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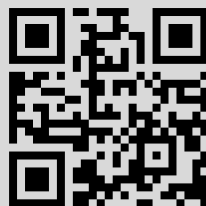
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Closeness to spheres of hypersurfaces with normal curvature bounded below

A. A. Borisenko and K. D. Drach

Abstract. For a Riemannian manifold M^{n+1} and a compact domain $\Omega \subset M^{n+1}$ bounded by a hypersurface $\partial\Omega$ with normal curvature bounded below, estimates are obtained in terms of the distance from O to $\partial\Omega$ for the angle between the geodesic line joining a fixed interior point O in Ω to a point on $\partial\Omega$ and the outward normal to the surface. Estimates for the width of a spherical shell containing such a hypersurface are also presented.

Bibliography: 9 titles.

Keywords: Riemannian manifold, sectional curvature, normal curvature of a hypersurface, comparison theorems, λ -convex hypersurface.

§ 1. Introduction

Blaschke [1] showed that if a complete regular hypersurface F^n has normal curvatures pinched between two positive constants k_1 and k_2 , so that $k_2 \geq k_n \geq k_1 > 0$, then for each point in F^n there exist supporting spheres with radii $1/k_1$ and $1/k_2$ that enclose the hypersurface and lie inside it, respectively. However, it turns out that we can also discuss the closeness of a surface to a sphere in the case when its normal curvatures are only bounded below.

Take a circle in a Euclidean plane. Obviously, the angle between the ray from its centre to a point on the circle and the outward normal at this point is zero. However, if instead of rays from the centre we consider rays with end-point at some fixed interior point O , then this angle does not vanish identically any longer; the same holds for arbitrary convex curves in the plane. On the other hand, the closer all such angles are to zero, the closer the curve to a circle and the point O to the centre of this circle, that is, the magnitude of this angle characterizes the closeness of the curve to a circle. This also provides motivation for the study of estimates for such angles in the general case. It proves that if the normal curvature of a hypersurface is bounded below, then for points O lying at a distance h from the boundary in the corresponding domain the angle between the radial direction from O and the normal cannot be very large.

More precisely, by comparison with constant curvature spaces, an estimate for a surface in $\mathbb{H}^{n+1}(-1)$ (the Lobachevskii space with constant sectional curvature -1) was derived in [2], provided that all the normal curvatures of the surface

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satisfy the inequality $k_n \geq 1$ or $k_n \geq \lambda$ with $\lambda < 1$. In [3] a similar estimate was derived in an Hadamard space M^{n+1} (a complete simply connected Riemannian manifold with nonpositive sectional curvature K) such that $-k_1^2 \leq K \leq 0$, provided that the normal curvatures of the surface satisfy the inequality $k_n > \lambda$ with $\lambda \leq k_1$. In [4] this estimate was extended to $k_n = k_1$.

Thus, the question of similar estimates in manifolds with nonpositive sectional curvature satisfying $0 \geq K \geq -k_1^2$, $k_1 \geq 0$, in the case when all the normal curvatures of the hypersurface satisfy $k_n \geq \lambda$ with $\lambda > k_1$, and of estimates for hypersurfaces with normal curvatures satisfying $k_n \geq \lambda \geq 0$ in manifolds with positive curvature remains open. For 2-dimensional manifolds such estimates were announced in [5]. These results and their generalization to many dimensions are presented in §2 and §3.

Another characteristic of closeness to a sphere is the width of a spherical shell into which the surface can be put. Clearly, the smaller this width, the closer this surface is to a sphere.

It was proved in [2] that a closed surface in a Lobachevskii space (with curvature -1) lies in a spherical shell of width $d \leq \ln 2$ if the normal curvatures of this hypersurface satisfy $k_n \geq 1$ at each point and in each direction. A similar estimate also holds in Hadamard manifolds (see [6]). In this paper we generalize these results to constant curvature spaces and to the general Riemannian case, under some other constraints on the normal curvature of the hypersurface.

§ 2. Preliminary observations and the statements of the main results

We look at a complete simply connected $(n + 1)$ -dimensional Riemannian manifold M^{n+1} . Let K_σ denote the sectional curvature of M^{n+1} at a point $P \in M^{n+1}$ in the direction of a 2-plane $\sigma \subset T_P M^{n+1}$. Let $\Omega \subset M^{n+1}$ be a compact closed domain such that its boundary $\partial\Omega$ is a C^2 -hypersurface.

Take a point $O \in \Omega$. Let $h := \text{dist}(O, \partial\Omega)$ be its distance from the boundary of the domain, and for an arbitrary point $P \in \partial\Omega$ let $\varphi := \varphi(P)$ denote the angle between the minimizing curve from O to P and the outward normal to $\partial\Omega$ at P (Fig. 1).

Here and throughout, $k_n = k_n(P, Y)$ denotes the normal curvature of the hypersurface $\partial\Omega$ at the point $P \in \partial\Omega$ in the direction of the vector $Y \in T_P \partial\Omega$.

It turns out that if the normal curvatures of $\partial\Omega$ in all directions are bounded below ($k_n \geq k_0$), then the angle φ cannot be very large. More precisely, we have the following theorem.

Theorem 1. *Let $M^{n+1}(c)$ be a complete simply connected Riemannian manifold with constant sectional curvature c and let Ω be a closed bounded domain in M such that $\partial\Omega$ is a regular C^2 hypersurface. Let $O \in \Omega$ be a point in the interior of the domain, $h = \text{dist}(O, \partial\Omega)$ be the distance from O to the hypersurface, and φ be the angle between the radial direction from O to a point in $\partial\Omega$ and the outward normal at this point. Then the following results hold.*

1. *If $c = 0$, that is, $M^{n+1}(c) = \mathbb{E}^{n+1}$ is a Euclidean space, and if the normal curvatures of the hypersurface in all directions satisfy $k_n \geq k_0 > 0$, then*

$$\cos \varphi \geq \sqrt{2hk_0 - h^2k_0^2} \geq hk_0. \tag{2.1}$$

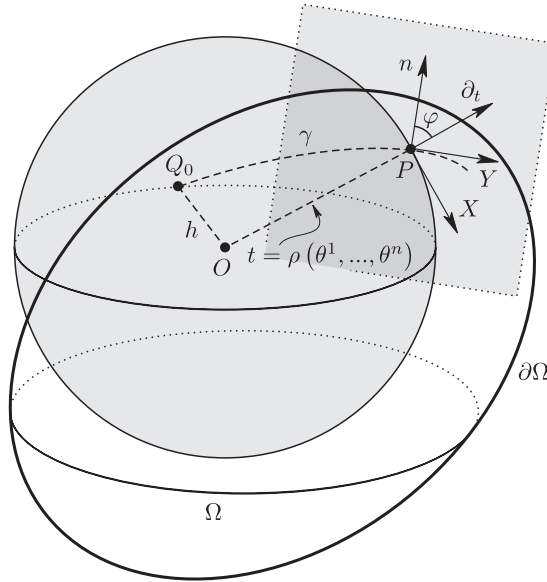


Figure 1

2. If $c = -k_1^2$, where $k_1 > 0$, that is, $M^{n+1}(c) = \mathbb{H}^{n+1}$ is the $(n + 1)$ -dimensional Lobachevskii space and if the normal curvatures of $\partial\Omega$ in all directions satisfy $k_n \geq k_0 > k_1$, then

$$\cos \varphi \geq \sqrt{1 - \frac{\sinh^2 k_1(R - h)}{\sinh^2 k_1 R}} \geq \frac{\sinh k_1 h}{\sinh k_1 R}. \tag{2.2}$$

3. If $c = k_1^2$, $k_1 > 0$, that is, $M^{n+1}(c) = S^{n+1}$ is an $(n + 1)$ -sphere with curvature k_1^2 , and the normal curvatures of the hypersurface $\partial\Omega$ in all directions satisfy $k_n \geq k_0 \geq 0$, then

$$\cos \varphi \geq \sqrt{1 - \frac{\sin^2 k_1(R - h)}{\sin^2 k_1 R}} \geq \frac{\sin k_1 h}{\sin k_1 R}. \tag{2.3}$$

In cases 2 and 3, R denotes the radius of a circle with curvature k_0 in a 2-plane with the corresponding curvature c .

