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Stabilization of the solution of a two-dimensional system of Navier–Stokes equations in an unbounded domain with several exits to infinity

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Abstract. The behaviour as $t \to \infty$ of the solution of the mixed problem for the system of Navier–Stokes equations with a Dirichlet condition at the boundary is studied in an unbounded two-dimensional domain with several exits to infinity. A class of domains is distinguished in which an estimate characterizing the decay of solutions in terms of the geometry of the domain is proved for exponentially decreasing initial velocities. A similar estimate of the solution of the first mixed problem for the heat equation is sharp in a broad class of domains with several exits to infinity.

Bibliography: 25 titles.

§ 1. Introduction

In the domain $D = (0, \infty) \times \Omega$, where Ω is an unbounded subdomain of $\mathbb{R}^2$, we consider the following problem:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu\Delta\mathbf{u} - \nabla p, \qquad \operatorname{div} \mathbf{u} = 0, \tag{1.1}$$

$$\mathbf{u}\big|_{x \in \partial\Omega} = 0, \qquad \mathbf{u}\big|_{t=0} = \boldsymbol{\varphi}(x). \tag{1.2}$$

Here $\mathbf{u}(t, x) = (u_1, u_2)$ and $p(t, x)$ are the unknown flow velocity and the pressure and $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ are the prescribed initial velocities.

Note that in the problems discussed here one can make the change of variables $\mathbf{u} = \nu\mathbf{v}$, $t = \tau/\nu$, $p = \nu^2 q$, bringing (1.1) to a similar system with $\nu = 1$.

In the past 10–15 years there have appeared numerous papers devoted to the research of the behaviour as $t \to \infty$ of the kinetic energy

$$\frac{1}{2}\int_\Omega \mathbf{u}^2(t, x)\, dx$$

(the $\mathbf{L}_2$-norm) of the flow of a fluid in an unbounded domain. A qualitative answer to the question of the convergence of the kinetic energy to zero in the case of a

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3-dimensional Cauchy problem was given by Kato [1] (for a strong solution) and Masuda [2] (for a weak solution). Moreover, the following estimate was established in [1]. If a solenoidal vector $\boldsymbol{\varphi}$ belongs to the intersection $\mathbf{L}_n(\mathbb{R}^n)\cap\mathbf{L}_r(\mathbb{R}^n)$, $r \in [1, n]$, and its norm $\|\boldsymbol{\varphi}\|_n$ is sufficiently small, then the Cauchy problem (1.1), (1.2) has a unique strong solution and $\|\mathbf{u}(t)\|_\alpha = O(t^{-\gamma})$, $\gamma = (n/r - n/\alpha)/2$, for $\alpha > r$ as $t \to \infty$. Here and throughout,

$$\|\mathbf{v}\|_{\alpha,Q} = \left(\sum_{i=1}^{n} \int_Q v_i^\alpha(x)\,dx\right)^{1/\alpha};$$

for $\alpha = 2$ and $Q = \Omega$ we shall drop the corresponding indices.

An estimate of the rate of decay of the kinetic energy for a weak solution of the n-dimensional Cauchy problem for the system (1.1) was established in [3] and refined in [4], [5]. We state here the result of [4]. If a solenoidal vector $\boldsymbol{\varphi}$ belongs to the intersection $\mathbf{L}_2(\mathbb{R}^n)\cap\mathbf{L}_r(\mathbb{R}^n)$, $n \geqslant 2$, $r \in [1, 2)$, then the Cauchy problem (1.1), (1.2) has a weak solution which decreases in the same manner as for the heat equation: $\|\mathbf{u}(t)\| = O(t^{-\gamma})$, $\gamma = (n/r - n/2)/2$. In [5] a similar estimate is proved for an arbitrary weak solution satisfying the energy inequality

$$\|\mathbf{u}(t)\|^2 + 2\nu \int_s^t \|\nabla\mathbf{u}(\tau)\|^2 d\tau \leqslant \|\mathbf{u}(s)\|^2$$

for $s = 0$, almost all $s > 0$, and all $t > s$. In the case of the problem in the exterior of a bounded domain results of this kind were obtained for $r \in (1, 2)$ in [6] (for $n = 3$) and [7] (for $n \geqslant 3$).

Thus, the non-linear terms and the pressure involved in the system (1.1) do not reduce the rate of decay of the flow of the fluid brought about by the heat operator in the system (1.1). Of course, the adhesion condition at the boundary (1.2) additionally slows down the flow; however, judging by the above results, this has no substantial effect on the behaviour of solutions of the exterior problem. On the other hand, we do not know whether the above results for the exterior problem are best possible.

The decay of the flow resulting from the adhesion of the fluid to the boundary of the domain is known to be perceptible in the case of a non-compact boundary. This is corroborated by the result of [8]. In particular, the following estimates are established in that paper in the case of rotation domains

$$\Omega(f) = \bigl\{x : x_1^2 + x_2^2 < f^2(x_3),\ x_3 > 0\bigr\} \tag{1.3}$$

defined by a non-decreasing function $f(r) \in C^3(0,\infty)$ such that

$$\lim_{r\to\infty} \frac{f(r)}{f(qr)} < \infty, \quad |f'| + |f''| + |f'''| \leqslant a_0, \qquad r \geqslant 1,$$

for some $q \in (0, 1)$. Let $r(t)$, $t > 0$, be the inverse of the increasing function $rf(r)$, $r > 0$. Let $\mathbf{u}(t, x)$ be the strong solution of the 3-dimensional problem (1.1), (1.2) in the domain $D = (0,\infty) \times \Omega(f)$ with solenoidal initial function $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$,

$$\boldsymbol{\varphi}(x) = 0 \quad \text{for } |x| > R_0, \tag{1.4}$$

satisfying the smallness condition of [9]. Then there exist positive constants $\varkappa$ and A_1 such that the following inequalities hold for all $x \in \Omega(f)$ and $t > 1$:

$$|\mathbf{u}(t,x)| + \|\nabla p(t)\| \leqslant A_1 \exp\biggl(\frac{-\varkappa r^2(t)}{t}\biggr), \qquad (1.5)$$
$$\|\mathbf{u}(t)\|_{\mathbf{W}_2^1(\Omega)} \leqslant A_1 t^{1/2} \exp\biggl(\frac{-\varkappa r^2(t)}{t}\biggr).$$

The constant $\varkappa$ here is independent of the initial function $\boldsymbol{\varphi}$.

An estimate of the rate of decay similar to (1.5) was established earlier [10] for solutions of the first mixed problem for the heat equation. Moreover, for domains of the form (1.3) and non-negative initial functions it was proved in the same paper that the estimate is sharp provided that the function f has a regular behaviour in a certain sense.

Thus, adhesion at the boundary results in the flow slowing down at least at the same rate as that due to the heat outflow across the boundary in the case of the first boundary condition for the heat equation.

Note that in papers devoted to the decay of the motion of a rotating fluid described by linear [11]–[15] or non-linear [16] equations (the Cauchy problem or the first boundary-value problem in a half-space) their authors study the phenomenon of the slow-down of the motion of a fluid brought about by its rotation, rather than by the adhesion condition at the boundary as in the present paper.

The proof of the estimates (1.5) for the solution of the 3-dimensional problem in [8] is based to a considerable extent on the following result of Heywood. In [9], for an arbitrary domain Ω of dimension $n = 3$ with boundary uniformly of class C^3 he proved the estimate $\sup_{x\in\Omega} |\mathbf{u}(t,x)| = O(t^{-1/2})$ as $t \to \infty$. (The term "boundary uniformly of class C^3" (see [9]) means the existence of positive constants d and b such that for an arbitrary point $\xi \in \partial\Omega$, in a local Cartesian system of coordinates, the intersection $\partial\Omega \cap \{|x-\xi| < d\}$ is the graph of a function with derivatives of the first three orders bounded by the constant b.) This result is slightly improved in [8] as follows:

$$\int_0^\infty \sup_{x\in\Omega} |\mathbf{u}(t,x)|^2\,dt < \infty. \qquad (1.6)$$

Recall that the unique solubility 'in the large' of the problem (1.1), (1.2) in the class $\mathbf{L}_4$ was proved in Ladyzhenskaya's paper [17]. In a joint work, Lions and Prodi [18] prove a uniqueness result for the weak solution.

Maremonti [19] established the following relations in the case of solenoidal initial velocities $\boldsymbol{\varphi} \in \mathbf{L}_p(\Omega) \cap \mathbf{L}_2(\Omega)$, $p \in (1,2]$, for the solution of the problem (1.1), (1.2) in an arbitrary domain $\Omega \subseteq \mathbb{R}^2$ with C^2-boundary:

$$\|\mathbf{u}(t)\|_{\mathbf{L}_2} + t^{1/2}\|\nabla\mathbf{u}(t)\|_{\mathbf{L}_2} + t\|\mathbf{u}_t(t)\|_{\mathbf{L}_2} = O(t^{-\alpha}) \quad \text{as } t\to\infty, \quad \alpha = \frac{1}{p} - \frac{1}{2},$$
$$\sup_{x\in\Omega} |\mathbf{u}(x,t)| = O(t^{-1/2-\alpha+\varepsilon}) \quad \text{for each } \varepsilon > 0, \quad \alpha = \frac{1}{p} - \frac{1}{2}.$$

In the case of a solenoidal initial vector $\boldsymbol{\varphi} \in \mathbf{L}_1(\Omega) \cap \mathbf{L}_2(\Omega)$ it is easy to derive (1.6) from the last relation.

The aim of the present paper is to obtain estimates of the form (1.5) in terms of geometric characteristics of the unbounded domain Ω with several exits to infinity. This problem has been partially solved in [20], where in the interiors of the parabolas

$$\Omega(\alpha) = \{x \in \mathbb{R}^2 : |x_2| < x_1^\alpha,\ x_1 > 1\} \tag{1.7}$$

the following estimate was proved for $\alpha \in \left(0, \frac{1}{2}\right)$:

$$|\mathbf{u}(t,x)| \leqslant A_1 \exp(-kt^{(1-\alpha)/(1+\alpha)}).$$

In the present paper we substantially extend the class of domains in which one can prove an estimate of the decrease of the flow of the fluid in the problem (1.1), (1.2) as $t \to \infty$. In particular, this class contains all parabolas $\Omega(\alpha)$ with $\alpha \in (0,1)$. By contrast with the 3-dimensional case the proof of our result is not based on relation (1.6) and does not use Maremonti's results.

Let Ω be a two-dimensional domain with k exits to infinity located along rays s_i, that is, a domain of the following form:

$$\Omega = \underset{0}{\Omega} \cup \biggl(\bigcup_{i=1}^k \underset{i}{\Omega}\biggr),$$

where the $\underset{i}{\Omega}$, $i = 1, \dots, k$, are disjoint unbounded simply connected domains and $\underset{0}{\Omega}$ is a bounded domain, which is not necessarily simply connected. We shall assume that if the Ox_1-axis is directed along a ray s_i, then the domain $\underset{i}{\Omega}$ lies in the half-plane $\{x_1 > 0\}$ and the $\underset{i}{\Omega}{}^r = \bigl\{x \in \underset{i}{\Omega} : x_1 < r\bigr\}$ are bounded simply connected domains for $r \geqslant P_i$. For a complete statement of the problem (1.1), (1.2) one must define the flows across the sections $S_i^r = \bigl\{x \in \underset{i}{\Omega} : x_1 = r\bigr\}$ of the domains $\underset{i}{\Omega}$. We set them equal to zero:

$$\int_{S_i^r} \frac{\partial \mathbf{u}}{\partial \mathbf{n}}\, dS = 0, \qquad i = 1, 2, \dots, k.$$

Let $\lambda_i(r)$ be the first eigenvalue of the operator $-\Delta$ in the domain $\underset{i}{\Omega}{}^r$ with a Neumann condition at the part of the boundary $\partial\underset{i}{\Omega} \cap \Omega$ and with a Dirichlet condition at the remaining part of the boundary:

$$\lambda_i(r) = \inf\biggl\{\int_{\underset{i}{\Omega}{}^r} |\nabla v|^2\, dx \biggl(\int_{\underset{i}{\Omega}{}^r} v^2\, dx\biggr)^{-1},\ v \in C_0^\infty\bigl(\underset{i}{\Omega}{}^r \cup \underset{0}{\Omega}\bigr)\biggr\}, \qquad r \geqslant P_i.$$

Obviously, the functions $\lambda_i(r)$, $r \geqslant P_i$, are non-increasing.

We shall use numbering such that

$$\lim_{r\to\infty} \lambda_i(r) = 0, \qquad i = 1, 2, \dots, s, \tag{1.8}$$

and $\lim_{r\to\infty}\lambda_i(r)>0$, $i=s+1,\dots,k$. Here it is possible that $s=k$. However, $s\geqslant 1$, since otherwise, as is well known, the solution decreases more rapidly than $e^{-\varepsilon t}$.

For the domain $Q=\underset{0}{\Omega}\cup\bigl(\bigcup_{i=s+1}^{k}\underset{i}{\Omega}\bigr)$ we require the inequality

$$\mu=\inf\biggl\{\int_Q|\nabla v|^2\,dx\biggl(\int_Q v^2\,dx\biggr)^{-1},\ v\in C_0^\infty(\Omega)\biggr\}>0. \tag{1.9}$$

It holds if the intersection $\partial Q\cap\partial\Omega$ is non-empty.

