T. S. Mardvilko, A. A. Pekarskii, Direct and inverse theorems of rational approximation in the Bergman space, *Sbornik: Mathematics*, 2011, Volume 202, Issue 9, 1327–1346

DOI: 10.1070/SM2011v202n09ABEH004189

Sbornik: *Mathematics* **202**:9 1327–1346

Matematicheskiĭ Sbornik **202**:9 77–96 DOI 10.1070/SM2011v202n09ABEH004189

Direct and inverse theorems of rational approximation in the Bergman space

T. S. Mardvilko and A. A. Pekarskiĭ

Abstract. For positive numbers p and μ let $A_{p,\mu}$ denote the Bergman space of analytic functions in the half-plane $\Pi := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. For $f \in A_{p,\mu}$ let $R_n(f)_{p,\mu}$ be the best approximation by rational functions of degree at most n. Also let $\alpha \in \mathbb{R}$ and $\tau > 0$ be numbers such that $\alpha + \mu = \frac{1}{\tau} - \frac{1}{p} > 0$ and $\frac{1}{p} + \mu \notin \mathbb{N}$. Then the main result of the paper claims that the set of functions $f \in A_{p,\mu}$ such that

$$\sum_{n=1}^{\infty} \frac{1}{n} (n^{\alpha+\mu} R_n(f)_{p,\mu})^{\tau} < \infty$$

is precisely the Besov space B^{α}_{τ} of analytic functions in Π.

Bibliography: 23 titles.

Keywords: direct and inverse theorems of rational approximation, Bernstein-type inequalities, Jackson-type inequalities, Bergman spaces, Besov spaces.

§ 1. Introduction. The main results

Let $L_p(\mathbb{R})$ denote the space of Lebesgue-measurable complex-valued functions f on $\mathbb{R}$ with finite quasinorm

$$\|f\|_{L_p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}, \qquad 0 < p < \infty.$$

We shall also consider $L_{p,\mu}(\Pi)$, the space of complex functions f in the half-plane

$$\Pi := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

which are measurable with respect to the planar Lebesgue measure m_2 and have a finite quasinorm

$$\|f\|_{L_{p,\mu}(\Pi)} = \left(\int_{\Pi} (\operatorname{Im} z)^{p\mu-1} |f(z)|^p \, dm_2(z) \right)^{1/p}, \qquad p > 0, \quad \mu > 0.$$

The subspace of $L_{p,\mu}(\Pi)$ consisting of the functions analytic in Π, the Bergman space, will be denoted by $A_{p,\mu} := A_{p,\mu}(\Pi)$. For $f \in A_{p,\mu}$ we shall set $\|f\|_{A_{p,\mu}} = \|f\|_{L_{p,\mu}(\Pi)}$.

AMS 2010 *Mathematics Subject Classification.* Primary 30E10, 30H20, 30H25.

For $0 < \tau < \infty$ and $\alpha \in \mathbb{R}$ we introduce the Besov space B^α_τ of analytic functions f in Π such that

$$\|f\|_{B^\alpha_\tau} = \|f^{(s)}\|_{A_{\tau,s-\alpha}} < \infty, \qquad s = [\alpha] + 1, \tag{1.1}$$

where $[\alpha]$ is the integer part of α. As usual, here $f^{(0)} = f$, $f^{(s)}$ for $s \geqslant 1$ is the sth derivative of f while for $s < 0$ $f^{(s)}$ is the $(-s)$th antiderivative of f uniquely defined by the condition $f^{(s)}(z) \to 0$ as $\operatorname{Im} z \to +\infty$. The functional $\|\cdot\|_{B^\alpha_\tau}$ on B^α_τ defines a quasinorm for $[\alpha] + 1 \leqslant 0$, that is, for $\alpha < 0$. On the other hand, if $[\alpha] + 1 \geqslant 1$, that is, $\alpha \geqslant 0$, then the functional $\|\cdot\|_{B^\alpha_\tau}$ is a semiquasinorm on B^α_τ because $\|f\|_{B^\alpha_\tau} = 0$ if and only if f is an algebraic polynomial of degree at most $[\alpha]$. In § 2 we show that for $\alpha \in [0, \frac{1}{\tau})$ the functional $\|\cdot\|_{B^\alpha_\tau}$ also is a quasinorm if we confine ourselves to the functions f such that $f(z) \to 0$ as $\operatorname{Im} z \to +\infty$. This is what we shall suppose throughout; that is, in the definition of the space B^α_τ for $\alpha \in [0, \frac{1}{\tau})$ we assume that $f(z) \to 0$ as $\operatorname{Im} z \to +\infty$. It is known that if $f \in B^\alpha_\tau$, then in (1.1) we can replace $s = [\alpha] + 1$ by any $s > \alpha$; the resulting quasinorms will be equivalent.

For the spaces B^α_τ we have the following continuous noncompact embedding (see, for example, [1] or [2]):

$$B^{\alpha_1}_{\tau_1} \hookrightarrow B^{\alpha_0}_{\tau_0} \quad \text{for } \alpha_1 - \alpha_0 = \frac{1}{\tau_1} - \frac{1}{\tau_0} > 0. \tag{1.2}$$

In particular,

$$B^\alpha_\tau \hookrightarrow A_{p,\mu} \quad \text{for } \alpha + \mu = \frac{1}{\tau} - \frac{1}{p} > 0. \tag{1.3}$$

Here and in what follows $X \hookrightarrow Y$ means a continuous embedding of the space X in the space Y.

Let $\mathscr{P}_n$, $n = 0, 1, 2, \dots$, be the set of algebraic polynomials of degree at most n; and $\mathscr{R}_n = \{\frac{p}{q} : p, q \in \mathscr{P}_n,\ q \not\equiv 0\}$ be the set of rational functions of degree at most n. For $f \in A_{p,\mu}$ we shall consider the best rational approximation

$$R_n(f)_{p,\mu} := R_n(f)_{A_{p,\mu}} = \inf\bigl\{\|f - r\|_{A_{p,\mu}} : r \in \mathscr{R}_n \cap A_{p,\mu}\bigr\}.$$

It is known that the set of rational functions is dense in $A_{p,\mu}$. Hence it follows from (1.3) that if $f \in B^\alpha_\tau$, then $R_n(f)_{p,\mu} \to 0$ as $n \to \infty$. The central result of this paper, Theorem 1, includes the direct and inverse theorems of rational approximation in $A_{p,\mu}$.

Theorem 1. *Let p, μ, τ be positive numbers and α be a real number such that $\alpha + \mu = \frac{1}{\tau} - \frac{1}{p} > 0$ and $\frac{1}{p} + \mu \notin \mathbb{N}$. Then $f \in A_{p,\mu}$ satisfies*

$$\sum_{n=1}^{\infty} \frac{1}{n} (n^{\alpha+\mu} R_n(f)_{p,\mu})^\tau < \infty$$

if and only if $f \in B^\alpha_\tau$.

The sufficiency part of Theorem 1 (the direct theorem) holds also for $\frac{1}{p} + \mu \in \mathbb{N}$. The necessity can be proved by a method from the paper [3], in which direct theorems of rational approximation in Hardy spaces were proved. We can also use the atomic decomposition of functions in the Bergman space (see [4] and [1]). In [4] and [1] atomic decomposition is used, in particular, to prove the direct theorem of rational approximation in the Bloch space. In the sufficiency part of the proof of Theorem 1 we have also established a Jackson-type inequality, Theorem 2.

Theorem 2. *Let α be a real number and p, μ, τ be positive numbers such that $\alpha+\mu=\frac{1}{\tau}-\frac{1}{p}>0$. If $f\in B_\tau^\alpha$, then*

$$R_n(f)_{A_{p,\mu}}\leqslant \frac{c}{n^{\alpha+\mu}}\|f\|_{B_\tau^\alpha},\qquad n=1,2,\dots,$$

where $c>0$ is independent of n and f.

The necessity part of Theorem 1 (the inverse theorem) is proved by Bernstein's method of the proof of inverse theorems of approximation theory; in our case it is based on Theorem 4 stated below. Also for $1<\tau<\infty$ we use the real interpolation method. Theorem 4 is proved with the use of Theorem 3, which also is of independent interest.

Theorem 3. *Let p and μ be positive numbers such that $\frac{1}{\lambda}=\frac{1}{p}+\mu\notin\mathbb{N}$. Then for $r\in\mathscr{R}_n\cap L_{p,\mu}(\Pi)$ and $n\geqslant 1$,*

$$\|r\|_{L_\lambda(\mathbb{R})}\leqslant cn^{\mu}\|r\|_{L_{p,\mu}(\Pi)},\qquad c=c(p,\mu)>0.$$

Theorem 4. *Let p and μ be positive numbers such that $\frac{1}{p}+\mu\notin\mathbb{N}$. Then for $\alpha>-\mu$, $\frac{1}{\tau}=\alpha+\mu+\frac{1}{p}$ and $r\in\mathscr{R}_n\cap A_{p,\mu}$, $n\geqslant 1$,*

$$\|r\|_{B_\tau^\alpha}\leqslant cn^{\alpha+\mu}\|r\|_{A_{p,\mu}},\qquad c=c(p,\mu,\alpha)>0.$$

The constraint $\frac{1}{p}+\mu\notin\mathbb{N}$ is essential in Theorems 3 and 4, as well as in the necessity part of Theorem 1.

