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Systems of Markov functions generated by graphs and the asymptotics of their Hermite-Padé approximants

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Abstract. The paper considers Hermite-Padé approximants to systems of Markov functions defined by means of directed graphs. The minimization problem for the energy functional is investigated for a vector measure whose components are related by a given interaction matrix and supported in some fixed system of intervals. The weak asymptotics of the approximants are obtained in terms of the solution of this problem. The defining graph is allowed to contain undirected cycles, so the minimization problem in question is considered within the class of measures whose masses are not fixed, but allowed to 'flow' between intervals. Strong asymptotic formulae are also obtained. The basic tool that is used is an algebraic Riemann surface defined by means of the supports of the components of the extremal measure. The strong asymptotic formulae involve standard functions on this Riemann surface and solutions of some boundary value problems on it. The proof depends upon an asymptotic solution of the corresponding matrix Riemann-Hilbert problem.

Bibliography: 40 titles.

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§ 1. Introduction

1.1. Markov functions and their Hermite-Padé approximants. Markov's paper [1] was devoted to power expansions of functions

$$f(z)=\sum_{k=0}^{\infty}\frac{c_k}{z^{k+1}}=\int_{\mathbb{R}}\frac{d\mu(x)}{z-x},\qquad c_k=\int_{\mathbb{R}}x^k\,d\mu(x),\quad \mu>0, \tag{1.1}$$

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which are Cauchy transforms of positive measures with compact support. In it Markov considered continued fractions representations of $f(z)$

$$f(z) \doteq \cfrac{c_0}{z - b_0 - \cfrac{a_1^2}{z - b_1 - \cfrac{a_2^2}{z - b_2 - \ddots}}}, \tag{1.2}$$

and proved that the convergents $\pi_n(z)$ to $f(z)$ of (1.2) converge uniformly on compact sets of the complex plane outside the interval E which supports the measure:

$$\operatorname{supp}\mu \subset E \Subset \mathbb{R} \quad \Rightarrow \quad \lim_{n\to\infty} \pi_n(z) = \int \frac{d\mu(x)}{z - x}, \quad z \in \overline{\mathbb{C}} \setminus E. \tag{1.3}$$

In the theory of rational approximations, functions of the form (1.1) are referred to as *Markov functions* (also known as *resolvent functions* or *Weyl functions* in the theory of operators). The Markov functions form a class of analytic functions that are useful for investigating rational approximations, and Markov's theorem (1.3) is the starting point for these studies. Rational functions obtained by truncating the continued fractions (1.2) at finite levels (convergents) are a particular case of Padé approximants.

We proceed to define a general construction of rational functions having a common denominator which furnish an approximation to the vector of power series

$$\vec{f} = (f_1, \dots, f_p), \qquad f_j(z) = \sum_{k=0}^{\infty} \frac{f_{j,k}}{z^{k+1}}, \quad j = 1, \dots, p. \tag{1.4}$$

A vector

$$\pi_{\vec{n}} = \left(\frac{Q_{\vec{n},1}}{P_{\vec{n}}}, \cdots, \frac{Q_{\vec{n},p}}{P_{\vec{n}}} \right), \qquad \vec{n} = (n_1, \dots, n_p) \in \mathbb{N}^p, \tag{1.5}$$

of rational functions with the common denominator $P_{\vec{n}}$ is said to be a *Hermite-Padé approximant* (of the second kind) with multi-index $\vec{n}$ to the vector of power series $\vec{f}$, provided that

$$P_{\vec{n}} \not\equiv 0, \qquad \deg P_{\vec{n}} \leqslant |\vec{n}| := n_1 + \cdots + n_p, \tag{1.6}$$

$$f_j(z)P_{\vec{n}}(z) - Q_{\vec{n},j}(z) =: R_{\vec{n},j}(z) = O\left(\frac{1}{z^{n_j+1}}\right) \quad \text{as } z \to \infty, \quad j = 1, \dots, p. \tag{1.7}$$

This construction was put forward by Hermite [2] in connection with his celebrated proof of the transcendence of e. For $p = 1$, the approximants (1.5) are known as *Padé approximants*. The relations (1.7) provide $|\vec{n}|$ homogeneous linear equations to determine the $|\vec{n}|+1$ coefficients of the polynomial (1.6). When each polynomial satisfying (1.7) has degree $\vec{n}$ (and so is uniquely defined up to a multiplicative constant), the multi-index is said to be *normal*, and the polynomial $P_{\vec{n}}$ is normalized as follows:

$$P_{\vec{n}}(z) = z^{|\vec{n}|} + \cdots . \tag{1.8}$$

The denominator $P_{\vec{n}}$ of the Hermite-Padé approximant to the system of Markov functions

$$f_j(z) = \widehat{\mu}_j(z) = \int_{E_j} \frac{d\mu_j(x)}{z-x}, \qquad E_j \subset \mathbb{R}, \quad j = 1, \dots, p, \tag{1.9}$$

satisfies the following orthogonality relations

$$\int_{E_j} P_{\vec{n}}(x) x^k \, d\mu_j(x) = 0, \qquad k = 0, 1, \dots, n_j - 1, \quad j = 1, \dots, p. \tag{1.10}$$

Polynomials satisfying the orthogonality relations (1.10) are also known as *multiple orthogonal polynomials*.

Apart from traditional applications to the theory of Diophantine approximations and to the theory of approximations of analytic functions (see [3]–[5]), Hermite-Padé approximants and multiple orthogonal polynomials prove useful in the spectral theory of higher-order nonsymmetric difference operators (see [6]–[8]). Recently a connection was discovered between them and the theory of random matrices (see [9]–[11]).

As distinct from conventional orthogonal polynomials ($p = 1$), the orthogonality relations (1.10) with $p > 1$ do not guarantee that the index n is normal, and so they do not guarantee the existence of a multiple orthogonal polynomial (1.8) of degree $|\vec{n}|$. Some general systems of Markov functions are known to have normal Hermite-Padé approximants. Among these, for example, are the following.

An *Angelesco system* [12] is defined by

$$\mathscr{A}: \quad \{\widehat{\mu}_j(z)\}_{j=1}^p, \quad \operatorname{supp} \mu_j \subset E_j: \quad \mathring{E}_k \cap \mathring{E}_j = \varnothing, \quad k \neq j, \quad k, j = 1, \dots, p. \tag{1.11}$$

It is easily ascertained (see (1.10)) that for an Angelesco system the polynomial $P_{\vec{n}}$ has n_j changes of sign in the interior $\mathring{E}_j$ of the interval E_j. This forces any arbitrary multi-index $\vec{n}$ to be normal.

A *Nikishin system* [13] is defined by means of the family of measures

$$\sigma := \{\sigma_j(x)\}_{j=1}^p, \qquad \operatorname{supp} \sigma_j \subset E_j, \quad E_j \cap E_{j-1} = \varnothing,$$

supported in the intervals $\{E_j\}_{j=1}^p$; this family in turn generates the vector of measures $\mu = \{\mu_j(x)\}_{j=1}^p$ as follows:

$$\begin{aligned}
&d\mu_1(x) := d\sigma_1(x), \\
&d\mu_2(x) := d\langle \sigma_1, \sigma_2 \rangle(x) := \left(\int_{E_2} \frac{d\sigma_2(t)}{x-t} \right) d\sigma_1(x), \\
&\qquad \dots \\
&d\mu_j(x) := d\langle \sigma_1, \sigma_2, \dots, \sigma_j \rangle := d\langle \sigma_1, \langle \sigma_2, \dots, \sigma_j \rangle \rangle, \qquad j = 3, \dots, p.
\end{aligned}$$

It is also worth noting that all the components of μ are supported in one interval: $\operatorname{supp} \mu_j \subset E_1$. The system of Markov functions $\{\widehat{\mu}_j(z)\}_{j=1}^p$ which corresponds to the vector μ is called a *Nikishin system*. So,

$$\mathscr{N}: \quad \{\widehat{\mu}_j(z)\}_{j=1}^p, \quad \operatorname{supp} \mu_j \subset E_1, \quad j = 1, \dots, p. \tag{1.12}$$

The conventional (see [13], [14]) condition for the normality of a multi-index $\vec{n}$ for the Hermite-Padé approximants to the Nikishin system is as follows:

$$n_k \leqslant n_j + 1 \quad \text{for } k > j. \tag{1.13}$$

Recently this condition was relaxed. In particular, in [15] all the indices $\vec{n}$ were shown to be normal for $p = 2$ and 3.

Weak asymptotics of Hermite-Padé approximants to Angelesco systems (that is, the nth root asymptotics and the limit measures of the distribution of poles), and therefore, the answer to the question of when they converge, were obtained in [16]. We note that the convergence or divergence of an Angelesco system (1.11) depends on the pattern of intervals $\{E_j\}_{j=1}^{p}$. Strong asymptotics of Hermite-Padé approximants to Angelesco systems (that is, asymptotics of the approximants themselves and the determination of the positions of individual poles for large $|\vec{n}|$) were obtained in [17]. The convergence of Hermite-Padé approximants to Nikishin systems (an analogue of Markov's theorem (1.3)) for $p = 2$ was established by Nikishin himself in [13]. As distinct from Angelesco systems, Hermite-Padé approximants for Nikishin systems always converge; this was shown in [18] for arbitrary p. Weak (strong) asymptotics of Hermite-Padé approximants to Nikishin systems have been examined in [19], [20] (in [21], respectively).

Generalized Nikishin systems (the so-called $\mathscr{GN}$-systems) of Markov functions $\{\widehat{\mu}_j(z)\}_{j=1}^{p}$ were introduced in [20] through the concept of a tree graph. Without going into the details of the definition (the process of generating systems of Markov functions by means of graphs will be discussed in detail further on), we note that the $\mathscr{GN}$-systems involve both Angelesco and Nikishin systems, as well as some mixed systems. For example, for $p = 3$, here are two such systems: the system

$$\begin{aligned} d\mu_1(x) &:= d\sigma_1(x), \qquad \operatorname{supp}\sigma_1 \subset E_1, \\ d\mu_2(x) &:= d\langle\sigma_1,\sigma_2\rangle(x) = \left(\int_{E_2} \frac{d\sigma_2(t)}{x-t}\right) d\sigma_1(x), \\ d\mu_3(x) &:= d\langle\sigma_1,\sigma_3\rangle(x) = \left(\int_{E_3} \frac{d\sigma_3(t)}{x-t}\right) d\sigma_1(x), \end{aligned} \tag{1.14}$$

with disjoint intervals $\{E_j\}_{j=1}^{3}$, and the system

$$\begin{aligned} d\mu_1(x) &:= d\sigma_1(x), \qquad \operatorname{supp}\sigma_1 \subset E_1, \\ d\mu_2(x) &:= d\sigma_2(x), \qquad \operatorname{supp}\sigma_2 \subset E_2, \\ d\mu_3(x) &:= d\langle\sigma_1,\sigma_3\rangle(x) = \left(\int_{E_3} \frac{d\sigma_3(t)}{x-t}\right) d\sigma_1(x) \end{aligned} \tag{1.15}$$

with $E_1 \cap E_2 = \varnothing$ and $E_1 \cap E_3 = \varnothing$. For Hermite-Padé approximants to a $\mathscr{GN}$-system, a condition for a multi-index $\vec{n}$ to be normal (similar to (1.13)) was determined in [20]. This paper also contains the solution to the problem of weak asymptotics.

The Hermite-Padé approximants to specific systems of Markov functions related to graphs with cycles were investigated in [3] in connection with applications to number theory.

The purpose of this paper is to extend the class of systems of Markov functions generated by graphs, and to obtain (weak and strong) asymptotics for the corresponding Hermite-Padé approximants.

The paper is structured as follows. In §§ 1.2–1.4, we give a number of definitions, and formulate our results. In § 1.2, we define a directed graph, to introduce systems of Markov functions (extending the $\mathscr{GN}$-systems of [20]), for which Hermite-Padé approximants will be investigated. Basically, we use the notation put forward in [20]; the key new feature here is that the graphs in question are not necessarily tree graphs as in [20]. This results in new effects in the asymptotic behaviour of the approximants, which in turn need new methods of justification. In § 1.3 we turn to a discussion of the results we have obtained on the weak asymptotics of approximants. The results will be formulated in terms of the solutions to the problem of minimizing the energy functional of a vector measure (whose components have support in some fixed system of intervals) involving an interaction matrix between measure components. Since the defining graph of the system of Markov functions is allowed to have undirected cycles, we cannot look at this problem in the class of measures with fixed masses; we have to allow the masses to 'flow' between the intervals (that is, we look at the extremal problem in the class of measures with linear relations between the masses); this is a novel feature in comparison with [20]. In § 1.4, we discuss our results on strong asymptotics. The key point here is the construction of the Riemann surface consisting of glued copies of the complex plane cut along the supports of the components of the extremal measure. The strong asymptotic formulae are given in terms of standard functions on this Riemann surface and in terms of solutions of some boundary valued problems on it. As regards the novelty of our results, we note that the strong asymptotics of Hermite-Padé approximants have not been studied before, even for $\mathscr{GN}$-systems defined by tree graphs. Our results on strong asymptotics are in agreement with Nuttall's general conjectures [22], which in turn gave impetus to our research.

We prove the results stated in the Introduction in the following sections. In § 2, we establish a theorem on the existence, uniqueness and equilibrium of a solution to the extremal problem for the energy functional of vector measures with interaction matrices between the components and with masses related by linear relations. In § 3, we prove a theorem on limit distribution of the zeros in the denominators of Hermite-Padé approximants. Strong asymptotic formulae are proved in § 4; this involves the use of the matrix Riemann-Hilbert method. Finally, in § 5, we discuss some new effects in the behaviour of Hermite-Padé approximants in detail, using some of the simplest examples of systems of functions generated by graphs with undirected cycles.

1.2. Graphs and the corresponding systems of Markov functions. Consider a directed multigraph with vertex set $\mathscr{V} := \{A, B, C, \dots\}$, $\#\mathscr{V} = p+1$ and with edges $\mathscr{E} := \{\alpha, \beta, \gamma, \dots\}$, $\#\mathscr{E} = m$. We suppose that

1) the graph is acyclic (that is, it contains no directed cycles);
2) there is a vertex O such that, for each vertex $A \in \mathscr{V}$ distinct from O, there is a directed path from O to A.

The vertex O is unique by condition 1).

We denote this graph by

$$\mathscr{G} := \mathscr{G}(\mathscr{V}, \mathscr{E}, O). \tag{1.16}$$

Let (A, B) be the set of edges connecting two adjacent vertices A and B.

The vertex set $\mathscr{V}$ can be equipped with a partial order relation as follows: we let $A \preccurlyeq B$ if either $A = B$ or there exists a directed path from A to B. In the latter case we shall also write $A \prec B$.

Given any vertex $A \in \mathscr{G}$, let

$$A_+ := \{B \in \mathscr{V} : \exists\, \alpha \in (A, B) \subset \mathscr{E}\} \quad \text{and} \quad A_- := \{B \in \mathscr{V} : \exists\, \alpha \in (B, A) \subset \mathscr{E}\}$$

be the sets of vertices nearest to A; also let

$$\mathscr{E}_{A+} := \{\alpha \in (A, B) : B \in A_+\} \quad \text{and} \quad \mathscr{E}_{A-} := \{\alpha \in (B, A) : B \in A_-\}$$

be the number of edges which go out from or come into A. We introduce the following relations on $\mathscr{E}$:

$$\begin{aligned}
&\alpha \to \beta \quad &&\Leftrightarrow \quad \exists\, A \in \mathscr{V}: \quad &&\alpha \in \mathscr{E}_{A-}, \quad \beta \in \mathscr{E}_{A+};\\
&\alpha \upuparrows \beta \quad &&\Leftrightarrow \quad \exists\, A, B \in \mathscr{V}: \quad &&\alpha, \beta \in (A, B);\\
&\alpha \leftrightarrow \beta \quad &&\Leftrightarrow \quad \exists\, A \in \mathscr{V}: \quad &&\alpha, \beta \in \mathscr{E}_{A-} \text{ or } \alpha, \beta \in \mathscr{E}_{A+}, \text{ but } \alpha \upuparrows \beta \text{ fails to hold.}
\end{aligned}$$

Following [20], we consider the system of Markov functions generated by the graph $\mathscr{G}$. To each edge α of the graph $\mathscr{G}$ we assign an interval $E_\alpha := [a_\alpha, b_\alpha]$ of the real axis $\mathbb{R}$ and a positive Borel measure σ_α with support in E_α; that is,

$$\forall\, \alpha \in \mathscr{E} \quad \longrightarrow \quad E_\alpha := [a_\alpha, b_\alpha] \subset \mathbb{R}, \quad \sigma_\alpha : \quad \sigma'_\alpha > 0 \quad \text{a.e. on } E_\alpha. \tag{1.17}$$

We also assume that if the edges α and β have a common vertex, then the corresponding intervals E_α and E_β do not overlap; that is,

$$\alpha \to \beta \vee \beta \to \alpha \vee \alpha \upuparrows \beta \vee \alpha \leftrightarrow \beta \quad \Rightarrow \quad E_\alpha \cap E_\beta = \varnothing. \tag{1.18}$$

Corresponding to each vertex $A \in \mathring{\mathscr{V}} := \mathscr{V} \setminus \{O\}$ there is a nonempty set $\mathscr{T}_A$ of paths $t_A = (\omega, \dots, \beta, \alpha)$ from the root vertex O to the vertex A; that is,

$$\begin{gathered}
\forall\, A \in \mathring{\mathscr{V}} \quad \longrightarrow \quad \mathscr{T}_A := \{t_A\}, \quad t_A := (\omega, \dots, \beta, \alpha) : \quad \omega \to \dots \to \beta \to \alpha,\\
\omega \in \mathscr{E}_{O+}, \qquad \alpha \in \mathscr{E}_{A-}.
\end{gathered}$$

To each chain of paths t_A of this type there is a corresponding measure μ_{t_A} defined by Nikishin's rule as follows:

$$\mu_{t_A}(x) = \langle \sigma_\omega, \dots, \sigma_\beta, \sigma_\alpha \rangle (x);$$

here,

$$d\langle \sigma_1, \sigma_2 \rangle (x) := \left(\int \frac{d\sigma_2(t)}{x - t} \right) d\sigma_1(x), \quad \dots, \quad d\langle \sigma_1, \sigma_2, \dots, \sigma_j \rangle := d\langle \sigma_1, \langle \sigma_2, \dots, \sigma_j \rangle \rangle.$$

To a vertex A we also assign the function

$$\widehat{\mu}_A(x) := \sum_{t_A \in \mathscr{T}_A} \int \frac{d\mu_{t_A}(t)}{x - t}. \tag{1.19}$$

Definition 1.1. The family of functions $\{\widehat{\mu}_A(x),\ A \in \mathring{\mathscr{V}}\}$ is called a *generalized Nikishin system* ($\mathscr{G}\mathscr{N}$-system) associated with the graph $\mathscr{G}$.

Remark 1.1. The concept of a generalized Nikishin system corresponding to a graph was introduced in [20]. In this paper, we discuss a wider class of graphs than those in [20], where only tree graphs defined by the condition

$$\mathscr{G}: \quad \forall\, A \in \mathring{\mathscr{V}} \quad \Rightarrow \quad \#\mathscr{E}_{A-} = 1 \tag{1.20}$$

were considered. Consequently, the class of $\mathscr{G}\mathscr{N}$-systems under discussion is larger than the class of generalized Nikishin systems corresponding to tree graphs of [20].

We give examples of various $\mathscr{G}\mathscr{N}$-systems. As has already been noted, Angelesco and Nikishin systems are $\mathscr{G}\mathscr{N}$-systems. An Angelesco system is generated by the tree graph of Fig. 1a)

$$\mathscr{G}: \quad \mathscr{E} = \mathscr{E}_{O+} \to \mathscr{A},$$

and a Nikishin system is generated by the tree graph of Fig. 1b):

$$\mathscr{G}: \quad \forall\, A \in \mathring{\mathscr{V}} \quad \Rightarrow \quad \#A_- = 1, \quad \#A_+ \leqslant 1 \quad \to \quad \mathscr{N}.$$

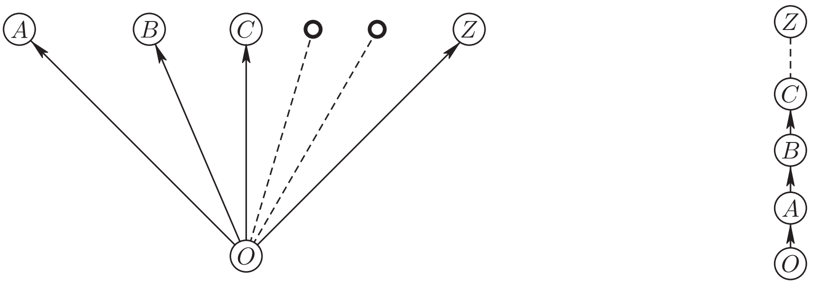


Figure 1. Graphs generating a) Angelesco and b) Nikishin systems.

Examples of tree graphs are shown in Fig. 2. The graphs generating the systems of Markov functions (1.14) and (1.15) are depicted in Figs. 2a) and 2b), respectively.

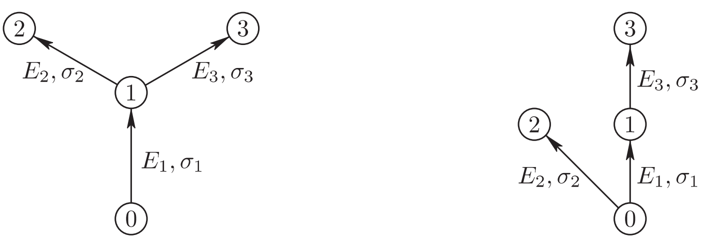


Figure 2. The tree graphs generating the systems of Markov functions (1.14) and (1.15).

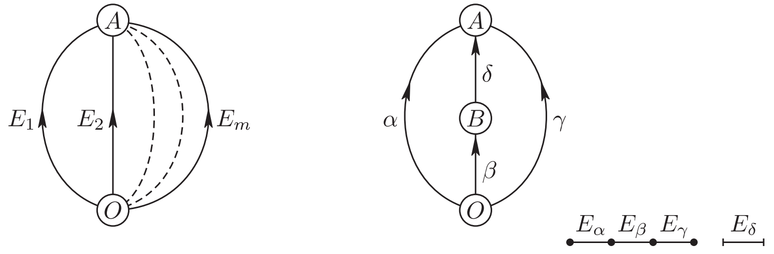


Figure 3. Graphs with undirected cycles.

