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Multiresolution analysis on zero-dimensional Abelian groups and wavelets bases

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Abstract. For a locally compact zero-dimensional group $(G, \dot{+})$, we build a multiresolution analysis and put forward an algorithm for constructing orthogonal wavelet bases. A special case is indicated when a wavelet basis is generated from a single function through contractions, translations and exponentiations.

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§ 1. Introduction

In recent years there has been considerable interest in the problem of constructing wavelet bases on locally compact zero-dimensional Abelian groups. In [1]–[3] these questions were examined on the Cantor dyadic group. Protasov and Farkov [4], [5] gave a characterization of the dyadic compactly supported wavelets on $\mathbb{R}_+$, and pointed out an algorithm for their construction. Protasov [6] studied approximative properties of dyadic wavelets put forward in [4]. Farkov [7], [8] pointed out a method for constructing compactly supported orthogonal wavelets on a locally compact Vilenkin group G with a constant generating sequence, and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis (MRA in the sequel) of $L_2(G)$. A good deal of studies was devoted to the construction of an MRA on the group of all p-adic numbers. Kozyrev [9], [10] found orthonormal p-adic wavelet bases consisting of eigenfunctions of p-adic pseudodifferential operators. Khrennikov, Shelkovich and Skopina [11]–[13] introduced the concept of a p-adic MRA with orthogonal refinable function, and described a general scheme for their creation. Riesz bases of wavelets over the p-adic number field were constructed in [14]. J. Benedetto and R. Benedetto [15] built wavelet bases on a locally compact Abelian group containing an open subgroup. The author [16] has put forward a scheme for constructing a Haar system on a compact zero-dimensional Abelian group. As distinct from previous papers, the Haar basis of [16] is generated from a single function through contractions, translations and exponentiations.

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We note that the same problem was solved in [8] and [13], namely, to construct an MRA and use it as a basis for constructing orthonormal bases for $L_2(G)$ through contractions and translations of several functions. In [8] this problem was considered for a Vilenkin group, and in [13], for the p-adic number field. In the present paper we examine the problem of constructing orthogonal wavelet bases on arbitrary locally compact zero-dimensional groups for which cosets of neighbouring subgroups have the same order, which is equal to some prime. To do so, we first construct an MRA from a given fixed refinable function with compactly supported Fourier transform. Next we will point out a way for constructing a refinable function for which the corresponding MRA generates an orthonormal basis consisting of contractions and translations of some functions. Such a refinable function is built from an arbitrary vector $(\lambda_0, \lambda_1, \dots, \lambda_{p^{s-1}-1})$ with complex coordinates satisfying the single condition: $|\lambda_j| = 1$ for all j. A special case will be indicated when an orthonormal basis is generated from a single function through contractions, translations and exponentiations.

§ 2. Locally compact zero-dimensional groups, topology and characters

We proceed to give basic notions and facts in the analysis on zero-dimensional groups. A more detailed account may be found in [17].

A topological group in which the connected component of 0 is 0 is usually referred to as a *zero-dimensional group*. If a separable locally compact group $(G, \dot{+})$ is zero-dimensional, then the topology on it can be generated by means of a descending sequence of subgroups. The converse statement holds for all topological groups (see [17], Ch. 1, § 3). So, for a locally compact group, we are going to say 'zero-dimensional group' instead of saying 'a group with topology generated by a sequence of subgroups'.

Let $(G, \dot{+})$ be a locally compact zero-dimensional Abelian group with topology generated by a countable system of open subgroups

$$\cdots \supset G_{-n} \supset \cdots \supset G_{-1} \supset G_0 \supset G_1 \supset \cdots \supset G_n \supset \cdots$$

with

$$\bigcup_{n=-\infty}^{+\infty} G_n = G \quad \text{and} \quad \bigcap_{n=-\infty}^{+\infty} G_n = \{0\}$$

(0 is the null element in the group G). Given any fixed $N \in \mathbb{Z}$, the subgroup G_N is a compact Abelian group with respect to the same operation $\dot{+}$ under the topology generated by the system of subgroups

$$G_N \supset G_{N+1} \supset \cdots \supset G_n \supset \cdots .$$

As each group G_n is compact, it follows that each quotient group G_n/G_{n+1} is finite (say, of order p_n). We may always assume that all the p_n are primes, for in fact, by Sylow's theorem (see [18]), the chain of subgroups can be refined so that the quotient groups G_n/G_{n+1} will be of prime order. In this case, a base of the topology is formed by all possible cosets $G_n \dot{+} g$, $g \in G$.

We further define the numbers $(\mathfrak{m}_n)_{n=-\infty}^{+\infty}$ as follows:

$$\mathfrak{m}_0 = 1, \qquad \mathfrak{m}_{n+1} = \mathfrak{m}_n \cdot p_n.$$

Clearly, for $n \geqslant 1$,

$$\mathfrak{m}_n = p_0 p_1 \cdots p_{n-1}, \qquad \mathfrak{m}_{-n} = \frac{1}{p_{-1} p_{-2} \cdots p_{-n}}.$$

The collection of all such cosets $G_n \dotplus g$, $n \in \mathbb{Z}$, along with the empty set form the semiring $\mathscr{K}$. On each coset $G_n \dotplus g$ we define a measure μ by $\mu(G_n \dotplus g) = \mu G_n = 1/\mathfrak{m}_n$. So if $n \in \mathbb{N}$ and $p_n = p$, we have $\mu G_n \cdot \mu G_{-n} = 1$. The measure μ can be extended from the semiring $\mathscr{K}$ onto the σ-algebra (for example, using the Carathéodory extension). This gives a translation invariant measure μ, which agrees on the Borel sets with the Haar measure on G. Further, let $\int_G f(x)\,d\mu(x)$ be the absolutely convergent integral with respect to the measure μ.

Given $n \in \mathbb{Z}$, consider an element $g_n \in G_n \setminus G_{n+1}$ and fix it. Then any $x \in G$ has a unique representation of the form

$$x = \sum_{n=-\infty}^{+\infty} a_n g_n, \qquad a_n = 0, \dots, p_n - 1, \tag{2.1}$$

the sum (2.1) containing a finite number of terms with negative subscripts; that is,

$$x = \sum_{n=N}^{+\infty} a_n g_n, \qquad a_n = 0, \dots, p_n - 1, \quad a_N \neq 0. \tag{2.2}$$

Classical examples of zero-dimensional groups are the Vilenkin groups and p-adic numbers (see [17], Ch. 1, § 2).

A direct sum of cyclic groups $Z(p_k)$ of order p_k, $k \in \mathbb{Z}$, is called a *Vilenkin group*. This means that the elements of a Vilenkin group are infinite sequences $x = (x_k)_{k=-\infty}^{+\infty}$ such that:

1) $x_k = 0, \dots, p_k - 1$;
2) only finitely many of the x_k with negative subscripts are distinct from zero;
3) the group operation $\dotplus$ is coordinatewise addition modulo p_k; that is,

$$x \dotplus y = (x_k \dotplus y_k), \qquad x_k \dotplus y_k = (x_k + y_k) \mod p_k.$$

The topology on such a group is generated by the chain of subgroups

$$G_n = \bigl\{x \in G : x = (\dots, 0, 0, \dots, 0, x_n, x_{n+1}, \dots),\ x_\nu = 0, \dots, p_\nu - 1,\ \nu \geqslant n\bigr\}.$$

It is easy to see that the subgroups G_n form a descending sequence. For g_n, we can take a sequence containing only zeros except for one at the nth position.

The group $\mathbb{Z}_p$ of all p-adic numbers (p is a prime) also consists of sequences $x = (x_k)_{k=-\infty}^{+\infty}$, $x_k = 0, \dots, p-1$ in which only finitely many x_k with negative subscripts are distinct from zero. However, the group operation in $\mathbb{Z}_p$ is differently defined. Namely, given elements

$$x = (\dots, 0, \dots, 0, x_N, x_{N+1}, \dots) \quad \text{and} \quad y = (\dots, 0, \dots, 0, y_N, y_{N+1}, \dots) \in \mathbb{Z}_p,$$

we again add them coordinatewise, but whereas in a Vilenkin group $x_n \dotplus y_n = (x_n + y_n) \mod p$ (that is, a 1 is not carried over to the next $(n+1)$th position), the corresponding p-adic summation has the property that the 1 occurring as a result of the addition of $x_n + y_n$ is carried over to the next $(n+1)$th position. We endow the group $\mathbb{Z}_p$ with the topology generated by the same system of subgroups G_n as for a Vilenkin group. Similarly, as a g_n, we may again take the same sequence.

Let X be the set of characters of a group $(G, \dot{+})$; it is itself a group with respect to multiplication. Also let $G_n^\perp = \{\chi \in X : \forall\, x \in G_n\ \chi(x) = 1\}$ be the annihilator of the group G_n. Each annihilator $G_n^\perp$ is a group with respect to multiplication, and the subgroups $G_n^\perp$ form an increasing sequence

$$\cdots \subset G_{-n}^\perp \subset \cdots \subset G_0^\perp \subset G_1^\perp \subset \cdots \subset G_n^\perp \subset \cdots \tag{2.3}$$

with

$$\bigcup_{n=-\infty}^{+\infty} G_n^\perp = X \quad \text{and} \quad \bigcap_{n=-\infty}^{+\infty} G_n^\perp = \{1\},$$

the quotient group $G_{n+1}^\perp / G_n^\perp$ having order p_n. The group of characters X may be equipped with the topology using the chain of subgroups (2.3), the family of cosets $G_n^\perp \cdot \chi$, $\chi \in X$, being taken as a base of the topology. The collection of such cosets, along with the empty set, forms the semiring $\mathscr{X}$. Given a coset $G_n^\perp \cdot \chi$, we define a measure ν on it by $\nu(G_n^\perp \cdot \chi) = \nu(G_n^\perp) = \mathfrak{m}_n$ (so that always $\mu(G_n)\nu(G_n^\perp) = 1$). The measure ν can be extended onto the σ-algebra of measurable sets in the standard way (for example, using Carathéodory's extension theorem). One then forms the absolutely convergent integral $\displaystyle\int_X F(\chi)\, d\nu(\chi)$ with respect to this measure.