Theorems 1–4 also have analogues in the disc. In the case $\mu+\frac{1}{p}<1$ analogues of Theorems 1–4 were earlier obtained by Dyn'kin (see [5]). Also for the disc, in the special case of $\mu=\frac{1}{p}$ and $p>2$ the necessity in Theorem 1 together with Theorems 3 and 4 were established by Misiuk (see [6]) simultaneously with Dyn'kin, but independently of him. Theorem 3 for $\mu=\frac{1}{p}$ and $0<p<1$ has recently been proved by the first author (see [7]), who has made in [7] an essential use of some results and tricks from [8]. This is where the methods used in [7] and this paper are considerably distinct from the methods of [5] and [6]. Our proofs of Theorems 3 and 4 are based on a further development of the methods of [8] and [7].

The central results of this paper have been announced in [9].

§ 2. The proof of the direct theorem and of the Jackson-type inequality

We agree that $c,c_1,c_2,\dots$ will denote positive quantities depending only on the parameters put in the parentheses. Sometimes the indication of the parameters will be suppressed.

In § 5 we present several results on the Bergman space $A_{p,\mu}(G)$ in a domain $G \neq \mathbb{C}$. In particular, by Lemma 10, for $f \in A_{p,\mu}(\Pi)$ we have

$$|f(z)| \leqslant \frac{c(p,\mu)}{(\operatorname{Im} z)^{\mu+1/p}} \|f\|_{A_{p,\mu}}, \qquad z \in \Pi. \tag{2.1}$$

It follows from (2.1) that for $f \in B_\tau^\alpha(\Pi)$ with $\alpha < \frac{1}{\tau}$ the condition $f(z) \to 0$ as $\operatorname{Im} z \to +\infty$ is well posed.

For the proof of the direct theorem and the Jackson-type inequality we shall use the atomic decomposition for functions in $A_{p,\mu}$ obtained by Koifman and Rochberg (see [4]). As concerns atomic decompositions and their applications, also see [1] and [2]. We state the results of [4] and [1] that we require in the convenient form of Theorems 5 and 6. To do this we introduce the following notation.

For each $\theta \in (0,1)$ consider the set $\mathscr{F}_\theta$ of closed squares

$$Q = \bigl\{z : \theta^{p+1} \leqslant \operatorname{Im} z \leqslant \theta^p;\, q\theta^p(1-\theta) \leqslant \operatorname{Re} z \leqslant (q+1)\theta^p(1-\theta)\bigr\}, \quad \text{where } p, q \in \mathbb{Z}.$$

Obviously, these squares have disjoint interiors and their union is Π. We shall number the squares in $\mathscr{F}_\theta$ in some way by positive integers: $\mathscr{F}_\theta = \{Q_k\}_{k=1}^\infty$. Let z_k be the centre of the square Q_k.

Theorem 5. *Let $\theta \in (0,1)$, $p > 0$, $\mu > 0$ and assume that*

$$\varkappa > \max\biggl\{1, \frac{1}{p}\biggr\} + \mu.$$

Then for any sequence $\{u_k\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty |u_k|^p < \infty$$

the series

$$\sum_{k=1}^\infty u_k \frac{(\operatorname{Im} z_k)^{\varkappa-(\mu+1/p)}}{(z-\overline{z}_k)^\varkappa}, \qquad z \in \Pi, \tag{2.2}$$

converges in $A_{p,\mu}$; furthermore, it converges uniformly and absolutely on compact subsets of Π. The sum $f(z)$ of the series (2.2) satisfies

$$\|f\|_{A_{p,\mu}}^p \leqslant c_1 \sum_{k=1}^\infty |u_k|^p, \qquad c_1 = c_1(p,\mu,\varkappa,\theta) > 0.$$

Theorem 6. *Let p, μ and $\varkappa$ be as in Theorem 5. Then there exists*

$$\theta_0 = \theta_0(p,\mu,\varkappa) \in (0,1)$$

such that for $\theta \in (\theta_0, 1)$ any function $f \in A_{p,\mu}$ can be represented by a series (2.2) with coefficients $\{u_k\}_{k=1}^\infty$ satisfying

$$\sum_{k=1}^\infty |u_k|^p \leqslant c_2 \|f\|_{A_{p,\mu}}^p, \qquad c_2 = c_2(p,\mu,\varkappa,\theta) > 0.$$

In particular, Theorems 5 and 6 imply the embeddings (1.2) and (1.3). With the help of these theorems we can readily show that for $f \in A_{p,\mu}$ the quasinorms $\|f\|_{A_{p,\mu}}$ and $\|f^{(s)}\|_{A_{p,\mu+s}}$, $s \in (-\mu, +\infty) \cap \mathbb{Z}$, are equivalent. Hence replacing $s = [\alpha] + 1$ in (1.1) by an arbitrary integer $s > \alpha$ we obtain an equivalent quasinorm in the space B^α_τ.

We shall also require the following Lemmas 1 and 2. Lemma 1 is a fragment of the proof of Theorem 1.3.1 in [10]. Lemma 2 is a consequence of the inequality

$$(x + y)^\theta \leqslant x^\theta + y^\theta, \quad \text{where } x \geqslant 0,\ y \geqslant 0,\ 0 < \theta \leqslant 1.$$

Lemma 1. *Let $0 < q < 1$ and let $\{a_n\}_{n=1}^\infty$ be a nonincreasing sequence of nonnegative numbers such that $\sum a_n^q < \infty$. Then*

$$\sum_{n=1}^\infty \biggl(\frac{1}{n}\sum_{k=n}^\infty a_k\biggr)^q \leqslant c(q)\sum_{n=1}^\infty a_n^q.$$

Lemma 2. *Assume that $0 < \alpha < \beta < \infty$ and let $\{a_k\}_{k=1}^\infty$ be a sequence of nonnegative numbers such that $\sum a_k^\alpha < \infty$. Then*

$$\biggl(\sum_{k=1}^\infty a_k^\beta\biggr)^{1/\beta} \leqslant \biggl(\sum_{k=1}^\infty a_k^\alpha\biggr)^{1/\alpha}.$$

The sufficiency part of Theorem 1 (the direct theorem), as well as Theorem 2 are obvious consequences of Theorem 7.

Theorem 7. *Let p, μ, τ be positive numbers and α a real number such that*

$$\alpha + \mu = \frac{1}{\tau} - \frac{1}{p} > 0. \tag{2.3}$$

If $f \in B^\alpha_\tau$, then

$$\sum_{n=1}^\infty \frac{1}{n}(n^{\alpha+\mu} R_n(f)_{p,\mu})^\tau \leqslant c\|f\|^\tau_{B^\alpha_\tau}, \qquad c = c(p, \mu, \tau, \alpha) > 0. \tag{2.4}$$

Proof. We set $s = \max\{0, [\alpha] + 1\}$, where $[\alpha]$ is the integer part of α. Next we take $\varkappa \in \mathbb{N}$ such that $\varkappa > 1 + \mu + s + \frac{1}{p}$. Obviously, we also have $\varkappa > \max\{1, \frac{1}{\tau}\} + s - \alpha$. Since $f \in B^\alpha_\tau$, it follows that $f^{(s)} \in A_{\tau,s-\alpha}$. By Theorem 6, for some $\theta_1 \in (0, 1)$ the function $f^{(s)}$ can be represented by a series

$$f^{(s)}(z) = \sum_{k=1}^\infty u_k \frac{(\operatorname{Im} z_k)^{\varkappa-(s-\alpha)-1/\tau}}{(z - \overline{z}_k)^\varkappa}, \qquad z \in \Pi. \tag{2.5}$$

(Here the $\{z_k\}_{k=1}^\infty$ are the centres of the corresponding squares $\{Q_k\}_{k=1}^\infty$ in $\mathscr{F}_{\theta_1}$.) Furthermore, the coefficients u_k of (2.5) satisfy

$$\sum_{k=1}^\infty |u_k|^\tau \leqslant c_1\|f^{(s)}\|^\tau_{A_{\tau,s-\alpha}} \leqslant c_2\|f\|^\tau_{B^\alpha_\tau}. \tag{2.6}$$

We shall assume that the squares Q_k are ordered so that $\{|u_k|\}_{k=1}^{\infty}$ is a nonincreasing sequence.

From (2.1) we see that

$$|f^{(s)}(z)| \leqslant \frac{c_3}{(\operatorname{Im} z)^{s-\alpha+1/\tau}} \|f\|_{B_\tau^\alpha}, \qquad z \in \Pi.$$

It follows from this and Theorem 5 that we can integrate (2.5) termwise s times along the vertical rays $[z, z+i\infty)$, $z \in \Pi$. In view of (2.3),

$$f(z) = b \sum_{k=1}^{\infty} u_k \frac{(\operatorname{Im} z_k)^{(\varkappa-s)-(\mu+1/p)}}{(z-\overline{z}_k)^{\varkappa-s}}, \qquad z \in \Pi, \tag{2.7}$$

where $b = (-1)^s \frac{(\varkappa-s-1)!}{(\varkappa-1)!}$.