Examples of graphs with undirected cycles are shown in Fig. 3. The graph in Fig. 3a) generates a single Markov function supported in several (m) intervals. The graph in Fig. 3b) generates two Markov functions. One of these, $\widehat{\mu}_A$, has its support in the union of three intervals $E := E_\alpha \cup E_\beta \cup E_\gamma$, while the other, $\widehat{\mu}_B$, has its support in the interval E_β; also,

$$d\mu_B(x) := d\sigma_\beta(x), \qquad \operatorname{supp}\sigma_\beta \subset E_\beta,$$
$$d\mu_A(x) := \begin{cases} d\sigma_\alpha(x) & \text{on } E_\alpha, \\ \left(\displaystyle\int_{E_\delta} \frac{d\sigma_\delta(t)}{x-t}\right) d\sigma_\beta(x) & \text{on } E_\beta, \\ d\sigma_\gamma(x) & \text{on } E_\gamma. \end{cases}$$

Thus graphs with undirected cycles are useful for representing Markov functions which have some part of their support in common (more precisely, one support contains the other), where the ratio of the weights in the common region of support is again a Markov function on some different interval.

We shall consider Hermite-Padé approximants to the $\mathscr{GN}$-system (1.19)

$$\vec{f} := \{\widehat{\mu}_A(x),\ A \in \mathring{\mathscr{V}}\},$$

which corresponds to the graph $\mathscr{G}(\mathscr{V}, \mathscr{E}, O)$ (see (1.16)). We fix a multi-index

$$\vec{n} := \{n_A,\ A \in \mathring{\mathscr{V}}\} : \quad n_A \leqslant n_B + 1, \quad \text{if } B \prec A. \tag{1.21}$$

Then there exists a polynomial $P_{\vec{n}} \not\equiv 0$ of degree $\deg P_{\vec{n}} \leqslant |\vec{n}| := \sum_{A \in \mathring{\mathscr{V}}} n_A$ such that

$$R_{\vec{n},A} := P_{\vec{n}}\widehat{\mu}_A - Q_{\vec{n},A} = O(z^{-n_A-1}), \qquad z \to \infty, \quad A \in \mathring{\mathscr{V}}, \tag{1.22}$$

where the $Q_{\vec{n},A}$ are some polynomials. This definition of Hermite-Padé approximants

$$\left\{\frac{Q_{\vec{n},A}}{P_{\vec{n}}},\ A \in \mathring{\mathscr{V}}\right\} \tag{1.23}$$

leads us to the following orthogonality relations:

$$\sum_{t_A \in \mathscr{T}_A} \int P_{\vec{n}}(x) x^k \, d\mu_{t_A}(x) = 0, \qquad k = 0, \dots, n_A - 1, \quad A \in \mathring{\mathscr{V}}. \tag{1.24}$$

To investigate these approximants, in addition to the functions of the second kind $R_{\vec{n},A}$, it proves useful to consider the functions $\Psi_{\vec{n},A}$, which are defined by induction with respect to the partial order on the graph:

$$\Psi_{\vec{n},O} = P_{\vec{n}}, \qquad \Psi_{\vec{n},B}(x) = \sum_{A\in B_-} \sum_{\alpha\in(A,B)} \int \frac{\Psi_{\vec{n},A}(t)\,d\sigma_\alpha(t)}{t-x}, \quad B \in \mathring{\mathscr{V}}. \tag{1.25}$$

In [20], the conventional condition (1.13) for the normality of a multi-index $\vec{n}$ for a Nikishin system was extended to a generalized Nikishin system generated by a tree graph (1.20) as follows:

$$n_A \leqslant n_B + 1, \quad \text{if } B \prec A. \tag{1.26}$$

It was further shown that the polynomial $P_{\vec{n}}$ has $|\vec{n}|$ simple zeros on the union of the intervals $\bigcup_{\alpha\in\mathscr{E}_{O+}} E_\alpha$. Consequently, the indices (1.26) are normal, and so the Hermite-Padé approximants are uniquely defined.

For an $\mathscr{G}\mathscr{N}$-system generated by an arbitrary graph $\mathscr{G}$ (see (1.16)), the problem of whether the indices are normal and of whether the approximants are unique require further investigation. It can be shown, however, than under the condition (1.26) any such $P_{\vec{n}}$ has at least $|\vec{n}| - g$ simple zeros on $\bigcup_{\alpha\in\mathscr{E}_{O+}} E_\alpha$, where g is the cyclomatic number (the number of independent undirected cycles) of the $\mathscr{G}$; that is

$$g = \#\mathscr{E} - \#\mathscr{V} + 1. \tag{1.27}$$

Some other conclusions regarding the normality and uniqueness of Hermite-Padé approximants to an arbitrary $\mathscr{G}\mathscr{N}$-systems will be stated below as corollaries to the asymptotic results.

Let $v = \{v_A,\, A \in \mathring{\mathscr{V}}\}$,

$$v_A > 0, \quad A \in \mathring{\mathscr{V}}, \quad \sum_{A\in\mathring{\mathscr{V}}} v_A = 1, \quad v_B \leqslant v_A, \quad \text{if } A \prec B,$$

be a fixed probability distribution on $\mathscr{V}$. Consider a sequence $\mathbf{N}$ of multi-indices $\vec{n} = \{n_A,\, A \in \mathring{\mathscr{V}}\}$ such that condition (1.26) holds and

$$\frac{n_A}{|\vec{n}|} \to v_A, \qquad A \in \mathring{\mathscr{V}}. \tag{1.28}$$

Our aim in this paper is to examine the asymptotic behaviour of $P_{\vec{n}}$ for $\vec{n} \in \mathbf{N}$.

1.3. Weak asymptotics. As weak asymptotics, in this paper we will examine the limit distributions of the zeros of the polynomial $P_{\vec{n}}$ and of the functions $\Psi_{\vec{n},A}$ in (1.25). A way to attack such problems was put forward in [16]. An Angelesco system was used to show that the components of the limit measure for the distributions of the zeros of $P_{\vec{n}}$ (with supports in the intervals $\{E_j\}_{j=1}^p$ of the system (1.11)) must be components of the extremal measure in the problem of minimizing the energy functional of a vector measure with some interaction matrix between the components of the measure (see below for the details). This approach was extended in [20] to generalized Nikishin systems generated by the tree graphs (1.20). This allowed the class of interaction matrices under consideration to be widened. Here, we adapt this approach to arbitrary $\mathscr{GN}$-systems (1.19). The novel feature that arises here (apart from the fact that, as we said, the class of interaction matrices is wider) is that the extremal problem of minimizing the energy functional is treated in the class of vector measures whose components masses are not fixed (as they were before), but are subject to some constraints. A general problem of this kind will be set up in § 1.3.1, where we shall also formulate Theorem 1.1, which concerns the existence, uniqueness and other features of the extremal vector measure. Then, in § 1.3.2, we will make the interaction matrices and linear relations specific to the masses of the components of vector measures corresponding to an arbitrary graph of the form (1.16) in relation to the general extremal problem. Here we shall also formulate Theorem 1.2 pertaining to the limit distribution of the zeros of the polynomial $P_{\vec{n}}$ and its successive Cauchy transforms, and give a corollary on the weak asymptotics of Hermite-Padé approximants.

1.3.1. *Equilibrium of the potentials of vector measures with interaction matrices and linearly related masses.* We begin by setting up a general energy minimization problem for a vector measure subject to some linear restrictions in regard to the masses of its components. As the initial data for the problem we have: a family of regular compact sets in the complex plane

$$\vec{E} = (E_1, \dots, E_m) \Subset \mathbb{C}^m,$$

a real symmetric nonnegative definite matrix

$$\mathscr{A} = (a_{kj})_{k,j=1}^{m} \in \mathbb{R}^{m\times m}, \qquad \mathscr{A} \geqslant 0,$$

a real $r \times m$ matrix of rank r

$$\mathscr{C} = (c_{kj})_{k,j=1}^{r,m} \in \mathbb{R}^{r\times m}, \qquad \operatorname{rank}\mathscr{C} = r,$$

and a nonzero vector

$$b = (b_1, \dots, b_r) \in \mathbb{R}^r, \qquad b \neq 0.$$

We shall assume in addition that the initial data satisfy the following conditions:

$$1)\ a_{jj} > 0; \quad 2)\ a_{kj} = 0 \text{ for } k \neq j, \ \text{ and } E_k \cap E_j \neq \varnothing,\ k, j = 1, \dots, m; \tag{1.29}$$

we also assume that the polytope

$$\left\{ x \in \mathbb{R}^m : \sum_{j=1}^{m} c_{kj} x_j = b_k,\ k = 1, \dots, r;\ x_j \geqslant 0,\ j = 1, \dots, m \right\} \tag{1.30}$$

is bounded and nonempty.

We require some notation from potential theory. Given a compact set $K \subset \mathbb{C}$, we denote by $\mathbf{M}(K)$ the set of all signed measures with finite variation, and by $\mathbf{M}^+(K)$, the set of all finite positive Borel measures ν whose support $S(\nu)$ lies in K. The function

$$V^{\nu}(z) = \int_K \ln\frac{1}{|z-t|}\, d\nu(t), \qquad z \in \mathbb{C},$$

is called the logarithmic potential of the measure ν; the integral

$$I(\nu_1,\nu_2)=\iint_{K\times K}\ln\frac{1}{|x-t|}\,d\nu_1(x)\,d\nu_2(t)$$

is called the mutual energy of two measures ν_1 and ν_2. The total variation (the mass) of a measure ν will be denoted by $|\nu|$.

For a finite collection of compact sets E, we define the set $\mathbf{M}^+(\vec{E})$ to be the Cartesian product of the sets $\mathbf{M}^+(E_j)$ over all $j=1,\dots,m$. So each element μ of the set $\mathbf{M}^+(\vec{E})$ is a collection of finite measures μ_j with $S(\mu_j)\subset E_j$.

Given a measure $\mu\in\mathbf{M}^+(\vec{E})$ with interaction matrix $\mathscr{A}$, the energy functional $J(\mu)$ and the vector potential $W^\mu=(W_1^\mu,\dots,W_m^\mu)$ are defined by

$$J(\mu)=\sum_{k,j=1}^{m}a_{kj}I(\mu_k,\mu_j)\quad\text{and}\quad W_k^\mu(x)=\sum_{j=1}^{m}a_{kj}V^{\mu_j}(x). \tag{1.31}$$

Finally, we introduce the class of measures with linear relations on the masses of the components

$$\mathbf{M}^+_{\mathscr{C},b}(\vec{E}):=\biggl\{\mu\in\mathbf{M}^+(\vec{E}):\sum_{j=1}^{m}c_{kj}|\mu_j|=b_k,\ k=1,\dots,r\biggr\} \tag{1.32}$$

(depending upon the initial parameters $\mathscr{C}$ and b) and consider the following energy minimization problem:

$$\begin{cases}J(\mu)\to\min,\\ \mu\in\mathbf{M}^+_{\mathscr{C},b}(\vec{E}).\end{cases} \tag{1.33}$$

We have the following result.

Theorem 1.1. 1) *There exists a unique measure λ, called extremal, which is a solution of problem* (1.33) (*that is, it minimizes the functional* (1.31) *subject to the linear relations* (1.32) *on the masses of the components*).

2 a) *There exists a set of constants $(l_1,\dots,l_r)$ such that the extremal measure λ gives the minimum of the Lagrangian*:

$$\begin{cases}\mathscr{L}(\mu):=J(\mu)+\displaystyle\sum_{k=1}^{r}l_k\sum_{j=1}^{m}c_{kj}|\mu_j|\to\min,\\ \mu\in\mathbf{M}^+(\vec{E}).\end{cases} \tag{1.34}$$

2 b) *If a measure $\lambda\in\mathbf{M}^+_{\mathscr{C},b}(\vec{E})$ is a solution of problem* (1.34) *for some set of constants $l_1,\dots,l_r$, then λ is the extremal measure.*

3) *The extremal measure λ is the unique measure within the class* (1.32) *which satisfies the following equilibrium conditions with relations on the equilibrium constants*:

$$\begin{cases}W_k^\lambda(x):=\displaystyle\sum_{j=1}^{m}a_{kj}V^{\lambda_j}(x)\begin{cases}=\varkappa_k, & x\in S(\lambda_k),\\ \geqslant\varkappa_k, & x\in E_k,\end{cases}\qquad k=1,\dots,m,\\ (\varkappa_1,\dots,\varkappa_m)\in\operatorname{Im}\mathscr{C}^T.\end{cases} \tag{1.35}$$

Remark 1.2. It worth pointing out that the linear relations between the masses of the components of the vector measures (1.32) are transformed in Theorem 1.1 into linear relations between the equilibrium constants $(\varkappa_1, \dots, \varkappa_m)$ of (1.35). In other words, if the measures (with the initial data $\mathscr{C}$ and b) are allowed to 'flow' between the compact sets $E_1, \dots, E_m$, this imposes additional relations on the equilibrium constants.

This remark can be illustrated by means of a trivial example. Suppose that the initial data correspond to an equilibrium measure on two disjoint intervals; that is,

$$\mathscr{A} := \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathscr{C} := \|1, 1\|, \qquad b := 1, \qquad E_1 \cap E_2 = \varnothing.$$

Then it is clear that

$$W_k^\lambda(x) := V^{\lambda_1}(x) + V^{\lambda_2}(x) = \varkappa_k, \quad k = 1, 2, \quad x \in E_1 \cup E_2, \quad \Rightarrow \quad \varkappa_1 = \varkappa_2.$$

We note that extremal problems (1.33) with relations on the masses (which first appeared in [23]) play a key role in the study of the asymptotic behaviour of Hermite-Padé approximants to general classes of analytic functions with branch points (see [24]).

1.3.2. *The equilibrium problem for graphs and the limit distribution of the poles of the approximants.* In the general extremal problem (1.33), we make the interaction matrices (1.31) and linear relations for the total masses (1.32) of the components of the vector measures specific; these in turn correspond to the limit measures for the distribution of the poles of the Hermite-Padé approximants and of the zeros of the functions (1.25) for the system of Markov functions (1.19) as generated by an arbitrary graph (1.16). From the graph $\mathscr{G}$ we construct a symmetric matrix $\mathscr{A} = (a_{\alpha\beta})$ as follows:

$$a_{\alpha\beta} = \begin{cases} 2 & \text{if } \alpha = \beta \text{ or } \alpha \upuparrows \beta, \\ 1 & \text{if } \alpha \leftrightarrow \beta, \\ -1 & \text{if } \alpha \to \beta \text{ or } \beta \to \alpha, \\ 0 & \text{if the edges } \alpha \text{ and } \beta \text{ have no common vertices.} \end{cases} \tag{1.36}$$

The restrictions on the measure masses are as follows:

$$\biggl\{\mu \in \mathbf{M}^+(\vec{E}) : \sum_{\alpha \in \mathscr{E}_{A-}} |\mu_\alpha| - \sum_{\beta \in \mathscr{E}_{A+}} |\mu_\beta| = v_A,\ A \in \mathring{\mathscr{V}}\biggr\}; \tag{1.37}$$

here $\{v_A > 0,\ A \in \mathring{\mathscr{V}}\}$, $\sum v_A = 1$, are given in (1.28), and by definition the sum over the empty set of indices is zero.

The matrix $\mathscr{A}$ is nonnegative definite. This follows either from the fact that $\mathscr{A}$ is the Gram matrix $a_{\alpha\beta} = (e_{A_2} - e_{A_1}, e_{B_2} - e_{B_1})$, where $\alpha \in (A_1, A_2)$, $\beta \in (B_1, B_2)$, and $\{e_A,\ A \in \mathscr{V}\}$ is the standard basis of $\mathbb{R}^{p+1}$, or from the equality

$$\sum_{\alpha,\beta \in \mathscr{E}} a_{\alpha\beta} x_\alpha x_\beta = \sum_{A \in \mathscr{V}} \biggl(\sum_{\alpha \in \mathscr{E}_{A-}} x_\alpha - \sum_{\beta \in \mathscr{E}_{A+}} x_\beta\biggr)^2.$$

The remaining conditions in the previous subsection concerning the matrix $\mathscr{A}$ are easily verified as are relations (1.32).

So there is a unique measure $\lambda = \{\lambda_\alpha,\ \alpha \in \mathscr{E}\}$ from the class (1.32) satisfying the equilibrium relations (1.35); that is,

$$W_\alpha^\lambda(x) := \sum_{\beta\in\mathscr{E}} a_{\alpha\beta} V^{\lambda_\beta}(x) \begin{cases} = \widetilde{\varkappa}_B - \widetilde{\varkappa}_A, & x \in S(\lambda_\alpha), \\ \geqslant \widetilde{\varkappa}_B - \widetilde{\varkappa}_A, & x \in E_\alpha, \end{cases} \tag{1.38}$$

where $\alpha \in (A, B) \subset \mathscr{E}$, and $\{\widetilde{\varkappa}_A,\ A \in \mathscr{V}\}$ is some distribution of constants over the vertices of the graph. As a result, the equilibrium constants $s\varkappa_\alpha := \widetilde{\varkappa}_B - \widetilde{\varkappa}_A$ are subject to g linear relations, and we can take $\widetilde{\varkappa}_O = 0$.

The limit distributions of the zeros of the polynomials $P_{\vec{n}}$ and of the functions $\Psi_{\vec{n},A}$ in (1.25) are represented in terms of the extremal measure λ. Suppose that $\alpha \in (A, B)$ and that $q_{\vec{n},\alpha}$ is a polynomial whose zeros, counted with multiplicities, are those of $\Psi_{\vec{n},A}$ on the interval E_α; that is,

$$q_{\vec{n},\alpha}(z) := \prod_{x:\Psi_{\vec{n},A}(x)=0,\, x\in E_\alpha} (z-x), \qquad \Bigl(\text{by convention: } \prod_{x\in\varnothing}(z-x) := 1\Bigr). \tag{1.39}$$

Let $\mu(q)$ be the equidistributed discrete measure of the mass $\deg q$ on the zeros of the polynomial q:

$$\mu(q) = \sum_{x:q(x)=0} \delta_x.$$

We have the following result.

Theorem 1.2. *For any $\alpha \in \mathscr{E}$, the limit relations*

$$\frac{1}{|\vec{n}|}\mu(q_{\vec{n},\alpha}) \to \lambda_\alpha$$

hold for $\vec{n} \in \mathbf{N}$ (see (1.28)). In particular,

$$\frac{1}{|\vec{n}|}\mu(P_{\vec{n}}) \to \sum_{\alpha\in O_+} \lambda_\alpha.$$

When we can show that the functions $\Psi_{\vec{n},A}(\prod_{\alpha\in A_+} q_{\vec{n},\alpha})^{-1}$ have no zeros outside $\bigcup_{\alpha\in\mathscr{E}_{A+}} E_\alpha$, we can set down asymptotic formulae for $\Psi_{\vec{n},A}$. This condition holds, for example, if the graph $\mathscr{G}$ is a tree. Theorem 1.3 has the following corollary (we suppose $P_{\vec{n}}$ to be normalized so that its leading coefficient is unity).

Corollary 1.1. *Suppose that $\Psi_{\vec{n},A}(x) \neq 0$ for $x \in \mathbb{C} \setminus \bigcup_{\alpha\in\mathscr{E}_{A-}\cup\mathscr{E}_{A+}} E_\alpha$, $A \in \mathscr{V}$, and some subsequence $\widetilde{\mathbf{N}} \subset \mathbf{N}$. Then the asymptotic formula*

$$\lim_{\vec{n}\in\widetilde{\mathbf{N}}} \frac{1}{|\vec{n}|} \ln|\Psi_{\vec{n},A}(x)| = \sum_{\alpha\in\mathscr{E}_{A-}} V^{\lambda_\alpha}(x) - \sum_{\alpha\in\mathscr{E}_{A+}} V^{\lambda_\alpha}(x) - \widetilde{\varkappa}_A,$$

$$x \in \mathbb{C} \setminus \bigcup_{\alpha\in\mathscr{E}_{A-}\cup\mathscr{E}_{A+}} E_\alpha$$

holds. In particular,

$$\lim_{\vec{n}\in\widetilde{\mathbf{N}}} \frac{1}{|\vec{n}|} \ln |P_{\vec{n}}(x)| = - \sum_{\alpha\in\mathscr{E}_{O+}} V^{\lambda_\alpha}(x), \qquad x \in \mathbb{C} \setminus \bigcup_{\alpha\in\mathscr{E}_{O+}} E_\alpha.$$

1.4. Strong asymptotics. In this section, we shall state a result on the strong asymptotics of the Hermite-Padé approximants (1.5)–(1.7) to the $\mathscr{G}\mathscr{N}$-system (1.19). In the case of strong asymptotics, we shall confine ourselves to diagonal sequences of Hermite-Padé approximants — these being the sequences with multi-indices $\vec{n} = \{n_A,\ A \in \mathring{\mathscr{V}}\}$:

$$n_A := n, \quad A \in \mathring{\mathscr{V}}, \quad |\vec{n}| = pn, \quad \#\mathring{\mathscr{V}} = p. \tag{1.40}$$

In addition to conditions (1.17) and (1.18) imposed on the initial data for the problem, in other words on the measures

$$\{\sigma_\alpha : S(\sigma_\alpha) \subset E_\alpha,\ \alpha \in \mathscr{E}\},$$

we shall assume that $d\sigma_\alpha(x) =: \rho_\alpha(x)\,dx$, where the weights ρ_α are continuous on the intervals $\mathring{E}_\alpha := (a_\alpha, b_\alpha)$, and that

$$\begin{cases} \rho_\alpha(x) =: \rho_\alpha^{(0)}(x)(a_\alpha - x)^{\delta(a_\alpha)}(x - b_\alpha)^{\delta(b_\alpha)}, & \delta(a_\alpha), \delta(b_\alpha) > -1, \\ \rho_\alpha^{(0)}(x) \in C[a_\alpha, b_\alpha]. \end{cases}$$

In addition, we shall require that the weight functions ρ_α be holomorphic in the interior of the supports of the components of the extremal measure λ associated with the problem (1.33), (1.36), (1.37) (with $v_A = 1/p$ in (1.37), by (1.40)):

$$\{S(\lambda_\alpha)\}_{\alpha\in\mathscr{E}} =: \{\cup_{\beta\in(\alpha)^*} E^*_\beta\}_{\alpha\in\mathscr{E}} \to \{E^*_\beta\}_{\beta\in\mathscr{E}^*} =: \vec{E}^*. \tag{1.41}$$

Here $(\alpha)^*$ is the set of connected components (intervals) E^*_β constituting the support $S(\lambda_\alpha)$ of λ_α; these generate the new set of intervals $\vec{E}^*$. Denoting the restriction of ρ_α to the components of the support by $\bigcup_{\beta\in(\alpha)^*} E^*_\beta$, we therefore require that

$$\begin{cases} \rho_\alpha\big|_{E^*_\beta}(x) =: w_\beta(x), & \beta \in (\alpha)^* \subset \mathscr{E}^*, \quad \alpha \in \mathscr{E}, \\ \rho_\alpha^{(0)}\big|_{E^*_\beta}(x) =: w_\beta^{(0)}(x); & w_\beta^{(0)}, \dfrac{1}{w_\beta^{(0)}} \in \mathscr{H}(E^*_\beta), \quad \beta \in \mathscr{E}^*. \end{cases} \tag{1.42}$$

Finally, we impose some restrictions connected with the geometry of the problem. When they are satisfied, they let us exclude the trivial case from consideration; more precisely, we assume that the intervals $\{E_\alpha\}_{\alpha\in\mathscr{E}}$ are arranged in a way that the following conditions hold for the equilibrium measure and its potential:

$$\begin{aligned} &\text{a) } \lambda'_\alpha \neq 0 \quad \text{on } S(\lambda_\alpha), \quad \alpha \in \mathscr{E}, \\ &\text{b) } W^\lambda_\alpha \neq \varkappa_\alpha \quad \text{on } E_\alpha \setminus S(\lambda_\alpha). \end{aligned} \tag{1.43}$$

In the next subsection, we set up the notation we need to formulate Theorem 1.3 on strong asymptotics, and then state the result.