A similar result holds when the ambient space is a Riemannian manifold with curvature having fixed sign.

Theorem 2. Let $\partial\Omega$ be a compact C^2 -hypersurface in a complete simply connected $(n + 1)$ -dimensional Riemannian manifold M^{n+1} that bounds a domain $\Omega \subset M^{n+1}$. Then the following results hold.

1. If the sectional curvatures K_σ of M^{n+1} on all the 2-planes σ satisfy $0 \geq K_\sigma \geq -k_1^2$, $k_1 > 0$, and the normal curvatures of $\partial\Omega$ in all directions satisfy $k_n \geq k_0 > k_1$, then (2.2) holds.

2. On the other hand, if the sectional curvatures of M^{n+1} satisfy $k_2^2 \geq K_\sigma \geq k_1^2$, $k_1 > 0$, Ω lies in a ball of radius $\pi/(2k_2)$, and the normal curvatures of $\partial\Omega$ in all directions satisfy $k_n \geq k_0 \geq 0$, then (2.3) holds.

Recall (see [3] and [4]) that a locally convex (not necessarily regular) hypersurface $\partial\Omega \subset M^{n+1}(c)$ in a constant curvature space is said to be λ -convex if for each point $P \in \partial\Omega$ there exists a sphere S_P with curvature λ such that in a neighbourhood of P the hypersurface lies to the convex side of S_P . The corresponding domain Ω is called a λ -convex domain.

Note that a C^k -hypersurface $\partial\Omega$, where $k \geq 2$, is λ -convex if and only if its normal curvatures in all directions at each point satisfy $k_n \geq \lambda$. Thus λ -convexity is a natural generalization of the property of having normal curvatures bounded below by λ .

In view of these definitions, we obtain the following result on the width of spherical shells in constant curvature spaces.

Theorem 3. *Let $\partial\Omega$ be a complete k_0 -convex hypersurface bounding a domain Ω in a complete simply connected $(n + 1)$ -dimensional manifold $M^{n+1}(c)$ with constant sectional curvature c . Then the following results hold.*

1. If $c = 0$ and $k_0 > 0$, then $\partial\Omega$ lies in a spherical shell with width

$$d \leq \frac{\sqrt{2} - 1}{k_0}. \tag{2.4}$$

2. If $c = k_1^2$, $k_1 > 0$ and $k_0 > 0$, then $\partial\Omega$ lies in a spherical shell with width

$$d \leq \frac{2}{k_1} \cos^{-1} \sqrt{\cos k_1 R} - R. \tag{2.5}$$

3. If $c = -k_1^2$, $k_1 > 0$ and $k_0 > k_1$, then $\partial\Omega$ lies in a spherical shell with width

$$d \leq \frac{2}{k_1} \cosh^{-1} \sqrt{\cosh k_1 R} - R. \tag{2.6}$$

In cases 2 and 3, R is the radius of a circle with curvature k_0 in a 2-plane with curvature c .

Remark 1. It is known that $R = (1/k_1) \cot^{-1}(k_0/k_1)$ on a 2-sphere with curvature k_1^2 , and that $R = (1/k_1) \coth^{-1}(k_0/k_1)$ in a 2-dimensional Lobachevskii space with curvature $-k_1^2$. Throughout we let R denote the radius of a circle with curvature k_0 in a plane with constant curvature c . Then we can write (2.5) and (2.6) as

$$d \leq \frac{1}{k_1} \left(2 \cos^{-1} \frac{\sqrt{k_0}}{\sqrt[4]{k_0^2 + k_1^2}} - \cot^{-1} \frac{k_0}{k_1} \right),$$

$$d \leq \frac{1}{k_1} \left(2 \cosh^{-1} \frac{\sqrt{k_0}}{\sqrt[4]{k_0^2 - k_1^2}} - \coth^{-1} \frac{k_0}{k_1} \right),$$

respectively.

Remark 2. The above bounds are sharp: we have equalities for the spindle-shaped surfaces described below.

We can generalize the bound for the width of a spherical shell to the case when the ambient Riemannian space has sectional curvature of fixed sign. Namely, we have the following result.

Theorem 4. *Let $\partial\Omega$ be a closed C^2 -hypersurface bounding a domain Ω in a complete simply connected $(n + 1)$ -dimensional Riemannian manifold M^{n+1} .*

1. *Assume that the sectional curvatures of M^{n+1} on all the 2-planes σ satisfy $k_2^2 \geq K_\sigma \geq k_1^2$, where $k_1, k_2 > 0$. Suppose that the hypersurface lies in a ball with radius π/k_2 which has the same centre as a ball inscribed in $\partial\Omega$. If the normal curvatures of $\partial\Omega$ in all directions satisfy $k_n \geq k_0 > 0$, then $\partial\Omega$ lies in a spherical shell with width (2.5).*

2. *Assume that $0 \geq K_\sigma \geq -k_1^2$ for every 2-plane, $k_1 > 0$. If all the normal curvatures of $\partial\Omega$ in all directions satisfy $k_n \geq k_0 > k_1$, then $\partial\Omega$ lies in a spherical shell with width (2.6).*

§ 3. The proofs of angle comparison theorems

3.1. Auxiliary results. We introduce the polar coordinate system with origin at $O \in \Omega$ in M^{n+1} . In these coordinates we can write the metric on the manifold as $ds^2 = dt^2 + g_{ij} d\theta^i d\theta^j$, $i, j = 1, \dots, n$, where t is the length parameter and the θ^i are angular variables.

We can assume that the regular hypersurface $\partial\Omega$ has equation $t = \rho(\theta^1, \dots, \theta^n)$. Indeed, this holds for convex hypersurfaces in the regularity region of the polar coordinate system. Then $\partial\Omega$ is the 0-level set of the function $F(t, \theta^1, \dots, \theta^n) = t - \rho(\theta^1, \dots, \theta^n)$.

For an arbitrary manifold N and a smooth function f the gradient vector field of this function is the unique vector field $\text{grad}_N f$ such that

$$\langle \text{grad}_N f, v \rangle = v(f) \quad \forall v \in TN.$$

Let Y be the gradient vector field of the distance function ρ on $\partial\Omega$ that measures the distance from points in $\partial\Omega$ to O :

$$Y = \text{grad}_{\partial\Omega} \rho.$$

It is known that the unit outward normal to $\partial\Omega$ has the form

$$n = \frac{\text{grad}_{M^{n+1}} F}{\|\text{grad}_{M^{n+1}} F\|}.$$

At each point in $\partial\Omega$ the vector $\partial_t = \partial/\partial t$ defines the radial direction; let φ be the angle between n and ∂_t .

The vectors $n(P)$, $\partial_t(P)$ and $Y(P)$ lie in the same 2-plane in $T_P M^{n+1}$ (see [3] and [4]). Let $X(P)$ be the unit vector orthogonal to $\partial_t(P)$ in this plane, and let k_n denote the normal curvature of $\partial\Omega$ at $P \in \partial\Omega$ in the direction of $Y = Y(P)$ (see Fig. 1).

Then the following result holds.

Lemma 1 (see [3] and [4]). *Let μ_n be the normal curvature of a sphere with radius ρ and centre O at a point $P \in \partial\Omega$ in the direction of X and assume that an integral trajectory of the vector field Y is parametrized by the distance t from O to the point on the curve. Then*

$$k_n(t) = \cos \varphi \mu_n(t) - \sin \varphi \frac{d\varphi}{dt}$$

on this trajectory.

To find connections between the curvature of a sphere and the curvature of the space we require the following lemma.

