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On centres of relatively free associative algebras with a Lie nilpotency identity

A. V. Grishin and S. V. Pchelintsev

Abstract. We study central polynomials of a relatively free Lie nilpotent algebra $F^{(n)}$ of degree n. We prove a product theorem, which generalizes the well-known results of Latyshev and Volichenko. We construct generalized Hall polynomials, by using which we prove that the core centre of the algebra $F^{(n)}$ is nontrivial for any $n \geqslant 5$. We obtain a number of special results when $n = 5$ and 6.

Bibliography: 27 titles.

Keywords: Lie nilpotency identity, centre of an algebra, core polynomial, proper polynomial, extended Grassmann algebra.

Introduction

Let $F = \operatorname{Ass}[X]$ be a free associative k-algebra over a countable set $X = \{x_1, \dots, x_n, \dots\}$ of free generators. As usual, $[x_1, \dots, x_n]$ denotes a commutator of length $n \geqslant 2$, that is, $[x_1, x_2] = x_1x_2 - x_2x_1$ and by induction $[x_1, \dots, x_n] = \bigl[[x_1, \dots, x_{n-1}], x_n\bigr]$ for $n \geqslant 3$. Throughout what follows, $T^{(n)}$ denotes the T-ideal generated by a commutator $[x_1, \dots, x_n]$. Let $F^{(n)} = F/T^{(n)}$ be a relatively free algebra with the identity $[x_1, \dots, x_n] = 0$, which is called the *Lie nilpotency identity of degree* n and is denoted by $\operatorname{LN}(n)$.

Latyshev was the first to study the algebras $F^{(3)}$ and $F^{(4)}$ in the 1960s (see [1] and [2]). In particular, he constructed an additive basis of the algebra $F^{(3)}$ and proved that the variety of associative algebras with the identity $\operatorname{LN}(4)$ over a field of characteristic 0 is a Specht variety. In 1978 Volichenko [3] constructed an additive basis of the algebra $F^{(4)}$ over a field of characteristic 0.

Currently there are a lot of papers devoted to studying the algebras $F^{(n)}$ from various viewpoints (see [1]–[15]). Here, the algebra $F^{(3)}$ plays a special role. On the one hand, if $\operatorname{char} k = 0$, then the algebra $F^{(3)}$ is isomorphic to a free algebra of the variety $\operatorname{var} G$ generated by the Grassmann algebra G which Kemer [16], [17] used in a key way in giving a positive solution of Specht's problem. On the other hand, if $\operatorname{char} k = p > 0$, then the algebra $F^{(3)}$ is the first and so far the only source of constructions of infinitely based T-spaces and T-ideals (see [5]–[7]).

The centres of the algebras $F^{(3)}$ and $F^{(4)}$ over a field of characteristic $p > 3$ were described in [12], [13]. The problem of describing the centres of the algebras $F^{(n)}$ for $n \geqslant 5$ was stated in the same papers.

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Non-associative algebras with the identity $\mathrm{LN}(n)$ are also of interest. Alternative algebras with the identity $\mathrm{LN}(5)$ were studied by Vaulin [18], [19]. In particular, he found the identities of a Grassmann algebra in this variety. Right-alternative algebras with the identity $\mathrm{LN}(n)$, $n \leqslant 6$, were studied in [20]–[22]. Finally, interesting results about varieties of right-alternative metabelian algebras with the identity $\mathrm{LN}(n)$ were recently obtained by Kuz'min [23].

This paper is devoted to studying the algebras $F^{(n)}$ for $n \geqslant 5$. It has six sections. Throughout what follows, $\operatorname{char} k \neq 2, 3$, if not stipulated otherwise.

In § 1 we prove Theorem 1: $T^{(m)}T^{(n)} \subseteq T^{(m+n-1)}$ if at least one of the numbers m or n is odd. This theorem is a natural extension of the well-known results of Latyshev [2] and Volichenko [3].

In § 2 we introduce the notion of the core $Z^*(F^{(n)})$ of the algebra $F^{(n)}$; this is the largest ideal of the algebra $F^{(n)}$ contained in the centre $Z(F^{(n)})$. Now, in 1970 Zhevlakov posed the question of whether core elements exist in a free alternative algebra $\mathrm{Alt}[X]$. Filippov's well-known and remarkable theorem [24] states that there exist nonzero core elements in a k-algebra $\mathrm{Alt}[x_1, \dots, x_n]$ of rank $n \geqslant 5$.

In § 2, for the Hall polynomials $h := [[x, y]^2, z]$ and $h' := [[x, y]^2, x]$, we prove the relations

$$h \in Z(F^{(5)}) \setminus Z^*(F^{(5)}), \qquad h' \in Z^*(F^{(5)}).$$

Throughout what follows, to simplify the notation we identify polynomials in the algebra F with their images in the algebras $F^{(n)}$.

We also prove Theorem 2 in § 2, namely $Z^*(F^{(n)}) \neq 0$ for any $n \geqslant 4$. In studying the algebra $F^{(2n+1)}$, an important role is played by the extended Grassmann algebra $E^{(n)}$. This was constructed in [11] where it was called a model algebra of the variety $\operatorname{var} F^{(2n+1)}$, since it satisfies the identity $\mathrm{LN}(2n+1)$. Note that every core element of the algebra $F^{(2n+1)}$ is an identity of the algebra $E^{(n)}$; this implies the question posed in [11] has a negative answer, namely, it is proved that for $n \geqslant 2$ the algebra $E^{(n)}$ has identities that do not follow from $\mathrm{LN}(2n+1)$.

In § 3 we study central and proper polynomials in two variables of the algebras $F^{(5)}$ and $F^{(6)}$ over a field k of characteristic 0. We prove that every polynomial $f(a, b)$ in two variables satisfies the following conditions:

a) if $f(a, b) \in Z(F^{(5)})$, then $f(a, b) \in T^{(4)}$;

b) if $f(a, b) \in Z(F^{(6)})$, then $f(a, b) \in Z^*(F^{(6)})$.

It is easy to see that $h' \in V^{(4)}$. In § 4 we prove that $h \notin V^{(4)}$, where $V^{(4)}$ is the T-space generated by a commutator of degree 4.

In § 5 we show that the algebra $E^{(2)}$ does not have an identity of degree $\leqslant 4$; consequently, the weak Hall polynomial h' is a core element of the algebra $F^{(5)}$ of the least possible degree.

In § 6 we state some unsolved problems.

§ 1. Products of T-ideals $T^{(m)}T^{(n)}$

In 1965 Latyshev [2] proved that the inclusion $T^{(m)}T^{(n)} \subseteq T^{(m+n-2)}$ holds for any positive integers $m, n \geqslant 2$. Later Volichenko [3] noticed that the stronger relation $T^{(m)}T^{(3)} \subseteq T^{(m+2)}$ holds for $n = 3$. Thus, the question arose: for which numbers m, n does the inclusion $T^{(m)}T^{(n)} \subseteq T^{(m+n-1)}$ hold?

