(t), t \in \mathbb{R}\}$ of this system that are bounded in the space \mathcal{F}^b and satisfy the energy inequality (2.6) on the whole of the time axis \mathbb{R} . The norm in \mathcal{F}^b is defined in the same way as the norm in \mathcal{F}_+^b (see (2.8)), replacing \mathbb{R}_+ by \mathbb{R} . It was proved in [23] that the trajectory attractor \mathfrak{A}_0 of the three-dimensional Navier-Stokes system coincides with the restriction of the kernel \mathcal{K}_0 of equation (2.1) to \mathbb{R}_+ :

$$\mathfrak{A}_0 = \Pi_+ \mathcal{K}_0. \tag{2.15}$$

Here, $\Pi_+ v(t) = \{v(t), t \geq 0\}$ for any function $\{v(t), t \in \mathbb{R}\}$. The set \mathcal{K}_0 is bounded in \mathcal{F}^b and compact in Θ^{loc} . The topology of Θ^{loc} is defined similarly to that of Θ_+^{loc} , replacing the intervals $(0, M)$ by $(-M, M)$.

§ 3. The well-posedness of the Cauchy problem and global attractors for general α -models

We fix $\alpha > 0$. We analyze the solutions (trajectories) of the system (1.1), (1.2)

$$\partial_t v = -\nu Av - PF(u, v) + g(x), \tag{3.1}$$

$$v = u + \alpha^2 Au, \tag{3.2}$$

together with the initial condition

$$u|_{t=0} = u_0. \tag{3.3}$$

The following lemma lets us extend the operator $PF(u, v)$ continuously to the larger spaces H^{-1} and H^{-2} by analogy with the extension to these spaces of the bilinear operator $B(u, v)$ in the Navier-Stokes system (2.1). We also introduce the operator

$$F_0(u, y) = F(u, u + y) - F(u, u). \tag{3.4}$$

Lemma 3.1. *Given the vector fields $u, v, w \in H$, the following inequalities hold:*

$$|\langle F(u, v), w \rangle| \leq c(|u|^{1/2} \|u\|^{1/2} \|v\| + \|u\| |v|^{1/2} \|v\|^{1/2} + |u|^{1/2} \|u\|^{3/2}) \|w\| \tag{3.5}$$

$$\forall u, v, w \in H^1;$$

$$|\langle F(u, v), w \rangle| \leq c(\|u\| \cdot |v| + \|u\| \cdot |u|) |Aw| \quad \forall u \in H^1, v \in H, w \in H^2, \tag{3.6}$$

$$|\langle F_0(u, y), w \rangle| \leq c\|u\| \cdot |y| \cdot |Aw| \quad \forall u \in H^1, y \in H, w \in H^2. \tag{3.7}$$

Proof. The operator $F(u, v)$ has the form of a sum of operator monomials (1.3) and so it is sufficient to prove these inequalities for operators with vector components of the form $u^k \partial_{x_j} v^n, v^k \partial_{x_j} u^n$ and $u^k \partial_{x_j} u^n$. Suppose that $u, v, w \in H^1$; then from Hölder’s inequality we obtain

$$\left| \int_{\mathbb{T}^3} u^k(x) \partial_{x_j} v^n(x) w^i(x) dx \right| \leq c \|u^k\|_{L_3} \|\partial_{x_j} v^n\|_{L_2} \|w^i\|_{L_6}.$$

Recall the following Sobolev inequalities in \mathbb{T}^3 :

$$\|\varphi\|_{L_3} \leq c \|\varphi\|_{L_2}^{1/2} \|\varphi\|_{H^1}^{1/2}, \quad \|\varphi\|_{L_6} \leq c \|\varphi\|_{H^1} \quad \forall \varphi \in H^1. \tag{3.8}$$

Then we conclude from the preceding inequality that

$$\left| \int_{\mathbb{T}^3} u^k \partial_{x_j} v^n w^i dx \right| \leq c |u|^{1/2} \|u\|^{1/2} \|v\| \|w\|.$$

Similar estimates are derived for $v^k \partial_{x_j} u^n$ and $u^k \partial_{x_j} u^n$. Inequality (3.5) is proved.

Suppose that $u \in H^1, v \in H, w \in H^2$. In the integral for the monomial $u^k \partial_{x_j} v^n$, we integrate by parts:

$$\begin{aligned} \int_{\mathbb{T}^3} u^k(x) \partial_{x_j} v^n(x) w^i(x) dx &= - \int_{\mathbb{T}^3} v^n(x) \partial_{x_j} (u^k(x) w^i(x)) dx \\ &= - \int_{\mathbb{T}^3} v^n(x) \partial_{x_j} u^k(x) w^i(x) dx - \int_{\mathbb{T}^3} v^n(x) u^k(x) \partial_{x_j} w^i(x) dx. \end{aligned}$$

We apply Hölder’s inequality, Sobolev’s inequalities (3.8) and also use

$$\|\varphi\|_{L_\infty} \leq c\|\varphi\|_{H^2} \quad \forall \varphi \in H^2,$$

and obtain the following estimates:

$$\begin{aligned} \left| \int_{\mathbb{T}^3} u^k \partial_{x_j} v^n w^i dx \right| &\leq \left| \int_{\mathbb{T}^3} v^n \partial_{x_j} u^k w^i dx \right| + \left| \int_{\mathbb{T}^3} v^n u^k \partial_{x_j} w^i dx \right| \\ &\leq c\|v^n\|_{L_2} \|\partial_{x_j} u^k\|_{L_2} \|w^i\|_{L_\infty} + c\|v^n\|_{L_2} \|u^k\|_{L_3} \|\partial_{x_j} w^i\|_{L_6} \\ &\leq c_1|v| \cdot \|u\| \cdot |Aw| + c_1|v| \cdot |u|^{1/2} \|u\|^{1/2} |Aw| \leq c_2|v| \cdot \|u\| \cdot |Aw|. \end{aligned} \tag{3.9}$$

The integrals in the form $\langle F(u, v), w \rangle$ corresponding to the monomials $v^k \partial_{x_j} u^n$ and $u^k \partial_{x_j} v^n$ are estimated directly:

$$\begin{aligned} \left| \int_{\mathbb{T}^3} v^k \partial_{x_j} u^n w^i dx \right| &\leq c\|v^k\|_{L_2} \|\partial_{x_j} u^n\|_{L_2} \|w^i\|_{L_\infty} \leq c_1|v| \cdot \|u\| \cdot |Aw|, \\ \left| \int_{\mathbb{T}^3} u^k \partial_{x_j} v^n w^i dx \right| &\leq c\|u^k\|_{L_2} \|\partial_{x_j} v^n\|_{L_2} \|w^i\|_{L_\infty} \leq c_1|u| \cdot \|v\| \cdot |Aw|. \end{aligned} \tag{3.10}$$

Inequality (3.6) is proved.

In order to verify (3.7), we use the representation (1.3), which implies that the components of the difference $F_0(u, y) = F(u, u + y) - F(u, u)$ are defined by the formula

$$F_0^i(u, y) = \sum_{k,j,n=1}^3 C_{kjn}^i u^k \partial_{x_j} y^n + D_{kjn}^i y^k \partial_{x_j} u^n, \quad i = 1, 2, 3, \tag{3.11}$$

that is, these are linear combinations of monomials only of the form $u^k \partial_{x_j} y^n$ and $y^k \partial_{x_j} u^n$. It remains to use inequalities (3.9) and (3.10) for $v(\cdot) = y(\cdot)$ and we obtain inequality (3.7).

The space of functions in which we construct a solution of problem (3.1)–(3.3) depends on the class of the α -model under consideration.

First we consider an α -model of class I, for which $\langle F(u, v), v \rangle = 0$ for all $u, v \in H^1$.

Suppose that we are given a function $u(\cdot) \in L_2(0, M; H^3) \cap L_\infty(0, M; H^2)$. The corresponding function v in equation (3.2) satisfies

$$v(\cdot) = (I + \alpha^2 A)u(\cdot) \in L_2(0, M; H^1) \cap L_\infty(0, M; H).$$

We consider weak solutions $v(x, t)$ of (3.1), (3.2), which belong to the space $L_2(0, M; H^1) \cap L_\infty(0, M; H)$. It follows from inequality (3.5) that

$$PF(u(\cdot), v(\cdot)) \in L_2(0, M; H^{-1}).$$

Furthermore, $Av(\cdot) \in L_2(0, M; H^{-1})$. Then it follows from equation (3.1) that $\partial_t v(\cdot) \in L_2(0, M; H^{-1})$, and therefore, $\partial_t u(\cdot) \in L_2(0, M; H^1)$. Equation (3.1) can be considered in the space of distributions $\mathcal{D}'(0, M; H^{-2})$ (see [13]). By the Lions-Magenes lemma, $u(\cdot) \in C_w([0, M]; H^2)$; therefore the initial condition (3.3) makes sense for $u_0 \in H^2$.