Lemma 2 (see [7], [8]). *Assume that the sectional curvatures of a Riemannian manifold M^{n+1} satisfy one of the following conditions.*

1) *For all 2-planes σ , $K_\sigma \geq k_1^2$, where $k_1 > 0$, and each sphere with radius t lies in the regularity region of the polar coordinate system with origin at the centre of the sphere;*

2) *$0 \geq K_\sigma \geq -k_1^2$, where $k_1 \geq 0$.*

Then the normal curvatures μ_n in all directions of spheres with radius t satisfy

$$\mu_n(t) \leq \mu_0(t),$$

where $\mu_0(t)$ is the geodesic curvature of a circle with radius t on a plane with constant curvature equal to k_1^2 in case 1) and to $-k_1^2$ in case 2).

Note that circles with radius t on 2-planes with constant curvature have geodesic curvatures $\mu_0(t)$ equal to:

$1/t$ in a Euclidean plane;

$k_1 \cot^{-1} k_1 t$ on a sphere with curvature k_1^2 ;

$k_1 \coth^{-1} k_1 t$ on a Lobachevskii plane with curvature $-k_1^2$.

To use this comparison lemma we investigate the magnitude of the angle φ for circles on surfaces with constant Gaussian curvature. The following result holds.

Lemma 3 (see [5]). *Let M^2 be a plane with constant curvature, γ be a circle with radius R , and O be a point lying at a distance h from the circle in the disc bounded by this circle. Then the angle φ between the geodesic curve from O to a point $\gamma(s)$ on the circle and the outward normal to the circle satisfies the following inequalities:*

1) *on a Euclidean plane*

$$\cos \varphi \geq \sqrt{\frac{2h}{R} - \frac{h^2}{R^2}} \geq \frac{h}{R};$$

2) *on the Lobachevskii space with curvature $-k_1^2$*

$$\cos \varphi \geq \sqrt{1 - \frac{\sinh^2 k_1 (R - h)}{\sinh^2 k_1 R}} \geq \frac{\sinh k_1 h}{\sinh k_1 R};$$

3) *on a sphere with curvature k_1^2 , provided that $R \leq \pi/(2k_1)$,*

$$\cos \varphi \geq \sqrt{1 - \frac{\sin^2 k_1 (R - h)}{\sin^2 k_1 R}} \geq \frac{\sin k_1 h}{\sin k_1 R}.$$

We also require the following result.

Lemma 4. *Let $f(x) \in C^1[a, b]$ and assume that $f(a) = 0$ and $f(b) < 0$. Then the set of points at which $f(x) < 0$ contains a point $x_0 \in (a, b)$ such that $f'(x_0) < 0$.*

3.2. The proofs of Theorems 1 and 2. In the two-dimensional case we have the following result.

Lemma 5. *Let γ be a C^k -regular closed curve embedded in a plane with constant Gaussian curvature, where $k \geq 2$, let O be a point lying at a distance h from γ in the domain bounded by this curve, and let φ be the angle between the radial direction from O to a point in γ and the outward normal at this point.*

1. *In the case of a Euclidean plane, if the curvature k of γ satisfies $k \geq k_0 > 0$, then (2.1) holds.*

2. *In the case of a Lobachevskii plane with curvature $K = -k_1^2$, $k_1 > 0$, if the curvature of the curve satisfies $k \geq k_0 > k_1$, then (2.2) holds.*

3. *In the case of a sphere with curvature $K = k_1^2$, if the curvature k of γ satisfies $k \geq k_0 \geq 0$, then (2.3) holds.*

Proof. We carry out the proof for all three cases simultaneously.

In the constant curvature plane we introduce the polar coordinate system with origin at O . Then the curvature k of the curve γ satisfies

$$k = \mu_0(t) \cos \varphi - \sin \varphi \frac{d\varphi}{dt}, \tag{3.1}$$

by Lemma 1, where $\mu_0(t)$ is the curvature of a circle in the plane with constant curvature k_0 .

Now we construct an object for comparison. In the constant curvature plane we look at a circle S with curvature k_0 . Taking a point O_1 lying at the distance h from the circle in its interior, we consider the polar system of coordinates with origin at O_1 . We shall denote the angle between the outward normal to the circle and the geodesic curve from O_1 to a point in S by β .

By Lemma 1

$$k_0 = \mu_0(t) \cos \beta - \sin \beta \frac{d\beta}{dt}. \tag{3.2}$$

We subtract equation (3.1) from (3.2) and use the assumption that $k \geq k_0$. Then we obtain

$$\mu_0(\cos \varphi - \cos \beta) - \sin \varphi \frac{d\varphi}{dt} + \sin \beta \frac{d\beta}{dt} = k - k_0 \geq 0. \tag{3.3}$$

We set $f(t) = \cos \varphi(t) - \cos \beta(t)$. Then it follows from (3.3) that this function satisfies

$$f' + \mu_0 f \geq 0, \quad f(h) = 0. \tag{3.4}$$

The condition $f(h) = 0$ holds because in both cases h is the distance between the origin and the curve, and therefore $\varphi(h) = \beta(h) = 0$.

We look at an arc of γ starting at the point Q_0 such that

$$\text{dist}(O, Q_0) = \text{dist}(O, \gamma) = h$$

and ending at some point Q_1 , such that $t(s)$ is increasing on this arc (on such arcs we can parametrize the curve by the distance to the origin). We shall perform the proof for this arc.

We continue the proof in the more complicated case 3.

If γ is not a circle or O is not the centre of the circle, then $h < \pi/(2k_1)$. Indeed, as $k \geq 0$ by assumption, the curve γ lies on a closed hemisphere, which gives us a bound for h . We claim that for t close to h we have $f(t) \geq 0$, and $f(t)$ is not identically equal to zero if the arc of γ is not an arc of a circle with curvature k_0 in a neighbourhood of Q_0 .

In fact, if t such that $f(t) < 0$ exists arbitrarily close to h , then in the set of such points, by Lemma 4 we can find t_0 close to h such that

$$f(t_0) < 0, \quad f'(t_0) < 0. \tag{3.5}$$

Since $h < \pi/(2k_1)$ and the point t_0 is arbitrarily close to h , from the inequality $\mu_0(h) > 0$ we obtain $\mu_0(t_0) > 0$. But then (3.5) contradicts (3.4).

Now we take t_1 close to h such that $f(t_1) > 0$. (We have shown that there exist such t_1 .) Consider the following Cauchy problem:

$$g' + \mu_0(t)g = 0, \quad g(t_1) = f(t_1) > 0. \tag{3.6}$$

In case 3, $\mu_0(t) = k_1 \cot k_1 t$, and then the solution of (3.6) has the form

$$g(t) = \frac{f(t_1) \sin k_1 t_1}{\sin k_1 t}.$$

Note that for $0 < t \leq \pi/(2k_1)$ the function $g(t)$ is positive; furthermore, it is increasing, that is, $g'(t) > 0$ for $\pi/(2k_1) \leq t < \pi/k_1$.

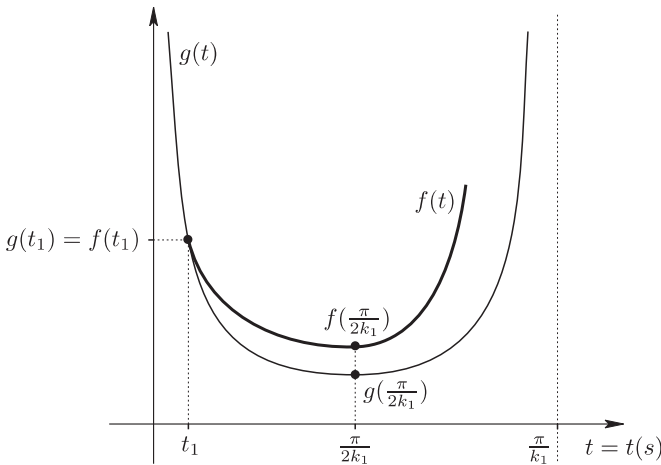


Figure 2

Now we compare solutions of inequality (3.4) and equation (3.6) which satisfy the same initial condition (Fig. 2). At points where $f - g < 0$, for $t_1 \leq t \leq \pi/(2k_1)$,

we have

$$(f - g)' \geq -\mu_0(f - g) > 0 \tag{3.7}$$

because $\mu_0(t) \geq 0$ on this interval. Since $f(t_1) - g(t_1) = 0$ (by (3.6)), it follows from Lemma 4 that the set of points at which $f - g < 0$ contains a point t_2 such that $f(t_2) - g(t_2) < 0$ and $f'(t_2) - g'(t_2) < 0$, which contradicts (3.7). Hence $f \geq g > 0$ for $t_1 \leq t \leq \pi/(2k_1)$.