1.4.1. *The Riemann surface and standard functions on it.* Let $\alpha \in \mathscr{E}$ be an arbitrary edge of the initial graph $\mathscr{G}(\mathscr{V}, \mathscr{E}, O)$ joining the vertices $A, B \in \mathscr{V}$. We associate the interval E_α and the support $S(\lambda_\alpha) \subset E_\alpha$ of the corresponding component of the extremal measure λ in problem (1.33) with this edge. Without going into detail, we will just mention that either $S(\lambda_\alpha)$ is the union of a finite number of disjoint intervals (we call these the components of $S(\lambda_\alpha)$, and use the notation in (1.41)) or it is the empty set. We build a new graph $\mathscr{G}^*(\mathscr{V}, \mathscr{E}^*, O)$. To each component $S(\lambda_\alpha)$ we assign an edge joining the vertices $A, B \in \mathscr{V}$ (if $S(\lambda_\alpha)$ is empty, it is not assigned any edge). So we have

$$\mathscr{G}(\mathscr{V}, \mathscr{E}, O) \to \{E_\alpha\}_{\alpha \in \mathscr{E}} \to \{S(\lambda_\alpha)\}_{\alpha \in \mathscr{E}} \to \{E^*_\beta\}_{\beta \in \mathscr{E}^*} \to \mathscr{G}^*(\mathscr{V}, \mathscr{E}^*, O). \tag{1.44}$$

Remark 1.3. Although some edges may be removed (when $S(\lambda_\alpha) = \varnothing$) or added (due to the multiple connectivity of $S(\lambda_\alpha)$), it is easily verified that the graph $\mathscr{G}^*$ remains a partially ordered set with a smallest element.

Using the graph $\mathscr{G}^*$ of (1.44) we shall define the *Riemann surface* $\mathfrak{R}$, which is a $(p+1)$-sheeted covering of the complex plane ($\#\mathscr{V} = p+1$). With each $A \in \mathscr{V}$ of the graph $\mathscr{G}^*$ we associate a replica of the complex plane cut along those intervals of $\{E^*_\beta\}$ which correspond to the edges $\beta \in \mathscr{E}^*_{A+} \cup \mathscr{E}^*_{A-}$. Sheets corresponding to vertices $A, B \in \mathscr{V}$, joined by the edge β, are connected crosswise over the interval E^*_β. Thus we obtain the compact Riemann surface

$$\mathscr{G}^* \quad \longrightarrow \quad \mathfrak{R} := \overline{\bigcup_{A \in \mathscr{V}} \mathfrak{R}_A} : \quad \begin{cases} \pi(\mathfrak{R}_A) := \overline{\mathbb{C}} \setminus \displaystyle\bigcup_{\beta \in \mathscr{E}^*_{A+} \cup \mathscr{E}^*_{A-}} E^*_\beta, \\ \pi_A^{-1}(E^*_{\beta\pm}) = \pi_B^{-1}(E^*_{\beta\mp}), \quad \beta \in (A,B)^*, \end{cases} \tag{1.45}$$

where π denotes the natural projection from $\mathfrak{R}$ onto $\mathbb{C}$, and π_A^{-1} is the lifting from the complex plane to the Ath sheet $\mathfrak{R}_A$ of the Riemann surface $\mathfrak{R}$. We note (see (1.27)) that the resulting surface has genus

$$g^* = \#\mathscr{E}^* - \#\mathscr{V} + 1.$$

We define the *homology basis* $\{\mathbf{a}_\mathfrak{R}, \mathbf{b}_\mathfrak{R}\}$ for the Riemann surface as follows:

$$\mathbf{a}_\mathfrak{R} := \{\mathbf{a}_j\}_{j=1}^{g^*}, \qquad \mathbf{b}_\mathfrak{R} := \{\mathbf{b}_j\}_{j=1}^{g^*}; \tag{1.46}$$

this is the set of cyclic cuts of $\mathfrak{R}$, which turn it into a simply connected domain. Suppose the vertex A of the graph $\mathscr{G}^*$ is entered by $m_A + 1$ edges,

$$\{\alpha_{A,j}\}_{j=0}^{m_A} := \mathscr{E}^*_{A-}.$$

We fix m_A of these edges, and make cuts of $\mathbf{b}_\mathfrak{R}$-cycles around the images in $\mathfrak{R}_A$ of the intervals $\{E^*_{\alpha_{A,j}}\}_{j=1}^{m_A}$ which correspond to these edges, where

$$\mathbf{b}_\mathfrak{R} := \bigcup_{A \in \mathscr{V}} \bigcup_{j=1}^{m_A} \mathbf{b}_{A,j}, \qquad \mathbf{b}_{A,j} := \pi_A^{-1}\{E^*_{\alpha_{A,j}+} \cup E^*_{\alpha_{A,j}-}\}. \tag{1.47}$$

Let $\mathbf{a}_\mathfrak{R}$-cycles be fixed arbitrarily, so that, when taken together with the $\mathbf{b}_\mathfrak{R}$-cycles of (1.47), they form the basis (1.46).

We define a *standard Abelian integral* (of the third kind) G on the Riemann surface $\mathfrak{R}$ as a function with purely imaginary periods on $\mathfrak{R}$ and analytic (bounded) on $\mathfrak{R}$ punctured at infinity on those sheets where G has logarithmic singularities,

$$G: \quad \begin{cases} \text{a) } G \in \mathscr{A}\Big(\mathfrak{R} \setminus \bigcup_{A \in \mathscr{V}} \{\infty^{(A)}\}\Big), \\ \text{b) } G(\zeta) = \begin{cases} -p \log \zeta + O(1) & \text{as } \zeta \to \infty^{(O)}, \\ \log \zeta + O(1) & \text{as } \zeta \to \infty^{(A)}, \quad A \in \mathring{\mathscr{V}}, \end{cases} \\ \text{c) } g := \operatorname{Re} G \text{ is univalent on } \mathfrak{R}. \end{cases} \tag{1.48}$$

Such a function G always exists (see, for example, [25]); it is defined by the relations (1.48) up to an additive constant, which will be fixed by introducing the additional normalization:

$$\text{c}') \ \sum_{A \in \mathscr{V}} g_A = 0.$$

Standard arguments (see, for example, [26] and [27]) tell us that the branches $\{g_A\}_{A \in \mathscr{V}}$ of the real part of the normalized Abelian integral G are connected with the potentials of the components of the equilibrium measure (1.32), (1.38) by the following linear relations:

$$\begin{aligned} g_A &= \sum_{\alpha \in \mathscr{E}^*_{A+}} V^{\lambda_\alpha} - \sum_{\beta \in \mathscr{E}^*_{A-}} V^{\lambda_\beta} + \gamma_A, \qquad A \in \mathring{\mathscr{V}}, \\ g_O &= \sum_{\alpha \in \mathscr{E}^*_{O+}} V^{\lambda_\alpha} - \sum_{A \in \mathring{\mathscr{V}}} \gamma_A; \end{aligned} \tag{1.49}$$

here the normalization constants $\{\gamma_A\}_{A \in \mathscr{V}}$ and the equilibrium constants $\{\widetilde{\varkappa}_A\}_{A \in \mathscr{V}}$ from (1.38) can also be expressed in terms of each other linearly.

In accordance with Proposition 3.2, the principal term of the asymptotics for the polynomial P_n and for the functions $\{\Psi_A\}_{A \in \mathscr{V}}$ is given by the function $\Phi := e^G$, where G is defined in (1.48). So,

$$\Phi(z) = \begin{cases} \dfrac{1}{C_O z^p} + \cdots, & z \to \infty^{(O)}, \\ \dfrac{z}{C_A} + \cdots, & z \to \infty^{(A)}, \quad A \in \mathring{\mathscr{V}}, \end{cases} \tag{1.50}$$

where $C_A = \exp\{-\gamma_A\}$, and so the following normalization of the branches of Φ at infinity is valid:

$$\prod_{A \in \mathscr{V}} \Phi_A(\infty) = 1.$$

The function Φ is not single-valued on $\mathfrak{R}$; the increase in the argument of Φ that occurs when the point z moves round both edges of the cut along the interval E^*_α is

$$\mathop{\triangle}_{\pi^{-1}(E^*_\alpha)} \arg \Phi = 2\pi |\lambda_\alpha|.$$

Thus if we know the values of the equilibrium measure (1.32), (1.38), we can ascertain the increase in the argument of Φ after encircling the $\mathbf{b}_{\mathfrak{R}}$-cycles (1.47). We set

$$\underset{\mathbf{b}_j}{\triangle} \arg \Phi =: 2\pi\omega_j, \qquad j = 1, \dots, g^*. \tag{1.51}$$

Let $\overrightarrow{\Omega}(\zeta)$ be the basis of normalized holomorphic Abelian integrals (of the first kind) on the Riemann surface $\mathfrak{R}$ (see [28]):

$$\overrightarrow{\Omega}(\zeta) := \{\Omega_k(\zeta)\}_{k=1}^{g^*}, \quad \zeta \in \mathfrak{R}: \quad \underset{\mathbf{a}_l}{\triangle}\Omega_k = \delta_{k,l}, \quad \underset{\mathbf{b}_l}{\triangle}\Omega_k = B_{k,l}, \quad \operatorname{Im}\|B_{k,l}\| > 0. \tag{1.52}$$

We recall the definition of the theta function. If the imaginary part of the matrix of parameters $\|B_{k,l}\|$ is positive definite, then the multiple series

$$\theta(u_1, \dots, u_{g^*}) := \sum_{n_1=-\infty,\, \dots,\, n_{g^*}=-\infty}^{+\infty,\dots,+\infty} \exp\left\{\pi i \sum_{\mu=1}^{g^*}\sum_{\nu=1}^{g^*} B_{\mu\nu} n_\mu n_\nu + 2\pi i \sum_{\nu=1}^{g^*} n_\nu u_\nu\right\}$$

converges uniformly and defines an entire function g^* of variables $\vec{u} := (u_1, \dots, u_{g^*})$. Let $\vec{e} \in \mathbb{C}^{g^*}$ be an arbitrary vector of constants. Taking the vector of Abelian integrals $\overrightarrow{\Omega}(\zeta)$ as a new variable in the series for θ and translating by $\vec{e}$, we arrive at the following function of one variable (acting on $\mathfrak{R}$)

$$\Theta^{(\vec{e})}(\zeta) := \theta\big(\vec{\Omega}(\zeta) - \vec{e}\big), \qquad \zeta \in \mathfrak{R}, \tag{1.53}$$

which is called a *theta function on the Riemann surface* $\mathfrak{R}$. The basic properties of $\Theta^{(\vec{e})}(\zeta)$ are as follows: it is holomorphic (analytic and single-valued) on $\mathfrak{R}$ cut along the $\mathbf{a}_{\mathfrak{R}}$-cycles, and it has g^* zeros (or vanishes identically); that is,

$$\begin{aligned} &\text{a) } \Theta^{(\vec{e})} \in \mathscr{H}\left(\mathfrak{R} \setminus \left\{\bigcup_{j=1}^{g^*} \mathbf{a}_j\right\}\right); \\ &\text{b) } \exists\, \{\dot{\zeta}_k\}_{k=1}^{g^*}: \quad \Theta^{(\vec{e})}(\dot{\zeta}_k) = 0, \quad k = 1, \dots, g^*. \end{aligned} \tag{1.54}$$

Varying the vector of constants in $\Theta^{(\vec{e})}(\zeta)$ varies the location of the zeros of the theta function on $\mathfrak{R}$:

$$\vec{e} \quad \longleftrightarrow \quad \{\dot{\zeta}_k\}_{k=1}^{g^*}. \tag{1.55}$$

1.4.2. *The Szegő function and the strong asymptotics formulae.* In the strong asymptotics formulae a significant role is played by the so-called Szegő function, which is a solution of the boundary value problem for an analytic function whose boundary values are dependent on the orthogonality weights. In our case, the boundary value problem is considered on the Riemann surface of (1.45):

$$\mathfrak{R} := \overline{\bigcup_{A\in\mathscr{V}} \mathfrak{R}_A} = \Big(\bigcup_{A\in\mathscr{V}} \mathfrak{R}_A\Big) \bigcup \Big(\bigcup_{\alpha\in\mathscr{E}^*} \partial\mathfrak{R}_\alpha\Big), \qquad \partial\mathfrak{R}_\alpha := \pi^{-1}\{E^*_{\alpha+} \cup E^*_{\alpha-}\};$$

here $\partial\mathfrak{R}_\alpha$ denotes a Jordan curve which separates the sheets $\mathfrak{R}_A$ and $\mathfrak{R}_B$ for $\alpha \in (B, A)$.

Let $(Gr)_\Omega$ be a complex Green's function for the domain Ω with a singularity at $\infty\in\Omega$ (its real part is harmonic in $\Omega\setminus\{\infty\}$, has a logarithmic singularity at $\infty\in\Omega$, and vanishes on the boundary $\partial\Omega$). The derivative $h_\Omega^{(0)}:=(Gr)'_\Omega$ is holomorphic on Ω,

$$h_\Omega^{(0)}\in\mathscr{H}(\Omega), \tag{1.56}$$

and, counting the zero at infinity, it has the same number of zeros in Ω as the connectivity of Ω.

Using the initial data $w=\{w_\alpha\}_{\alpha\in\mathscr{E}^*}$ (see (1.17) and (1.42)), on the contours $\partial\mathfrak{R}(E^*):=\bigcup_{\alpha\in\mathscr{E}^*}\partial\mathfrak{R}_\alpha$ we define the following analogues of 'trigonometric weights' for $\alpha\in\mathscr{E}^*$

$$\widetilde{w}:=\{\widetilde{w}_\alpha\},\qquad \widetilde{w}_\alpha:=\begin{cases} i\dfrac{1}{(\widetilde{h}_B^{(0)})_-}w_\alpha, & \alpha\in(O,B)^*,\\[2ex] -\dfrac{\widetilde{h}_A^{(0)}}{(\widetilde{h}_B^{(0)})_-}w_\alpha, & \alpha\in(A,B)^*,\quad A\ne O;\end{cases} \tag{1.57}$$

here $(\,\cdot\,)_-$ means the boundary values as one approaches the boundary (E^*_α in this case) from the right, and

$$\widetilde{h}_A^{(0)}:=h^{(0)}_{\Omega_{A_-}},\qquad \Omega_{A_-}:=\overline{\mathbb{C}}\setminus\bigcup_{\beta\in\mathscr{E}^*_{A-}}E^*_\beta,\quad A\in\mathring{\mathscr{V}}. \tag{1.58}$$

The initial data for the weight $\widetilde{w}_\alpha$ on the contour $\partial\mathfrak{R}_\alpha$ means that the same function (1.57) is defined on both edges of the cut E^*_α.

By definition, a *Szegő function* is a function which is piecewise holomorphic on $\mathfrak{R}$, and is a solution of the following homogeneous Riemann boundary value problem (see [29]):

$$\mathscr{F}:=\{\mathscr{F}_A\}_{A\in\mathscr{V}}:\quad\begin{cases} 1)\ \mathscr{F}\in\mathscr{H}\Big(\mathfrak{R}\setminus\Big(\bigcup\limits_{\alpha\in\mathscr{E}^*}\partial\mathfrak{R}_\alpha\bigcup\mathbf{a}_\mathfrak{R}\Big)\Big),\\ 2)\ \mathscr{F}_+=\mathscr{F}_-\widetilde{w}_\alpha\ \text{ on }\partial\mathfrak{R}_\alpha,\quad \alpha\in\mathscr{E}^*,\\ 3)\ \prod\limits_{A\in\mathscr{V}}\mathscr{F}_A(\infty)=1.\end{cases} \tag{1.59}$$

Below (see § 4.4.1), we shall examine the boundary value problem (1.59) in more detail; in particular, we shall use meromorphic (Cauchy) differentials on $\mathfrak{R}$ to obtain an integral representation of its solution. We note that the Szegő function (see condition 1) in (1.59)) is discontinuous across the contours of $\mathbf{a}_\mathfrak{R}$-cycles. Hence it has nontrivial $\mathbf{b}_\mathfrak{R}$-periods, which we denote

$$\mathop{\triangle}_{\mathbf{b}_k}\arg\mathscr{F}=:2\pi c_w^{(k)},\qquad k=1,\dots,g^*. \tag{1.60}$$

We now proceed to formulate the result on strong asymptotics.

Consider the set $\{z_k^*\}_{k=1}^{g^*}$ of all finite zeros of the derivatives of the Green's functions (1.56) in the domains

$$\Omega_{A-}:=\overline{\mathbb{C}}\setminus\bigcup_{\beta\in\mathscr{E}^*_{A-}}E^*_\beta,\qquad A\in\mathring{\mathscr{V}},$$

and lift these zeros to the corresponding sheets (that is, to $\mathfrak{R}_A$) of the Riemann surface. We have

$$\begin{aligned} &\{z_k^*\}_{k=1}^{g^*}: \quad \exists\, A(k): \quad h^0_{\Omega_{A-}}(z_k^*)=0, \quad A\in\mathring{\mathscr{V}}, \\ &\{\zeta_k^*\}_{k=1}^{g^*}: \quad \zeta_k^* := \pi^{-1}_{A(k)}(z_k^*), \quad k=1,\dots,g^*. \end{aligned} \tag{1.61}$$

It is easily verified that these points are equal in number to the genus of $\mathfrak{R}$.

We choose a vector of constants $\vec{e}$ so that the theta function (1.53) vanishes at these points of $\mathfrak{R}$; that is,

$$\vec{e}: \quad \Theta^{(\vec{e})}(\zeta_k^*)=0, \quad k=1,\dots,g^*. \tag{1.62}$$

Let

$$\vec{c}_{n,w} := (n\omega_1 + c_w^{(1)},\dots, n\omega_g^* + c_w^{(g^*)}) \tag{1.63}$$

be the vector of constants consisting of $\mathbf{b}_{\mathfrak{R}}$-periods of the Abelian integral G of (1.51) and of the Szegő function $\mathscr{F}$ of (1.60). Also let Λ be the set of indices $\{n\}$ such that the theta function given by the vector of constants $(\vec{e}-\vec{c}_{n,w})$ has no zeros at points of $\mathfrak{R}$ which lie over infinity; that is,

$$\Lambda := \{n\}: \quad \Theta^{(\vec{e}-\vec{c}_{n,w})}(\zeta)\neq 0, \quad \pi(\zeta)=\infty. \tag{1.64}$$

Finally, we set

$$T^{(\vec{e},\vec{c}_{n,w})}(\zeta) := \frac{\Theta^{(\vec{e}-\vec{c}_{n,w})}(\zeta)}{\Theta^{(\vec{e})}(\zeta)} =: \{T_A\}_{A\in\mathscr{V}}. \tag{1.65}$$

We have the following theorem.

Theorem 1.3. *Let $\{P_n\}$ be the sequence of denominators of the diagonal (1.40) Hermite-Padé approximants (1.5)–(1.7) to the generalized Nikishin system (1.19), as generated by a $\mathscr{G}$-graph (1.16), for which the intervals $\{E_\alpha\}_{\alpha\in\mathscr{E}}$ are subject to the condition (1.43), and the measures are subject to the initial conditions (1.17), (1.18) and the condition (1.42). Then, for $\vec{n}=(n,\dots,n)$, $n\in\Lambda$, as $n\to\infty$, the asymptotic formula*

$$P_n(z) = \left(\frac{C_O}{\Phi_O(z)}\right)^n \frac{\mathscr{F}_O(z)}{\mathscr{F}_O(\infty)}\,\frac{T_O(z)}{T_O(\infty)}\left(1+O\left(\frac{1}{n}\right)\right)$$

holds uniformly in z on compact sets $K\Subset\mathbb{C}\setminus\{\bigcup_{\alpha\in\mathscr{E}^*_{O+}} E^*_\alpha\}$ of the complex plane, and the asymptotic formula

$$\begin{aligned} P_n(x) = \Bigg[&\left(\frac{C_O}{\Phi_{O+}(x)}\right)^n \frac{\mathscr{F}_{O+}(x)}{\mathscr{F}_O(\infty)}\,\frac{T_{O+}(x)}{T_O(\infty)} \\ &+ \left(\frac{C_O}{\Phi_{O-}(x)}\right)^n \frac{\mathscr{F}_{O-}(x)}{\mathscr{F}_O(\infty)}\,\frac{T_{O-}(x)}{T_O(\infty)}\Bigg]\left(1+O\left(\frac{1}{n}\right)\right) \end{aligned}$$

holds uniformly in x on compact sets $K\Subset\bigcup_{\alpha\in\mathscr{E}^*_{O+}} E^*_\alpha$ which are interior to the system of intervals. Here, the right-hand sides of the asymptotic formulae contain branches of the functions (1.50), (1.59) and (1.65) from the sheet O of the Riemann surface (1.45) which corresponds to the root of the graph $\mathscr{G}^*$ of (1.44).

The proof we give of this theorem in § 4 depends upon the method used for the matrix Riemann-Hilbert problem [30]. On the one hand, this compels us to impose rather stringent analyticity conditions, but on the other hand, it gives the global asymptotic pattern of $\{P_n\}$, including the local asymptotics near the end-points of the intervals of $\bigcup_{\alpha\in\mathscr{E}^*_{O+}} E^*_\alpha$. We also obtain the strong asymptotics of the functions of the second kind $\Psi_{n,A}$, $A\in\mathscr{V}$.

§ 2. The proof of Theorem 1.1

1) We shall need some properties of the energy functional; for a proof we refer the reader to [31].

First, the principle of descent holds for the energy functional $I(\,\cdot\,)$; that is, if $\mu_n\xrightarrow{*}\mu$, then $\varliminf_{n\to\infty} I(\mu_n)\geqslant I(\mu)$.