The value $\chi(g)$ of the character χ at an element $g \in G$ will be denoted by (χ, g). The Fourier transform $\widehat{f}$ of an $f \in L_2(G)$ is defined as follows

$$\widehat{f}(\chi) = \int_G f(x)\overline{(\chi, x)}\, d\mu(x) = \lim_{n\to+\infty} \int_{G_{-n}} f(x)\overline{(\chi, x)}\, d\mu(x),$$

the limit being in the norm of $L_2(X)$. For $f \in L_2(G)$, Plancherel's formula is valid:

$$f(x) = \int_X \widehat{f}(\chi)(\chi, x)\, d\nu(\chi) = \lim_{n\to+\infty} \int_{G_n^\perp} \widehat{f}(\chi)(\chi, x)\, d\nu(\chi);$$

here the limit also signifies the convergence in the norm of $L_2(G)$.

Endowed with this topology, the group of characters X is a zero-dimensional locally compact group; there is, however, a dual situation: every element $x \in G$ is a character of the group X, and G_n is the annihilator of the group $G_n^\perp$. Below (Definition 2.1), we shall consider a dilation operator on a group G. However, we have been able to define such an operator only in the case when $p_n = p$ for any $n \in \mathbb{Z}$. Thus in what follows we shall only consider groups G for which $p_n = p$. The translation of the argument of a function f by an element $g \in G$ will be denoted by $f_{\dot{+}g}$; that is, $f_{\dot{+}g}(x) = f(x \dot{+} g)$. As regards the operation $\dot{+}$, we additionally assume that

$$pg_n = \alpha_1 g_{n+1} \dot{+} \alpha_2 g_{n+2} \dot{+} \cdots \dot{+} \alpha_s g_{n+s}; \tag{2.4}$$

here, $\alpha_1, \alpha_2, \dots, \alpha_s$ are fixed numbers. It is worth noting that if $pg_n = 0$, then G is a Vilenkin group, and if $pg_n = g_{n+1}$, then G is the p-adic number group.

Lemma 2.1. *If* $\varphi \in L_2(G)$, *then* $\widehat{\varphi}_{\dot{-}h}(\chi) = \overline{(\chi, h)}\widehat{\varphi}(\chi)$.

Proof. We obtain

$$\begin{aligned}\widehat{\varphi}_{\dot{-}h}(\chi) &= \int_G \varphi(x \dot{-} h)\overline{(\chi, x)}\, d\mu(x) = \int_G \varphi(x \dot{-} h)\overline{(\chi, x \dot{-} h \dot{+} h)}\, d\mu(x) \\ &= \overline{(\chi, h)} \int_G \varphi(x \dot{-} h)\overline{(\chi, x \dot{-} h)}\, d\mu(x) = \overline{(\chi, h)} \int_G \varphi(x)\overline{(\chi, x)}\, d\mu(x) \\ &= \overline{(\chi, x)}\widehat{\varphi}(\chi)\end{aligned}$$

using the properties of characters and by the invariance of the integral. The lemma herewith follows.

We set

$$H_n = \left\{ q \in G : \ q = \sum_{j=N}^{n-1} a_j g_j, \ N \in \mathbb{Z}, \ a_j = 0, \dots, p-1 \right\}.$$

If G is a Vilenkin group, then H_n is a group. This is not so in the general case (for example, if G is the group of p-adic numbers). However, it is worth noting that H_n is always a countable set.

Lemma 2.2. *Let* $g, h \in H_0$. *Then*

$$\int_{G_0^\perp} \overline{(\chi, g)}(\chi, h)\, d\nu(\chi) = \delta_{g,h} = \begin{cases} 1, & g = h, \\ 0, & g \neq h. \end{cases}$$

Proof. We look upon elements $x \in G$ as characters of the group X; let $\widetilde{x} = x\big|_{G_0^\perp}$ be the restrictions of these characters to the group $G_0^\perp$. Then

$$\widetilde{x} = a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} a_{-N}g_{-N} \in H_0$$

(here we take into account that $(G_0^\perp, g_k) = 1$ for $k \geqslant 0$). Hence H_0 is the group of characters of the compact group $G_0^\perp$, and so the elements of H_0 (or, more precisely, their restrictions to $G_0^\perp$) form an orthonormal system in $L_2(G_0^\perp)$.

Corollary 2.1. *The following equality holds*: $\int_{G_0^\perp} (\chi, x)\, d\nu(\chi) = \mathbf{1}_{G_0}(x)$.

Lemma 2.3. *Let* $\varphi \in L_2(G)$. *Suppose that* $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$. *Then the translations* $(\varphi(x \dot{-} h))_{h \in H_0}$ *form an orthonormal system in* $L_2(G)$.

Proof. Using Plancherel's formula and Lemma 2.1 gives

$$\begin{aligned}\int_G \varphi(x \dot{-} h)\overline{\varphi(x \dot{-} g)}\, d\mu(x) &= \int_X \widehat{\varphi}_{\dot{-}h}(\chi)\overline{\widehat{\varphi}_{\dot{-}g}(\chi)}\, d\nu(\chi) \\ &= \int_X \overline{(\chi, h)}\widehat{\varphi}(\chi)(\chi, g)\overline{\widehat{\varphi}(\chi)}\, d\nu(\chi) = \int_{G_0^\perp} \overline{(\chi, h)}(\chi, g)\, d\nu(\chi) = \delta_{h,g}.\end{aligned}$$

Definition 2.1. We define the map $A\colon G\to G$ by $Ax:=\sum_{n=-\infty}^{+\infty}a_ng_{n-1}$, where $x\in G$, $x=\sum_{n=-\infty}^{+\infty}a_ng_n$. As any element $x\in G$ can be uniquely expanded as $x=\sum a_ng_n$, the map $A\colon G\to G$ is one-to-one onto. It is called a *dilation operator* if $A(x\dot{+}y)=Ax\dot{+}Ay$ for all $x,y\in G$.

We note that if G is a Vilenkin group ($p\cdot g_n=0$) or is the group of all p-adic numbers ($p\cdot g_n=g_{n+1}$), then A is an additive operator and hence a dilation operator. Moreover, the operator A is additive if the condition (2.3) is satisfied. It is also clear that $AG_n=G_{n-1}$.

Lemma 2.4. *Let $f\in L(G)$ and let A be a dilation operator. Then*

$$\int_G f(Ax)\,d\mu(x)=\frac{1}{p}\int_G f(x)\,d\mu(x). \tag{2.5}$$

Proof. The equality (2.5) will plainly be true if

$$f(x)=\lambda\cdot\mathbf{1}_{G_n\dot{+}g}(x).$$

Therefore (2.5) holds for step functions, and therefore for an $f\in L(G)$. The proof of Lemma 2.4 is complete.

We use the notation $f_{A\,\cdot\,\dot{+}g}(x)=f(Ax\dot{+}g)$ by analogy with $f_{\dot{+}g}$.

Lemma 2.5. *Let $\varphi\in L_2(G)$ and let A be a dilation operator. Then*

$$\widehat{\varphi}_{A\,\cdot\,\dot{-}g}(\chi)=\overline{(\chi,A^{-1}g)}\widehat{\varphi}(\chi A^{-1}),$$

where χA^{-1} is the character defined by $\chi A^{-1}(x)=\chi(A^{-1}x)$.

Proof. Using Lemma 2.4 gives

$$\begin{aligned}\widehat{\varphi}_{A\,\cdot\,\dot{-}g}(\chi)&=\int_G\varphi(Ax\dot{-}g)\overline{(\chi,x)}\,d\mu(x)=\int_G\varphi(Ax\dot{-}g)\overline{(\chi,A^{-1}Ax)}\,d\mu(x)\\&=\frac{1}{p}\int_G\varphi(x\dot{-}g)\overline{(\chi,A^{-1}x)}\,d\mu(x)\\&=\frac{1}{p}\int_G\varphi(x\dot{-}g)\overline{(\chi,A^{-1}x\dot{-}A^{-1}g\dot{+}A^{-1}g)}\,d\mu(x)\\&=\frac{1}{p}\overline{(\chi,A^{-1}g)}\int_G\varphi(x\dot{-}g)\overline{(\chi A^{-1},x\dot{-}g)}\,d\mu(x)\\&=\frac{1}{p}\overline{(\chi,A^{-1}g)}\int_G\varphi(x)\overline{(\chi A^{-1},x)}\,d\mu(x)=\frac{1}{p}\overline{(\chi,A^{-1}g)}\widehat{\varphi}(\chi A^{-1}).\end{aligned}$$

§ 3. Multiresolution analysis on a locally compact zero-dimensional group

Our main objective is to construct orthogonal wavelet bases for $L_2(G)$. For this we shall use a multiresolution analysis on the group G as follows.

Definition 3.1. A family of closed subspaces V_n, $n \in \mathbb{Z}$, is said to be a multiresolution analysis (MRA) of $L_2(G)$ if the following axioms are satisfied:

1) $V_n \subset V_{n+1}$;
2) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L_2(G)$ and $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$;
3) $f(x) \in V_n \iff f(Ax) \in V_{n+1}$ (A is a dilation operator);
4) $f(x) \in V_0 \implies f(x \dot{-} h) \in V_0$ for all $h \in H_0$;
5) there exists a function $\varphi \in L_2(G)$ such that the system $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal basis for V_0.

The function φ occurring in Axiom 5) is called a *refinable function*.

