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Hermite-Padé approximation of exponential functions

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Abstract. The paper is concerned with diagonal Hermite-Padé polynomials of the first kind for the system of exponentials $\{e^{\lambda_j z}\}_{j=0}^{k}$ with arbitrary distinct complex parameters $\{\lambda_k\}_{j=0}^{k}$. An asymptotic formula for the remainder term is established and the location of the zeros is described. For real parameters the asymptotics are found and the extremal properties are described. The theorems obtained supplement the well-known results due to Borwein, Wielonsky, Saff, Varga and Stahl.

Bibliography: 43 titles.

Keywords: system of exponentials, Padé polynomials, Hermite-Padé polynomials, asymptotic equalities, the Laplace method, the saddle-point method.

§ 1. Introduction

In recent years there has been rapid growth of interest in Hermite-Padé approximations to exponential functions and their generalizations — in particular, in problems of approximation of analytic functions [1]–[3], problems of analytic continuation [4], [5], in applications to random matrices [6]–[8], operator theory [9], [10], Diophantine approximations including the irrationality measure of numbers [11], [12], in proofs of transcendence [12], [13], and in investigations of the algebraic nature of mathematical constants [14] (for more details, see the surveys [4], [5], [12], [15]–[17]).

The construction of such approximants is due to Charles Hermite in connection with the arithmetic properties of the number e. Ever since, Hermite-Padé approximants to exponential functions have attracted a great deal of attention from both classical authors (Hilbert, Klein, Lindemann, Mahler, Siegel) and famous modern mathematicians, and they continue to do so.

We shall adopt the terminology of [5], [18] and [19].

By *diagonal Hermite-Padé approximants of the second kind for the system of exponentials* $\{e^{jz}\}_{j=1}^{k}$ we shall mean the family of rational functions

$$\pi_{n,n}^{j}(z; e^{j\xi}) = \frac{P_n^j(z)}{Q_n(z)}, \qquad j = 1, 2, \dots, k,$$

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where the polynomials $P_n^1, P_n^2, \dots, P_n^k$, Q_n (known as the *diagonal Hermite-Padé polynomials of the second kind for the system of exponentials* $\{e^{jz}\}_{j=1}^k$) have degree at most kn and are found from the conditions

$$Q_n(z)e^{jz} - P_n^j(z) = O(z^{kn+n+1}), \qquad z \to 0. \tag{1.1}$$

The rational fractions $\{\pi_{n,n}^j(z; e^{j\xi})\}_{j=1}^k$ first appeared in Hermite's well-known paper [20], where he proved the transcendence of e. Analogues of Hermite fractions for systems of exponentials $\{e^{\lambda_p z}\}_{p=1}^k$, where λ_p are distinct algebraic numbers, were given by Lindemann (see [21]), who used them, in particular, to prove the transcendence of π. Aptekarev [22] proved that the rational functions $\pi_{n,n}^j(z; e^{\lambda_j \xi})$ converge uniformly to $e^{\lambda_j z}$ on compact subsets of $\mathbb{C}$ for systems of exponentials $\{e^{\lambda_j z}\}_{j=1}^k$ with arbitrary nonzero distinct complex coefficients λ_j in the exponents of the exponentials. For $k = 1$ this result is well-known and is due to Padé [23]. Starovoitov [24]–[26] described the asymptotic behaviour of the difference $e^{\lambda_j z} - \pi_{n,n}^j(z; e^{\lambda_j \xi})$ in the case when the λ_j are arbitrary distinct nonzero real or purely imaginary numbers (see also [27]).

Some time afterwards, Hermite [28] introduced the polynomials $A_0, A_1, \dots, A_k$ (which we shall call *diagonal Hermite-Padé polynomials of the first kind for the system of exponentials* $\{e^{jz}\}_{j=1}^k$) of degree at most $n-1$, not all identically equal to zero, for which

$$\sum_{p=0}^{k} A_p(z)e^{pz} = O(z^{kn+n-1}), \qquad z \to 0. \tag{1.2}$$

Based on the properties of Hermite-Padé polynomials of the first kind, as described in [28], Mahler [29] found another proof of the transcendence of e.

In the one-dimensional setting, Padé [23] posed the general problem of finding polynomials that satisfy equalities (1.1) and (1.2); the polynomials constructed in both cases were found to agree. In the multivariate setting $k \geqslant 2$, polynomials and Hermite-Padé approximants of the first and second kind for arbitrary systems of analytic functions have become the subject of intensive and systematic study after the appearance of the papers [13], [29], [30] by Mahler (the definition of Hermite-Padé approximants of the first kind can be found in [19], for example). (For an account of the contribution of other researchers in the development of the formal theory, see [15], [16], [31].) As we have already pointed out, both types of approximants, which are clearly distinct in the multivariate case, have numerous applications in various branches of analysis.

When $k = 1$ we obtain the classical Padé approximants to the exponential function. In this case, Padé's theorem states that the Padé polynomials

$$A_0(z) = -P_{n-1}^1(z) \quad \text{and} \quad A_1(z) = Q_{n-1}(z)$$

with the normalization $A_1(0) = 1$ satisfy the asymptotic equalities

$$A_0(z) = -e^{z/2}\biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr), \qquad A_1(z) = e^{z/2}\biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr)$$

as $n \to \infty$ locally uniformly in $z \in \mathbb{C}$ (that is, on compact subsets of $\mathbb{C}$).

With the help of explicit formulae, Borwein [32] found asymptotics for the diagonal Hermite-Padé polynomials of the first kind for the system $\{e^{pz}\}_{p=0}^{k}$ with $k=2$. This result was extended by Wielonsky [33] to arbitrary k. An analogue of Borwein's theorem for the system of exponentials $\{e^{\lambda_p z}\}_{p=0}^{2}$ with arbitrary distinct real parameters $\lambda_0<\lambda_1<\lambda_2$ was proved in [34] .

Our paper is concerned with certain properties of diagonal Hermite-Padé polynomials of the first kind for systems of exponentials $\{e^{\lambda_p z}\}_{p=0}^{k}$ with distinct arbitrary complex parameters $\{\lambda_p\}_{p=0}^{k}$. In particular, for the polynomials $\{A_n^p\}_{p=0}^{k}$, $\deg A_n^p \leqslant n-1$, satisfying the conditions

$$R_n(z)=\sum_{p=0}^{k} A_n^p(z)e^{\lambda_p z}=O(z^{kn+n-1}), \qquad z\to 0, \tag{1.3}$$

we give the asymptotics of the remainder term R_n. For real parameters $\lambda_0<\lambda_1<\dots<\lambda_k$ we find the asymptotics of A_n^p. We show that when the parameters λ_p in the exponents of the exponentials are real, normalized and appropriately transformed polynomials $\{A_{n+1}^p\}_{p=0}^{k}$ are solutions of the following extremal problem:

Given n, find the polynomials a_n^p, $p=0,1,\dots,k$, of degree at most n, where a_n^k is monic, that minimize the expression

$$E_n=E_n(\lambda_0,\lambda_1,\dots,\lambda_k;\rho)=\min_{\{a_n^p(z)\}_{p=0}^{k}}\left\|\sum_{p=0}^{k} a_n^p(z)e^{\lambda_p z}\right\|_{\rho}. \tag{1.4}$$

Here $\|h\|_\rho=\max\{|h(z)|: z\in D_\rho\}$, $D_\rho=\{z:|z|\leqslant\rho\}\subset\mathbb{C}$.

Our ultimate aim is to find the asymptotic law of decrease of the sequence $\{E_n\}_{n=1}^{\infty}$.

For $\lambda_p=p$, $p=0,1,\dots,k$, with $k=2$ and $\rho=1$ this problem was posed and solved by Borwein [32]. Wielonsky [33] examined the case $k\geqslant 2$ and $\rho<\pi/k$. Earlier Trefethen [35] and Braess [36] found the solution for $k=1$ for a disc and an interval.

One of the main results in this paper is as follows.

Theorem 1. *Let $\lambda_0<\lambda_1<\dots<\lambda_k$ be arbitrary real numbers and let $\rho<\pi/(\lambda_k-\lambda_0)$. Then, as $n\to\infty$,*

$$E_n\sim\frac{n!\,\lambda^{n+1}}{(kn+n+k)!}\rho^{kn+n+k},$$

where

$$\lambda=\prod_{p=0}^{k-1}(\lambda_k-\lambda_p).$$

All the main results in this paper, including Theorem 1, were obtained by analyzing the asymptotic properties of integral representations of the remainder term R_n and the polynomials A_n^p. The asymptotic properties of Hermite-Padé approximants of the second kind to exponential functions were investigated (with the help of Laplace's method) in [24]–[26]. In our approach, Laplace's method is combined with the saddle-point method; both rely on a further refinement of Wielonsky's method, which he outlined in his fundamental paper [33] (see also [24]–[26]).