We give a complete answer to this question in our paper.

1.1. Auxiliary lemmas. Throughout what follows, $V^{(n)}$ denotes the T-space generated by the commutator $[x_1,\dots,x_n]$. Note that $V^{(2)}=[F,F]$, $V^{(n)}=[V^{(n-1)},F]$, and $T^{(2)}=F'$ is the commutator subalgebra of the algebra F, $T^{(n)}=V^{(n)}\circ F$, where $x\circ y=xy+yx$ is the Jordan product of elements x, y.

Throughout this section we assume that $v_i\in V^{(i)}$ and $x,y,z,t,a,b\in F$.

Lemma 1. *For any $x\in F$, $[V^{(m)},x][V^{(n-1-m)},x]\subseteq T^{(n)}$.*

Proof. If $a=v_m$, $b=v_{n-1-m}$, then

$$[a,x]\circ[x,b]=\bigl[[a,x]\circ x,b\bigr]-[a,x,b]\circ x=\bigl[a,x^2,b\bigr]-[a,x,b]\circ x\in T^{(n)},$$

since $[V^{(m+1)},V^{(n-m-1)}]\subseteq V^{(n)}$. This obviously yields the required result.

As usual we denote the inner derivation of the algebra F defined by an element a by $D_a\colon x\to[x,a]$.

Lemma 2. *The inclusion $T^{(i)}D_xD_y\subseteq T^{(i+2)}$ holds.*

Proof. By applying the Leibniz rule and Lemma 1 we obtain

$$\begin{aligned}(v_ia)D_xD_y&=(v_iD_xD_y)a+v_i(aD_xD_y)+(v_iD_x)(aD_y)+(v_iD_y)(aD_x)\\&\in V^{(i+2)}F+V^{(i)}V^{(3)}+T^{(i+2)}\subseteq T^{(i+2)}.\end{aligned}$$

The proof of the following lemma is presented for the completeness of the exposition.

Lemma 3 (see [3]). *If $n\geqslant 4$, $T^{(n-2)}T^{(3)}+T^{(3)}T^{(n-2)}\subseteq T^{(n)}$.*

Proof. We represent the arguments in several steps, working in the algebra $F^{(n)}$.

1°. $[v_{n-3},x]\circ[x,y,z]=0$. Setting $v_{n-2}=v_{n-3}D_x$, by Lemma 1 we have

$$0=[v_{n-3},x^2]D_yD_z=(v_{n-2}\circ x)D_yD_z=v_{n-2}\circ(xD_yD_z)=[v_{n-3},x]\circ[x,y,z].$$

2°. The element $f=[v_{n-3},x]\circ[y,z,t]$ is skew-symmetric with respect to x, y, z, t.

The fact that f is skew-symmetric with respect to x, y, z follows from part 1°; that it is skew-symmetric with respect to x, t follows from Lemma 1.

3°. $V^{(n-2)}V^{(3)}=0$. First we observe that $v_{n-2}\circ[x,y,y]=0$ by part 2°. Since, by the Jacobi identity, $3[a,b,c]$ is a linear combination of elements of the form $[x,y,y]$, we have the equation $V^{(n-2)}\circ V^{(3)}=0$. It remains to note that $[V^{(n-2)},V^{(3)}]=0$. Part 3° gives the required result.

Corollary 1. *The inclusion $T^{(3)}T^{(2)}+[T^{(3)},F]\subseteq T^{(4)}$ holds.*

1.2. Theorem on the products $T^{(m)}T^{(n)}$.

Theorem 1. *If one of the numbers $m,n\geqslant 2$ is odd, then*

$$T^{(m)}T^{(n)}\subseteq T^{(m+n-1)}.$$

Proof. We assume that m is odd and proceed by induction on m. The base of induction for $m = 3$ is true by Lemma 3. Assuming the induction hypothesis $T^{(m)}T^{(n)} \subseteq T^{(m+n-1)}$, by Lemmas 1 and 2 we have modulo $T^{(m+n+1)}$

$$\begin{aligned}0 &\equiv (v_m v_n)D_y D_z\\ &= (v_m D_y D_z)v_n + v_m(v_n D_y D_z) + (v_m D_y)(v_n D_z) + (v_m D_z)(v_n D_y)\\ &\equiv (v_m D_y D_z)v_n,\end{aligned}$$

which completes the proof.

Later we will show (see Lemma 6) that for even $m, n \geqslant 2$ we have

$$T^{(m)}T^{(n)} \not\subset T^{(m+n-1)}.$$

§ 2. The algebra $E^{(m)}$ and the core of the algebra $F^{(2m+1)}$

2.1. The extended Grassmann algebra $E^{(m)}$. Recall the construction of the algebras $E^{(m)}$ introduced in [11]. Let E be an associative algebra with unity 1 over a field k defined by a set of generators e_m ($m \in \mathbb{N}$), θ_{ij} ($i, j \in \mathbb{N}$, $1 \leqslant i \leqslant j$) and by the defining relations

$$e_i e_j + e_j e_i = \theta_{ij}, \qquad [\theta_{ij}, e_m] = 0.$$

Let Θ be the ideal of the algebra E generated by the elements θ_{ij}. The *extended Grassmann algebra of multiplicity* m is defined to be the quotient algebra $E^{(m)} = E/\Theta^m$; it was proved in [11] that the algebra $E^{(m)}$ satisfies the identity $\mathrm{LN}(2m+1)$.

Note that $E^{(1)} = G$ is the ordinary Grassmann algebra.

We claim that the algebra E has an additive basis consisting of the elements

$$v(\dots, \theta_{ij}, \dots)e_{i_1} \cdots e_{i_n},$$

where $v(\dots, \theta_{ij}, \dots)$ are commutative-associative monomials in the variables indicated and $1 \leqslant i_1 < \dots < i_n$.

Let $B = \{b_1, b_2, \dots\}$ be a basis of a space V, on which a symmetric bilinear form is defined with $q(e_i, e_j) = \theta_{ij} \cdot 1$ if $i \leqslant j$. Consider the Clifford algebra $\mathrm{Cl}(V, q)$ of the space V over the field of rational functions $k(\theta_{ij} \mid 1 \leqslant i \leqslant j)$ in the variables indicated (see [25]). Let V^* denote the subalgebra over the field k in $\mathrm{Cl}(V, q)$ generated by the set V. If ξ is a homomorphism $E \to V^*$ extending the map $e_i \to b_i$, then the elements $(v(\dots, \theta_{ij}, \dots)e_{i_1} \cdots e_{i_n})\xi$ are linearly independent over the field k.