Using an MRA, we shall build functions ψ_ν, $\nu = 1, \dots, p-1$, whose contractions and translations $\psi_\nu(A^j x \dot{-} h)$ form an orthogonal basis for $L_2(G)$. Next we will follow the conventional approach. Let $\varphi(x) \in L_2(G)$, and suppose that $(\varphi(x \dot{-} h))_{h\in H_0}$ is an orthonormal system in $L_2(G)$. For the function φ and the dilation operator A, we define the linear subspaces $L_j = (\operatorname{span} \varphi(A^j x \dot{-} h))_{h\in H_0}$ and closed subspaces $V_j = \overline{L_j}$. If the subspaces V_j form an MRA, then the function φ is said to *generate* an MRA of $L_2(G)$. We shall look for a function $\varphi \in L_2(G)$ that generates an MRA of $L_2(G)$ as a solution of the refinement equation

$$\varphi(x) = \sum_{h\in H} c_h \varphi(Ax \dot{-} h), \tag{3.1}$$

where $H \subset H_0$. We shall assume straight away that $H \subset H_0$ is a finite set since the resulting efficient algorithm for constructing wavelet bases will apply only to finite sets H. If G is a Vilenkin group, then H_0 is a group, and so Axioms 1) and 4) are automatically satisfied. In the general case, H_0 is not a group since $pg_n \neq 0$; hence additional conditions are required for Axioms 1) and 4) to hold. In [13], for the case of the p-adic number group, it was proposed to use the condition that $\operatorname{supp} \widehat{\varphi}(\chi) \subset G_0^\perp$. If the more stringent condition $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$ is imposed, then by Lemma 2.3, the system of translations $(\varphi(x \dot{-} h))_{h\in H_0}$ is orthonormal. Therefore we shall look for a function φ with $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$.

Lemma 3.1. *If* $\operatorname{supp} \widehat{\varphi} \subset G_n^\perp$, *then the function* φ *is periodic with period* g_n.

Proof. For the Fourier transform, we have

$$\begin{aligned}\widehat{\varphi}_{\dot{+}g_n}(\chi) &= \int_G \varphi_{\dot{+}g_n}(x)\overline{(\chi, x)}\, d\mu(x) = \int_G \varphi(x \dot{+} g_n)\overline{(\chi, x)}\, d\mu(x)\\ &= \overline{(\chi, \dot{-}g_n)} \int_G \varphi(x \dot{+} g_n)\overline{(\chi, x \dot{+} g_n)}\, d\mu(x) = \overline{(\chi, \dot{-}g_n)}\widehat{\varphi}(\chi).\end{aligned}$$

1) If $\chi \notin \operatorname{supp} \widehat{\varphi}$, then $\widehat{\varphi}(\chi) = 0$. Hence $\widehat{\varphi}_{\dot{+}g_n}(x) = \widehat{\varphi}(\chi) = 0$.
2) If $\chi \in \operatorname{supp} \widehat{\varphi} \subset G_n^\perp$, then $\chi(g_n) = 1$. It follows that $\chi(\dot{-}g_n) = 1$, and so, $\widehat{\varphi}_{\dot{+}g_n}(\chi) = \widehat{\varphi}(\chi)$.

Thus, $\widehat{\varphi}_{\dot{+}g_n}(\chi) = \widehat{\varphi}(\chi)$ for all $\chi \in G$. By Plancherel's theorem,

$$\varphi_{\dot{+}g_n}(x) = \int_X \widehat{\varphi}_{\dot{+}g_n}(\chi)(\chi, x)\, d\mu(\chi) = \int_X \widehat{\varphi}(\chi)(\chi, x)\, d\nu(\chi) = \varphi(x).$$

Consequently, $\varphi(x \dot{+} g_n) = \varphi(x)$.

Corollary 3.1. *If* $\operatorname{supp}\widehat{\varphi} \subset G_n^{\perp}$, *then* φ *is periodic with any period* g_s, $s \geqslant n$.

Corollary 3.2. *Suppose that* $\operatorname{supp}\widehat{\varphi}\subset G_n^{\perp}$ *and that* φ *is continuous. Then* $\varphi(x) = \mathrm{const}$ *on each coset* $G_n \dot{+} q$.

Lemma 3.2. *If* $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^{\perp}}(\chi)$, *then the system*

$$(p^{n/2}\varphi(A^n x \dot{-} h))_{h\in H_0} \tag{3.2}$$

is an orthonormal basis for V_n.

Proof. It suffices to show that the system (3.2) is orthonormal. By Lemma 2.3, the system $\varphi(x \dot{-} h)$ is an orthonormal basis for V_0, and hence

$$\int_G \varphi(x \dot{-} h)\overline{\varphi}(x \dot{-} q)\, d\mu(x) = \delta_{h,q}, \qquad h, q \in H_0.$$

So by Lemma 2.4,

$$\int_G \varphi(A^n x \dot{-} h)\overline{\varphi}(A^n x \dot{-} q)\, d\mu(x) = \frac{1}{p^n}\int_G \varphi(x \dot{-} h)\overline{\varphi}(x \dot{-} q)\, d\mu(x) = \frac{1}{p^n}\,\delta_{hq}.$$

Lemma 3.3. *Suppose that* $\operatorname{supp}\widehat{\varphi} \subset G_0^{\perp}$ *and* $\varphi(x)$ *satisfies equation* (3.1). *Then* $V_j \subset V_{j+1}$ *for any* $j \in \mathbb{Z}$.

Proof. By the definition of φ,

$$\varphi(x \dot{-} q_0) = \sum_{h\in H} c_h\varphi(A(x \dot{-} q_0) \dot{-} h)$$

for any $q_0 \in H_0$. Since $pg_n = \alpha_1 g_{n+1} \dot{+} \cdots \dot{+} \alpha_s g_{n+s}$, it follows that

$$\begin{aligned} Aq_0 \dot{+} h &= b_{-2}g_{-2} \dot{+} b_{-3}g_{-3} \dot{+} \cdots \dot{+} b_{-M}g_{-M} \dot{+} a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} b_{-N}g_{-N} \\ &= \beta_{s-1}g_{s-1} \dot{+} \beta_{s-2}g_{s-2} \dot{+} \cdots \dot{+} \beta_0 g_0 \dot{+} \cdots \dot{+} \beta_{-m}g_{-m}, \\ &\qquad m = \max(M, N), \qquad p_j = 0, \ldots, p-1. \end{aligned}$$

By Corollary 3.1, $\varphi(x \dot{+} g_j) = \varphi(x)$ for all $j \geqslant 0$. Hence

$$\varphi(Ax \dot{-} Aq_0 \dot{-} h) = \varphi(Ax \dot{-} (\beta_{-1}g_{-1} \dot{+} \beta_{-2}g_{-2} \dot{+} \cdots \dot{+} \beta_{-m}g_{-m})) = \varphi(Ax \dot{-} q), \quad q \in H_0.$$

Consequently,

$$\varphi(x \dot{-} q_0) = \sum_{h\in H} c_h\varphi(Ax \dot{-} q_h), \qquad q_h \in H_0.$$

This means that $L_0 \subset L_1$, and hence $V_0 \subset V_1$. Then it is clear that $V_j \subset V_{j+1}$ for any $j \in \mathbb{Z}$.

Lemma 3.4. *Let* $\varphi \in L_2(G)$ *be a solution of the equation* (3.1). *Suppose that* $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^{\perp}}(\chi)$. *Then* $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(G)$ *if and only if*

$$\bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}(\,\cdot\, A^{-j}) = X.$$

Proof. 1) We claim that $\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ is translation invariant. First, we prove that V_j is invariant under translations by $h\in H_j$. In fact, let $f\in L_j$. Then

$$f(x)=\sum_{h_j\in H_j}\beta_{h_j}\varphi(A^j(x\dot{-}h_j)),$$

and hence, for $h\in H_j$,

$$f(x\dot{-}h)=\sum_{h_j\in H_j}\beta_{h_j}\varphi(A^jx\dot{-}A^jh_j\dot{-}A^jh).$$

Since $A^jh, A^jh_j\in H_0$, it follows that

$$\begin{aligned}A^jh&=a_{-1}g_{-1}\dot{+}a_{-2}g_{-2}\dot{+}\cdots\dot{+}a_{-N}g_{-N},\\ A^jh_j&=b_{-1}g_{-1}\dot{+}b_{-2}g_{-2}\dot{+}\cdots\dot{+}b_{-N}g_{-N}.\end{aligned}$$

Hence $A^jh_j\dot{+}A^jh\in H_{s-1}$, and since φ is periodic with any period g_k, $k\geqslant 0$, we have $f(x\dot{-}h)\in L_j$. Now let $f\in V_j=\overline{L_j}$. Then there exists a sequence (f_n) such that $f_n\in L_j$ and $\|f_n-f\|_2\to 0$. The subspace L_j being invariant under translations by $h_n\in H_j$ implies that $f_n(\,\cdot\,\dot{+}h)\in L_j$. Hence

$$\int_G|f(x\dot{+}h)-f_n(x\dot{+}h)|^2\,d\mu(x)=\int_G|f(x)-f_n(x)|^2\,d\mu(x)=\|f-f_n\|_2^2\to 0$$

by the invariance of the integral. This means that $f(\,\cdot\,\dot{+}h)\in V_j$. We have thus proved that V_j is invariant under translations by $h\in H_j$.