We have $f(\pi/(2k_1)) - g(\pi/(2k_1)) \geq 0$. For $t > \pi/(2k_1)$ we also have $\mu_0(t) < 0$. Thus it follows from (3.7) that $f' - g' \geq 0$, that is, $f' \geq g' > 0$. Hence for $t \geq \pi/(2k_1)$, on the piece of γ where $t = t(s)$ is monotonically increasing, f is monotonically increasing too. Since $f(\pi/(2k_1)) \geq 0$, we have $f \geq 0$ on this part of γ to the right of $\pi/(2k_1)$.

Thus we have shown that $\cos \varphi(t) \geq \cos \beta(t)$ on the arc under consideration. Using the estimate for $\cos \beta(t)$ from Lemma 3 we obtain the result of Lemma 5 in this case. Obviously, the regular curve γ is a union of several arcs of this type, which only differ in their minimum distances h_i from the point O . Finding an estimate for the angle φ on every piece of γ on which $t = t(s)$ is monotonic, we obtain a bound for the closed curve. Here we use that the right-hand sides of the estimates in Lemma 3 are monotone increasing in h . As the minimum distance on each arc satisfies $h_i \geq h$, the estimate $\cos \varphi(t) \geq \cos \beta(t)$ holds also for h in place of h_i . Moreover, if $h_i > \pi/(2k_1)$, then $f(h_i) = 0$, and it follows from (3.4) that $f' > 0$ and $f > 0$ on this arc of the curve.

In this way we complete the proof of assertion 3. Assertions 1 and 2 are proved in a similar way.

Proof of Theorems 1 and 2. We introduce the polar coordinate system with origin O . Then the arc length element can be written as $ds^2 = dt^2 + g_{ij} d\theta^i d\theta^j$. By the constraints on the normal curvature the hypersurface $\partial\Omega$ is embedded, convex, and compact. Moreover, in all cases, this hypersurface lies in the regularity region of this system of coordinates. Hence we can assume that $\partial\Omega$ is the 0-level set of the function $F(t, \theta) = t - \rho(\theta)$.

Let γ be an integral trajectory of the vector field $Y = \text{grad}_{\partial\Omega} \rho$ and let $Q_0 \in \gamma$ be the point lying at distance h from O ; $Y = 0$ at this point and $\varphi(Q_0) = 0$. Let $P \in \gamma$ be a point at a distance h_1 from O such that the distance from O is a monotone function on the arc Q_0P of γ . Then γ can be parametrized by the parameter $t \in (h; h_1]$ measuring the distance from O .

By Lemma 1, at points in γ we have

$$k_n(t) = \cos \varphi(t)\mu_n(t) - \sin \varphi(t) \frac{d\varphi}{dt}. \tag{3.8}$$

Just as in the two-dimensional case, we look at a circle S with curvature k_0 lying in a 2-plane with constant curvature (equal to 0 in the Euclidean, to k_1^2 in the spherical, and to $-k_1^2$ in the hyperbolic case). Let $\overline{Q_0}$ be a point on S , and O_1 be a point inside S lying at the distance h from $\overline{Q_0}$ on the geodesic curve orthogonal to S at $\overline{Q_0}$; let β be the angle between the geodesic curve from O_1 to a point in S and the outward normal to S at this point. Then equality (3.2) holds.

Subtracting (3.2) from (3.8), from the assumptions of Theorems 1 and 2, using Lemma 2 we obtain the differential inequality (3.4).

The minimum distance h from points in γ to the point O is less than $\pi/(2k_1)$. Indeed, by the assumptions in Theorem 2, part 2 (the argument is even simpler in the other cases), the hypersurface $\partial\Omega$ lies in a ball of radius $\pi/(2k_2)$, which is no larger than $\pi/(2k_1)$. So if $h = \pi/(2k_1)$, then a fortiori $k_1 = k_2$ and $\partial\Omega$ lies in a constant curvature space, where it is a totally geodesic sphere with centre O , or, in the 2-dimensional case, it is a closed geodesic.

Now we can repeat the calculations in Lemma 5 and complete the proof of Theorems 1 and 2.

§ 4. The proofs of the bounds for the width of spherical shells

4.1. Auxiliary results required for the proof of Theorem 3.

Lemma 6. *Let $\partial\Omega$ be a complete k_0 -convex hypersurface in a space $M^{n+1}(c)$ with constant curvature c , such that*

- 1) *if $c = 0$ or $c = k_1^2 > 0$, then $k_0 > 0$;*
- 2) *if $c = -k_1^2$ then $k_0 > k_1$.*

Then $\partial\Omega$ is an embedded convex hypersurface, and at each point $P \in \partial\Omega$ it has a supporting sphere with radius R that contains this hypersurface in its interior.

Remark 3. In Lemma 6 we understand by R the radius of a circle with curvature k_0 in a 2-dimensional manifold $M^2(c)$ with constant sectional curvature c .

Proof of Lemma 6. For a C^k -smooth hypersurface, where $k \geq 2$, the required result is an immediate consequence of [9].

Now let $\partial\Omega$ be a k_0 -convex irregular hypersurface. For sufficiently small τ we look at the outer equidistant surfaces $\partial\Omega_\tau$, which are $\varepsilon(\tau)$ -convex with $\varepsilon(\tau) > 0$ (in case 1) or $\varepsilon(\tau) > k_1$ (in case 2)), where $\varepsilon(\tau) \rightarrow k_0 - 0$ as $\tau \rightarrow 0$. It is known that $\partial\Omega_\tau$ is a $C^{1,1}$ -smooth hypersurface. Moreover, $\partial\Omega_\tau$ is a limit of regular hypersurfaces $\partial\Omega_{\tau,\delta}$ with normal curvatures satisfying $k_n \geq \varepsilon(\tau) - \nu(\delta)$, where $\varepsilon(\tau) - \nu(\delta) > 0$ (in case 1) or $\varepsilon(\tau) - \nu(\delta) > k_1$ (in case 2)) and $\nu(\delta) \rightarrow 0 + 0$ as $\delta \rightarrow 0$. By the regular case, which we have already considered, $\partial\Omega_{\tau,\delta}$ lies in a sphere with radius $R_{\tau,\delta}$ and curvature $\varepsilon(\tau) - \nu(\delta)$, which is a supporting sphere at an arbitrary point $P_{\tau,\delta} \in \partial\Omega_{\tau,\delta}$. Taking the limit as $\tau \rightarrow 0$ and $\delta \rightarrow 0$ we see that the sphere with radius $R = \lim_{\tau,\delta \rightarrow 0} R_{\tau,\delta}$ supporting $\partial\Omega$ at the point $P = \lim_{\tau,\delta \rightarrow 0} P_{\tau,\delta} \in \partial\Omega$ contains the whole hypersurface. This holds for any point P and proves the lemma in the irregular case.

We make an observation, which we use below. Let A and B be points in $\Omega \subset M^{n+1}(c)$ and let $M^2(c)$ be a totally geodesic subspace of $M^{n+1}(c)$ containing A and B (it exists because the sectional curvature is constant). Then $M^2(c)$ contains precisely two circles with radius R passing through A and B . These points partition each of the circles into two arcs, a longer one and a shorter one. Throughout what follows we call the shorter arc of a circle with radius R passing through points A and B the *shorter arc of a radius- R circle for A and B* .