The functional $I(\,\cdot\,,\cdot\,)$ is a bilinear form on the linear space $\mathbf{M}(K)$ of all signed measures (charges) $\delta=\delta_+-\delta_-$ on a compact set K with $I(\delta_++\delta_-)<\infty$. The second fact that we require is that this form is positive definite. More precisely, under the (technical) assumption that $K\subset\{z:|z|<1\}$, we have $I(\delta)\geqslant 0$ for any signed measure $\delta\in\mathbf{M}(K)$, and moreover, if $I(\delta)=0$, then $\delta=0$. In other words, $I(\,\cdot\,,\cdot\,)$ defines an inner product on $\mathbf{M}(K)$ by $(\delta_1,\delta_2):=I(\delta_1,\delta_2)$.

We prove the existence of the extremal measure. The principle of descent also holds for the functional $J(\mu)=\sum_{k,j}a_{kj}I(\mu^k,\mu^j)$ since it is valid for $a_{jj}I(\mu^j)$ and the nondiagonal terms are continuous, as $E_k\cap E_j=\varnothing$ for $a_{kj}\neq 0$ (see (1.29)).

Let $\mu_n\in\mathbf{M}^+_{\mathscr{C},b}(\vec{E})=\{\mu\in\mathbf{M}^+(\vec{E}):\sum_{j=1}^m c_{kj}|\mu_j|=b_k,\ k=1,\dots,r\}$ be a minimizing sequence, that is,

$$J(\mu_n)\to J_0:=\inf\{J(\mu):\mu\in\mathbf{M}^+_{\mathscr{C},b}(E)\}.$$

As the polytope (1.30) is bounded, thus the masses of the measures from $\mathbf{M}^+_{\mathscr{C},b}(\vec{E})$ are also bounded; hence the set $\mathbf{M}^+_{\mathscr{C},b}(\vec{E})$ is compact in the weak topology. From the sequence μ_n we choose a convergent subsequence $\mu_{n_k}\to\lambda\in\mathbf{M}^+_{\mathscr{C},b}(\vec{E})$, $n\in\Lambda$. On the one hand, $J(\lambda)\geqslant J_0$. On the other hand, by semi-continuity, $J_0\geqslant J(\lambda)$. Hence λ is an extremal measure.

To verify the uniqueness, observe that $J(\mu)$ is a convex functional, as $\mathscr{A}$ is nonnegative definite. If λ and λ' were two extremal measures, this would imply that $(\lambda+\lambda')/2$ is also an extremal measure: $J((\lambda+\lambda')/2)\leqslant (J(\lambda)+J(\lambda'))/2=J_0$. But since $J(\lambda-\lambda')+J(\lambda+\lambda')=2J(\lambda)+2J(\lambda')$, we have $J(\lambda-\lambda')=0$, and so $(\lambda-\lambda')\mathscr{A}=0$. Consider the charge $\nu=\lambda-\lambda'$: $\nu=(\nu_1,\dots,\nu_m)$, $S(\nu_k)\subset E_k$, $k=1,\dots,m$. Since $\nu\mathscr{A}=0$, the relation

$$-\,a_{kk}\nu_k=\sum_{j=1,\,j\neq k}^{m} a_{kj}\nu_j \tag{2.1}$$

holds for all k. In the last sum, we have either $a_{kj}=0$ or $E_k\cap E_j=\varnothing$ (see (1.29)). Hence, the supports of the charges in the right- and left-hand sides of (2.1) are disjoint, and $a_{kk}\nu_k=0$. Since $a_{kk}>0$, we have $\nu_k=0$. As a result, $\lambda=\lambda'$, and so the extremal measure is unique. This proves assertion 1) of Theorem 1.1.

2) In essence, assertions a) and b) are just variants of the Kuhn-Tucker theorem [32]. We proceed to prove that this is indeed so.

Let λ be the extremal measure. Suppose that $J(\lambda) = J_0$, and consider the following set in $\mathbb{R}^{r+1}$:

$$\Xi := \biggl\{(t_0,\dots,t_r) =: \mathbf{t} \biggm| \exists\, \mu \in \mathbf{M}^+(E) : J(\mu) \leqslant t_0 + J_0, \\ \sum_{j=1}^{m} c_{kj}|\mu_j| = b_k + t_k,\ k = 1,\dots,r\biggr\}.$$

It is readily verified that this set is nonempty, convex, and has no common points with the ray

$$\Upsilon := \{(\omega_0, 0, \dots, 0) : \omega_0 < 0\}.$$

By the finite-dimensional separation theorem, there is a nonzero vector $(l_0, l_1, \dots, l_r) \in \mathbb{R}^{r+1}$ such that

$$\inf_{\Xi} \sum_{j=0}^{r} l_j t_j \geqslant \sup_{\omega_0<0} l_0\omega_0 \geqslant 0,$$

and so,

$$\sum_{j=0}^{r} l_j t_j \geqslant 0 \quad \forall\, \mathbf{t} \in \Xi. \tag{2.2}$$

We claim that $l_0 > 0$. Substituting $(1, 0, \dots, 0) \in \Xi$ in (2.2) gives $l_0 \geqslant 0$. Suppose that $l_0 = 0$. Then, for any $x \in \mathbb{R}^m_+$,

$$\sum_{k=1}^{r} l_k \sum_{j} c_{kj} x_j \geqslant \sum_{k=1}^{r} l_k \sum_{j} c_{kj}|\lambda_j|.$$

It follows that $\sum_{k=1}^{r} l_k \sum_j c_{kj}|\lambda_j| = 0$ and $\sum_{k=1}^{r} l_k c_{kj} \geqslant 0$ for each j. But this is possible only when $l_1 = \dots = l_r = 0$, contradicting the assumption $l \neq 0$. Hence, $l_0 > 0$, and thus we can take $l_0 = 1$.

Suppose now that $\mu \in \mathbf{M}^+(\vec{E})$. Then

$$J(\mu) - J_0 + \sum_k l_k \biggl(\sum_j c_{kj}|\mu_j| - b_k\biggr) \geqslant 0;$$

that is, $\mathscr{L}(\mu) \geqslant \mathscr{L}(\lambda)$.

Conversely, if a measure $\lambda \in \mathbf{M}^+_{\mathscr{C},b}(\vec{E})$ satisfies

$$\mathscr{L}(\mu) \geqslant \mathscr{L}(\lambda) \quad \forall\, \mu \in \mathbf{M}^+(\vec{E}),$$

then, for every $\mu \in \mathbf{M}^+_{\mathscr{C},b}(\vec{E})$, we have $J(\mu) \geqslant J(\lambda)$. This proves assertion 2).

3) We claim that the equilibrium conditions (1.35) are equivalent to the Lagrangian minimization problem (1.34).

Suppose that, for $\lambda \in \mathbf{M}^+_{\mathscr{C},b}(\vec{E})$,

$$\mathscr{L}(\lambda) \leqslant \mathscr{L}(\mu) \quad \forall\, \mu \in \mathbf{M}^+(\vec{E}), \tag{2.3}$$

and consider the charge $\nu^{(k)} = (0, \dots, \nu, \dots, 0) \in \mathbf{M}^+(\vec{E})$ whose only nonzero component is its kth component. Then

$$\begin{aligned}\mathscr{L}(\lambda + \varepsilon\nu^{(k)}) - \mathscr{L}(\lambda) &= 2\varepsilon \sum_j a_{kj} I(\lambda_j, \nu) + \varepsilon \sum_j l_j c_{jk} |\nu| + O(\varepsilon^2)\\ &= \varepsilon \int \Bigl(2W_k^\lambda + \sum_j l_j c_{jk}\Bigr)\, d\nu + O(\varepsilon^2).\end{aligned}$$

We set

$$\varkappa_k = -\frac{1}{2} \sum_j l_j c_{jk}. \tag{2.4}$$

We will show that $W_k^\lambda \geqslant \varkappa_k$ on E_k. Since $W_k^\lambda - a_{kk}V^{\lambda_k}$ is continuous on E_k, which is a regular compact set, it suffices to show that $W_k^\lambda \geqslant \varkappa_k$ quasi-everywhere on E_k. Assume on the contrary that there is an $E \subset E_k$, $\operatorname{cap} E > 0$, on which $W_k^\lambda < \varkappa_k$, and consider a (scalar) positive measure $\nu \in \mathbf{M}^+(E_k)$. Then $\mathscr{L}(\lambda + \varepsilon\nu^{(k)}) - \mathscr{L}(\lambda) < 0$, which contradicts (2.3) for sufficiently small $\varepsilon > 0$. Assume now that $W_k^\lambda(x_0) > \varkappa_k$ at $x_0 \in S(\lambda_k)$. Then since W_k^λ is lower semi-continuous, there is a neighbourhood $U(x_0)$ in which $W_k^\lambda > \varkappa_k$. Also, $\lambda(U(x_0)) > 0$, since $x_0 \in S(\lambda_k)$. We now pick a negative measure ν with support in $U(x_0)$ and choose an $\varepsilon > 0$ so that $\lambda_k + \varepsilon\nu$ is a positive measure. This again contradicts (2.3).

Conversely, assume that the equilibrium relations (1.35) hold. Then the constants $l_1, \dots, l_r$ satisfying (2.4) are known. Suppose that $\mathscr{L}(\mu) < \mathscr{L}(\lambda)$ for some $\mu \in \mathbf{M}^+(\vec{E})$. Then, by convexity, $\mathscr{L}((1-\varepsilon)\lambda + \varepsilon\mu) < \mathscr{L}(\lambda)$, and also $(1-\varepsilon)\lambda + \varepsilon\mu \in \mathbf{M}^+(\vec{E})$. Hence the derivative of $\mathscr{L}$ in the direction $\mu - \lambda$ at λ is negative, that is,

$$2\sum_{j,k} a_{jk} I(\lambda_j, \mu_k - \lambda_k) + \sum_j l_j \sum_k c_{jk}(|\mu_k| - |\lambda_k|) < 0.$$

The last inequality holds if and only if

$$2\sum_{j,k} a_{jk} I(\lambda_j, \mu_k) + \sum_j l_j \sum_k c_{jk}(|\mu_k|) < \mathscr{L}(\lambda). \tag{2.5}$$

Integrating the kth equilibrium condition over λ_k and summing over k, this gives $\mathscr{L}(\lambda) = 0$. Integrating the kth equilibrium relation over μ_k, we obtain

$$2\sum_j a_{jk} I(\lambda_j, \mu_k) + \sum_j l_j c_{jk}(|\mu_k|) \geqslant 0,$$

and hence, summing over k, we arrive at a contradiction to (2.5). The proof of Theorem 1.1 is complete.

§ 3. The proof of Theorem 1.2

3.1. Auxiliary facts about functions of the second kind.

Proposition 3.1. *Under condition* (1.26), *the order of the zero of the function* $\Psi_{\vec n,A}$ *at infinity is not lower than that of the function* $R_{\vec n,A}$ *in* (1.22). *Moreover,*

$$\Psi_{\vec n,A}(z)=O(z^{-n_A-1})\quad \text{as } z\to\infty,\quad A\in\mathring{\mathscr V}.$$

Proof. The functions of the second kind $R_{\vec n,A}$ can be written as follows:

$$\begin{aligned}R_{\vec n,A}(z)&=\sum_{t_A\in\mathscr T_A}\int\frac{P_{\vec n}(x)\,d\mu_{t_A}(x)}{z-x}\\&=\sum_{(\alpha_1\to\cdots\to\alpha_k)\in\mathscr T_A}\int\frac{P_{\vec n}(x_1)\,d\sigma_{\alpha_1}(x_1)\cdots d\sigma_{\alpha_k}(x_k)}{(z-x_1)(x_1-x_2)\cdots(x_{k-1}-x_k)},\end{aligned}$$

while the functions $\Psi_{\vec n,A}$, defined inductively, have the following representation

$$\Psi_{\vec n,A}(z)=\sum_{(\alpha_1\to\cdots\to\alpha_k)\in\mathscr T_A}\int\frac{P_{\vec n}(x_1)\,d\sigma_{\alpha_1}(x_1)\cdots d\sigma_{\alpha_k}(x_k)}{(x_1-x_2)\cdots(x_{k-1}-x_k)(x_k-z)}.$$

Adding these two equations yields

$$R_{\vec n,A}(z)+\Psi_{\vec n,A}(z)=\sum_{(\alpha_1\to\cdots\to\alpha_k)\in\mathscr T_A}\int\frac{(x_k-x_1)P_{\vec n}(x_1)\,d\sigma_{\alpha_1}(x_1)\cdots d\sigma_{\alpha_k}(x_k)}{(z-x_1)(x_1-x_2)\cdots(x_{k-1}-x_k)(x_k-z)}.$$

Substituting $-(x_1-x_2)-\cdots-(x_{k-1}-x_k)$ for (x_k-x_1) in the numerator we obtain

$$\begin{aligned}&R_{\vec n,A}(z)+\Psi_{\vec n,A}(z)\\&=\sum_{(\alpha_1\to\cdots\to\alpha_k)\in\mathscr T_A}\sum_{j=1}^{k-1}(-1)^{k-j}\int\frac{P_{\vec n}(x)\,d\langle\sigma_{\alpha_1},\dots,\sigma_{\alpha_j}\rangle(x)}{z-x}\int\frac{d\langle\sigma_{\alpha_k},\dots,\sigma_{\alpha_{j+1}}\rangle(x)}{x-z},\end{aligned}$$

and rearranging, this gives

$$R_{\vec n,A}(z)+\Psi_{\vec n,A}(z)=\sum_{B\prec A}\biggl(R_{\vec n,B}(z)\sum_{(\beta_1\to\cdots\to\beta_l)\in\mathscr T_{BA}}(-1)^l\int\frac{d\langle\sigma_{\beta_l},\dots,\sigma_{\beta_1}\rangle(x)}{x-z}\biggr).$$

Here $\mathscr T_{BA}$ is the set of all paths between the vertices B and A. In view of (1.26), this gives $R_{\vec n,A}(z)+\Psi_{\vec n,A}(z)=O(z^{-n_A-1})$ as $z\to\infty$. The proof of Proposition 3.1 is complete.

We set (see (1.39))

$$q_{B+}:=\prod_{C\in B_+}\prod_{\beta\in(B,C)}q_\beta=\prod_{\beta\in\mathscr E_{B+}}q_\beta,\qquad q_{B-}:=\prod_{A\in B_-}\prod_{\alpha\in(A,B)}q_\alpha=\prod_{\alpha\in\mathscr E_{B-}}q_\alpha,\tag{3.1}$$

and correspondingly,

$$m_\beta:=\deg q_\beta,\qquad m_{B\pm}:=\deg q_{B\pm}.$$

Here if $B_+=\varnothing$, then we put $q_{B+}=1$ and $m_{B+}=0$.

Proposition 3.2. 1) *Suppose that* $B \in \mathring{\mathscr{V}}$. *Then*

$$\sum_{A\in B_-}\sum_{\alpha\in(A,B)}\int \Psi_{\vec{n},A}(x)x^k\frac{d\sigma_\alpha(x)}{q_{B+}(x)}=0, \qquad k=0,\dots,m_{B+}+n_B-1. \tag{3.2}$$

2) *Let* $d_B := m_{B-}-m_{B+}-n_B+\operatorname{ind}B_- - 1$, *where*

$$\operatorname{ind}B_- := \#\{\alpha : \alpha\in(A,B),\, A\in B_-\}.$$

Then

$$d_B \geqslant 0, \qquad B\in\mathring{\mathscr{V}}, \qquad \sum_{B\in\mathring{\mathscr{V}}} d_B \leqslant g. \tag{3.3}$$

Proof. Using Proposition 3.1 and the definition of q_{B+} (see (3.1) and (1.39)) we have

$$\frac{\Psi_{\vec{n},B}(z)}{q_{B+}(z)} = O\biggl(\frac{1}{z^{m_{B+}+n_B+1}}\biggr) \quad \text{as } z\to\infty, \qquad \frac{\Psi_{\vec{n},B}}{q_{B+}} \in \mathscr{H}\biggl(\mathbb{C}\setminus\bigcup_{\alpha\in\mathscr{E}_{B-}}E_\alpha\biggr),$$

and so by Cauchy's theorem and the definition of $\Psi_{\vec{n},B}$ (see (1.25)),

$$0=\oint_{(\infty)} z^k\frac{\Psi_{\vec{n},B}(z)}{q_{B+}(z)}\,dz=\oint_{(\infty)}\frac{z^k}{q_{B+}(z)}\sum_{A\in B_-}\sum_{\alpha\in(A,B)}\int\frac{\Psi_{\vec{n},A}(x)\,d\sigma_\alpha(x)}{x-z},$$

$k=0,\dots,m_{B+}+n_B-1$. Now applying Fubini's theorem, integrating with respect to z, and invoking Cauchy's integral formula we obtain (3.2).

Further, we note that by (3.1) the function $\Psi_{\vec{n},A}$ has exactly m_{B-} zeros on $\bigcup_{\alpha\in\mathscr{E}_{B-}}E_\alpha$ for $A\in B_-$. If we take into account that the weight function may change sign in the gaps between the intervals, and then apply the orthogonality relations (3.2), we obtain a lower bound for m_{B-}; namely $m_{B-}\geqslant m_{B+}+n_B-\operatorname{ind}B_-+1$, and so $\delta_B\geqslant 0$. Adding these inequalities over all $B\in\mathring{\mathscr{V}}$ gives

$$\sum_{B\in\mathring{\mathscr{V}}}\delta_B=\sum_{B\in\mathring{\mathscr{V}}}(m_{B-}-m_{B+}-n_B+\operatorname{ind}B_-)-\#\mathring{\mathscr{V}}=\sum_{B\in\mathring{\mathscr{V}}}(m_{B-}-m_{B+})-|\vec{n}|+g\geqslant 0,$$

and now, since

$$\sum_{B\in\mathring{\mathscr{V}}}(m_{B-}-m_{B+})=m_{O+},$$

we obtain a lower bound for the number of zeros of $P_{\vec{n}}$ on $\bigcup_{\alpha\in\mathscr{E}_{O+}}E_\alpha$:

$$m_{O+}\geqslant|\vec{n}|-\#\mathscr{E}+\#\mathring{\mathscr{V}}=|\vec{n}|-g. \tag{3.4}$$

On the other hand, $|\vec{n}|\geqslant\deg P_{\vec{n}}\geqslant m_{O+}$, so that $\sum_{B\in\mathring{\mathscr{V}}}d_B\leqslant g$, and thus

$$m_{B+}+n_B-\operatorname{ind}B_-+1+g\geqslant m_{B-}\geqslant m_{B+}+n_B-\operatorname{ind}B_-+1. \tag{3.5}$$

The proof of Proposition 3.2 is complete.

3.2. Auxiliary facts about the asymptotic behavior of quasi-orthogonal polynomials. In this subsection we will show that the theorem due to Gonchar and Rakhmanov in [33] about the asymptotic behaviour of polynomials orthogonal with respect to variable weights also remains true for quasi-orthogonal polynomials.

We first introduce some notation. Let $E=\bigcup E_j$ be the disjoint union of a finite number of intervals on the real axis, and let $\mathscr{F}(E)$ be the set of functions from E to $(-\infty,+\infty]$, which are lower semi-continuous and weakly approximatively continuous on E. We say that a sequence f_n from $\mathscr{F}(E)$ is convergent to $f\in\mathscr{F}(E)$ as $n\to\infty$ (written $f_n\xrightarrow{\mathscr{F}}f$), if the following two conditions hold: a) f_n converges to f with respect to the Lebesgue measure on E; and b) the inequality $\min_\Delta f_n\to\min_\Delta f$ holds for any interval $\Delta\subset E$.

Consider the polynomial $Q_n(x)=x^n+\cdots$, orthogonal with respect to the weight function $e^{-nf_n(x)}$ on E:

$$\int Q_n(x)x^l e^{-nf_n(x)}\,d\sigma(x)=0,\qquad l=0,\dots,n-1.$$

Here $\sigma(x)$ is a finite positive Borel measure, $\sigma'(x)>0$ a.e. on E, and $f_n\in\mathscr{F}(E)$. Suppose that $f_n\xrightarrow{\mathscr{F}}f$, $n\in\Lambda$, and that a unit measure λ on E is an equilibrium measure in the field f: $(2V^\lambda+f)(x)=w:=\min_E(2V^\lambda+f)$, $x\in S(\lambda)$. It was shown in [33] that

$$\frac{1}{n}\mu(Q_n)\to\lambda,\qquad n\in\Lambda, \tag{3.6}$$

$$\biggl(\int Q_n^2e^{-nf_n(x)}\,d\sigma(x)\biggr)^{1/n}\to e^{-w},\qquad n\in\Lambda. \tag{3.7}$$

It is clear that a corollary of this result is an upper estimate for the functions of the second order for $x\notin E$, namely $\overline{\lim}\,|R_n(x)|^{1/n}\leqslant e^{V^\lambda(x)-w}$, where

$$R_n(x)=\int\frac{Q_n(t)e^{-nf_n(t)}\,d\sigma(t)}{x-t}=\frac{1}{Q_n(x)}\int\frac{Q_n^2(t)e^{-nf_n(t)}\,d\sigma(t)}{x-t}.$$

We will now consider the sequence of quasi-orthogonal polynomials

$$p_n=x^n+\cdots:\qquad\int p_n(x)x^le^{-nf_n(x)}\,d\sigma(x)=0,\quad l=0,\dots,n-1-g,$$

where $g>0$ and is independent of n. Assuming the above conditions on the weight functions, we shall prove the following:

Proposition 3.3. 1) *The polynomials p_n satisfy*

$$\frac{1}{n}\mu(p_n)\to\lambda,\qquad n\in\Lambda;$$

2) *Suppose that all zeros of p_n lie on E. Let E' be an arbitrary finite union of intervals, $E'\cap E=\varnothing$. Then*

$$-\frac{1}{n}\ln\biggl|\int\frac{p_n(t)e^{-nf_n(t)}\,d\sigma(t)}{t-x}\biggr|\xrightarrow{\mathscr{F}(E')}-V^\lambda(x)+w,\qquad n\in\Lambda.$$

Proof. 1) We claim that the counting measure of p_n does not differ very much from that of the orthogonal polynomial; that is, we will show that their distribution functions satisfy the inequality

$$\left|\int_{-\infty}^{x} d\mu(Q_n) - \int_{-\infty}^{x} d\mu(p_n)\right| \leqslant g. \tag{3.8}$$

Combined with (3.6), this will give the required asymptotics.