We assume moreover that there exist absolutely continuous non-decreasing positive functions $l_i(r)$, $r>0$, $i=1,\dots,s$, such that for $r\geqslant P_i$ the domains $\omega_i(r)=\underset{i}{\Omega}{}_r^{r+l_i(r)}=\underset{i}{\Omega}{}^{r+l_i(r)}\setminus\overline{\underset{i}{\Omega}{}^r}$ satisfy the Condition D below. It is known that for each bounded domain Q with Lipschitz boundary the equation

$$\operatorname{div}\mathbf{v}=g,\qquad x\in Q,\quad g\in L_2(Q),\quad \int_Q g\,dx=0,$$

has a solution $\mathbf{v}\in\overset{\circ}{\mathbf{W}}{}_2^1(Q)$ satisfying the estimate (see [21] and also [22])

$$\|\nabla\mathbf{v}\|_Q\leqslant d_1(Q)\|g\|_Q.$$

Condition D states that the constant d_1 in this inequality can be taken the same for each domain $\omega_i(r)$, $r\geqslant P_i$:

$$\|\nabla\mathbf{v}\|_{\omega_i(r)}\leqslant d_1\|g\|_{\omega_i(r)},\qquad i=1,\dots,s. \tag{1.10}$$

As follows from [22], this condition holds, for instance, if the domains $\omega_i(r)$, $r\geqslant P_i$, are uniformly star-shaped relative to some balls B_i. The uniformity means that the ratios $\operatorname{diam}\omega_i(r)/\operatorname{diam}B_i$ are bounded by a constant independent of $i=1,\dots,s$ and $r\geqslant P_i$. If $\Omega(\alpha)$ is a domain of the form (1.7) with some $\alpha\in(0,1)$, then obviously, the domains $\omega(r)$, $r\geqslant P$, become uniformly star-shaped for sufficiently large P if we take $l(r)=r^\alpha$. Hence such a domain satisfies our assumption.

We impose additionally the following regularity conditions on the functions l_i. There exist quantities $\alpha\in(0,1)$ and $q_i\in(0,1)$ such that

$$\frac{l_i(r)}{l_i(q_ir)}<q_i^{-\alpha},\qquad r\geqslant P_i. \tag{1.11}$$

Roughly speaking, this is a restriction from above on the growth of the functions $l_i(r)$.

We define the functions $r_i(t)$, $t>P_il_i(P_i)$, as the inverses of the increasing functions $rl_i(r)$, $r>P_i$. Obviously, $r_i(t)$ is increasing, tends to infinity, and satisfies the equalities

$$\frac{t}{l_i^2(r_i(t))}=\frac{r_i(t)}{l_i(r_i(t))}=\frac{r_i^2(t)}{t}\,. \tag{1.12}$$

Assume that there exists $\delta \in (0,1]$ such that

$$\lim_{r\to\infty} \frac{r^{1-\delta}}{\max_i l_i(r)} = 0. \tag{1.13}$$

Let $\boldsymbol{\varphi}$ be an initial function in $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ that is a limit of solenoidal functions with compact support and satisfies the condition

$$\|\boldsymbol{\varphi}(x)\|_{\underset{i}{\Omega}_r} \leqslant e^{-cr^{\delta}}, \qquad r \geqslant P, \quad i=1,\dots,s, \tag{1.14}$$

with some positive constants c and P, where $\underset{i}{\Omega}_r = \underset{i}{\Omega} \setminus \underset{i}{\Omega}^r$.

Theorem 1. *Let Ω be a two-dimensional domain with boundary uniformly of class C^3 and let l_i be functions satisfying* (1.11), (1.13), *and Condition* D; *let $\boldsymbol{\varphi}$ be a solenoidal initial function in $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ satisfying condition* (1.14). *Then there exist $\varkappa$, A_2, and T such that the solution of the problem* (1.1), (1.2) *satisfies the following estimates for all $x \in \Omega$ and $t > T$:*

$$|\mathbf{u}(t,x)| + \|\mathbf{u}(t)\| \leqslant A_2 \exp\biggl(-t\min_i\biggl\{\lambda_i\biggl(\frac{r_i(t)}{\varkappa}\biggr),\ \varkappa^2 l_i^{-2}(r_i(t))\biggr\}\biggr), \tag{1.15}$$

$$\begin{aligned}&\|\nabla\mathbf{u}(t)\| + \|D^2\mathbf{u}(t)\| + \|\nabla p(t)\| \\ &\qquad \leqslant A_2 \exp\biggl(-t\min_i\biggl\{\lambda_i\biggl(\frac{r_i(t)}{\varkappa}\biggr),\ \varkappa^2 l_i^{-2}(r_i(t))\biggr\}\biggr),\end{aligned} \tag{1.16}$$

where A_2 depends only on the constants d and b in the definition of a boundary uniformly in the class C^3, $\|\boldsymbol{\varphi}\|$, and $\|\nabla\boldsymbol{\varphi}\|$, and $\varkappa$ depends only on q_i and α in inequality (1.11) *and on d_1 in inequality* (1.10).

If the domains $\omega_i(r)$ are uniformly star-shaped, then the estimate (1.15) can be brought to the following form:

$$|\mathbf{u}(t,x)| \leqslant A_2 \exp\bigl(-\widetilde{\varkappa} t \min_i\{l_i^{-2}(r_i(t))\}\bigr). \tag{1.17}$$

For domains with several exits to infinity such that each exit 'tongue' is isometric to a domain of the form $\Omega(\alpha)$, the estimate of the solution of the heat equation similar to (1.17) is sharp. For in an individual 'tongue' $\underset{i}{\Omega}$ the following estimate is sharp according to [10]:

$$|\underset{i}{\mathbf{u}}(t,x)| \leqslant A_2 \exp(-\widetilde{\varkappa} t l_i^{-2}(r_i(t))).$$

Next, one has the maximum principle $\mathbf{u}(t,x) \geqslant \underset{i}{\mathbf{u}}(t,x) \geqslant 0$ for non-negative initial functions. Hence $\mathbf{u}(t,x) \geqslant \max_i \underset{i}{\mathbf{u}}(t,x)$, which demonstrates the sharpness of the estimate of the form (1.17) for solutions of the heat equation.

§ 2. Existence of a solution and its properties.

As pointed out in the introduction, the existence of a solution 'in the large' to the problem (1.1), (1.2) was proved in [17]. However, we require several additional differential properties of this solution, which can be more easily proved at the stage of the construction of Galerkin approximations. Hence we must repeat here parts of the known proofs in a slightly modified form adapted to our aims.

For an arbitrary domain $\Omega \subset \mathbb{R}^2$ let $\overset{\circ}{\mathbf{J}}{}^\infty(\Omega)$ be the set of smooth solenoidal vector-valued functions with compact support $\mathbf{v}\colon \Omega \to \mathbb{R}^2$ (in what follows we shall talk about vectors in place of vector-valued functions). Next, let $\overset{\circ}{\mathbf{J}}{}^\infty(D_a^b)$ be the set of smooth solenoidal vectors $\mathbf{v}(t,x)\colon D_a^b \to \mathbb{R}^2$ in the cylinder $D_a^b = (a,b) \times \Omega$, $\operatorname{div}_x \mathbf{v} = 0$. We denote by $\overset{\circ}{\mathbf{J}}(\Omega)$ the completion of the set $\overset{\circ}{\mathbf{J}}{}^\infty(\Omega)$ in the norm $\|\mathbf{v}\|$, and by $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$ the completion of the same set in the norm $\|\mathbf{v}\| + \|\nabla \mathbf{v}\|$, and we define the space of solenoidal functions $\mathbf{H}_0(\Omega)$ to be the completion of $\overset{\circ}{\mathbf{J}}{}^\infty(\Omega)$ in the norm $\|\nabla \mathbf{u}\|$. We define the space $\overset{\circ}{\mathbf{J}}{}_2^{0,1}(D_a^b)$ as the closure of the set $\overset{\circ}{\mathbf{J}}{}^\infty(D_{a-1}^{b+1})$ in the space $\mathbf{W}_2^{0,1}(D_a^b)$.

For arbitrary vectors $\mathbf{u}(x)$, $\mathbf{v}(x)$ we set

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{2} u_i v_i, \qquad \mathbf{u}^2 = \mathbf{u} \cdot \mathbf{u},$$

$$\nabla \mathbf{u} : \nabla \mathbf{v} = \sum_{i,j=1}^{2} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}, \qquad |\nabla \mathbf{u}|^2 = \nabla \mathbf{u} : \nabla \mathbf{u},$$

$$|D_x^2 \mathbf{u}|^2 = \sum_{i=1}^{2} \sum_{|\alpha|=2} \left| \frac{\partial^2 u_i}{\partial x^\alpha} \right|^2.$$

We use Galerkin's method to prove the existence of a generalized solution of the problem (1.1), (1.2) with initial function $\boldsymbol{\varphi}(x) \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$ satisfying condition (1.4). Next, having established certain properties of that solution we shall prove the existence of a solution for an arbitrary initial function $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$.

A *generalized solution of the problem* (1.1), (1.2) in $D^T = (0,T) \times \Omega$ is a vector-valued function $\mathbf{u} \in \overset{\circ}{\mathbf{J}}{}^{0,1}(D^T)$ satisfying the integral identity

$$\int_0^T \int_\Omega (\nabla \mathbf{u} : \nabla \mathbf{v} - \mathbf{u} \cdot \mathbf{v}_t - u_k \mathbf{u} \cdot \mathbf{v}_{x_k})\, dx\, dt = \int_\Omega \boldsymbol{\varphi} \cdot \mathbf{v}(0,x)\, dx \tag{2.1}$$

for each function $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^\infty(D_{-1}^T)$. Here we mean summation over the repeating index $k = 1, 2$.

First we prove the existence of a generalized solution in a bounded domain Ω and then consider the case of an unbounded domain. Then the operator $\widetilde{\Delta} = P\Delta$, where $P\colon \mathbf{L}_2(\Omega) \to \overset{\circ}{\mathbf{J}}(\Omega)$ is the orthogonal projection, has a complete orthonormal system (in $\mathbf{L}_2(\Omega)$) of eigenfunctions $\mathbf{a}^l \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$, $l \in \mathbb{N}$. If the domain Ω has boundary of class C^2, then $\mathbf{a}^l \in \mathbf{W}_2^2(\Omega)$, $l \in \mathbb{N}$ ([23], Chapter III, § 17, Theorem 17.1).

We seek approximate solutions in the form

$$\mathbf{u}^n(x,t) = \sum_{l=1}^{n} c_{ln}(t)\mathbf{a}^l(x),$$

where the functions $c_{ln}(t)$ are defined by the conditions

$$(\mathbf{u}_t^n, \mathbf{a}^l) - (u_k^n \mathbf{u}^n, \mathbf{a}_{x_k}^l) + (\mathbf{u}_{x_k}^n, \mathbf{a}_{x_k}^l) = 0, \qquad l = 1, 2, \dots, n, \tag{2.2}$$

and the initial data

$$c_{ln}\big|_{t=0} = (\mathbf{a}^l, \boldsymbol{\varphi}), \qquad l = 1, 2, \dots, n. \tag{2.3}$$

Here $(\mathbf{u}, \mathbf{v})$ is the scalar product in $\mathbf{L}_2(\Omega)$. Equalities (2.2) make up a system of differential equations of the following form for the c_{ln}:

$$\frac{dc_{ln}(t)}{dt} + \sum_{i=1}^{n} a_{li} c_{in}(t) - \sum_{i,p=1}^{n} a_{lip} c_{in}(t) c_{pn}(t) = 0, \qquad l = 1, 2, \dots, n, \tag{2.4}$$

where the a_{li} and the a_{lip} are constant scalars.

To prove the unique solubility of the system (2.4) under conditions (2.3) we shall find an a priori estimate holding for all $t \geqslant 0$. We multiply each equality (2.2) by the corresponding $c_{ln}(t)$ and sum over l ranging from 1 to n. After simple transformations we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^n\|^2 + (\mathbf{u}_{x_k}^n, \mathbf{u}_{x_k}^n) = 0. \tag{2.5}$$

Integrating this equality with respect to t from t_0 to t we obtain

$$\|\mathbf{u}^n(t)\|^2 + 2\int_{t_0}^{t} \|\nabla \mathbf{u}^n(\tau)\|^2 \, d\tau = \|\mathbf{u}^n(t_0)\|^2, \tag{2.6}$$

which for $t_0 = 0$ gives us the estimate of the function $\|\mathbf{u}^n(t)\|$ by the quantity $\|\mathbf{u}^n(0)\| \leqslant \|\boldsymbol{\varphi}\|$.