The following lemma holds.

Lemma 7. *Let $\partial\Omega$ be a complete k_0 -convex hypersurface in a complete simply connected space $M^{n+1}(c)$ with constant sectional curvature c , which bounds a domain Ω*

(where $k_0 > 0$ if $c = 0$ or $c = k_1^2 > 0$ and $k_0 > k_1$ if $c = -k_1^2 < 0$). Then the shorter arc of any radius- R circle for arbitrary points $A, B \in \Omega$ lies in Ω .

Proof. Assume that the statement fails and that there exist $A, B \in \Omega$ and a shorter arc ω for A and B which does not lie entirely in Ω . We look at the intersection of Ω with the two-dimensional subspace $M^2(c)$ containing A, B and ω . Let $\gamma = M^2(c) \cap \partial\Omega$ be a curve in this intersection.

It is known that the intersection of a λ -convex surface and a 2-subspace is a λ -convex curve. Thus γ is k_0 -convex.

Let A_1 and B_1 be the points of intersection of ω and γ . Then letting ω_1 denote the part of ω lying between A_1 and B_1 and letting γ_1 denote the part of γ confined between ω and the chord AB we see that ω_1 and γ_1 are convex curves lying to the same side of the geodesic curve joining A_1 to B_1 .

Let C_1 be a point on γ_1 distinct from A_1 and B_1 . Since γ is a k_0 -convex closed curve, the circle δ with radius R supporting γ at C_1 contains γ in its interior by Lemma 6. Assume that δ intersects ω_1 in points X_1 and Y_1 . Note that if X_1 or Y_1 coincides with A_1 or B_1 for every C_1 , then $\omega_1 \equiv \gamma_1$ and we arrive at a contradiction to the assumption that ω_1 does not lie in Ω . Hence we can assume that $X_1 \neq A_1$ and $Y_1 \neq B_1$.

However, ω_1 is the shorter arc of a radius- R circle, so the arc δ_1 of δ confined between ω_1 and the chord A_1B_1 is less than half the circle δ . On the other hand, by convexity, δ_1 and ω_1 lie to the same side of the geodesic X_1Y_1 .

We have proved in this way that for the fixed points X_1 and Y_1 there exist two different shorter arcs of radius- R circles lying to the same side of the geodesic X_1Y_1 , which is impossible. This proves Lemma 7.

4.2. Finding bounds for special spindle-shaped hypersurfaces. As above, we consider a complete simply connected Riemannian manifold $M^{n+1}(c)$ with constant sectional curvature c . As previously, R denotes the radius of a circle with curvature k_0 in a 2-dimensional manifold $M^2(c)$.

We now construct a key object that we need for estimates of the width of spherical shells.

For fixed points $P, Q \in M^{n+1}(c)$ we look at the special class of *spindle-shaped hypersurfaces* $v(P, Q)$ obtained by rotating the shorter arc of a radius- R circle for P and Q about the geodesic curve l joining these points.

Note that $v(P, Q)$ is a k_0 -convex surface of revolution. Then for any 2-plane $M^2(c)$ containing P and Q the intersection $M^2(c) \cap v(P, Q)$ is a curve γ formed by two shorter arcs of radius- R circles symmetric relative to the geodesic PQ . We call such curves *lunes* or *curvilinear digons*.

Let $O \in l$ be a point lying at equal distance from P and Q . Since $v(P, Q)$ is the surface of revolution of a circle arc, O is the centre of a sphere S with radius r inscribed in $v(P, Q)$. Then $\omega := M^2(c) \cap S$ is a circle with centre O and radius r inscribed in γ .

As $v(P, Q)$ is constructed by rotating the shorter arc of a radius- R circle, the circumscribed sphere S_1 of $v(P, Q)$ has centre O and radius $\rho := OP = OQ$. In a similar way, $\omega_1 = M^2(c) \cap S_1$ is the circle with radius ρ and centre O circumscribed about γ .

Here it is obvious that given the radius of the inscribed sphere, for fixed R we can uniquely recover the points P and Q and therefore the hypersurface $v(P, Q)$ (because an arc and its end-points are uniquely recovered from the radius R of the circle and the height r of the circular segment).

Thus we can consider the class of spindle-shaped surfaces so defined. They are parametrized by r ; note that $r \in [0, R]$. By construction, each hypersurface in this class can be put in a spherical shell with width $d = d(r) = \rho(r) - r$ (for ρ can also be uniquely recovered from r).

The following lemma is true.

Lemma 8. *The following estimates for the width $d = d(r)$ of spherical shells hold in the class of spindle-shaped hypersurfaces in constant curvature spaces:*

- 1) (2.4) in Euclidean space;
- 2) (2.5) in the spherical space $S^{n+1}(k_1^2)$;
- 3) (2.6) in the Lobachevskii space $\mathbb{H}^{n+1}(-k_1^2)$.

The proof is by direct calculation of the function $d(r) = \rho(r) - r$ and determining its maximum by means of standard calculus.

Remark 4. As $k_1 \rightarrow 0$, the metrics in $S^{n+1}(k_1^2)$ and $\mathbb{H}^{n+1}(-k_1^2)$ approach the Euclidean metric. It is easy to show that the bounds (2.5) and (2.6) tend to (2.4) as $k_1 \rightarrow 0$.

4.3. The proof of Theorem 3. We shall prove the theorem in all cases simultaneously, indicating differences between these where necessary.

The regular case. Let $\partial\Omega$ be a C^2 -hypersurface.

Let B be a ball with centre O and radius r inscribed in $\partial\Omega$. Let

$$\rho_1 = \max \text{dist}(O, \partial\Omega)$$

be the maximum distance from O to points in the hypersurface. Then $\partial\Omega$ obviously lies in a spherical shell with width $d = \rho_1 - r$.

Let $\rho(r)$ be the radius of the circumscribed sphere in Lemma 8. We claim that

$$d = \rho_1 - r \leq \rho(r) - r \tag{4.1}$$

for all $r \in [0, R]$.

Suppose that this inequality fails and let

$$d = \max \text{dist}(O, \partial\Omega) - r > \rho(r) - r. \tag{4.2}$$

Let $P' \in \partial\Omega$ be a point such that $\max \text{dist}(O, \partial\Omega) = \text{dist}(O, P')$ (Fig. 3). Then by (4.2) there is a point P on the geodesic curve OP' , between O and P' , such that

$$\text{dist}(O, P) = \rho(r). \tag{4.3}$$

Let Q be the point symmetric to P in O . We look at the spindle-shaped hypersurface $v(P, Q)$.

Then B is also inscribed in $v(P, Q)$. We take the hyperplane π orthogonal to OP and passing through O . Let $D = \pi \cap B$ be the ‘equatorial’ ball.

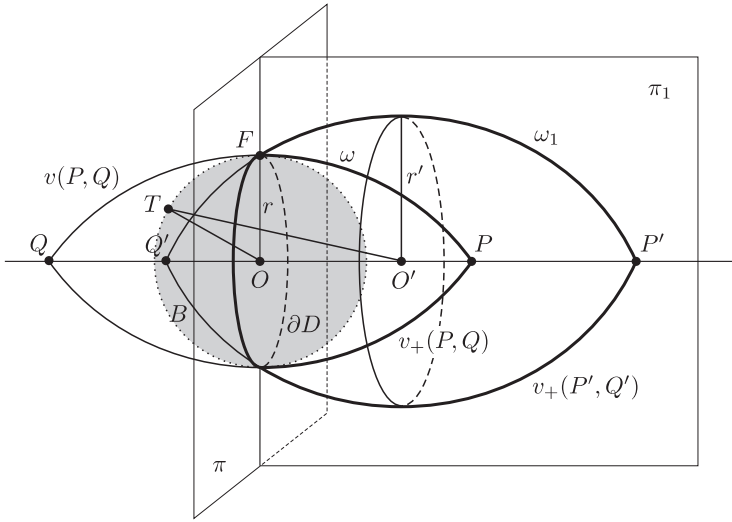


Figure 3

Let F be a point on ∂D . Then the shorter arc ω of a radius- R circle for the points P and F , which lies on $v(P, Q)$, also lies in Ω by Lemma 7. As $F \in \partial D$ can be arbitrary, the part $v_+(P, Q)$ of the spindle-shaped hypersurface $v(P, Q)$ lying in the same half-space with respect to π as P and P' lies in Ω .