Consider the sequence of polynomials $\{Q_k^{(n)} = x^k + \cdots\}$ orthogonal with respect to the measure $e^{-nf_n(x)}d\sigma(x)$. These polynomials satisfy the three-term recurrence relations

$$xQ_k^{(n)}(x) = Q_{k+1}^{(n)}(x) + a_k^{(n)}Q_k^{(n)}(x) + b_k^{(n)}Q_{k-1}^{(n)}(x),$$

where

$$b_k^{(n)} = \int (Q_k^{(n)}(x))^2 e^{-nf_n(x)}\, d\sigma(x)\left(\int (Q_{k-1}^{(n)}(x))^2 e^{-nf_n(x)}\, d\sigma(x)\right)^{-1} \neq 0.$$

As a result, the polynomial $Q_{k-1}^{(n)}$ can be expressed in terms of $Q_{k+1}^{(n)}$ and $Q_k^{(n)}$, and continuing the process, it follows that $Q_{k-i}^{(n)}$ can be expressed in terms of $Q_{k+1}^{(n)}$ and $Q_k^{(n)}$ with polynomial coefficients. Using the orthogonality relations, it follows that the polynomial $p_n(x)$ lies in the linear span of $Q_n^{(n)}, \dots, Q_{n-g}^{(n)}$. Consequently, $p_n(x) = a_n(x)Q_n^{(n)}(x) + b_n(x)Q_{n-1}^{(n)}(x)$, where $\deg a_n \leqslant \max(0, g-2)$ and $\deg b_n \leqslant g-1$. We know that the zeros of $Q_n^{(n)} \equiv Q_n$ and $Q_{n-1}^{(n)}$ interlace. Hence, if $x_1 < x_2$ are two successive zeros of $Q_n(x)$ and $b_n(x) \neq 0$ on $(x_1, x_2]$, then $p_n(x)$ has a zero on $[x_1, x_2)$. Since $\deg b_n \leqslant g-1$, this gives (3.8).

2) We now claim that if all the zeros of p_n lie on E, then

$$\left(\int p_n^2(x)e^{-nf_n(x)}\, d\sigma(x)\right)^{1/n} \to e^{-w}, \qquad n \in \Lambda.$$

We recall that all the polynomials in question have leading coefficient one. Hence $p_n = Q_n + c_{1,n}Q_{n-1}^{(n)} + \cdots + c_{g,n}Q_{n-g}^{(n)}$. Since the zeros of all these polynomials lie in the compact set E (and thus are uniformly bounded), by Viète's theorem we have $c_{i,n} = O(n^i)$, and so $\overline{\lim}\, |c_{i,n}|^{1/n} \leqslant 1$. Therefore (see (3.7)), for $n \in \Lambda$,

$$\left(\int p_n^2 e^{-nf_n}\, d\sigma\right)^{1/n} = \left(\int Q_n^2 e^{-nf_n}\, d\sigma + \sum_{j=1}^{g} c_{j,n}^2 \int (Q_{n-j}^{(n)})^2 e^{-nf_n}\, d\sigma\right)^{1/n} \to e^{-w}.$$

We set

$$r_n(x) := \int \frac{p_n(t)e^{-nf_n(t)}\, d\sigma(t)}{x-t}.$$

Then the following upper bound is valid for $x \in E'$:

$$\varlimsup_{n\to\infty} |r_n(x)|^{1/n} \leqslant e^{V^{\lambda}(x)-w}.$$

In fact, this bound is a consequence of the representation $r_n = R_n + \sum_{j=1}^{g} c_{j,n} R^{(n)}_{n-i}$ and a similar bound for the functions

$$R^{(n)}_{n-i}(x) := \int \frac{Q^{(n)}_{n-i}(t)e^{-nf_n(t)}\,d\sigma(t)}{x-t}.$$

By what has been proved in assertion 1), on E' we have

$$-\frac{1}{n}\ln|p_n(x)| \rightrightarrows V^{\lambda}(x), \qquad n\in\Lambda.$$

Therefore it suffices to show that

$$-\frac{1}{n}\ln|(r_np_n)(x)| \xrightarrow{\mathscr{F}} w.$$

Consider

$$(r_np_n)(x) = \int \frac{p_n^2(t)e^{-nf_n(t)}\,d\sigma(t)}{x-t} + \int \frac{p_n(t)(p_n(x)-p_n(t))e^{-nf_n(t)}\,d\sigma(t)}{x-t}.$$

Using the orthogonality relations for p_n, we see that the second integral is a polynomial, say $s_{n,g-1}(x)$, of degree at most $g-1$. We set $g' = 2[g/2]+2$, and consider the divided difference of order $g'-1$:

$$(r_np_n)(x_1;\dots;x_{g'}) = \sum_j \frac{(r_np_n)(x_j)}{\prod_{k\neq j}(x_j-x_k)}.$$

We have

$$(r_np_n)(x_1;\dots;x_{g'}) = \int \frac{p_n^2(t)e^{-nf_n(t)}\,d\sigma(t)}{(t-x_1)\cdots(t-x_{g'})}. \tag{3.9}$$

Consider points $x_1,\dots,x_{g'}$ lying in one connected component of E'. Then the measure

$$\frac{d\sigma(t)}{(t-x_1)\cdots(t-x_{g'})}$$

has constant sign on E, and hence $|(r_np_n)(x_1;\dots;x_{g'})|^{1/n}\to e^{-w}$. Therefore,

$$\max_{1\leqslant j\leqslant g'}\{|r_np_n|(x_j)\}^{1/n}\to e^{-w}. \tag{3.10}$$

We claim that the sequence in question converges in measure. In fact, we fix an interval E_1'; this is a connected component of E'. Let $\delta := \operatorname{dist}(E_1',E) > 0$. Suppose that $\varepsilon>0$ and that $x_1,\dots,x_{g'}\in e_\varepsilon := \{x\in E_1' : |r_np_n|(x) < (e^{-w}-\varepsilon)^n\}$. Then there is an N such that the inequality $\int p_n^2e^{-nf_n}\,d\sigma \geqslant (e^{-w}-\varepsilon/2)^n$ holds for any $n>N$, $n\in\Lambda$. Now $|(r_np_n)(x_1;\dots;x_{g'})| \geqslant \delta^{-g'}(e^{-w}-\varepsilon/2)^n$ follows by (3.9). Further, since

$$|(r_np_n)(x_1;\dots;x_{g'})| \leqslant \frac{g'\max_j|r_np_n|(x_j)}{\min_{k,j:k\neq j}|x_k-x_j|^{g'-1}}$$

we see that

$$\min_{k,j:k\neq j}|x_k-x_j|\leqslant\left(g'\delta^{g'}\left(\frac{e^{-w}-\varepsilon}{e^{-w}-\varepsilon/2}\right)^n\right)^{1/(g'-1)}=:\varepsilon_1(n).$$

Since this is valid for any choice of g' points from e_ε, we have $|e_\varepsilon|\leqslant 2(g'-1)\varepsilon_1(n)\to 0$. This proves that the sequence converges in measure. Proposition 3.3 now follows if we appeal to (3.10).

Remark 3.1. 1) Proposition 3.3 remains valid if $\deg p_n=n'$ for $n'\leqslant n$, provided the normalization $p_n(x)=x^{n'}+\cdots$ is unchanged.

2) Throughout this subsection we can assume that the measure $d\sigma(x)$ has fixed sign on each interval E_j, since we can always change to a measure which is positive on the whole E, at the cost of losing a finite number of orthogonality relations.

3.3. Proof of Theorem 1.2. We rewrite the orthogonality relations (3.2) as

$$\sum_{A\in B_-}\sum_{\alpha\in(A,B)}\int q_{B-}(x)x^k\frac{\Psi_{\vec n,A}(x)\,d\sigma_\alpha(x)}{q_{A+}(x)q_{B+}(x)}\,\frac{q_{A+}(x)}{q_{B-}(x)}=0; \tag{3.11}$$

here $k=0,\dots,m_{B-}-2+\operatorname{ind}B_--d_B$ and $0\leqslant d_B\leqslant g$.

We have

$$\frac{\Psi_{\vec n,B}(x)}{q_{B+}(x)}=\sum_{A\in B_-}\sum_{\alpha\in(A,B)}\int\frac{\Psi_{\vec n,A}(t)\,d\sigma_\alpha(t)}{(t-x)q_{B+}(t)}. \tag{3.12}$$

The proof of this formula is like that of Proposition 3.2.

Consider an arbitrary limit point.

$$\frac{1}{|\vec n|}\mu(q_{\vec n,\alpha})\to\mu_\alpha,\qquad \vec n\in\Lambda,\quad \alpha\in\mathscr{E}.$$

We note that, for the limit point, (3.5) follows from (1.37). Since the polynomial q_{B-} is quasi-orthogonal with respect to the variable weight function, which has fixed sign on the connected components of the set $\bigcup_{\alpha\in\mathscr{E}_{B-}}E_\alpha$ (at most g orthogonality relations being omitted), using (3.11) and Proposition 3.3 (proceeding inductively, from the root O upwards) this gives

$$\sum_{\beta\in\mathscr{E}_{B-}}2V^{\mu_\beta}(x)-\lim\frac{1}{|\vec n|}\ln\left|\frac{\Psi_{\vec n,A}(x)}{q_{A+}(x)}\,\frac{q_{A+}(x)}{q_{B-}(x)q_{B+}(x)}\right|\begin{cases}=\widetilde\varkappa_B, & x\in\bigcup\limits_{\beta\in\mathscr{E}_{B-}}S(\mu_\beta),\\ \geqslant\widetilde\varkappa_B, & x\in\bigcup\limits_{\beta\in\mathscr{E}_{B-}}E_\beta,\end{cases} \tag{3.13}$$

$$-\frac{1}{|\vec n|}\ln\left|\frac{\Psi_{\vec n,B}(x)}{q_{B+}(x)}\right|\quad\xrightarrow{\mathscr{F}}-\sum_{\beta\in\mathscr{E}_{B-}}V^{\mu_\beta}(x)+\widetilde\varkappa_B,\qquad x\in\Bigl\{\bigcup_{\alpha\in\mathscr{E}}E_\alpha\Bigr\}\setminus\Bigl\{\bigcup_{\beta\in\mathscr{E}_{B-}}E_\beta\Bigr\}. \tag{3.14}$$

Substituting (3.14) in (3.13) we obtain (1.38). In fact, for any $\alpha\in(A,B)$,

$$2\sum_{\beta\in\mathscr{E}_{B-}}V^{\mu_\beta}(x)-\sum_{\beta\in\mathscr{E}_{A-}}V^{\mu_\beta}(x)+\widetilde{\varkappa}_A$$
$$+\sum_{\beta\in\mathscr{E}_{A+}}V^{\mu_\beta}(x)-\sum_{\beta\in\mathscr{E}_{B+}}V^{\mu_\beta}(x)-\sum_{\beta\in\mathscr{E}_{B-}}V^{\mu_\beta}(x)\begin{cases}=\widetilde{\varkappa}_B, & x\in S(\mu_\alpha),\\ \geqslant\widetilde{\varkappa}_B, & x\in E_\alpha.\end{cases}$$

Hence

$$\sum_{\beta\in\mathscr{E}_{B-}}V^{\mu_\beta}(x)+\sum_{\beta\in\mathscr{E}_{A+}}V^{\mu_\beta}(x)-\sum_{\beta\in\mathscr{E}_{A-}}V^{\mu_\beta}(x)-\sum_{\beta\in\mathscr{E}_{B+}}V^{\mu_\beta}(x)$$
$$=2\sum_{\beta\in(A,B)}V^{\mu_\beta}(x)+\sum_{\beta\in\mathscr{E}_{B-},\,\beta\notin(A,B)}V^{\mu_\beta}(x)+\sum_{\beta\in\mathscr{E}_{A+},\,\beta\notin(A,B)}V^{\mu_\beta}(x)$$
$$-\sum_{\beta\in\mathscr{E}_{A-}}V^{\mu_\beta}(x)-\sum_{\beta\in\mathscr{E}_{B+}}V^{\mu_\beta}(x)\begin{cases}=\widetilde{\varkappa}_B, & x\in S(\mu_\alpha),\\ \geqslant\widetilde{\varkappa}_B, & x\in E_\alpha.\end{cases}$$

Finally, as the solution of the equilibrium problem (1.38) is unique, so is the limit point for the sequence of discrete measures in question. Theorem 1.2 now follows.

§ 4. The matrix Riemann-Hilbert problem and the proof of Theorem 1.3

Our aim here is to prove Theorem 1.3. The idea underlying our approach is to find an asymptotic solution of the matrix Riemann-Hilbert boundary value problem (as $n\to\infty$), which in turn is equivalent to the problem of determining the Hermite-Padé approximants (1.22):

$$R_{\vec n,A}:=P_{\vec n}\widehat{\mu}_A-Q_{\vec n,A}=O(z^{-n_A-1}),\quad z\to\infty,\quad A\in\mathring{\mathscr{V}},\quad \vec n:=\{n_B=n,\ B\in\mathring{\mathscr{V}}\}. \tag{4.1}$$

Here $P_{\vec n}\not\equiv0$ is a polynomial of degree $\deg P_{\vec n}\leqslant|\vec n|=\sum_{B\in\mathring{\mathscr{V}}}n_B$, and $Q_{\vec n,A}$ are some polynomials.

A reformulation of the definition of multiple orthogonal polynomials in terms of the matrix Riemann-Hilbert problem was put forward in [34] specifically for Angelesco systems (1.10), (1.11). In the present paper the matrix boundary value problem, as set forth in [27], is adapted for Hermite-Padé approximants to generalized Nikishin systems. That the method of steepest descent is useful in finding asymptotic solutions for 2×2 Riemann-Hilbert problems (as $n\to\infty$) was first demonstrated in [35]. We extend this approach to matrix boundary value problems of arbitrary dimension, $\#\mathscr{V}\times\#\mathscr{V}=(p+1)\times(p+1)$. To treat matrices of large dimension requires the use of new techniques, which were developed in [23] for 3×3 boundary value problems. Use is also made of methodological studies [36] to discuss the effects which occur in Riemann surfaces of positive genus.

4.1. Setting up the matrix Riemann-Hilbert problem. We introduce the following notation for multi-indices (see (4.1)):

$$n^{(O)}:=\vec n=\{n_B=n,\ B\in\mathring{\mathscr{V}}\},$$
$$n^{(A)}:=\{n_A=n-1,\ n_B=n,\ A\in\mathring{\mathscr{V}},\ B\in\mathring{\mathscr{V}}\setminus A\}.$$

The index $n^{(O)}$ being normal is equivalent to saying that $\deg P_{n^{(O)}}(z) = |n^{(O)}|$. In other words, a polynomial can be normalized as in (1.64); that is,

$$P_{n^{(O)}}(z) = z^{|n^{(O)}|} + \cdots = z^{pn} + \cdots . \tag{4.2}$$

In this case it can also be shown that $R_{n^{(A)},A}$ has a zero of order n at infinity. This, in turn, implies (see Proposition 3.1) that the condition for a function of the second order[1] $\Psi_{n^{(A)},A}$ to be as follows

$$\Psi_{\vec{n},O} = P_{\vec{n}}, \qquad \Psi_{\vec{n},A}(x) = \frac{1}{2\pi i} \sum_{B\in A_-} \sum_{\alpha\in(B,A)} \int \frac{\Psi_{\vec{n},B}(t)\, d\sigma_\alpha(t)}{t-x}$$

is equivalent to having a zero of order n at infinity. We set

$$c_O := 1, \quad c_A : \quad \lim_{z\to\infty} c_A \Psi_{n^{(A)},A}(z) z^n = 1, \quad A \in \mathring{\mathscr{V}}. \tag{4.3}$$

We order the vertices $\{A\}_{A\in\mathscr{V}}$ arbitrarily, and consider the matrix function

$$Y(z) := \{Y_{AB}(z)\}_{A,B\in\mathscr{V}}$$

of dimension $(p+1)\times(p+1)$, whose elements are defined as follows:

$$Y_{AB} := c_A \Psi_{n^{(A)},B}, \qquad A, B \in \mathscr{V}. \tag{4.4}$$

Assuming the intervals to be directed from a_α to b_α, we have, by the Plemelj-Sokhotskiĭ formulae,

$$\Psi_{\vec{n},A+}(x) - \Psi_{\vec{n},A-}(x) = \Psi_{\vec{n},B}(x)\rho_\alpha(x), \qquad x \in E_\alpha := (a_\alpha, b_\alpha), \quad \alpha \in (B,A).$$

Now, in view of (4.2) and (4.3), we see that the function Y is the unique solution of the following boundary value problem:

1) $Y(z)$ is a matrix function of dimension $(p+1)\times(p+1)$, analytic on $\mathbb{C}\setminus E$ ($E := \bigcup_{\alpha\in\mathscr{E}} E_\alpha$), and with continuous boundary values from the left and right on $\mathring{E} = \bigcup_{\alpha\in\mathscr{E}} \mathring{E}_\alpha$, which in turn are related by the jump matrix W (defined in part 2) below). The function $Y(z)$ behaves at infinity as described in part 3). The behaviour of $Y(z)$ near the end-points of E_α is outlined in part 4).

2) The function $Y(z)$ has a multiplicative jump on E, which can be written $Y_+ = Y_- W$, where the elements of the matrix W on each E_α, $\alpha\in\mathscr{E}$, are as follows:

$$W_{AB}\big|_{E_\alpha} =: W^\alpha_{AB}(x) := \begin{cases} \rho_\alpha(x) & \text{if } \alpha\in(A,B)\subset\mathscr{E}, \\ 1 & \text{if } A = B, \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

3) At infinity the function $Y(z)$ behaves like a diagonal matrix (say, D):

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) D(z); \tag{4.6}$$

here

$$D_{OO}(z) = z^{pn}, \qquad D_{AA}(z) = z^{-n}, \quad A \in \mathring{\mathscr{V}}.$$

[1]It is convenient to change the normalization in (1.25).

4) We will now describe the behaviour of the function $Y(z)$ near the end-points of E. Let a be an end-point of the interval E_α, $\alpha \in (A,B) \subset \mathscr{E}$. Then the asymptotic behaviour as $z \to a$ of the column of the matrix Y which corresponds to the index C is as follows:

$$Y_{AC}(z) = \begin{cases} O(\kappa(z)) & \text{if } C = B, \\ O(1) & \text{if } C \neq B. \end{cases} \tag{4.7}$$

Here (see the notation (1.42)), for $a \in \{a_\alpha^*, b_\alpha^*\}_{\alpha \in \mathscr{E}^*}$,

$$\kappa(z) = \begin{cases} |z-a|^{\delta(a)} & \text{if } -1 < \delta(a) < 0, \\ \log|z-a| & \text{if } \delta(a) = 0, \\ 1 & \text{if } \delta(a) > 0, \end{cases}$$

and $\kappa(z) = 1$ for $a \notin \{a_\alpha^*, b_\alpha^*\}_{\alpha \in \mathscr{E}^*}$.

Our objective is an asymptotic solution of this boundary value problem as $n \to \infty$. To find it we utilize a series of equivalent transformations, which will enable us to pass from the boundary value problem for the matrix function Y to the boundary value problem for the matrix function $\mathscr{J} \in \mathscr{H}(\bar{\mathbb{C}} \setminus \Sigma)$, $\mathscr{J}(\infty) = 1$, for which the jump on the curves of discontinuity Σ converges uniformly as $n \to \infty$ to the identity matrix

$$\mathscr{J}_+ = \mathscr{J}_- \widetilde{I}_n \text{ on } \Sigma, \qquad \widetilde{I}_n \rightrightarrows I \quad \text{as } n \to \infty.$$

A standard argument (see [30]) shows that

$$\mathscr{J} \rightrightarrows I \text{ in } \overline{\mathbb{C}} \quad \text{as } n \to \infty. \tag{4.8}$$

Inverting the above transformations, returning to the matrix Y, and taking into account (4.8) and (4.4), we obtain asymptotic formulae for $P_{\vec{n}}$ and $\Psi_{\vec{n},A}$, $A \in \mathring{\mathscr{V}}$.

4.2. The Riemann surface and normalization of the Riemann-Hilbert boundary value problem at infinity. The Riemann surface $\mathfrak{R}$ we introduced in (1.45) is crucial in constructing the sequences of transformations of the original boundary value problem for the function Y (see (4.4)) into the boundary value problem for the function $\mathscr{J}$ (see (4.8)).

We will now specify the arrangement of the projections of the $\mathbf{a}_{\mathfrak{R}}$-cycles (see (1.46)) onto $\mathbb{C}$. After traversing g^* intervals $\{E_j^*\}_{j=1}^{g^*} \subset \{E_\alpha^*\}_{\alpha \in \mathscr{E}^*}$, which are fixed by the choice of the $\mathbf{b}_{\mathfrak{R}}$-cycles, see (1.47), the corresponding branches $\{\Phi_A(z)\}_{A \in \mathscr{V}}$, which are the exponentials of the Abelian integral Φ (see (1.50)), have nontrivial periods (see (1.51)). It follows that the branches of Φ have jumps where they intersect the projection of the cycles $\mathbf{a}_{\mathfrak{R}} := \{\mathbf{a}_j\}_{j=1}^{g^*}$. Hence we can express this set of discontinuities of $\{\Phi_A(z)\}_{A \in \mathscr{V}}$ in $\mathbb{C}$ as follows:

$$\widetilde{a} := \bigcup_{j=1}^{g^*} \widetilde{a}^{(j)} := \bigcup_{j=1}^{g^*} \bigcup_k \widetilde{a}_k^{(j)}. \tag{4.9}$$

Here $\widetilde{a}_k^{(j)}$ is a Jordan arc in the upper half-plane, which connects a pair of intervals from E^*, and two different branches of $\Phi_{A(k)}$ and $\Phi_{B(k)}$, $A \neq B$, are discontinuous on each $\widetilde{a}_k^{(j)}$. Therefore, for fixed j, we have defined the functions $\rho^{\pm}$

$$A = \rho^+(k), \qquad B = \rho^-(k) \tag{4.9$_1$}$$

so that the portion of the cycle $\mathbf{a}_j$ from the sheet $\mathfrak{R}_A$ is projected into $\widetilde{a}_k^{(j)}$ preserving the orientation, while the corresponding portion of $\mathbf{a}_j$ from the sheet $\mathfrak{R}_B$ is carried to the same $\widetilde{a}_k^{(j)}$ with a change of orientation.