Since $p > \tau$, from (2.6) and Lemma 2 we obtain that $\sum |u_k|^p$ is convergent. Hence by Theorem 5 the series (2.7) converges in $A_{p,\mu}$. For $n \in \mathbb{N}$ let $r_n(z) = 0$ for $n < \varkappa - s$ and

$$r_n(z) = b \sum_{1 \leqslant k \leqslant n/(\varkappa-s)} u_k \frac{(\operatorname{Im} z_k)^{(\varkappa-s)-(\mu+1/p)}}{(z-\overline{z}_k)^{\varkappa-s}} \quad \text{for } n \geqslant \varkappa - s.$$

Obviously, $r_n \in \mathscr{R}_n \cap A_{p,\mu}$ and therefore $R_n(f)_{p,\mu} \leqslant \|f - r_n\|_{A_{p,\mu}}$. For an estimate of the last quasinorm we use Theorem 5. This yields

$$R_n(f)_{p,\mu} \leqslant c_5 \biggl(\sum_{k>n/(\varkappa-s)} |u_k|^p\biggr)^{1/p}. \tag{2.8}$$

Now let $a_k = |u_{[k/(\varkappa-s)]+1}|$, $k = 1, 2, \dots$. The sequence $\{a_k\}_{k=1}^{\infty}$ is nonnegative, nonincreasing and

$$\sum_{k=1}^{\infty} a_k^\tau \leqslant c_6 \|f\|_{B_\tau^\alpha}^\tau \tag{2.9}$$

by (2.6). It also follows from (2.8) that

$$R_n(f)_{p,\mu} \leqslant c_7 \biggl(\sum_{k=n}^{\infty} |a_k|^p\biggr)^{1/p}. \tag{2.10}$$

By (2.3) we have $\frac{\tau}{p} < 1$, so we deduce (2.4) from Lemma 1 and relations (2.9) and (2.10):

$$\sum_{n=1}^{\infty} \frac{1}{n} (n^{\alpha+\mu} R_n(f)_{p,\mu})^\tau \leqslant c_8 \sum_{n=1}^{\infty} \biggl(\frac{1}{n} \sum_{k=n}^{\infty} |a_k|^p\biggr)^{\tau/p} \leqslant c_9 \sum_{n=1}^{\infty} a_n^\tau \leqslant c_{10} \|f\|_{B_\tau^\alpha}^\tau.$$

The proof is complete.

§ 3. Comparing the quasinorms of a rational function with respect to the linear and planar measures

Here we prove Theorem 3. We require the Hardy space $H_p = H_p(\Pi)$ in the half-plane Π. By definition (see [11]) an analytic function f in Π belongs to H_p, $0<p<\infty$, if

$$\|f\|_{H_p} := \sup_{y>0} \|f(\,\cdot\,+iy)\|_{L_p(\mathbb{R})} < \infty.$$

As is known, if $f\in H_p$, then for almost all $x\in\mathbb{R}$ a limit $\lim f(z) =: f(x)$ exists for $z\in\Pi$ approaching x along paths nontangential to $\mathbb{R}$, and we have $\|f\|_{H_p} = \|f\|_{L_p(\mathbb{R})}$. For $f\in H_p$ we also point out the inequality

$$|f(z)| \leqslant \frac{c(p)}{(\operatorname{Im} z)^{1/p}} \|f\|_{H_p}, \qquad z\in\Pi. \tag{3.1}$$

Results similar to the conditional Lemma 3 below can also be found in [8] and [7].

Lemma 3. *Let p, μ and λ be positive numbers such that $\frac{1}{\lambda} = \mu + \frac{1}{p}$. Assume that for some $p = p_1 < 1$*

$$\|r\|_{H_\lambda} \leqslant c_1(p,\mu) n^\mu \|r\|_{A_{p,\mu}} \tag{3.2}$$

for each function $r\in\mathscr{R}_n\cap A_{p,\mu}$, $n\geqslant 1$, which has real coefficients and no poles outside $\mathbb{R}$. Then for all $p = kp_1$, $k\in\mathbb{N}$, and all $r\in\mathscr{R}_n\cap L_{p,\mu}(\Pi)$, $n\geqslant 1$,

$$\|r\|_{L_\lambda(\mathbb{R})} \leqslant c_2(p,\mu) n^\mu \|r\|_{L_{p,\mu}(\Pi)} \tag{3.3}$$

with some constant $c_2(p,\mu)$ in place of $c_1(p,\mu)$.

Proof. We start with the case $k=1$. Let $r\in\mathscr{R}_n\cap L_{p,\mu}(\Pi)$ be a rational function with some poles outside $\mathbb{R}$. Let b_+ and b_- be the Blaschke products with poles coinciding (with multiplicities) with the poles of r in the half-planes $\operatorname{Im} z>0$ and $\operatorname{Im} z<0$, respectively. For example, if r has no poles in $\operatorname{Im} z<0$ then we set $b_-(z)\equiv 1$. Obviously, r cannot have a pole at ∞. For each fixed $\zeta\in T$, $T:=\{\zeta : |\zeta|=1\}$, all the poles of the rational function

$$r(z,\zeta) := r(z)(b_+(z) - \zeta b_-(z))^{-1}$$

lie in $\mathbb{R}$; its degree coincides with that of r. It is easy to verify (or see [12], Theorem 1.7 and [7], Lemma 1) that

$$J(p,z) := \int_{\mathrm{T}} |b_+(z) - \zeta b_-(z)|^{-p}\,|d\zeta|$$

is a bounded function in $\mathbb{C}$, so that $J(p,z)\leqslant \omega(p)$ for $z\in\mathbb{C}$ with positive $\omega(p)$ depending only on $p=p_1\in(0,1)$. Using Fubini's theorem we obtain

$$\begin{aligned}\int_T \biggl(\int_\Pi (\operatorname{Im} z)^{p\mu-1} |r(z,\zeta)|^p\,dm_2(z)\biggr)\,|d\zeta| \\ = \int_\Pi (\operatorname{Im} z)^{p\mu-1}|r(z)|^p J(p,z)\,dm_2(z) \leqslant \omega(p)\|r\|^p_{L_{p,\mu}(\Pi)}.\end{aligned}$$

Hence there exists $\zeta_0 \in T$ such that the function $r_0(z) := r(z,\zeta_0)$ satisfies

$$\|r_0\|_{A_{p,\mu}} \leqslant c_3(p)\|r\|_{L_{p,\mu}(\Pi)}, \qquad c_3(p) = \left(\frac{\omega(p)}{2\pi}\right)^{1/p}.$$

We set $r_1(x) = \operatorname{Re} r_0(x)$ and $r_2(x) = \operatorname{Im} r_0(x)$, $x \in \mathbb{R}$. Then for the r_j, $j = 1, 2$, we have

$$\|r_j\|_{A_{p,\mu}} \leqslant c_3(p)\|r\|_{L_{p,\mu}(\Pi)}. \tag{3.4}$$

By assumption the r_j satisfy (3.2). In view of (3.2), (3.4) and the obvious inequality $|b_+(x) - \zeta_0 b_-(x)| \leqslant 2$, $x \in \mathbb{R}$, we obtain

$$\begin{aligned}\|r_j(b_+ - \zeta_0 b_-)\|_{L_\lambda(\mathbb{R})} &\leqslant 2\|r_j\|_{L_\lambda(\mathbb{R})} \leqslant 2c_1(p,\mu)n^\mu\|r_j\|_{A_{p,\mu}}\\ &\leqslant 2c_1(p,\mu)c_3(p)n^\mu\|r\|_{L_{p,\mu}(\Pi)}, \qquad j = 1, 2.\end{aligned} \tag{3.5}$$

Since $r = (r_1 + ir_2)(b_+ - \zeta_0 b_-)$ and $0 < \lambda < 1$, by (3.5) inequality (3.3) holds for each $r \in \mathscr{R}_n \cap L_{p,\mu}(\Pi)$ with the constant $c_2(p,\mu) = 2^{1/\lambda}c_1(p,\mu)c_3(p)$ in place of $c_1(p,\mu)$.

Now we show that (3.3) holds for the exponents $p = p_k := kp_1$, $k = 2, 3, \dots$. Let $r \in \mathscr{R}_n \cap L_{p,\mu}(\Pi)$. Then $r^k \in \mathscr{R}_{nk} \cap L_{p_1,k\mu}(\Pi)$ and therefore, by the case $k = 1$ already considered,

$$\|r^k\|_{L_{\sigma_k}(\mathbb{R})} \leqslant c_2(p_1,k\mu)(nk)^{k\mu}\|r^k\|_{L_{p_1,k\mu}(\Pi)}, \qquad \frac{1}{\sigma_k} := k\mu + \frac{1}{p_1}. \tag{3.6}$$

Next we set $\frac{1}{\lambda_k} = \mu + \frac{1}{kp_1}$ and observe that

$$\|r^k\|_{L_{\sigma_k}(\mathbb{R})} = \|r\|^k_{L_{\lambda_k}(\mathbb{R})}, \qquad \|r^k\|_{L_{p_1,k\mu}(\Pi)} = \|r\|^k_{L_{p_k,\mu}(\Pi)}.$$

Thus, (3.6) yields the required inequality

$$\|r\|_{L_{\lambda_k}(\mathbb{R})} \leqslant c_4(p_k,\mu)n^\mu\|r\|_{L_{p_k,\mu}(\Pi)}$$

with the constant $c_4(p_k,\mu) = k^\mu c_2(p_1,k\mu)$. The proof is complete.

Now we follow the pattern used in [8] and [7]. To do this we introduce further notation. Let n and l be positive integers such that $n \geqslant l+1$. We shall denote the set of functions $r \in \mathscr{R}_n$ with real coefficients and poles only on $\mathbb{R}$ such that each pole has multiplicity at most l and

$$r(z) = O(z^{-l-1}) \quad \text{as } z \to \infty$$

by $\mathscr{R}_n^l$.