We now state the existence and uniqueness theorem for weak solutions of the Cauchy problem for an α -model of class I.

Theorem 3.1. *Suppose that $g \in H$ and the operator $F(u, v)$ belongs to class I (see (1.6)). Then for any $u_0 \in H^2$ and for any $M > 0$ the problem (3.1)–(3.3) has a unique solution in the space of functions*

$$u(\cdot) \in L_2(0, M; H^3) \cap L_\infty(0, M; H^2), \quad \partial_t u(\cdot) \in L_2(0, M; H^1);$$

furthermore, the corresponding function $v(\cdot) = (I + \alpha^2 A)u(\cdot)$ satisfies

$$v(\cdot) \in L_2(0, M; H^1) \cap L_\infty(0, M; H), \quad \partial_t v(\cdot) \in L_2(0, M; H^{-1}),$$

and the following energy identity holds:

$$\frac{1}{2} \frac{d}{dt} |v(t)|^2 + \nu \|v(t)\|^2 = \langle g, v(t) \rangle, \quad t \geq 0, \tag{3.12}$$

in which the function $|v(t)|^2$, $t \geq 0$, is absolutely continuous; identity (3.12) holds for almost all $t \geq 0$ and the function $v(\cdot)$ belongs to $C([0, M]; H)$.

We now consider an α -model of class II. By definition

$$\langle F(u, v), u \rangle = 0 \quad \text{for all } u, v \in H^1.$$

We fix a function

$$u(\cdot) \in L_2(0, M; H^2) \cap L_\infty(0, M; H^1).$$

The corresponding function is

$$v(\cdot) = (I + \alpha^2 A)u(\cdot) \in L_2(0, M; H) \cap L_\infty(0, M; H^{-1}).$$

From (3.6) we obtain $B(u(\cdot), v(\cdot)) \in L_2(0, M; H^{-2})$. Furthermore, obviously, $Av \in L_2(0, M; H^{-2})$. Consequently, if $v(\cdot)$ satisfies (3.1), then $\partial_t v \in L_2(0, M; H^{-2})$ and therefore, $\partial_t u \in L_2^{\text{loc}}(0, M; H)$. We consider equation (3.1) in the space of distributions $\mathcal{D}'(0, M; H^{-2})$. It follows from the Lions-Magenes lemma that $u(\cdot) \in C_w([0, M]; H^1)$, that is, the initial condition (3.3) makes sense for $u_0 \in H^1$.

For class II we introduce the following auxiliary function:

$$w(\cdot) = (I + \alpha^2 A)^{1/2} u(\cdot) \implies v(\cdot) = (I + \alpha^2 A)^{1/2} w(\cdot).$$

Then, obviously,

$$w(\cdot) \in L_2(0, M; H^1) \cap L_\infty(0, M; H), \quad \partial_t w(\cdot) \in L_2(0, M; H^{-1}).$$

The function $w(x, t)$ occupies an ‘intermediate’ position between $u(x, t)$ and $v(x, t)$. Note that

$$|w(t)|^2 = |u(t)|^2 + \alpha^2 \|u(t)\|^2, \quad \|w(t)\|^2 = \|u(t)\|^2 + \alpha^2 |Au(t)|^2, \tag{3.13}$$

$$|v(t)|^2 = |w(t)|^2 + \alpha^2 \|w(t)\|^2, \quad \|v(t)\|^2 = \|w(t)\|^2 + \alpha^2 |Aw(t)|^2. \tag{3.14}$$

We have the following theorem on the existence and uniqueness of solutions of the Cauchy problem for α -models of class II.

Theorem 3.2. *Suppose that $g \in H$ and the operator $F(u, v)$ belongs to class II (see (1.7)). Then for any $u_0 \in H^1$ and any $M > 0$ the problem (3.1)–(3.3) has a unique solution in the space*

$$u(\cdot) \in L_2(0, M; H^2) \cap L_\infty(0, M; H^1), \quad \partial_t u(\cdot) \in L_2(0, M; H);$$

furthermore, the functions $v(\cdot) = (I + \alpha^2 A)u(\cdot)$, $w(\cdot) = (I + \alpha^2 A)^{1/2}u(\cdot)$ satisfy

$$\begin{aligned} v(\cdot) &\in L_2(0, M; H) \cap L_\infty(0, M; H^{-1}), & \partial_t v(\cdot) &\in L_2(0, M; H^{-2}), \\ w(\cdot) &\in L_2(0, M; H^1) \cap L_\infty(0, M; H), & \partial_t v(\cdot) &\in L_2(0, M; H^{-1}), \end{aligned}$$

and the energy identity

$$\frac{1}{2} \frac{d}{dt} |w(t)|^2 + \nu \|w(t)\|^2 = \langle g, u(t) \rangle, \quad t \geq 0, \tag{3.15}$$

holds for the function $w(\cdot)$. The function $|w(t)|^2$, $t \geq 0$, is absolutely continuous, (3.15) holds for almost all $t \geq 0$, and the function $w(\cdot)$ belongs to $C([0, M]; H)$.

Remark 3.1. Note that for α -models of class II we did not succeed in proving the energy identity for the function $v(\cdot) = (I + \alpha^2 A)u(\cdot)$ (as we did for class I), but only for the intermediate function $w(\cdot) = (I + \alpha^2 A)^{1/2}u(\cdot)$, which is smoother. Therefore the convergence of α -models of class II to the exact 3D Navier-Stokes system turns out to be weaker than that for α -models of class I (see § 4 and § 5).

Theorems 3.1 and 3.2 are proved in a fairly standard way, using the Galerkin approximation method. The proof repeats the well-known arguments for the existence and uniqueness theorem for a solution for the 2D Navier-Stokes system (see [19], [35] and [36]). We also note that the proof of Theorem 3.1 for the Leray α -model of class I was given in [8], and the proof of Theorem 3.2 for the LANS- α model of class II can be found in [5]. For general α -models of these classes the proof is similar.

We now prove identities (3.12) and (3.15) for α -models of class I and class II.

For class I we take the inner product in H of equation (3.1) with the function $v(t)$ and use the fact that $v(\cdot) \in L_2(0, M; H^1)$ and $\partial_t v(\cdot) \in L_2(0, M; H^{-1})$. We now apply Lemma 1.2 from Ch. 3 in [35], which implies that the function $|v(t)|^2$ is absolutely continuous, almost everywhere differentiable, and

$$\frac{d}{dt} |v(t)|^2 = 2\langle v(t), \partial_t v(t) \rangle.$$

For class I, from property (1.6) we obtain

$$\langle PF(u(t), v(t)), v(t) \rangle = 0$$

for $t \geq 0$, since, as already mentioned, $PF(u(\cdot), v(\cdot)) \in L_2(0, M; H^{-1})$. It remains to use the identity $\langle v(t), Av(t) \rangle = \|v(t)\|^2$ and obtain (3.12).

For class II we take the inner product in H of equation (3.1) with the function $u(t)$ and use the fact that $u \in L_2(0, M; H^2)$ and $\partial_t u(\cdot) \in L_2(0, M; H)$. Consequently,

$$A^{1/2}u \in L_2(0, M; H^1), \quad \partial_t A^{1/2}u(\cdot) \in L_2(0, M; H^{-1}).$$

Then from the same lemma we obtain

$$\begin{aligned} \frac{d}{dt}|u(t)|^2 &= 2\langle u, \partial_t u \rangle, \\ \frac{d}{dt}\|u(t)\|^2 &= 2\langle A^{1/2}u, \partial_t A^{1/2}u \rangle = 2\langle Au, \partial_t u \rangle = 2\langle u, \partial_t Au \rangle. \end{aligned}$$

For class II we use property (1.7) $F(u(\cdot), v(\cdot)) \in L_2(0, M; H^{-2})$ and obtain

$$\langle PF(u(t), v(t)), u(t) \rangle = 0$$

for $t \geq 0$. To complete the proof of identity (3.15) we use equations (3.13), (3.14) and apply the identity $(Av, u) = \|u(t)\|^2 + \alpha^2|Au(t)|^2$.