§ 2. Preliminary results

In this and the next section, the λ_p are arbitrary distinct complex numbers and $|\lambda_0| \leqslant |\lambda_1| \leqslant \cdots \leqslant |\lambda_k|$.

Polynomials $A_n^0, A_n^1, \dots, A_n^k$ satisfying equalities (1.3) can be obtained by solving a linear system of $kn+n-1$ homogeneous equations with $kn+n$ unknown coefficients. In this case, a nontrivial solution always exists. Moreover, such nontrivial solutions can be written down explicitly. Indeed, let C_p be the boundary of a disc with centre at λ_p and whose radius is so small that all the remaining λ_j lie in the complement of this disc; let C_∞ be the boundary of a disc with centre at the origin and whose radius is so large that all the λ_j, $j=0,1,2,\dots,k$, lie in its interior. Using Cauchy's residue theorem it is easy to show that the functions

$$A_n^p(z) = \frac{e^{-\lambda_p z}}{2\pi i} \int_{C_p} \frac{e^{\xi z}\,d\xi}{[\varphi(\xi)]^n}, \qquad 0 \leqslant p \leqslant k, \tag{2.1}$$

$$R_n(z) = \frac{1}{2\pi i} \int_{C_\infty} \frac{e^{\xi z}\,d\xi}{[\varphi(\xi)]^n}, \tag{2.2}$$

where $\varphi(\xi) = (\xi-\lambda_0)(\xi-\lambda_1)\cdots(\xi-\lambda_k)$, satisfy (1.3) and all other conditions.

Next, we shall consider the normalized function $\widetilde{R}_{n-1}$ obtained by dividing R_n by the leading coefficient of the polynomial A_n^k. In order to find its value, setting $p=k$ in (2.1) we differentiate it $n-1$ times. As a result, the value of the leading coefficient of A_n^k agrees with that of the integral

$$\frac{1}{2\pi i(n-1)!} \int_{C_k} \frac{d\xi}{(\xi-\lambda_k)(\xi-\lambda_0)^n(\xi-\lambda_1)^n\cdots(\xi-\lambda_{k-1})^n},$$

which, after evaluation by Cauchy's integral formula, is found to be

$$\frac{1}{(n-1)!\prod_{p=0}^{k-1}(\lambda_k-\lambda_p)^n} = \frac{\lambda^{-n}}{(n-1)!}.$$

We give several assertions without proof which we require in the sequel in a convenient form (see [37], Ch. VII, § 43 and § 45).

Assertion 1 (Laplace's method). *Let $f(x)$ and $S(x)$ be continuous functions on $[a,b]$, where $S(x)$ is real and $f(x)$ may assume complex values. We set*

$$I_n = \int_a^b f(x) e^{nS(x)}\,dx.$$

Suppose that the absolute maximum of $S(x)$ on $[a,b]$ is attained at a point $x_0 \in (a,b)$ (that is, $S(x) < S(x_0)$ for $x \neq x_0$ and $S''(x_0) \neq 0$) and that $f(x)$ and $S(x)$ are both infinitely differentiable near x_0. Then, if $f(x_0) \neq 0$, the asymptotic equality

$$I_n = \sqrt{-\frac{2\pi}{nS''(x_0)}}\, e^{nS(x_0)}\biggl(f(x_0) + O\biggl(\frac{1}{n}\biggr)\biggr)$$

holds as $n \to +\infty$.

Assertion 2 (the saddle-point method). *Suppose that the functions $f(z)$ and $S(z)$ are regular in some domain G containing a piecewise smooth curve γ and let*

$$F_n = \int_\gamma f(\xi)e^{nS(\xi)}\,d\xi.$$

Suppose that $\max\{\operatorname{Re} S(\xi) : \xi \in \gamma\}$ *is attained only at a point z_0 which is an interior point of the contour γ and is a saddle point; that is, $S'(z_0) = 0$, $S''(z_0) \neq 0$. Suppose further that near z_0 the contour γ passes through both sectors in which* $\operatorname{Re} S(\xi) < \operatorname{Re} S(z_0)$ *(see* [37], *Ch.* VII, § 45*). If $f(z_0) \neq 0$, then*

$$F_n = \sqrt{-\frac{2\pi}{nS''(z_0)}}\,e^{nS(z_0)}\left(f(z_0) + O\left(\frac{1}{n}\right)\right) \tag{2.3}$$

as $n \to \infty$.

The branch of the root function in (2.3) *is chosen from the conditions*

$$\arg\sqrt{-\frac{1}{S''(z_0)}} = \varphi_0,$$

where φ_0 is the angle between the tangent to the curve l at z_0 and the positive direction of the real axis and l is the path of steepest descent passing through z_0, that is, the following conditions are satisfied on l near z_0: $\operatorname{Im} S(z) = \operatorname{Im} S(z_0)$ *for $z \in l$;* $\operatorname{Re} S(z) < \operatorname{Re} S(z_0)$ *for $z \in l$, $z \neq z_0$.*

Two sequences $\{\alpha_n\}$ and $\{\beta_n\}$, which both tend either to zero or to infinity, are called equivalent ($\alpha_n \sim \beta_n$) if $\lim_{n\to\infty} \alpha_n/\beta_n = 1$ as $n \to \infty$.

§ 3. Asymptotic behaviour of the remainder term R_n

Theorem 2. *Let $\{\lambda_p\}_{p=0}^k$ be arbitrary distinct complex numbers. Then*

$$R_n(z) \sim \frac{\exp\{\frac{\lambda_0+\lambda_1+\dots+\lambda_k}{k+1}z\}}{(kn+n-1)!}z^{kn+n-1} \tag{3.1}$$

as $n \to \infty$ uniformly in z on compact subsets of $\mathbb{C}$.

Proof. We assume without loss of generality that $\lambda_0 = 0$. The general case can be reduced to this by multiplying (1.3) by $e^{-\lambda_0 z}$.

Since $R_n(0) = 0$, (3.1) is true with $z = 0$. We take an arbitrary fixed $z \neq 0$ and change to $z = nw$ in (2.2). This gives

$$R_n(nw) = \frac{1}{2\pi i}\int_{C_\infty}\frac{d\xi}{[e^{-\xi w}\varphi(\xi)]^n}. \tag{3.2}$$

We will find the critical points of the function $\psi(\xi) = e^{-\xi w}\varphi(\xi)$ (the zeros of $\psi'(\xi)$). These are the roots of the equation

$$w\varphi(\xi) = \varphi'(\xi),$$

which can be written as

$$w=\frac{1}{\xi}+\frac{1}{\xi-\lambda_1}+\cdots+\frac{1}{\xi-\lambda_k}\,. \tag{3.3}$$

The contour C_∞ encloses all the λ_p. We seek a critical point on the contour C_∞ lying sufficiently far from the origin. More precisely, we assume that the distance of the critical point from the origin is greater than $2|\lambda_k|$. In this case, changing to $\zeta=1/\xi$, we expand the right-hand side of (3.3) in a power series

$$w=(k+1)\zeta+(\lambda_1+\lambda_2+\cdots+\lambda_k)\zeta^2+(\lambda_1^2+\lambda_2^2+\cdots+\lambda_k^2)\zeta^3+\cdots. \tag{3.4}$$

Inverting the series (3.4) using the Lagrange-Bürmann formulae (see [37], Ch. V, § 31) and returning to the previous variable ξ, we find the behaviour of the critical point ξ_0 with respect to the values of w; in view of the change $z=nw$ the latter lie in a sufficiently small neighbourhood of the origin:

$$\xi_0=\frac{k+1}{w}+\frac{\lambda_1+\lambda_2+\cdots+\lambda_k}{k+1}+O(w). \tag{3.5}$$

Let us now define the contour C_∞ so that it passes through ξ_0, surrounds all the points $\lambda_0,\lambda_1,\dots,\lambda_k$, and furthermore, the absolute value of the function $\psi(\xi)$ on C_∞ attains its minimum at the unique point ξ_0. To this end we consider the level lines of the functions $\varphi(\xi)$ and $e^{-w\xi}$ that pass through the point ξ_0,

$$L=\bigl\{\xi\in\mathbb{C}:|\varphi(\xi)|=|\varphi(\xi_0)|\bigr\}\quad\text{and}\quad L_1=\bigl\{\xi\in\mathbb{C}:|e^{-w\xi}|=|e^{-w\xi_0}|\bigr\}.$$

Note that L is a lemniscate, while L_1 is a straight line through ξ_0 making an angle $\arg(i/w)$ with the positive direction of the abscissa axis. Writing the equation of the lemniscate L,

$$\left|\varphi(\xi_0)+\frac{\varphi'(\xi_0)}{1!}(\xi-\xi_0)+\cdots+\frac{\varphi^{(k+1)}(\xi_0)}{(k+1)!}(\xi-\xi_0)^{(k+1)}\right|=|\varphi(\xi_0)|,$$

and taking the fact that $\varphi'(\xi_0)=w\varphi(\xi_0)$ into account, it is easily seen that the slope of the tangent to L at the point ξ_0 is $\tan(\arg(i/w))$. So L_1 is tangent to L at ξ_0.