To demonstrate the required properties of the generalized solution we derive several further inequalities for the Galerkin approximations:

$$\frac{d}{dt}\|\nabla \mathbf{u}^n\|^2 + \|\widetilde{\Delta}\mathbf{u}^n\|^2 \leqslant \|(u_k^n \mathbf{u}^n)_{x_k}\|^2, \tag{2.7}$$

$$\|\mathbf{u}_t^n\|^2 \leqslant 2\|(u_k^n \mathbf{u}^n)_{x_k}\|^2 + 2\|\widetilde{\Delta}\mathbf{u}^n\|^2, \tag{2.8}$$

$$\|\widetilde{\Delta}\mathbf{u}^n\|^2 \leqslant 2\|\mathbf{u}_t^n\|^2 + 2\|(u_k^n \mathbf{u}^n)_{x_k}\|^2, \tag{2.9}$$

$$\frac{d}{dt}\|\mathbf{u}_t^n\|^2 + \|\nabla \mathbf{u}_t^n\|^2 \leqslant \sum_{k=1}^{2} \|(u_k^n \mathbf{u}^n)_t\|^2. \tag{2.10}$$

To obtain (2.7) we multiply each equality (2.2) by the corresponding coefficient λ_l and use the formulae $\lambda^l \mathbf{a}^l = \widetilde{\Delta}\mathbf{a}^l = P\Delta\mathbf{a}^l$. Then

$$(\mathbf{u}_t^n, P\Delta\mathbf{a}^l) - (u_k^n\mathbf{u}^n, (\widetilde{\Delta}\mathbf{a}^l)_{x_k}) + (\mathbf{u}_{x_k}^n, (\widetilde{\Delta}\mathbf{a}^l)_{x_k}) = 0. \tag{2.11}$$

In view of the simple relations

$$\begin{aligned}(\mathbf{u}_t^n, P\Delta\mathbf{a}^l) &= (P\mathbf{u}_t^n, \Delta\mathbf{a}^l) = (\mathbf{u}_t^n, \Delta\mathbf{a}^l) = -(\nabla\mathbf{u}_t^n, \nabla\mathbf{a}^l),\\ (\mathbf{u}_{x_k}^n, (\widetilde{\Delta}\mathbf{a}^l)_{x_k}) &= -(\mathbf{u}_{x_kx_k}^n, P^2\Delta\mathbf{a}^l) = -(P\Delta\mathbf{u}^n, P\Delta\mathbf{a}^l),\end{aligned}$$

equalities (2.11) can be written as follows:

$$-(\nabla\mathbf{u}_t^n, \nabla\mathbf{a}^l) + ((u_k^n\mathbf{u}^n)_{x_k}, \widetilde{\Delta}\mathbf{a}^l) - (\widetilde{\Delta}\mathbf{u}^n, \widetilde{\Delta}\mathbf{a}^l) = 0.$$

Multiplying them by c_{ln} and summing for l from 1 to n we obtain

$$-(\nabla\mathbf{u}_t^n, \nabla\mathbf{u}^n) + ((u_k^n\mathbf{u}^n)_{x_k}, \widetilde{\Delta}\mathbf{u}^n) - (\widetilde{\Delta}\mathbf{u}^n, \widetilde{\Delta}\mathbf{u}^n) = 0. \tag{2.12}$$

It is easy to deduce (2.7) from this inequality.

Next, we multiply (2.2) by $\dfrac{d}{dt}c_{ln}$ and add the resulting equalities with l going from 1 to n; then

$$(\mathbf{u}_t^n, \mathbf{u}_t^n) - (u_k^n\mathbf{u}^n, (\mathbf{u}_t^n)_{x_k}) + (\mathbf{u}_{x_k}^n, (\mathbf{u}_t^n)_{x_k}) = 0.$$

Integrating by parts we bring this equality to the following form:

$$(\mathbf{u}_t^n, \mathbf{u}_t^n) + ((u_k^n\mathbf{u}^n)_{x_k}, \mathbf{u}_t^n) - (\Delta\mathbf{u}^n, P\mathbf{u}_t^n) = 0.$$

Then

$$\begin{aligned}\|\mathbf{u}_t^n\|^2 &= -((u_k^n\mathbf{u}^n)_{x_k}, \mathbf{u}_t^n) + (\widetilde{\Delta}\mathbf{u}^n, \mathbf{u}_t^n)\\ &\leqslant \left(\|(u_k^n\mathbf{u}^n)_{x_k}\|^2 + \frac{1}{4}\|\mathbf{u}_t^n\|^2\right) + \|\widetilde{\Delta}\mathbf{u}^n\|^2 + \frac{1}{4}\|\mathbf{u}_t^n\|^2.\end{aligned}$$

This yields inequality (2.8).

It is easy to obtain from (2.12) the equality

$$(\mathbf{u}_t^n, \widetilde{\Delta}\mathbf{u}_n) + ((u_k^n\mathbf{u}^n)_{x_k}, \widetilde{\Delta}\mathbf{u}^n) - (\widetilde{\Delta}\mathbf{u}^n, \widetilde{\Delta}\mathbf{u}^n) = 0.$$

Hence

$$\begin{aligned}\|\widetilde{\Delta}\mathbf{u}^n\|^2 &= (\mathbf{u}_t^n, \widetilde{\Delta}\mathbf{u}^n) + ((u_k^n\mathbf{u}^n)_{x_k}, \widetilde{\Delta}\mathbf{u}^n)\\ &\leqslant \|\mathbf{u}_t^n\|^2 + \frac{1}{4}\|\widetilde{\Delta}\mathbf{u}^n\|^2 + \|(u_k^n\mathbf{u}^n)_{x_k}\|^2 + \frac{1}{4}\|\widetilde{\Delta}\mathbf{u}^n\|^2.\end{aligned}$$

This yields inequality (2.9).

We differentiate (2.2) with respect to t:

$$(\mathbf{u}_{tt}^n, \mathbf{a}^l) - ((u_k^n \mathbf{u}^n)_t, \mathbf{a}_{x_k}^l) + ((\mathbf{u}_{x_k}^n)_t, \mathbf{a}_{x_k}^l) = 0,$$

multiply the results by $\dfrac{d}{dt} c_{ln}$ and sum for l from 1 to n; then

$$(\mathbf{u}_{tt}^n, \mathbf{u}_t^n) - ((u_k^n \mathbf{u}^n)_t, (\mathbf{u}_t^n)_{x_k}) + ((\mathbf{u}_{x_k}^n)_t, (\mathbf{u}_{x_k}^n)_t) = 0.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2 + \|\nabla \mathbf{u}_t^n\|^2 = ((u_k^n \mathbf{u}^n)_t, (\mathbf{u}_t^n)_{x_k}) \leqslant \frac{1}{2}\sum_{k=1}^{2} \|(u_k^n \mathbf{u}^n)_t\|^2 + \frac{1}{2}\|\nabla \mathbf{u}_t^n\|^2,$$

which immediately yields (2.10).

We shall bring (2.7)–(2.10) to the following form:

$$\frac{d}{dt}\|\nabla \mathbf{u}^n\|^2 + \frac{3}{4}\|\widetilde{\Delta} \mathbf{u}^n\|^2 \leqslant 2c_1\|\boldsymbol{\varphi}\|\,\|\nabla \mathbf{u}^n\|^3 + 4c_1^2\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^4, \tag{2.13}$$

$$\|\mathbf{u}_t^n\|^2 \leqslant 4c_1\|\boldsymbol{\varphi}\|\,\|\nabla \mathbf{u}^n\|^3 + \frac{5}{2}\|\widetilde{\Delta} \mathbf{u}^n\|^2 + 8c_1^2\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^4, \tag{2.14}$$

$$\|\widetilde{\Delta} \mathbf{u}^n\|^2 \leqslant 4\|\mathbf{u}_t^n\|^2 + 8c_1\|\boldsymbol{\varphi}\|\,\|\nabla \mathbf{u}^n\|^3 + 16c_1^2\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^4, \tag{2.15}$$

$$\frac{d}{dt}\|\mathbf{u}_t^n\|^2 + \frac{1}{2}\|\nabla \mathbf{u}_t^n\|^2 \leqslant 2^7\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^2\|\mathbf{u}_t^n\|^2. \tag{2.16}$$

(Here and in what follows the constants c_1–c_4 depend only on (d, b) and on the quantities involved in the definition of a boundary uniformly of class C^3.) To this end we shall prove the inequalities

$$\|(u_k^n \mathbf{u}^n)_{x_k}\|^2 \leqslant 2c_1\|\boldsymbol{\varphi}\|\,\|\nabla \mathbf{u}^n\|^3 + \frac{1}{4}\|\widetilde{\Delta} \mathbf{u}^n\|^2 + 4c_1^2\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^4, \tag{2.17}$$

$$\|(u_k^n \mathbf{u})_t^n\|^2 \leqslant 128\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^2\|\mathbf{u}_t^n\|^2 + \frac{1}{2}\|\nabla \mathbf{u}_t^n\|^2, \qquad k = 1, 2. \tag{2.18}$$

To prove (2.17) it is sufficient to find an estimate for the expression on the right-hand side of the inequality

$$\|u_k^n \mathbf{u}_{x_k}^n\|^2 = \int_\Omega (u_k^n \mathbf{u}_{x_k}^n)^2\,dx \leqslant \left(\int_\Omega (u_k^n)^4\,dx\right)^{1/2} \left(\int_\Omega (\mathbf{u}_{x_k}^n)^4\,dx\right)^{1/2}.$$

An estimate of the first factor is provided by the well-known inequality ([24], Chapter I, § 1, Lemma 1) for functions $u \in \overset{\circ}{W}{}_2^1(\Omega)$:

$$\int_\Omega u^4\,dx \leqslant 2\int_\Omega u^2\,dx \int_\Omega |\nabla u|^2\,dx. \tag{2.19}$$

This inequality cannot be applied directly to $\mathbf{u}_{x_k}$, therefore we consider an extension $\mathbf{w} \in \mathbf{W}_2^1(\mathbb{R}^n)$ of this function such that

$$\int_{\mathbb{R}^n} \mathbf{w}^2\,dx < c_2 \int_\Omega (\mathbf{u}^n_{x_k})^2\,dx,$$
$$\int_{\mathbb{R}^n} |\nabla \mathbf{w}|^2\,dx \leqslant c_2 \int_\Omega \big((\mathbf{u}^n_{x_k})^2 + |\nabla \mathbf{u}^n_{x_k}|^2\big)\,dx.$$

Then

$$\int_\Omega (\mathbf{u}^n_{x_k})^4\,dx \leqslant \int_{R^n} \mathbf{w}^4\,dx \leqslant 2\int_{R^n} \mathbf{w}^2\,dx \int_{R^n} |\nabla \mathbf{w}|^2\,dx$$
$$\leqslant 2c_2^2 \|\mathbf{u}^n_{x_k}\|^2(\|\mathbf{u}^n_{x_k}\|^2 + \|\nabla \mathbf{u}^n_{x_k}\|^2).$$

Combining this with the above inequalities we obtain

$$\|(u_k^n \mathbf{u}^n)_{x_k}\|^2 \leqslant 2c_2 \|\mathbf{u}^n\|\,\|\nabla \mathbf{u}^n\|^2(\|\nabla \mathbf{u}^n\| + \|\nabla \mathbf{u}^n_{x_k}\|). \tag{2.20}$$

Let $\mathbf{w} \in \mathbf{H}_0(\Omega)$ and $\mathbf{f} \in \overset{\circ}{\mathbf{J}}(\Omega)$ be functions satisfying the identity

$$\int_\Omega \nabla \mathbf{w} : \nabla \mathbf{v}\,dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,dx$$

for all $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^\infty(\Omega)$. Then the function $\mathbf{f}$ is uniquely determined by $\mathbf{w}$ and we can consider the operator $\mathbf{f} = \widetilde{\Delta}\mathbf{w}$. The following inequalities, which are similar to the ones established in [9] for $n = 3$, hold in this case:

$$\|D^2 \mathbf{w}\| \leqslant c_3(\|\widetilde{\Delta}\mathbf{w}\| + \|\nabla \mathbf{w}\|), \tag{2.21}$$
$$\sup_{x\in\Omega} |\mathbf{w}(x)| \leqslant c_4(\|\widetilde{\Delta}\mathbf{w}\| + \|\nabla \mathbf{w}\|). \tag{2.22}$$

With the help of (2.21) we can give an estimate of $\|\nabla \mathbf{u}_{x_k}\|$ and then (2.20) can be continued as follows:

$$\|(u_k^n \mathbf{u}^n)_{x_k}\|^2 \leqslant 2c_2 \|\mathbf{u}^n\|\,\|\nabla \mathbf{u}^n\|^2((1+c_3)\|\nabla \mathbf{u}^n\| + c_3\|\widetilde{\Delta}\mathbf{u}^n\|)$$
$$\leqslant 2c_2(1+c_3)\|\mathbf{u}^n\|\,\|\nabla \mathbf{u}^n\|^3 + \frac{1}{4}\|\widetilde{\Delta}\mathbf{u}^n\|^2 + 4c_2^2(1+c_3)^2\|\mathbf{u}^n\|^2\|\nabla \mathbf{u}^n\|^4,$$

which proves (2.17) in view of the inequality $\|\mathbf{u}^n\| \leqslant \|\boldsymbol{\varphi}\|$.

For the proof of (2.18) it is sufficient to find an estimate of the expression on the right-hand side of the relation

$$\|u_k^n \mathbf{u}_t^n\|^2 \leqslant \int_\Omega (u_k^n \mathbf{u}_t^n)^2\,dx \leqslant \bigg(\int_\Omega (u_k^n)^4\,dx\bigg)^{1/2}\bigg(\int_\Omega (\mathbf{u}_t^n)^4\,dx\bigg)^{1/2}.$$

The estimate of the first factor is provided by (2.19). Note that $\mathbf{u}_t^n$ has trace zero at the boundary $\partial\Omega$, therefore the same inequality can be applied to this function. Thus,

$$\|u_k^n \mathbf{u}_t^n\|^2 \leqslant \left(4\int_\Omega (u_k^n)^2\,dx \int_\Omega (\nabla u_k^n)^2\,dx\right)^{1/2} \left(4\int_\Omega (\mathbf{u}_t^n)^2\,dx \int_\Omega (\nabla \mathbf{u}_t^n)^2\,dx\right)^{1/2}$$
$$\leqslant 4\|\boldsymbol{\varphi}\|\,\|\nabla \mathbf{u}^n\|\,\|\mathbf{u}_t^n\|\,\|\nabla \mathbf{u}_t^n\|,$$

which yields (2.18). Using formulae (2.17) and (2.18) it is now easy to deduce inequalities (2.13)–(2.16) from (2.7)–(2.10).