Let Q' be the second point of intersection of the geodesic curve PQ and the shorter arc ω_1 of the radius- R circle for P' and F that bulges towards PQ .

Then it is obvious that ∂D lies on the spindle-shaped surface $v(P', Q')$, and the part $v_+(P', Q')$ lies in Ω by Lemma 7.

Moreover, ω and ω_1 must be disjoint. Since the same holds for any point F , it follows that $v_+(P, Q)$ lies inside $v_+(P', Q')$ and these surfaces intersect in ∂D .

Note that all the shorter arcs of radius- R circles which lie on $v(P, Q)$ and join P to ∂D are orthogonal to geodesic curves from O to points in ∂D . As $v_+(P, Q)$ lies inside $v_+(P', Q')$, the angle between $P'F$ and the geodesic OF is larger than $\pi/2$. Hence the radius r' of the ball B' inscribed in $v(P', Q')$ is larger than r and its centre O' lies between O and P' .

By the construction of $v(P', Q')$, all the points in $v(P', Q')$ whose distance from O' is equal to r' lie on rays from O' orthogonal to OP . Since $O'F$ is not orthogonal to OP , it follows that $|O'F| = \text{dist}(O', \partial D) > r'$ (here $|\cdot|$ is the length of the geodesic interval in the corresponding space).

Let $T \in \partial B_-$, where B_- is the part of the ball B lying in the different half-space from the points P and P' relative to the plane π . The angle $O'OT$ of the geodesic triangle $\triangle OO'T$ is larger than $\pi/2$, so $|O'T| > |O'F| > r'$ by the cosine theorem. Since T can be arbitrary, we have $B' \subset B_- \cap v_+(P', Q') \subset \Omega$.

Thus we have found a ball in Ω which has a radius larger than the inscribed ball: a contradiction, which proves (4.1). Now Lemma 8 yields estimates (2.4)–(2.6). The proof in the smooth case is complete.

The irregular case. Let $\partial\Omega$ be an arbitrary complete k_0 -convex hypersurface. Our arguments will be similar to the ones in the proof of Lemma 6. Let $\partial\Omega_\tau$ be the outer equidistant $C^{1,1}$ -smooth convex hypersurface lying at distance τ from $\partial\Omega$. The hypersurface $\partial\Omega_\tau$ is $\varepsilon(\tau)$ -convex and $\lim_{\tau \rightarrow 0} \varepsilon(\tau) = k_0$. We approximate $\partial\Omega_\tau$ by C^k -smooth hypersurfaces $\partial\Omega_{\tau,\delta}$, $k \geq 2$, with normal curvatures satisfying $k_n \geq \varepsilon(\tau) - \nu(\delta)$, where $\nu(\delta) \rightarrow 0 + 0$ as $\delta \rightarrow 0$. For such hypersurfaces we proved the bound above. Passing to the limit as $\tau, \varepsilon \rightarrow 0$ and bearing in mind that $\lim_{\tau,\delta \rightarrow 0} R_{\tau,\delta} = R$, we obtain the required bounds in the general case.

The proof of Theorem 3 is complete.

4.4. Auxiliary results required for the proof of Theorem 4. Let $M^{n+1}(c)$ be a complete simply connected Riemannian manifold with constant sectional curvature c . Let Ω be a compact convex domain in it, with boundary $\partial\Omega$ which is a closed C^2 -hypersurface. Let $O \in \Omega$ be an interior point of the domain and let $P \in \partial\Omega$ be a point such that $\text{dist}(O, P) = \text{dist}(O, \partial\Omega)$. Let $\varphi(Q)$ denote the angle between the geodesic OQ from O to some point $Q \in \partial\Omega$ and the outward normal to $\partial\Omega$ at Q . Let $S_P \subset M^{n+1}(c)$ be a sphere passing through P and orthogonal to OP such that O lies in the corresponding ball B_P and let $\beta(\bar{Q})$ denote the angle between the geodesic $O\bar{Q}$ going through some point $\bar{Q} \in S_P$ and the outward normal to the sphere at this point.

Lemma 9. *In the above notation, if*

$$\varphi(Q) \leq \beta(\bar{Q})$$

for some points $Q \in \partial\Omega$ and $\bar{Q} \in S_P$ such that the intervals OQ and $O\bar{Q}$ of geodesic lines have the same length, then S_P is tangent to the hypersurface $\partial\Omega$ at P and the whole of Ω lies in the ball B_P .

Proof. We introduce the polar system of coordinates with origin O on $M^{n+1}(c)$. Then the metric has the expression $ds^2 = dt^2 + g_{ij} d\theta^i d\theta^j$, where t is the distance from the origin and $\theta^1, \dots, \theta^n$ are the coordinate variables on the standard Euclidean sphere S^n . We can assume that the point P has coordinates $(h, 0, \dots, 0)$, where $h = \text{dist}(O, Q) = \text{dist}(O, \partial\Omega)$.

In a neighbourhood of P we can define our surface and the sphere by explicit equations. Assume that $\partial\Omega$ is given by $t = f(\theta^1, \dots, \theta^n)$ and S_P is given by $t = \rho(\theta^1, \dots, \theta^n)$.

Let $Q \in \partial\Omega$ and $\bar{Q} \in S_P$ be points in this neighbourhood such that $OQ = O\bar{Q}$. Then the corresponding outward normals $N_{\partial\Omega}(Q)$ and $N_{S_P}(\bar{Q})$ to the surfaces at these points have the following expressions:

$$N_{\partial\Omega}(Q) = \frac{\partial_t - g^{ij} \frac{\partial f}{\partial \theta^i} \partial_{\theta^j}}{\sqrt{1 + |\nabla f|_{\partial\Omega}^2}}, \quad N_{S_P}(\bar{Q}) = \frac{\partial_t - g^{ij} \frac{\partial \rho}{\partial \theta^i} \partial_{\theta^j}}{\sqrt{1 + |\nabla \rho|_{S_P}^2}}, \quad (4.4)$$

where all the derivatives are taken at Q or \bar{Q} , respectively; $\partial_t, \partial_{\theta^i}, i = 1, \dots, n$, is the coordinate basis in the corresponding tangent space $T_Q M^{n+1}(c)$ or $T_{\bar{Q}} M^{n+1}(c)$, and we have set $\partial f = \frac{\partial f}{\partial \theta^i} \partial_{\theta^i}$ and $\partial \rho = \frac{\partial \rho}{\partial \theta^i} \partial_{\theta^i}$ (throughout, we assume summation over repeating indices).

By (4.4) the cosines of the angles between the radial directions $\partial_t(Q)$ and $\partial_t(\bar{Q})$ at Q and \bar{Q} and the corresponding normals are expressed by

$$\cos \varphi(Q) = \frac{1}{\sqrt{1 + |\nabla f|_{\partial\Omega}^2}}, \quad \cos \beta(\bar{Q}) = \frac{1}{\sqrt{1 + |\nabla \rho|_{S_P}^2}}. \tag{4.5}$$

Finally, since $\varphi(Q) \leq \beta(\bar{Q})$ by assumption, at the corresponding points we obtain

$$|\nabla f|_{\partial\Omega}^2 \leq |\nabla \rho|_{S_P}^2. \tag{4.6}$$

We claim that

$$f(\theta^1, \dots, \theta^n) \leq \rho(\theta^1, \dots, \theta^n) \tag{4.7}$$

for all $(\theta^1, \dots, \theta^n) \in S^n$, and that equality is only attained for the closed domain B_P , containing $(0, \dots, 0)$. Then the result of the lemma follows from the choice of the origin.