We now proceed to prove that $\overline{\bigcup_{j\in\mathbb{Z}} V_j}$ is invariant under any translations. First suppose that $f\in\bigcup_{j\in\mathbb{Z}} V_j$. Since $V_j\subset V_{j+1}$, there exists a $j_1\in\mathbb{Z}$ such that $f\in V_j$ for $j\geqslant j_1$. By the above, $f(\,\cdot\,\dot{+}h_j)\in V_j$ for $h_j\in H_j$, $j\geqslant j_1$. Given an arbitrary $h\in G$, we have $h=\sum_{l=-k}^{+\infty}a_lg_l$. Consider the sequence h_j defined by

$$h_j=a_{-k}g_{-k}\dot{+}a_{-k+1}g_{-k+1}\dot{+}\cdots\dot{+}a_{j-1}g_{j-1}.$$

It is clear that $h_j\to h$ as $j\to+\infty$. Since $h_j\in H_j$, it follows that $f(x\dot{+}h_j)\in V_j$ for all $j\geqslant j_1$, and hence $f(x\dot{+}h_j)\in\bigcup_{\nu\in\mathbb{Z}} V_\nu$ for all $j\geqslant j_1$. Further, since $f\in L_2(G)$,

$$\|f(\,\cdot\,\dot{+}h)-f(\,\cdot\,\dot{+}h_j)\|_2\to 0;$$

that is, $f(\,\cdot\,\dot{+}h)\in\overline{\bigcup_{j\in\mathbb{Z}} V_j}$.

Now suppose that $f\in\overline{\bigcup_{j\in\mathbb{Z}} V_j}$. Then there exists a sequence $f_n\in\bigcup_{j\in\mathbb{Z}} V_j$ such that $\|f-f_n\|_2\to 0$. By what has just been proved, $f_n(\,\cdot\,\dot{+}h)\in\bigcup_{j\in\mathbb{Z}} V_j$. Therefore,

$$\|f(\,\cdot\,\dot{+}h)-f_n(\,\cdot\,\dot{+}h)\|_2=\|f-f_n\|_2\to 0;$$

that is, $f(\,\cdot\,\dot{+}h)\in\bigcup_{j\in\mathbb{Z}} V_j$.

2) We now proceed to prove the assertion of the lemma. We set $Y=\overline{\bigcup_{j\in\mathbb{Z}} V_j}$. Since Y is invariant under any translations by $h\in G$, an application of Wiener's theorem shows that $\widehat{Y}=L_2(X_1)$, where $X_1\subset X$. Also since

$$Y=L_2(G)\quad\Longleftrightarrow\quad\widehat{Y}=L_2(X),$$

we have

$$Y = L_2(G) \iff X = X_1$$

modulo null sets. Therefore it suffices to show that

$$X = X_1 \iff \bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}(\,\cdot\, A^{-j}) = X.$$

We set $\varphi_j(x) = \varphi(A^j x)$, $X_0 = \bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}_j$; we claim that $X_0 = X_1$ modulo null sets. Since $\varphi_j \in V_j$, we have $\operatorname{supp}\widehat{\varphi}_j \subset X_1$ because $\varphi_j \in Y$ and $\widehat{Y}$ consists of functions defined on X_1. Hence

$$X_0 = \bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}_j \subset X_1.$$

It will be shown that $X_1 \setminus X_0$ is a null set. Given an $f \in V_j$, we have

$$f = \lim f_n, \qquad f_n = \sum_{h\in H_0} d_h \varphi(A^j x \dot{-} h),$$

where f_n is a finite sum. Hence, for all $\chi \in X_1 \setminus X_0$,

$$\begin{aligned}\widehat{f}_n(\chi) &= \sum_{h\in H_0} d_n \int_G \varphi(A^j x \dot{-} h)\overline{(\chi, x)}\, d\mu(x)\\ &= \sum_{h\in H_0} d_n \overline{(\chi, A^{-j}h)} \int_G \varphi(A^j(x \dot{-} A^{-j}h))\overline{(\chi, x \dot{-} A^{-j}h)}\, d\mu(x)\\ &= \sum_{h\in H_0} d_n \overline{(\chi, A^{-j}h)} \int_G \varphi(A^j x)\overline{(\chi, x)}\, d\mu(x) = 0;\end{aligned}$$

that is, $\widehat{f}_n(\chi) = 0$ for all $\chi \in X_1 \setminus X_0$. As a result, $\widehat{f}(\chi) = 0$ a.e. on $X_1 \setminus X_0$ for $f \in V_j$ and so throughout $f \in Y$. This means that $L_2(X_0) = L_2(X_1)$, and hence $\nu(X_1 \setminus X_0) = 0$. But $\operatorname{supp}\widehat{\varphi}_j = \operatorname{supp}\widehat{\varphi}(\,\cdot\, A^{-j})$, so it follows that

$$\bigcup_{j\in\mathbb{Z}} \operatorname{supp}\overline{\varphi}(\,\cdot\, A^{-j}) = \bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}_j = X_0.$$

Since $X \supset X_1 \supset X_0$, it is found that $X = X_1$ modulo null sets. This means that $X_0 = X$ modulo null sets. This is equivalent to saying that $\bigcup_{j\in\mathbb{Z}} \operatorname{supp}\widehat{\varphi}(\,\cdot\, A^{-j}) = X$ modulo null sets. The proof of Lemma 3.4 is complete.

Remark. See [19; Appendix A8] for the formulation and proof of Wiener's theorem for $L_2(\mathbb{R}^m)$. Only slight modifications are required for a zero-dimensional group.

Corollary 3.1. *Let $\varphi \in L_2(G)$ be a solution of the equation (3.1). Suppose that $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$. Then $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L_2(G)$.*

Proof. The condition $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$ implies that $\operatorname{supp}\widehat{\varphi}(\,\cdot\, A^{-j}) = G_j^\perp$, and now, since

$$\bigcup_{j\in\mathbb{Z}} G_j^\perp = X,$$

it remains to apply Lemma 3.4.

Lemma 3.5. *Let $\varphi \in L_2(G)$ and let $(\varphi(x \dot{-} h))_{h\in H_0}$ be an orthonormal system. Then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.*

Proof. Suppose that $f \in L_2(G)$. Then, for $j > 0$, using the equality

$$\int_G f(A\,\cdot\,)\,d\mu = \frac{1}{p}\int_G f\,d\mu$$

and taking into account the orthonormality of $(\varphi(x \dot{-} h))_{h\in H_0}$, gives

$$\frac{1}{p^j}\sum_{h\in H_0}\biggl|\int_G f(x)\overline{\varphi(A^{-j}x \dot{-} h)}\,d\mu(x)\biggr|^2 = \sum_{h\in H_0}\biggl|\int_G f(A^jx)\overline{\varphi(x \dot{-} h)}\,d\mu(x)\biggr|^2$$
$$\leqslant \|f(A^j\,\cdot\,)\|_2^2 = \int_G |f(A^jx)|^2\,d\mu(x) = \frac{1}{p^j}\int_G |f(x)|^2\,d\mu(x) \to 0 \quad \text{as } j\to+\infty. \tag{3.3}$$

The system $(p^{j/2}\varphi(A^jx \dot{-} h))_{h\in H_0}$ being an orthonormal basis for V_j implies that, for $f \in V_j$,

$$\|f\|_2 = \biggl(\sum_{h\in H_0} |(f, p^{j/2}\varphi(A^j\,\cdot\,\dot{-}h))|^2\biggr)^{1/2}. \tag{3.4}$$

Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Then, from (3.3) and (3.4) it follows that $\|f\|_2 = 0$; that is, $f = 0$ a.e.

Lemma 3.6. *Suppose that* $\operatorname{supp}\widehat{\varphi} \subset G_0^{\perp}$. *Then the condition $f \in V_0$ implies that $f(\,\cdot\,\dot{-} y) \in V_0$ for $g \in H_0$; that is, Axiom* 4) *holds.*

Proof. Let $g, h \in H_0$ and let

$$g = a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} a_{-N}g_{-N}, \qquad h = b_{-1}g_{-1} \dot{+} b_{-2}g_{-2} \dot{+} \cdots \dot{+} b_{-N}g_{-N}.$$

Then

$$g \dot{+} h = (\widetilde{a}_sg_s \dot{+} \widetilde{a}_{s-1}g_{s-1} \dot{+} \cdots \dot{+} \widetilde{a}_0g_0) \dot{+} (\widetilde{a}_{-1}g_{-1} \dot{+} \widetilde{a}_{-2}g_{-2} \dot{+} \cdots \dot{+} \widetilde{a}_{-N}g_{-N}) = \widetilde{g} \dot{+} \widetilde{h},$$

where $\widetilde{h} \in H_0$.

Since $\operatorname{supp}\widehat{\varphi} \subset G_0^{\perp}$, it follows that φ is periodic with any period g_j, $j \geqslant 0$. Therefore, taking into consideration that $\widetilde{h} \in H_0$, we obtain

$$\sum_{h\in H_0} c_h\varphi(x \dot{-} g \dot{-} h) = \sum_{\widetilde{h}\in H_0} c_{\widetilde{h}}\varphi(x \dot{-} \widetilde{h}),$$

and so Axiom 4) holds.

Lemma 3.7. *Let $\varphi \in L_2(G)$. Then $f \in V_n$ if and only if $f(Ax) \in V_{n+1}$.*

This immediately follows from the equality

$$\int_G \biggl|f(Ax) - \sum_{h\in H_0} c_h\varphi(A^nAx \dot{-} h)\biggr|^2\,d\mu(x)$$
$$= \frac{1}{p}\int_G \biggl|f(x) - \sum_{h\in H_0} c_h\varphi(A^nx \dot{-} h)\biggr|^2\,d\mu(x).$$

Combining Lemmas 3.3–3.7 gives us the following result.

Theorem 3.1. *Let* $\varphi \in L_2(G)$ *be a solution of equation* (3.1). *Suppose that* $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$. *Then* φ *generates an MRA of* $L_2(G)$.