2.2. Core elements and identities of the algebras $E^{(m)}$. We call the set

$$Z^*(F^{(n)}) = \bigl\{z \in Z(F^{(n)}) \mid (\forall\, x \in F^{(n)})\ zx \in Z(F^{(n)})\bigr\}$$

the *core of the algebra* $F^{(n)}$. It is easy to see that the core coincides with the largest ideal of the algebra $F^{(n)}$ contained in its centre. Elements of the core $Z^*(F^{(n)})$ are called *core* elements.

Proposition 1. *Every core element of the algebra $F^{(2m+1)}$ is an identity of the algebra $E^{(m)}$.*

Proof. The algebra $E^{(m)}$ has zero core $Z^*(E^{(m)})$. In fact, if $0 \neq f \in Z^*(E^{(m)})$, then in the algebra $E^{(m)}$ the element $[f \cdot e_N, e_{N+1}]$ is nonzero for a sufficiently large number N, which contradicts the fact that f is a core element.

2.3. The Hall polynomials. We consider the following polynomials:

$h(a,b,c) := [[a,b]^2, c]$ (the Hall polynomial);

$h'(a,b) := [[a,b]^2, b]$ (the weak Hall polynomial).

Lemma 4. *The following relations hold*:

a) $h(a,b,c) \in Z(F^{(5)}) \setminus Z^*(F^{(5)})$;

b) $0 \neq h'(a,b) \in Z^*(F^{(5)})$.

Proof. a) Since $[a,b]^2 = 0$ in the algebra $F^{(3)}$, it follows that $[a,b]^2 \in T^{(3)}$. Then by Lemma 2 we have

$$[a,b]^2 D_z D_t \in T^{(3)} D_z D_t \subseteq T^{(5)}.$$

Thus, we have proved that the Hall polynomial is central.

We now prove that the Hall polynomial is nonzero in the algebra $E^{(2)}$. We conduct calculations in the algebra E modulo Θ^2:

$$f = [e_1, e_2] \circ [e_1, e_3 e_4] = (2e_1 e_2 - \theta_{12}) \circ (e_1 e_3 e_4 - e_3 e_4 e_1).$$

Since

$$e_3 e_4 e_1 = e_3(-e_1 e_4 + \theta_{14}) = -e_3 e_1 e_4 + \theta_{14} e_3 = e_1 e_3 e_4 - \theta_{13} e_4 + \theta_{14} e_3,$$

it follows that

$$f = [e_1, e_2] \circ [e_1, e_3 e_4] = (2e_1 e_2 - \theta_{12}) \circ (\theta_{13} e_4 - \theta_{14} e_3) = 2(e_1 e_2) \circ (\theta_{13} e_4 - \theta_{14} e_3).$$

Next, taking the equation

$$e_4 e_1 e_2 = -e_1 e_4 e_2 + \theta_{14} e_2 = e_1 e_2 e_4 - \theta_{24} e_1 + \theta_{14} e_2$$

into account, we obtain

$$(e_1 e_2) \circ e_4 = e_1 e_2 e_4 + e_4 e_1 e_2 = 2e_1 e_2 e_4 - \theta_{24} e_1 + \theta_{14} e_2.$$

Similarly, $(e_1 e_2) \circ e_3 = 2e_1 e_2 e_3 - \theta_{23} e_1 + \theta_{13} e_2$. Consequently,

$$f = 2(e_1 e_2) \circ (\theta_{13} e_4 - \theta_{14} e_3) = 2e_1 e_2 e_4 \theta_{13} - 2e_1 e_2 e_3 \theta_{14},$$
$$[f, e_5] = 2[e_1 e_2 e_4, e_5]\theta_{13} - 2[e_1 e_2 e_3, e_5]\theta_{14} = 4e_1 e_2 e_4 e_5 \theta_{13} - 4e_1 e_2 e_3 e_5 \theta_{14}.$$

Of course, this implies that the Hall polynomial is not annihilated by any power of the commutator subalgebra of the algebra $E^{(2)}$, in particular, $h \notin Z^*\big(E^{(2)}\big)$.

b) We set $(x,y,z)^+ := (x \circ y) \circ z - x \circ (y \circ z)$. It is well known and is easy to verify that

$$(x,y,z)^+ = \big[y, [x,z]\big]. \tag{2.1}$$

We present the rest of the arguments as a sequence of steps.

1°. Since $[V^{(2)}, V^{(2)}, F] = [V^{(2)}, F, V^{(2)}] \subseteq T^{(5)}$, every Jordan associator $(u, v, x)^+$, $(u, x, v)^+$, $(x, u, v)^+$ containing elements $u, v \in V^{(2)}$ and $x \in F$ is equal to zero by equation (2.1).

2°. As before, let $w = [a, b]$. Then

$$f := [w^2, a] \circ [x, y] = [w, a] \circ w \circ [x, y] = -w \circ [b, a, a] \circ [x, y].$$

3°. $[x^2, a, a] = [x, a, a] \circ x + 2[x, a]^2$.

4°. $w \circ [x, a, a] \circ [x, y] = 0$.

This follows because, applying the identity indicated in part 3°, by Lemmas 1 and 3 we have

$$\begin{aligned} &w \circ [x, a, a] \circ [x, y] \\ &\quad = w \circ \big\{ \big[x \circ [x, y], a, a\big] - x \circ \big[[x, y], a, a\big] - 2[x, a] \circ [x, y, a] \big\} \\ &\quad = w \circ \big[[x^2, y], a, a\big] - w \circ \big[[x, y], a, a\big] \circ x - 2(w \circ [x, a]) \circ [x, y, a] \\ &\quad \in ([a, b] \circ [V^{(3)}, a]) \circ F + T^{(3)} T^{(3)} = 0. \end{aligned}$$

5°. $f = -w \circ [b, a, a] \circ [x, y] = w \circ [x, a, a] \circ [b, y] \in T^{(3)} T^{(3)} = 0$ by Lemma 3.

6°. $\big[[a, b]^2, a\big] \neq 0$ in $F^{(5)}$. In fact, the element $h'(a, b)$ has degree 5 and is not contained in $V^{(5)}$, since in the universal enveloping algebra for the free Lie algebra $\mathrm{Lie}[a, b]$ the element $[a, b][a, b, a]$ and the commutators of degree 5 are linearly independent by the Poincaré-Birkhoff-Witt theorem (PBW theorem); see [26].

2.4. Generalized Hall polynomials. Let $y_1, \ldots, y_n, z_1, \ldots, z_n$ be a set of distinct generators different from a, b, c. Then the *generalized Hall polynomials* of degree $2n + 5$ are defined to be the polynomials

$$H_n = [h, y_1, z_1, \ldots, y_n, z_n], \qquad H'_n = [h', y_1, z_1, \ldots, y_n, z_n];$$

further, the Hall polynomials h and h' coincide with H_0 and H'_0, respectively.

The following lemma is proved similarly to Lemma 4.

Lemma 5. *For any $n \geqslant 0$ the relations*

$$0 \neq H_n \in Z(F^{(2n+5)}) \cap T^{(2n+4)}, \qquad 0 \neq H'_n \in Z^*(F^{(2n+5)})$$

hold.