Let $\widetilde{C}$ and $\widetilde{\Phi}$ be the diagonal matrices (see (1.50)):

$$\widetilde{C} := \operatorname{diag}\{C_A^n\}_{A\in\mathscr{V}}, \qquad \widetilde{\Phi}(z) := \operatorname{diag}\{\Phi_A^n(z)\}_{A\in\mathscr{V}}.$$

Consider the matrix function

$$Z := \widetilde{C} Y \widetilde{\Phi}. \tag{4.10}$$

The matrix Z is a solution of the following boundary value problem:

1) $Z(z) \in \mathscr{H}(\mathbb{C} \setminus (E \cup \widetilde{a}))$;

2) $Z_+ = Z_- J$ on $E \cup \widetilde{a}$, where the elements of the jump matrix $\{J_{AB}(z)\}_{A,B\in\mathscr{V}}$ on the intervals E_α^*, $\alpha \in \mathscr{E}$, are as follows:

$$J_{AB} = \begin{cases} w_\alpha & \text{for } A, B : \alpha \in (A, B) \subset \mathscr{E}^*, \\ \dfrac{\Phi_{A,+}^n}{\Phi_{A,-}^n} & \text{if } A = B \in \mathscr{V}, \\ 0 & \text{otherwise;} \end{cases} \tag{4.10$_1$}$$

the jump on the set $E_\alpha \setminus S(\lambda_\alpha)$ has the representation:

$$J_{AB} = \begin{cases} \rho_\alpha \dfrac{\Phi_B^n}{\Phi_A^n} & \text{for } A, B : \alpha \in (A, B) \subset \mathscr{E}, \\ 1 & \text{if } A = B \in \mathscr{V}, \\ 0 & \text{otherwise.} \end{cases}$$

The jump of J on $\widetilde{a}_k^{(j)}$ (see (4.9)) is the diagonal matrix with two nonzero elements J_{AA} and J_{BB}, $A(k) \neq B(k)$:

$$J_{AA} = \frac{\Phi_{A,+}^n}{\Phi_{A,-}^n} = e^{2\pi i n\omega_j}, \quad J_{BB} = \frac{\Phi_{B,+}^n}{\Phi_{B,-}^n} = e^{-2\pi i n\omega_j}, \qquad j = 1, \dots, g^*;$$

3) $Z(z) = I + O(1/z)$ as $z \to \infty$;

4) the behaviour at the end-points of E is given by (4.7) and conforms with the way Φ branches.

4.3. 'Opening local lenses'. Along with each interval E_α^*, we consider two smooth curves E_α^{*+} and E_α^{*-}, with the same end-points as E_α^*, and which are located in the upper and lower half-plane, respectively. Let $L_\alpha^{(+)}$ and $L_\alpha^{(-)}$ be the lens-shaped regions between E_α^* and E_α^{*+} and E_α^* and E_α^{*-}, respectively. For all the edges α of the graph $\mathscr{G}^*(\mathscr{V},\mathscr{E}^*,O)$, we consider the matrix functions $D_\alpha(z)$, $\alpha\in\mathscr{E}^*$, with elements

$$D_{\alpha,BA}=\begin{cases}\dfrac{1}{w_\alpha(x)}\,\dfrac{\Phi_A^n}{\Phi_B^n} & \text{for } A,B:\alpha\in(A,B),\\ 1 & \text{if } A=B,\\ 0 & \text{otherwise.}\end{cases}\tag{4.11}$$

Also, we consider the matrix function $\widehat{Z}$:

$$\widehat{Z}(z)=\begin{cases}ZD_\alpha^{-1} & \text{in } L_\alpha^{(+)},\ \alpha\in\mathscr{E}^*,\\ ZD_\alpha & \text{in } L_\alpha^{(-)},\ \alpha\in\mathscr{E}^*,\\ Z & \text{otherwise.}\end{cases}\tag{4.12}$$

The matrix function $\widehat{Z}$ is holomorphic in $\overline{\mathbb{C}}\setminus\Sigma$, and its jump $\widehat{Z}_+=\widehat{Z}_-\widehat{J}$ on

$$\Sigma:=(E\cup\widetilde{a})\bigcup_{\alpha\in\mathscr{E}^*}E_\alpha^{*+}\cup E_\alpha^{*-}$$

is as follows

$$\widehat{J}=\begin{cases}D_\alpha & \text{on } E_\alpha^{*+}\cup E_\alpha^{*-},\ \alpha\in\mathscr{E}^*,\\ \widetilde{W}_\alpha & \text{on } E_\alpha^*,\ \alpha\in\mathscr{E}^*,\\ J & \text{on } (E\cup\widetilde{a})\setminus E^*.\end{cases}\tag{4.12$_1$}$$

Here $E^*:=\bigcup_{\alpha\in\mathscr{E}^*}E_\alpha^*$, and the matrix $\widetilde{W}_\alpha$ is block diagonal,

$$\widetilde{W}_{\alpha,AB}(x)=\begin{cases}w_\alpha(x) & \text{for } A,B\colon \alpha\in(A,B),\\ -\dfrac{1}{w_\alpha(x)} & \text{for } A,B\colon \alpha\in(B,A),\\ 1 & \text{if } A=B,\ \alpha\notin\mathscr{E}_{A-}^*\cup\mathscr{E}_{A+}^*,\\ 0 & \text{otherwise.}\end{cases}\tag{4.13}$$

The jump on the contours $\widetilde{a}$ has the same value inside and outside the lenses because $JD_+=D_-J$, which holds on $\widetilde{a}$. It is also worth noting that the normalization at infinity is preserved by the transformation $Z\longrightarrow\widehat{Z}$; that is,

$$\widehat{Z}(z)=I+O\left(\frac{1}{z}\right)\quad\text{as } z\to\infty.$$

4.4. The limiting external boundary value problem. By referring to the explicit representation (4.11) of the jump of D_α on the outer boundary of the lenses $E_\alpha^{*+}\cup E_\alpha^{*-}$ as $\alpha\in\mathscr{E}^*$, we see (in view of part a) of (1.43)) that, away from the end-points, this jump converges to the identity matrix as $n\to\infty$. In view of (1.38), (1.49) and part a) of (1.43), the same holds for the jump J on $E\setminus E^*$. Therefore, in order to change from the boundary value problem for the function $\widehat{Z}$ in (4.12) to the boundary value problem for the function $\mathscr{J}$ in (4.8), we must solve the following limiting boundary value problem:

$$X: \quad \begin{cases} X \in \mathscr{H}\big(\overline{\mathbb{C}} \setminus \{E^* \cup \widetilde{a}\}\big), \\ X_+ = X_- \begin{cases} \widetilde{W}_\alpha & \text{on } E^*_\alpha, \\ J & \text{on } \widetilde{a}, \end{cases} \\ X(z) = I + O\bigg(\dfrac{1}{z}\bigg) \quad \text{as } z \to \infty. \end{cases} \tag{4.14}$$

Here we recall that the matrix $\widetilde{W}_\alpha$, $\alpha \in \mathscr{E}^*$, is defined in (4.13), and the jump on the projection of $\mathbf{a}_{\mathfrak{R}}$-cycles (see (4.9)) has the following representation:

$$J\big|_{\widetilde{a}_k^{(j)}} := J_k^{(j)} := \operatorname{diag}\{\tau_C^{(j,k)}\}_{C \in \mathscr{V}}, \qquad j = 1, \dots, g^*,$$

where

$$\tau_C^{(j,k)} := \begin{cases} e^{2\pi i n \omega_j} & \text{for } C = \rho^+(k), \\ e^{-2\pi i n \omega_j} & \text{for } C = \rho^-(k), \\ 1 & \text{otherwise}. \end{cases}$$

The solution of this problem is a key new ingredient in obtaining and proving the asymptotic formulae of Theorem 1.3.

4.4.1. *The Szegő function* (*the solution of the boundary value problem* (1.59)). Let $d\widehat{\omega}_{\xi_1\xi_2}(\zeta)$ be the Cauchy differential (see [29]), single-valued on $\widehat{\mathfrak{R}} := \mathfrak{R} \setminus \mathbf{a}_{\mathfrak{R}}$, with simple poles at the points ξ_1 and ξ_2 with residues $+1$ and -1, respectively,

$$\frac{d}{d\zeta}\widehat{\omega}_{\xi_1\xi_2}(\zeta) = \begin{cases} \dfrac{1}{\zeta - \xi_1} + O(1) & \text{as } \zeta \to \xi_1, \\ \dfrac{-1}{\zeta - \xi_2} + O(1) & \text{as } \zeta \to \xi_2. \end{cases} \tag{4.15}$$

We need some notation. For any point on the Riemann surface, the function $S\colon \mathfrak{R} \to \mathscr{V}$ assigns the number of the sheet on which it is located; that is,

$$S(\xi) = B \quad \Leftrightarrow \quad \xi \in \mathfrak{R}_B.$$

We also set $\widetilde{\xi}^{(A)}$ to be the point on $\mathfrak{R}_A$ which has the same projection onto $\overline{\mathbb{C}}$ as the point ξ, but which lies on a different sheet $A \neq S(\xi)$; that is,

$$\widetilde{\xi}^{(A)}(\xi): \quad \xi \in \mathfrak{R}, \quad \widetilde{\xi}^{(A)} \in \mathfrak{R}_A: \quad \pi(\xi) = \pi(\widetilde{\xi}^{(A)}), \quad \xi \neq \widetilde{\xi}.$$

On $\mathfrak{R}$ we define a meromorphic (Cauchy) differential as follows:

$$dM_\xi(\zeta) := \sum_{A \in \mathscr{V} \setminus S(\xi)} d\widehat{\omega}_{\xi\widetilde{\xi}^{(A)}}(\zeta). \tag{4.16}$$

Proposition 4.1. *The function*

$$\mathscr{F}(\xi) := \exp\biggl\{\frac{1}{(p+1)2\pi i}\int_{\partial\mathfrak{R}(E^*)} \ln\widetilde{w}(\zeta)\,dM_\xi(\zeta)\biggr\}, \qquad \xi\in\widehat{\mathfrak{R}}\setminus\partial\mathfrak{R}(E^*), \quad (4.17)$$

which is piecewise holomorphic in $\widehat{\mathfrak{R}}\setminus\partial\mathfrak{R}(E^*)$, is a solution of the boundary value problem (1.59), *and its $\mathbf{b}_{\mathfrak{R}}$-periods* (1.60) *are as follows*:

$$c_w^{(j)} := -\frac{p}{(p+1)2\pi i}\int_{\partial\mathfrak{R}(E^*)} \ln\widetilde{w}(\zeta)\,d\Omega_j(\zeta), \qquad j=1,\dots,g. \qquad (4.18)$$

Proof. We first recall the notation used in (4.17) and (4.18).

The function $\widetilde{w}$ is the 'trigonometric' weight (1.57), $\{d\Omega_j(\zeta)\}_{j=1}^g$ is the basis of normalized holomorphic Abelian differentials of the first kind corresponding to (1.52), $\partial\mathfrak{R}(E^*) := \bigl(\bigcup_{\alpha\in\mathscr{E}^*}\partial\mathfrak{R}_\alpha\bigr)$, and $(p+1)=\#\mathscr{V}$. So (4.17) defines a Szegő function associated with the vector weight w (see (1.42)) on E^* (see (1.41)).

We claim that the following basic properties of the boundary value problem (1.59) are satisfied with the function $\mathscr{F}$ as defined by (4.17) and (4.18):

$$\begin{cases} 1)\ \displaystyle\prod_{A\in\mathscr{V}}\mathscr{F}_A(z)\equiv 1, \quad z\in\bar{\mathbb{C}}, \\ 2)\ \mathscr{F}_+=\mathscr{F}_-\widetilde{w}_\alpha \text{ on } \partial\mathfrak{R}_\alpha, \quad \alpha\in\mathscr{E}^*, \\ 3)\ \mathscr{F}_+=\mathscr{F}_-e^{2\pi i c_w^{(j)}} \text{ on } \mathbf{a}_j, \quad j=1,\dots,g. \end{cases}.$$

Property 1) holds because all the residues for the differential

$$dM_\xi(\zeta)+\sum_{A\in\mathscr{V}\setminus S(\xi)} dM_{\widetilde{\xi}(A)}(\zeta), \qquad \zeta\in\widehat{\mathfrak{R}},$$

are zero (see (4.15) and (4.16)).

We will prove property 2). By (4.17), in view of 1),

$$\frac{1}{2\pi i}\int_{\partial\mathfrak{R}(E^*)}\ln\widetilde{w}(\zeta)\,dM_\xi(\zeta) = p\ln\mathscr{F}(\xi) - \sum_{A\in\mathscr{V}\setminus S(\xi)}\ln\mathscr{F}(\widetilde{\xi}^{(A)}) = (p+1)\ln\mathscr{F}(\xi).$$

Substituting the expression

$$\ln\widetilde{w}_\alpha = \ln\mathscr{F}_+ - \ln\mathscr{F}_-,$$

which holds on $\partial\mathfrak{R}_\alpha$, into the left-hand side and making use of the Plemelj-Sokhotskiĭ formulae (or Cauchy's residue theorem), we obtain an identity which proves property 2).

Property 3) follows from the well-known Riemann relation (see [29]):

$$t\in\mathbf{a} \quad\Rightarrow\quad d\widehat{\omega}_{t_+p}(\zeta)-d\widehat{\omega}_{t_-p}(\zeta) = -2\pi i\,d\Omega_k(\zeta), \qquad k=1,\dots,g.$$

4.4.2. *The limiting external problem with weight-independent jumps on E^* (stating the problem)*. Let

$$F(z) := \operatorname{diag}\{\mathscr{F}_A(z)\}_{A\in\mathscr{V}}, \quad z\in\mathbb{C}, \qquad F_\infty := F(\infty). \tag{4.19}$$

We define the function $\widetilde{X}$ (see (4.14)) by

$$\widetilde{X} := F_\infty^{-1} X F. \tag{4.20}$$

Then this function satisfies the following boundary value problem:

$$\begin{cases} \widetilde{X} \in H\bigl(\overline{\mathbb{C}}\setminus\{E^*\cup\widetilde{a}\}\bigr), \\ \widetilde{X}_+ = \widetilde{X}_- H \quad \text{on } E^*\cup\widetilde{a}, \\ \widetilde{X}_\infty = I. \end{cases} \tag{4.21}$$

The jump matrix H on E^* satisfies

$$H\big|_{E^*_\alpha} = F_-^{-1}\widetilde{W}_\alpha F_+ =: \{H_{\alpha,AB}\}_{A,B\in\mathscr{V}}, \qquad \alpha\in\mathscr{E}^*,$$

and so, taking the boundary behaviour of the components of the Szegő vector function for $x\in E^*_\alpha$ into account, this gives

$$H_{\alpha,AB}(x) := \begin{cases} w_\alpha(x)\dfrac{\mathscr{F}_{B+}(x)}{\mathscr{F}_{A-}(x)} = \dfrac{w_\alpha(x)}{\widetilde{w}_\alpha(x)} & \text{for } A,B\colon \alpha\in(A,B), \\ \dfrac{(-1)}{w_\alpha(x)}\,\dfrac{\mathscr{F}_{A+}(x)}{\mathscr{F}_{B-}(x)} = -\dfrac{\widetilde{w}_\alpha(x)}{w_\alpha(x)} & \text{for } A,B\colon \alpha\in(B,A), \\ 1 & \text{if } A=B,\ \alpha\notin\mathscr{E}^*_{A-}\cup\mathscr{E}^*_{A+}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.22}$$

Recalling the notation for 'trigonometric' weights (1.57), we obtain

$$\frac{w_\alpha}{\widetilde{w}_\alpha} := \begin{cases} -i(\widetilde{h}^0_B)_-, & \alpha\in(O,B), \\ -\dfrac{(\widetilde{h}^0_B)_-}{\widetilde{h}^0_A}, & \alpha\in(A,B), \quad A\ne O. \end{cases} \tag{4.23}$$

For the jump matrix H on $\widetilde{a}$, we have, for $j=1,\dots,g^*$,

$$H\big|_{\widetilde{a}^{(j)}_k} := \operatorname{diag}\{\varsigma^{(j,k)}_C\}_{C\in\mathscr{V}}, \qquad \varsigma^{(j,k)}_C := \begin{cases} e^{2\pi i(n\omega_j+c^{(j)}_w)} & \text{for } C=\rho^+(k), \\ e^{-2\pi i(n\omega_j+c^{(j)}_w)} & \text{for } C=\rho^-(k), \\ 1 & \text{otherwise}. \end{cases} \tag{4.24}$$

This reduces our problem (4.14) to problem (4.21)–(4.24), for which the jump across the intervals E^* is described by the standard function (independent of the weight w) $h^0 := \{h^0_A\}_{A\in\mathring{\mathscr{V}}}$, and the jump across the projection of an $\mathbf{a}_{\mathfrak{R}}$-cycle is constant along each $\pi(\mathbf{a}_j)$, $j=1,\dots,g^*$.

To solve this problem we first build a function having jump H on E^* and no discontinuities on $\widetilde{a}$. Then we use the Riemann theta function, which is holomorphic in $\widehat{\mathfrak{R}}$ (see (1.54)), and so has no discontinuities on the $\mathbf{b}_{\mathfrak{R}}$-cycles, so it satisfies the boundary conditions on $\widetilde{a}$.

4.4.3. *The limiting external problem with weight-independent jumps* (*the solution of the auxiliary problem*). We recall (see (1.61)) that the set $\{\zeta_l^*\}_{l=1}^{g^*}$ denotes the points on $\mathfrak{R}$, obtained by lifting all finite zeros of the Green's functions (1.56) for the domains $\Omega_{A-} := \overline{\mathbb{C}} \setminus \bigcup_{\beta\in A_-} E_\beta^*$, $A \in \mathring{\mathscr{V}}$, to the corresponding sheets of the Riemann surface $\mathfrak{R}$ (that is, on $\mathfrak{R}_A$).

1. On $\mathfrak{R}$ we define a family of single-valued rational functions $\{\chi^{(A)}\}_{A\in\mathring{\mathscr{V}}}$ whose divisors dominate the divisor on $\mathfrak{R}$:

$$\chi^{(A)} \in \mathscr{M}(\mathfrak{R}): \quad \frac{(\infty^{(O)})}{(\infty^{(A)}\zeta_1^* \dots \zeta_g^*)} \,\Big|\, \chi^{(A)}, \qquad A \in \mathring{\mathscr{V}}. \tag{4.25}$$

In other words, we have fixed g^*+1 poles (one pole at $\infty^{(A)}$ and g poles at the points $\{\zeta_l^*\}_{l=1}^{g^*}$), and fixed one zero at $\infty^{(O)}$. The remaining g^* zeros (say, $\{\zeta_l^{(A)}\}_{l=1}^{g^*}$) will take up some assigned positions, so that χ is single-valued on $\mathfrak{R}$. So

$$\chi^{(A)}(\zeta_l^{(A)}) = 0, \qquad l = 1, \dots, g^*. \tag{4.26}$$

These functions are defined up to a multiplicative constant, which is fixed by

$$\chi^{(A)}(\xi) = \xi + \cdots, \qquad \xi \to \infty^{(A)}. \tag{4.27}$$

2. We solve the auxiliary problem by letting the jump $H\big|_{\widetilde{a}} := I$ in (4.21); that is, we shall find a function $\widetilde{\widetilde{X}}$ such that

$$\widetilde{\widetilde{X}}: \quad \begin{cases} \widetilde{\widetilde{X}} \in \mathscr{H}(\mathbb{C} \setminus E^*), \\ \widetilde{\widetilde{X}}_+ = \widetilde{\widetilde{X}}_- H \quad \text{on } E^*, \\ \widetilde{\widetilde{X}}(\infty) = I. \end{cases} \tag{4.28}$$

We set (see (1.58))

$$\begin{gathered} \widetilde{\widetilde{X}} := \{x_{AB}\}_{A,B\in\mathscr{V}}: \\ x_{OO} := 1, \quad x_{AO} := i\chi_O^{(A)}, \quad x_{OB} := ih_B^0, \quad x_{AB} := -h_B^0\chi_B^{(A)}, \quad A, B \in \mathring{\mathscr{V}}. \end{gathered} \tag{4.29}$$

It follows from the definitions of h_A^0 (see (1.57), (1.58)) and $\chi^{(A)} = \{\chi_B^{(A)}\}_{B\in\mathscr{V}}$ (see (4.25)–(4.27)) that the function $\widetilde{\widetilde{X}}$ of (4.29) is holomorphic and normalized at ∞.

3. We check the boundary conditions on E_α^*, $\alpha \in (A,B)^* \subset \mathscr{E}^*$. If $\alpha \in (O,B)^*$, then

$$\forall C \in \mathscr{V} \quad \longrightarrow \qquad x_{CO+} = x_{CB-}\left(\frac{-i}{h_{B-}^0}\right), \qquad x_{CB+} = x_{CO-}(-ih_{B-}^0).$$

Substituting (4.29), we obtain

$$C = O \quad \longrightarrow \quad 1 = -ih_{B-}^0\left(\frac{-i}{h_{B-}^0}\right), \qquad ih_{B+}^0 = 1 \cdot (-i)(-h_{B+}^0),$$

$$C \in \mathring{\mathscr{V}} \quad \longrightarrow \quad i\chi_{O+}^{(C)} = -h_{B-}^0\chi_{B-}^{(C)}\left(\frac{-i}{h_{B-}^0}\right), \qquad -h_{B+}^0\chi_{B+}^{(C)} = i\chi_{O-}^{(C)}(-i)(-h_{B+}^0).$$

Similarly, for $\alpha \in (A,B)$, $A \neq O$,

$$\forall C \in \mathscr{V} \quad \longrightarrow \qquad x_{CA+} = x_{CB-}\left(\frac{h^0_A}{h^0_{B-}}\right), \qquad x_{CB+} = x_{CA-}\left(\frac{-h^0_{B-}}{h^0_A}\right).$$

Substituting (4.29), this gives

$$C = O \quad \longrightarrow \quad ih^0_{A+} = ih^0_{B-}\left(\frac{h^0_A}{h^0_{B-}}\right), \quad ih^0_{B+} = ih^0_A\left(\frac{h^0_{B+}}{h^0_A}\right),$$

$$C \in \mathring{\mathscr{V}} \quad \longrightarrow \quad -h^0_{A+}\chi^{(C)}_{A+} = -h^0_{B-}\chi^{(C)}_{B-}\left(\frac{h^0_A}{h^0_{B-}}\right), \quad -h^0_{B+}\chi^{(C)}_{B+} = -h^0_A\chi^{(C)}_{A-}\left(\frac{h^0_{B+}}{h^0_A}\right).$$

Thus, $\widetilde{\widetilde{X}}$ of (4.29) is a solution of the auxiliary boundary value problem (4.28).

4.4.4. *Solving the limiting external boundary value problem.* We modify the function $\widetilde{\widetilde{X}}$ of (4.29) so as to preserve the other properties of problem (4.28) while the modified function acquires constant jumps H on the projections of $\mathbf{a}_{\mathfrak{R}}$-cycles $\widetilde{a}$ (see (4.24)):

$$H\big|_{\widetilde{a}^{(j)}_k} := \operatorname{diag}\{\varsigma^{(j,k)}_C\}_{C\in\mathscr{V}}, \qquad \varsigma^{(j,k)}_C := \begin{cases} e^{2\pi i c^{(j)}_{n,w}} & \text{for } C = \rho^+(k), \\ e^{-2\pi i c^{(j)}_{n,w}} & \text{for } C = \rho^-(k), \\ 1 & \text{otherwise.} \end{cases}$$

Here $j = 1,\dots,g^*$ and

$$\vec{c}_{n,w} := (n\omega_1 + c^{(1)}_w, \dots, n\omega_g + c^{(g^*)}_w) =: (c^{(1)}_{n,w}, \dots, c^{(g^*)}_{n,w})$$

(see (1.51), (1.63), (4.18)). Such a transformation can be done using the theta function (1.53) on the Riemann surface $\mathfrak{R}$.