§ 4. Refinement equation and its solutions

Given a fixed $s \geqslant 1$, we set

$$H_0^{(s)} = \{a_{-1}g_{-1} \dotplus a_{-2}g_{-2} \dotplus \dots \dotplus a_{-s}g_{-s} : a_j = 0, \dots, p-1\},$$

and consider the following refinement equation

$$\varphi(x) = \sum_{h \in H_0^{(s)}} \beta_h \varphi(Ax \dot{-} h). \tag{4.1}$$

From this equation for the Fourier transform $\widehat{\varphi}$ we have

$$\begin{aligned}\widehat{\varphi}(\chi) &= \sum_{h \in H_0^{(s)}} \beta_h \int_G \varphi(Ax \dot{-} AA^{-1}h)\overline{(\chi, x)}\, d\mu(x) \\ &= \sum_{h \in H_0^{(s)}} \beta_h \overline{(\chi, A^{-1}h)} \int_G \varphi(Ax)\overline{(\chi, x)}\, d\mu(x) \\ &= \sum_{h \in H_0^{(s)}} \beta_h \overline{(\chi, A^{-1}h)} \int_G \varphi(Ax)\overline{(\chi, A^{-1}Ax)}\, d\mu(x) \\ &= \frac{1}{p} \sum_{h \in H_0^{(s)}} \beta_h \overline{(\chi, A^{-1}h)} \int_G \varphi(x)\overline{(\chi, A^{-1}x)}\, d\mu(x) = m_0(\chi)\widehat{\varphi}(\chi A^{-1}),\end{aligned}$$

where

$$m_0(\chi) = \frac{1}{p} \sum_{h \in H_0^{(s)}} \beta_h \overline{(\chi, A^{-1}h)}. \tag{4.2}$$

Hence the refinement equation can be written in terms of the Fourier transform as follows:

$$\widehat{\varphi}(\chi) = m_0(\chi)\widehat{\varphi}(\chi A^{-1}). \tag{4.3}$$

The function $m_0(\chi)$ is called a *mask* for equation (4.1). Knowing a mask, it is possible to recover the Fourier transform $\widehat{\varphi}$ from the value of $\widehat{\varphi}(1)$. Another name for a 'mask' is a 'symbol'.

Lemma 4.1. *Let* $\varphi \in L_2(G)$ *be a solution of the refinement equation* (4.1). *Suppose that* $\widehat{\varphi}(\chi)$ *is continuous at the point* $\chi_0 \equiv 1$ *and that* $\widehat{\varphi}(1) \neq 0$. *Then*

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(1) \prod_{k=0}^{\infty} m_0(\chi A^{-k}). \tag{4.4}$$

Proof. By repeated application of the equality

$$\widehat{\varphi}(\chi) = m_0(\chi)\widehat{\varphi}(\chi A^{-1}),$$

we have, for a positive integer N,

$$\widehat{\varphi}(\chi) = \prod_{k=0}^{N} m_0(\chi A^{-k})\widehat{\varphi}(\chi A^{-N-1}).$$

The result stated now follows since $\chi A^{-N-1} \to \chi_0 \equiv 1$ as $N \to +\infty$ and $\widehat{\varphi}$ is continuous.

The converse is also true.

Lemma 4.2. *Let $\widehat{\varphi}$ be given by (4.4), and let $m_0(1) = 1$. Then*

$$\widehat{\varphi}(\chi) = m_0(\chi)\widehat{\varphi}(\chi A^{-1}).$$

Also, if $\chi \in G^{\perp}_{-s+l+1}$, $l \geqslant 0$, then

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(1) \prod_{j=0}^{l-1} m_0(\chi A^{-j}); \tag{4.5}$$

in particular, for $l = 0$ (that is, when $\chi \in G^{\perp}_{-s+1}$) we have $\widehat{\varphi}(\chi) = 1$.

Proof. Putting χA^{-1} for χ in (4.4) we obtain

$$\widehat{\varphi}(\chi A^{-1}) = \widehat{\varphi}(1) \prod_{k=0}^{\infty} m_0(\chi A^{-k-1}).$$

Multiplying both sides by $m_0(\chi)$ and taking into account (4.4) gives (4.3).

Let $\chi \in G^{\perp}_{-s+l+1}$, $l \geqslant 0$. We need to verify equality (4.5). Since $\chi \in G^{\perp}_{-s+l+1}$, we have $\chi(G_{-s+l+1}) = 1$. In view of $A^{-1}(G_n) = G_{n+1}$, this gives $\chi A^{-l-1}(G_{-s}) = 1$. If $h \in H_0^{(s)} \subset G_{-s}$, then

$$(\chi A^{-l}, A^{-1}h) = (\chi A^{-l-1}, h) = 1,$$

and so

$$m_0(\chi A^{-l}) = \frac{1}{p} \sum_{h \in H_0^{(s)}} \beta_h \overline{(\chi A^{-l}, A^{-1}h)} = \frac{1}{p} \sum_{h \in H_0^{(s)}} \beta_h = m_0(1) = 1.$$

But the condition $\chi \in G^{\perp}_{-s+l+1}$ implies that $\chi \in G^{\perp}_{-s+j+1}$ for $j \geqslant l$, whence $m_0(\chi A^{-j}) = 1$ for $j \geqslant l$. Hence,

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(1) \prod_{j=0}^{l-1} m_0(\chi A^{-j}) \prod_{j=l}^{\infty} m_0(\chi A^{-j}) = \widehat{\varphi}(1) \prod_{j=0}^{l-1} m_0(\chi A^{-j})$$

if $\chi \in G^{\perp}_{-s+l+1}$ and $l \geqslant 0$. For $l = 0$, this entails $\widehat{\varphi}(\chi) = \widehat{\varphi}(1)$ for $\chi \in G^{\perp}_{-s+1}$.

Lemma 4.3. *Assume that the hypotheses of Lemma 4.2 are satisfied by $\widehat{\varphi}$. Then $\widehat{\varphi}$ is constant on each of the cosets of the subgroup $G^{\perp}_{-s+1}$.*

Proof. Let $\chi \in G^{\perp}_{-s+l+1}$, $l \geqslant 1$. We write $G^{\perp}_{-s+l+1}$ as a union of cosets:

$$G^{\perp}_{-s+l+1} = \bigsqcup_{\alpha_{-s+1}=0}^{p-1} \cdots \bigsqcup_{\alpha_{-s+l}=0}^{p-1} G^{\perp}_{-s+l} r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-s+l}^{\alpha_{-s+l}}.$$

By Lemma 4.2, the following holds on each of the cosets:

$$\widehat{\varphi}(G^{\perp}_{-s+1} r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-s+l}^{\alpha_{-s+l}}) = \widehat{\varphi}(1) \prod_{j=0}^{l-1} m_0(G^{\perp}_{-s+1} r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-s+l}^{\alpha_{-s+l}} A^{-j})$$

$$= \widehat{\varphi}(1) \frac{1}{p^l} \prod_{j=0}^{l-1} \sum_{h \in H_0^{(s)}} \beta_h \overline{(G^{\perp}_{-s+1} r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-s+l}^{\alpha_{-s+l}}, A^{-j-1}h)}.$$

Since $h \in G_{-s}$, we have $A^{-j-1}h \in G_{-s+j+1}$, and hence $\overline{(G^{\perp}_{-s+1}, A^{-j-1}h)} = 1$. This gives

$$\widehat{\varphi}(G^{\perp}_{-s+1} r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-s+l}^{\alpha_{-s+l}}) = \widehat{\varphi}(1) \frac{1}{p^l} \prod_{j=0}^{l-1} \sum_{h \in H_0^{(s)}} \beta_h \overline{(r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-s+l}^{\alpha_{-s+l}}, A^{-j-1}h)} = \mathrm{const}.$$

Theorem 4.1. *Let $\widehat{\varphi}$ be given by (4.4), and let $\widehat{\varphi}(1) = 1$. Suppose that $|m_0(G_0^{\perp})| = 1$ and $m_0(G_1^{\perp} \setminus G_0^{\perp}) = 0$. Then $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^{\perp}}(\chi)$.*

Proof. 1) By Lemma 4.2,

$$\widehat{\varphi}(\chi) = m_0(\chi)\widehat{\varphi}(\chi A^{-1}).$$

If $m_0(G_1^{\perp} \setminus G_0^{\perp}) = 0$, we find immediately that $\widehat{\varphi}(G_1^{\perp} \setminus G_0^{\perp}) = 0$. If $\chi \in G_2^{\perp} \setminus G_1^{\perp}$, then $\chi A^{-1} \in G_1^{\perp} \setminus G_0^{\perp}$, and so $\widehat{\varphi}(\chi A^{-1}) = 0$, implying $\widehat{\varphi}(\chi) = 0$. Proceeding by induction, we obtain $\operatorname{supp} \widehat{\varphi} \subset G_0^{\perp}$.

2) Suppose that $|m_0(G_0^{\perp})| = 1$. By Lemma 4.2,

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(1) \prod_{j=0}^{l-1} m_0(\chi A^{-j}),$$

provided that $\chi \in G^{\perp}_{-s+l+1}$. We put $l = s-1$. Then $\chi \in G_0^{\perp}$, and so

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(1) \prod_{j=0}^{s-2} m_0(\chi A^{-j}).$$

Since $\chi \in G_0^{\perp}$, we have $\chi A^{-j} \in G^{\perp}_{-j} \subset G_0^{\perp}$, and hence $|\widehat{\varphi}(\chi)| = |\widehat{\varphi}(1)| = 1$.