We now prove the estimate

$$\|\nabla \mathbf{u}^n(t)\|^2 \leqslant A(\|\boldsymbol{\varphi}\|, \|\nabla\boldsymbol{\varphi}\|) \qquad \text{for all } t \in [0,\infty). \tag{2.23}$$

Applying the inequality

$$2c_1\|\boldsymbol{\varphi}\|\,\|\nabla \mathbf{u}_n\|^3 \leqslant \|\nabla \mathbf{u}^n\|^2 + c_1^2\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^4$$

to the right-hand side of (2.13) we obtain

$$\frac{d}{dt}\|\nabla \mathbf{u}^n\|^2 + \frac{3}{4}\|\widetilde{\Delta}\mathbf{u}^n\|^2 \leqslant \|\nabla \mathbf{u}^n\|^2 + 5c_1^2\|\boldsymbol{\varphi}\|^2\|\nabla \mathbf{u}^n\|^4. \tag{2.24}$$

Dropping the second term on the left-hand side and dividing the inequality by $\|\nabla \mathbf{u}^n\|^2$ we integrate the result with respect to t from y to t:

$$\ln\|\nabla \mathbf{u}^n\|^2\Big|_y^t \leqslant t - y + 5c_1^2\|\boldsymbol{\varphi}\|^2 \int_y^t \|\nabla \mathbf{u}^n\|^2\,d\tau, \qquad t > y \geqslant 0.$$

We now use (2.6) for the estimate of the second term on the right-hand side of the last inequality. We obtain

$$\|\nabla \mathbf{u}^n(t)\|^2 \leqslant \|\nabla \mathbf{u}^n(y)\|^2 \exp\left(t - y + \frac{5}{2}c_1^2\|\boldsymbol{\varphi}\|^4\right). \tag{2.25}$$

After that, we integrate the result again, with respect to y from δ_0 to t:

$$\|\nabla \mathbf{u}^n(t)\|^2(t-\delta_0) \leqslant \int_{\delta_0}^t \|\nabla \mathbf{u}^n(y)\|^2 \exp\left(t - y + \frac{5}{2}c_1^2\|\boldsymbol{\varphi}\|^4\right)dy$$
$$\leqslant \exp\left(t - \delta_0 + \frac{5}{2}c_1^2\|\boldsymbol{\varphi}\|^4\right)\int_{\delta_0}^t \|\nabla \mathbf{u}^n(y)\|^2\,dy.$$

We use inequality (2.6) for an estimate of the integral. Then we obtain

$$\|\nabla \mathbf{u}^n(t)\|^2 \leqslant \frac{1}{2(t-\delta_0)}\exp\left(t - \delta_0 + \frac{5}{2}c_1^2\|\boldsymbol{\varphi}\|^4\right)\|\mathbf{u}^n(\delta_0)\|^2, \qquad t \geqslant \delta_0. \tag{2.26}$$

From (2.25) with $y = 0$ we also deduce the inequality

$$\|\nabla \mathbf{u}^n(t)\|^2 \leqslant \|\nabla \boldsymbol{\varphi}\|^2 \exp\biggl(t + \frac{5}{2} c_1^2 \|\boldsymbol{\varphi}\|^4\biggr). \tag{2.27}$$

The estimate (2.23) for $t \in [0,1]$ now follows from (2.27) and for $t > 1$ from (2.26) with $\delta_0 = t - 1$ and (2.6).

In what follows we approximate the unbounded domain Ω with boundary uniformly of class C^3 by a sequence of bounded domains Ω_m, $\bigcup_m \Omega_m = \Omega$, each with boundary of class C^3 with constants d and b independent of m. If condition (1.4) holds, then we can select Ω_m such that $\overline{\Omega_m} \supset \operatorname{supp} \boldsymbol{\varphi}$. In this case the solution of the problem (1.1), (1.2) in the domain Ω can be obtained as the weak limit (in appropriate spaces) of a subsequence $\mathbf{u}^m$ of the solutions in the bounded domains Ω_m with the same initial function $\boldsymbol{\varphi}$. Each solution $\mathbf{u}^m$ can in its turn be obtained as the limit as $n \to \infty$ of a subsequence of the Galerkin approximations $\mathbf{u}^{m,n}$. (Earlier we dropped the index m for compactness.)

We claim that the generalized solution of the problem (1.1), (1.2) has the following properties:

$$\mathbf{u}_t \in L^2(0, \infty; \overset{\circ}{\mathbf{J}}(\Omega)), \tag{2.28}$$

$$D^2 \mathbf{u} \in L^2(0, \infty; \mathbf{L}_2(\Omega)), \tag{2.29}$$

$$\nabla \mathbf{u}_t \in L^2(\varepsilon, \infty; \mathbf{L}_2(\Omega)) \quad \text{for each } \varepsilon > 0. \tag{2.30}$$

First, we establish these properties for the Galerkin approximations. We shall prove the following estimate

$$\int_0^\infty \|\widetilde{\Delta} \mathbf{u}^n\|^2 \, dt < \infty. \tag{2.31}$$

To this end we integrate (2.24) with respect to time from t to ∞. Using (2.23) we obtain

$$\|\nabla \mathbf{u}^n\|^2 \Big|_t^\infty + \frac{3}{4} \int_t^\infty \|\widetilde{\Delta} \mathbf{u}^n\|^2 \, dt \leqslant \int_t^\infty \|\nabla \mathbf{u}^n\|^2 (1 + 5 c_1^2 \|\boldsymbol{\varphi}\|^2 A) \, dt.$$

Hence inequality (2.6) yields

$$\frac{3}{4} \int_t^\infty \|\widetilde{\Delta} \mathbf{u}^n\|^2 \, dt \leqslant c_5 \|\mathbf{u}^n(t)\|^2 + \|\nabla \mathbf{u}^n(t)\|^2. \tag{2.32}$$

Here and in what follows the constants c_5–c_7 can depend only on (d, b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla \boldsymbol{\varphi}\|$. Now, applying (2.6) and (2.23) we deduce (2.31). Inequalities (2.21), (2.31), and (2.6) give us (2.29).

From (2.14) we obtain

$$\|\mathbf{u}_t^n\|^2 \leqslant \|\nabla \mathbf{u}^n\|^2 + 12 c_1^2 \|\boldsymbol{\varphi}\|^2 \|\nabla \mathbf{u}^n\|^4 + \frac{5}{2} \|\widetilde{\Delta} \mathbf{u}^n\|^2. \tag{2.33}$$

Integrating with respect to time from 0 to t and applying (2.23) and (2.6) we see that

$$\int_0^t \|\mathbf{u}_\tau^n\|^2\,d\tau \leqslant \int_0^t \|\nabla\mathbf{u}^n\|^2\,d\tau + 12c_1^2\|\boldsymbol{\varphi}\|^2 A\int_0^t \|\nabla\mathbf{u}^n\|^2\,d\tau + \frac{5}{2}\int_0^t \|\widetilde{\Delta}\mathbf{u}^n\|^2\,d\tau$$
$$\leqslant \frac{1}{2}\|\boldsymbol{\varphi}\|^2 + 6c_1^2\|\boldsymbol{\varphi}\|^4 A + \frac{5}{2}\int_0^\infty \|\widetilde{\Delta}\mathbf{u}^n\|^2\,d\tau.$$

Hence it follows by (2.31) that

$$\mathbf{u}_t \in L^2(0,\infty;\mathbf{L}_2(\Omega)). \tag{2.34}$$

We now claim that (2.28) holds. We obtain the solution of the problem (1.1), (1.2) as a weak limit of the Galerkin approximations $\mathbf{u}^{m,n}$ constructed for the sequence of bounded domains $\Omega_m \subset \Omega$. Each Galerkin approximation satisfies (2.28). After the weak limit transition property (2.28) still holds for the limit function, the solution.

We claim that

$$\int_\varepsilon^\infty \|\nabla\mathbf{u}_t^n\|^2 \leqslant c_A, \tag{2.35}$$

where the constant c_A depends only on ε, (d,b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla\boldsymbol{\varphi}\|$. In fact, from (2.16) we obtain the differential inequality

$$\frac{d}{dt}\|\mathbf{u}_\tau^n\|^2 \leqslant 2^7\|\boldsymbol{\varphi}\|^2\|\nabla\mathbf{u}^n\|^2\|\mathbf{u}_\tau^n\|^2.$$

It shows that

$$\|\mathbf{u}_t^n(t)\|^2 \leqslant \|\mathbf{u}_t^n(t_0)\|^2 \exp\biggl(2^7\|\boldsymbol{\varphi}\|^2\int_{t_0}^t \|\nabla\mathbf{u}^n\|^2\,d\tau\biggr). \tag{2.36}$$

In view of (2.6), we have

$$\|\mathbf{u}_t^n(t)\|^2 \leqslant c_6\|\mathbf{u}_t^n(t_0)\|^2, \qquad t \geqslant t_0. \tag{2.37}$$

Integrating this inequality with respect to $t_0 \in [0,\varepsilon]$ and using (2.34) we see that

$$\|\mathbf{u}_t^n(t)\|^2 \leqslant \frac{c_7}{\varepsilon}, \qquad t \geqslant \varepsilon. \tag{2.38}$$

We now integrate (2.16) with respect to τ from ε to t, which shows that

$$\|\mathbf{u}_t^n(t)\|^2 + \frac{1}{2}\int_\varepsilon^t \|\nabla\mathbf{u}_t^n\|^2\,d\tau \leqslant 2^7\|\boldsymbol{\varphi}\|^2\int_\varepsilon^t \|\nabla\mathbf{u}^n\|^2\|\mathbf{u}_t^n\|^2\,d\tau + \|\mathbf{u}_t^n(\varepsilon)\|$$
$$\leqslant 2^7\|\boldsymbol{\varphi}\|^2 A\int_\varepsilon^t \|\mathbf{u}_t^n\|^2\,d\tau + \|\mathbf{u}_t^n(\varepsilon)\|.$$

Hence, using (2.34) and (2.38) we obtain (2.35) and (2.30).

Theorem. *Let Ω be a domain with boundary uniformly of class C^3 and let $\boldsymbol{\varphi}$ lie in $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$. Then the solution of the problem* (1.1), (1.2) *satisfies the inequalities*

$$\|\nabla \mathbf{u}(T)\|^2 \leqslant b_1 \|\mathbf{u}(t)\|^2, \qquad T \geqslant t+1, \tag{2.39}$$
$$\|D^2 \mathbf{u}(T)\|^2 \leqslant b_2 \|\mathbf{u}(t)\|^2, \qquad T \geqslant t+2, \tag{2.40}$$
$$\|\nabla p(T)\|^2 \leqslant b_3 \|\mathbf{u}(t)\|^2, \qquad T \geqslant t+2, \tag{2.41}$$
$$|\mathbf{u}(T,x)| \leqslant b_4 \|\mathbf{u}(t)\|, \quad x \in \Omega, \qquad T \geqslant t+2, \tag{2.42}$$

in which $t \geqslant 0$. Here the constants b_1–b_4 can depend only on (d,b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla \boldsymbol{\varphi}\|$.

Proof. We start with the proof of relations (2.39)–(2.42) for the Galerkin approximations under the assumption that the initial function $\boldsymbol{\varphi}$ satisfies condition (1.4). From (2.26) for $\delta_0 = t-1$ we obtain the inequality

$$\|\nabla \mathbf{u}^n(t)\|^2 \leqslant b_1 \|\mathbf{u}^n(t-1)\|^2, \qquad t \geqslant 1.$$

Since $\|\mathbf{u}^n(t)\|^2$ is a non-increasing function, it follows that

$$\|\nabla \mathbf{u}^n(t)\|^2 \leqslant b_1 \|\mathbf{u}^n(t-1)\|^2 \leqslant b_1 \|\mathbf{u}^n(s-1)\|^2, \qquad t \geqslant s \geqslant 1.$$

This immediately yields (2.39).

Because $\|\mathbf{u}^n(t)\|$ is a non-increasing function, it follows from (2.32) and (2.39) that

$$\int_t^\infty \|\widetilde{\Delta} \mathbf{u}^n(t)\|^2\,dt < \widetilde{b}_1 \|\mathbf{u}^n(t-1)\|, \qquad t \geqslant 1. \tag{2.43}$$

Here the constant $\widetilde{b}_1$ (as well as the constants $\widetilde{b}_2$–$\widetilde{b}_6$ below) can depend only on (d,b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla \boldsymbol{\varphi}\|$.

We integrate (2.33) to obtain

$$\int_t^\infty \|\mathbf{u}_t^n\|^2\,ds \leqslant \int_t^\infty \left(\|\nabla \mathbf{u}^n\|^2 + 12c_1^2 \|\boldsymbol{\varphi}\|^2 \|\nabla \mathbf{u}^n\|^4 + \frac{5}{2} \|\widetilde{\Delta} \mathbf{u}^n\|^2 \right) ds.$$

Applying (2.23), (2.43), and (2.6) to the last inequality we obtain

$$\begin{aligned} \int_t^\infty \|\mathbf{u}_t^n\|^2\,ds &\leqslant \int_t^\infty \|\nabla \mathbf{u}^n\|^2 (1 + 12c_1^2 \|\boldsymbol{\varphi}\|^2 A)\,ds + \frac{5}{2} \int_t^\infty \|\widetilde{\Delta} \mathbf{u}^n\|^2\,ds \\ &\leqslant \widetilde{b}_2 \|\mathbf{u}^n(t)\|^2 + \frac{5}{2} \widetilde{b}_1 \|\mathbf{u}^n(t-1)\|^2 \leqslant \widetilde{b}_3 \|\mathbf{u}^n(t-1)\|^2. \end{aligned} \tag{2.44}$$

We replace t by $t+1$ in (2.37) and integrate with respect to $t_0 \in [t, t+1]$. In combination with (2.44) this yields

$$\|\mathbf{u}_t^n(t+1)\|^2 \leqslant c_6 \int_t^{t+1} \|\mathbf{u}_t^n(s)\|^2\,ds \leqslant \widetilde{b}_4 \|\mathbf{u}^n(t-1)\|^2, \qquad t \geqslant 1.$$

In particular,

$$\|\mathbf{u}_t^n(t)\|^2 \leqslant \widetilde{b}_4 \|\mathbf{u}^n(t-2)\|^2, \qquad t \geqslant 2. \tag{2.45}$$

Using now relations (2.15), (2.23), and (2.39) it is easy to conclude that

$$\|\widetilde{\Delta}\mathbf{u}^n(t)\|^2 \leqslant 4\|\mathbf{u}_t^n\|^2 + \widetilde{b}_5\|\nabla\mathbf{u}^n\|^2 \leqslant 4\|\mathbf{u}_t^n\|^2 + \widetilde{b}_6\|\mathbf{u}^n(t-1)\|^2.$$

We apply (2.45) to this inequality:

$$\|\widetilde{\Delta}\mathbf{u}^n(t)\|^2 \leqslant 4\widetilde{b}_4\|\mathbf{u}^n(t-2)\|^2 + \widetilde{b}_6\|\mathbf{u}^n(t-1)\|^2.$$

Since $\|\mathbf{u}^n(t)\|$ is a non-increasing function, this yields (2.40).