We define (see (1.55)) vectors of constants e^A, $A \in \mathscr{V}$,

$$\begin{aligned} e^O:& \quad \Theta^{(e^O)}(\zeta^*_l) = 0, \\ e^A:& \quad \Theta^{(e^A)}(\zeta^{(A)}_l) = 0, \quad A \in \mathring{\mathscr{V}}, \end{aligned} \qquad l = 1,\dots,g^*, \tag{4.30}$$

so that the theta function $\Theta^{(e^O)}$ vanishes at the points (1.61) (the 'lifts' to $\mathfrak{R}$ of the finite zeros of the derivatives of the Green's functions (1.56) for the regions Ω_{A-}, $A \in \mathring{\mathscr{V}}$) and so that $\Theta^{(e^A)}$ vanishes at the finite zeros (4.26) of the function $\chi^{(A)}$, $A \in \mathring{\mathscr{V}}$ (see (4.25)), which is rational on $\mathfrak{R}$.

Consider the family of meromorphic functions on $\widehat{\mathfrak{R}}$ (see (1.53))

$$T^{(A)}(\zeta) := \frac{\Theta^{(e^A - \vec{c}_{n,w})}(\zeta)}{\Theta^{(e^A)}(\zeta)}, \qquad \zeta \in \widehat{\mathfrak{R}}, \quad A \in \mathscr{V}. \tag{4.31}$$

The reason it is useful to consider ratios of theta functions is that the argument of the ratio undergoes a constant jump on the $\mathbf{a}_{\mathfrak{R}}$-cycles (see [28] and [29]):

$$T^{(A)}_+ = T^{(A)}_- e^{2\pi i c^{(j)}_{n,w}} \quad \text{on } \mathbf{a}_j,\ j = 1,\dots,g*,\ A \in \mathscr{V}. \tag{4.32}$$

Let

$$T_\infty := \operatorname{diag}\{T_A^{(A)}(\infty)\}_{A\in\mathscr{V}} \tag{4.33}$$

be the diagonal matrix of constants. Here, as usual, the lower subscript denotes the branch of the function (it indicates the sheet of $\mathfrak{R}$ from which the values are taken).

Now we can find a solution of the problem (4.21). Let

$$\widehat{X} := \{\widehat{x}_{AB}\}_{A,B\in\mathscr{V}}: \quad \widehat{x}_{AB} := T_B^{(A)} x_{AB}, \qquad A, B \in \mathscr{V};$$

here the x_{AB} are elements of the matrix function $\widetilde{\widetilde{X}}$ (see (4.29)), and $\{T_B^{(A)}\}_{B\in\mathscr{V}}$ are branches of the ratio of theta functions (4.31). Consider the function

$$\widetilde{X}(z) := T_\infty^{-1}\widehat{X}. \tag{4.34}$$

We shall verify that it gives a solution of problem (4.21). The first condition (holomorphicity) follows from (4.30), (4.25), and the third condition (normalization), is a consequence of (4.33) and part 3) of (4.28).

We first check the jumps on $\widetilde{a} := \bigcup_{j=1}^{g^*}\bigcup_k \widetilde{a}_k^{(j)}$ (see (4.9)). On $\widetilde{a}_k^{(j)}$ we have

$$\widetilde{X}_+ := T_\infty^{-1}\{x_{AB}T_{B+}^{(A)}\}_{A,B\in\mathscr{V}} = T_\infty^{-1}\{x_{AB}T_{B-}^{(A)}\varsigma_B^{(j,k)}\}_{A,B\in\mathscr{V}};$$

here we have used (4.32), the notation in (4.24) and the fact that for $B = \rho^-(k)$ the projection of the portion of the **a**-cycle from the sheet $\mathfrak{R}_B$ to $\mathbb{C}$ reverses its orientation. This gives

$$\widetilde{X}_+ := \widetilde{X}_- \operatorname{diag}\{\varsigma_B^{(j,k)}\}_{B\in\mathscr{V}} \quad \text{on } \widetilde{a}_k^{(j)} \subset \mathbb{C}.$$

We now consider the jump on E^* (the projections of the **b**-cycle). The functions $T^{(A)}$ (see (4.31)) are continuous on the **b**-cycles (as well as $\chi^{(A)}$, see (4.25)); that is,

$$T_{B\pm}^{(A)} = T_{C\mp}^{(A)}, \quad \chi_{B\pm}^{(A)} = \chi_{C\mp}^{(A)} \quad \text{on } E_\alpha^*, \quad \alpha \in (B, C).$$

Since the function T occurs in the matrix function $\widetilde{X}$ of (4.34)) to just the same extent as χ (see (4.29)),

$$\begin{aligned} \widehat{x}_{OO} &:= T_O^{(O)}, & \widehat{x}_{OB} &:= ih_B^0 T_B^{(O)}, \\ \widehat{x}_{AO} &:= i\chi_O^{(A)}T_O^{(A)}, & \widehat{x}_{AB} &:= -h_B^0\chi_B^{(A)}T_B^{(A)}, \end{aligned}$$

it follows that the matrix $\widetilde{X}$ has the same jump on E^* as the matrix $\widetilde{\widetilde{X}}$ (see (4.28)).

Hence the matrix function (4.34) furnishes a solution of the problem (4.21). To sum up, we see that the function

$$X := F_\infty \widetilde{X} F^{-1} \tag{4.35}$$

gives a solution of the external limiting boundary value problem (4.14), where F_∞, F are given in (4.19), and $\widetilde{X}$ in (4.34).

4.5. Local boundary value problems. We now return to the boundary value problem for the matrix function $\widehat{Z}$ defined in (4.12). Since the jumps in the D_α (see (4.11)) on $E^{*\pm}_\alpha$, $\alpha \in \mathscr{E}^*$, and the jump in J (see the boundary value problem for Z in (4.10)) on $E \setminus E^*$ do not converge uniformly to the identity matrix near the end-points of the intervals of E^*, this suggests considering local boundary value problems near each of the end-points.

Suppose that $e \in \{a^*_\alpha, b^*_\alpha\}_{\alpha \in \mathscr{E}^*}$. We introduce the following notation:

$$\alpha := \alpha(e) \quad \Leftrightarrow \quad e = a^*_\alpha \quad \text{or} \quad e = b^*_\alpha.$$

Consider a neighbourhood O_e of the point e. We set

$$\Sigma_e := E^*_e \cup E^{*+}_e \cup E^{*-}_e,$$
$$E^*_e := E^*_{\alpha(e)} \cap O_e, \qquad E^{*\pm}_e := E^{*\pm}_{\alpha(e)} \cap O_e, \qquad E_e := (E \setminus E^*_e) \cap O_e.$$

We need solutions of the following two local boundary value problems.

For $e \in \{a^*_\alpha, b^*_\alpha\}_{\alpha \in \mathscr{E}^*} \cap \{a_\alpha, b_\alpha\}_{\alpha \in \mathscr{E}} =: \{e\}$, we have

$$\begin{cases} U_e \in H^{(p+1)\times(p+1)}(O_e \setminus \Sigma_e), \\ U_{e_j+} = U_{e_j-}\widehat{J} \quad \text{on } \Sigma_e, \\ U_{e_j} = \left(I + O\left(\dfrac{1}{n}\right)\right) X \text{ uniformly on } \partial O_{e_j} \text{ as } n \to \infty, \end{cases} \tag{4.36}$$

and for $e \in \{a^*_\alpha, b^*_\alpha\}_{\alpha \in \mathscr{E}^*} \setminus \{a_\alpha, b_\alpha\}_{\alpha \in \mathscr{E}} =: \{e^*\}$,

$$\begin{cases} U_e \in H^{(p+1)\times(p+1)}(O_e \setminus \{\Sigma_e \cup E_e\}), \\ U_{e_j+} = U_{e_j-}\widehat{J} \quad \text{on } \Sigma_e \cup E_e, \\ U_{e_j} = \left(I + O\left(\dfrac{1}{n}\right)\right) X \text{ uniformly on } \partial O_e \text{ as } n \to \infty, \end{cases} \tag{4.37}$$

where the jump matrices $\widehat{J}$ are defined in (4.12_1), and the elements of the matrix $U_e(z)$ have the same limit behaviour as $z \to e$ as $\widehat{Z}(z)$.

We note that in view of condition (1.18), the branch points of the Riemann surface $\mathfrak{R}$ are of square-root type, and hence the matrix solutions of local boundary value problems (4.36) and (4.37) consist of two blocks: a 2×2 block and the identity matrix. Therefore, solutions of the boundary value problems (4.36), (4.37) can be obtained on the basis of the available solutions of the corresponding boundary value problems for 2×2 matrices.

Analysis of the problem (4.36), (4.37) depends upon what happens to the derivative λ' of the equilibrium measure (1.38) near the end-point e. We know (for an end-point from $\{e^*\}$ we refer the reader to [37], for example; the case of $\{e\}$ can be analyzed in a similar way) that

$$\lambda'(z) = (z-e)^{n-1/2} m(z), \qquad m(z) \in \mathscr{H}(O_e), \quad m(e) \neq 0,$$

where n takes values, depending on the character of the end-point, as follows:

$$\begin{aligned} \{e\} \quad &\Rightarrow \quad n = 0, 1, 2, \dots, \\ \{e^*\} \quad &\Rightarrow \quad n = 2k+1, \quad k = 0, 1, \dots . \end{aligned} \tag{4.38}$$

Also, for principal (regular) cases (cases of general position),

$$\{e\} \quad \Rightarrow \quad n=0, \qquad \{e^*\} \quad \Rightarrow \quad n=1,$$

the explicit form of the solutions of the boundary value problems (4.36), (4.37) for 2×2 matrices is known (see [38], [39]); for the remaining (singular) cases of (4.38), existence theorems are also available (see [38], [40]).

We will write down the explicit solutions to problems (4.36) and (4.37) in the regular cases. To do this we need to adapt the corresponding solutions in [38] and [39] to our notation and dimensions. We start with the problem (4.36).

If we fix

$$e \in \{e\} := \{a_\alpha^*, b_\alpha^*\}_{\alpha\in\mathscr{E}^*} \cap \{a_\alpha, b_\alpha\}_{\alpha\in\mathscr{E}},$$

we also fix

$$\alpha \in \mathscr{E}: \quad \alpha := \alpha(e) \quad \text{and} \quad B, C \in \mathscr{V}: \quad \alpha \in (B,C).$$

The matrices representing a solution U_e have a nontrivial block

$$U_e^{BC} := \begin{pmatrix} U_{e,BB} & U_{e,BC} \\ U_{e,CB} & U_{e,CC} \end{pmatrix}$$

that is 2×2, while all the remaining entries in U_e are zeros apart from the ones on the main diagonal. The block U_e^{BC} is as follows:

$$U_e^{BC} = E_e^{BC} V_e^{BC} A_e^{BC},$$

where

$$A_e^{BC} = \operatorname{diag}\biggl\{\biggl(\frac{\Phi_B^{-n}}{\Phi_C^{-n}} w_\alpha^{1/2}\biggr)^{-1}, \frac{\Phi_B^{-n}}{\Phi_C^{-n}} w_\alpha^{1/2}\biggr\},$$

$$E_e^{BC} := \frac{1}{2} X^{BC} \operatorname{diag}(w_\alpha^{1/2}, w_\alpha^{-1/2}) M_e \operatorname{diag}\biggl(\sqrt{\pi n\varphi}, \frac{1}{\sqrt{\pi n\varphi}}\biggr).$$

Here X^{BC} is the corresponding block of the matrix (4.35), and

$$M_{a_\alpha} := \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}, \qquad M_{b_\alpha} := \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \qquad \varphi(z) = \log\biggl(\frac{\Phi_B(z)}{\Phi_C(z)}\biggr).$$

To write down an expression for V_e^{BC}, we introduce the matrices

$$\Psi_{a_\alpha} := \begin{pmatrix} I_{\delta(a_\alpha)}\Bigl(\dfrac{n\varphi}{2}\Bigr) & \dfrac{i}{\pi} K_{\delta(a_\alpha)}\Bigl(\dfrac{n\varphi}{2}\Bigr) \\ n\pi i\varphi I'_{\delta(a_\alpha)}\Bigl(\dfrac{n\varphi}{2}\Bigr) & -n\varphi K'_{\delta(a_\alpha)}\Bigl(\dfrac{n\varphi}{2}\Bigr) \end{pmatrix},$$

and, similarly, Ψ_{b_α} (which is of the same form as Ψ_{a_α}, with a_α changed to b_α, and the second column multiplied by -1). Here I and K are modified Bessel functions, and the exponentials $\delta(e)$ were introduced in (1.42). We partition the neighbourhood O_e into the sectors $O_e^{(\pm)}$, bounded by $\partial O_e^{(\pm)} := E_\alpha^{*\pm} \cap E_\alpha^*$ and consider the sector $O_e^* := O_e \setminus \{\overline{O_e^{(+)} \cup O_e^{(-)}}\}$.

In the sectors $O_e^* \cup O_e^{(+)} \cup O_e^{(-)}$ the matrices V_e^{BC} are given by

$$V_{a_\alpha}^{BC} := \begin{cases} \Psi_{a_\alpha} & \text{in } O_{a_\alpha}^*, \\ \Psi_{a_\alpha} \begin{pmatrix} 1 & 0 \\ e^{\delta(a_\alpha)\pi i} & 1 \end{pmatrix} & \text{in } O_{a_\alpha}^{(+)}, \\ \Psi_{a_\alpha} \begin{pmatrix} 1 & 0 \\ -e^{-\delta(a_\alpha)\pi i} & 1 \end{pmatrix} & \text{in } O_{a_\alpha}^{(-)}, \end{cases}$$

$$V_{b_\alpha}^{BC} := \begin{cases} \Psi_{b_\alpha} & \text{in } O_{b_\alpha}^*, \\ \Psi_{b_\alpha} \begin{pmatrix} 1 & 0 \\ -e^{-\delta(b_\alpha)\pi i} & 1 \end{pmatrix} & \text{in } O_{b_\alpha}^{(+)}, \\ \Psi_{b_\alpha} \begin{pmatrix} 1 & 0 \\ e^{\delta(b_\alpha)\pi i} & 1 \end{pmatrix} & \text{in } O_{b_\alpha}^{(-)}. \end{cases}$$

Similarly, for $e \in \{e^*\} := \{a_\alpha^*, b_\alpha^*\}_{\alpha \in \mathscr{E}^*} \setminus \{a_\alpha, b_\alpha\}_{\alpha \in \mathscr{E}}$, the nontrivial block U_e^{BC} of the solution U_e of problem (4.37) is as follows

$$U_e^{BC} = E_e^{BC} V_e^{BC} A_e^{BC},$$

where

$$\begin{aligned} A_e^{BC} &:= \operatorname{diag}\left\{ \left(\frac{\Phi_B^{-n}}{\Phi_C^{-n}} w_\alpha^{1/2} \right)^{-1}, \frac{\Phi_B^{-n}}{\Phi_C^{-n}} w_\alpha^{1/2} \right\}, \\ E_e^{BC} &:= -\frac{\sqrt{\pi}}{2} e^{i\pi/6} X^{BC} \operatorname{diag}\{w_\alpha^{1/2}, w_\alpha^{-1/2}\} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \\ &\qquad \times \operatorname{diag}\left\{ \left(\frac{3n}{2}\varphi \right)^{-1/6}, \left(\frac{3n}{2}\varphi \right)^{1/6} \right\}. \end{aligned}$$

To obtain an expression for V_e^{BC}, we introduce the matrices

$$\Psi_e := \begin{pmatrix} \operatorname{Ai}\left(\left(\frac{3n}{2}\varphi \right)^{2/3} \right) & \operatorname{Ai}\left(\epsilon_3^2 \left(\frac{3n}{2}\varphi \right)^{2/3} \right) \\ \operatorname{Ai}'\left(\left(\frac{3n}{2}\varphi \right)^{2/3} \right) & \epsilon_3^2 \operatorname{Ai}'\left(\epsilon_3^2 \left(\frac{3n}{2}\varphi \right)^{2/3} \right) \end{pmatrix} \widetilde{\sigma},$$

$$\widetilde{\Psi}_e := \begin{pmatrix} \operatorname{Ai}\left(\left(\frac{3n}{2}\varphi \right)^{2/3} \right) & -\epsilon_3^2 \operatorname{Ai}\left(\epsilon_3^2 \left(\frac{3n}{2}\varphi \right)^{2/3} \right) \\ \operatorname{Ai}'\left(\left(\frac{3n}{2}\varphi \right)^{2/3} \right) & -\operatorname{Ai}'\left(\epsilon_3^2 \left(\frac{3n}{2}\varphi \right)^{2/3} \right) \end{pmatrix} \widetilde{\sigma},$$

where Ai is the Airy function, and

$$\widetilde{\sigma} := \operatorname{diag}(e^{i\pi/6}, e^{-i\pi/6}), \qquad \epsilon_3 = e^{2\pi i/3}.$$

We split up the sector O_e^* into sectors $O_e^{*(\pm)}$, bounded by $\partial O_e^{*(\pm)} := E_\alpha^{*\pm} \cap E_e$.

In sectors $O_e^{*(+)} \cup O_e^{*(-)} \cup O_e^{(+)} \cup O_e^{(-)}$ the matrices V_e^{BC} are given by

$$V_{b^*} := \begin{cases} \Psi_e & \text{in } O_e^{*(+)}, \\ \Psi_e \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} & \text{in } O_e^{(+)}, \\ \widetilde{\Psi}_e \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{in } O_e^{(-)}, \\ \widetilde{\Psi}_e & \text{in } O_e^{*(-)}. \end{cases}$$

4.6. The final transformations and asymptotic formulae. Consider the function

$$\mathscr{J} := \begin{cases} \widehat{Z}X^{-1} & \text{in } \overline{\mathbb{C}} \setminus \Big\{ \bigcup\limits_{\alpha\in\mathscr{E}^*} \Big(O_{a_\alpha} \bigcup O_{b_\alpha} \Big) \Big\}, \\ \widehat{Z}U_e^{-1} & \text{in } O_e,\ e \in \{a_\alpha, b_\alpha\}_{\alpha\in\mathscr{E}^*}. \end{cases} \tag{4.39}$$

An analysis of the problems for the functions $\widehat{Z}$, X and U_e (see (4.12), (4.14) and (4.36), (4.37)) shows that

$$\mathscr{J} \in H(\overline{\mathbb{C}} \setminus \widetilde{\Sigma}), \qquad \widetilde{\Sigma} := \mathring{E}^{*\pm} \cup \partial\widetilde{O} \cup \mathring{E},$$

where

$$\mathring{E}^{*\pm} := \bigcup_{\alpha\in\mathscr{E}^*} \{(E_\alpha^{*+} \cup E_\alpha^{*-}) \setminus (O_{a_\alpha^*} \cup O_{b_\alpha^*})\}, \qquad \partial\widetilde{O} := \bigcup_{\alpha\in\mathscr{E}^*} (\partial O_{a_\alpha} \cup \partial O_{b_\alpha}),$$

$$\mathring{E} := \bigcup_{\alpha\in\mathscr{E}} (E_\alpha \setminus S(\lambda_\alpha)) \setminus \bigcup_{\alpha\in\mathscr{E}^*} (O_{a_\alpha} \cup O_{b_\alpha}).$$

The jump on the curves of discontinuity converges uniformly to the identity matrix as $n \to \infty$,

$$\mathscr{J}_+ = \mathscr{J}_- \widetilde{I}_n \quad \text{on } \widetilde{\Sigma}, \qquad \widetilde{I}_n \rightrightarrows I \quad \text{as } n \to \infty,$$

and now $J(\infty) = 1$.

Thus our objective (see (4.8)) has been reached, and so we conclude that

$$\mathscr{J} \rightrightarrows I \quad \text{in } \overline{\mathbb{C}} \quad \text{as } n \to \infty. \tag{4.40}$$

We invert the equivalent transformations which took us from Y to $\mathscr{J}$. Substituting (4.4) in (4.10) and further in (4.12) we obtain the following results for $\widehat{Z} = \{\widehat{Z}_{A,B}\}_{A,B\in\mathscr{V}}$. Outside E^*, we have

$$\widehat{Z}_{A,B} = c_A \Psi_{n^{(A)},B} C_A^n \Phi_B^n \quad \text{in } \overline{\mathbb{C}} \setminus (L_\alpha^{(+)} \cup L_\alpha^{(-)}), \quad \alpha \in B_- \cup B_+. \tag{4.41}$$

Now we consider neighbourhoods of interior points of E^*. Let $E_\alpha^* \subset E^*$ be fixed. Suppose that $\alpha \in (C, D)$. Then, by (4.12), (4.13), on $L_\alpha^{(\pm)}$ we have:

$$\widehat{Z}_{A,B} = \begin{cases} c_A \Psi_{n^{(A)},B} C_A^n \Phi_B^n, & B \neq C, \\ c_A \Psi_{n^{(A)},B} C_A^n \Phi_B^n \mp c_A \Psi_{n^{(A)},D} C_A^n \Phi_D^n \dfrac{\Phi_B^n}{\Phi_D^n w_\alpha}, & B = C; \end{cases} \tag{4.42}$$

that is, the elements of the Cth column of the matrix $\widehat{Z}$ (see (4.41)) in a neighbourhood of interior points of the interval E^*_α, $\alpha \in (C,D)$, have the following representation

$$\widehat{Z}_{A,C}(z) = c_A\Psi_{n^{(A)},C}(z)C_A^n\Phi_C^n(z) \mp c_A\Psi_{n^{(A)},D}(z)C_A^n\Phi_C^n(z)\frac{1}{w_\alpha(z)}, \qquad z \in L_\alpha^{(\pm)}.$$

Also, it is worth noting that on the interval E^*_α, $\alpha \in (C,D)$, the function $\Psi_{n^{(A)},C}$ is holomorphic, and the function $\Psi_{n^{(A)},D}$ assumes different boundary values according to whether we approach from above or below.