A) We shall prove the lemma for $n = 1$. It is sufficient to show that

$$f(\theta) \leq \rho(\theta) \tag{4.8}$$

for each $\theta \in S^1$.

In the polar coordinate system on $M^2(c)$,

$$g^{-1}(t, \theta) = g_{11}^{-1}(t, \theta) = \frac{1}{\text{sc}^2 k_1 t},$$

where

$$\text{sc } k_1 t = \begin{cases} \sin k_1 t & \text{if } c = k_1^2 > 0, \\ t & \text{if } c = 0, \\ \sinh k_1 t & \text{if } c = -k_1^2 < 0. \end{cases}$$

Thus $g^{-1}(t, \theta)$ is positive and independent of the magnitude of the angle. Hence for θ_1 and θ_2 such that $f(\theta_1) = \rho(\theta_2)$, from (4.6) we obtain

$$f'^2(\theta_1) \leq \rho'^2(\theta_2). \tag{4.9}$$

In the case of a circle S_P , $\rho(\theta)$ is known to be a strictly increasing function on $[0, \pi]$, unless S_P has radius h . If its radius is equal to h , then $\rho \equiv h$, and it follows from (4.9) that $f \equiv h$, so that (4.8) holds.

Since $h = f(0)$ is the minimum distance, the function $f(\theta)$ is also strictly increasing in some right neighbourhood of zero $[0, \tilde{\theta})$, $\tilde{\theta} < \pi$.

In fact, if $f \equiv h$ in a neighbourhood of zero, then (4.8) holds. But if f takes the value h in every right neighbourhood of zero, but is not constant, then we look at the arc of the curve between two close points P_1 and P_2 such that $f(P_1) = h$ and $f(P_2) \neq h$. By convexity, f is strictly increasing in some neighbourhood of P_1 on the arc joining P_1 and P_2 . Then we take $P = P_1$.

Since $f(0) = \rho(0) = h$, we can take $\tilde{\theta}$ such that for each $\theta_2 \in [0, \tilde{\theta})$ we can find $\theta_1 \in [0, \tilde{\theta})$ such that $f(\theta_1) = \rho(\theta_2)$. Hence

$$0 < f'(\theta_1) \leq \rho'(\theta_2) \tag{4.10}$$

in this neighbourhood, from (4.9).

We set $\tilde{h} := f(\tilde{\theta})$. In view of inequality (4.10) above, the inverse functions $\theta = f^{-1}(t)$ and $\theta = \rho^{-1}(t)$ are defined on $[h; \tilde{h})$. If $t_0 := f(\theta_1) = \rho(\theta_2)$, then by (4.10)

$$(f^{-1})'(t_0) = \frac{1}{f'(\theta_1)} \geq \frac{1}{\rho'(\theta_2)} = (\rho^{-1})'(t_0) > 0. \tag{4.11}$$

Hence f^{-1} is increasing more rapidly than ρ^{-1} , and since $f^{-1}(h) = \rho^{-1}(h) = 0$, it follows that $\theta_1 = f^{-1}(t_0) \geq \rho^{-1}(t_0) = \theta_2$. If inequality (4.11) is strict at some point then $\theta_1 > \theta_2$. However, then, as f is monotonic, we obtain

$$f(\theta_2) < f(\theta_1) = \rho(\theta_2),$$

and since $\theta_2 \in [0, \tilde{\theta})$ is arbitrary, this implies (4.8) on the interval in question. If in (4.11) equality holds at each point in $[h; \tilde{h})$, then the curve coincides with an arc of the circle S_P at all points.

Similar arguments for the left neighbourhood of zero ensure that S_P is locally a supporting circle at P and $\partial\Omega$ lies locally inside S_P or coincides with this circle on an arc containing P . We shall show that the same holds for all $\theta \in S^1$.

Assume the converse. As $\partial\Omega \subset B_P$ locally, the curve $\partial\Omega$ must go outside S_P . Let $\theta_0 \in [0, 2\pi]$ be the first point where $\partial\Omega$ intersects S_P and extends outside. Then $f(\theta_0) = \rho(\theta_0)$. By the hypotheses of the lemma, for the corresponding angles at the point $Q_0 = (f(\theta_0), \theta_0) = (\rho(\theta_0), \theta_0) \in \partial\Omega \cap S_P$ we have

$$\varphi(Q_0) \leq \beta(Q_0),$$

in contradiction to the assumption that the curve goes outside the circle.

Thus we have arrived at a contradiction, which proves (4.8) and therefore also the lemma for $n = 1$.

B) The case $n \neq 1$. We take a 2-dimensional totally geodesic submanifold $M^2(c)$ of $M^{n+1}(c)$ which contains the geodesic interval OP . It intersects S_P in a circle and $\partial\Omega$ in a plane curve. Let $\tilde{\varphi}(Q)$ and $\tilde{\beta}(\bar{Q})$ be the corresponding angles between the geodesics OQ and $O\bar{Q}$ and the normals at $Q \in \partial\Omega$ and $\bar{Q} \in S_P$ to the curves in intersection. As $\tilde{\varphi}(Q) \leq \varphi(Q)$, $\beta(\bar{Q}) = \tilde{\beta}(\bar{Q})$ and $\varphi(Q) \leq \beta(Q)$, the angles $\tilde{\varphi}$ and $\tilde{\beta}$ satisfy the assumptions of the lemma. Hence we can apply the arguments in part A), which show that the curve lies in the disc bounded by the circle. However, the same holds for any $M^2(c)$, so $\partial\Omega \subset B_P$.

Lemma 9 is proved.

4.5. The proof of Theorem 4. Let O be the centre of a ball B with radius r inscribed in $\partial\Omega$. We look at the domain $D = \exp_O^{-1}(\Omega)$ in the tangent space $T_O M^{n+1}$. Then $\partial D = \exp_O^{-1}(\partial\Omega)$.

Let $\bar{O} \in M^{n+1}(c)$ be a point in a manifold with constant sectional curvature c . Identifying the tangent spaces $T_O M^{n+1}$ and $T_{\bar{O}} M^{n+1}(c)$ by an isometry we define $\bar{\Omega}$ to be the domain $\exp_{\bar{O}} D$. Then $\partial\bar{\Omega} = \exp_{\bar{O}}(\partial D)$. We also set $\bar{B} := \exp_{\bar{O}} B$ to be a ball of radius r .

We introduce the polar coordinate systems with origins at O and \bar{O} on the manifolds M^{n+1} and $M^{n+1}(c)$, respectively. Then the metrics in these spaces can be expressed as

$$M^{n+1}: ds^2 = dt^2 + g_{ij} d\theta^i d\theta^j, \quad M^{n+1}(c): ds^2 = dt^2 + G_{ij} d\theta^i d\theta^j,$$

where, as in Lemma 9, t is the length parameter and $\theta^1, \dots, \theta^n$ are the coordinate variables on the standard Euclidean sphere S^n .

If the sectional curvatures K of M^{n+1} are negative, $0 \geq K \geq -k_1^2$, then such a system of coordinates is regular everywhere apart from O (see [8]), and if the sectional curvatures of M^{n+1} are positive, $k_2^2 \geq K \geq k_1^2 > 0$, then the coordinate system in question is certainly regular in the ball of radius π/k_2 punctured at the centre. Thus by the hypotheses of the theorem the domain $\Omega \subset M^{n+1}$ lies in the regularity region of the polar coordinate system which we have introduced on M^{n+1} .