In the proof of Theorems 3.1 and 3.2 we also derive a priori estimates necessary for solutions, which can be conveniently stated in the form of consequences of the energy identities (3.12) and (3.15). These are proved by straightforward integration of these equations, using Poincaré’s inequality (0.1). We state these estimates in the same form for both classes of models.

Corollary 3.1. *Let $u(t)$ be a solution of the problem (3.1)–(3.3). We introduce the function $z(t) = v(t) = (I + \alpha^2 A)u(t)$ for class I, and $z(t) = w(t) = (I + \alpha^2 A)^{1/2}u(t)$ for class II. Then the following inequalities hold:*

$$|u(t)|^2 \leq |z(t)|^2 \leq |z(0)|^2 e^{-\nu\lambda_1 t} + \frac{|g|^2}{\nu^2\lambda_1^2}, \tag{3.16}$$

$$\nu \int_t^{t+1} \|u(s)\|^2 ds \leq \nu \int_t^{t+1} \|z(s)\|^2 ds \leq |z(0)|^2 e^{-\nu\lambda_1 t} + \frac{|g|^2}{(\nu\lambda_1)^2} + \frac{|g|^2}{\nu\lambda_1}. \tag{3.17}$$

Note that estimates (3.16) and (3.17) do not depend explicitly on α .

To give a complete picture we also state the smoothing property for solutions of α -models; the proof is fairly standard.

Proposition 3.1. *If $u(\cdot)$ is a solution of the problem (3.1)–(3.3), for class I the function is $z(t) = (I + \alpha^2 A)u(t)$, whilst for class II the function is $z(t) = (I + \alpha^2 A)^{1/2}u(t)$, then $z(t)$ satisfies the inequality*

$$t\|z(t)\|^2 + \nu \int_0^t s|Az(s)|^2 ds \leq \Psi(\alpha, t, |z(0)|, |g|) \quad \forall t > 0. \tag{3.18}$$

In this inequality, $\Psi(\alpha, z, r_1, r_2)$ is a monotonically increasing function in every argument z, r_1, r_2 , and $\Psi(\alpha, z, r_1, r_2) \rightarrow +\infty$ as $\alpha \rightarrow 0$.

To conclude this section we construct the global attractors for general α -models in both classes for a fixed $\alpha > 0$.

We construct a semigroup $\{S_\alpha(t)\}$ in H by the following rule. For an α -model of class I, for any $z_0 \in H$ we set $S_\alpha(t)z_0 = (I + \alpha^2 A)u(t)$, where $u(t)$ is the solution of problem (3.1)–(3.3) with the initial condition $u_0 = (I + \alpha^2 A)^{-1}z_0$. For an α -model of class II, for an arbitrary $z_0 \in H$ we set $S_\alpha(t)z_0 = (I + \alpha^2 A)^{1/2}u(t)$, where $u(t)$ is the solution of problem (3.1)–(3.3) with the initial condition $u_0 = (I + \alpha^2 A)^{-1/2}z_0$. It follows from Theorems 3.1 and 3.2 that the semigroup $\{S_\alpha(t)\}$ is well defined in H for α -models in both classes.

It follows from inequality (3.16) that the ball B_{R_0} in the space H with radius $R_0 = 2|g|/\nu\lambda_1$ is an absorbing set for the semigroup we have constructed. The set $P = S_\alpha(1)B_{R_0}$ is also absorbing, and from (3.18) we conclude that this set is bounded in the space H^1 , that is, the semigroup under consideration has a compact absorbing set in H . It is easy to verify that this semigroup is continuous in H .

It follows from the facts listed above that there exists a global attractor \mathcal{A}_α in the space H for the semigroup $\{S_\alpha(t)\}$, that is, the set \mathcal{A}_α is compact in H , $S_\alpha(t)\mathcal{A}_\alpha = \mathcal{A}_\alpha$ for $t \geq 0$, and $\text{dist}_H(S_\alpha(t)B, \mathcal{A}_\alpha) \rightarrow 0$ as $t \rightarrow \infty$ for any set of initial data $B = \{z_0\}$ bounded in H . Here we used the general theorem on global attractors of semigroups (see, for example, [18], [19] or [23]). Note that for some concrete α -models of classes I and II global attractors were constructed in [5], [8] and [10]–[12].

Finally we point out that it is also possible to construct a global attractor with respect to the more smooth variable $w(\cdot)$ for an α -model of class I, as it is for class II; this global attractor obviously coincides with $(I + \alpha^2)^{-1/2}\mathcal{A}_\alpha$ and is bounded in the space H^2 , while for an α -model of class II its global attractor \mathcal{A}_α with respect to the variable $w(\cdot)$ is merely bounded in H^1 . This shows that there is a substantial difference between models of different classes with respect to the degree of smoothing of the exact 3D Navier-Stokes system, which these models are designed to approximate. There is a similar difference in the comparison of their trajectory attractors in the limit as $\alpha \rightarrow 0+$ (see § 4 and § 5).

§ 4. The convergence of solutions of α -models

We need estimates of the derivatives $\partial_t v(\cdot)$ and $\partial_t w(\cdot)$ for solutions of the α -models under consideration that do not depend explicitly on time, similar to the estimates obtained for the functions $v(\cdot)$ and $w(\cdot)$ in Corollary 3.1.

Proposition 4.1. *Let $u(\cdot)$ be a solution of problem (3.1)–(3.3) and let the function be $v(\cdot) = (I + \alpha^2 A)u(\cdot)$. Then for class I,*

$$\left(\int_t^{t+1} \|\partial_t v(s)\|_{-2}^2 ds \right)^{1/2} \leq C_1 |v(0)|^2 e^{-\nu\lambda_1 t} + R_1^2, \tag{4.1}$$

and for class II,

$$\left(\int_t^{t+1} \|\partial_t w(s)\|_{-2}^2 ds \right)^{1/2} \leq \left(\int_t^{t+1} \|\partial_t v(s)\|_{-2}^2 ds \right)^{1/2} \leq C_1 |w(0)|^2 e^{-\nu\lambda_1 t} + R_1^2, \tag{4.2}$$

where $w(t) = (I + \alpha^2 A)^{1/2}u(t)$, the constant C_1 depends on λ_1 and ν , while R_1 depends on λ_1, ν and $|g|$, and the quantities C_1 and R_1 are independent of α .

Proof. We verify (4.2) for class II. The first inequality for $\partial_t w(\cdot)$ and $\partial_t v(\cdot)$ follows from (3.14). We further use inequality (3.6), from which it follows that

$$\|PF(u, v)\|_{-2} \leq c(\|u\| \cdot |v| + |u| \cdot \|u\|) \quad \forall u \in H^1, v \in H. \tag{4.3}$$

Substituting in the solution $u(t)$ of problem (3.1)–(3.3) and the corresponding function $v = u + \alpha^2 Au$, in view of (3.13) we obtain

$$\begin{aligned} \|PF(u(t), v(t))\|_{-2} &\leq 2c(|u(t)| \cdot \|u(t)\| + \alpha \|u(t)\| \alpha |Au|) \\ &\leq 2c(|u(t)|^2 + \alpha^2 \|u(t)\|^2)^{1/2} (\|u(t)\|^2 + \alpha^2 |Au|)^{1/2} = 2c|w(t)| \cdot \|w(t)\|. \end{aligned} \tag{4.4}$$

We now apply estimate (3.16), with $|z(t)| = |w(t)|$ for class II, and obtain

$$\|PF(u(t), v(t))\|_{-2}^2 \leq 4c^2 \left\{ |w(0)|^2 e^{-\nu\lambda_1 t} + \frac{|g|^2}{(\nu\lambda_1)^2} \right\} \|w(t)\|^2.$$

We integrate this inequality over the segment $[t, t + 1]$, use the second inequality in (3.17), with $\|z(t)\| = \|w(t)\|$ for class II, and obtain

$$\int_t^{t+1} \|PF(u(s), v(s))\|_{-2}^2 ds \leq 4c^2 \frac{1}{\nu} \left\{ |w(0)|^2 e^{-\nu\lambda_1 t} + \frac{|g|^2}{(\nu\lambda_1)^2} + \frac{|g|^2}{\nu\lambda_1} \right\}^2.$$

Consequently,

$$\left(\int_t^{t+1} \|PF(u(s), v(s))\|_{-2}^2 ds \right)^{1/2} \leq C |w(0)|^2 e^{-\nu\lambda_1 t} + R^2, \tag{4.5}$$

where $C = 2c\nu^{-1/2}$ and $R^2 = |g|^2/(\nu\lambda_1)^2 + |g|^2/\nu\lambda_1$.