2.5. The core of the algebra $F^{(n)} (n \geqslant 4)$. It follows from the results in [1] and [3] that if $\operatorname{char} k = 0$, then

$$Z(F^{(3)}) = [F^{(3)}, F^{(3)}], \qquad Z^*(F^{(3)}) = 0,$$
$$Z(F^{(4)}) = T^{(3)} + [F^{(4)}, F^{(4)}]^2, \qquad Z^*(F^{(4)}) = T^{(3)}.$$

Lemma 6. *The element $aD_b^{2n-1} \cdot [x_1, y_1] \cdots [x_N, y_N] \notin T^{(2n+1)}$. Furthermore,*

$$T^{(2m)} T^{(2n)} \not\subset T^{(2n+2m-1)}.$$

Proof. It is sufficient to show that $aD_b^{2n-1} \notin \Theta^n$ in the algebra E.

We set $a = e_1$, $b = e_2$, $\theta = \theta_{12}$, $\eta = \theta_{22}$. We use induction to prove that

$$aD_b^{2m} = 2^m \eta^{m-1}(\eta a - \theta b). \tag{2.2}$$

First we verify the induction hypothesis for $m = 1$:

$$aD_b^2 = -\bigl[b, [a, b]\bigr] = -(a, b, b)^+ = -2(\theta b - \eta a) = 2(\eta a - \theta b).$$

Using the induction hypothesis we obtain

$$aD_b^{2(m+1)} = 2^m[\eta^{m-1}(\eta a - \theta b), b, b] = 2^m \eta^m [a, b, b] = 2^{m+1}\eta^m(\eta a - \theta b),$$

which proves the induction step.

Finally, using equation (2.2) we obtain

$$\begin{aligned} aD_b^{2n-1} &= [aD_b^{2(n-1)}, b] = 2^{n-1}\eta^{n-2}[(\eta a - \theta b), b] = 2^{n-1}\eta^{n-1}[a, b] \\ &= 2^n \eta^{n-1}(2ab - \theta) \equiv 2^{n+1}\eta^{n-1}ab \pmod{\Theta^n}, \end{aligned}$$

that is, $aD_b^{2n-1} \notin \Theta^n$.

Since by what was proved above we have the representations

$$\begin{aligned} e_1 D_{e_2}^{2m-1} &= 2^{m+1}\theta_{22}^{m-1} e_1 e_2 + \theta^{(m)}, \quad \text{where} \quad \theta^{(m)} \in \Theta^m, \\ e_3 D_{e_4}^{2n-1} &= 2^{n+1}\theta_{44}^{n-1} e_3 e_4 + \theta^{(n)}, \quad \text{where} \quad \theta^{(n)} \in \Theta^n, \end{aligned}$$

it follows that

$$e_1 D_{e_2}^{2m-1} \cdot e_3 D_{e_4}^{2n-1} \equiv 2^{m+n+2}\theta_{22}^{m-1}\theta_{44}^{n-1} e_1 e_2 e_3 e_4 \pmod{\Theta^{(m+n-1)}}.$$

Therefore,

$$T^{(2m)}T^{(2n)} \not\subset T^{(2n+2m-1)}.$$

Note that Lemma 6 shows that the restrictions in Theorem 1 are essential.

Furthermore, Theorem 1 and Lemmas 4, 5 imply the following.

Theorem 2. *For any $n \geqslant 4$, $Z^*(F^{(n)}) \neq 0$.*

Thus, for even $n \geqslant 4$ the algebra $F^{(n)}$ contains a core element of degree $n-1$, for odd $n \geqslant 4$ the algebra $F^{(n)}$ contains a core element of degree n. Later we will prove that the algebra $F^{(5)}$ does not contain core elements of degree 4. Obviously, the algebra $F^{(n)}$ does not contain central elements of degree $\leqslant n-2$.

§ 3. Proper and central polynomials in two variables in the algebras $F^{(5)}$ and $F^{(6)}$

Throughout this section we assume that $\operatorname{char}(k) = 0$. Recall that a polynomial $f \in F$ is said to be *proper* if $\partial f/\partial x_i = 0$ for any i. Commutators in generators (Lie monomials) are proper polynomials. Now, the proper polynomials form a subalgebra F_0 of the algebra F, which is generated by Lie monomials. Moreover, the PBW theorem describes an additive basis of the algebra F consisting of standard monomials in basis elements of a free Lie algebra (see [26]).

3.1. Central polynomials in two variables in $F^{(5)}$.

Lemma 7. *Let A be an algebra generated by elements a, b and satisfying the identities* LN(5) *and h'. Then $A'^3 = 0$, where A' is the commutator subalgebra of the algebra A.*

Proof. We present our proof in several steps, taking

$$T^{(m)} = T^{(m)}(A), \qquad V^{(m)} = V^{(m)}(A).$$

We see from part 1° in Lemma 4 that the Jordan associators $(u, v, x)^+$, $(u, x, v)^+$ and $(x, u, v)^+$ containing elements $u, v \in V^{(2)}$ and $x \in A$ are equal to 0.

1°. $w^2 \in Z(A)$, $w^3 = 0$, where $w = [a, b]$.

The weak Hall polynomial h' implies that $w^2 \in Z(A)$ and

$$0 = [[a, b^2] \circ w, a] = [(b \circ w) \circ w, a] = 2[b \circ w^2, a] = -4w^3.$$

We need the following two representations of the ideals A' and $T^{(m)}$, which are trivial to verify.

2°. $A' = wA + T^{(3)} = Aw + T^{(3)}$ and $(A')^2 \subset T^{(3)}$.

3°. $T^{(m)} = \sum_{c\in\{a,b\}} A[V^{(m-1)}, c] + T^{(m+1)}$.

4°. $T^{(4)} \cdot A' = 0$. Indeed, using parts 2° and 3°, by Lemmas 1 and 3 we obtain

$$\begin{aligned} T^{(4)} \cdot A' &= \left(\sum_{c\in\{a,b\}} A[V^{(3)}, c] \right) \cdot (wA + T^{(3)}) \\ &\subseteq \sum_{c\in\{a,b\}} A[V^{(3)}, c]wA + T^{(4)}T^{(3)} = 0. \end{aligned}$$

5°. $[T^{(3)}, A'] = 0$. Based on Corollary 1 and parts 2° and 4° we have

$$\begin{aligned} [T^{(3)}, A'] &= [T^{(3)}, Aw + T^{(3)}] = [T^{(3)}, Aw] = [V^{(3)}A, Aw] = [V^{(3)}, Aw]A \\ &\subseteq [V^{(3)}, A]wA \subseteq T^{(4)} \cdot A' = 0. \end{aligned}$$

6°. $A'^3 = 0$. Indeed, similarly to the above we have

$$A'^2 = (Aw + T^{(3)})(wA + T^{(3)}) \subseteq T^{(3)}w + Aw^2;$$

consequently, $A'^3 \subseteq (T^{(3)}w + Aw^2)(wA + T^{(3)}) = 0$, since $w^2 \in T^{(3)}$.