Theorem 4.2. *Let $\widehat{\varphi}$ be given by (4.4), and let $\widehat{\varphi}(1) = 1$. Suppose that $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^{\perp}}(\chi)$. Then*

$$m_0(G_1^{\perp} \setminus G_0^{\perp}) = 0, \qquad |m_0(G_0^{\perp})| = 1.$$

Proof. First of all we note that it follows from the condition $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^\perp}(\chi)$ that $\widehat{\varphi}$ is the Fourier transform of some function $\varphi \in L_2(G)$. Further, since

$$\widehat{\varphi}_{\dot{-}h}(\chi) = \overline{(\chi, h)}\widehat{\varphi}(\chi),$$

we have

$$\int_G \varphi(x)\overline{\varphi(x \dot{-} h)}\,d\mu(x) = \int_X \widehat{\varphi}(\chi)\overline{\widehat{\varphi}_{\dot{-}h}(\chi)}\,d\nu(\chi)$$
$$= \int_X |\widehat{\varphi}(\chi)|^2(\chi, h)\,d\nu(\chi) = \int_{G_0^\perp} |\widehat{\varphi}(\chi)|^2(\chi, h)\,d\nu(\chi).$$

Using Lemma 4.3 with $l = s-1$ this gives

$$\int_G \varphi(x)\overline{\varphi(x \dot{-} h)}\,d\mu(x) = \sum_{\alpha_{-s+1}=0}^{p-1} \cdots \sum_{\alpha_{-1}=0}^{p-1} \int_{G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}}\cdots r_{-1}^{\alpha_{-1}}} |\widehat{\varphi}(\chi)|^2(\chi, h)\,d\nu(\chi)$$
$$= \sum_{\alpha_{-1}=0}^{p-1} \cdots \sum_{\alpha_{-s+1}=0}^{p-1} |\widehat{\varphi}(G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha_{-1}})|^2 \int_{G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}}\cdots r_{-1}^{\alpha_{-1}}} (\chi, h)\,d\nu(\chi).$$

If $\chi \in G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha_{-1}}$ and $h \in H_0^{(s-1)}$, then $(\chi, h) = (r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha-1}, h)$. Hence

$$\int_G \varphi(x)\overline{\varphi(x \dot{-} h)}\,d\mu(x)$$
$$= \sum_{\alpha_{-1}=0}^{p-1} \cdots \sum_{\alpha_{-s+1}=0}^{p-1} |\widehat{\varphi}(G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha_{-1}})|^2 \frac{1}{p^{s-1}} (r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha_{-1}}, h). \tag{4.6}$$

Consider (4.6) with

$$h \in H_0^{(s-1)} = \{a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} a_{-s+1}g_{-s+1}\}.$$

In this case relations (4.6) may be looked upon as a system of linear equations in the unknowns $|\widehat{\varphi}(G_{-s+1}^\perp r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha_{-1}})|^2$ with coefficients

$$b_{j,k} = (r_{-s+1}^{\alpha_{-s+1}} \cdots r_{-1}^{\alpha_{-1}}, a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} a_{-s+1}g_{-s+1}).$$

This is a system of order p^{s-1}. We claim that $\det(b_{j,k}) \neq 0$.

Consider the characters

$$\chi_k = r_{-s+1}^{\alpha_{-s+1}} r_{-s+2}^{\alpha_{-s+2}} \cdots r_{-2}^{\alpha_{-2}} r_{-1}^{\alpha_{-1}}$$

of the group G_{-s+1}. On each coset $G_0 \dot{+} h_j$ (where $h_j = a_{-1}g_{-1} \dot{+} a_{-2}g_{-2} \dot{+} \cdots \dot{+} a_{-s+1}g_{-s+1}$), whose union is G_{-s+1}, the character χ_k remains constant and equal to $(r_{-s+1}^{\alpha_{-s+1}} r_{-s+2}^{\alpha_{-s+2}} \cdots r_{-2}^{\alpha_{-2}} r_{-1}^{\alpha_{-1}}, h_j)$. Since the characters χ_k are orthogonal on the group G_{-s+1}, it follows that the columns of the matrix $(b_{j,k}) = (\chi_k, h_j)_{j,k=0}^{p^{s-1}-1}$ are orthogonal. This forces the matrix $\left(\dfrac{1}{\sqrt{p^{s-1}}}\,b_{j,k}\right)$ to be unitary. Consequently, system (4.6) has a unique solution, which is

$$|\widehat{\varphi}(G^{\perp}_{-s+1}r^{\alpha_{-s+1}}_{-s+1}\cdots r^{\alpha_{-1}}_{-1})|^2=1.$$

We claim that $|m_0(G^{\perp}_0)|=1$. We write the equality (4.5) of Lemma 4.2 for $l=1$ as follows:

$$\widehat{\varphi}(\chi)=m_0(\chi),\qquad \chi\in G^{\perp}_{-s+2},\quad -s+2\leqslant 0.$$

Since $|\widehat{\varphi}(G^{\perp}_0)|=1$ and $G^{\perp}_{-s+2}\subset G^{\perp}_0$, it follows that $|m_0(G^{\perp}_{-s+2})|=1$. Assuming that $-s+3\leqslant 0$, we write equality (4.5) for $l=2$ as follows:

$$\widehat{\varphi}(\chi)=m_0(\chi)m_0(\chi A^{-1}),\qquad \chi\in G^{\perp}_{-s+3}. \tag{4.7}$$

If $\chi\in G^{\perp}_{-s+3}\setminus G^{\perp}_{-s+2}$, then $\chi A^{-1}\in G^{\perp}_{-s+2}$, and hence $|m_0(\chi A^{-1})|=1$. Now since $|\widehat{\varphi}(\chi)|=1$ for $\chi\in G^{\perp}_{-s+3}$, the equality (4.7) yields $|m_0(G^{\perp}_{-s+3}\setminus G^{\perp}_{-s+2})|=1$. Hence $|m_0(G^{\perp}_{-s+3})|=1$. A continuation of this process, with the use of equality (4.5) with $l=\overline{3,s-1}$, shows that $|m_0(G^{\perp}_0)|=1$. Writing (4.5) with $l=s$, and taking into account the inclusion $\operatorname{supp}\widehat{\varphi}\subset G^{\perp}_0$, it is found that $m_0(G^{\perp}_1\setminus G^{\perp}_0)=0$.

This completes the proof of Theorem 4.2.

§ 5. Construction of wavelet bases

As usual, W_n stands for the orthogonal complement of V_n in V_{n+1}; that is, $V_{n+1}=V_n\otimes W_n$ and $V_n\perp W_n$ ($n\in\mathbb{Z}$, and $\otimes$ denotes the direct sum).

It is readily seen that

1) $f\in W_n \iff f(Ax)\in W_{n+1}$;
2) $W_n\perp W_k$ for $k\neq n$;
3) $\otimes W_n=L_2(G)$, $n\in\mathbb{Z}$.

From Lemmas 4.1–4.3 and Theorems 4.1, 4.2 we derive an algorithm for constructing wavelet bases.

Step 1. Build a function $m_0(\chi)$ which is constant on the cosets of $G^{\perp}_{-s+1}$ as follows. We write the equality

$$m_0(\chi)=\frac{1}{p}\sum_{h\in H_0^{(s)}}\beta_h\overline{(\chi,A^{-1}h)} \tag{5.1}$$

as a system of linear equations. To do so, we assign the number

$$j=a_{-1}+a_{-2}p+\cdots+a_{-s}p^{s-1},\qquad 0\leqslant j\leqslant p^s-1,$$

to the element

$$h=a_{-1}g_{-1}\dotplus a_{-2}g_{-2}\dotplus\cdots\dotplus a_{-s}g_{-s}\in H_0^{(s)}.$$

Since the function $m_0(\chi)$ is constant on any coset of the subgroup $G^{\perp}_{-s+1}$, that is, on the sets

$$G^{\perp}_{-s+1}r^{\alpha_{-s+1}}_{-s+1}r^{\alpha_{-s+2}}_{-s+2}\cdots r^{\alpha_{-1}}_{-1}r^{\alpha_0}_0\subset G^{\perp}_1,$$

we can pick a character χ_k in each of these cosets, where

$$k = \alpha_{-s+1} + \alpha_{-s+2}p + \cdots + \alpha_{-1}p^{s-2} + \alpha_0 p^{s-1}.$$

Then (5.1) can be written as the system

$$m_0(\chi_k) = \frac{1}{p}\sum_{j=0}^{p^s-1} \beta_j \overline{(\chi_k, A^{-1}h_j)}, \qquad k = 0, \dots, p^s - 1, \tag{5.2}$$

in the unknowns β_j. We consider the characters χ_k on the subgroup G_{-s+1}, on which they are orthogonal. Since $A^{-1}h_j$ all lie in G_{-s+1}, it follows that the matrix $p^{-s/2}(\chi_k, A^{-1}h_j)$ is unitary, and so the system (5.2) has a unique solution for each finite sequence $(m_0(\chi_k))_{k=0}^{p^s-1}$.

Step 2. Take $m_0(\chi_k)$ so that $|m_0(\chi_k)| = 1$ for $k = 0, \dots, p^{s-1} - 1$ and $m_0(\chi_k) = 0$ for $k = p^{s-1}, \dots, p^s - 1$. Solving the system (5.2) yields β_j. Thus, we have constructed a function $m_0(\chi)$ that is constant on cosets to $G_{-s+1}^{\perp}$ and such that $|m_0(G_0^{\perp})| = 1$ and $m_0(G_1^{\perp} \setminus G_0^{\perp}) = 0$. It is worth noting that the numbers $m_0(\chi_k)$ for $k = 0, \dots, p^{s-1} - 1$ are just the λ_k of which we spoke in the introduction.

Step 3. We set $\widehat{\varphi}(1) = 1$ and build $\widehat{\varphi}(\chi)$ using (4.4). By Theorem 4.1, $|\widehat{\varphi}(\chi)| = \mathbf{1}_{G_0^{\perp}}(\chi)$, and hence the function φ generates an MRA.