From the estimate

$$\begin{aligned} \|Av\|_{-2}^2 &= |v|^2 = |u + \alpha^2 Au|^2 \leq (|u| + \alpha^2 |Au|)^2 \\ &\leq 2(|u|^2 + \alpha^4 |Au|^2) \leq 2(|u|^2 + \alpha^2 |Au|^2) \end{aligned}$$

for $\alpha \leq 1$ we conclude that

$$\int_t^{t+1} \|Av(s)\|_{-2}^2 ds \leq 2 \int_t^{t+1} (|u(s)|^2 + \alpha^2 |Au(s)|^2) ds \tag{4.6}$$

$$= 2 \int_t^{t+1} \|w(s)\|^2 ds \leq C |w(0)|^2 e^{-\nu\lambda_1 t} + R^2 \tag{4.7}$$

for suitable quantities C and R . Here we have once more used (3.17) for $z(\cdot) = w(\cdot)$.

The derivative $\partial_t v(\cdot)$ satisfies equation (3.1), therefore the second inequality in (4.2) for $\partial_t v(\cdot)$ is obtained from estimates (4.5) and (4.7).

Inequality (4.1) for class I is proved in a similar fashion if we extend the estimates (4.4) and (4.6) using the inequalities $|w(t)| \leq |v(t)|$ and $\|w(t)\| \leq \|v(t)\|$ (see (3.13)) and then use the stronger inequalities (3.16) and (3.17) established for class I.

We now combine the inequalities in Proposition 4.1 and Corollary 3.1 in order to obtain estimates for solutions of α -models in the norm of the space \mathcal{F}_+^b (see (2.8)).

Note that the inequalities

$$|u| \leq |w| \leq |v|, \quad \|u\| \leq \|w\| \leq \|v\|, \quad \|\partial_t u\|_{-2} \leq \|\partial_t w\|_{-2} \leq \|\partial_t v\|_{-2}$$

(see (3.13) and (3.14)) imply that

$$\|u(\cdot)\|_{\mathcal{F}_+^b} \leq \|w(\cdot)\|_{\mathcal{F}_+^b} \leq \|v(\cdot)\|_{\mathcal{F}_+^b}. \tag{4.8}$$

Proposition 4.2. *Let $u(t)$ be a solution of the α -model (3.1), (3.2). Let $z(t) = v(t) = (I + \alpha^2 A)u(t)$ for class I, and $z(t) = w(t) = (I + \alpha^2 A)^{1/2}u(t)$ for class II. Then the function $z(\cdot)$ belongs to \mathcal{F}_+^b and the following inequality holds*

$$\|T(h)u(\cdot)\|_{\mathcal{F}_+^b} \leq \|T(h)z(\cdot)\|_{\mathcal{F}_+^b} \leq C_2|z(0)|^2 \exp(-\nu\lambda_1 h) + R_2^2. \tag{4.9}$$

The quantities C_2 and R_2 are independent of α .

By analogy with the trajectory space \mathcal{K}_0^+ for the 3D Navier-Stokes system, which was introduced in §2, we define the trajectory space \mathcal{K}_α^+ for a given α -model of class I or II.

Definition 4.1. For class I, the space \mathcal{K}_α^+ consists of all functions

$$\mathcal{K}_\alpha^+ = \{v_\alpha(t) = (I + \alpha^2 A)u_\alpha(t), t \geq 0 \mid u_\alpha(0) \in H^2\},$$

where $u_\alpha(t)$ is the solution of problem (3.1)–(3.3) with the initial condition $u_\alpha(0) \in H^2$.

For class II the trajectory space \mathcal{K}_α^+ consists of all functions

$$\mathcal{K}_\alpha^+ = \{w_\alpha(t) = (I + \alpha^2 A)^{1/2}u_\alpha(t), t \geq 0 \mid u_\alpha(0) \in H^1\},$$

where $u_\alpha(t)$ is the solution of problem (3.1)–(3.3) with the initial condition $u_\alpha(0) \in H^1$.

We need the energy identities (3.12) and (3.15), which we rewrite in an equivalent integral form similar to the energy inequality (2.7) for the 3D Navier-Stokes system.

Proposition 4.3. *In class I, for any function $v \in \mathcal{K}_\alpha^+$ we have*

$$-\frac{1}{2} \int_0^\infty |v(t)|^2 \psi'(t) dt + \nu \int_0^\infty \|v(t)\|^2 \psi(t) dt = \int_0^\infty \langle g, v(t) \rangle \psi(t) dt \tag{4.10}$$

for any $\psi \in C_0^\infty(\mathbb{R}_+)$, and in class II, for any function $w \in \mathcal{K}_\alpha^+$ we have

$$-\frac{1}{2} \int_0^\infty |w(t)|^2 \psi'(t) dt + \nu \int_0^\infty \|w(t)\|^2 \psi(t) dt = \int_0^\infty \langle g, u(t) \rangle \psi(t) dt. \tag{4.11}$$

We consider the topological space Θ_+^{loc} introduced in §2. Recall that $\mathcal{F}_+^b \subset \Theta_+^{\text{loc}}$.

Lemma 4.1. *Let the sequences $\{u_n(\cdot)\} \subset \mathcal{F}_+^b$ and $\{\alpha_n\} \subset (0, 1)$ be such that $\alpha_n \rightarrow 0$. Set $z_n = (I + \alpha_n^2 A)u_n$ or $z_n = (I + \alpha_n^2 A)^{1/2}u_n$. If it is known that the sequence $\{z_n(\cdot)\}$ is bounded in \mathcal{F}_+^b and $z_n(\cdot) \rightarrow V(\cdot)$ in the topology of Θ_+^{loc} , where $V(\cdot) \in \mathcal{F}_+^b$, then $\{u_n(\cdot)\}$ is also bounded in \mathcal{F}_+^b and $u_n(\cdot) \rightarrow V(\cdot)$ in the topology of Θ_+^{loc} .*

Proof. The fact that the sequence $\{u_n(\cdot)\}$ is bounded in \mathcal{F}_+^b follows from inequality (4.8).

We now prove that $u_n(\cdot) \rightarrow V(\cdot)$ in Θ_+^{loc} . Consider the case $z_n = (I + \alpha_n^2 A)u_n$. By hypothesis, $z_n(\cdot) \rightarrow V(\cdot)$ in Θ_+^{loc} . Since $\{u_n(\cdot)\}$ is bounded in \mathcal{F}_+^b and any ball in \mathcal{F}_+^b is compact in the topology of Θ_+^{loc} , by passing to a subsequence we can assume that $u_n(\cdot) \rightarrow U(\cdot)$ in Θ_+^{loc} for some function $U \in \mathcal{F}_+^b$. In particular, for any $M > 0$

we have the weak convergence $u_n(\cdot) \rightharpoonup U(\cdot)$ in $L_2(0, M; H^1)$ (see (2.11)) and, consequently, $Au_n(\cdot) \rightharpoonup AU(\cdot)$ weakly in $L_2(0, M; H^{-1})$. By the hypothesis of the lemma, $\alpha_n \rightarrow 0$. Then, obviously, $\alpha_n^2 Au_n(\cdot) \rightharpoonup 0$ weakly in $L_2(0, M; H^{-1})$. Recall that

$$u_n + \alpha_n^2 Au_n = z_n.$$

Passing to the weak limit on both sides of the equation we have obtained, we find that $U(\cdot) = V(\cdot)$. Consequently, $u_n(\cdot) \rightharpoonup V(\cdot)$ in Θ_+^{loc} .

The case $z_n = (I + \alpha_n^2 A)^{1/2} u_n$ was proved in [31], Lemma 3.1.

We now state the main theorem of this section. For an arbitrary α -model of class I or II, we denote by $z_n(\cdot)$ elements of its trajectory space $\mathcal{K}_{\alpha_n}^+$.

