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Trigonometric Padé approximants for functions with regularly decreasing Fourier coefficients

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Abstract. Sufficient conditions describing the regular decrease of the coefficients of a Fourier series $f(x)=a_0/2+\sum a_n\cos kx$ are found which ensure that the trigonometric Padé approximants $\pi^t_{n,m}(x;f)$ converge to the function f in the uniform norm at a rate which coincides asymptotically with the highest possible one. The results obtained are applied to problems dealing with finding sharp constants for rational approximations.

Bibliography: 31 titles.

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§ 1. Introduction

We shall consider real continuous 2π-periodic functions f expanded in a convergent Fourier series:

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx), \tag{1.1}$$

where the Fourier coefficients a_k and b_k are real numbers. For convenience we can write the Fourier series (1.1) in the complex form

$$f(x)=\sum_{k=-\infty}^{\infty}c_ke^{ikx}, \tag{1.2}$$

where we set $c_k=(a_k-ib_k)/2$, $c_0=a_0/2$, $c_{-k}=\bar{c}_k$.

The sequence $\{c_k\}_{k=-\infty}^{\infty}$ contains all the information about f, therefore in principle various properties of the function can be described directly in terms of the coefficients of its Fourier series (1.1) or (1.2). In the framework of this paper we investigate the approximation properties of continuous functions f with sufficiently regularly decreasing sequences of Fourier coefficients $\{c_k\}_{k=-\infty}^{\infty}$.

Let $\mathscr{R}^t_{n,m}$ be the class of trigonometric rational functions $r(x)=p_n(x)/q_m(x)$ where the numerator $p_n(x)$ and denominator $q_m(x)$ are trigonometric polynomials with real coefficients such that $\deg p_n\leqslant n$, $\deg q_m\leqslant m$, $q_m\not\equiv 0$. We define the *best uniform trigonometric rational approximations to* f to be:

$$\mathrm{R}^t_{n,m}(f):=\inf\{\|f-r\|:r\in\mathscr{R}^t_{n,m}\}=\|f-r^*_{n,m}\|,$$

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where $\|g\| = \max\{|g(x)| : x \in [0, 2\pi]\}$ and $r^*_{n,m}$ is the trigonometric function of best uniform approximation of f by fractions from $\mathscr{R}^t_{n,m}$. It is well known that $r^*_{n,m}$ is uniquely defined (see, for instance, [1]).

By a *trigonometric Padé approximant* to a function f we shall mean a rational function $\pi^t_{n,m}(x) = \pi^t_{n,m}(x; f) = p^t_n(x)/q^t_m(x)$ in $\mathscr{R}^t_{n,m}$, where the numerator and denominator satisfy the condition

$$q^t_m(x)f(x) - p^t_n(x) = \sum_{k=n+m+1}^{\infty} (\widetilde{a}_k \cos kx + \widetilde{b}_k \sin kx), \tag{1.3}$$

where $\widetilde{a}_k$ and $\widetilde{b}_k$ are real numbers.

We shall consider certain matrices and determinants whose entries are equal to Fourier coefficients of the function f. To define them, to each $k \in \mathbb{Z}$ and each real x we assign the row matrices

$$C_k = \|c_{k-j}\|, \quad E(x) = \|e^{ijx}\|, \qquad j = -m, \dots, m.$$

Here and throughout, the symbol i in the expressions e^{ijx}, e^{ikx} and e^{ix} is the imaginary unit. Let

$$d_{n,m}(x) = \det \begin{bmatrix} C_{n+m} \\ \dots \\ C_{n+1} \\ E(x) \\ C_{-n-1} \\ \dots \\ C_{-n-m} \end{bmatrix}.$$

We denote by $d_{n,m,k}$ the determinant obtained from $d_{n,m}$ by replacing the row $E(x)$ by C_k, and we denote by $\Delta_{n,m}$ the determinant of order $2m$ obtained from $d_{n,m}(x)$ by deleting the $(m+1)$st row and $(m+1)$st column.

Proposition 1. *Let f be a function defined by a series (1.2). Then the trigonometric Padé approximant $\pi^t_{n,m}(\,\cdot\,; f)$ exists for all positive integers n and m. If $\Delta_{n,m} \neq 0$, then the ratio $\pi^t_{n,m}(x; f) = p^t_n(x; f)/q^t_m(x; f)$ is uniquely determined; its numerator and denominator are defined by*

$$p^t_n(x) = \sum_{k=-n}^{n} d_{n,m,k} e^{ikx}, \qquad q^t_m(x) = d_{n,m}(x). \tag{1.4}$$