Step 4. We set $m_l(\chi) = m_0(\chi r_0^{-l})$, $l = 1, \dots, p-1$. Clearly, $m_l(\chi)$ may be written as

$$\begin{aligned} m_l(\chi) &= \sum_{h \in H_0^{(s)}} \beta_h \overline{(\chi r_0^{-l}, A^{-1}h)} = \sum_{h \in H_0^{(s)}} \beta_h \overline{(r_0^{-l}, A^{-1}h)}\,\overline{(\chi, A^{-1}h)} \\ &= \sum_{h \in H_0^{(s)}} \beta_h^{(l)} \overline{(\chi, A^{-1}h)}, \end{aligned} \tag{5.3}$$

where $\beta_h^{(l)} = \beta_h \overline{(r_0^{-l}, A^{-1}h)}$. It is also easily checked that $|m_l(G_0^{\perp} r_0^l)| = 1$ and $|m_l(G_0^{\perp} r_0^{\nu})| = 0$ for $\nu \neq l$.

Step 5. Consider the functions

$$\psi_l(x) = \sum_{h \in H_0^{(s)}} \beta_h^{(l)} \varphi(Ax \dot{-} h), \qquad \overline{1, p-1}.$$

Theorem 5.1. *The functions $\psi_l(x \dot{-} h)$, where $l = 1, \dots, p-1$, $h \in H_0$, form an orthonormal basis for W_0.*

Proof. a) We claim that $(\varphi(\,\cdot - g^{(1)}), \psi_l(\,\cdot - g^{(2)})) = 0$ for any $g^{(1)}, g^{(2)} \in H_0$. Since

$$\widehat{\varphi}_{\cdot\, -h}(\chi) = \overline{(\chi, h)}\widehat{\varphi}(\chi), \qquad \widehat{\varphi}_{A\,\cdot\, -g}(\chi) = \frac{1}{p}\,\overline{(\chi, A^{-1}g)}\widehat{\varphi}(\chi A^{-1}),$$

it follows that

$$(\varphi(\cdot - g^{(1)}), \psi_l(\cdot - g^{(2)})) = \int_G \varphi(x \dot{-} g^{(1)})\overline{\psi_l(x \dot{-} g^{(2)})}\, d\mu(x)$$
$$= \sum_{h\in H_0^{(s)}} \overline{\beta}_h^{(l)} \int_G \varphi(x \dot{-} g^{(1)})\overline{\varphi}(Ax \dot{-} Ag^{(2)} \dot{-} h)\, d\mu(x)$$
$$= \frac{1}{p}\sum_{h\in H_0^{(s)}} \overline{\beta}_h^{(l)} \int_X \widehat{\varphi}(\chi)\overline{(\chi, g^{(1)})}\,\overline{\widehat{\varphi}(\chi A^{-1})}(\chi, g^{(2)})(\chi, A^{-1}h)\, d\nu(\chi)$$
$$= \int_X \widehat{\varphi}(\chi)\overline{\widehat{\varphi}(\chi A^{-1})}\,\overline{(\chi, g^{(1)})}(\chi, g^{(2)})\frac{1}{p}\sum_{h\in H_0^{(s)}} \overline{\beta}_h^{(l)}(\chi, A^{-1}h)\, d\nu(\chi)$$
$$= \int_X \widehat{\varphi}(\chi)\overline{\widehat{\varphi}(\chi A^{-1})}\,\overline{(\chi, g^{(1)})}(\chi, g^{(2)})\overline{m_l(\chi)}\, d\nu(\chi) = 0$$

because $\operatorname{supp}\widehat{\varphi}(\chi) = G_0^{\perp}$ and $m_l(G_0^{\perp}) = 0$, $l = 1, \dots, p-1$. Similarly,

$$(\psi_l(\cdot \dot{-} g^{(1)}), \psi_k(\cdot \dot{-} g^{(2)})) = 0$$

for $k \neq l$, $\quad k, l = 1, \dots, p-1$.

b) We verify that $(\psi_l(\cdot \dot{-} g^{(1)}), \psi_l(\cdot \dot{-} g^{(2)})) = 0$, provided that $g^{(1)}, g^{(2)} \in H_0$ and $g^{(1)} \neq g^{(2)}$. First of all we note that it follows from the equality

$$0 = \int_G \varphi(x \dot{-} g^{(1)})\overline{\varphi(x \dot{-} g^{(2)})}\, d\mu(x) = \int_X \widehat{\varphi}_{\cdot \dot{-} g^{(1)}}(\chi)\overline{\widehat{\varphi}_{\cdot \dot{-} g^{(2)}}(\chi)}\, d\nu(\chi)$$
$$= \int_X \overline{(\chi, g^{(1)})}\widehat{\varphi}(\chi)(\chi, g^{(2)})\overline{\widehat{\varphi}(\chi)}\, d\nu(\chi) = \int_{G_0^{\perp}} \overline{(\chi, g^{(1)})}(\chi, g^{(2)})\, d\nu(\chi)$$

that the elements $g \in H_0$ form an orthogonal system on $G_0^{\perp}$, and hence on the cosets $G_0^{\perp} r_0^l$, $l = 1, \dots, p-1$. Therefore it follows from the equality

$$\int_G \psi_l(x \dot{-} g^{(1)})\overline{\psi_l(x \dot{-} g^{(2)})}\, d\mu(x) = \int_{G_0^{\perp} r_0^l} \overline{(\chi, g^{(1)})}(\chi, g^{(2)})\, d\nu(\chi)$$

that the system of translates $(\psi_l(x \dot{-} g))_{g\in H_0}$ is orthogonal on G (with fixed $l = 1, \dots, p-1$). Consequently, the linear span of the system

$$(\psi_l(x \dot{-} h))_{l=\overline{1,p-1},\, h\in H_0}$$

is orthogonal to V_0, and so the system $(\psi(\cdot \dot{-} h))$ is orthonormal.

c) We claim that any function $f \in W_0$ can be expanded uniquely in a series in $(\psi_l(x \dot{-} g))_{g\in H_0, l=\overline{1,p-1}}$. First we note that $\widehat{\psi}_l(\chi) = \frac{1}{p}\widehat{\varphi}(\chi A^{-1})m_l(\chi)$. In fact,

$$\widehat{\psi}_l(\chi) = \int_G \psi_l(\chi)\overline{(\chi, x)}\, d\mu(x) = \int_G \sum_{h\in H_0^{(s)}} \beta_h^{(l)}\varphi(Ax \dot{-} h)\overline{(\chi, x)}\, d\mu(x)$$
$$= \sum_{h\in H_0^{(s)}} \beta_h^{(l)} \int_G \varphi(Ax \dot{-} h)\overline{(\chi, x)}\, d\mu(x) = \sum_{h\in H_0^{(s)}} \beta_h^{(l)} \frac{1}{p}\overline{(\chi, A^{-1}h)}\widehat{\varphi}(\chi A^{-1})$$
$$= \frac{1}{p}\,\widehat{\varphi}(\chi A^{-1}) \sum_{h\in H_0^{(s)}} \beta_h^{(l)}\overline{(\chi, A^{-1}h)} = \frac{1}{p}\,\widehat{\varphi}(\chi A^{-1})m_l(\chi). \tag{5.4}$$

Further, if $f \in V_1$ and

$$f = \sum_{h \in H_0} a_h \varphi(Ax \dot{-} h), \tag{5.5}$$

then

$$\widehat{f}(\chi) = \frac{1}{p}\, Q(\chi)\widehat{\varphi}(\chi A^{-1}), \tag{5.6}$$

where

$$Q(\chi) = \sum_{h \in H_0} a_h \overline{(\chi, A^{-1}h)} \in L_2(G_1^{\perp}); \tag{5.7}$$

the series in the last equality converges in the norm of $L_2(G_1^{\perp})$. The converse is also true: if $\widehat{f}(\chi)$ is given by (5.6) and $Q(x)$ is as in (5.7), then f is the sum of the series (5.5).

We take $u(x) \in W_0$, and verify that u is the sum of the series in the system

$$(\psi_l(x \dot{-} h))_{l=\overline{1,p-1},\, h \in H_0}.$$

Suppose that $v(x) \in V_0$. Then $f := v(x) + u(x) \in V_0 \otimes W_0 = V_1$, and hence

$$f = \sum_{h \in H_0} \alpha_h \varphi(Ax \dot{-} h).$$

Then (see (5.3)–(5.5))

$$\begin{aligned}
\widehat{f}(\chi) &= \frac{1}{p}\, Q(\chi)\widehat{\varphi}(\chi A^{-1}) = \frac{1}{p}\, Q(\chi)\mathbf{1}_{G_1^{\perp}}(\chi)\widehat{\varphi}(\chi A^{-1}) = \frac{1}{p}\, Q(\chi)\sum_{l=0}^{p-1} |m_l(\chi)|^2 \widehat{\varphi}(\chi A^{-1}) \\
&= \frac{1}{p}\, Q(\chi)\overline{m_0(\chi)}\, \underbrace{m_0(\chi)\widehat{\varphi}(\chi A^{-1})}_{=\widehat{\varphi}(\chi)} + \sum_{l=1}^{p-1} Q(\chi)\overline{m_l(\chi)}\, \underbrace{\frac{1}{p}\widehat{\varphi}(\chi A^{-1}) m_l(\chi)}_{=\widehat{\psi}_l(\chi)} \\
&= \frac{1}{p} Q(\chi)\overline{m_0(\chi)}\widehat{\varphi}(\chi) + \sum_{l=1}^{p-1} Q(\chi)\overline{m_l(\chi)}\widehat{\psi}_l(\chi).
\end{aligned} \tag{5.8}$$

The last equality implies that

$$f(x) = \sum_{h \in H_0} b_h \varphi(x \dot{-} h) + \sum_{l=1}^{p-1} \sum_{h \in H_0} b_h^{(l)} \psi_l(x \dot{-} h). \tag{5.9}$$

We proceed to prove that this is indeed so.