We shall prove that inequalities (2.39) and (2.40), established so far for the Galerkin approximations $\mathbf{u}^n$, hold also for the solution of the problem (1.1), (1.2). We fix $t > 0$ and select a subsequence $\mathbf{u}^n$ such that there exists a limit

$$\lim_{n\to\infty} \|\mathbf{u}^n(t)\| = L.$$

Using equalities (2.6) for the Galerkin approximations $\mathbf{u}^n$ and a similar identity for the solution $\mathbf{u}$ we shall show that $L \leqslant \|\mathbf{u}(t)\|$. In fact,

$$\begin{aligned}\lim_{n\to\infty} \|\mathbf{u}^n(t)\|^2 &= \lim_{n\to\infty}\left(\|\mathbf{u}^n(0)\|^2 - 2\int_0^t \|\nabla\mathbf{u}^n(\tau)\|^2\,d\tau\right)\\ &\leqslant \|\boldsymbol{\varphi}\|^2 - 2\int_0^t \|\nabla\mathbf{u}(\tau)\|^2\,d\tau = \|\mathbf{u}(t)\|^2.\end{aligned}$$

Here we used the weak convergence of the subsequence $\mathbf{u}^n$ to the function $\mathbf{u}$ in the space $L_2(0,t;\overset{\circ}{\mathbf{J}}{}^1(\Omega_m))$ as $n\to\infty$. It is known that the norm of the limit function

$$\int_0^t \|\nabla\mathbf{u}(\tau)\|^2\,d\tau$$

has the estimate

$$\varliminf_{n\to\infty} \int_0^t \|\nabla\mathbf{u}^n\|^2\,d\tau.$$

First, we prove inequality (2.39) for the solution $\mathbf{u}^m$ in the bounded domain Ω_m. Assume that it fails on a subset E of $[t+1,\infty)$ of positive measure. Then

$$\int_E \|\nabla\mathbf{u}^m(s)\|^2\,ds > \|\mathbf{u}^m(t)\|^2 \int_E b_1\,ds. \tag{2.46}$$

On the other hand, integrating inequality (2.39) for the Galerkin approximations we obtain

$$\int_E \|\nabla\mathbf{u}^{m,n}(s)\|^2\,ds \leqslant \|\mathbf{u}^{m,n}(t)\|^2 \int_E b_1\,ds.$$

Passing to the limit as $n\to\infty$ we obtain

$$\int_E \|\nabla\mathbf{u}^m(s)\|^2\,ds \leqslant \varliminf_{n\to\infty}\int_E \|\nabla\mathbf{u}^{m,n}(s)\|^2\,ds \leqslant L^2\int_E b_1\,ds,$$

which contradicts (2.46). Hence inequality (2.39) holds for $\mathbf{u}^m$ for almost all $T > t$.

In a perfectly similar fashion one carries out the transition as $m\to\infty$ in inequality (2.39) from the functions $\mathbf{u}^m$ to the solution of (1.1), (1.2). Finally, let $\boldsymbol{\varphi}^i$ be a sequence of functions in $\overset{\circ}{\mathbf{J}}{}^{\infty}(\Omega)$ convergent to $\boldsymbol{\varphi}$. Then for the corresponding solutions $\mathbf{u}^i$ of the problem (1.1), (1.2) we have energy identity (2.6), properties (2.28)–(2.30), and the estimates (2.39)–(2.42). Hence among the functions $\mathbf{u}^i$ we can select a subsequence weakly convergent in the appropriate spaces. It is easy to see that its limit is a solution of the problem (1.1), (1.2) with initial function $\boldsymbol{\varphi}$, and the above properties and inequalities hold also for this solution.

The proof of inequality (2.40) is carried out in a similar fashion to (2.39).

Since $\|\widetilde{\Delta}\mathbf{u}\| = \|P\Delta\mathbf{u}\| \leqslant \|D^2\mathbf{u}\|$, inequality (2.42) is a consequence of (2.40) and (2.22).

It is known that a solution with properties (2.28)–(2.30) is also a solution almost everywhere [24]. To prove (2.41) we observe that

$$\begin{aligned}\|\mathbf{u}(T)\cdot\nabla\mathbf{u}(T)\| &\leqslant \max_{x\in\Omega}|\mathbf{u}(T,x)|\,\|\nabla\mathbf{u}(T)\| \leqslant b_4 b_1^{1/2}\|\mathbf{u}(t)\|^2\\ &\leqslant b_4 b_1^{1/2}\|\boldsymbol{\varphi}\|\,\|\mathbf{u}(t)\|, \qquad T\geqslant t+2. \end{aligned}\tag{2.47}$$

Next, from (1.1) we obtain the equality

$$\|\nabla p(T)\| \leqslant \|(1-P)(\nu\Delta\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{u})\|.$$

Inequality (2.41) is now a consequence of (2.40) and (2.47). The proof is complete.

§ 3. Behaviour of the solution at infinity

In this section we prove Theorem 1 for solenoidal initial functions $\boldsymbol{\varphi}$ in $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ with condition (1.4), and after that establish the result of Theorem 1 also for solenoidal initial functions $\boldsymbol{\varphi}$ in $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ with condition (1.14).

3.1. Behaviour of the solution as $|x|\to\infty$. First of all, it is easy to see from (2.42) that the solution $\mathbf{u}$ of the problem (1.1), (1.2) belongs to the space $\mathbf{L}_\infty(\Omega)$ for each $t>0$.

We claim that the following estimate holds for each $\varepsilon\in(0,1)$:

$$\int_0^t \|\mathbf{u}(\tau)\|_{\infty,\Omega}^2\,d\tau \leqslant C_1 + C_2 t^{\varepsilon}. \tag{3.1}$$

Here the constant C_2 depends only on (d,b), $\|\boldsymbol{\varphi}\|$, and ε, and C_1 depends only on (d,b), $\|\boldsymbol{\varphi}\|$, $\|\nabla\boldsymbol{\varphi}\|$. We use Sobolev's inequality [25]

$$|\mathbf{v}(x)| \leqslant C_K(\|\mathbf{v}\|_K + \|D^2\mathbf{v}\|_K), \tag{3.2}$$

which holds in each cone K for the function $\mathbf{v}\in\mathbf{W}_2^2(K)$; the point x is the vertex of the cone. Since the boundary of Ω is uniformly of class C^3, each point $x\in\Omega$ is a vertex of a small cone lying in Ω, the same for all points x. The size of this cone depends on the constants (d,b). By (3.2) we obtain

$$\int_0^t |\mathbf{u}(\tau,x)|^2\,d\tau \leqslant 2C_K^2\int_0^t \|\mathbf{u}\|_K^2\,d\tau + 2C_K^2\int_0^t \|D^2\mathbf{u}\|^2\,d\tau. \tag{3.3}$$

By Hölder's inequality,

$$\|\mathbf{u}(t)\|_K^2 = \int_K \mathbf{u}^2\,dx \leqslant \left(\int_K 1^q\,dx\right)^{1/q}\left(\int_\Omega \mathbf{u}^{2p}\,dx\right)^{1/p} = C_3(K)\|\mathbf{u}\|_{2p}^2. \qquad (3.4)$$

For the function $\mathbf{u} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ in the two-dimensional case we have the inequality ([23], Chapter II, § 3, inequality (3.1))

$$\|\mathbf{u}\|_{2p,\Omega} \leqslant \chi_1\|\nabla\mathbf{u}\|^{\varrho}\|\mathbf{u}\|^{1-\varrho}, \qquad \varrho = 1 - \frac{1}{p}\,, \qquad (3.5)$$

where χ_1 depends only on $p \in [1,\infty)$. We shall find an estimate of the integral on the right-hand side of (3.3) by applying to it (3.4), (3.5), and after that Hölder's inequality and (2.6):

$$\begin{aligned}\int_0^t \|\mathbf{u}(\tau)\|_K^2\,d\tau &\leqslant C_3(K,p)\int_0^t \|\mathbf{u}\|_{2p}^2\,d\tau \leqslant C_4(K,p)\int_0^t \|\nabla\mathbf{u}\|^{2\varrho}\|\mathbf{u}\|^{2-2\varrho}\,d\tau \\ &\leqslant C_4(K,p)\|\boldsymbol{\varphi}\|^{2-2\varrho}\int_0^t \|\nabla\mathbf{u}\|^{2\varrho}\,d\tau \\ &\leqslant C_5\left(\int_0^t 1^p\,d\tau\right)^{1/p}\left(\int_0^t \|\nabla\mathbf{u}\|^{2\varrho\cdot\frac{1}{\varrho}}\,d\tau\right)^{\varrho} \leqslant C_6 t^{1/p}. \qquad (3.6)\end{aligned}$$

The constant C_6 depends only on K, $\|\boldsymbol{\varphi}\|$, and p. Hence, applying (3.6) and (2.29) to (3.3) we obtain (3.1) with $\varepsilon = 1/p$.

Theorem 2. *Let i be one of the integers $1, 2, \dots, s$. Let Ω be a domain with boundary uniformly of class C^3, let l_i be a function satisfying* (1.11) *and Condition* D, *and assume that the initial function $\boldsymbol{\varphi}$ in $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$ satisfies the condition*

$$\boldsymbol{\varphi} = 0 \quad \text{for } x \in \underset{i}{\Omega}{}^{\infty}_{R_0}. \qquad (3.7)$$

Then there exist positive quantities $\Gamma_i > 1$, γ_i, and A_3 such that the following inequality holds for all $t > 0$ for the solution of the problem (1.1), (1.2):

$$\int_{\underset{i}{\Omega}{}^{\infty}_{R-1}} \mathbf{u}^2(t,x)\,dx < tA_3 \exp\left(\int_0^t \|\mathbf{u}(\tau)\|_{\infty,\Omega}^2\,d\tau + \frac{\Gamma_i t}{l_i^2(R)} - \frac{2\gamma_i R}{l_i(R)}\right), \quad R \geqslant \frac{R_0}{q_i^2}\,, \quad (3.8)$$

where R_0 is sufficiently large. The constant Γ_i depends only on q_i and α in inequality (1.11), *while γ_i also depends on d_1 in inequality* (1.10); *A_3 depends only on (d,b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla\boldsymbol{\varphi}\|$, but not on R_0.*

Since $R/l_i(R) \to \infty$ as $R \to \infty$, the estimate (3.8) characterizes the decrease of the solution as $x_1 \to \infty$ in the domain $\underset{i}{\Omega}$.

We carry out the proof of the theorem with the use of Lemmas 1 and 2 below. We set

$$M(t) = \sup_{x\in\Omega} \mathbf{u}^2(t,x), \qquad g(t,r) = M(t) + \frac{2}{l_i^2(r)}\,.$$

In view of (3.1), we have the simple inequality

$$\int_0^t g(\tau,r)\,d\tau \leqslant C_1 + C_2 t^{\varepsilon} + \frac{2t}{l_i^2(r)}\,. \tag{3.9}$$

We select a system of coordinates such that the Ox_1-axis is directed along the ray s_i and define a cut-off function $\eta(x)$ with support in $\underset{i}{\Omega}{}_r$ by the equality

$$\eta(x) = \xi\biggl(\frac{x_1 - r}{l_i(r)}\biggr),$$

where $\xi(r)$ is a continuous function vanishing for $r<0$, equal to 1 for $r>1$, and linear in the remaining interval. Then the gradient of η has its support in $\underset{i}{\Omega}{}_r^{r+l_i(r)}$ and

$$\frac{\partial \eta}{\partial x_1} = \frac{1}{l_i(r)}\,. \tag{3.10}$$

Moreover, for $r \geqslant P_i$, in view of the monotonicity of l_i, we have the inequality

$$-\frac{\partial \eta}{\partial r} = \frac{1}{l_i(r)} + \frac{x_1 - r}{l_i^2(r)} l_i'(r) \geqslant \frac{1}{l_i(r)}\,, \qquad x \in \omega_i(r). \tag{3.11}$$

We introduce our notation:

$$\theta(t) = \exp\biggl(-\int_0^t M(\tau)\,d\tau\biggr)$$

and

$$H(t,r) = \theta(t)\biggl(\int_{\underset{i}{\Omega}} \eta|\nabla u(t,x)|^2\,dx + \int_0^t \int_{\underset{i}{\Omega}} \eta u_t^2\,dx\,d\tau\biggr). \tag{3.12}$$

We can point out the inequality

$$H(t,r) \leqslant b_5, \qquad t>0, \quad r \geqslant P_i, \tag{3.13}$$

which follows from (2.23) and (2.28), in which b_5 depends only on (d,b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla\boldsymbol{\varphi}\|$.

Lemma 1. *Under the assumptions of Theorem* 2 *there exists* $\beta>1$ *such that the following inequality holds for all* $t>0$ *and* $r \geqslant \max\{R_0, P_i\}$:

$$H(t,r) \leqslant -\beta l_i(r)\biggl(H_r(t,r) + \int_0^t g(\tau,r) H_r(\tau,r)\,d\tau\biggr). \tag{3.14}$$

Here the subscript r *denotes the derivative*; *the constant* β *depends only on* d_1 *in inequality* (1.10).