Theorem 4.1. *Consider some α -model of class I or II with trajectory spaces \mathcal{K}_{α}^+ . Suppose that $z_n(\cdot) \subset \mathcal{K}_{\alpha_n}^+$, $n \in \mathbb{N}$, is a sequence of functions such that $\{z_n(\cdot)\}$ is bounded in \mathcal{F}_+^b and $\alpha_n \rightarrow 0$, and it is known that $z_n(\cdot) \rightarrow V(\cdot)$ in the topology of Θ_+^{loc} . Then the function $V(x, t)$ is a Leray-Hopf weak solution of the 3D Navier-Stokes system, that is, the solution satisfies the energy inequality (2.7) and, consequently, $V(\cdot) \in \mathcal{K}_0^+$, where \mathcal{K}_0^+ is the trajectory space of the Navier-Stokes system.*

Proof. We prove the theorem for class I. Let $v_n(\cdot) = z_n(\cdot)$ be the given sequence of trajectories in $\mathcal{K}_{\alpha_n}^+$. By the hypothesis of the theorem,

$$\|v_n(\cdot)\|_{\mathcal{F}_+^b} \leq C \quad \forall n \in \mathbb{N}. \tag{4.12}$$

Therefore the given convergence $v_n(\cdot) \rightarrow V(\cdot)$ in Θ_+^{loc} implies $\|V(\cdot)\|_{\mathcal{F}_+^b} \leq C$, that is, $V \in \mathcal{F}_+^b$.

Let $u_n(\cdot)$ denote the solution of the system (3.1), (3.2) that corresponds to the function $v_n(\cdot)$. Then it follows from Lemma 4.1 that $u_n(\cdot)$ is bounded in \mathcal{F}_+^b and

$$u_n(\cdot) \rightarrow V(\cdot) \quad \text{in the topology of } \Theta_+^{\text{loc}}. \tag{4.13}$$

We claim that $V(\cdot)$ is a weak solution of the 3D Navier-Stokes system on any interval $(0, M)$. The function $v_n(\cdot)$ satisfies the equation

$$\partial_t v_n = -\nu Av_n - PF(u_n, v_n) + g(x) \tag{4.14}$$

in the space $\mathcal{D}'(0, M; H^{-2})$, and $v_n = u_n + \alpha_n^2 Au_n$. By the hypothesis of the theorem,

$$v_n(\cdot) \rightarrow V(\cdot) \tag{4.15}$$

weakly in $L_2(0, M; H^1)$, weak-* in $L_\infty(0, M; H)$, and

$$\partial_t v_n(\cdot) \rightarrow \partial_t V(\cdot) \tag{4.16}$$

weakly in $L_2(0, M; H^{-2})$. Then the convergences (4.15) and (4.16) also hold in the weaker topology of the space $\mathcal{D}'(0, M; H^{-2})$. It follows from (4.15) that

$$Av_n(\cdot) \rightarrow AV(\cdot) \tag{4.17}$$

weakly in $L_2(0, M; H^{-1})$, and so also in $\mathcal{D}'(0, M; H^{-2})$. Consequently, in order to obtain

$$\partial_t V = -\nu AV - B(V, V) + g(x) \tag{4.18}$$

from (4.14), it is sufficient to prove that

$$PF(u_n, v_n) \rightarrow B(V, V) \quad \text{in } \mathcal{D}'(0, M; H^{-2}). \tag{4.19}$$

We substitute in $F(u, u + y) = F_0(u, y) + F(u, u)$ (see (3.4)). Then

$$PF(u_n, v_n) = PF(u_n, u_n + \alpha_n^2 Au_n) = PF_0(u_n, \alpha_n^2 Au_n) + PF(u_n, u_n).$$

It follows from condition (1.4) that $PF(u_n, u_n) = B(u_n, u_n)$, that is,

$$PF(u_n, v_n) = PF_0(u_n, \alpha_n^2 Au_n) + B(u_n, u_n). \tag{4.20}$$

It follows from inequality (3.5) that

$$\|PF_0(u_n, \alpha_n^2 Au_n)\|_{-2} \leq c \|u_n\| \cdot |\alpha_n^2 Au_n| = c \alpha_n^2 \|u_n\| \cdot |Au_n|. \tag{4.21}$$

By the Cauchy-Bunyakovskii inequality we conclude that

$$\alpha_n \int_0^M \|u_n(t)\| |\alpha_n |Au_n(t)| dt \leq \alpha_n \left(\int_0^M \|u_n(t)\|^2 dt \right)^{1/2} \left(\int_0^M \alpha_n^2 |Au_n(t)|^2 dt \right)^{1/2}. \tag{4.22}$$

Recall that

$$\|v\|^2 = \|u + \alpha^2 Au\|^2 = \|u\|^2 + 2\alpha^2 |Au|^2 + \alpha^4 \|Au\|^2.$$

Therefore it follows from (4.9) (see also the second inequality in (3.17)) that

$$\int_0^M (\|u_n(t)\|^2 + \alpha_n^2 |Au_n(t)|^2) dt \leq \int_0^M \|v_n(t)\|^2 dt \leq C_1 \quad \forall n \in \mathbb{N}, \tag{4.23}$$

since $v_n(\cdot)$ is bounded in \mathcal{F}_+^b by hypothesis. Consequently,

$$\int_0^M \|PF_0(u_n(t), \alpha_n^2 Au_n(t))\|_{-2} dt \leq \alpha_n C_2 \rightarrow 0, \quad \alpha_n \rightarrow 0,$$

by (4.21)–(4.23), that is, $PF_0(u_n, \alpha_n^2 Au_n) \rightarrow 0$ strongly in $L_1(0, M; H^{-2})$, and a fortiori in $\mathcal{D}'(0, M; H^{-2})$.

We now find the limit in $\mathcal{D}'(0, M; H^{-2})$ of the second summand $B(u_n, u_n)$ in (4.20). It follows from (4.13) that $u_n(t) \rightarrow V(t)$ weakly in $L_2(0, M; H^1)$ and the sequence $\{u_n(\cdot)\}$ is bounded in this space. Furthermore, $\partial_t u_n(t) \rightarrow \partial_t V(t)$ weakly in $L_2(0, M; H^{-2})$ and $\{\partial_t u_n(\cdot)\}$ is also bounded there. Applying Aubin’s compactness theorem (see [38], [39], [13]) we see that $u_n(t) \rightarrow V(t)$ strongly in $L_2(0, M; H)$. Recall that $L_2(0, M; H) \subset L_2(\mathbb{T}^3 \times [0, M])^3$; therefore we can assume that

$$u_n(x, t) \rightarrow V(x, t) \quad \text{as } n \rightarrow \infty, \quad \text{for almost all } (x, t) \in \mathbb{T}^3 \times [0, M]. \tag{4.24}$$

The operator $B(u, u)$ can be written in the form

$$B(u_n, u_n) = P \sum_{j=1}^3 \partial_{x_j} (u_n^j u_n). \tag{4.25}$$

It follows from (4.24) that for almost all $(x, t) \in \mathbb{T}^3 \times [0, M]$,

$$u_n^j(x, t) u_n(x, t) \rightarrow V^j(x, t) V(x, t), \quad n \rightarrow \infty. \tag{4.26}$$

Recall that $\{u_n(\cdot)\}$ is bounded in $L_2(0, M; H^1)$ and in $L_\infty(0, M; H)$. Then from Sobolev’s inequality

$$\|u\|_{L_4} \leq c \|u\|^{3/4} |u|^{1/4} \quad \forall u \in H^1$$

we conclude that the sequence $\{u_n^j(\cdot) u_n(\cdot)\}$ is bounded in $L_{4/3}(0, M; H)$, and also in the space $L_{4/3}(\mathbb{T}^3 \times [0, M])^3$. We apply the well-known result on weak convergence in the space L_p (Lemma 1.3 in Ch. 1 of [13]) and deduce from (4.26) that

$$u_n^j(\cdot) u_n(\cdot) \rightharpoonup V^j(\cdot) V(\cdot) \quad \text{as } n \rightarrow \infty, \quad \text{weakly in } L_{4/3}(\mathbb{T}^3 \times [0, M])^3,$$

and it converges weakly in $L_{4/3}(0, M; H)$. Then from (4.25) we obtain that

$$B(u_n, u_n) \rightharpoonup B(V, V) \quad \text{weakly in } L_{4/3}(0, M; H^{-1}),$$

and hence also in the space $\mathcal{D}'(0, M; H^{-2})$. We have proved the convergence (4.19) and we have established that the function $V(\cdot)$ satisfies the 3D Navier-Stokes system (4.18). It remains to verify that $V(\cdot)$ is a Leray-Hopf weak solution, that is, it satisfies the energy inequality (2.7).