Let Π_m^l, $m \geqslant 2$, $l \geqslant 1$, denote the set of piecewise-polynomial functions φ on $\mathbb{R}$, with compact support, of degree at most $l-1$ and with m free nodes. Namely, $\varphi \in \Pi_m^l$ if there exist m points (nodes of φ) $-\infty < x_1 < x_2 < \dots < x_m < +\infty$ such that $\varphi|_{(x_k,x_{k+1})} \in \mathscr{P}_{l-1}$, $k = 1, 2, \dots, m-1$, and $\varphi(x) = 0$ for $x \in \mathbb{R} \setminus [x_1, x_m]$. The quantities $\varphi(x_k)$, $k = 1, 2, \dots, m$, can be arbitrary. For convenience we assume that $\Pi_1^l = \{0\}$.

Lemma 4 (see [8]). *If $r \in \mathscr{R}_n^l$, then there exists a real-valued function $\varphi \in \Pi_m^l$, $m \leqslant n$, such that*

$$r(z) = \frac{l!}{\pi} \int_{\mathbb{R}} \frac{\varphi(t)\,dt}{(t-z)^{l+1}}, \qquad z \in \mathbb{C} \setminus \operatorname{supp} \varphi.$$

This φ has nodes at poles of r.

Lemma 5 (see [8]). *For $1 < q < \infty$, and $k \in \mathbb{N}$, let $\lambda = (k + \frac{1}{q})^{-1}$ and let $\varphi \in \Pi_m^l$ and*

$$f(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(t)\,dt}{t-z}, \qquad z \in \Pi.$$

Then $f^{(k)} \in H_\lambda$ and

$$\|f^{(k)}\|_{H_\lambda} \leqslant c(k,q,l) \cdot m^k \cdot \|\varphi\|_{L_q(\mathbb{R})}.$$

We shall also use the Bergman and Smirnov spaces $A_{p,\mu}(G)$ and $E_p(G)$ in a bounded domain G with boundary ∂G which is a rectifiable Jordan curve. We give the definitions of these spaces and some relevant inequalities in § 5. Using the method of the proof of Theorem 10 we can also establish the following well-known embedding. Let p, q and μ be positive numbers such that

$$\mu + \frac{1}{p} - \frac{1}{q} =: l \in \mathbb{N}, \qquad p < q. \tag{3.7}$$

If f is analytic in Π, $f(z) \to 0$ as $\operatorname{Im} z \to +\infty$ and $f^{(l)} \in A_{p,\mu}(\Pi)$, then $f \in H_q(\Pi)$ and we have

$$\|f\|_{H_q} \leqslant c(p,q,\mu) \|f^{(l)}\|_{A_{p,\mu}}. \tag{3.8}$$

Let $l \in \mathbb{N}$, $\varepsilon > 0$, let f be an analytic function in Π and $f(z) = O((\operatorname{Im} z)^{-l-\varepsilon})$ as $\operatorname{Im} z \to \infty$. Then we denote the lth antiderivative of f determined by the condition $f^{(-l)}(z) \to 0$ as $\operatorname{Im} z \to \infty$ by $f^{(-l)}$. At the beginning of § 2 we showed that $f^{(-l)}$ is well defined for $f \in A_{p,\mu}$ in the case when $\mu + \frac{1}{p} > l$. It follows from (3.1) that $f^{(-l)}$ is well defined for $f \in H_p$ in the case when $\frac{1}{p} > l$.

Lemma 6. *Let p, q, μ and l be as in (3.7) and let $f \in A_{p,\mu}(\Pi)$. Then $f^{(-l)} \in H_q$ and for each $k \in \mathbb{N}$ there exists $\psi \in \Pi_k^l$ such that*

$$\| \operatorname{Im} f^{(-l)} - \psi\|_{L_q(\mathbb{R})} \leqslant \frac{c}{k^{l-\mu}} \|f\|_{A_{p,\mu}},$$

where the positive c is independent of f and k.

Proof. By (3.8), for $k=1$ we can set $\psi \equiv 0$. Now we assume that $k \geqslant 2$ and $f \not\equiv 0$. It follows from (3.8) that there exist $x_1, x_k \in \mathbb{R}$, $x_1 < x_k$, such that

$$\|f^{(-l)}\|_{L_q(\mathbb{R}\setminus(x_1,x_k))} \leqslant \frac{1}{k^{1/p}} \|f\|_{A_{p,\mu}}. \tag{3.9}$$

Next we can find $x_2, x_3, \dots, x_{k-1}$, $x_1 < x_2 < x_3 < \dots < x_{k-1} < x_k$, such that

$$\int_{D_j} (\operatorname{Im} z)^{p\mu-1} |f(z)|^p \, dm_2(z) \leqslant \frac{2}{k} \int_D (\operatorname{Im} z)^{p\mu-1} |f(z)|^p \, dm_2(z), \tag{3.10}$$

where $D \subset \Pi$ is the square with side (x_1, x_k) and $D_j \subset D$ is the square with side $J_j := (x_j, x_{j+1})$, $j = 1, 2, \dots, k-1$. Let $\Delta_j \subset D_j$ be the equilateral triangle with base J_j. From Lemma 12 and (3.10) we obtain the following relation:

$$\|f\|^p_{A_{p,\mu}(\Delta_j)} \leqslant \frac{c_1}{k} \int_D (\operatorname{Im} z)^{p\mu-1} |f(z)|^p \, dm_2(z). \tag{3.11}$$

Let $p_j(z)$ be the $(l-1)$th order Taylor polynomial of $f^{(-l)}$ at Δ_j. Using Theorem 10 with Remark 2 and also relation (3.11) we obtain

$$\|f^{(-l)} - p_j\|_{L_q(J_j)} \leqslant \frac{c_2}{k^{1/p}} \|f\|_{A_{p,\mu}(\Pi)}. \tag{3.12}$$

Now we define a function $\psi \in \Pi^l_k$ by setting $\psi(x) = \operatorname{Im} p_j(x)$ for $x \in J_j$, $j = 1, 2, \dots, k-1$, and $\psi(x) = 0$ for $x \in \mathbb{R} \setminus [x_1, x_k]$. This is the required function; indeed, from (3.9) and (3.12) we obtain

$$\|\operatorname{Im} f^{(-l)} - \psi\|^q_{L_q(\mathbb{R})} \leqslant \frac{c_3}{k^{q/p-1}} \|f\|^q_{A_{p,\mu}(\Pi)} = \frac{c_3}{k^{q(l-\mu)}} \|f\|^q_{A_{p,\mu}(\Pi)}.$$

The proof is complete.

Proof of Theorem 3. By Lemma 3 it is sufficient to establish Theorem 3 for $p \in (0,1)$ such that $\mu + \frac{1}{p} \notin \mathbb{N}$ and a rational function $r \in \mathscr{R}_n \cap A_{p,\mu}(\Pi)$ with real coefficients and no poles outside $\mathbb{R}$. We can readily see that such r belong to $\mathscr{R}^l_n$ with $l = [\mu + \frac{1}{p}]$. In view of our assumptions, we can find $q \in (1, +\infty)$ such that $\mu + \frac{1}{p} = l + \frac{1}{q}$. Using Lemma 4 we see that there exists a real function $\varphi \in \Pi^l_m$, $m \leqslant n$, such that

$$r^{(-l)}(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{\varphi(t)\, dt}{t - z}, \qquad z \in \Pi. \tag{3.13}$$

By M. Riesz's theorem $r^{(-l)} \in H_q(\Pi)$. Obviously, $\operatorname{Im} r^{(-l)}(z)$ is the Poisson integral of φ in Π. Hence $\operatorname{Im} r^{(-l)}(x) = \varphi(x)$ at all points $x \in \mathbb{R}$ except for the poles of r (the nodes of φ).

Consider the least positive integer ν such that $2^\nu \geqslant m$. By Lemma 6, for each $s = 0, 1, 2, \dots, \nu - 1$ there exists a real function $\psi_s \in \Pi^l_{2^s}$ such that

$$\|\varphi - \psi_s\|_{L_q(\mathbb{R})} \leqslant \frac{c_1}{(2^s)^{l-\mu}} \|r\|_{A_{p,\mu}(\Pi)}. \tag{3.14}$$

We set $\varphi_s = \psi_s - \psi_{s-1}$, $s = 1, 2, \dots, \nu$, and also set $\psi_\nu = \varphi$. Recall that $\varphi_0 \equiv 0$. Obviously, $\varphi_s \in \Pi^l_{2^{s+1}}$ and

$$\varphi = \varphi_1 + \varphi_2 + \dots + \varphi_\nu \tag{3.15}$$

on $\mathbb{R}$. From (3.14) we also see that

$$\|\varphi_s\|_{L_q(\mathbb{R})} \leqslant \frac{c_2}{(2^s)^{l-\mu}} \|r\|_{A_{p,\mu}(\Pi)}. \tag{3.16}$$

Now we can find a lower bound for $\|r\|_{L_\lambda(\mathbb{R})}$. Indeed, by (3.13) and (3.15)

$$\|r\|^\lambda_{L_\lambda(\mathbb{R})} = \|r\|^\lambda_{H_\lambda(\Pi)} = \|(r^{(-l)})^{(l)}\|^\lambda_{H_\lambda(\Pi)} \leqslant \sum_{s=1}^{\nu} \left\| \frac{l!}{\pi} \int_{\mathbb{R}} \frac{\varphi_s(t)\, dt}{(t-z)^{l+1}} \right\|^\lambda_{H_\lambda(\Pi)}.$$

We continue the estimate using Lemma 5 for $k=l$ and inequality (3.16). Then we obtain

$$\|r\|^{\lambda}_{L_\lambda(\mathbb{R})} \leqslant c_3 \sum_{s=1}^{\nu} \frac{(2^{s+1})^{l\lambda}}{(2^s)^{\lambda(l-\mu)}} \|r\|^{\lambda}_{A_{p,\mu}(\Pi)} = c_4 \|r\|^{\lambda}_{A_{p,\mu}(\Pi)} \sum_{s=1}^{\nu} (2^{\lambda\mu})^s$$
$$\leqslant c_5 (2^{\nu})^{\lambda\mu} \|r\|^{\lambda}_{A_{p,\mu}(\Pi)} \leqslant c_6 n^{\lambda\mu} \|r\|^{\lambda}_{A_{p,\mu}(\Pi)}.$$

The proof is complete.