Proposition 2. *Every central polynomial in two variables $f(a, b)$ for the algebra $F^{(5)}$ is contained in the* T*-ideal $T^{(4)}$.*

Proof. An arbitrary polynomial $f(a, b)$ can be represented in the form

$$f(a, b) = \sum_{i,j} f_{i,j} a^i b^j,$$

where $f_{i,j}$ are proper polynomials.

If $f(a, b)$ is central, then applying the operators $\partial/\partial a$, $\partial/\partial b$ the requisite number of times we obtain that $f_{i,j}$ is also central. Using homogeneity considerations we can assume that the $f_{i,j}$ are homogeneous polynomials. Therefore we can assume without loss of generality that $f(a,b)$ is a homogeneous proper polynomial. Since the free associative algebra F does not have any nonzero central elements, we have $\deg f \geqslant 4$.

It follow from Lemma 7 that f is a consequence of the weak Hall polynomial h' in $F^{(5)}$ if $\deg f \geqslant 6$. If, however, $\deg f = 5$, then f is a linear combination of elements of the form $[a_1,a_2,a_3][a_4,a_5]$, where $a_1,\dots,a_5 \in \{a,b\}$. But every such element in the algebra $F^{(5)}$ is proportional to h'. Note that $h' \in T^{(4)}$.

Finally, if $\deg f = 4$, then f is a linear combination of elements of the form $[a_1,a_2,a_3,a_4]$ and $[a_1,a_2][a_3,a_4]$, where $a_1,\dots,a_4 \in \{a,b\}$. Note that $[a_1,a_2][a_3,a_4]$ is proportional to the element w^2, where $w = [a,b]$. It remains to observe that $[V^{(4)},x]=0$ and $[w^2,x] = h(a,b,x) \neq 0$ by Lemma 4.

3.2. Proper polynomials in two variable in $F^{(5)}$.

Proposition 3. *The relation* $[a,b]^3 \notin T^{(5)}$ *holds.*

Proof. Let $w = [a,b] \in V^{(2)}$. Then

$$\begin{aligned}[a,b^2,x,y,z] &= [w\circ b,x,y,z] \in [V^{(2)}\circ F,x,y,z] \subseteq [V^{(3)}\circ F + V^{(2)}\circ V^{(2)},y,z]\\ &\subseteq [V^{(4)}\circ F + V^{(3)}\circ V^{(2)},z] \subseteq V^{(5)}\circ F + V^{(4)}\circ V^{(2)} + V^{(3)}\circ V^{(3)}.\end{aligned}$$

This implies that every proper polynomial of degree 6 contained in the ideal $T^{(5)}$ can be represented in the form of a linear combination of the elements u_6, u_4u_2 and u_3v_3, where u_i, v_i are commutators of degree i. By applying the PBW theorem to the free Lie algebra $\mathrm{Lie}[a,b]$ we find that $[a,b]^3 \notin T^{(5)}$.

Proposition 4. a) *Proper central polynomials in two variables of degrees* 5 *and* 6 *in the algebra* $F^{(5)}$ *are exhausted by elements of the form*

$$\big[[a,b]^2,a\big], \qquad [a,b]^3.$$

b) *Proper polynomials in two variables of degree* $\geqslant 7$ *are identities in the algebra* $F^{(5)}$.

Proof. a) In essence this part was proved in Proposition 2.

b) Let $a_1,a_2,\ldots \in \{a,b\}$. Commutators of the form $[a_1,a_2,\dots,a_m]$, where $m \geqslant 2$, are said to be *regular*. It suffices to show that a product $\pi := v_1v_2\cdots v_l$ of regular commutators $v_1,v_2,\dots,v_l$ in which

$$\sum_{i=1}^{l}\deg(v_i) \geqslant 7, \qquad 4 \geqslant \deg(v_1) \geqslant \cdots \geqslant \deg(v_i) \geqslant 2$$

is zero.

If $\deg(v_1) = 2$, then $\pi = w^l$, where $w = [a,b]$ and $l \geqslant 4$. Since $w^2 \in T^{(3)}$, it follows that $\pi \in (T^{(3)})^2 \subseteq T^{(5)}$ by Lemma 3.

If $\deg(v_1) = 3$, then either $\deg(v_2) = 3$, or $\deg(v_2) = \deg(v_3) = 2$; therefore, again $\pi \in (T^{(3)})^2 \subseteq T^{(5)}$.

The case $\deg(v_1) = 4$ is considered in a similar fashion.

3.3. Central polynomials in two variables in $F^{(6)}$.

Lemma 8. *In the algebra $F^{(6)}$ both $\big[[a,b]^2,b,b\big]\neq 0$ and $\big[[a,b]^3,b\big]\neq 0$.*

Proof. We verify the second relation, since the first is obvious by the PBW theorem. To do this it is enough to show that every polynomial of the form $[a,b^2,x,y,z,t]$ is contained in the space $V^{(6)}\circ F+V^{(5)}\circ V^{(2)}+V^{(4)}\circ V^{(3)}$. Let $w=[a,b]\in V^{(2)}$; then

$$\begin{aligned}[a,b^2,x,y,z,t]&=[w\circ b,x,y,z,t]\in[V^{(2)}\circ F,x,y,z,t]\\&\subseteq[V^{(3)}\circ F+V^{(2)}\circ V^{(2)},y,z,t]\subseteq[V^{(4)}\circ F+V^{(3)}\circ V^{(2)},z,t]\\&\subseteq[V^{(5)}\circ F+V^{(4)}\circ V^{(2)}+V^{(3)}\circ V^{(3)},t]\\&\subseteq V^{(6)}\circ F+V^{(5)}\circ V^{(2)}+V^{(4)}\circ V^{(3)}.\end{aligned}$$

Proposition 5. *Every central polynomial in two variables $f(a,b)$ for the algebra $F^{(6)}$ is a core polynomial.*

Proof. Let

$$f(a,b)=\sum_{i,j}f_{i,j}a^ib^j,$$

where the $f_{i,j}$ are proper polynomials.

Following Proposition 2, we can assume that $f(a,b)$ is a homogeneous proper polynomial and $\deg f\geqslant 5$. If $\deg f=5$ and $f\notin T^{(5)}$, then we can assume that $f=[a,b,b][a,b]$. But $[a,b,b][a,b]\notin Z(F^{(6)})$ by Lemma 8. Therefore a proper polynomial of degree 5 is central only if it is contained in $V^{(5)}$.