According to [38], Ch. III, § 3.3, for sufficiently small $|w|$ the lemniscate L is a Jordan analytic curve, which encloses all the zeros of $\varphi(\xi)$; the straight line L_1 decomposes the plane into two half-planes, one of which (the half-plane Ω) contains L. In the half-plane Ω the absolute value of $e^{-w\xi}$ is greater than that of $e^{-w\xi_0}$. The lemniscate L decomposes the plane into two connected domains (interior and exterior). If ξ lies in the exterior domain, then $|\varphi(\xi)|>|\varphi(\xi_0)|$.

We now construct the required contour C_∞, taking account of possible deformations of the contour of integration in (3.2). To this end we take a closed interval from L_1 with centre at ξ_0 and connect its ends by a smooth Jordan curve which lies in the half-plane Ω and encircles L. The contour C_∞ is the required one.

Note that the equation $\varphi'(\xi)=0$ has k roots; they all lie in a convex polygon containing all the roots of $\varphi(\xi)$; that is, if η is a root of $\varphi'(\xi)$, then

$$\eta=m_0\lambda_0+m_1\lambda_1+\cdots+m_k\lambda_k$$

(see [39], Part III, Ch. 1, § 3, Exercise 31), where $m_p \geqslant 0$, $m_0+m_1+\dots+m_k=1$. It follows that $|\eta| \leqslant |\lambda_k|$. We have $\xi_0 \to \infty$ as $w \to 0$, the remaining k roots of the equation $w\varphi(\xi) = \varphi'(\xi)$ being sufficiently close to the roots of the equation $\varphi'(\xi) = 0$. Hence they all lie in the disc with centre at the origin and radius $2|\lambda_k|$. Consequently, the contour C_∞ contains a unique critical point ξ_0 of the function $\psi(\xi)$.

By the argument principle, as the point ξ describes the contour C_∞ in the positive direction, the variation in the argument of $\varphi(\xi)$ is $2(k+1)\pi$. Hence C_∞ can be decomposed into two contours C^j_∞, $j=0,1$, where the increment in the argument of $\varphi(\xi)$ on the contour C^1_∞ is $(2k+1)\pi$. It can be assumed without loss of generality that ξ_0 lies inside the contour C^0_∞ and that $-\pi/2 \leqslant \arg\varphi(\xi) \leqslant \pi/2$ if $\xi \in C^0_\infty$; if not, we can multiply and divide the right-hand side of (3.2) by $e^{in\alpha}$, where the real number α is chosen so that $-\pi/2 \leqslant \arg(e^{i\alpha}\varphi(\xi)) \leqslant \pi/2$, and then consider the function $e^{i\alpha}\varphi(\xi)$ instead of $\varphi(\xi)$. (Here and below, i is the imaginary unit.)

Consider the function

$$S(\xi) = w\xi - \ln\varphi(\xi), \qquad \xi \in C^0_\infty,$$

where $\ln\varphi(\xi) = \ln|\varphi(\xi)| + i\arg_0\varphi(\xi)$ is the single-valued branch of the logarithm for which $\arg_0\varphi(\xi) \in [-\pi/2, \pi/2]$. Note that $S(\xi)$ is the restriction to $C^0_\infty \subset G$ of the single-valued analytic function $S(\xi)$ defined in a simply connected domain G not containing any zeros of $\varphi(\xi)$. In this domain,

$$S'(\xi) = w - \frac{\varphi'(\xi)}{\varphi(\xi)} = w - \frac{1}{\xi} - \frac{1}{\xi-\lambda_1} - \dots - \frac{1}{\xi-\lambda_k},$$

$$S''(\xi) = \frac{1}{\xi^2} + \frac{1}{(\xi-\lambda_1)^2} + \dots + \frac{1}{(\xi-\lambda_k)^2},$$

and hence, $S'(\xi_0) = 0$ and $S''(\xi_0) \neq 0$.

For any $\xi \in C_\infty$,

$$\frac{1}{|\psi(\xi)|^n} = \exp\bigl\{n(\operatorname{Re}(w\xi) - \ln|\varphi(\xi)|)\bigr\},$$

the function $\operatorname{Re}(w\xi) - \ln|\varphi(\xi)|$ attains its maximum on C_∞ at a unique point ξ_0. Consider the integrals

$$F_j(n) = \frac{1}{2\pi i}\int_{C^j_\infty} \frac{d\xi}{[e^{-\xi w}\varphi(\xi)]^n}, \qquad j=0,1.$$

Arguing as in the proof of the inequalities (8) in Ch. VII, § 45 in [37], it is readily seen that

$$|F_1(n)| \leqslant c|e^{n(S(\xi_0)-\delta)}|, \tag{3.6}$$

where $c, \delta > 0$ are constants. The integral $F_0(n)$ can be written as

$$F_0(n) = \frac{1}{2\pi i}\int_{C^0_\infty} e^{nS(\xi)}\,d\xi.$$

Taking account of the fact that $\max\{\operatorname{Re}S(\xi) : \xi \in C^0_\infty\}$ is attained at a unique point ξ_0, which is a simple saddle point interior to the contour C^0_∞, we apply the saddle-point method (Assertion 2) to find the asymptotics of this integral. As a result, we have

$$F_0(n) = \frac{1}{2\pi i}\sqrt{\frac{-2\pi}{nS''(\xi_0)}}\, e^{nS(\xi_0)}\left(1 + O\left(\frac{1}{n}\right)\right).$$

It follows from (3.6) that the absolute value of the integral $F_1(n)$ is exponentially small as $n \to \infty$ compared to that of $e^{nS(\xi_0)}$. Hence, the principal contribution to the asymptotics of $R_n(nw)$ comes from the integral $F_0(n)$. Consequently,

$$R_n(nw) = \frac{1}{2\pi i}\sqrt{\frac{-2\pi}{nS''(\xi_0)}}\, e^{nS(\xi_0)}\left(1 + O\left(\frac{1}{n}\right)\right). \tag{3.7}$$

The point ξ_0 lies sufficiently far from the origin and so

$$\begin{aligned} S(\xi_0) &= w\xi_0 - (k+1)\ln\xi_0 - \ln\left(1 - \frac{\lambda_1}{\xi_0}\right) - \cdots - \ln\left(1 - \frac{\lambda_k}{\xi_0}\right) \\ &= w\xi_0 + (k+1)\ln\frac{1}{\xi_0} + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{\xi_0} + O\left(\frac{1}{\xi_0^2}\right). \end{aligned}$$

As a result, using (3.5),

$$S(\xi_0) = k + 1 + (k+1)\ln\frac{w}{k+1} + \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{k+1} w + O(w^2).$$

Hence

$$e^{nS(\xi_0)} = e^{(k+1)n}\left(\frac{w}{k+1}\right)^{(k+1)n} \exp\left\{\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{k+1} nw\right\}(1 + O(nw^2)).$$

Changing from w to z, as $n \to \infty$, we have

$$e^{nS(\xi_0)} = e^{(k+1)n}\left(\frac{z}{(k+1)n}\right)^{(k+1)n} \exp\left\{\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{k+1} z\right\}\left(1 + O\left(\frac{z^2}{n}\right)\right). \tag{3.8}$$

From the above equality for $S''(\xi)$ it follows that

$$S''(\xi_0) = \frac{1}{\xi_0^2}\left(k + 1 + 2\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_k}{\xi_0} + O\left(\frac{1}{\xi_0^2}\right)\right).$$