§ 4. Bernstein-type inequalities for the quasinorms of derivatives of rational functions and the proof of the inverse theorem

Several special cases of Theorem 8 below were considered in [5]–[7].

Theorem 8. *Let p and μ be positive numbers such that $\mu + \frac{1}{p} \notin \mathbb{N}$, $s \in \mathbb{N}$ and $\frac{1}{\lambda} = \mu + s + \frac{1}{p}$. If $r \in \mathscr{R}_n \cap A_{p,\mu}(\Pi)$, then*

$$\|r^{(s)}\|_{H_\lambda} \leqslant c(p,\mu,s) n^{\mu+s} \|r\|_{A_{p,\mu}}.$$

Proof. Since $r \in A_{p,\mu}$, it follows that $r^{(s)} \in A_{p,\mu+s}$ and

$$\|r^{(s)}\|_{A_{p,\mu+s}} \leqslant c_1(p,\mu,s) \|r\|_{A_{p,\mu}} \tag{4.1}$$

(see § 2). It is also obvious that $r^{(s)}$ is a rational function of degree at most $(s+1)n$. It remains to apply Theorem 3 with $\mu+s$ in place of μ to $r^{(s)}$ and to use inequality (4.1).

We order the squares in $\mathscr{F}_{1/2}$ (see § 2) by numbering them: $\mathscr{F}_{1/2} = \{Q_k\}_{k=1}^{\infty}$. Let $d(Q_k)$ (in accordance with § 5.1) be the length of the diagonal of Q_k.

Lemma 7. *Let $f \in H_p(\Pi)$, $0 < p < \infty$. Then the quantities*

$$\delta_k(f,p) := d(Q_k)^{1/p} \|f\|_{C(Q_k)}, \qquad k = 1,2,3,\dots,$$

form an infinitesimal sequence. Moreover, if the squares Q_k in $\mathscr{F}_{1/2}$ are ordered so that the above sequence is nonincreasing, then

$$\delta_k(f,p) \leqslant \frac{c(p)}{k^{1/p}} \|f\|_{H_p}, \qquad k = 1,2,3,\dots.$$

For a disc Lemma 7 can be found in [13], [5] and [14]. In a half-plane its proof is similar.

Proof of Theorem 4. Let $s = \max\{1, [\alpha]+1\}$, $\frac{1}{\lambda} = \mu + s + \frac{1}{p}$, and also let

$$\delta_k = \delta_k(r^{(s)}, \lambda) = d(Q_k)^{1/\lambda} \|r^{(s)}\|_{C(Q_k)}, \qquad k = 1,2,3,\dots.$$

From inequality (4.1) and Lemma 11 we obtain

$$\sum_{k=1}^{\infty} \delta_k^p \leqslant c_1 \|r\|^p_{A_{p,\mu}}.$$

Hence $\{\delta_k\}_{k=1}^{\infty}$ is an infinitesimal sequence and we can assume that the squares Q_k in $\mathscr{F}_{1/2}$ are numbered so that this sequence is nonincreasing. Then

$$\delta_k \leqslant \frac{c_2}{k^{1/p}} \|r\|_{A_{p,\mu}}, \qquad k = 1, 2, 3, \dots . \tag{4.2}$$

Using Theorem 8 and Lemma 7 we also find

$$\delta_k = c_3 \frac{n^{\mu+s}}{k^{1/\lambda}} \|r\|_{A_{p,\mu}}, \qquad k = 1, 2, 3, \dots . \tag{4.3}$$

From (4.2) and (4.3) we obtain

$$\sum_{k=1}^{\infty} \delta_k^{\tau} \leqslant c_4 n^{\tau(\alpha+\mu)} \|r\|_{A_{p,\mu}}^{\tau}.$$

Applying Lemma 11 again we see that the last inequality is tantamount to the statement of Theorem 4.

In the necessity part of the proof of Theorem 1 (the inverse theorem) we shall use the method of real interpolation. For necessary information the reader can consult [10].

For $\theta \in (0, 1)$ and $0 < q \leqslant \infty$ let $(\,\cdot\,,\cdot\,)_{\theta,q}$ denote the Peetre interpolation functor.

Now we introduce the approximation space $\mathfrak{R}_q^{\beta}$, $\beta > 0$, $q > 0$, of functions $f \in A_{p,\mu}(\Pi)$ which have a finite quasinorm

$$\|f\|_{\mathfrak{R}_q^{\beta}} = \left[\sum_{k=1}^{\infty} \frac{1}{k} (k^{\beta} R_n(f)_{p,\mu})^q\right]^{1/q}.$$

In our notation for $\mathfrak{R}_q^{\beta}$ we drop p and μ because we fix these parameters throughout this section.

Lemma 8. *For $0 < \beta_0 < \beta_1 < \infty$ and $0 < \theta < 1$ let $\beta = (1-\theta)\beta_0 + \theta\beta_1$ and let $q, q_0, q_1 \in (0, \infty)$. Then*

$$(\mathfrak{R}_{q_0}^{\beta_0}, \mathfrak{R}_{q_1}^{\beta_1})_{\theta,q} = \mathfrak{R}_q^{\beta}.$$

This is a special case of Theorem 7.1.8 in [10].

Lemma 9. *For $1 < \tau_0 < \tau_1 < \infty$, $\alpha_0 < \frac{1}{\tau_0}$, $\alpha_1 < \frac{1}{\tau_1}$, $\alpha_0 \neq \alpha_1$ and $0 < \theta < 1$ let*

$$\alpha = (1-\theta)\alpha_0 + \theta\alpha_1 \quad \textit{and} \quad \frac{1}{\tau} = \frac{1-\theta}{\tau_0} + \frac{\theta}{\tau_1}.$$

Then $(B_{\tau_0}^{\alpha_0}, B_{\tau_1}^{\alpha_1})_{\theta,\tau} = B_{\tau}^{\alpha}$.

Lemma 9 is a special case of Theorem 6.4.5 in [10]. The reader should not be confused by the fact that [10] treats Besov spaces from a different standpoint from our paper.

Proof of Theorem 1. Necessity. Following our notation we must prove that

$$\mathfrak{R}_{\tau}^{\alpha+\mu} \hookrightarrow B_{\tau}^{\alpha}. \tag{4.4}$$

Using Theorem 4 and Bernstein's method of the proof of inverse theorems of approximation theory we can easily see that

$$\mathfrak{R}^{\alpha+\mu}_{\min\{\tau,1\}} \hookrightarrow B^{\alpha}_{\tau}. \tag{4.5}$$

The embeddings (4.4) and (4.5) coincide for $0 < \tau \leqslant 1$.

Now let $\tau > 1$. We find $-\mu < \alpha_0 < \alpha_1 < \infty$ and $0 < \theta < 1$ such that $\alpha = (1-\theta)\alpha_0 + \theta\alpha_1$ and $\frac{1}{\tau_1} = \alpha_1 + \mu + \frac{1}{p} < 1$. Then we also have $\frac{1}{\tau_0} = \alpha_0 + \mu + \frac{1}{p} < 1$ and

$$\alpha + \mu = (1-\theta)(\alpha_0 + \mu) + \theta(\alpha_1 + \mu), \qquad \frac{1}{\tau} = \frac{1-\theta}{\tau_0} + \frac{\theta}{\tau_1}.$$

By (4.5),

$$\mathfrak{R}^{\alpha_s+\mu}_1 \hookrightarrow B^{\alpha_s}_{\tau_s}, \qquad s = 0, 1.$$

Hence

$$(\mathfrak{R}^{\alpha_0+\mu}_1, \mathfrak{R}^{\alpha_1+\mu}_1)_{\theta,\tau} \hookrightarrow (B^{\alpha_0}_{\tau_0}, B^{\alpha_1}_{\tau_1})_{\theta,\tau}.$$

It remains to use Lemmas 8 and 9. The proof of Theorem 1 is complete.