Note that $v_n(\cdot)$ satisfies the energy identity (4.10):

$$-\frac{1}{2} \int_0^M |v_n(t)|^2 \psi'(t) dt + \nu \int_0^M \|v_n(t)\|^2 \psi(t) dt = \int_0^M \langle g, v_n(t) \rangle \psi(t) dt \tag{4.27}$$

for any $\psi \in C_0^\infty(0, M)$. Now suppose that $\psi(t) \geq 0$ for $t \in [0, M]$. By Aubin’s theorem it follows from (4.15) and (4.16) that $v_n(t) \rightarrow V(t)$ strongly in $L_2(0, M; H)$. Then, obviously,

$$\int_0^M \langle g, v_n(t) \rangle \psi(t) dt \rightarrow \int_0^M \langle g, V(t) \rangle \psi(t) dt. \tag{4.28}$$

Furthermore, for real functions we have $|v_n(t)| \rightarrow |V(t)|$ strongly in $L_2(0, M)$ as $n \rightarrow \infty$. In particular, passing if necessary to a subsequence, we can assume that

$$|v_n(t)| \rightarrow |V(t)| \quad \text{for almost all } t \in [0, M].$$

Consider the functions $|v_n(t)|^2 \psi'(t)$, $t \in [0, M]$. It follows from inequality (4.12) (see also (3.16)) that these functions have an integrable majorant on $[0, M]$. From the Lebesgue dominated convergence theorem we conclude that

$$\int_0^M |v_n(t)|^2 \psi'(t) dt \rightarrow \int_0^M |V(t)|^2 \psi'(t) dt. \tag{4.29}$$

Note that $v_n(t)\sqrt{\psi(t)} \rightharpoonup V(t)\sqrt{\psi(t)}$ weakly in the space $L_2(0, M; H^1)$ (this follows from (4.13)). Consequently,

$$\int_0^M \|V(t)\|^2 \psi(t) dt \leq \liminf_{n \rightarrow \infty} \int_0^M \|v_n(t)\|^2 \psi(t) dt. \tag{4.30}$$

Using relations (4.28)–(4.30) we pass to the limit in (4.27) and obtain the required inequality

$$-\frac{1}{2} \int_0^M |V(t)|^2 \psi'(t) dt + \nu \int_0^M \|V(t)\|^2 \psi(t) dt \leq \int_0^\infty \langle g, V(t) \rangle \psi(t) dt \tag{4.31}$$

for any $\psi \in C_0^\infty(0, M)$, $\psi \geq 0$.

We have proved that $V(\cdot) \in \mathcal{K}_0^+$, where \mathcal{K}_0^+ is the trajectory space of the Navier-Stokes system. This gives the proof of Theorem 4.1 for models of class I.

For α -models of class II the proof of Theorem 4.1 is given in [31] for the special case of the LANS- α model. But that proof only uses the characteristic properties (1.4) and (1.7) of models of class II, the a priori estimate (4.9), and inequalities (3.6) and (3.7). Therefore the arguments in that paper can be extended in a straightforward fashion to the general case of α -models of class II. Thus, the proof of Theorem 4.1 is complete.

§ 5. The convergence of trajectories of α -models to the trajectory attractor of the 3D Navier-Stokes system

We consider the trajectory attractor \mathfrak{A}_0 of the three-dimensional Navier-Stokes system (2.1), which was constructed in §2. Recall that $\mathfrak{A}_0 \subset \mathcal{K}_0^+$, the set \mathfrak{A}_0 is bounded in \mathcal{F}_+^b and compact in Θ_+^{loc} .

Let \mathcal{K}_0 denote the kernel of the system (2.1) consisting of all bounded (in the norm of \mathcal{F}^b) complete weak solutions $\{v(t), t \in \mathbb{R}\}$ of this system that are bounded in the space \mathcal{F}^b and satisfy the energy inequality (2.6) on the entire time axis \mathbb{R} . It was shown in §2 that $\mathfrak{A}_0 = \Pi_+ \mathcal{K}_0$.

Consider the trajectory space \mathcal{K}_α^+ of some α -model (3.1), (3.2) of class I or II for $\alpha > 0$. Recall that for models of class I the space \mathcal{K}_α^+ consists of the trajectories $z_\alpha(t) = v_\alpha(t) = (I + \alpha^2 A)u_\alpha(t)$, where $u_\alpha(t)$ is the solution of problem (3.1)–(3.3) with some initial condition in H^2 , and for models of class II the space \mathcal{K}_α^+ consists of the trajectories $z_\alpha(t) = w_\alpha(t) = (I + \alpha^2 A)^{1/2}u_\alpha(t)$, where the solution $u_\alpha(t)$ has initial condition in H^1 . In what follows we do not distinguish between the classes of the α -models under consideration.

Consider the shift operators $T(h)$, $h \geq 0$, acting on \mathcal{K}_α^+ according to the formula $T(h)z(t) = z(t+h)$, $t \geq 0$. Let $B_\alpha = \{z_\alpha(t), t \geq 0\}$ denote families of trajectories in \mathcal{K}_α^+ that are bounded with respect to the norm of \mathcal{F}_+^b . We state and prove the main theorem of the paper.

Theorem 5.1. *Suppose a family of sets $\{B_\alpha\}_{\alpha \in (0,1]}$, $B_\alpha = \{z_\alpha(t), t \geq 0\} \subset \mathcal{K}_\alpha^+$, that are uniformly bounded (with respect to α) in \mathcal{F}_+^b is given:*

$$\|B_\alpha\|_{\mathcal{F}_+^b} \leq R \quad \forall \alpha \in (0, 1]. \tag{5.1}$$

Then the family of shifted trajectories $\{T(h)B_\alpha\}_{\alpha \in (0,1]}$ converges to the trajectory attractor $\mathfrak{A}_0 = \Pi_+ \mathcal{K}_0$ of the Navier-Stokes system in the topology of Θ_+^{loc} as $h \rightarrow +\infty$ and as $\alpha \rightarrow 0+$:

$$T(h)B_\alpha \rightarrow \mathfrak{A}_0, \quad h \rightarrow +\infty, \quad \alpha \rightarrow 0+ \quad \text{in } \Theta_+^{\text{loc}}, \tag{5.2}$$

that is, for any neighbourhood $\mathcal{O}(\mathfrak{A}_0)$ of the set \mathfrak{A}_0 in the topology of Θ_+^{loc} there exist $h_1 = h_1(\mathcal{O})$ and $\alpha_1 = \alpha_1(\mathcal{O})$ such that

$$T(h)B_\alpha \subset \mathcal{O}(\mathfrak{A}_0) \quad \forall h \geq h_1, \quad \forall \alpha, \quad 0 < \alpha \leq \alpha_1.$$

Proof. Suppose that this is not the case, and relation (5.2) does not hold. Then there exist a neighbourhood $\mathcal{O}(\mathfrak{A}_0)$ in Θ_+^{loc} and sequences $\alpha_n \rightarrow 0$ and $h_n \rightarrow +\infty$ such that

$$T(h_n)B_{\alpha_n} \not\subset \mathcal{O}(\mathfrak{A}_0).$$

Consequently, there are trajectories $z_{\alpha_n}(\cdot) \in B_{\alpha_n}$ such that the functions

$$Z_{\alpha_n}(t) = T(h_n)z_{\alpha_n}(t) = z_{\alpha_n}(t + h_n)$$

do not belong to the neighbourhood $\mathcal{O}(\mathfrak{A}_0)$:

$$Z_{\alpha_n}(\cdot) \notin \mathcal{O}(\mathfrak{A}_0). \tag{5.3}$$

The function $Z_{\alpha_n}(t)$ is defined on the half-line $[-h_n, +\infty)$. Consider the corresponding function $u_{\alpha_n}(\cdot)$ which is a solution of the α -model (3.1), (3.2) for $\alpha = \alpha_n$ with respect to the variable u (that is, $u_{\alpha_n}(t) = (I + \alpha_n^2 A)^{-1} Z_{\alpha_n}(t)$ for class I, and $u_{\alpha_n}(t) = (I + \alpha_n^2 A)^{-1/2} Z_{\alpha_n}(t)$ for class II). Obviously, $u_{\alpha_n}(t)$ is a solution of the system (3.1), (3.2) on the half-line $[-h_n, +\infty)$, since (3.1), (3.2) is an autonomous system and $Z_{\alpha_n}(t)$ is the reverse time shift of the function $z_{\alpha_n}(t)$ by the time interval h_n . Furthermore, from the condition of uniform boundedness (5.1) we conclude that

$$\begin{aligned} \sup_{t \geq -h_n} |Z_{\alpha_n}(t)| + \left(\sup_{t \geq -h_n} \int_t^{t+1} \|Z_{\alpha_n}(s)\|^2 ds \right)^{1/2} \\ + \sup_{t \geq -h_n} \left(\int_t^{t+1} \|\partial_t Z_{\alpha_n}(s)\|_{-2}^2 ds \right)^{1/2} \leq R. \end{aligned} \tag{5.4}$$