Since $K_\sigma \geq c$, using the standard comparison techniques (see [7]), for the inverse matrices (g^{ij}) and (G^{ij}) and for any covector $a(a_1, \dots, a_n)$, we obtain

$$g^{ij}a_ia_j \geq G^{ij}a_ia_j. \tag{4.12}$$

Assume that $\partial\Omega$ is explicitly defined by $t = f(\theta^1, \dots, \theta^n)$. By construction, $\partial\bar{\Omega}$ is given by the same equation. If N and \bar{N} are the unit outward normals at two points $Q \in \partial\Omega$ and $\bar{Q} \in \partial\bar{\Omega}$ corresponding to each other by means of the isometry of tangent spaces, then by analogy with Lemma 9 they can be written as

$$N(Q) = \frac{\partial_t - g^{ij} \frac{\partial f}{\partial \theta^i} \partial_{\theta^j}}{\sqrt{1 + |\nabla f|_{\partial\Omega}^2}} \quad \text{and} \quad \bar{N}(\bar{Q}) = \frac{\partial_t - G^{ij} \frac{\partial f}{\partial \theta^i} \partial_{\theta^j}}{\sqrt{1 + |\nabla f|_{\partial\bar{\Omega}}^2}}, \tag{4.13}$$

where $\partial f = \frac{\partial f}{\partial \theta^i} \partial_{\theta^i}$ is a tangent vector to the sphere S^n .

Then by (4.13) the cosines of the angles $\varphi(Q)$ and $\bar{\varphi}(\bar{Q})$ between the radial direction ∂_t and the corresponding outward normals N and \bar{N} have the expressions

$$\cos \varphi(Q) = \frac{1}{\sqrt{1 + |\nabla f|_{\partial\Omega}^2}}, \quad \cos \bar{\varphi}(\bar{Q}) = \frac{1}{\sqrt{1 + |\nabla f|_{\partial\bar{\Omega}}^2}}.$$

In view of (4.12), it follows from these relations that at the corresponding points we have

$$\cos \varphi(Q) \leq \cos \bar{\varphi}(\bar{Q}). \tag{4.14}$$

Let $P \in \partial\Omega \cap B$ be a point of tangency of an inscribed ball B with radius r and the hypersurface $\partial\Omega$, $\text{dist}(O, \partial\Omega) = \text{dist}(O, P) = r$, and let $\bar{P} \in \partial\bar{\Omega}$ be the point corresponding to it by means of the isometry, so that $\text{dist}(\bar{O}, \partial\bar{\Omega}) = \text{dist}(\bar{O}, \bar{P}) = r$, $\bar{P} \in \partial\bar{\Omega} \cap \bar{B}$. We take the sphere $S_{\bar{P}}$ in the manifold $M^{n+1}(c)$ which has curvature k_0 , passes through \bar{P} , and is orthogonal to the geodesic curve $\bar{O}\bar{P}$, so that \bar{O} lies in the corresponding ball $B_{\bar{P}}$.

As above, for any $Q_0 \in S_{\bar{P}}$ we let $\beta(Q_0)$ denote the angle between the radial direction and the outward normal at this point. For points $Q \in \partial\Omega$ and $Q_0 \in S_{\bar{P}}$ such that $\text{dist}(O, Q) = \text{dist}(\bar{O}, Q_0)$ it follows from Theorem 1 that

$$\cos \beta(Q_0) \leq \cos \varphi(Q). \tag{4.15}$$

From (4.14) and (4.15), for any points $\bar{Q} \in \partial\bar{\Omega}$ and $Q_0 \in S_{\bar{P}}$ such that

$$\text{dist}(\bar{O}, \bar{Q}) = \text{dist}(\bar{O}, Q_0),$$

we obtain

$$\cos \beta(Q_0) \leq \cos \bar{\varphi}(\bar{Q}).$$

However, $S_{\bar{P}}$ is then a supporting sphere for $\partial\bar{\Omega}$ by Lemma 9 and the whole of $\bar{\Omega}$ lies in the ball $B_{\bar{P}}$:

$$\bar{\Omega} \subset B_{\bar{P}}. \tag{4.16}$$

Obviously, such an inclusion holds for any $\bar{P} \in \partial\bar{\Omega} \cap \bar{B}$.

Next we look at the domain

$$\mathcal{C} = \bigcap_{\bar{P} \in \partial\bar{\Omega} \cap \bar{B}} B_{\bar{P}}.$$

By construction, $\partial\mathcal{C}$ is a complete k_0 -convex hypersurface. Furthermore, by (4.16),

$$\bar{\Omega} \subset \mathcal{C}. \tag{4.17}$$

Now we use arguments similar to the ones in the proof of Theorem 3 to show that the ball \bar{B} is also inscribed in the hypersurface $\partial\mathcal{C}$.

Indeed, since B is inscribed in $\partial\Omega$, it is known that the set $\partial\Omega \cap B$ cannot lie on an open hemisphere of ∂B . By construction, the same can be said about $\partial\bar{\Omega} \cap \bar{B}$. Now assume that the claim fails and that \bar{B} is not inscribed in $\partial\mathcal{C}$. Then there exists a ball $B_1 \subset \mathcal{C}$ with the same radius, but distinct from \bar{B} . Let O_1 be the centre of B_1 .

Let π_0 and π_1 denote totally geodesic n -dimensional submanifolds of $M^{n+1}(c)$ passing through \bar{O} and O_1 , respectively, and orthogonal to the geodesic curve $\bar{O}O_1$. For an arbitrary point $\bar{P} \in \pi_0 \cap \bar{B}$, let $P_1 \in \pi_1 \cap B_1$ be a point such that P_1, O_1, \bar{P} and \bar{O} lie in some 2-plane $M^2(c)$, and the geodesic intervals P_1O_1 and $\bar{P}\bar{O}$ lie to the same side of O_1O . Since $\partial\mathcal{C}$ is a k_0 -convex hypersurface, it follows from Lemma 7 that any shorter arc of a curvature- k_0 circle for the points \bar{P} and P_1 lies in the corresponding domain \mathcal{C} . We take an arc s of this type which makes angles larger than $\pi/2$ with the geodesic curves $\bar{O}\bar{P}$ and O_1P_1 . Since we can do this for any $\bar{P} \in \pi_0 \cap \bar{B}$, the part of $\partial\bar{B}$ lying in the same half-space as O_1 with respect to π_0 contains no points in $\partial\mathcal{C}$ and hence no points in $\partial\bar{\Omega} \cap \bar{B}$. We see that some points in $\bar{P} \in \partial\bar{\Omega} \cap \bar{B}$ must lie on the equatorial circle $\pi_0 \cap \partial\bar{B}$. But the supporting curvature- k_0 spheres at these points are orthogonal to the geodesic curve $\bar{O}\bar{P}$, by construction. Hence the arc s does not lie in this sphere, which contradicts the construction of \mathcal{C} and the fact that s lies in \mathcal{C} . This contradiction proves that the ball \bar{B} with radius r is inscribed in $\partial\mathcal{C}$.

Now since $\partial\mathcal{C}$ is a complete k_0 -convex surface, the width $\max \text{dist}(\bar{O}, \partial\mathcal{C}) - r$ of a spherical shell, which clearly contains the hypersurface $\partial\mathcal{C}$, satisfies the bounds in Theorem 3.

By (4.17), $\max \text{dist}(\bar{O}, \partial\bar{\Omega}) - r \leq \max \text{dist}(\bar{O}, \partial\mathcal{C}) - r$. On the other hand, $\max \text{dist}(\bar{O}, \partial\bar{\Omega}) - r = \max \text{dist}(O, \partial\Omega) - r$, by construction. Hence

$$\max \text{dist}(O, \partial\Omega) - r \leq \max \text{dist}(\bar{O}, \partial\mathcal{C}) - r,$$

and from Theorem 3 we obtain the required bounds for the width of a spherical shell in the case of a hypersurface in a Riemannian manifold with nonconstant curvature.

Theorem 4 is proved.

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Alexander A. Borisenko
Sumy State University, Ukraine
E-mail: aborisenk@gmail.com

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Translated by N. KRUSHILIN

Kostiantyn D. Drach
Kharkiv National University, Ukraine
E-mail: kostya.drach@gmail.com