Hence, using (3.5),

$$S''(\xi_0) = \frac{w^2}{k+1}(1 + O(w)),$$

and so

$$\sqrt{\frac{-1}{S''(\xi_0)}} = \sqrt{\frac{-(k+1)}{w^2}}\,(1 + O(w)).$$

Taking account of the fact that the angle φ_0 is $\arg(i/w)$ for the contour C^0_∞ and changing to the variable z, we finally obtain

$$\sqrt{\frac{-1}{S''(\xi_0)}} = \sqrt{k+1}\,\frac{i}{w}(1+O(w)) = i\sqrt{k+1}\,\frac{n}{z}\biggl(1+O\biggl(\frac{z}{n}\biggr)\biggr). \tag{3.9}$$

From (3.7)–(3.9) it follows that

$$R_n(z) = \sqrt{\frac{(k+1)n}{2\pi}}\biggl(\frac{e}{(k+1)n}\biggr)^{(k+1)n} \times \exp\biggl\{\frac{\lambda_1+\lambda_2+\cdots+\lambda_k}{k+1}z\biggr\}z^{kn+n-1}\biggl(1+O\biggl(\frac{1}{n}\biggr)\biggr).$$

Hence, using Stirling's formula, we have proved the asymptotic equality (3.1) with any fixed complex number z.

That the asymptotics in (3.1) are uniform follows from Vitali's theorem and since the sequences of functions

$$(kn+n-1)!\,\exp\biggl\{-\frac{\lambda_1+\lambda_2+\cdots+\lambda_k}{k+1}z\biggr\}\frac{R_n(z)}{z^{kn+n-1}}, \qquad n=n_0,n_0+1,\dots,$$

are uniformly bounded in absolute value on compact subsets of $\mathbb{C}$. Indeed,

$$|R_n(nw)| \leqslant \frac{1}{2\pi}\int_\alpha^\beta \exp\bigl\{n(\operatorname{Re}(w\zeta(t)) - \ln|\varphi(\zeta(t))|)\bigr\}|\zeta'(t)|\,dt,$$

where the contour of integration C_∞ is the same and is parametrized by the real parameter $t\in[\alpha,\beta]$. Denoting the closed interval corresponding to the parametrization of the contour C^0_∞ by $[\alpha_1,\beta_1]$, for sufficiently large n we have

$$|R_n(nw)| \leqslant \frac{1}{\pi}\int_{\alpha_1}^{\beta_1} \exp\bigl\{n\operatorname{Re}S(\zeta(t))\bigr\}|\zeta'(t)|\,dt. \tag{3.10}$$

To find the asymptotics of the integral in (3.10) we use the Laplace method (Assertion 1). As a result, we have

$$\int_{\alpha_1}^{\beta_1} e^{n\operatorname{Re}S(\zeta(t))}|\zeta'(t)|\,dt = \sqrt{\frac{-2\pi}{n[\operatorname{Re}S(\zeta(t))]''_{t=t_0}}}\,e^{n\operatorname{Re}S(\xi_0)}|\zeta'(t_0)|\biggl(1+O\biggl(\frac{1}{n}\biggr)\biggr), \tag{3.11}$$

where t_0 is chosen so that $\zeta(t_0)=\xi_0$. In a sufficiently small neighbourhood of the point $\xi_0=x_0+iy_0$ the curve C^0_∞ is given by the parametric equation

$$\zeta(t)=x(t)+iy(t), \qquad t\in[-\tau,\tau], \quad \tau>0,$$

where

$$\begin{aligned} x(t)&=\beta t+x_0, \qquad & y(t)&=\alpha t+y_0, \qquad & w&=\alpha+i\beta,\\ t_0&=0, & \zeta(0)&=\xi_0, & |\zeta'(t_0)|&=|w|. \end{aligned}$$

Now, $\operatorname{Re} S(\zeta(t))$ has a local maximum at t_0 and so elementary calculations show that

$$-[\operatorname{Re} S(\zeta(t))]''_{t=0} = \sum_{p=0}^{k} \frac{|w|^2}{|\xi_0 - \lambda_p|^2} - 2\sum_{p=0}^{k} \biggl[\frac{\operatorname{Im}(w(\xi_0 - \lambda_p))}{|\xi_0 - \lambda_p|^2}\biggr]^2.$$

Hence, using (3.5) and the easily verified relation

$$2[\operatorname{Im}\{w(\xi_0 - \lambda_p)\}]^2 = |w|^2\,|\xi_0 - \lambda_p|^2 - \operatorname{Re}\{w^2(\xi_0 - \lambda_p)^2\},$$

it is readily shown that, for sufficiently large n,

$$-[\operatorname{Re} S(\zeta(t))]''_{t=0} = \frac{|w|^4}{k+1}(1 + O(w)).$$

Changing to the variable z and taking (3.8)–(3.11) into account, for sufficiently large n we arrive at the required inequality

$$|R_n(z)| \leqslant \frac{2\,|z|^{kn+n-1}}{(kn+n-1)!}\biggl|\exp\biggl\{\frac{\lambda_1 + \lambda_2 + \dots + \lambda_k}{k+1}\,z\biggr\}\biggr|.$$

This proves Theorem 2.

§ 4. Proof of Theorem 1

Following Trefethen [35] and Braess [36], let us consider a translation of Hermite-Padé polynomials of the first kind and of degree n. Let $\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_k$ be arbitrary real numbers,

$$\begin{aligned} \widetilde{a}_n^p(z) &= n!\,\lambda^{n+1} A_{n+1}^p(z - z_n), &\qquad 0 \leqslant p \leqslant k\,, \\ \widetilde{R}_n(z) &= n!\,\lambda^{n+1} R_{n+1}(z - z_n), &\qquad E_n^* = \|\widetilde{R}_n\|_\rho, \end{aligned} \tag{4.1}$$

where

$$z_n = \frac{\lambda_0 + \lambda_1 + \lambda_2 + \dots + \lambda_k}{k+1}\,\frac{\rho^2}{kn+n+k}\,,$$

the factor $n!\,\lambda^{n+1}$ in the above formulae normalizing the polynomial $\widetilde{a}_n^k$ to be monic.

We prove Theorem 1 using the following two lemmas.

Lemma 1. *If $n \to \infty$, then*

$$E_n^* \sim \frac{n!\,\lambda^{n+1}}{(kn+n+k)!}\,\rho^{kn+n+k}. \tag{4.2}$$

Proof. From Theorem 2, in view of the equivalence

$$(z - z_n)^{kn+n+k} \sim z^{kn+n+k} \exp\biggl\{-\frac{\lambda_0 + \lambda_1 + \lambda_2 + \dots + \lambda_k}{k+1}\,\frac{\rho^2}{z}\biggr\},$$

it follows that for $|z| = \rho$

$$|R_{n+1}(z - z_n)| \sim \frac{\rho^{kn+n+k}}{(kn+n+k)!}$$

as $n \to \infty$. Now (4.2) follows from the definition of E_n^* (see (4.1)). The proof of Lemma 1 is complete.

Lemma 2. *If $\rho < \pi/(\lambda_k - \lambda_0)$, then $E_n = E_n^*$ for sufficiently large n.*

Proof. We use the method in [32] and [33]. It suffices to show that $E_n^* \leqslant E_n$ for large n. Suppose that this is not so. Then $E_n < E_n^*$, and hence there exist polynomials a_n^p, $p = 0, 1, \dots, k$, where $\deg a_n^p \leqslant n$ and a_n^k is monic, such that

$$\biggl\| \sum_{p=0}^{k} a_n^p(z) e^{\lambda_p z} \biggr\| < \biggl\| \sum_{p=0}^{k} \widetilde{a}_n^p(z) e^{\lambda_p z} \biggr\|.$$

Hence for sufficiently large n and $|z| = \rho$,

$$\biggl| \sum_{p=0}^{k} a_n^p(z) e^{\lambda_p z} \biggr| < \biggl| \sum_{p=0}^{k} \widetilde{a}_n^p(z) e^{\lambda_p z} \biggr|.$$

Consequently, Rouché's theorem implies that the function

$$\sum_{p=0}^{k} (a_n^p(z) - \widetilde{a}_n^p(z)) e^{\lambda_p z} \tag{4.3}$$

has at least $kn + n + k$ zeros in D_ρ. But this is not so. In fact, consider the polynomials $b_n^p = a_n^p - \widetilde{a}_n^p$, $p = 0, 1, \dots, k$. Let h be the sum of the degrees of these polynomials. It is known (see [39], Part III, Ch. 4, § 4, Exercise 206) that the function

$$\sum_{p=0}^{k} b_n^p(z) e^{\lambda_p z}$$

can have at most $h + k + (\lambda_k - \lambda_0)\rho/\pi$ zeros in the disc D_ρ. In our setting, $h \leqslant (k+1)n - 1$ and $\rho < \pi/(\lambda_k - \lambda_0)$. Hence, the function (4.3) can have at most $kn + n + k - 1$ zeros in D_ρ. This contradiction proves Lemma 2.