It follows from the inequality thus obtained that the sequence $\{Z_{\alpha_n}(\cdot)\}$ is weakly compact in the space

$$\Theta_{-M,M} := L_2(-M, M; H^1) \cap L_\infty(-M, M; H) \cap \{u \mid \partial_t u \in L_2(-M, M; H^{-2})\}$$

for every M , if α_n is considered with indices n such that $h_n \geq M$. Consequently, for every fixed $M > 0$ we can find a subsequence $\{\alpha_{n'}\} \subset \{\alpha_n\}$ such that $\{Z_{\alpha_{n'}}(\cdot)\}$ converges weakly in $\Theta_{-M,M}$. Applying the standard Cantor diagonal procedure we construct a function $V(t)$, $t \in \mathbb{R}$, and a subsequence $\{\alpha_{n''}\} \subset \{\alpha_n\}$ such that

$$Z_{\alpha_{n''}}(\cdot) \rightarrow V(\cdot) \quad \text{weakly in } \Theta_{-M,M} \quad \text{as } n'' \rightarrow \infty \quad \text{for any } M > 0. \tag{5.5}$$

From (5.4) we obtain an inequality for the limit function $V(t)$, $t \in \mathbb{R}$:

$$\sup_{t \in \mathbb{R}} |V(t)| + \left(\sup_{t \in \mathbb{R}} \int_t^{t+1} \|V(s)\|^2 ds \right)^{1/2} + \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|\partial_t V(s)\|_{-2}^2 ds \right)^{1/2} \leq R, \tag{5.6}$$

and, in particular,

$$V(\cdot) \in \mathcal{F}^b = L^b_2(\mathbb{R}; H^1) \cap L_\infty(\mathbb{R}; H) \cap \{u \mid \partial_t u \in L^b_2(\mathbb{R}; H^{-2})\}.$$

We now apply Theorem 4.1, in which we can assume that the trajectories $z_n(t) = Z_{\alpha_n}(t)$ under consideration are defined on the half-axis $[-M, +\infty)$ instead of $[0, +\infty)$ (the equations are autonomous) and belong to the trajectory space $\mathcal{K}_{\alpha_n}^{-M}$ (instead of $\mathcal{K}_{\alpha_n}^0 := \mathcal{K}_{\alpha_n}^+$). Then from (5.5) and (5.6) we conclude that $V(t)$ is a weak solution of the three-dimensional Navier-Stokes system for all $t \in \mathbb{R}$, and $V(t)$ satisfies the energy inequality on the entire time axis, that is, $V(\cdot) \in \mathcal{K}_0$, where \mathcal{K}_0 is the kernel of the system (2.1). But $\Pi_+ \mathcal{K}_0 = \mathfrak{A}_0$ and, consequently, $\Pi_+ V(\cdot) \in \mathfrak{A}_0$. At the same time, from (5.5) we find that

$$\Pi_+ Z_{\alpha_{n''}}(\cdot) \rightarrow \Pi_+ V(\cdot) \quad \text{in } \Theta_+^{\text{loc}} \quad \text{as } n'' \rightarrow \infty,$$

and, in particular, for large n''

$$\Pi_+ Z_{\alpha_{n''}}(\cdot) \in \mathcal{O}(\Pi_+ V) \subseteq \mathcal{O}(\mathfrak{A}_0).$$

This contradicts (5.3). Consequently, (5.2) is true and Theorem 5.1 is proved.

Note that a similar picture is observed if, instead of trajectories $z_\alpha(t)$ in \mathcal{K}_α^+ , we consider the corresponding smoothed solutions $u_\alpha(t)$ of the system (3.1), (3.2); for class I these are the functions $u_\alpha(t) = (I + \alpha^2 A)^{-1} z_\alpha(t)$, and for class II, $u_\alpha(t) = (I + \alpha^2 A)^{-1/2} z_\alpha(t)$. Then to a set of trajectories $B_\alpha \subset \mathcal{K}_\alpha^+$ there will correspond the set of smoothed solutions $\tilde{B}_\alpha = (I + \alpha^2 A)^{-1} B_\alpha$ or $\tilde{B}_\alpha = (I + \alpha^2 A)^{-1/2} B_\alpha$.

Corollary 5.1. *If the hypotheses of Theorem 5.1 hold, then*

$$T(h)\tilde{B}_\alpha \rightarrow \mathfrak{A}_0, \quad h \rightarrow +\infty, \quad \alpha \rightarrow 0+ \quad \text{in } \Theta_+^{\text{loc}}, \tag{5.7}$$

where the set is $\tilde{B}_\alpha = (I + \alpha^2 A)^{-1} B_\alpha$ for class I, and $\tilde{B}_\alpha = (I + \alpha^2 A)^{-1/2} B_\alpha$ for class II. Here, $\{B_\alpha\}_{\alpha \in (0,1]}$ is an arbitrary bounded family of trajectories in \mathcal{K}_α^+ satisfying (5.1).

To prove (5.7) we apply (5.2) and Lemma 4.1.

If we use the inclusions (2.13), as for (2.14), we can obtain strong convergence in the Hausdorff metric in these spaces.

Corollary 5.2. *For any $M > 0$ and $\delta \in (0, 1]$,*

$$\begin{aligned} \text{dist}_{L_2(0,M;H^{1-\delta})}(T(h)B_\alpha, \mathfrak{A}_0) &\rightarrow 0, \\ \text{dist}_{C([0,M];H^{-\delta})}(T(h)B_\alpha, \mathfrak{A}_0) &\rightarrow 0, \quad h \rightarrow +\infty, \quad \alpha \rightarrow 0+, \end{aligned} \tag{5.8}$$

where $\{B_\alpha\}_{\alpha \in (0,1]}$ is an arbitrary bounded family of sets in \mathcal{K}_α^+ .

We now analyze the behaviour of trajectory attractors of general α -models as $\alpha \rightarrow 0+$.

We fix $\alpha > 0$ and consider some α -model of class I or II. We analyze the action of the translation semigroup $\{T(h)\}$ on the trajectory space \mathcal{K}_α^+ of this model. It is easy to verify that the space \mathcal{K}_α^+ is closed in the topology of Θ_+^{loc} . It follows from (4.9) that $\mathcal{K}_\alpha^+ \subset \mathcal{F}_+^{\text{b}}$ and there exists an absorbing set of the semigroup $\{T(h)\}$ on \mathcal{K}_α^+ that is bounded in \mathcal{F}_+^{b} and compact in Θ_+^{loc} , and the diameter of this absorbing set is independent of α . Then, similarly to § 2, for $\alpha > 0$ the trajectory attractor \mathfrak{A}_α is constructed for the α -model under consideration, that is, $\mathfrak{A}_\alpha \subset \mathcal{K}_\alpha^+$, the set \mathfrak{A}_α is bounded in \mathcal{F}_+^{b} , compact in Θ_+^{loc} , strictly invariant, so that $T(h)\mathfrak{A}_\alpha = \mathfrak{A}_\alpha$ for all $h \geq 0$, and

$$T(h)B_\alpha \rightarrow \mathfrak{A}_\alpha \quad \text{as } h \rightarrow +\infty$$

in the topology of Θ_+^{loc} for any bounded set $B_\alpha \subset \mathcal{K}_\alpha^+$. In addition, $\mathfrak{A}_\alpha = \Pi_+ \mathcal{K}_\alpha$, where \mathcal{K}_α is the kernel of the system (1.1), (1.2) (which for the class I is described using the functions $v(\cdot)$, and for the class II using the functions $w(\cdot)$). Finally, it follows from (4.9) that the trajectory attractors \mathfrak{A}_α are uniformly bounded in \mathcal{F}_+^{b} (with respect to $\alpha \in (0, 1]$):

$$\|\mathfrak{A}_\alpha\|_{\mathcal{F}_+^{\text{b}}} \leq R_2, \quad \text{where } R_2 \text{ is independent of } \alpha \in (0, 1]. \tag{5.9}$$

We next establish a simple connection between the trajectory attractor \mathfrak{A}_α of some α -model and its global attractor \mathcal{A}_α , which was constructed at the end of § 3.