The functions $Q(\chi)$ lie in $L_2(G_1^{\perp}) \subset L_2(G_0^{\perp})$, and the function $m_0(\chi)$ is bounded on $G_0^{\perp}$. Therefore $\frac{1}{p}Q(\chi)\overline{m_0(\chi)} \in L_2(G_0^{\perp})$, and hence

$$\frac{1}{p}\, Q(\chi)\overline{m_0(\chi)} = \sum_{h \in H_0} b_h \overline{(\chi, h)},$$

because by Lemma 2.2 the restrictions of the elements $h \in H_0$ to $G_0^\perp$ form an orthonormal basis for $L_2(G_0^\perp)$. It follows that the function $\frac{1}{p}\,Q(\chi)\overline{m_0(\chi)}\widehat{\varphi}(\chi)$ is the Fourier transform of

$$v(x) = \sum_{h\in H_0} b_h\varphi(x \dot{-} h).$$

Consider the functions $Q(\chi)\overline{m_l(\chi)}\,\overline{\psi}_l(\chi)$. Since the $Q(\chi)$ lie in $L_2(G_1^\perp)$ and $m_l(\chi) = m_0(\chi r_0^{-l})$ is bounded on $G_0^\perp r^l$, $m_l(G_1^\perp \setminus G_0^\perp r_0^l) = 0$, it is found that $Q(\chi)\overline{m_l(\chi)} \in L_2(G_0^\perp r_0^l)$, and hence $\overline{Q(\chi)}m_l(\chi) \in L_2(G_0^\perp r_0^l)$. We look upon elements $h \in H_0$ as functions on $(G_0^\perp r_0^l)$. Since the restrictions of the elements $h \in H_0$ to $G_0^\perp$ form an orthonormal basis for $L_2(G_0^\perp)$, it follows that the elements $h(\chi r_0^{-l})$ form an orthonormal basis for $L_2(G_0^\perp r_0^l)$.

Therefore, for each $l = 1, \dots, p-1$:

i) $\overline{Q(\chi)}m_l(\chi) = \sum_{h\in H_0} \overline{b_h^{(l)}}(r_0^l, h)(\chi r_0^{-l}, h)$, $\sum_{h\in H_0} |\overline{b_h^{(l)}}(r_0^l, h)|^2 < \infty$;

ii) $U^{(l)}(x) = \sum_{h\in H_0} b_h^{(l)}\psi_l(x \dot{-} h) \in L_2(G)$;

iii) $\widehat{U}^{(l)}(\chi) = \widehat{\psi}_l(\chi)Q(\chi)\overline{m_l(\chi)}$.

Hence (5.8) gives (5.9).

By the uniqueness of the representation of f as $f = v + u$, $v \in V_0$, $u \in W_0$, it follows that

$$u(x) = \sum_{l=1}^{p-1}\sum_{h\in H_0} b_h^{(l)}\psi_l(x \dot{-} h).$$

This means that the system $\psi_l(x \dot{-} h)$ forms a basis for W_0.

The proof of Theorem 5.1 is complete.

Step 6. Since the subspaces $(V_j)_{j\in\mathbb{Z}}$ form an MRA of $L_2(G)$, it follows that the functions

$$\psi_l(A^n x \dot{-} h), \qquad l = \overline{1, p-1}, \quad n \in \mathbb{Z}, \quad h \in H_0,$$

form a complete orthogonal system in $L_2(G)$.

Example. Let $s = 1$. Then the mask m_0 is as follows:

$$m_0(\chi) = \frac{1}{p}\sum_{h\in H_0^{(1)}} \beta_h\overline{(\chi, A^{-1}h)};$$

it is also constant on cosets of the subgroup $G_0^\perp$. We choose $m_0(\chi)$ so that $|m_0(G_0^\perp)| = 1$ and $m_0(\chi) = 0$ for $\chi \in G_1^\perp \setminus G_0^\perp$. Then (5.2) becomes

$$m_0(\chi_k) = \frac{1}{p}\sum_{j=0}^{p-1} \beta_j\overline{(\chi_k, A^{-1}h_j)}. \tag{5.10}$$

In this case $h_j = jg_{-1}, A^{-1}h_j = jg_0, \chi_k = r_0^k$, $j, k = 0, \dots, p-1$, so equality (5.10) assumes the form

$$m_0(r_0^k) = \frac{1}{p}\sum_{j=0}^{p-1} \beta_j\overline{(r_0, g_0)^{k\cdot j}}.$$

Letting $m_0(r_0^k) = 0$, $k = 1, \dots, p-1$, $m_0(1) = 1$, we arrive at the system

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \frac{1}{\sqrt{p}} \begin{pmatrix} a_{0,0} & a_{0,1} & \dots & a_{0,p-1} \\ a_{1,0} & a_{1,1} & \dots & a_{1,p-1} \\ \dots & \dots & \dots & \dots \\ a_{p-1,0} & a_{p-1,1} & \dots & a_{p-1,p-1} \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix}, \tag{5.11}$$

where

$$a_{k,j} = \frac{1}{\sqrt{p}} \overline{(r_0, g_0)^{kj}}.$$

The matrix $(a_{k,j})$ being unitary, the system (5.11) has a unique solution $\beta_0 = \beta_1 = \dots = \beta_{p-1} = 1$. We use the formulae

$$\beta_j^{(l)} = \beta_j \overline{(r_0^{-l}, jg_0)} = (r_0, g_0)^{jl}, \qquad l = \overline{1, p-1},$$

to calculate the coefficients $\beta_j^{(l)}$, and set

$$\psi_l(x) = \sum_{j=0}^{p-1} (r_0, g_0)^{jl} \varphi(A(x \dot{-} jg_0)).$$

It remains to find the function φ. We set $\widehat{\varphi}(1) = 1$. Since

$$\widehat{\varphi}(\chi) = \widehat{\varphi}(1) \prod_{k=0}^{\infty} m_0(\chi A^{-k}),$$

it follows that $\widehat{\varphi}(G_0^\perp) = 1$ and $\widehat{\varphi}(\chi) = 0$ for $\chi \notin G_0^\perp$. Hence by Corollary 2.1,

$$\varphi(x) = \int_X \widehat{\varphi}(\chi)(\chi, x)\, d\nu(\chi) = \int_{G_0^\perp} (\chi, x)\, d\nu(\chi) = \begin{cases} 1, & x \in G_0, \\ 0, & x \notin G_0. \end{cases}$$

Putting $\varphi(x)$ in the expression for ψ_l, gives

$$\psi_l(x) = r_0^l(x) \mathbf{1}_{G_0}(x), \qquad l = 1, \dots, p-1.$$

Thus we have constructed an orthonormal basis, which is generated from a single function $r_0(x)\mathbf{1}_{G_0}(x)$ through contractions, translations and exponentiations.

The resulting functions ψ_l are complex-valued. We indicate a method of obtaining an orthonormal basis consisting of real functions. To do so, we note that the function $r_0(x)$ is constant on the cosets $G_1 \dot{+} jg_0$, $j = 1, \dots, p-1$, and takes values from the set of pth roots of unity. It is no restriction to assume that $r_0(G_1 \dot{+} jg_0) = e^{2\pi i j/p}$. Hence

$$\psi_l(G_1 \dot{+} jg_0) = e^{2\pi i j l/p} = \cos \frac{2\pi}{p} jl + i \sin \frac{2\pi}{p} jl.$$

Direct calculations show that each of the functions $\operatorname{Re} \psi_l(x)$ is orthogonal to the functions $\operatorname{Im} \psi_m(x)$ for $m = 1, \dots, p-1$ and to the functions $\operatorname{Re} \psi_m(x)$ for $m \neq l$ and

$m \neq p-l$. Similarly, any function $\operatorname{Im}\psi_l(x)$ is orthogonal to the functions $\operatorname{Re}\psi_m(x)$ and to the $\operatorname{Im}\psi_m(x)$ for $m \neq l$ and $m \neq p-l$. Therefore from one system $(\psi_1(x), \psi_2(x), \dots, \psi_{p-1}(x))$ it is possible to construct $4^{(p-1)/2}$ systems of real functions $(\psi_1^{(j)}, \psi_2^{(j)}, \dots, \psi_{p-1}^{(j)})$, $j = 1, \dots, 4^{(p-1)/2}$, some of which can be the same.

For $p = 2$, this gives one function

$$\psi_1(x) = \begin{cases} 1, & x \in G_1, \\ -1, & x \in G_0 \setminus G_1. \end{cases}$$

This is the classical Haar function.

For $p = 3$, we obtain four systems, of which only two are different:

$$\psi_1^{(1)}(x) = \begin{cases} \sqrt{2}, & x \in G_1, \\ -\dfrac{1}{\sqrt{2}}, & x \in G_1 \dot{+} g_0, \\ -\dfrac{1}{\sqrt{2}}, & x \in G_1 \dot{+} 2g_0, \\ 0, & x \notin G_0, \end{cases} \qquad \psi_2^{(1)}(x) = \begin{cases} 0, & x \in G_1, \\ \sqrt{\dfrac{3}{2}}, & x \in G_1 \dot{+} g_0, \\ -\sqrt{\dfrac{3}{2}}, & x \in G_1 \dot{+} 2g_0, \\ 0, & x \notin G_0, \end{cases}$$

the other two being symmetric to these,

$$\psi_1^{(2)}(x) = \begin{cases} \sqrt{2}, & x \in G_1, \\ -\dfrac{1}{\sqrt{2}}, & x \in G_1 \dot{+} g_0, \\ -\dfrac{1}{\sqrt{2}}, & x \in G_1 \dot{+} 2g_0, \\ 0, & x \notin G_0, \end{cases} \qquad \psi_2^{(2)}(x) = \begin{cases} 0, & x \in G_1, \\ -\sqrt{\dfrac{3}{2}}, & x \in G_1 \dot{+} g_0, \\ \sqrt{\dfrac{3}{2}}, & x \in G_1 \dot{+} 2g_0, \\ 0, & x \notin G_0. \end{cases}$$

In the case when G is the p-adic number field, these systems were obtained in [13].

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