We proceed to the asymptotics. Since, by (4.39) and (4.40),

$$\widehat{Z} = \left(I + O\left(\frac{1}{n}\right)\right)X \quad \text{as } n \to \infty,$$

we arrive at the following asymptotic formulae (outside E^*):

$$\begin{cases} P_n(z) = (C_O\Phi_O(z))^{-n}X_{OO}(z)\left(1 + O\left(\frac{1}{n}\right)\right), & z \in K \Subset \overline{\mathbb{C}} \setminus \bigcup\limits_{\alpha\in\mathscr{E}_{O+}}, \\ c_A\Psi_{n^{(A)},B}(z) = (C_A\Phi_B(z))^{-n}X_{AB}(z)\left(1 + O\left(\frac{1}{n}\right)\right), & z \in K \Subset \overline{\mathbb{C}} \setminus \bigcup\limits_{\alpha\in\mathscr{E}_{B+}\cup\mathscr{E}_{B-}}. \end{cases}$$

Also, for the E^*_α in question $\alpha \in (C,D)$, so it follows from (4.42) that

$$c_A\Psi_{n^{(A)},D\pm}(z) = (C_A\Phi_{D\pm}(z))^{-n}X_{AD}(z)\left(1 + O\left(\frac{1}{n}\right)\right),$$

In addition, for $c_A\Psi_{n^{(A)},C}$ on E^*_α we have, by (4.42),

$$\begin{aligned} c_A\Psi_{n^{(A)},C} &= \left(\frac{c_A\Psi_{n^{(A)},D+}}{w_\alpha} + \frac{X_{A,C+}}{(C_A\Phi_{C+})^n}\right)\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= \left(\frac{X_{A,D+}}{w_\alpha(C_A\Phi_{D+})^n} + \frac{X_{A,C+}}{(C_A\Phi_{C+})^n}\right)\left(1 + O\left(\frac{1}{n}\right)\right), \end{aligned}$$

when approaching from above. Since (see (4.14), (4.13)) on E^*_α, where $\alpha \in (C,D)$, we have

$$X_{AD+} = w_\alpha X_{AC-}, \qquad \Phi_{D+} = \Phi_{C-},$$

thus uniformly for $x \in K \Subset E^*_\alpha$, $\alpha \in C_+$,

$$c_A\Psi_{n^{(A)},C} = \left(\frac{X_{A,C-}}{(C_A\Phi_{C-})^n} + \frac{X_{A,C+}}{(C_A\Phi_{C+})^n}\right)\left(1 + O\left(\frac{1}{n}\right)\right).$$

As a particular case, the asymptotics

$$P_n(x) = \left\{(C_O\Phi_{O+}(x))^{-n}X_{OO+}(x) + (C_O\Phi_{O-}(x))^{-n}X_{OO-}(x)\right\}\left(1 + O\left(\frac{1}{n}\right)\right)$$

is uniform for $x \in K \Subset E^*_\alpha$, $\alpha \in O_+$. It is worth recalling that the matrix X can be written explicitly in terms of Szegő functions (see (4.17)) and in terms of ratios of theta functions (4.31) through formulae (4.35), (4.19), (4.34) and (4.25).

This completes the proof of Theorem 1.3.

§ 5. Examples

The Hermite-Padé problem for a system of functions on arbitrary graphs gives rise to two new effects, which are not found in tree graphs. First, the energy functional may attain its minimum at the boundary of the admissible set of measures; that is, some components of the extremal measure may be zero. Second, in special cases there may exist a subsequence of non-normal indices. We illustrate these special features by two concrete examples.

5.1. The triangle graph. Consider the graph on the vertices A, B, O (O is the least (root) vertex) joined by three edges α, β and γ (see Fig. 4). To the edges of the graph we assign nonoverlapping intervals E_α, E_β and E_γ of the real axis. On these intervals we have positive Borel measures σ_α, σ_β and σ_γ such that the derivative of the absolutely continuous component σ'_κ, for $\kappa \in \mathscr{E} = \{\alpha, \beta, \gamma\}$, satisfies $\sigma'_\kappa > 0$ a.e. on E_κ with respect to the Lebesgue measure.

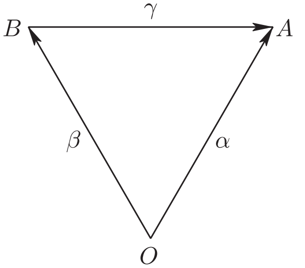


Figure 4. The triangle graph.

Corresponding to the vertices A and B of the graph there is a generalized system of Nikishin functions:

$$f_A(z) = \widehat{\sigma}_\alpha(z) + \int_{E_\beta} \frac{\widehat{\sigma}_\gamma(x)\,d\sigma_\beta(x)}{z-x}, \tag{5.1}$$

$$f_B(z) = \widehat{\sigma}_\beta(z); \tag{5.2}$$

here, $\widehat{\sigma}_\kappa(x) = \displaystyle\int_{E_\kappa} \frac{d\sigma_\kappa(t)}{x-t}$.

Given a fixed diagonal multi-index (n, n), consider the Hermite-Padé approximants to this system; these are the rational functions $(Q_{n,A}/P_n,\ Q_{n,B}/P_n)$ with common denominator P_n with $\deg P_n \leqslant 2n$ and such that, as $z \to \infty$,

$$R_{n,A} := f_A P_n - Q_{n,A} = O(z^{-n-1}), \tag{5.3}$$

$$R_{n,B} := f_B P_n - Q_{n,B} = O(z^{-n-1}). \tag{5.4}$$

These conditions are equivalent to the orthogonality relations for $j = 0, \dots, n-1$:

$$\int P_n(x) x^j\, d\sigma_\alpha(x) + \int P_n(x) x^j \widehat{\sigma}_\gamma(x)\, d\sigma_\beta(x) = 0, \tag{5.5}$$

$$\int P_n(x) x^j\, d\sigma_\beta(x) = 0. \tag{5.6}$$

This gives the following relations for the function of the second kind $R_{n,B}$:

$$\int R_{n,B}(x)x^j\,d\sigma_\gamma(x) = -\int P_n(x)x^j\widehat{\sigma}_\gamma(x)\,d\sigma_\beta(x), \qquad j=0,\dots,n. \tag{5.7}$$

In fact, in view of the orthogonality relations (5.6) and Fubini's theorem,

$$\begin{aligned}\int_{E_\gamma} R_{n,B}(x)x^j\,d\sigma_\gamma(x) &= \int_{E_\gamma}\int_{E_\beta}\frac{P_n(t)\,d\sigma_\beta(t)}{x-t}x^j\,d\sigma_\gamma(x)\\ &= \int_{E_\gamma}\int_{E_\beta}P_n(t)\frac{x^j-t^j}{x-t}\,d\sigma_\beta(t)\,d\sigma_\gamma(x)+\int_{E_\gamma}\int_{E_\beta}\frac{P_n(t)t^j\,d\sigma_\beta(t)}{x-t}\,d\sigma_\gamma(x)\\ &= \int_{E_\gamma}\frac{d\sigma_\gamma(x)}{x-t}P_n(t)t^j\,d\sigma_\beta(t) = -\int_{E_\beta}P_n(t)t^j\widehat{\sigma}_\gamma(t)\,d\sigma_\beta(t).\end{aligned} \tag{5.8}$$

Hence

$$\int P_n(x)x^j\,d\sigma_\alpha(x) - \int R_{n,B}(x)x^j\,d\sigma_\gamma(x) = 0, \qquad j=0,\dots,n-1. \tag{5.9}$$

Let $p_{n,\gamma}$ be a normalized polynomial (with leading coefficient one) formed from the zeros of $R_{n,B}$ on E_γ (if there are no zeros, then we set $p_{n,\gamma}=1$), $m_\gamma = \deg p_{n,\gamma}$. Then using Cauchy's integral formula, from (5.4) we obtain

$$\int_{E_\beta} P_n(x)x^j\frac{d\sigma_\beta(x)}{p_{n,\gamma}(x)} = 0, \qquad j=0,\dots,m_\gamma+n-1, \tag{5.10}$$

and so

$$\frac{R_{n,B}(z)}{p_{n,\gamma}(z)} = \int\frac{P_n(x)\,d\sigma_\beta(x)}{(z-x)p_{n,\gamma}(x)}. \tag{5.11}$$

Let $p_{n,\alpha}$ and $p_{n,\beta}$ be the polynomials (of degrees m_α and m_β) formed from the zeros of P_n on the intervals E_α and E_β, respectively. Then conditions (5.9) and (5.10) force $m_\alpha+m_\gamma \geqslant n-1$ and $m_\beta \geqslant m_\gamma+n$. Hence $m_\alpha+m_\beta \geqslant 2n-1$. We write the orthogonality relations (5.9) as follows:

$$\int_{E_\alpha} p_{n,\alpha}(x)p_{n,\gamma}(x)x^j\frac{P_n(x)\,d\sigma_\alpha(x)}{p_{n,\alpha}(x)p_{n,\gamma}(x)} - \int_{E_\gamma} p_{n,\alpha}(x)p_{n,\gamma}(x)x^j\frac{R_{n,B}(x)\,d\sigma_\gamma(x)}{p_{n,\gamma}(x)p_{n,\alpha}(x)} = 0. \tag{5.12}$$

Let $\mu_{n,\alpha}$, $\mu_{n,\beta}$ and $\mu_{n,\gamma}$ be the zero counting measures associated with the polynomials $p_{n,\alpha}$, $p_{n,\beta}$ and $p_{n,\gamma}$, respectively. Consider a subsequence $\Lambda\subset\mathbb{N}$ such that these measures have some weak limits: $\mu_{n,\alpha}/n\to\lambda_\alpha$, $\mu_{n,\beta}/n\to\lambda_\beta$, $\mu_{n,\gamma}/n\to\lambda_\gamma$. By the Gonchar-Rakhmanov theorem on the weak asymptotics of polynomials which are orthogonal with respect to a variable weight (see [33], [20]), and using the orthogonality conditions (5.12) and (5.10) and representations (5.11) we obtain the following equilibrium relations for the potentials of the limit measures:

$$2V^{\lambda_\alpha}(x)+V^{\lambda_\beta}(x)+V^{\lambda_\gamma}(x)\begin{cases}=\varkappa_\alpha, & x\in S(\lambda_\alpha),\\ \geqslant\varkappa_\alpha, & x\in E_\alpha,\end{cases} \tag{5.13}$$

$$2V^{\lambda_\beta}(x)+V^{\lambda_\alpha}(x)-V^{\lambda_\gamma}(x)\begin{cases}=\varkappa_\beta, & x\in S(\lambda_\beta),\\ \geqslant\varkappa_\beta, & x\in E_\beta,\end{cases} \tag{5.14}$$

$$2V^{\lambda_\gamma}(x)+V^{\lambda_\alpha}(x)-V^{\lambda_\beta}(x)\begin{cases}=\varkappa_\gamma:=\varkappa_\alpha-\varkappa_\beta, & x\in S(\lambda_\gamma),\\ \geqslant\varkappa_\gamma, & x\in E_\gamma.\end{cases} \tag{5.15}$$

In fact, (5.14) follows from (5.10), relation (5.13) follows from (5.12) on the interval E_α, and (5.15) is a consequence of (5.12) on the interval E_γ; here we have taken into account the asymptotics

$$\frac{1}{n}\ln\left|\frac{R_{n,B}(z)}{p_{n,\gamma}(z)}\right| \quad\rightrightarrows\quad V^{\lambda_\beta}(z)-\varkappa_\beta,$$

which hold uniformly outside of E_β. This asymptotics follows from (5.11). As there is a unique vector measure $(\lambda_\alpha,\lambda_\beta,\lambda_\gamma)$ that satisfies these equilibrium relations and the conditions $|\lambda_\beta|-|\lambda_\gamma|=1$, $|\lambda_\alpha|+|\lambda_\gamma|=1$, it follows that

$$\frac{1}{n}(\mu_{n,\alpha},\mu_{n,\beta},\mu_{n,\gamma})\to(\lambda_\alpha,\lambda_\beta,\lambda_\gamma)\quad\text{and}\quad n\to\infty.$$

Formally, two kinds of degenerate solutions of the equilibrium problem are possible: $|\lambda_\alpha|=0$ and $|\lambda_\gamma|=0$. In the first case, $(\lambda_\beta,\lambda_\gamma)$ is an equilibrium measure with Nikishin interaction matrix, and the inequality $V^\beta+V^\gamma\geqslant\varkappa_\beta+\varkappa_\gamma$ holds on E_α. In the second case, $(\lambda_\alpha,\lambda_\beta)$ is an equilibrium measure with Angelesco interaction matrix and $V^\beta(x)-V^\alpha(x)\leqslant\varkappa_\beta-\varkappa_\alpha$ holds for $x\in E_\gamma$.

For a Nikishin system, the equilibrium potentials can be expressed in terms of the branches of the algebraic function $\Phi(z)$,

$$\Phi_O\in\mathscr{H}(\mathbb{C}\setminus E_\beta),\qquad \Phi_O(z)=z^{-2}\left(1+O\left(\frac{1}{z}\right)\right)\ \text{as } z\to\infty, \tag{5.16}$$

$$\Phi_B\in\mathscr{H}(\mathbb{C}\setminus(E_\gamma\cup E_\beta)),\qquad \Phi_A(z)=c_Bz\left(1+O\left(\frac{1}{z}\right)\right)\ \text{as } z\to\infty, \tag{5.17}$$

$$\Phi_A\in\mathscr{H}(\mathbb{C}\setminus E_\gamma),\qquad \Phi_A(z)=c_Az\left(1+O\left(\frac{1}{z}\right)\right)\ \text{as } z\to\infty, \tag{5.18}$$

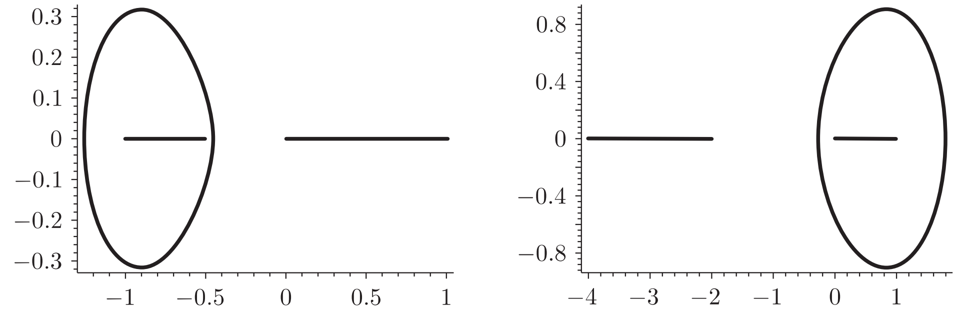
as follows:

$$V^\beta=\ln|\Phi_O|, \tag{5.19}$$

$$V^\beta-V^\gamma=-\ln|\Phi_B|+\varkappa_\beta, \tag{5.20}$$

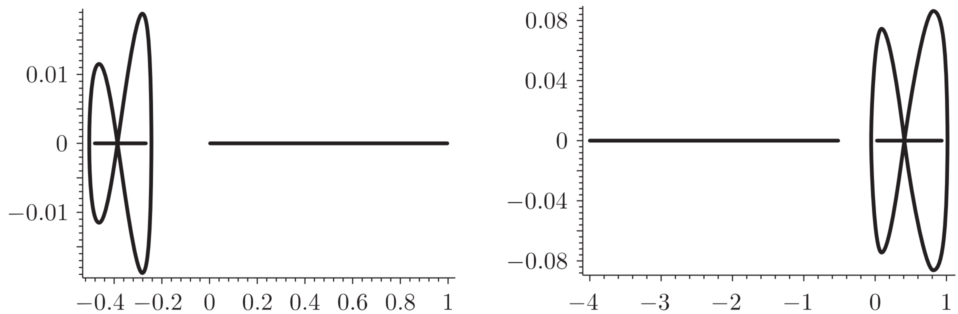
$$V^\gamma=-\ln|\Phi_A|+\varkappa_\beta+\varkappa_\gamma. \tag{5.21}$$

Also, $V^\beta+V^\gamma\geqslant\varkappa_\beta+\varkappa_\gamma$ if and only if $|\Phi_0|\geqslant|\Phi_2|$. However, as was shown in [21], the inequality $|\Phi_2|>|\Phi_1|>|\Phi_0|$ holds for all $z\in\mathbb{C}\setminus(E_\gamma\cup E_\beta)$. Hence this boundary solution is not realized.



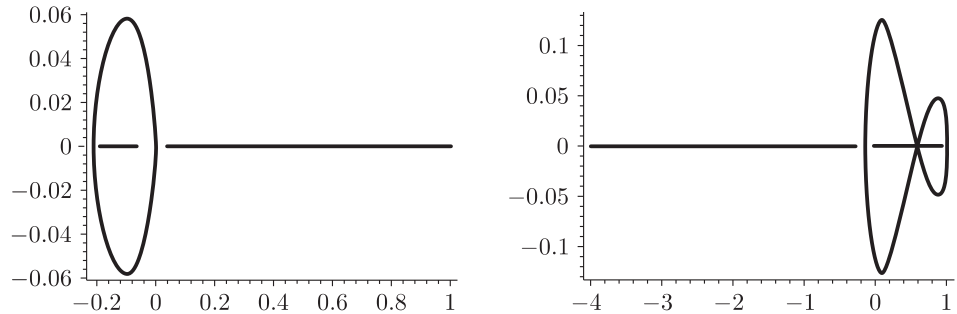
a) b)

Figure 5. Critical trajectories for the Angelesco system: $E_\alpha = [0,1]$; $E_\beta = [-1,-1/2]$ (Fig. a)), $E_\beta = [-4,-2]$ (Fig. b)).



a) b)

Figure 6. Critical trajectories for the Angelesco system: $E_\alpha = [0,1]$; $E_\beta = [-1/2,-1/4]$ (Fig. a)), $E_\beta = [-4,-1/2]$ (Fig. b)).



a) b)

Figure 7. Critical trajectories for the Angelesco system in the case of collision: $E_\alpha^* = [0,1]$, $E_\beta = [-0.2,-0.0716]$ (Fig. a)); $E_\alpha = [0,1]$, $E_\beta^* = [-4,-1/7]$ (Fig. b)).

We now consider the case when the measure λ_γ is identically zero. It is known that the equilibrium potentials for the Angelesco problem can be expressed in terms of the branches of the three-valued algebraic function $\Phi(z)$ as follows:

$$V^\alpha + V^\beta = \ln|\Phi_O|, \tag{5.22}$$
$$V^\alpha = -\ln|\Phi_A| + \varkappa_\alpha, \tag{5.23}$$
$$V^\beta = -\ln|\Phi_B| + \varkappa_\beta; \tag{5.24}$$

here, Φ_O, Φ_A and Φ_B are branches of the function $\Phi(z)$ such that

$$\Phi_O \in \mathscr{H}(\mathbb{C}\setminus(E_\alpha^* \cup E_\beta^*)), \qquad \Phi_O(z) = z^{-2}\biggl(1+O\biggl(\frac{1}{z}\biggr)\biggr) \text{ as } z\to\infty, \tag{5.25}$$
$$\Phi_A \in \mathscr{H}(\mathbb{C}\setminus E_\alpha^*), \qquad \Phi_A(z) = c_A z\biggl(1+O\biggl(\frac{1}{z}\biggr)\biggr) \text{ as } z\to\infty, \tag{5.26}$$
$$\Phi_B \in \mathscr{H}(\mathbb{C}\setminus E_\beta^*), \qquad \Phi_B(z) = c_B z\biggl(1+O\biggl(\frac{1}{z}\biggr)\biggr) \text{ as } z\to\infty. \tag{5.27}$$

Hence, the condition $V^\beta - V^\alpha \leqslant \varkappa_\beta - \varkappa_\alpha$ is equivalent to saying that $|\Phi_A| \leqslant |\Phi_B|$. The curves on which different pairs of branches of the function Φ are equal in absolute value are depicted in Figs. 5–7. We shall now indicate regions in which $|\Phi_A| < |\Phi_B|$. In Figs. 5, 6b) and in Fig. 7, the measure $|\lambda_\gamma| = 0$ when the interval E_γ lies in the region which contains the point $1+0$. In Fig. 6a), the measure $|\lambda_\gamma| = 0$ when the interval E_γ lies in the complement of the region containing the point $-0,5-0$.

5.2. The quadrangular graph. Consider the graph in Fig. 8. Corresponding to the edges α, β, γ and δ of this graph there are intervals E_α, E_β, E_γ, E_δ and measures σ_α, σ_β, σ_γ, σ_δ acting on them. The intervals E_α, E_β and E_γ are disjoint, and so are the intervals E_β, E_γ and E_δ. To the vertex set of this graph there

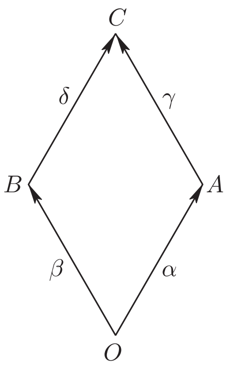


Figure 8. The quadrangular graph.

corresponds the system of functions

$$f_A = \widehat{\sigma}_\alpha, \qquad f_B = \widehat{\sigma}_\beta, \tag{5.28}$$

$$f_C(z) = \int_{E_\alpha} \frac{\widehat{\sigma}_\gamma(x)\,d\sigma_\alpha(x)}{z-x} + \int_{E_\beta} \frac{\widehat{\sigma}_\delta(x)\,d\sigma_\beta(x)}{z-x}. \tag{5.29}$$

Consider the Hermite-Padé approximants to this system of functions for the multi-index (n, n, n). The polynomial P_n of degree $\deg P_n \leqslant 3n$ satisfies the following orthogonal relations for $j = 0, \dots, n-1$:

$$\int P_n(x) x^j\,d\sigma_\alpha(x) = \int P_n(x) x^j\,d\sigma_\beta(x) = 0, \tag{5.30}$$

$$\int P_n(x) x^j \widehat{\sigma}_\gamma(x)\,d\sigma_\alpha(x) + \int P_n(x) x^j \widehat{\sigma}_\delta(x)\,d\sigma_\beta(x) = 0. \tag{5.31}$$

As in the previous case, $\deg P_n \geqslant 3n-1$. We shall show that in general the indices may fail to be normal.

Consider the symmetric case:

$$E_\beta = -E_\alpha, \quad E_\gamma = -E_\delta, \qquad d\sigma_\beta(-x) = -d\sigma_\alpha(x), \quad d\sigma_\gamma(-x) = -d\sigma_\delta(x).$$

In this case, there is an even polynomial P_n satisfying (5.32) and (5.33). In fact, for an even polynomial it suffices to place $[3n/2]$ conditions:

$$\int P_n(x) x^j\,d\sigma_\alpha(x) = 0, \qquad j = 0, \dots, n-1, \tag{5.32}$$

$$\int P_n(x) x^j \widehat{\sigma}_\gamma(x)\,d\sigma_\alpha(x) + \int P_n(x) x^j \widehat{\sigma}_\delta(x)\,d\sigma_\beta(x) = 0, \qquad j = 1, 3, \dots, 2\left[\frac{n}{2}\right] - 1. \tag{5.33}$$

The number of unknown coefficients of even powers of the polynomial P_n of degree $2[3n/2]$ is $[3n/2]+1$. Hence there exists an even orthogonal polynomial, and so the indices of the form $(2k+1, 2k+1, 2k+1)$ are not normal.

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