Proposition 5.1. *The following identity holds:*

$$\mathfrak{A}_\alpha = \{z(t) = S_\alpha(t)z_0, t \geq 0 \mid z_0 \in \mathcal{A}_\alpha\}.$$

Using Proposition 3.1 it can be shown that for a fixed $\alpha > 0$ the trajectory attractor \mathfrak{A}_α is bounded (but not uniformly with respect to α !) in the space

$$\mathcal{F}_+^{\text{b},s} = \{z(\cdot) \mid z(\cdot) \in L_2^{\text{b}}(\mathbb{R}_+; H^2) \cap L_\infty(\mathbb{R}_+; H^1), \partial_t z(\cdot) \in L_2^{\text{b}}(\mathbb{R}_+; H^{-1})\},$$

and \mathfrak{A}_α is compact and attracts bounded families of trajectories in \mathcal{K}_α^+ in the strong topology of the space

$$\Theta_+^{\text{loc},s} = \{z(\cdot) \mid z(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+; H^2) \cap L_\infty^{\text{loc}}(\mathbb{R}_+; H^1), \partial_t z(\cdot) \in L_2^{\text{loc}}(\mathbb{R}_+; H^{-1})\}.$$

Of course, these strong properties are not preserved as $\alpha \rightarrow 0+$, since in the limit there is the three-dimensional Navier-Stokes system, for which all these questions are closely related to solution of the Millennium problem for the 3D Navier-Stokes system. Nevertheless, the following weak result holds.

Corollary 5.3. *The trajectory attractors \mathfrak{A}_α for α -models of both classes converge as $\alpha \rightarrow 0+$ in the topology of Θ_+^{loc} to the trajectory attractor \mathfrak{A}_0 of the 3D Navier-Stokes system:*

$$\mathfrak{A}_\alpha \rightarrow \mathfrak{A}_0, \quad \alpha \rightarrow 0+. \tag{5.10}$$

Furthermore, for classes I and II we have, respectively,

$$(I + \alpha^2 A)^{-1} \mathfrak{A}_\alpha \rightarrow \mathfrak{A}_0, \quad (I + \alpha^2 A)^{-1/2} \mathfrak{A}_\alpha \rightarrow \mathfrak{A}_0, \quad \alpha \rightarrow 0+. \tag{5.11}$$

In fact, we substitute $B_\alpha = \mathfrak{A}_\alpha$ into (5.2), use the strict invariance of the sets \mathfrak{A}_α , $T(h)\mathfrak{A}_\alpha = \mathfrak{A}_\alpha$ for $h \geq 0$, and obtain (5.10). By Theorem 5.1, relation (5.11) follows from (5.7).

In conclusion, similarly to Corollary 5.2 we can derive the following.

Corollary 5.4. *For any $M > 0$ and $\delta \in (0, 1]$,*

$$\text{dist}_{L_2(0, M; H^{1-\delta})}(\mathfrak{A}_\alpha, \mathfrak{A}_0) \rightarrow 0, \quad \text{dist}_{C([0, M]; H^{-\delta})}(\mathfrak{A}_\alpha, \mathfrak{A}_0) \rightarrow 0, \quad \alpha \rightarrow 0+. \tag{5.12}$$

§ 6. Minimal limits of the trajectory attractors \mathfrak{A}_α as $\alpha \rightarrow 0+$

Let \mathfrak{A}_α be the trajectory attractor of some α -model, $0 < \alpha \leq 1$. In §5 it was proved that $\mathfrak{A}_\alpha \subset B_{R_2}$, where B_{R_2} is the ball in \mathcal{F}_+^b with radius R_2 independent of α (see (5.9)).

Clearly, the trajectory attractor \mathfrak{A}_0 of the exact 3D Navier-Stokes system is also contained in B_{R_2} . Recall that the ball B_{R_2} equipped with the topology of Θ_+^{loc} is a metrizable space. We denote the corresponding metric on B_{R_2} by $\rho(\cdot, \cdot)$, and the metric space itself by B_ρ . Using this notation we can restate Corollary 5.3 as follows:

$$\text{dist}_\rho(\mathfrak{A}_\alpha, \mathfrak{A}_0) \rightarrow 0 \quad \text{as } \alpha \rightarrow 0+, \tag{6.1}$$

where, as usual, $\text{dist}_\rho(X, Y)$ denotes the (nonsymmetric) Hausdorff distance from a set X to a set Y in the metric ρ (see (0.2)). Note that the limit relation (6.1) is stronger than relations (5.12).

Recall that the set $\mathfrak{A}_0 \subset B_\rho$ is closed in this metric space B_ρ .

Definition 6.1. Let \mathfrak{A}_{\min} be the minimal closed subset of \mathfrak{A}_0 that satisfies (6.1), that is,

$$\lim_{\alpha \rightarrow 0+} \text{dist}_\rho(\mathfrak{A}_\alpha, \mathfrak{A}_{\min}) = 0$$

and \mathfrak{A}_{\min} is contained in every closed subset $\mathfrak{A}' \subseteq \mathfrak{A}_0$ such that

$$\lim_{\alpha \rightarrow 0+} \text{dist}_\rho(\mathfrak{A}_\alpha, \mathfrak{A}') = 0.$$

The set \mathfrak{A}_{\min} is called the *minimal limit of the trajectory attractors \mathfrak{A}_α as $\alpha \rightarrow 0+$* for the α -model under consideration.

One can prove that the set \mathfrak{A}_{\min} exists, is unique, and is defined by the formula

$$\mathfrak{A}_{\min} = \bigcap_{0 < \delta \leq 1} \left[\bigcup_{0 < \alpha \leq \delta} \mathfrak{A}_\alpha \right]_\rho.$$

In [31] this was established for the concrete LANS- α model, but the proof is easily extended to any α -model in either class.

We now state the final theorem of the paper.

Theorem 6.1. *The minimal limit \mathfrak{A}_{\min} of the trajectory attractors \mathfrak{A}_α as $\alpha \rightarrow 0+$ for any α -model is a connected component of the trajectory attractor \mathfrak{A}_0 , and the set \mathfrak{A}_{\min} is strictly invariant under the translation semigroup $\{T(h)\}$, that is,*

$$T(h)\mathfrak{A}_{\min} = \mathfrak{A}_{\min} \quad \forall h \geq 0.$$

Proof of Theorem 6.1 is given in [31] for the LANS- α model, but this proof extends word-for-word to general α -models of class I or II.

Note that the question of whether the trajectory attractor \mathfrak{A}_0 of the exact 3D Navier-Stokes system itself is connected remains unsolved. Furthermore, several years ago Professor Mark Vishik stated the following conjecture: to *different* 3D α -models of fluid dynamics (the LANS- α model, the Leray α -model, and others) there may correspond *different* minimal limits \mathfrak{A}_{\min} of their trajectory attractors \mathfrak{A}_α as $\alpha \rightarrow 0+$, which form *different* connected components of the attractor \mathfrak{A}_0 of the three-dimensional Navier-Stokes system.

In conclusion we point out that for α -models of class II we had no success in proving the convergence of trajectories in terms of the functions $v_\alpha(t) = (I + \alpha^2 A)u_\alpha(t)$, as we had for models of class I, since for class II we have proved only weaker a priori estimates for the functions $w_\alpha(t) = (I + \alpha^2 A)^{1/2}u_\alpha(t)$. We could conclude that α -models of class I (for example, the Leray α -model) ensure ‘stronger’ approximation of the original 3D Navier-Stokes system than α -models of class II (for example, the LANS- α model).

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