§ 5. Asymptotics of the polynomials A_n^p

In this section $\{\lambda_p\}_{p=0}^k$ are distinct real numbers. In what follows it will be assumed without loss of generality that $0 = \lambda_0 < \lambda_1 < \dots < \lambda_k$. The general case reduces to this one.

First, we introduce our notation. Let $\{x_j\}_{j=1}^k$ be the zeros of the polynomial φ'. It is clear that the x_j are real numbers and $x_j \in (\lambda_{j-1}, \lambda_j)$, $j = 1, 2, \dots, k$. We next assume that G is a simply connected domain such that $\{x_j\}_{j=1}^k \subset G \subset \mathbb{C} \setminus \{\lambda_p\}_{p=0}^k$. Then (see [37], Ch. IV, § 24, Example 6) the function

$$S(\xi) = -\ln \varphi(\xi),$$

where

$$S(x_1) = -\ln |\varphi(x_1)| \quad \text{if} \quad \varphi(x_1) > 0,$$
$$S(x_1) = -\ln |\varphi(x_1)| - i\pi \quad \text{if} \quad \varphi(x_1) < 0,$$

is a single-valued analytic function in G. The values of S are calculated using the formula

$$S(\xi) = -\ln|\varphi(\xi)| - i[\operatorname{Im} S(x_1) + \Delta_\gamma \arg\varphi(\xi)],$$

where the curve γ lies in G and joins the points x_1 and ξ and $\Delta_\gamma \arg\varphi(\xi)$ is the increment in the argument of $\varphi(\xi)$ along γ.

If $\xi \in G$, then

$$S'(\xi) = -\frac{\varphi'(\xi)}{\varphi(\xi)} = -\frac{1}{\xi} - \frac{1}{\xi-\lambda_1} - \dots - \frac{1}{\xi-\lambda_k},$$
$$S''(\xi) = -\frac{\varphi''(\xi)\varphi(\xi) - [\varphi'(\xi)]^2}{\varphi^2(\xi)} = \frac{1}{\xi^2} + \frac{1}{(\xi-\lambda_1)^2} + \dots + \frac{1}{(\xi-\lambda_k)^2},$$

and hence $S'(x_j) = 0$ and $S''(x_j) = -\varphi''(x_j)/\varphi(x_j) > 0$, $j = 1, 2, \dots, k$.

Taking the positive value of the root function, we set

$$B_n(x_j) = \sqrt{\frac{1}{2\pi n S''(x_j)}}\, e^{nS(x_j)}, \qquad j = 1, 2, \dots, k.$$

Theorem 3. *If $z \in \mathbb{C}$ is fixed and $n \to \infty$, then*

$$A_n^0(z) = B_n(x_1) e^{x_1 z}\biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr), \tag{5.1}$$

$$\begin{aligned} A_n^p(z) = {}& B_n(x_{p+1}) e^{(x_{p+1}-\lambda_p)z}\biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr) \\ & - B_n(x_p) e^{(x_p-\lambda_p)z}\biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr), \qquad 1 \leqslant p \leqslant k-1, \end{aligned} \tag{5.2}$$

$$A_n^k(z) = -B_n(x_k) e^{(x_k-\lambda_k)z}\biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr). \tag{5.3}$$

Proof. Equality (5.1) will be proved using the integral representation

$$A_n^0(z) = \frac{1}{2\pi i}\int_{C_0} \frac{e^{\xi z}\, d\xi}{[\varphi(\xi)]^n}. \tag{5.4}$$

To this end we deform the contour of integration C_0 in (5.4) into a rectangle R in the half-plane

$$\{z : -\infty < \operatorname{Re} z < \lambda_1\},$$

with vertices at points $A(-a', -r)$, $B(-a', r)$, $C(a, r)$, $D(a, -r)$, where r is a sufficiently large positive number, $a \in (0, \lambda_1)$ and $a' > 0$. We have

$$|\varphi(a + it)| = \prod_{j=0}^{k} \sqrt{(a-\lambda_j)^2 + t^2} > |\varphi(a)|, \qquad t \in [-r, r] \setminus \{0\},$$

and hence the minimum of the function $|\varphi(\xi)|$ is attained at a unique point a on the vertical closed interval between the points C and D. Similarly, on the vertical interval between the points A and B the minimum of the function $|\varphi(\xi)|$

is attained at a unique point $-a'$. On the remaining two horizontal intervals, for sufficiently large r, $|\varphi(\xi)|$ exceeds the values of $|\varphi(\xi)|$ at both $-a'$ and a. Indeed, if $r > 2\max\{a', \lambda_k\}$, then for $t \in [-a, a]$

$$|\varphi(t \pm ir)| = \prod_{j=0}^{k} \sqrt{(t-\lambda_j)^2 + r^2} > \max\{|\varphi(a)|, |\varphi(-a')|\}.$$

Now we specify a' and a. We set $a = x_1$, and take a' such that $|\varphi(-a')| > |\varphi(a)|$. Such a choice is possible, since $|\varphi(t)| \to +\infty$ as $t \to -\infty$, $t \in \mathbb{R}$.

For an arbitrary interval $[L, N]$, we take the positive direction to be from L to N and define

$$F_n^{[L,N]}(z) = \frac{1}{2\pi i} \int_{[L,N]} \frac{e^{\xi z}\, d\xi}{[\varphi(\xi)]^n}.$$

A domain G can be chosen to contain $[D, C]$. Hence,

$$F_n^{[D,C]}(z) = \frac{1}{2\pi i} \int_{[D,C]} e^{\xi z} e^{nS(\xi)}\, d\xi.$$

By the choice of the point a, the maximum of the function $\operatorname{Re} S(\xi)$ on the interval $[D, C]$ is attained at a unique point x_1, which is a simple saddle point. Hence the asymptotics of the integral $F_n^{[D,C]}$ can be found using the saddle-point method (Assertion 2). As a result, we have

$$F_n^{[D,C]}(z) = \frac{1}{2\pi i} \sqrt{\frac{-2\pi}{nS''(x_1)}}\, e^{nS(x_1)} e^{x_1 z} \left(1 + O\left(\frac{1}{n}\right)\right). \tag{5.5}$$

We choose a branch of the root in (5.5) with due regard to the fact that $\varphi_0 = \pi/2$ in this setting. Hence, we finally obtain

$$F_n^{[D,C]}(z) = B_n(x_1) e^{x_1 z} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{5.6}$$

as $n \to \infty$.

Similar arguments apply to the integral $F_n^{[B,A]}$. Taking into account the choice of $-a'$, it is easy to check that

$$|F_n^{[B,A]}(z)| \leqslant \theta |e^{n(S(x_1)-\delta)}|,$$

where θ and δ are positive constants. This means that, as $n \to \infty$, the absolute value of the integral $F_n^{[B,A]}$ is exponentially small compared with that of $e^{nS(x_1)}$. This also holds for the integrals $F_n^{[C,B]}$ and $F_n^{[A,D]}$. Therefore, the principal contribution to the asymptotics of A_n^0 comes from the integral over the interval $[D, C]$. Consequently, (5.1) follows from (5.6).

Equality (5.3) is proved using the same argument, the only difference being that when applying the saddle-point method to the corresponding integral the branch of the root function is chosen using the condition that $\varphi_0 = -\pi/2$.