Remark 1. The constraint $\frac{1}{p} + \mu \notin \mathbb{N}$ in Theorem 3 is essential, as we see in the example of rational functions of degree $2l$,

$$r_\varepsilon(z) = [(z+i)(z+i\varepsilon)]^{-l}, \qquad l := \frac{1}{p} + \mu \in \mathbb{N},$$

depending on the parameter $\varepsilon \in (0, \frac{1}{2}]$. Easy calculations (the case $\mu = \frac{1}{p}$ was considered in [7]) show that for $\varepsilon \in (0, \frac{1}{2}]$

$$\|r_\varepsilon\|_{L_\lambda(\mathbb{R})} \asymp \left(\log\frac{1}{\varepsilon}\right)^{1/p+\mu}, \qquad \|r_\varepsilon\|_{A_{p,\mu}(\Pi)} \asymp \left(\log\frac{1}{\varepsilon}\right)^{1/p}.$$

Since $\|r_\varepsilon\|_{L_\lambda(\mathbb{R})}$ grows much more rapidly than $\|r_\varepsilon\|_{A_{p,\mu}(\Pi)}$ as $\varepsilon \to +0$, Theorem 3 fails for $\frac{1}{p} + \mu \in \mathbb{N}$.

In a similar way we can show that the constraint $\frac{1}{p} + \mu \notin \mathbb{N}$ is essential in Theorems 4 and 8 as well. Hence the condition $\frac{1}{p} + \mu \notin \mathbb{N}$ is essential also for the necessity in Theorem 1.

§ 5. Appendix. Several results on Bergman and Smirnov spaces

The aim of this section is to prove auxiliary results on the Bergman and Smirnov spaces. We used these results in our proofs of the central results of this paper. We establish Theorems 9 and 10 below in greater generality than required for the proof of Theorem 3 since they are of independent interest.

5.1. The Bergman space in a domain and a Whitney-type decomposition. Let $G \subset \mathbb{C}$ be a domain distinct from $\mathbb{C}$; ∂G the boundary of G; $\overline{G} = G \cup \partial G$ the closure of G; and let $\rho(z, \partial G)$ be the distance from the point z to the boundary ∂G. In § 1, for positive p and μ we introduced the Lebesgue spaces $L_{p,\mu}(\Pi)$ and the Bergman spaces $A_{p,\mu}(\Pi)$. In a similar way we define the Lebesgue spaces $L_{p,\mu} = L_{p,\mu}(G)$ and the Bergman spaces $A_{p,\mu} = A_{p,\mu}(G)$ in G. Here $\operatorname{Im} z$ must be replaced by $\rho(z, \partial G)$.

Lemma 10. *If* $f \in A_{p,\mu}(G)$, *then*

$$|f(z)| \leqslant c(p,s,\mu)\frac{\|f\|_{A_{p,\mu}}}{(\rho(z,\partial G))^{\mu+1/p}}, \qquad z \in G.$$

The proof leans on the fact that $|f(z)|^p$ is a subharmonic function (see, for instance, Proposition 1.1 in [12]).

Using Lemma 10 we can prove in particular that $A_{p,\mu}(G)$ is a complete space. If G is a Jordan domain with rectifiable boundary, then it follows from Theorem 1 in [15] that the set of algebraic polynomials is dense in $A_{p,\mu}(G)$.

Let K and Γ be subsets of $\mathbb{C}$ and $\rho(K,\Gamma)$ the distance between K and Γ, so that $\rho(K,\Gamma) = \inf\{|z-\xi| : z \in K,\ \xi \in \Gamma\}$. Let $d(K)$ and $d_0(K)$ denote the diameters of the smallest closed disc containing K and of the largest open disc lying in K, respectively.

Let G be a domain distinct from $\mathbb{C}$. A family $\mathscr{Z}$ of simply connected closed domains Q with piecewise smooth boundaries is called a *Whitney-type decomposition* of G if it satisfies the following conditions:

(i) two domains in $\mathscr{Z}$ can meet only at boundary points;

(ii) the union of all the domains in $\mathscr{Z}$ is G;

(iii) there exist constants c_1 and c_2 such that for each domain $Q \subset \mathscr{Z}$

$$c_1\rho(Q,\partial G) \leqslant d_0(Q) \leqslant d(Q) \leqslant c_2\rho(Q,\partial G). \tag{5.1}$$

One example of such a partitioning of G is provided by Whitney squares (see [16], Ch. VI) with $c_1 = \frac{\sqrt{2}}{8}$ and $c_2 = 1$. If $G = \Pi$, then the family $\mathscr{F}_\theta$, $0 < \theta < 1$, constructed in § 2 is a Whitney-type decomposition with $c_1 = \frac{1}{\theta} - 1$ and $c_2 = \sqrt{2}(\frac{1}{\theta} - 1)$.

If K is a compact subset of $\mathbb{C}$, then let $C(K)$ denote the set of continuous complex-valued functions on K. For $f \in C(K)$ we set $\|f\|_{C(K)} = \max_{z\in K}|f(z)|$.

Lemma 11. *Let* p *and* μ *be positive numbers and let* $f \in A_{p,\mu}(G)$. *Then*

$$c_3\|f\|^p_{A_{p,\mu}} \leqslant \sum_{Q\in\mathscr{Z}} (d(Q)^{\mu+1/p}\|f\|_{C(Q)})^p \leqslant c_4\|f\|^p_{A_{p,\mu}},$$

where c_3 *and* c_4 *depend on* p, μ *and the constants* c_1 *and* c_2 *in* (5.1).

The proof (similarly to the case of Lemma 10) leans on the fact that $|f(z)|^p$ is subharmonic. For a disc Lemma 11 was proved in [17].

Lemma 12. *Let* p, a *and* μ *be positive numbers,* D *the open square with vertices at* 0, a, ia *and* $a+ia$ *and let* $\Delta \subset D$ *be the equilateral triangle with base* $(0,a)$. *If* f *is an analytic function in* D *and* $(\operatorname{Im} z)^{p\mu-1}|f(z)|^p$ *is integrable in* D, *then*

$$\int_\Delta \rho(z,\partial\Delta)^{p\mu-1}|f(z)|^p\,dm_2(z) \leqslant c(p,\mu)\int_D (\operatorname{Im} z)^{p\mu-1}|f(z)|^p\,dm_2(z). \tag{5.2}$$

Proof. Since $\rho(z,\partial\Delta) \leqslant \operatorname{Im} z$ for $z \in \Delta$, inequality (5.2) holds for $p\mu \geqslant 1$ with $c(p,\mu) = 1$. Thus we assume in what follows that $p\mu < 1$. For convenience we shall consider $a = 2$ because the general case reduces to this after a change of the variable of integration in (5.2).

Let z_0 be the centre of the triangle Δ and Δ_0 the triangle with vertices at z_0, 0 and 2; let $\Lambda = \Delta \setminus \Delta_0$ and let Λ_+ and Λ_- be the right and left halves of Λ cut by the line $x = 1$.

Obviously, $\rho(z, \partial\Delta) = \operatorname{Im} z$ for $z \in \Delta_0$, so to prove (5.2) it is sufficient to verify that

$$\int_{\Lambda_\pm} \rho(z, \partial\Delta)^{p\mu-1} |f(z)|^p \, dm_2(z) \leqslant c_1(p, \mu) \int_D (\operatorname{Im} z)^{p\mu-1} |f(z)|^p \, dm_2(z). \qquad (5.3)$$

For example, we prove (5.3) for Λ_-. We represent the triangle $\Lambda_- \setminus \{0\}$ as the union of closed trapezia T_k with bases on the lines $x = \frac{1}{2^{k-1}}$ and $x = \frac{1}{2^k}$, $k = 1, 2, 3, \dots$, and with lateral sides lying on the lateral sides of Λ_-. Using the same method as in the proof of Lemma 11 we readily verify that

$$\sum_{k=1}^{\infty} (d(T_k))^{p\mu+1} \|f\|^p_{C(T_k)} \leqslant c_2(p, \mu) \int_D (\operatorname{Im} z)^{p\mu-1} |f(z)|^p \, dm_2(z). \qquad (5.4)$$

It is an immediate consequence of (5.4) that (5.3) for Λ_-. This proves the lemma.

5.2. Quasiconformal reflection. Here we shall assume that G is a bounded simply connected domain with boundary ∂G which is a Lavrent'ev curve, that is, a rectifiable Jordan curve such that for any points $\xi_1, \xi_2 \in \partial G$ we have

$$|\Gamma(\xi_1, \xi_2)| \leqslant \varkappa |\xi_1 - \xi_2|, \qquad (5.5)$$

where $\varkappa > 1$ is a constant and $|\Gamma(\xi_1, \xi_2)|$ is the length of $\Gamma(\xi_1, \xi_2)$, the shortest of the two arcs of ∂G with end-points at ξ_1 and ξ_2.

We present the required facts from the theory of quasiconformal mappings (see [18], [19]). We shall assume that $0 \in G$. Let * denote a quasiconformal involution of $\overline{\mathbb{C}}$ such that: $\xi^{**} = \xi$ for all $\xi \in \overline{\mathbb{C}}$, $0^* = \infty$, $\xi^* = \xi$ for $\xi = \partial G$. Such a map * is not unique; we can choose it so that

(i) for each neighbourhood $U \subset G$ of 0 the map $\xi \mapsto \xi^*$ takes $\mathbb{C} \setminus (U \cup U^*)$ quasiconformally into itself;

(ii) the map $\xi \mapsto \xi^*$ is continuously differentiable in $\mathbb{C} \setminus (0 \cup \partial G)$ and for all $\xi \in \mathbb{C} \setminus (U \cup U^* \cup \partial G)$ we have

$$\left| \frac{\partial \xi^*}{\partial \xi} \right| \leqslant c_1, \qquad c_2 \leqslant \left| \frac{\partial \xi^*}{\partial \overline{\xi}} \right| \leqslant \frac{1}{c_2},$$

where c_1 and c_2 depend only on $\varkappa$ in (5.5) and the neighbourhood $U \subset G$ of 0.