Let $\deg f\geqslant 6$. By Theorem 1 we have $V^{(i)}V^{(j)}\subseteq Z(F^{(6)})$ if $i+j\geqslant 6$, and $T^{(i)}T^{(j)}=0$ if $i+j\geqslant 7$. Since $\big[[a,b]^3,b\big]\neq 0$ by Lemma 8, it is easy to see that it is sufficient to verify the following relations:

$$g_1,g_2,g_3\in Z^*(F^{(6)}),$$

where $[ab^m]=aD_b^m$ and $g_1=[ab^3][a,b]$, $g_2=[ab^2][a,b]^2$, $g_3=[a,b]^4$.

We verify each of the three relations:

$$g_1[x,y]=[ab^3][x,y][a,b]=\big(-\big[[ab^2],x\big][b,y]+t^{(5)}\big)[a,b]=0,$$

where $t^{(5)}\in T^{(5)}$,

$$g_2[x,y]=[ab^2][a,b]^2[x,y]\in T^{(3)}T^{(3)}T^{(2)}\subseteq T^{(5)}T^{(2)}=0,$$
$$g_3[x,y]=[a,b]^4[x,y]\in T^{(3)}T^{(3)}T^{(2)}=0.$$

§ 4. The Hall polynomials and the T-space $V^{(4)}$

It is easy to see that $h'(x,y)\in V^{(4)}$, where $V^{(4)}$ is the T-space generated by a commutator of degree 4. Indeed,

$$\big[[x,y]^2,x\big]=\big[[x,y],[x,y]\circ x\big]=\big[[x,y],[x^2,y]\big].$$

Proposition 6. *The element $h(x,y,z)$ satisfies the relation $h(x,y,z)\notin V^{(4)}$.*

Proof. Suppose the opposite, that $h(x,y,z)\in V^{(4)}$.

We observe that an element of degree 5 in $V^{(4)}$ is a linear combination of commutators of the form $[a,pq,b,c]$. Setting $w=[a,p]$ we have

$$\begin{aligned}[a,p^2,b,c] &= \big[w\circ p,b,c\big] = \big[[w,b]\circ p + w\circ[p,b],c\big]\\ &= [w,b,c]\circ p + [w,b]\circ[p,c] + [w,c]\circ[p,b] + w\circ[p,b,c].\end{aligned}$$

Consequently, $h(x,y,z)$ is a linear combination of elements of the form

$$\begin{aligned}[a,pq,b,c] &= [a,p,b,c]q + [a,q,b,c]p + [a,p,b][q,c] + [a,q,b][p,c]\\ &\quad + [a,p,c][q,b] + [a,q,c][p,b] + [a,p][q,b,c] + [a,q][p,b,c]\end{aligned}$$

and commutators of length 5.

We write down the necessary elements of the form $[a,pq,b,c]$ in the variables x, y, z, with degrees 2, 2 and 1 respectively.

a) If z is in the first position of the tuple (a,p,q,b,c), then we obtain four elements

$$[z,x^2,y,y],\quad [z,y^2,x,x],\quad [z,xy,x,y],\quad [z,xy,y,x].$$

b) If z is the second element of the tuple (a,p,q,b,c), then we can assume that $a=x$ and $q=y$; in this case we obtain the two elements

$$[x,zy,x,y],\quad [x,zy,y,x].$$

c) If $b=z$, then we have two more elements

$$[x,y^2,z,x],\quad [y,x^2,z,y].$$

d) If $c=z$, then we obtain the two elements

$$[x,y^2,x,z],\quad [y,x^2,y,z].$$

Thus, for suitable scalars $\lambda_1,\dots,\lambda_{10}$ we have the congruence modulo $V^{(5)}$

$$\begin{aligned}&\lambda_1\big([z,x,y,y]x + 2[z,x,y][x,y] + [x,y,y][z,x]\big)\\
&\quad + \lambda_2\big([z,y,x,x]y + 2[z,y,x][y,x] + [y,x,x][z,y]\big)\\
&\quad + \lambda_3\big([z,x,x,y]y + [z,y,x,y]x + [z,y,x][x,y]\\
&\qquad + [z,x,y][y,x] + [y,x,y][z,x] + [x,x,y][z,y]\big)\\
&\quad + \lambda_4\big([z,x,y,x]y + [z,y,y,x]x + [z,x,y][y,x]\\
&\qquad + [z,y,x][x,y] + [x,y,x][z,y]\big)\\
&\quad + \lambda_5\big([x,z,x,y]y + [x,y,x,y]z + [x,y,x][z,y]\\
&\qquad + 2[x,y,y][z,x] + 2[z,x,y][x,y]\big)\\
&\quad + \lambda_6\big([x,z,y,x]y + [x,y,y,x]z + [x,z,y][y,x] + [x,y,y][z,x]\\
&\qquad + [x,y,x][z,y] + [z,y,x][x,y]\big)\\
&\quad + \lambda_7\big([x,y,z,x]y + [x,y,z][y,x] + [x,y,x][y,z] + [y,z,x][x,y]\big)\\
&\quad + \lambda_8\big([y,x,z,y]x + [y,x,z][x,y] + [y,x,y][x,z] + [x,z,y][y,x]\big)\end{aligned}$$

$$
\begin{aligned}
&+ \lambda_9\big([x,y,x,z]y + [x,y,x][y,z] + [x,y,z][y,x] + [y,x,z][x,y]\big)\\
&+ \lambda_{10}\big([y,x,y,z]x + [y,x,y][x,z] + [y,x,z][x,y] + [x,y,z][y,x]\big)\\
\equiv\ & [x,y,z][x,y].
\end{aligned}
$$

Using the PBW theorem, we write down a system of linear equations by comparing the coefficients of the same basis products.

1) By applying the operator $\partial/\partial x$ we obtain

$$\lambda_1[z,x,y,y] + \lambda_3[z,y,x,y] + \lambda_4[z,y,y,x] + \lambda_8[y,x,z,y] + \lambda_{10}[y,x,y,z] = 0.$$

In what follows, to keep our notation concise we shall write $[abcd]$ instead of $[a,b,c,d]$. Now,

$$
\begin{gathered}
[y,x,z,y] = [z,[x,y],y] = [zxyy] - [zyxy],\\
[y,x,y,z] = [z,[x,y,y]] = [z,[x,y],y] - [z,y,[x,y]] = [zxyy] - 2[zyxy] + [zyyx],
\end{gathered}
$$

and so

$$
\begin{gathered}
\lambda_1[zxyy] + \lambda_3[zyxy] + \lambda_4[zyyx] + \lambda_8([zxyy] - [zyxy])\\
+ \lambda_{10}([zxyy] - 2[zyxy] + [zyyx]) = 0,
\end{gathered}
$$

that is,

$$
\begin{gathered}
\lambda_1[zxyy] + \lambda_8[zxyy] + \lambda_{10}[zxyy] + \lambda_3[zyxy] - \lambda_8[zyxy]\\
- 2\lambda_{10}[zyxy] + \lambda_4[zyyx] + \lambda_{10}[zyyx] = 0.
\end{gathered}
$$