We now proceed with the proof of (3.2). Let $z \in \mathbb{C}$ be fixed. Writing the polynomial A_n^p, $1 \leqslant p \leqslant k-1$, in the form (2.1), we deform the contour of integration C_p into a rectangle R^* in the domain $\{z : \lambda_{p-1} < \operatorname{Re} z < \lambda_{p+1}\}$, with vertices at the points $A^*(a', -r)$, $B^*(a', r)$, $C^*(a, r)$, $D^*(a, -r)$, where r is a sufficiently large positive number, $a' \in (\lambda_{p-1}, \lambda_p)$ and $a \in (\lambda_p, \lambda_{p+1})$. As a result, on the vertical interval between D^* and C^* the function $|\varphi(\xi)|$ attains minimum at a unique point a, while on $[B^*, A^*]$ its minimum is attained at a unique point a'. For sufficiently large r ($r > 2\lambda_k$), the values of $|\varphi(\xi)|$ on the remaining two horizontal intervals $[B^*, C^*]$ and $[A^*, D^*]$ exceed its values at the points a' and a. Putting $a' = x_p$ and $a = x_{p+1}$, we see that the principal contribution to the asymptotics of A_n^p comes from the integrals over $[B^*, A^*]$ and $[D^*, C^*]$. Arguing as above, we have

$$F_n^{[D^*,C^*]}(z) = \frac{e^{-\lambda_p z}}{2\pi i}\sqrt{\frac{-2\pi}{nS''(x_{p+1})}}\, e^{nS(x_{p+1})} e^{x_{p+1}z}\left(1 + O\left(\frac{1}{n}\right)\right), \tag{5.7}$$

$$F_n^{[B^*,A^*]}(z) = \frac{e^{-\lambda_p z}}{2\pi i}\sqrt{\frac{-2\pi}{nS''(x_p)}}\, e^{nS(x_p)} e^{x_p z}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{5.8}$$

as $n \to \infty$. The branch of the root function in (5.7) is chosen using the condition $\varphi_0 = \pi/2$; in choosing the branch of the root in (5.8), we note that $\varphi_0 = -\pi/2$. Now (5.2) is secured by (5.7) and (5.8). The proof of Theorem 3 is complete

Corollary 1. *If $n \to \infty$, then*

$$\begin{aligned}
&A_n^0(0) = B_n(x_1)\left(1 + O\left(\frac{1}{n}\right)\right),\\
&A_n^p(0) = B_n(x_{p+1})\left(1 + O\left(\frac{1}{n}\right)\right) - B_n(x_p)\left(1 + O\left(\frac{1}{n}\right)\right), \quad 1 \leqslant p \leqslant k-1,\\
&A_n^k(0) = -B_n(x_k)\left(1 + O\left(\frac{1}{n}\right)\right).
\end{aligned} \tag{5.9}$$

It follows from (5.9) that $A_n^0(0) \neq 0$ and $A_n^k(0) \neq 0$ for sufficiently large n. For such n we look at two sequences of normalized polynomials

$$\widetilde{A}_n^0(z) = \frac{A_n^0(z)}{A_n^0(0)}, \qquad \widetilde{A}_n^k(z) = \frac{A_n^k(z)}{A_n^k(0)}.$$

To define analogous sequences for $1 \leqslant p \leqslant k-1$, we consider three possible cases, each of which can be realized for certain systems of exponentials.

A) $|\varphi(x_p)| \neq |\varphi(x_{p+1})|$. We let $\widetilde{x}_p$ denote a point in the pair x_p, x_{p+1} such that

$$\min\bigl\{|\varphi(x_p)|, |\varphi(x_{p+1})|\bigr\} = |\varphi(\widetilde{x}_p)|.$$

Then for sufficiently large n we have $A_n^p(0) \neq 0$, and hence the sequence $\widetilde{A}_n^p(z) = A_n^p(z)/A_n^p(0)$ is defined.

B) $\varphi(x_{p+1}) = -\varphi(x_p)$ and $S''(x_{p+1}) \neq S''(x_p)$. For large n, we have $A_n^p(0) \neq 0$, and hence the sequence $\widetilde{A}_n^p(z) = A_n^p(z)/A_n^p(0)$ is defined.

C) $\varphi(x_{p+1}) = -\varphi(x_p)$ and $S''(x_{p+1}) = S''(x_p)$. We have $(-1)^{k+p+1}\varphi(x_p) > 0$, and so

$$e^{nS(x_p)} = (-1)^{n(k+p+1)} e^{-n\ln|\varphi(x_p)|},$$
$$e^{nS(x_{p+1})} = (-1)^{n(k+p+1)+n} e^{-n\ln|\varphi(x_p)|}.$$

As a result,

$$A_n^p(0) = (-1)^{n(k+p+1)} \sqrt{\frac{1}{2\pi n S''(x_p)}}\, e^{-n\ln|\varphi(x_p)|}\big((-1)^n - 1\big)\left(1 + O\left(\frac{1}{n}\right)\right).$$

Hence $A_{2n+1}^p(0) \neq 0$ for sufficiently large n, and so the sequence of polynomials $\widetilde{A}_{2n+1}^p(z) = A_{2n+1}^p(z)/A_{2n+1}^p(0)$ is defined.

The derivative of the polynomial A_n^p can be written as

$$\frac{dA_n^p}{dz}(z) = \frac{e^{-\lambda_p z}}{2\pi i} \int_{C_p} (\xi - \lambda_p) \frac{e^{\xi z}\, d\xi}{[\varphi(\xi)]^n}. \tag{5.10}$$

Proceeding in a similar way to that used in finding the asymptotic behaviour of A_n^p, we apply the saddle-point method to the integral on the right of (5.10), with $z = 0$, to obtain

$$\frac{dA_n^1}{dz}(0) = B_n(x_{p+1})(x_{p+1} - \lambda_p)\left(1 + O\left(\frac{1}{n}\right)\right) - B_n(x_p)\,(x_p - \lambda_p)\left(1 + O\left(\frac{1}{n}\right)\right).$$

Hence, under our assumptions,

$$\frac{dA_{2n}^p}{dz}(0) = (-1)^{n(k+p+1)} \sqrt{\frac{1}{2\pi n S''(x_p)}}\, e^{-n\ln|\varphi(x_p)|}(x_{p+1} - x_p)\left(1 + O\left(\frac{1}{n}\right)\right),$$

and the sequence of polynomials $\widetilde{A}_{2n}^p(z) = A_{2n}^p(z)/(A_{2n}^p)'(0)$ is defined.

Theorem 4. *If $n \to \infty$, then*

$$\widetilde{A}_n^0(z) \rightrightarrows e^{x_1 z}, \qquad \widetilde{A}_n^k(z) \rightrightarrows e^{(x_k - \lambda_k)z} \tag{5.11}$$

locally uniformly in z.

If $1 \leqslant p \leqslant k-1$, then as $n \to \infty$,

$$\widetilde{A}_n^p(z) \rightrightarrows e^{(\widetilde{x}_p - \lambda_p)z} \tag{5.12}$$

in case A);

$$\widetilde{A}_{2n}^p(z) \rightrightarrows \left(\frac{e^{(x_{p+1}-\lambda_p)z}}{\sqrt{S''(x_{p+1})}} - \frac{e^{(x_p-\lambda_p)z}}{\sqrt{S''(x_p)}}\right)\left(\frac{1}{\sqrt{S''(x_{p+1})}} - \frac{1}{\sqrt{S''(x_p)}}\right)^{-1}, \tag{5.13}$$

$$\widetilde{A}_{2n+1}^p(z) \rightrightarrows \left(\frac{e^{(x_{p+1}-\lambda_p)z}}{\sqrt{S''(x_{p+1})}} + \frac{e^{(x_p-\lambda_p)z}}{\sqrt{S''(x_p)}}\right)\left(\frac{1}{\sqrt{S''(x_{p+1})}} + \frac{1}{\sqrt{S''(x_p)}}\right)^{-1} \tag{5.14}$$

in case B);

$$\widetilde{A}_{2n}^p(z) \rightrightarrows \frac{1}{x_{p+1} - x_p}\big(e^{(x_{p+1}-\lambda_p)z} - e^{(x_p-\lambda_p)z}\big), \tag{5.15}$$

$$\widetilde{A}_{2n+1}^p(z) \rightrightarrows \frac{1}{2}\big(e^{(x_{p+1}+\lambda_p)z} + e^{(x_p-\lambda_p)z}\big) \tag{5.16}$$

in case C). *All the convergences are locally uniform in z.*

Proof. The pointwise convergence in (5.11)–(5.16) is secured by Theorem 3. It remains to show that in each of the cases A), B) and C) the polynomials $\widetilde{A}_n^p$ with $0 \leqslant p \leqslant k$ converge uniformly on compact subsets of $\mathbb{C}$ to the corresponding functions. For example, we prove this result for $\widetilde{A}_n^0$.