Note that for each $D \subset \overline{\mathbb{C}}$ we set $D^* = \{\xi^* : \xi \in D\}$. In what follows we assume that we have picked the map * satisfying conditions (i) and (ii). For example, if G is the disc $|\xi| < 1$, then $\xi^* = \frac{1}{\overline{\xi}}$.

We shall use a partitioning $\mathscr{Z} = \{Q_k\}_{k=0}^{\infty}$ of the domain G into Whitney-type domains Q_k in which $Q_0 = \{\xi \in G : |\xi| \leqslant \frac{1}{2}\rho(0, \partial G)\}$. Such a partitioning is easy to construct with the use of Whitney squares.

It follows from the properties of the map * listed above and relations (5.1) that the infinitesimal sequences $\rho(Q_k, \partial G)$, $\rho(Q_k^*, \partial G)$, $d(Q_k)$, $d(Q_k^*)$, $d_0(Q_k)$ and $d_0(Q_k^*)$, where $k = 1, 2, 3, \dots$, all have the same order.

5.3. The integral representation. Here we assume that the domain G and the map * are as described in § 5.2. We look at the closed curves

$$\Gamma_s = \left\{\xi^* : |\xi| = \frac{1}{2^{2-s}}\rho(0, \partial G)\right\},$$

$s = 0, 1$, and let Ω_0 and Ω_1 be the closed two-connected domains lying between the curves Γ_0 and Γ_1 and between Γ_1 and ∂G, respectively. Let $\Omega = \Omega_0 \cup \Omega_1$. Consider a smooth function $\eta\colon \Omega \mapsto [0, 1]$ such that $\eta(\xi) = 1$ for $\xi \in \Omega_1$ and $\eta(\xi) = 0$ for $\xi \in \Gamma_0$. Its existence can readily be demonstrated using averaging in the sense of Steklov. Then the image of Ω under the map $\theta(\xi) := \xi^* \cdot \eta(\xi)$ coincides with $\overline{G}$, and we have $\theta(\xi) = \xi^*$ for $\xi \in \Omega_1$.

Theorem 9. *Let f be an analytic function in G, $f^{(s)} \in A_{1,s}(G)$, $s \in \mathbb{N}$, and*

$$T_{s-1}(z) = T_{s-1}(z, f) = \sum_{k=0}^{s-1} \frac{f^{(k)}(0)}{k!} z^k.$$

Then for $z \in G$

$$f(z) = T_{s-1}(z) - \frac{1}{\pi(s-1)!} \int_\Omega f^{(s)}(\theta(\xi))(\xi - \theta(\xi))^{s-1} \frac{\partial\theta(\xi)}{\partial\bar{\xi}} \frac{dm_2(\xi)}{\xi - z}.$$

This is proved in the same way as Lemma 2.3 in [20]. The only difference is that the density of the algebraic polynomials in the Smirnov space $E_{1/(s+1)}(G)$ was used in that lemma, while here we use the fact that they are dense in $A_{1,s}(G)$.

The next lemma is well known (see, for instance, [21] or [20]).

Lemma 13. *Let G be a domain satisfying the above conditions and $\beta > 0$. Then for each $\xi \in \mathbb{C} \setminus \partial G$*

$$\int_{\partial G} \frac{|dz|}{|\xi - z|^{\beta+1}} \leqslant \frac{c}{\rho(\xi, \partial G)^\beta},$$

where the positive c depends only on β and the constant $\varkappa$ in condition (5.5).

5.4. The Smirnov space. For a rectifiable curve Γ in $\mathbb{C}$ and $0 < p < \infty$ let $L_p(\Gamma)$ denote the Lebesgue space of measurable complex-valued functions f on Γ with finite quasinorm

$$\|f\|_{L_p} = \|f\|_{L_p(\Gamma)} = \left(\int_\Gamma |f(z)|^p\, |dz|\right)^{1/p}.$$

Let $G = G_+$ be a simply connected bounded domain with boundary ∂G that is a rectifiable Jordan curve; let $G_- := \overline{\mathbb{C}} \setminus \overline{G}$. For $0 < p < \infty$ let $E_p^\pm = E_p(G_\pm)$ be the Smirnov space of analytic functions in $G_\pm$ (for the definition and properties of $E_p^\pm$ see [22] and [23]). If G is a disc in $|z| < 1$, then $E_p^\pm$ coincides with the Hardy space $H_p^\pm$. Many properties of $H_p^\pm$ also hold for $E_p^\pm$. In particular, if $f \in E_p^\pm$, then for almost all $\xi \in \partial G$ there exists a limit $\lim f(z) =: f(\xi)$ as z in $G_\pm$ converges to ξ along paths nontangential to ∂G. Correspondingly, we view functions in $E_p^\pm$ as defined in $G_\pm$ and almost everywhere on ∂G. The quasinorm of $f \in E_p^\pm$ is defined by $\|f\|_{E_p^\pm} = \|f\|_{L_p(\partial G)}$. Note that $f(\infty) = 0$ for $f \in E_p^-$.

The next Lemma 14 follows from David's theorem (see, for example, [21]) on singular integrals of Cauchy type.

Lemma 14. *Let G be a domain with boundary ∂G which is a Lavrent'ev curve and let $g \in L_u(\partial G)$, $1 < u < \infty$. Then there exists a unique pair of functions $g_\pm \in E_u(G_\pm)$ such that $g = g_+ + g_-$ a.e. on ∂G and $\|g_\pm\|_{E_u^\pm} \leqslant c\|g\|_{L_u}$, where the positive c depends only on u and the constant $\varkappa$ in (5.5).*

In Lemma 15 we use a construction described in § 5.2. This lemma follows from Lemma 2.1 in [20] and properties of the domains Q_k and Q_k^*, $k = 0, 1, 2, \dots$.

Lemma 15. *Let $h \in E_u(G_-)$, $0 < u < \infty$, and*

$$\delta_0 = \|h\|_{C(Q_0^*)}, \qquad \delta_k = (d(Q_k^*))^{1/u}\|h\|_{C(Q_k^*)}, \quad k = 1, 2, 3, \dots .$$

Then

(i) $\delta_0 \leqslant c_1(G, u)\|h\|_{E_u^-}$;

(ii) $\{\delta_k\}_{k=1}^\infty$ *is an infinitesimal sequence and if the Q_k, $k = 1, 2, 3, \dots$, are ordered so that this sequence is nonincreasing, then*

$$\delta_k \leqslant \frac{c_2(G, u)}{k^{1/u}}\|g\|_{E_u^-}, \qquad k = 1, 2, 3, \dots .$$

5.5. The embedding theorem. Now everything is ready for the proof of Theorem 10 used in the proof of Theorem 3, one of the central results of this paper.

Theorem 10. *Let p, q and μ be positive numbers such that*

$$0 < p < q < \infty, \qquad \mu + \frac{1}{p} - \frac{1}{q} =: l \in \mathbb{N}$$

and let G be a simply connected bounded domain whose boundary is a Lavrent'ev curve. If f is an analytic function in G and $f^{(l)} \in A_{p,\mu}(G)$, then $f \in E_q(G)$. Furthermore, if $f(z_0) = f'(z_0) = \dots = f^{(l-1)}(z_0) = 0$ at a point $z_0 \in G$, then

$$\|f\|_{E_q} \leqslant c\|f^{(l)}\|_{A_{p,\mu}}, \tag{5.6}$$

where the positive c is independent of f.

Remark 2. An analysis of the proof of Theorem 10 below demonstrates that if z_0 is the centre of the largest disc inscribed in $\overline{G}$ (or one of the centres if there are several such discs), then we can assume that the constant in (5.6) depends only on p, q, μ and the constant $\varkappa$ in (5.5).

Proof of Theorem 10. We shall assume without loss of generality that $0 \in G$ and $z_0 = 0$. Since the set of algebraic polynomials is dense in E_q and $A_{p,\mu}$, it is sufficient to prove (5.6) for any algebraic polynomial f such that $f(0) = f'(0) = \dots = f^{(l-1)}(0) = 0$. For f satisfying these constraints the inclusions $f \in E_q$ and $f^{(l)} \in A_{p,\mu}$ are obvious. We shall use Theorem 9 and the sets Q_k, $k = 0, 1, 2, \dots$, in § 5.2.