For $m = 0$ the approximants $\pi^t_{n,0}(\,\cdot\,; f)$, $n = 0, 1, 2, \dots$, coincide with the partial Fourier sums of f. Formulae (1.4) also hold if we set the determinant of order zero to be equal to 1 by definition. For $m > 0$ the assertion of Proposition 1 that the trigonometric Padé approximant $\pi^t_{n,m}(\,\cdot\,; f)$ exists and is unique and has a representation in the form (1.4) is not new (see, for instance, [2]–[4]). As is known, trigonometric polynomials $p^t_n(x)$ and $q^t_m(x)$ and the fraction $\pi^t_{n,m}(\,\cdot\,; f)$, which is their ratio, are not uniquely defined by equality (1.3). In what follows we shall assume that p^t_n and q^t_m are defined by (1.4). Then

$$L^t_{n,m}(x; f) := q^t_m(x)f(x) - p^t_n(x) = \sum_{k=n+m+1}^{\infty} (\widetilde{c}_k e^{ikx} + \widetilde{c}_{-k} e^{-ikx}), \tag{1.5}$$

where $\widetilde{c}_k = d_{n,m,k}$ for all $k \in \mathbb{Z}$. Since f is a real-valued function, it follows that $\overline{c}_k = c_{-k}$. In this case $\overline{d_{n,m}(x)} = d_{n,m}(x)$ and $\overline{d_{n,m,k}} = d_{n,m,-k}$. This means that the trigonometric polynomials p_n^t and q_m^t are also real.

In a similar way, repeating [5], we define trigonometric Padé approximants in the sense of Baker. Namely a *trigonometric Padé-Baker approximant* for a function f defined by (1.1) is a rational function $\widehat{\pi}_{n,m}^t(x;f) = \widehat{p}_n^t(x)/\widehat{q}_m^t(x)$ in $\mathscr{R}_{n,m}^t$ which is analytic on $[0,2\pi]$, representable by its Fourier series and has the greatest possible order of osculation (in terms of the number of free parameters) with the series (1.1), that is,

$$f(x) - \widehat{\pi}_{n,m}^t(x;f) = \sum_{k=n+m+1}^{\infty} (\widehat{a}_k \cos kx + \widehat{b}_k \sin kx),$$

where $\widehat{a}_k$ and $\widehat{b}_k$ are real numbers. On a formal level the definitions of trigonometric Padé and Padé-Baker approximants are similar to the corresponding definitions in the algebraic case. However, as regards their content, the differences between these notions in the trigonometric case are more significant than in the algebraic case. In many respects the situation here is analogous to the one which exists for some other generalizations of the classical case (see, for example, the survey [6]): the various generalizations lead to different rational constructions.

In fact, consider the Weierstrass function, which can be represented by the lacunary Fourier series

$$f(x) = \sum_{i=0}^{\infty} q^i \cos(2k+1)^i x, \qquad 0 < q < 1. \tag{1.6}$$

In the classical setting there exists a close connection between the C-table (with entries equal to the Hadamard determinants $C(n/m)$) and the Padé-Baker table $[\widehat{\pi}_{n,m}(\,\cdot\,;f)]_{n,m=0}^{\infty}$ (see [5], Ch. 1, § 1.4). In particular, in the C-table zeros form square blocks, and if $C(n/m) \neq 0$, then for such n and m the Padé-Baker approximant $\widehat{\pi}_{n,m}(\,\cdot\,;f)$ is well defined and coincides with the classical Padé approximant $\pi_{n,m}(\,\cdot\,;f)$. In the periodic case an analogue of the C-table is the Δ-table of values of the determinants $\Delta_{n,m}$. If we set $q = 1/4$ and $k = 1$ in (1.6), then the Δ-table of f contains the following fragment (the precise values of the entries were calculated 'by hand'; these calculations were duplicated using the Mathcad software package):

m	n									
	0	1	2	3	4	5	6	7	8	9
0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$
1	0	$\neq 0$	0	$\neq 0$	0	0	0	$\neq 0$	0	$\neq 0$
2	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	0	$\neq 0$	0	0	$\neq 0$
3	0	$\neq 0$	0	$\neq 0$	0	0	0	0	0	$\neq 0$
4	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	0	0	0	$\neq 0$
5	0	$\neq 0$	0	$\neq 0$	0	0	0	0	0	$\neq 0$
6	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	0	0