If we assume that $|z| \leqslant \rho$ and $\xi \in R$, then the absolute value of $e^{\xi z}$ is majorized by $M = e^{4\rho \max\{a', \lambda_k\}}$. Taking (5.4) into account, in the case under consideration we have

$$|A_n^0(z)| \leqslant \frac{M}{\pi} \int_\alpha^\beta e^{-n \ln |\varphi(\zeta(t))|} |\zeta'(t)|\, dt \tag{5.17}$$

provided that the contour of integration R is the same and is parameterized by the real parameter $t \in [\alpha, \beta]$. For large n, inequality (5.17) also holds if we replace R by the interval $[D, C]$. Assume that $[D, C]$ is parametrized by $t \in [\alpha_1, \beta_1]$. To find the asymptotics of the integral in (5.17) we use Laplace's method (Assertion 1). As a result,

$$\begin{aligned} &\int_{\alpha_1}^{\beta_1} e^{n \operatorname{Re} S(\zeta(t))} |\zeta'(t)|\, dt \\ &\qquad = \sqrt{\frac{-2\pi}{n[\operatorname{Re} S(\zeta(t))]''_{t=t_0}}}\, e^{n \operatorname{Re} S(x_1)} |\zeta'(t_0)| \biggl(1 + O\biggl(\frac{1}{n}\biggr)\biggr) \end{aligned} \tag{5.18}$$

as $n \to \infty$, where t_0 is chosen so that $\zeta(t_0) = x_1$. It is easily seen that

$$[\operatorname{Re} S(\zeta(t))]''_{t=t_0} = -S''(x_1)|\zeta'(t_0)|^2.$$

Hence, using (5.9) and (5.18), we obtain the inequality $|\widetilde{A}_n^0(z)| \leqslant 2M$ for sufficiently large n, from which it follows that the sequence $\{\widetilde{A}_n^0(z)\}_{n=1}^\infty$ is uniformly bounded in absolute value in the disc $\{z : |z| \leqslant \rho\}$. Now by Vitali's theorem this sequence converges uniformly to the function $e^{x_1 z}$ on any compact subset of the disc $\{z : |z| \leqslant \rho\}$. Similar arguments also apply to the other sequences in Theorem 4. The proof of Theorem 4 is complete.

§ 6. Illustrative examples

6.1. Consider the system of exponentials $\{e^{\lambda_p z}\}_{p=0}^2$, where $0 = \lambda_0 < \lambda_1 < \lambda_2$. Let

$$p = \sqrt{\lambda_1^2 - \lambda_1\lambda_2 + \lambda_2^2}\,, \qquad h = 5\lambda_1\lambda_2 - 2\lambda_1^2 - 2\lambda_2^2.$$

Easy calculations show that

$$\begin{aligned} x_1 &= \frac{\lambda_1 + \lambda_2 - p}{3}, & x_2 &= \frac{\lambda_1 + \lambda_2 + p}{3}, \\ \varphi(x_1) &= \frac{(\lambda_1 + \lambda_2)h + 2p^3}{27}\,, & \varphi(x_2) &= \frac{(\lambda_1 + \lambda_2)h - 2p^3}{27}\,, \\ S''(x_1) &= \ln \frac{54p}{(\lambda_1 + \lambda_2)h + 2p^3}\,, & S''(x_2) &= \ln \frac{-54p}{(\lambda_1 + \lambda_2)h - 2p^3}\,. \end{aligned}$$

The following result is a consequence of Theorem 3.

Corollary 2. *If $n \to \infty$, then*

$$A_n^0(z) = B_n(x_1)e^{x_1 z}\left(1 + O\left(\frac{1}{n}\right)\right),$$
$$A_n^1(z) = B_n(x_2)e^{(x_2-\lambda_1)z}\left(1 + O\left(\frac{1}{n}\right)\right) - B_n(x_1)e^{(x_1-\lambda_1)z}\left(1 + O\left(\frac{1}{n}\right)\right),$$
$$A_n^2(z) = -B_n(x_2)e^{(x_2-\lambda_2)z}\left(1 + O\left(\frac{1}{n}\right)\right).$$

In this example, as was the case in examples in [32] and [33], only cases A) and C) are realized. Moreover, case C) is realized with $h = 0$, that is, when $\lambda_2 = 2\lambda_1$.

Putting $\lambda_2 = 2\lambda_1$, we have

$$S(x_1) = \ln\left(\frac{27}{2p^3}\right), \qquad S(x_2) = \ln\left(\frac{27}{2p^3}\right) + i\pi, \qquad S''(x_1) = S''(x_2) = \frac{27}{p^2},$$
$$A_n^1(0) = \sqrt{\frac{p^2}{54\pi n}}\left(\frac{27}{2p^3}\right)^n [(-1)^n - 1]\left(1 + O\left(\frac{1}{n}\right)\right).$$

Hence $A_{2n+1}^1(0) \neq 0$ for sufficiently large n. Next, it is an easy consequence of the now familiar arguments that

$$\frac{dA_{2n}^1}{dz}(0) = \sqrt{\frac{p^2}{108\pi n}}\left(\frac{27}{2p^3}\right)^{2n}(x_2 - x_1)\left(1 + O\left(\frac{1}{n}\right)\right).$$

The following corollary is a consequence of Theorem 4 in the case under consideration.

Corollary 3. *As $n \to \infty$*

$$\widetilde{A}_n^0(z) \rightrightarrows e^{x_1 z}, \qquad \widetilde{A}_n^2(z) \rightrightarrows e^{(x_2-\lambda_2)z}.$$

If $\lambda_2 \neq 2\lambda_1$, then

$$\widetilde{A}_n^1(z) \rightrightarrows e^{(x_2-\lambda_1)z},$$

and if $\lambda_2 = 2\lambda_1$, then

$$\widetilde{A}_{2n+1}^1(z) \rightrightarrows \frac{1}{2}\left(e^{(x_2-\lambda_1)z} + e^{(x_1-\lambda_1)z}\right), \qquad \widetilde{A}_{2n}^1(z) \rightrightarrows \frac{1}{x_2 - x_1}\left(e^{(x_2-\lambda_1)z} - e^{(x_1-\lambda_1)z}\right).$$

For purposes of comparison, we reformulate a similar result from [24] on the asymptotics of Hermite-Padé approximants of the second kind in terms of the quantities now involved (see also [27] and [40], where the method of the Riemann-Hilbert matrix problem was applied in the case $\lambda_1 = -1$, $\lambda_2 = 1$ to derive very precise asymptotics for the quadratic diagonal approximants and the Hermite-Padé polynomials of the first and second kind with a rescaled independent variable).

Theorem 5. *Let $\pi^j_{n,n}(z;e^{\lambda_j\xi})$, $j=1,2$, be the Hermite-Padé approximants of the second kind for the family $\{e^{\lambda_1 z}, e^{\lambda_2 z}\}$, where λ_1 and λ_2 are distinct nonzero real numbers. Then, as $n\to\infty$,*

$$e^{\lambda_1 z}-\pi^1_{n,n}(z;e^{\lambda_1\xi}) = B^*_n(x_1;z)e^{(\lambda_1-x_1)z}\biggl(1+O\biggl(\frac{1}{n}\biggr)\biggr),$$
$$e^{\lambda_2 z}-\pi^2_{n,n}(z;e^{\lambda_2\xi}) = B^*_n(x_1;z)e^{(\lambda_2-x_1)z}\biggl(1+O\biggl(\frac{1}{n}\biggr)\biggr)$$
$$+(-1)^n B^*_n(x_2;z)\,e^{(\lambda_2-x_2)z}\biggl(1+O\biggl(\frac{1}{n}\biggr)\biggr),$$

locally uniformly in z, where

$$B^*_n(x_j;z)=\frac{z^{3n+1}}{(3n)!}e^{(\lambda_1+\lambda_2)z/3}\sqrt{\frac{2\pi}{nS''(x_j)}}\,e^{-nS(x_j)}, \qquad j=1,2.$$

Assume now that

$$\lambda_0=0, \qquad \lambda_1=1, \qquad \lambda_2=1+\varepsilon, \quad 0<\varepsilon\leqslant 1.$$

For $0<\varepsilon<1$ Theorem 3 implies that

$$A^0_n(z)\sim\sqrt{\frac{2p^3+(2+\varepsilon)h}{108\pi pn}}\biggl(\frac{27}{2p^3+(2+\varepsilon)h}\biggr)^n e^{(2+\varepsilon-p)z/3},$$
$$A^1_n(z)\sim(-1)^n\sqrt{\frac{2p^3-(2+\varepsilon)h}{108\pi pn}}\biggl(\frac{27}{2p^3-(2+\varepsilon)h}\biggr)^n e^{(-1+\varepsilon+p)z/3},$$
$$A^2_n(z)\sim(-1)^n\sqrt{\frac{2p^3-(2+\varepsilon)h}{108\pi pn}}\biggl(\frac{27}{2p^3-(2+\varepsilon)h}\biggr)^n e^{(-1-2\varepsilon+p)z/3}.$$