First we look at the case $1 < q < \infty$. Let q' be the conjugate exponent of q: $\frac{1}{q'} + \frac{1}{q} = 1$. By the duality $(L_q(\partial G))' = L_{q'}(\partial G)$ there exists a function $g \in L_{q'}(\partial G)$ such that $\|g\|_{L_{q'}} = 1$ and

$$\|f\|_{E_q(G)} = \|f\|_{L_q(\partial G)} = \int_{\partial G} f(z)g(z)\,dz. \tag{5.7}$$

By Lemma 14 there exist functions $g_\pm \in E_q(G_\pm)$ such that $\|g_\pm\|_{E_q^\pm} \leqslant c_1$ and $g = g_+ + g_-$ a.e. on ∂G. Since $fg_+ \in E_1(G_+)$, it follows by the generalized Cauchy theorem (see [22] and [23]) that $\int_{\partial G} f(z)g_+(z)\,dz = 0$. Hence in view of (5.7),

$$\|f\|_{E_q} = \int_{\partial G} f(z)g_-(z)\,dz.$$

From this equality, Theorem 9 for $s = l$, Fubini's theorem and Cauchy's integral formula we obtain

$$\|f\|_{E_q} = \frac{2i}{l!}\int_\Omega f^{(l)}(\theta(\xi))g_-(\xi)(\xi-\theta(\xi))^{l-1}\frac{\partial\theta(\xi)}{\partial\overline{\xi}}\,dm_2(\xi). \tag{5.8}$$

Now we use Lemma 15 for $h = g_-$ and $u = q'$. We assume that the domains $\{Q_k\}_{k=1}^\infty$ are ordered so that the sequence $\{\delta_k\}_{k=1}^\infty$ is nonincreasing. Since the relations hold uniformly in $\xi \in Q_k$ and $k = 1,2,3,\dots$, we see from (5.8) that

$$\|f\|_{E_q} \leqslant c_1(d(Q_0))^{l+1/q}\|f^{(l)}\|_{C(Q_0)} + c_2\sum_{k=1}^\infty (d(Q_k))^{l+1/q}\|f^{(l)}\|_{C(Q_k)}\cdot k^{-1/q'}.$$

Note that it follows from the condition $l + \frac{1}{q} = \mu + \frac{1}{p}$ and Lemma 10 that the first term on the right-hand side of the last inequality has the estimate $c_3\|f^{(l)}\|_{A_{p,\mu}}$. Hence to complete the discussion of the case $1 < q < \infty$ it is sufficient to verify the inequality

$$\sum_{k=1}^\infty (d(Q_k))^{\mu+1/p}\|f^{(l)}\|_{C(Q_k)}\cdot k^{-1/q'} \leqslant c_4\|f^{(l)}\|_{A_{p,\mu}}. \tag{5.9}$$

For $0 < p \leqslant 1 < q < \infty$ it follows from Lemmas 2 and 11, while if $1 < p < q < \infty$, then we apply Hölder's inequality with exponents p and p', $\frac{1}{p'} + \frac{1}{p} = 1$, to the left-hand side of (5.9) and use Lemma 11 again.

Now we consider the case $0 < q \leqslant 1$. Let $\nu = [\frac{1}{q}] + 1$. Then for $z \in \overline{G}$ we have

$$f(z) = -\frac{\nu!}{\pi(l+\nu-1)!}\int_\Omega f^{(l)}(\theta(\xi))(\xi-\theta(\xi))^{l+\nu-1}\frac{\partial\theta(\xi)}{\partial\overline{\xi}}\,\frac{dm_2(\xi)}{(\xi-z)^{\nu+1}}. \tag{5.10}$$

To deduce (5.10) for $z \in G$, we must apply Theorem 9 to $f^{(-\nu)}$ (that is, to the νth antiderivative of f) and $s = l + \nu$, and then differentiate ν times the resulting integral representation for $f^{(-\nu)}$. Then equality (5.6) for $z \in \overline{G}$ follows from the continuity of f in $\overline{G}$ and the fact that the integrand on the right-hand side of (5.10) is uniformly bounded in $z \in G$ and $\xi \in \Omega$.

We pick points $\xi_0 \in \Omega_0$ and $\xi_k \in Q_k^*$, $k = 1,2,\dots$. Then we see from (5.10) that for $z \in \overline{G}$

$$|f(z)| \leqslant c_5\sum_{k=0}^\infty (d(Q_k))^{l+\nu+1}\|f^{(l)}\|_{C(Q_k)}\cdot\frac{1}{|\xi_k - z|^{\nu+1}}. \tag{5.11}$$

From the construction of the sets Q_k and the choice of the ξ_k we obtain

$$d(Q_k) \asymp \rho(\xi_k, \partial G) \quad \text{for } k = 1,2,3,\dots .$$

Hence bearing in mind that $0 < q \leqslant 1$ and $q(\nu + 1) > 1$ and using (5.11) and Lemma 13 we obtain

$$\int_{\partial G} |f(z)|^q \, |dz| \leqslant c_6 \sum_{k=0}^{\infty} ((d(Q_k))^{l+1/q} \|f^{(l)}\|_{C(Q_k)})^q. \tag{5.12}$$

Since $0 < p < q$, with the help of Lemmas 2 and 11 we deduce (5.6) from (5.12). Now the proof of Theorem 10 is complete.

Bibliography

[1] R. Rochberg, "Decomposition theorems for Bergman spaces and their applications", *Operators and function theory* (Lancaster 1984), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 153, Reidel, Dordrecht 1985, pp. 225–277.

[2] K. Zhu, *Spaces of holomorphic functions in the unit ball*, Grad. Texts in Math., vol. 226, Springer-Verlag, New York 2005.

[3] A. A. Pekarskiĭ, "Classes of analytic functions determined by best rational approximations in H_p", *Mat. Sb.* **127(169)**:1(5) (1985), 3–20; English transl. in *Math. USSR-Sb.* **55**:1 (1986), 1–18.

[4] R. R. Coifman and R. Rochberg, "Representation theorems for holomorphic and harmonic functions in L^p", *Representation theorems for Hardy spaces*, Astérisque, vol. 77, Soc. Math. France, Paris 1989, pp. 11–66.

[5] E. Dyn'kin, "Rational functions in Bergman spaces", *Complex analysis, operators, and related topics*, Oper. Theory Adv. Appl., vol. 113, Birkhüser, Basel 2000, pp. 77–94.

[6] V. R. Misiuk, "Bernstein-type inequalities for derivatives of rational functions with respect to the planar measure", *Tr. Inst. Mat. Nats. Akad. Nauk Belarusi* **9** (2001), 105–108. (Russian)

[7] T. S. Mardvilko, "An inequality for the quasinorms of a rational function with respect to the linear and planar measures and its applications", *Vestsī Nats. Akad. Navuk Belarusī Ser. Fīz. Mat. Navuk* **1** (2010), 41–48. (Russian)

[8] A. A. Pekarskiĭ and H. Stahl, "Bernstein type inequalities for derivatives of rational functions in L_p spaces for $p < 1$", *Mat. Sb.* **186**:1 (1995), 119–130; English transl. in *Sb. Math.* **186**:1 (1995), 121–131.

[9] T. S. Mardvilko and A. A. Pekarskiĭ, "Rational approximation in the Bergman space in a half-plane", *Current problems of function theory and their applications*, 15th Saratov Winter School, Saratov State University Publishing House, Saratov 2010, pp. 113–115. (Russian)

[10] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin–Heidelberg–New York 1976.

[11] J. B. Garnett, *Bounded analytic functions*, Pure Appl. Math., vol. 96, Academic Press, New York–London 1981.

[12] H. Hedenmalm, B. Korenblum and K. Zhu, *Theory of Bergman spaces*, Grad. Texts in Math., vol. 199, Springer-Verlag, New York 2000.

[13] A. A. Pekarskiĭ, "Generalized rational approximation in the disc", *Vestsī Nats. Akad. Navuk BSSR Ser. Fīz.-Mat. Navuk* **6** (1990), 9–14. (Russian)

[14] A. A. Pekarskii, "Approximation by rational functions with free poles", *East J. Approx.* **13**:3 (2007), 227–319; "Corrigendum", *East J. Approx.* **13**:4 (2007), 483.

[15] J. E. Brennan, "Point evaluations, invariant subspaces and approximation in the mean by polynomials", *J. Funct. Anal.* **34**:3 (1979), 407–420.

[16] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser., vol. 30, Princeton Univ. Press, Princeton, NJ 1970.

[17] F. A. Šamojan (Shamoyan), "Embedding theorems and a characterization of traces in the spaces $H^p(U^n)$, $0 < p < \infty$", *Mat. Sb.* **107(149)**:3(11) (1978), 446–462; English transl. in *Math. USSR-Sb.* **35**:5 (1979), 709–725.

[18] L. V. Ahlfors, *Lectures on quasiconformal mappings*, Van Nostrand, Toronto, ON–New York–London 1966.

[19] V. I. Belyĭ, "Modern methods of the geometric theory of functions of a complex variable in approximation problems", *Algebra i Analiz* **9**:3 (1997), 3–40; English transl. in *St. Petersburg Math. J.* **9**:3 (1998), 421–453.

[20] A. A. Pekarskiĭ, "Rational approximation of functions with derivatives in a V. I. Smirnov space", *Algebra i Analiz* **13**:2 (2001), 165–190; English transl. in *St. Petersburg Math. J.* **13**:2 (2002), 281–300.

[21] G. David, "Opérateurs intégraux singuliers sur certaines courbes du plan complexe", *Ann. Sci. École Norm. Sup.* (4) **17**:1 (1984), 157–189.

[22] I. I. Privalov, *Boundary properties of analytic functions*, GITTL, Moscow–Leningrad 1950. (Russian)

[23] P. L. Duren, *Theory of H^p spaces*, Academic Press, New York–London 1970.

T. S. Mardvilko
Belarus State University of Informatics and Radioelectronics, Minsk
E-mail: mardvilko@mail.ru

Received 17/MAY/10
Translated by N. KRUZHILIN

A. A. Pekarskiĭ
Belarus State University, Minsk
E-mail: pekarskii@gmail.com