Proof. Let F be the set of $t>0$ such the $\mathbf{u}_t(t,x) \in \overset{\circ}{\mathbf{J}}(\Omega)$. By property (2.28) the measure of its complement $(0,\infty)\setminus F$ is zero. We fix $t \in F$.

Since $\mathbf{u}(x,t)$ is a generalized solution satisfying (2.29), it is a solution of the system (1.1) almost everywhere. We consider the scalar product of the Navier–Stokes equation and the function $\eta\mathbf{u}_t$ and integrate over $\underset{i}{\Omega}$, which yields

$$\int_{\underset{i}{\Omega}} \mathbf{u}_t^2\eta\,dx + \int_{\underset{i}{\Omega}} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\eta\mathbf{u}_t\,dx = \int_{\underset{i}{\Omega}} \Delta\mathbf{u}\cdot(\eta\mathbf{u}_t)\,dx - \int_{\underset{i}{\Omega}} \nabla p\cdot\eta\mathbf{u}_t\,dx. \tag{3.15}$$

We process the integral on the right-hand side using (3.10):

$$\begin{aligned}
-\int_{\underset{i}{\Omega}} \Delta\mathbf{u}\cdot(\eta\mathbf{u}_t)\,dx &= \int_{\underset{i}{\Omega}} \nabla\mathbf{u} : \nabla(\eta\mathbf{u}_t)\,dx \\
&= \int_{\underset{i}{\Omega}} \eta\nabla\mathbf{u} : \nabla\mathbf{u}_t\,dx + \int_{\underset{i}{\Omega}} \sum_{j=1}^{2} (u_j)_t \frac{\partial u_j}{\partial x_1}\,\frac{\partial \eta_i}{\partial x_1}\,dx \\
&= \int_{\underset{i}{\Omega}} \eta\nabla\mathbf{u} : \nabla\mathbf{u}_t\,dx + \int_{\omega_i(r)} \sum_{j=1}^{2} (u_j)_t \frac{\partial u_j}{\partial x_1}\,\frac{1}{l_i(r)}\,dx.
\end{aligned}$$

We can now bring (3.15) to the following form:

$$\begin{aligned}
\int_{\underset{i}{\Omega}} \mathbf{u}_t^2\eta\,dx + \int_{\underset{i}{\Omega}} \eta\nabla\mathbf{u} : \nabla\mathbf{u}_t\,dx &= -\int_{\underset{i}{\Omega}} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\eta\mathbf{u}_t\,dx \\
&\quad - \int_{\omega_i(r)} \left(\sum_{j=1}^{2} (u_j)_t \frac{\partial u_j}{\partial x_1}\,\frac{1}{l_i(r)} - p(\mathbf{u}_t\cdot\nabla\eta)\right) dx.
\end{aligned} \tag{3.16}$$

We shall find estimates of the terms on the right-hand side:

$$\left|\int_{\underset{i}{\Omega}} (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\eta\mathbf{u}_t\,dx\right| \leqslant \int_{\underset{i}{\Omega}} \frac{\eta}{2}\big(\mathbf{u}^2|\nabla\mathbf{u}|^2 + \mathbf{u}_t^2\big)\,dx \leqslant \int_{\underset{i}{\Omega}} \frac{\eta}{2}\big(M(t)|\nabla\mathbf{u}|^2 + \mathbf{u}_t^2\big)\,dx.$$

Next,

$$\left|\int_{\omega_i(r)} \sum_{j=1}^{2} (u_j)_t \frac{\partial u_j}{\partial x_1}\,\frac{1}{l_i(r)}\,dx\right| \leqslant \int_{\omega_i(r)} \frac{1}{2}\left(\frac{|\nabla\mathbf{u}|^2}{l_i^2(r)} + |\mathbf{u}_t|^2\right) dx. \tag{3.17}$$

For the estimate of the remaining integral we shall prove that $\displaystyle\int_{\omega_i(r)} (\mathbf{u}_t\nabla\eta)\,dx = 0$. Since $\mathbf{u}_t(t,x) \in \overset{\circ}{\mathbf{J}}(\Omega)$, it is sufficient to prove this equality for the vectors $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^{\infty}(\Omega)$.

By Gauss's divergence theorem,

$$0 = \int_{\underset{i}{\Omega}{}_{r_1}^{r_2}} \operatorname{div}\mathbf{v}\,dx = \int_{\partial\underset{i}{\Omega}{}_{r_1}^{r_2}} \mathbf{v}\cdot n\,ds = \int_{x_1=r_2} v_1\,ds - \int_{x_1=r_1} v_1\,ds.$$

Hence, taking the section $x_1 = r_2$ outside the support of $\mathbf{v}$ we obtain

$$\int_{x_1=r_1} v_1\,ds = \int_{x_1=r_2} v_1\,ds = 0 \qquad \text{for each } r_1 > 0.$$

We have thus proved that

$$\int_{\omega_i(r)} v_1\,dx = \int_r^{r+l_i(r)} dr_1 \int_{x_1=r_1} v_1\,ds = 0.$$

Consequently,

$$\int_{\omega_i(r)} (\mathbf{v}\cdot\nabla\eta)\,dx = \int_{\omega_i(r)} \mathbf{v}\cdot\left(\frac{1}{l_i(r)}\,,0\right)dx = \frac{1}{l_i(r)}\int_{\omega_i(r)} v_1\,dx = 0. \tag{3.18}$$

Thus there exists a vector $\mathbf{w} \in \overset{\circ}{\mathbf{W}}{}^1_2(\omega_i(r))$ such that $\operatorname{div}\mathbf{w} = (\mathbf{u}_t\cdot\nabla\eta)$ and, in view of (1.10),

$$\|\nabla\mathbf{w}\|_{\omega_i(r)} \leqslant d_1\|\mathbf{u}_t\nabla\eta\|_{\omega_i(r)} \leqslant \frac{d_1}{l_i(r)}\|\mathbf{u}_t\|_{\omega_i(r)}.$$

Friedrichs's inequality enables us to prove the estimate

$$\|\mathbf{w}\|_{\omega_i(r)} \leqslant C\|\mathbf{u}_t\|_{\omega_i(r)}.$$

Here the constant C differs from d_1 by an absolute coefficient.

Now, for the remaining integral on the right-hand side of (3.16) we can write the following chain of relations:

$$\begin{aligned}
&\left|\int_{\omega_i(r)} p(\mathbf{u}_t\cdot\nabla\eta)\,dx\right| = \left|\int_{\omega_i(r)} p\operatorname{div}\mathbf{w}\,dx\right| = \left|\int_{\omega_i(r)} \nabla p\cdot\mathbf{w}\,dx\right| \\
&\quad= \left|\int_{\omega_i(r)} (\mathbf{u}_t - \Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u})\mathbf{w}\,dx\right| \\
&\quad= \left|\int_{\omega_i(r)} ((\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u})\cdot\mathbf{w} + \nabla\mathbf{u}:\nabla\mathbf{w})\,dx\right| \\
&\quad\leqslant \int_{\omega_i(r)} \left(\frac{|\mathbf{u}_t|^2 + |(\mathbf{u}\cdot\nabla)\mathbf{u}|^2}{2} + \mathbf{w}^2 + |\nabla\mathbf{u}:\nabla\mathbf{w}|\right)dx \\
&\quad\leqslant \int_{\omega_i(r)} \left(\frac{1}{2}\left(\mathbf{u}_t^2 + M(t)|\nabla\mathbf{u}|^2\right) + C^2\mathbf{u}_t^2 + \frac{1}{2}\left(\frac{|\nabla\mathbf{u}|^2}{l_i^2(r)} + |\nabla\mathbf{w}|^2 l_i^2(r)\right)\right)dx \\
&\quad\leqslant \int_{\omega_i(r)} \left(\frac{1}{2}\left(\mathbf{u}_t^2 + M(t)\nabla\mathbf{u}^2\right) + C^2\mathbf{u}_t^2 + \frac{1}{2}\left(\frac{|\nabla\mathbf{u}|^2}{l_i^2(r)} + d_1^2\mathbf{u}_t^2\right)\right)dx.
\end{aligned}$$

We substitute the estimates so obtained in (3.16):

$$\begin{aligned}
\int_{\underset{i}{\Omega}} \mathbf{u}_t^2\eta\,dx + \int_{\underset{i}{\Omega}} \eta\nabla\mathbf{u}:\nabla\mathbf{u}_t\,dx &\leqslant \int_{\underset{i}{\Omega}} \frac{\eta}{2}(M(t)|\nabla\mathbf{u}|^2 + \mathbf{u}_t^2)\,dx \\
&\quad+ \int_{\omega_i(r)} \left(\frac{|\nabla\mathbf{u}|^2}{l_i^2(r)} + \frac{1}{2}M(t)|\nabla\mathbf{u}|^2 + \beta\mathbf{u}_t^2\right)dx.
\end{aligned}$$

Hence

$$\int_{\Omega_i} \mathbf{u}_t^2 \eta\, dx + \frac{d}{dt}\int_{\Omega_i} \eta |\nabla \mathbf{u}|^2\, dx$$
$$\leqslant \int_{\Omega_i} M(t)\eta|\nabla \mathbf{u}|^2\, dx + \int_{\omega_i(r)} \left(\left(\frac{2}{l_i^2(r)} + M(t)\right)|\nabla \mathbf{u}|^2 + \beta \mathbf{u}_t^2\right) dx.$$

In view of the inequality $\eta\nabla\boldsymbol{\varphi} \equiv 0$, for $r \geqslant R_0$ we have $\int_{\Omega_i} \eta|\nabla \mathbf{u}(0,x)|^2\, dx = 0.$ Hence integration with respect to t over the interval $(0,T)$ produces the inequality

$$\int_0^T \int_{\Omega_i} \eta \mathbf{u}_t^2\, dx\, dt + \int_{\Omega_i} \eta|\nabla \mathbf{u}(T,x)|^2\, dx \leqslant \int_0^T \int_{\Omega_i} \eta M(t)|\nabla \mathbf{u}|^2\, dx\, dt + h(T), \quad (3.19)$$

where we use the notation

$$h(T) = h_1(T) + h_2(T) = \int_0^T \int_{\omega_i(r)} g(t,r)|\nabla \mathbf{u}|^2\, dx\, dt + \beta \int_0^T \int_{\omega_i(r)} \mathbf{u}_t^2\, dx\, dt.$$

By (3.19), for the function

$$z(T) = \int_0^T \int_{\Omega_i} \eta M(t)|\nabla \mathbf{u}|^2\, dx\, dt$$

we obtain the differential inequality

$$z' = M(T)\int_{\Omega_i} \eta|\nabla \mathbf{u}(T,x)|^2\, dx \leqslant M(T)(z(T) + h(T)).$$

On solving it we see that

$$z(T) \leqslant \int_0^T \exp\left(\int_t^T M(\tau)\, d\tau\right) M(t)h(t)\, dt$$
$$= \int_0^T M(t)(h_1(t) + h_2(t)) \exp\left(\int_t^T M(\tau)\, d\tau\right) dt. \quad (3.20)$$

Integration by parts yields easily the equality

$$\int_0^T M(t)h_1(t)\exp\left(\int_t^T M(\tau)\, d\tau\right) dt = -h_1(T) + \int_0^T h_1'(t)\exp\int_t^T M(\tau)\, d\tau\, dt.$$

Combining it with (3.20) and substituting in (3.19) we see that

$$\int_0^T \int_{\Omega_i} \eta \mathbf{u}_t^2\, dx\, dt + \int_{\Omega_i} \eta|\nabla \mathbf{u}(T,x)|^2\, dx$$
$$\leqslant \int_0^T \frac{\theta(t)}{\theta(T)}(M(t)h_2(t) + h_1'(t))\, dt + h_2(T).$$

Multiplying by $\theta(T)$ and using the notation (3.12) we obtain

$$\begin{aligned}H(T,r) &\leqslant \beta\theta(T)\int_0^T\int_{\omega_i(r)}\mathbf{u}_t^2\,dx\,dt\\&\quad+\int_0^T\theta(t)\biggl(M(t)\beta\int_0^t\int_{\omega_i(r)}\mathbf{u}_t^2\,dx\,dt+\int_{\omega_i(r)}g(t,r)|\nabla\mathbf{u}|^2\,dx\biggr)\,dt\\&\leqslant \beta\theta(T)\int_0^T\int_{\omega_i(r)}\mathbf{u}_t^2\,dx\,d\tau\\&\quad+\int_0^T\theta(t)g(t,r)\biggl(\beta\int_0^t\int_{\omega_i(r)}\mathbf{u}_t^2\,dx\,d\tau+\int_{\omega_i(r)}|\nabla\mathbf{u}|^2\,dx\biggr)\,dt.\qquad(3.21)\end{aligned}$$

Relation (3.11) now gives us the inequality

$$-l_i(r)\frac{\partial H(t,r)}{\partial r}\geqslant\theta(t)\biggl(\int_{\omega_i(r)}|\nabla\mathbf{u}|^2\,dx+\int_0^t\int_{\omega_i(r)}\mathbf{u}_t^2\,dx\,dt\biggr).$$

Hence the assertion of the Lemma is a consequence of (3.21).