We see that the zero entries of a Δ-table do not always form square blocks; moreover, if $\Delta_{n,m} \neq 0$, then the trigonometric Padé approximants $\pi^t_{n,m}(\,\cdot\,;f)$ and $\widehat{\pi}^t_{n,m}(\,\cdot\,;f)$ do not necessarily coincide when the second of these is well-defined. For example, $\Delta_{0,2} \neq 0$, but it is easy to calculate that

$$\pi^t_{0,2}(x;f) = \frac{0}{5 - 8\cos 2x} \equiv 0$$

is not a trigonometric Padé-Baker approximant, though the polynomials $p^t_0(x) = 0$ and $q^t_2(x) = 5 - 8\cos 2x$ satisfy condition (1.3). In a similar way $\Delta_{6,6} \neq 0$, but

$$\pi^t_{6,6}(x;f) = \frac{80\cos x + 75\cos 3x + 52\cos 5x}{64 + 128\cos 4x - 32\cos 6x}$$

is not representable by a Fourier series since the denominator vanishes at some point in $[0, 2\pi]$. For this reason $\pi^t_{6,6}(x;f)$ and $\widehat{\pi}^t_{6,6}(x;f)$ cannot coincide.[1]

If $m = 0$, then the Padé approximants $\pi^t_{n,0}(\,\cdot\,;f)$ and $\widehat{\pi}^t_{n,0}(\,\cdot\,;f)$ always exist and are identically equal to the corresponding partial sum of the Fourier series (1.1). By a theorem due to Bernstein (see [8]), the trigonometric polynomial of best uniform approximation of order at most n for the Weierstrass function (1.6) is equal to the corresponding partial Fourier sum of the series (1.6):

$$r^*_{n,0}(x;f) = \pi^t_{n,0}(x;f) = \widehat{\pi}^t_{n,0}(x;f).$$

We can add to Bernstein's theorem. Namely, the following result holds.

Proposition 2. *Let $f(x) = \sum_{i=0}^{\infty} q^i \cos(2k+1)^i x$ be a Weierstrass function. For fixed n let s be determined by the conditions*

$$(2k+1)^s \leqslant n \leqslant (2k+1)^{s+1} - 1.$$

If $0 \leqslant m \leqslant 2k(2k+1)^s - 1$ and $0 \leqslant i + j \leqslant 2k(2k+1)^s - 1$, then there exists $\widehat{\pi}^t_{(2k+1)^s+i,\,j}(x;f)$ and

$$r^*_{n,m}(x;f) = \widehat{\pi}^t_{(2k+1)^s+i,\,j}(x;f) = \pi^t_{(2k+1)^s+i,\,j}(x;f) = \pi^t_{(2k+1)^s,\,0}(x,f), \qquad (1.7)$$
$$\mathrm{R}^t_{n,m}(f) = \frac{q^{s+1}}{1-q}.$$

Proof. We shall show that $\pi^t_{(2k+1)^s,\,0}(x,f) = \sum_{i=0}^{s} q^i \cos(2k+1)^i x$ is the trigonometric rational function of best approximation in the class $\mathscr{R}^t_{n,m}$. To do this it is sufficient to prove (see [1], Ch. 7, § 3, Theorem 2.10) that it has $2(n+m-d)+2$ points of alternance, where $d = \min\{n-(2k+1)^s; m\}$. We shall consider two cases. Let $d = m$, that is,

$$0 \leqslant m \leqslant n - (2k+1)^s \leqslant (2k+1)^{s+1} - 1 - (2k+1)^s = 2k(2k+1)^s - 1.$$

[1] The referee drew our attention to [7], where Gibbs phenomenon for generalized Padé approximants of the sign function $\operatorname{sgn} x$ is investigated and where, in particular, explicit expressions for the fractions $\pi^t_{n,m}(x;s)$ and $\widehat{\pi}^t_{n,m}(x;s)$, where $s(x) = \operatorname{sgn}(\cos x)$, are found, which demonstrate that these fractions are different for all $m \geqslant 1$.

Then $2(n+m-d)+2=2(n+1)$. At the points

$$x_i=\frac{i\pi}{(2k+1)^{s+1}},\qquad i=0,1,\dots,2(2k+1)^{s+1}-1,$$

we have

$$f(x_i)-\pi^t_{(2k+1)^s,\,0}(x_i,f)=(-1)^i\,\frac{q^{s+1}}{1-q}.$$

Since $2(n+1)\leqslant 2(2k+1)^{s+1}$, there exist sufficiently many points of alternance. If $d=n-(2k+1)^s$, then

$$2(n+m-d)+2=2(m+(2k+1)^s)+2.$$

In view of the constraints on m in the hypotheses,

$$2(m+(2k+1)^s)+2\leqslant 2(2k+1)^{s+1}.$$

Hence $r^*_{n,m}(x;f)=\pi^t_{(2k+1)^s,\,0}(x;f)$. It remains to prove that the Padé approximant $\widehat{\pi}^t_{(2k+1)^s+i,\,j}(x;f)$ exists for $0\leqslant i+j\leqslant 2k(2k+1)^s-1$ and coincides with $\pi^t_{(2k+1)^s+i,\,j}(x;f)$ and $\pi^t_{(2k+1)^s,\,0}(x;f)$. For this it is sufficient to show that

$$(2k+1)^s+i+j+1\leqslant(2k+1)^{s+1}.$$

This is indeed so because

$$(2k+1)^s+i+j+1\leqslant(2k+1)^s+2k(2k+1)^s=(2k+1)^{s+1}.$$

The proof of Proposition 2 is complete.