Therefore,

$$\lambda_1 + \lambda_8 + \lambda_{10} = 0, \tag{4.1}$$

$$\lambda_3 - \lambda_8 - 2\lambda_{10} = 0, \tag{4.2}$$

$$\lambda_4 + \lambda_{10} = 0. \tag{4.3}$$

2) By applying the operator $\partial/\partial y$ we obtain

$$
\begin{aligned}
&\lambda_2[z,y,x,x] + \lambda_3[z,x,x,y] + \lambda_4[z,x,y,x]\\
&\qquad - \lambda_5[z,x,x,y] - \lambda_6[z,x,y,x] + \lambda_7[x,y,z,x] + \lambda_9[x,y,x,z] = 0.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
&\lambda_2[zyxx] + \lambda_3[zxxy] + \lambda_4[zxyx] - \lambda_5[zxxy] - \lambda_6[zxyx]\\
&\qquad + \lambda_7[zyxx] - \lambda_7[zxyx]) + \lambda_9[zyxx] - 2\lambda_9[zxyx] + \lambda_9[zxxy] = 0;
\end{aligned}
$$

therefore,

$$\lambda_2 + \lambda_7 + \lambda_9 = 0, \tag{4.4}$$

$$\lambda_3 - \lambda_5 + \lambda_9 = 0, \tag{4.5}$$

$$\lambda_4 - \lambda_6 - \lambda_7 - 2\lambda_9 = 0. \tag{4.6}$$

3) Applying the operator $\partial/\partial z$ we obtain

$$\lambda_5[x,y,x,y]+\lambda_6[x,y,y,x]=0.$$

Since $[x,y,x,y]=[x,y,y,x]$, it follows that

$$\lambda_5+\lambda_6=0. \tag{4.7}$$

4) We compare the coefficients of the element $[x,y,y][z,x]$:

$$\lambda_1+2\lambda_5+\lambda_6+\lambda_8+\lambda_{10}=0. \tag{4.8}$$

5) We compare the coefficients of the element $[y,x,x][z,y]$:

$$\lambda_2-\lambda_4-\lambda_5-\lambda_6+\lambda_7+\lambda_9=0. \tag{4.9}$$

6) We compare the coefficient of the elements $[z,x,y][x,y]$ and $[z,y,x][x,y]$:

$$\begin{aligned}(2\lambda_1-\lambda_3-\lambda_4+2\lambda_5+\lambda_6+\lambda_8)[z,x,y][x,y]&\\ +(2\lambda_2+\lambda_3+\lambda_4+\lambda_6-\lambda_7)[z,y,x][y,x]&\\ +(\lambda_7+\lambda_8+2\lambda_9+2\lambda_{10}+1)[z,[x,y]][x,y]&=0.\end{aligned}$$

Therefore we have

$$\begin{aligned}(2\lambda_1-\lambda_3-\lambda_4+2\lambda_5+\lambda_6+\lambda_8+(\lambda_7+\lambda_8+2\lambda_9+2\lambda_{10}+1))[z,x,y][x,y]&\\ +(2\lambda_2+\lambda_3+\lambda_4+\lambda_6-\lambda_7-(\lambda_7+\lambda_8+2\lambda_9+2\lambda_{10}+1))[z,y,x][y,x]&=0.\end{aligned}$$

Thus, we have two more equations:

$$2\lambda_1-\lambda_3-\lambda_4+2\lambda_5+\lambda_6+\lambda_7+2\lambda_8+2\lambda_9+2\lambda_{10}+1=0, \tag{4.10}$$

$$2\lambda_2+\lambda_3+\lambda_4+\lambda_6-2\lambda_7-\lambda_8-2\lambda_9-2\lambda_{10}-1=0. \tag{4.11}$$

It is easy to verify that the system of equations (4.1)–(4.11) is inconsistent.

§ 5. Identities of the algebra $E^{(2)}$

Lemma 9. *The element* $[a,x,x,b][x,c]\neq 0$ *is skew-symmetric with respect to linear variables in the algebra* $F^{(5)}$.

Proof. First,

$$[a,x,x,b][x,c]=-[a,x,x,x][b,c]\neq 0 \quad \text{in } E^{(2)}.$$

Second,

$$[a,x,x,b][x,b]=0,$$

since $[x,y,z,b][t,b]=0$ by Lemma 1. Third,

$$2[a,x,x,a][x,c]=[a,x,x,c]\circ[a,x]=\big[[a,x,x]\circ[a,x],c\big]=[h'(a,x),c]=0.$$

Lemma 10. *The extended Grassmann algebra* $E^{(2)}$ *does not satisfy a nontrivial identity of degree at most* 4.

Proof. Note that due to the restrictions on the characteristic we can assume that the question is about proper multilinear identities $f = 0$. Obviously, in $E^{(2)}$ there are no identities of degrees 2 or 3.

Suppose that an identity $f = 0$ of degree 4 holds in $E^{(2)}$. Then f has the form

$$f(y, x_1, x_2, x_3) = \sum_{\sigma \in S_3} \alpha_\sigma [y, x_{1\sigma}, x_{2\sigma}, x_{3\sigma}] + \sum_{\sigma \in A_3} \beta_\sigma [y, x_{1\sigma}] \circ [x_{2\sigma}, x_{3\sigma}],$$

where S_3 and A_3 are the symmetric and alternating groups of degree 3, respectively.

We claim that all the scalars β_σ are equal to 0. Suppose not. Since $f \in Z(F^{(5)})$, it follows that

$$[y, a] \circ [b, c] + \lambda [y, b] \circ [c, a] + \mu [y, c] \circ [a, b] \in Z(E^{(2)}).$$

Since the Hall polynomial is nonzero in $E^{(2)}$, we obtain $\lambda = \mu = 1$. Then

$$g(y, x_1, x_2, x_3, x) := \sum_{\sigma \in A_3} [[y, x_{1\sigma}][x_{2\sigma}, x_{3\sigma}], x] = 0.$$

In particular, $g(y, x_1, x_2, x_3, y) = 0$. We claim that this identity cannot hold in the algebra $E^{(2)}$. We conduct calculations in the algebra E assuming that $\theta_{ij} = 0$ $(i \neq j)$:

$$[e_1, e_2][e_3, e_4] = [e_1, e_3][e_4, e_2] = [e_1, e_4][e_2, e_3] = 4e_1e_2e_3e_4.$$