Lemma 2. *Under the assumptions of Theorem* 2 *let* $\beta>1$ *be the quantity in Lemma* 1. *Then there exists a constant* A_3 *such that for all* $r_0\geqslant\max\{R_0,P_i\}$, $t>0$ *the solution of the problem* (1.1), (1.2) *satisfies the inequality*

$$H(t,r)\leqslant A_3\exp\biggl(\frac{2t}{l_i^2(r_0)}-\int_{r_0}^r\frac{d\rho}{4\beta l_i(\rho)}+\int_0^t\|\mathbf{u}(t)\|_{\infty,\Omega}^2\,d\tau\biggr),\qquad r\geqslant r_0,\quad(3.22)$$

with A_3 *dependent only on* (d,b), $\|\boldsymbol{\varphi}\|$, *and* $\|\nabla\boldsymbol{\varphi}\|$.

Proof. We fix $r_0\geqslant\max\{R_0,P_i\}$ and let $y(\zeta)$ be the solution of the Cauchy problem $r'=\beta l_i(r)$, $r(0)=r_0$. We set $h(t,\zeta)=H(t,y(\zeta))$. The function $g(t,r)$ is non-increasing in r. Hence fixing $r=r_0$ we can write inequality (3.14) as follows:

$$h(t,\zeta)\leqslant -h_\zeta(t,\zeta)-\int_0^t g(\tau,r_0)h_\zeta(\tau,\zeta)\,d\tau,\qquad\zeta\geqslant0.$$

Integrating this inequality with respect to ζ we see that

$$\int_\zeta^\infty h(t,\rho)\,d\rho\leqslant h(t,\zeta)+\int_0^t g(\tau,r_0)h(\tau,\zeta)\,d\tau,\qquad\zeta\geqslant0.\qquad(3.23)$$

Repeating the integration and using induction on n we obtain the inequality

$$\int_0^\infty\frac{\zeta^{n-1}}{(n-1)!}h(t,\zeta)\,d\zeta<\sum_{j=0}^n C_n^j(G^jh)(t,0),\qquad(3.24)$$

where G is the integral operator

$$(Gh)(t,\zeta)=\int_0^t g(\tau,r_0)h(\tau,\zeta)\,d\tau$$

and the C_n^j are binomial coefficients.

Since h is a non-negative function non-increasing in the second variable, we can write

$$\int_{\zeta/2}^{\zeta} \rho^{n-1} h(t,\rho)\,d\rho \geqslant \int_{\zeta/2}^{\zeta} \left(\frac{\zeta}{2}\right)^{n-1} h(t,\zeta)\,d\rho = \left(\frac{\zeta}{2}\right)^{n} h(t,\zeta). \tag{3.25}$$

The binomial coefficients have the estimate 2^n, and taking b_5 in (3.13) for an estimate of h we see that the right-hand side of (3.24) does not exceed

$$2^n b_5 \exp\biggl(\int_0^t g(\tau,r_0)\,d\tau\biggr).$$

From (3.25) and the last estimate we obtain

$$\begin{aligned}\frac{1}{(n-1)!}\left(\frac{\zeta}{2}\right)^{n} h(t,\zeta) &\leqslant \int_{\zeta/2}^{\zeta} \frac{\rho^{n-1} h(t,\rho)\,d\rho}{(n-1)!}\\ &< \int_0^{\infty} \frac{\zeta^{n-1} h(t,\zeta)\,ds}{(n-1)!} \leqslant 2^n b_5 \exp\biggl(\int_0^t g(\tau,r_0)\,d\tau\biggr).\end{aligned}$$

Hence

$$h(t,\zeta) \leqslant \left(\frac{2}{\zeta}\right)^{n} (n-1)!2^n b_5 \exp\biggl(\int_0^t g(\tau,r_0)\,d\tau\biggr).$$

Using now Stirling's formula we derive a consequence of inequality (3.24):

$$h(t,\zeta) \leqslant b_5 \left(\frac{4}{\zeta}\right)^{n} \frac{n!}{n} \exp\biggl(\int_0^t g(\tau,r_0)\,d\tau\biggr) \leqslant k_1 \left(\frac{4n}{\zeta e}\right)^{n} \exp\biggl(\int_0^t g(\tau,r_0)\,d\tau\biggr).$$

We set $n = [\zeta/4] \geqslant \zeta/4 - 1$:

$$h(t,\zeta) \leqslant k_1 \exp\biggl(\int_0^t M(\tau)\,d\tau + \frac{2t}{l_i^2(r_0)} + 1 - \frac{\zeta}{4}\biggr), \qquad \zeta \geqslant 0.$$

Returning to the variable r we arrive at (3.22). The proof of Lemma 2 is complete.

We claim that for r larger than some quantity D_i we have the inequality

$$1 + l_i(r) \leqslant r(q_i^{-1} - 1), \qquad r \geqslant D_i. \tag{3.26}$$

In fact, let n be a positive integer such that $q_i P_i < q_i^n r \leqslant P_i$. Then by (1.11),

$$l_i(r) < q_i^{-n\alpha} l_i(q_i^n r) < \left(\frac{r}{q_i P_i}\right)^{\alpha} l_i(P_i) = C_i r^{\alpha}. \tag{3.27}$$

This yields (3.26) since $\alpha \in (0,1)$.

We shall deduce from (3.22) the required inequality

$$\int_0^T \int_{\Omega_{i\,R-1}^{\infty}} \mathbf{u}_t^2(t,x)\,dx\,dt \leqslant A_3 \exp\biggl(\frac{\Gamma_i T}{l_i^2(R)} - \frac{\gamma_i R}{l_i(R)} + 2\int_0^T \|\mathbf{u}(t)\|_{\infty,\Omega}^2\,d\tau\biggr) \tag{3.28}$$

for $R \geqslant \max\{R_0, P_i, D_i\}/q_i^2$ and all $T > 0$. To this end we set $R = r_0/q_i^2$ and $r = q_i R$. From (3.26) we conclude that $R \geqslant 1 + l_i(r) + r$ or $R - 1 \geqslant r + l_i(r)$. Then $\eta(x, r) = 1$ for $x_1 > R - 1$, therefore by (3.12) we obtain

$$\int_0^T \int_{\substack{\Omega^\infty_{R-1} \\ i}} \mathbf{u}_t^2(t, x)\, dx\, dt \leqslant H(T, r) \exp\biggl(\int_0^T M(t)\, dt\biggr). \tag{3.29}$$

Since the functions $l_i(r)$ are non-decreasing, it follows by (1.11) that

$$l_i(r_0) > q_i^{2\alpha} l_i(R), \qquad \int_{r_0}^r \frac{d\rho}{4\beta l_i(\rho)} \geqslant \gamma_i \frac{R}{l_i(R)}, \qquad r_0 \geqslant P_i.$$

Combining now (3.29) and (3.22) we obtain the estimate (3.28).

Proof of Theorem 2. We now proceed directly to the proof of Theorem 2. We can assume without loss of generality that $R_0 \geqslant \max\{P_i, D_i\}$. In view of the initial condition and the Newton–Leibniz formula, the inequalities

$$|u_j(t, x)| \leqslant \int_0^t \left|\frac{\partial u_j(\tau, x)}{\partial t}\right| d\tau, \qquad j = 1, 2,$$

hold for almost all $x \in \underset{i}{\Omega}{}^\infty_{R_0}$. Hence by the Cauchy–Schwarz–Bunyakovskĭ inequality

$$\int_{\substack{\Omega^\infty_{R-1} \\ i}} \mathbf{u}^2(t, x)\, dx \leqslant t \int_0^t \int_{\substack{\Omega^\infty_{R-1} \\ i}} \mathbf{u}_t^2\, dx\, d\tau, \qquad R \geqslant R_0 + 1.$$

Then (3.8) is seen to be a simple consequence of (3.28).

3.2. Proof of Theorem 1. First, we carry out the proof for an initial function satisfying condition (3.7) for $i = 1, 2, \dots, s$. It is technically more convenient to consider, in place of the function $r_i(t)$ defined in the introduction, the functions $R_i(t)$ defined by the equalities

$$t = \frac{\gamma_i}{2\Gamma_i} R_i l_i(R_i). \tag{3.30}$$

Note that $r_i(t) < R_i(t)$ for $\gamma_i < 2\Gamma_i$ and $R_i(t) < 2\Gamma_i r_i(t)/\gamma_i$. Let $\varkappa = \min_{i \leqslant s}\{\gamma_i/(2\Gamma_i)\}$. Then we can write $R_i(t) < r_i(t)/\varkappa$ and $\lambda_i(R_i(t)) \geqslant \lambda_i(r_i(t)/\varkappa)$; the second inequality holds because λ_i is non-increasing. Obviously,

$$\frac{t}{l_i^2(r_i(t))} > \frac{t}{l_i^2(R_i(t))}.$$

On the other hand, by (1.12) and (3.30) we obtain

$$\frac{t}{l_i^2(r_i(t))} = \frac{r_i^2(t)}{t} < \frac{R_i^2(t)}{t} = \frac{4\Gamma_i^2 t}{\gamma_i^2 l_i^2(R_i(t))}.$$

Thus,

$$\frac{t}{l_i^2(R_i(t))} \leqslant \frac{t}{l_i^2(r_i(t))} \leqslant \frac{t}{\varkappa^2 l_i^2(R_i(t))}. \tag{3.31}$$

We shall now show that

$$\lim_{t\to\infty} \frac{t^{1-\varepsilon}}{l_i^2(r_i(t))} = \infty \tag{3.32}$$

for sufficiently small positive ε. By (3.27) we obtain $t = r_i(t)l_i(r_i(t)) < C_i r_i^{1+\alpha}(t)$, and therefore

$$r_i(t) > \left(\frac{t}{C_i}\right)^{1/(1+\alpha)}.$$

Hence by (1.12),

$$\frac{t^{1-\varepsilon}}{l_i^2(r_i(t))} = \frac{r_i^2(t)}{t^{1+\varepsilon}} > \frac{(t/C_i)^{2/(1+\alpha)}}{t^{1+\varepsilon}}\,.$$

This immediately shows the existence of $\varepsilon > 0$ suitable for (3.32). We fix one such ε.

We choose a large number T such that the following inequalities hold for all $t \geqslant T$:

$$R_i(t) > \frac{\max\{R_0, P_i, D_i\}}{q_i^2}, \qquad t > R_i(t),$$

$$\min_{i\leqslant s} \lambda_i(R_i(t)) < \mu, \tag{3.33}$$

$$\frac{\Gamma_i t}{l_i^2(R_i(t))} > C_2 t^{\varepsilon}. \tag{3.34}$$

Here the constant R_0 is as in condition (3.7), μ is as in (1.9), and C_2 is as in (3.1). We fix arbitrary $t \geqslant T$.

Using (3.1) and our choice of T we can write inequality (3.8) as follows:

$$\begin{aligned}\int_{\Omega^{\infty}_{i\,R_i-1}} \mathbf{u}^2(\tau,x)\,dx &\leqslant \tau A_4 \exp\biggl(C_2\tau^{\varepsilon} + \frac{\Gamma_i\tau}{l_i^2(R_i)} - 2\gamma_i\frac{R_i}{l_i(R_i)}\biggr)\\ &\leqslant tA_4 \exp\biggl(\frac{2\Gamma_i t}{l_i^2(R_i)} - 2\gamma_i\frac{R_i}{l_i(R_i)}\biggr)\end{aligned}$$

for each $\tau \leqslant t$. Here and in what follows $R_i = R_i(t)$. Note that by (3.30),

$$\frac{2\Gamma_i t}{l_i^2(R_i)} = \gamma_i\frac{R_i}{l_i(R_i)}\,. \tag{3.35}$$

Hence

$$\int_{\Omega^{\infty}_{i\,R_i-1}} \mathbf{u}^2(\tau,x)\,dx \leqslant tA_4\exp\biggl(-\gamma_i\frac{R_i}{l_i(R_i)}\biggr) \equiv \delta_i \quad \text{for each } \tau \leqslant t. \tag{3.36}$$

Here A_4 (as well as A_5 below) depends only on (d,b), $\|\boldsymbol{\varphi}\|$, and $\|\nabla\boldsymbol{\varphi}\|$. We consider the function $\xi(r)$ equal to 1 for $r < R-1$, to 0 for $r > R$ and linear on the remaining interval; then for an appropriate function $\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ we have

$$\begin{aligned}\lambda_i(R_i)\int_{\underset{i}{\Omega}^{R_i-1}} \mathbf{v}^2(x)\,dx &\leqslant \lambda_i(R_i)\int_{\underset{i}{\Omega}^{R_i}} \xi^2(x_1)\mathbf{v}^2(x)\,dx \\ &\leqslant \int_{\underset{i}{\Omega}^{R_i}} |\nabla(\xi\mathbf{v})|^2\,dx \leqslant 2\int_{\underset{i}{\Omega}^{R_i}} \xi^2|\nabla\mathbf{v}|^2\,dx + 2\int_{\underset{i}{\Omega}^{R_i}_{R_i-1}} \mathbf{v}^2\,dx \\ &\leqslant 2\int_{\underset{i}{\Omega}} |\nabla\mathbf{v}|^2\,dx + 2\delta_i, \qquad \tau \in (0,t). \end{aligned} \tag{3.37}$$

Using (3.36), (3.37), (3.33), (1.9) and identity (2.6) for the solution $\mathbf{u}$, we obtain a differential inequality for the absolutely continuous function $E(\tau) = \|\mathbf{u}(\tau)\|^2$:

$$\begin{aligned}\min_i \lambda_i(R_i)\biggl(E(\tau) - \sum_{i=1}^s \delta_i\biggr) &\leqslant \mu\int_Q \mathbf{u}^2(\tau,x)\,dx + \sum_{i=0}^s \lambda_i(R_i)\int_{\underset{i}{\Omega}^{R_i-1}} \mathbf{u}^2(\tau,x)\,dx \\ &\leqslant \int_Q |\nabla\mathbf{u}|^2\,dx + 2\sum_{i=1}^s \int_{\underset{i}{\Omega}} |\nabla\mathbf{u}|^2\,dx + 2\sum_{i=1}^s \delta_i \leqslant 2\|\nabla\mathbf{u}(\tau)\|^2 + 2\sum_{i=1}^s \delta_i \\ &= 2\sum_{i=1}^s \delta_i - \frac{d}{d\tau}E(\tau), \qquad \tau \in [0,t]. \end{aligned}$$