A comparison of these expressions shows that the principal terms of the asymptotic formulae for the values of the polynomials $A^1_n(z)$ and $A^2_n(z)$ at z differ by a factor $e^{\varepsilon z}$, which tends to 1 as $\varepsilon\to 0$ locally uniformly. With $\varepsilon=1$ Theorem 3 yields asymptotic equalities which agree with the corresponding assertions in [32] and [33]:

$$A^0_n(z)\sim\frac{1}{3\sqrt{2\pi n}}\biggl(\frac{3\sqrt{3}}{2}\biggr)^n e^{(1-1/\sqrt{3})z},$$
$$A^1_n(z)\sim(-1)^n\frac{1}{3\sqrt{2\pi n}}\biggl(\frac{3\sqrt{3}}{2}\biggr)^n\bigl(e^{z/\sqrt{3}}+(-1)^{n-1}e^{-z/\sqrt{3}}\bigr),$$
$$A^2_n(z)\sim(-1)^{n-1}\frac{1}{3\sqrt{2\pi n}}\biggl(\frac{3\sqrt{3}}{2}\biggr)^n e^{(-1+1/\sqrt{3})z}.$$

Comparing the previous asymptotic equalities with those given in this case by Theorem 5 we find that

$$e^z - \pi^1_{n,n}(z;e^\xi) \sim \frac{z^{3n+1}}{(3n)!} e^z \sqrt{\frac{2\pi}{9n}} \left(\frac{2}{3\sqrt{3}}\right)^n e^{z/\sqrt{3}}, \tag{6.1}$$

$$e^{\lambda_2 z} - \pi^2_{n,n}(z;e^{2\xi}) \sim \frac{z^{3n+1}}{(3n)!} e^{2z} \sqrt{\frac{2\pi}{9n}} \left(\frac{2}{3\sqrt{3}}\right)^n \left(e^{z/\sqrt{3}} + (-1)^n e^{-z/\sqrt{3}}\right).$$

The next result follows from Theorem 5 with $0 < \varepsilon < 1$.

Corollary 4. *Let $\pi^1_{n,n}(z;e^\xi)$ and $\pi^2_{n,n}(z;e^{(1+\varepsilon)\xi})$ be the Hermite-Padé approximants of the second kind for the family $\{e^z, e^{(1+\varepsilon)z}\}$. Then, as $n \to \infty$,*

$$e^z - \pi^1_{n,n}(z;e^\xi) \sim \frac{z^{3n+1}}{(3n)!} \sqrt{\frac{\pi(2p^3 + (2+\varepsilon)h)}{27pn}} \left(\frac{2p^3 + (2+\varepsilon)h}{27}\right)^n e^{(3+p)z/3},$$

$$e^{(1+\varepsilon)z} - \pi^2_{n,n}(z;e^{(1+\varepsilon)\xi}) \sim \frac{z^{3n+1}}{(3n)!} \sqrt{\frac{\pi(2p^3 + (2+\varepsilon)h)}{27pn}} \left(\frac{2p^3 + (2+\varepsilon)h}{27}\right)^n e^{(3+3\varepsilon+p)z/3}.$$

Corollary 4 asserts that for small values of ε the asymptotics of the corresponding deviations of Hermite-Padé approximants of the second kind differ insignificantly and tend to a common value as $\varepsilon \to 0$. In addition, the factor $(2p^3 + (2+\varepsilon)h)/27)^n$, which depends on ε and governs the principal term in the asymptotic formula, tends to $(2/(3\sqrt{3}\,))^{2n}$. This is rather surprising in view of (6.1), because according to Corollary 4 the rate of approximation of the function e^z by Hermite-Padé approximants increases substantially (by almost a factor of $(2/(3\sqrt{3}\,))^n$).

6.2. We next give an example in which case B) is realized. To this end we look at the system of exponentials $\{e^{\lambda_p z}\}_{p=0}^3$, where

$$\lambda_0 = 0, \quad \lambda_1 = 1 - \varepsilon, \quad \lambda_2 = 2 + \varepsilon, \quad \lambda_3 = 3, \qquad 0 \leqslant \varepsilon < 1.$$

With these values of the parameters

$$x_1 = \frac{3}{2} - \frac{1}{2}\sqrt{9 - 2(1-\varepsilon)(2+\varepsilon)}\,, \qquad x_2 = \frac{3}{2}\,,$$

$$x_3 = \frac{3}{2} + \frac{1}{2}\sqrt{9 - 2(1-\varepsilon)(2+\varepsilon)}\,,$$

$$\varphi(x_1) = \varphi(x_3) = -\frac{1}{4}(1-\varepsilon)^2(2+\varepsilon)^2, \qquad \varphi(x_2) = \frac{9}{4}(0,5+\varepsilon)^2.$$

Hence, for $\varepsilon = \frac{3}{2}\sqrt{2} - 2 \in (0,1)$,

$$x_1 = \frac{3}{2} - \frac{3}{2}\sqrt{2 - \sqrt{2}}\,, \qquad x_2 = \frac{3}{2}\,, \qquad x_3 = \frac{3}{2} + \frac{3}{2}\sqrt{2 - \sqrt{2}}\,,$$

$$\varphi(x_2) = -\varphi(x_1) = -\varphi(x_3) = \frac{81}{8}\left(\frac{3}{2} - \sqrt{2}\right),$$

whilst

$$\varphi''(x_2) = -18 + 9\sqrt{2}\,, \qquad \varphi''(x_1) = \varphi''(x_3) = 36 - 18\sqrt{2}\,.$$

Hence,

$$S''(x_1) = S''(x_3) = \frac{16}{9}(2 - \sqrt{2}\,), \qquad S''(x_2) = \frac{32}{9}\,.$$

As a result, case B) is realized with $p = 1$ and $p = 2$. For example Theorem 4 with $p = 1$ implies that

$$\widetilde{A}^1_{2n}(z) \rightrightarrows \frac{e^{3(\sqrt{2}-1)\,z/2}}{1 - \sqrt{1 - \sqrt{2}/2}} \biggl[e^{-3\sqrt{2-\sqrt{2}}\,z/2} - \sqrt{1 - \frac{\sqrt{2}}{2}}\,\biggr],$$

$$\widetilde{A}^1_{2n+1}(z) \rightrightarrows \frac{e^{3(\sqrt{2}-1)\,z/2}}{1 + \sqrt{1 - \sqrt{2}/2}} \biggl[e^{-3\sqrt{2-\sqrt{2}}\,z/2} + \sqrt{1 - \frac{\sqrt{2}}{2}}\,\biggr].$$

§ 7. Location of the zeros of the polynomials A_n^p

Szegő [41] studied the behaviour of the zeros of Taylor polynomials for power series related to exponential functions. Saff and Varga [42] examined the location of the zeros of Padé approximants to the exponential function, and in particular, ascertained the boundary of the annulus containing the zeros of the Padé polynomials. Stahl [18] studied the location of the zeros of the diagonal Hermite-Padé polynomials of the first and second kinds, transformed by rescaling the independent variable, for system of exponentials $\{1, e^z, e^{2z}\}$. He showed that these zeros lie on special arcs in the complex plane (see also [27] and [40]). Wielonsky [33] proved an analogue of Saff and Varga's theorem for Hermite-Padé polynomials A_p satisfying (1.2).

The next theorem supplements and extends results due to Saff, Varga, Stahl and Wielonsky.

Theorem 6. *Let $\{\lambda_p\}_{p=0}^k$ be arbitrary distinct complex numbers. Then, for $n \geqslant 2$, $k \geqslant 1$, the zeros of A_n^p (that is, the zeros of the Hermite-Padé polynomials of the first kind for the system of exponentials $\{e^{\lambda_p z}\}_{p=0}^k$) lie in the disc $\{z : |z| < R_n^p\}$, where*

$$R_n^p = 2\biggl(n - \frac{1}{3}\biggr) \sum_{\substack{j=0 \\ j \neq p}}^{k} \frac{1}{|\lambda_p - \lambda_j|}\,.$$

The proof of Theorem 6 depends on the method in [33]. We leave the details to the reader.

Some of the results in this paper were announced in [43].

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