Therefore,

$$g(e_1, e_2, e_3, e_4, e_1) = 12e_1e_2e_3e_4.$$

Next, taking the equations $[e_ie_j, e_p] = 0$ if the indices i, j, p are distinct, the fact that $2e_1^2 = \theta_{11}$, and

$$[e_1e_2e_3e_4, e_1] = \theta_{11}e_2e_3e_4$$

into account, we obtain

$$g(e_1, e_2, e_3, e_4, e_1) = 12\theta_{11}e_2e_3e_4 \neq 0.$$

Thus, f has the form

$$f(y, x_1, x_2, x_3) = \sum_{\sigma \in S_3} \alpha_\sigma [y, x_{1\sigma}, x_{2\sigma}, x_{3\sigma}],$$

that is,

$$\begin{aligned} f(y, x_1, x_2, x_3) = \alpha_1 [y, x_1, x_2, x_3] + \beta_1 [y, x_1, x_3, x_2] + \alpha_2 [y, x_2, x_1, x_3] \\ + \beta_2 [y, x_2, x_3, x_1] + \alpha_3 [y, x_3, x_1, x_2] + \beta_3 [y, x_3, x_2, x_1]. \end{aligned}$$

Since $f[x_1, x_2] = 0$, it follows that

$$\big(\alpha_1 [y, x_1, x_2, x_3] + \alpha_2 [y, x_2, x_1, x_3]\big)[x_1, x_2] = 0.$$

Consequently, by Lemma 9,

$$(\alpha_1 + \alpha_2)[y, x_1, x_1, x_3][x_1, x_2] = 0, \qquad \alpha_1 + \alpha_2 = 0.$$

Similarly, $\beta_1 + \alpha_3 = 0$, $\beta_2 + \beta_3 = 0$. Thus,

$$f(y, x_1, x_2, x_3) = \alpha[y, [x_1, x_2], x_3] + \beta[y, [x_1, x_3], x_2] + \gamma[y, [x_2, x_3], x_1],$$

where $\alpha = \alpha_1$, $\beta = \beta_1$, $\gamma = \beta_2$. Then

$$f(x_1, x_1, x_1, x_3) = (\beta + \gamma)[x_1, [x_1, x_3], x_1];$$

therefore, $\beta + \gamma = 0$ by Lemma 6. Similarly,

$$\alpha + \beta = \beta + \gamma = 0.$$

Then

$$\alpha + \beta + \gamma = 0, \qquad \alpha = \beta = \gamma = 0;$$

therefore, $f = 0$. The lemma is proved.

As a by-product we have proved two corollaries.

Corollary 2. *An element of degree* 4 *is central in the algebra* $F^{(5)}$ *only if it is contained in the* T*-space* $V^{(4)}$.

Corollary 3. *The algebra* $F^{(5)}$ *does not contain nonzero core elements of degree* 4, *that is*, *the weak Hall polynomial* h' *is a core element of the least possible degree*.

§ 6. Some unsolved problems

In the preceding sections we have presented results which deal mainly with the centres $Z(F^{(n)})$ and $Z^*(F^{(n)})$ for $n = 5, 6$.

In the general case, a complete description of the centres of the algebras $F^{(n)}$ has not been obtained, but it is possible to give a partial description (that is, to find a fairly substantial part of the centre) under certain restrictions on the characteristic.

To do this we recall the following facts.

In the case of characteristic $p > 0$, the following definition plays an important role (see [11]). Let W_p be the T-space in $F^{(n)}$ generated by all p-words, that is, monomials in which every variable occurs with multiplicity p. Note that the T-space W_p is a subalgebra of the algebra $F^{(n)}$. If $p \geqslant n > 2$, then we have the equation

$$W_p = D_p \oplus CD_p,$$

where $D_p = \{x_i^p\}^T$ is the T-space generated by the pth power of a variable (a subalgebra isomorphic to the algebra of commutative polynomials in a countable set of variables), and

$$CD_p = W_p \cap T^{(2)}.$$

We list the basic known results about centres.

1. $Z(F^{(3)}) = V^{(2)}$ if $\operatorname{char} k = 0$;
   $Z(F^{(3)}) = D_p \oplus CD_p$ if $\operatorname{char} k = p > 0$.

2. $Z(F^{(4)}) = T^{(3)} + (V^{(2)})^2$ if $\operatorname{char} k = 0$;
   $Z(F^{(4)}) = (T^{(3)} + CD_p^2) \oplus D_p$ if $\operatorname{char} k = p > 3$.

3. $Z(F^{(5)}) \supseteq V^{(4)} + (h')^T + \{h\}^T$ if $\operatorname{char} k = 0$;
   $Z(F^{(5)}) \supseteq (V^{(4)} + (h')^T + \{h\}^T) \oplus D_p$ if $\operatorname{char} k = p > 5$;
   here, $\{h\}^T$ is the T-space generated by the polynomial h.

4. $Z(F^{(6)}) \supseteq T^{(5)} + Z(F^{(5)})V^{(2)} + Z(F^{(3)})V^{(4)}$ if $\operatorname{char} k = 0$;
   $Z(F^{(6)}) \supseteq (T^{(5)} + Z(F^{(5)})V^{(2)} + Z(F^{(3)})V^{(4)}) \oplus D_p$ if $\operatorname{char} k = p > 6$.

5. $Z(F^{(2m+5)}) \supseteq V^{(2m+4)} + (H'_m)^T + \{H_m\}^T$ if $\operatorname{char} k = 0$;
   $$Z(F^{(2m+5)}) \supseteq (V^{(2m+4)} + (H'_m)^T + \{H_m\}^T) \oplus D_p \quad \text{if } \operatorname{char} k = p > 2m+5;$$
   here, H'_m and H_m are the generalized Hall polynomials of degree $2m + 5$.

6. $$Z(F^{(2m+6)}) \supseteq T^{(2m+5)} + Z(F^{(2m+5)})V^{(2)} + Z(F^{(2m+3)})V^{(4)} + \dots + Z(F^{(3)})V^{(2m+4)} \quad \text{if } \operatorname{char} k = 0;$$
   $$Z(F^{(2m+6)}) \supseteq (T^{(2m+5)} + Z(F^{(2m+5)})V^{(2)} + Z(F^{(2m+3)})V^{(4)} + \dots + Z(F^{(3)})V^{(2m+4)}) \oplus D_p \quad \text{if } \operatorname{char} k = p > 2m+6.$$

The proofs of parts 1 and 2 can be found in [12] and [13], respectively. The key result for finding central polynomials in $F^{(n)}$ are Theorem 1 and Lemma 5. In the case of characteristic p, the arguments are completely analogous; only the T-space D_p is added, which is contained in the centre due to the identity $[x^p, y] = 0$ in the algebra $F^{(n)}$ when $p \geqslant n$.

We draw the reader's attention to some unsolved problems.

1) Is it true that the equations hold in parts 3–6?
2) It is easy to see that $H'_n \in V^{(2n+4)}$. Is it true that $H_n \notin V^{(2n+4)}$?
3) Is it true that $Z(F^{(5)}) \subseteq T^{(4)}$ and $Z^*(F^{(6)}) = T^{(5)}$?
4) Is it true that $Z^*(F^{(5)}) = T(E^{(2)}) = (\mathrm{LN}(5), h')^T$?

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