For the increasing function $E(\tau)$ we have the estimate

$$E(t) \leqslant \biggl(1 + \frac{2}{\min_i \lambda_i(R_i)}\biggr)\sum_{i=1}^s \delta_i + E(0)\exp\bigl(-t\min_i \lambda_i(R_i)\bigr). \tag{3.38}$$

Since the width of the domain $\underset{i}{\Omega}^r$ is at most r, we can conclude from Friedrichs's inequality that $\lambda_i(r) \geqslant r^{-2}$. By our choice of T we obtain $R_i(t) < t$, therefore $\lambda_i(R_i(t)) \geqslant t^{-2}$. Inequality (3.38) now assumes the following form:

$$E(t) \leqslant (1 + 2t^2)\sum_{i=1}^s \delta_i + E(0)\exp\bigl(-t\min_i \lambda_i(R_i)\bigr).$$

Taking account of (3.35) and the notation (3.36) we obtain

$$\|\mathbf{u}(t)\| \leqslant s(1+2t^2)tA_4\exp\biggl(\frac{-2\Gamma_i t}{\max_i l_i^2(R_i)}\biggr) + E(0)\exp\bigl(-t\min_i \lambda_i(R_i)\bigr).$$

In view of relations (3.34), (3.31), and $\Gamma_i > 1$ we can write this as follows:

$$\|\mathbf{u}(t)\| \leqslant A_5\exp\biggl(-t\min_i\biggl\{\lambda_i\biggl(\frac{r_i(t)}{\varkappa}\biggr), \varkappa^2 l_i^{-2}(r_i(t))\biggr\}\biggr). \tag{3.39}$$

Hence it follows by (2.42) that (1.15) holds for $t \geqslant T$ for the solutions of the problem (1.1), (1.2) with initial function $\boldsymbol{\varphi}$ in $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$ satisfying condition (3.7) for

$i = 1, 2, \ldots, s$. Note that the pressure and the derivatives of $\mathbf{u}$ also satisfy estimates of the kind (1.16) in view of inequalities (2.41) and (2.40).

We now prove the theorem for the initial function $\boldsymbol{\varphi}$ satisfying condition (1.14). For the proof we select sufficiently large r and represent the function $\boldsymbol{\varphi}$ as a sum of solenoidal functions: $\boldsymbol{\varphi} = \boldsymbol{\varphi}_1 + \sum_{i=1}^{s} \boldsymbol{\psi}_i$, such that $\boldsymbol{\varphi}_1(x) = 0$ for $x \in \bigcup_{i=1}^{s} \underset{i}{\Omega}{}_r^{\infty}$, $\|\boldsymbol{\varphi}_1\|_{\mathbf{W}_2^1(\Omega)} \leqslant A_6 \|\boldsymbol{\varphi}\|_{\mathbf{W}_2^1(\Omega)}$, $\|\boldsymbol{\psi}_i\| \leqslant A_6 e^{-c(q_i r)^{\delta}}$, where A_6 depends only on d_1. To this end we fix an integer i in the set $\{1, 2, \ldots, s\}$ and construct the function $\boldsymbol{\psi}_i$. Let r_i^* be a quantity such that $r_i^* + l_i(r_i^*) = r$ and $\eta(x, r) = \xi((x_1 - r_i^*)/l_i(r_i^*))$. Then the vector $\eta(x)\boldsymbol{\varphi}(x)$ fails to be solenoidal only in the domain $\omega_i(r_i^*)$, and we have

$$\operatorname{div} \eta\boldsymbol{\varphi} = (\boldsymbol{\varphi} \cdot \nabla\eta), \qquad x \in \omega_i(r_i^*).$$

Using relation (1.14) it is easy to establish a relation of the form (3.18) for the function $(\boldsymbol{\varphi} \cdot \nabla\eta)$. Hence there exists a vector $\mathbf{w}_i(x, r) \in \overset{\circ}{\mathbf{W}}{}_2^1(\omega_i(r_i^*))$ such that $\operatorname{div} \mathbf{w}_i = (\boldsymbol{\varphi} \cdot \nabla\eta)$ and

$$\|\nabla \mathbf{w}\|_{\omega_i(r_i^*)} \leqslant d_1 \|\boldsymbol{\varphi} \nabla\eta\|_{\omega_i(r_i^*)} \leqslant \frac{d_1}{l_i(r_i^*)} \|\boldsymbol{\varphi}\|_{\omega_i(r_i^*)}. \tag{3.40}$$

Using Friedrichs's inequality we obtain the estimate

$$\|\mathbf{w}_i\|_{\omega_i(r_i^*)} \leqslant d_1 \|\boldsymbol{\varphi}\|_{\omega_i(r_i^*)} \leqslant d_1 \exp(-c(r_i^*)^{\delta}). \tag{3.41}$$

It remains to set $\boldsymbol{\psi}_i(x, r) = \eta(x, r)\boldsymbol{\varphi}(x) - \mathbf{w}_i(x, r)$ and use inequality (3.26), which shows that $r = r_i^* + l(r_i^*) \leqslant r_i^*/q_i$.

We shall now prove that $\boldsymbol{\psi}_i \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$, $i = 1, 2, \ldots, s$. It is well known that in a bounded domain with Lipschitz boundary each solenoidal function from the space $\overset{\circ}{\mathbf{W}}{}_2^1$ belongs also to the space $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$. Thus, if the domain Ω is unbounded, but the solenoidal function $\mathbf{v} \in \overset{\circ}{\mathbf{W}}{}_2^1$ has bounded support, then $\mathbf{v}$ also belongs to the space $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$. Moreover, it follows from inequalities (3.40), (3.41) that $\mathbf{w}_i(x, R) \to 0$ as $R \to \infty$ in the space $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$. Hence $\boldsymbol{\psi}_i(x, R) \to 0$ as $R \to \infty$ in $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$. The vectors $\boldsymbol{\psi}_i(x, r) - \boldsymbol{\psi}_i(x, R)$ are solenoidal and have bounded supports, therefore they belong to the space $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$. The function $\boldsymbol{\psi}_i(x, r)$, which is their limit as $R \to \infty$, also belongs to this space. Since the initial function is the limit of solenoidal functions with compact support, it belongs to $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$. Hence $\boldsymbol{\varphi}_1$ also lies in the space $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$.

Let $\mathbf{u}(t, x)$ be a solution of the problem (1.1), (1.2) with initial function $\boldsymbol{\varphi}$ satisfying condition (1.14) and let $\mathbf{u}^1(t, x)$ be the solution of (1.1), (1.2) with initial function $\boldsymbol{\varphi}_1$. We claim that the quantity $\|\mathbf{u} - \mathbf{u}^1\|$ is small.

We subtract from identity (2.1) for $\mathbf{u}$ the same identity for $\mathbf{u}^1$. In the resulting equality we set

$$\Phi = \begin{cases} \mathbf{u} - \mathbf{u}^1 \equiv \mathbf{v} & \text{for } 0 \leqslant t \leqslant t_1; \\ 0 & \text{for } \ t > t_1, \end{cases}$$

which yields

$$\int_0^{t_1}\int_\Omega\bigl(\mathbf{v}_t\mathbf{v}+\mathbf{v}_{x_k}\mathbf{v}_{x_k}-(u_k\mathbf{v}+v_k\mathbf{u}^1)\mathbf{v}_{x_k}\bigr)\,dx\,dt=0.$$

We can transform this equality to the form

$$\frac{1}{2}\|\mathbf{v}(x,t)\|^2\Big|_0^{t_1}+\int_0^{t_1}\|\nabla\mathbf{v}\|^2\,dt-\int_0^{t_1}\int_\Omega v_k\mathbf{u}^1\mathbf{v}_{x_k}\,dx\,dt=0, \tag{3.42}$$

bearing in mind that $\operatorname{div}\mathbf{u}=0$ ([24], Chapter VI, § 1, equality (5)). We now set $\sigma(t)=\|\nabla\mathbf{u}^1(t)\|$ and find an estimate of the last integral in (3.42):

$$\begin{aligned}
\left|\int_\Omega v_k\mathbf{u}^1\mathbf{v}_{x_k}\,dx\right|&=\left|\int_\Omega v_k\mathbf{u}^1_{x_k}\mathbf{v}\,dx\right|\leqslant\left(\int_\Omega|\nabla\mathbf{u}^1|^2\,dx\int_\Omega\mathbf{v}^4\,dx\right)^{1/2}\\
&\leqslant\sqrt{2}\,\sigma(t)\left(\sum_j\int_\Omega v_j^4\,dx\right)^{1/2}\leqslant\sqrt{2}\,\sigma(t)\sum_j\left(2\int_\Omega v_j^2\,dx\int_\Omega\nabla v_j^2\,dx\right)^{1/2}\\
&\leqslant 2\sigma(t)\|\mathbf{v}\|\,\|\nabla\mathbf{v}\|\leqslant\sigma^2(t)\|\mathbf{v}\|^2+\|\nabla\mathbf{v}\|^2.
\end{aligned}$$

Next, we substitute it in (3.42):

$$\frac{1}{2}\|\mathbf{v}(t_1)\|^2+\int_0^{t_1}\|\nabla\mathbf{v}\|^2\,dt\leqslant\frac{1}{2}\sum_{i=1}^s\|\boldsymbol{\psi}_i\|+\int_0^{t_1}(\sigma^2(t)\|\mathbf{v}\|^2+\|\nabla\mathbf{v}\|^2)\,dt.$$

Hence

$$\|\mathbf{v}(t_1)\|^2\leqslant A_3 s\exp\bigl(-c(r\min_i q_i)^\delta\bigr)+2\int_0^{t_1}\sigma^2(t)\|\mathbf{v}\|^2\,dt.$$

Denoting the right-hand side by $y(t_1)$ we can write the last inequality as follows:

$$\frac{dy(t_1)}{dt_1}\leqslant 2\sigma^2(t_1)y(t_1).$$

This yields

$$y(t)\leqslant y(0)e^{2\int_0^t\sigma^2(\tau)\,d\tau}.$$

From the identity of the kind (2.6) for $\mathbf{u}^1$ we deduce the estimate

$$\int_0^t\sigma^2(\tau)\,d\tau\leqslant\|\boldsymbol{\varphi}_1\|^2\leqslant A_7\|\boldsymbol{\varphi}\|^2,$$

where we can set $A_7=2$ for sufficiently large r.

Hence we can write

$$y(t)\leqslant A_8\exp\bigl(-c(r\min_i q_i)^\delta\bigr).$$

We fix $t > T$. Then for $r = \min_i\{r_i(t)q_i^2\}$ condition (3.7) holds with $R_0 = r$, therefore we can find an estimate of the norm $\|\mathbf{u}^1\|$ with the help of (3.39):

$$\begin{aligned}\|\mathbf{u}(t)\| &\leqslant \|\mathbf{u}^1\| + \|\mathbf{u} - \mathbf{u}^1\| \\ &\leqslant 2\|\boldsymbol{\varphi}_1\| \exp\biggl(-t \min_i\biggl\{\lambda_i\biggl(\frac{r_i(t)}{\varkappa}\biggr), \varkappa^2 l_i^{-2}(r_i(t))\biggr\}\biggr) \\ &\qquad + A_8 \exp\bigl(-c(\min_i q_i^3 \min_i r_i(t))^\delta\bigr).\end{aligned}$$

Let $\widetilde{c} = c \min_i q_i^{3\delta}$. Note that both $\min_i r_i(t)$ and $\max_i l_i(r_i(t))$ are attained for the same index j because $t = r_i l_i(r_i)$. Hence

$$\begin{aligned}\frac{\varkappa^2 t}{\max_i l_i^2(r_i(t))} - \widetilde{c}\min_i r_i^\delta(t) &= \frac{\varkappa^2 t}{l_j^2(r_j(t))} - \widetilde{c} r_j^\delta(t) = \frac{\varkappa^2 r_j(t)}{l_j(r_j(t))} - \widetilde{c} r_j^\delta(t) \\ &= r_j^\delta(t)\biggl(\frac{\varkappa^2 r_j^{1-\delta}(t)}{l_j(r_j(t))} - \widetilde{c}\biggr) < 0.\end{aligned}$$

The last inequality is a consequence of (1.13) with sufficiently large t. This brings us to the result of Theorem 1.

Consider now the case when the domains $\omega_i(r)$ are uniformly star-shaped. We fix some i. It follows from the definition of uniform star-shapedness that

$$\operatorname{diam}\omega_i(r) \leqslant \widetilde{d}\operatorname{diam} B_i \leqslant \widetilde{d} l_i(r),$$

where the constant $\widetilde{d}$ is independent of $i = 1, 2, \dots, k$ and $r \geqslant P_i$. Hence we have Friedrichs's inequality

$$\int_{\omega_i^r} \mathbf{u}^2\,dx \leqslant \widetilde{d}^2 l_i^2(r) \int_{\omega_i^r} |\nabla\mathbf{u}|^2\,dx$$

for $\mathbf{u} \in C_0^\infty(\Omega)$. Covering the domain $\underset{i}{\Omega}^R$ by domains of the form $\omega_i(r)$ we see that

$$\int_{\underset{i}{\Omega}^R} \mathbf{u}^2\,dx \leqslant \widetilde{d}^2 l_i^2(R) \int_{\underset{i}{\Omega}^R} |\nabla\mathbf{u}|^2\,dx.$$

Hence the estimates $\lambda_i(R) \geqslant \widetilde{d}^{-2} l_i^{-2}(R)$ must hold. By (1.15) this yields (1.17).

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