We proceed to the statement of the problem. The sequence of Fourier coefficients $\{c_k\}_{k=-\infty}^{\infty}$ of the series (1.2) determines the coefficients of the numerator and denominator of the fraction $r^*_{n,m}(\,\cdot\,;f)$ uniquely. However, at present the problem of obtaining explicit expressions for these coefficients in the general case seems, for all practical purposes, to be unsolvable. Even for $m=0$ the only nonelementary function for which we can find them is the Weierstrass function. For the fraction $\widehat{\pi}^t_{n,m}(\,\cdot\,;f)$ the situation is similar. On the other hand the precise values of the coefficients of the numerator and denominator of the classical trigonometric Padé approximant $\pi^t_{n,m}(\,\cdot\,;f)$ can be calculated using (1.4). Our immediate aim is to find conditions on the coefficients of the Fourier series (1.2) ensuring that the trigonometric Padé approximants $\pi^t_{n,m}(\,\cdot\,;f)$ (for $m=0$ the $\pi^t_{n,0}(\,\cdot\,;f)$ are partial sums of the Fourier series (1.2)) approximate the function f in the uniform norm at a rate asymptotically coinciding with the highest possible. In this case, as $n+m\to\infty$, the infinitesimals $\|f-r^*_{n,m}\|$ and $\|f-\pi^t_{n,m}\|$ will be equivalent.[2] Hence by describing the decreasing asymptotic behaviour of the sequence $\|f-\pi^t_{n,m}\|$ in terms of the Fourier coefficients of the series (1.1) we shall solve the problem of best rational approximation of f in a certain sense.

[2] We say that infinitesimals $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are *equivalent* ($\alpha_n\sim\beta_n$) if $\alpha_n/\beta_n\to 1$ as $n\to\infty$.

This approach to the analysis of approximation properties of periodic functions was initiated by Rusak, and the general statement of the problem is due to him. Classes of functions f for which the trigonometric Padé approximants $\pi^t_{n,m}(\,\cdot\,;f)$ have the extremal properties indicated above were first discovered by Berezkina [9] and Ta Hong Quang [10]. In the algebraic case similar problems were investigated by Levin and Lubinsky (see [11]–[14]), in connection with the analysis of results due to Saff (see [15], [16]) on rational approximation of the exponential function. In [11]–[14], [17]–[19] the authors found broad classes of entire functions for which the classical Padé approximants $\pi_{n,m}(\,\cdot\,;f)$ approximate f in the unit disc at a rate asymptotically equal to the highest possible, that is, the so-called Saff phenomenon occurs (see [19] for details). To date the most general results in the algebraic case have been obtained in [20]. In this paper, in § 2 and § 3 we prove analogues of the main results in [20] for trigonometric Padé approximants.

First we introduce the requisite function classes. In what follows we consider continuous 2π-periodic even functions f. Then

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.8}$$

and the coefficients $c_k = c_{-k} = a_k/2$ of the Fourier series in the complex form (1.2) are real numbers. With these constraints $\widetilde{c}_k = \widetilde{c}_{-k} = d_{n,m,k} = d_{n,m,-k}$. Therefore, taking (1.5) into account we obtain

$$L^t_{n,m}(x;f) = \sum_{k=1}^{\infty} \widetilde{c}_{n+m+k}\bigl(e^{i(n+m+k)x} + e^{-i(n+m+k)x}\bigr). \tag{1.9}$$

We use $\mathrm{T}^{\alpha}_{\beta}(q)$, $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$, to denote the set of functions f representable in the form (1.8) with nonzero Fourier coefficients $\{a_n\}_{n=0}^{\infty}$ satisfying the following restrictions: for each j, $1 \leqslant j \leqslant m(n)$, where

$$\lim_{n\to\infty} \frac{(m(n))^{2+\beta}}{n} = 0, \tag{1.10}$$

there exists a sequence of real numbers $\{b^{(j)}_k\}_{k=1}^{\infty}$, such that

$$|b^{(j)}_k| \leqslant (cj^{\beta})^k, \qquad c = \mathrm{const}, \tag{1.11}$$

uniformly for $k = 1, 2, \dots$, and, for $n \geqslant n_0$,

$$\frac{a_{n+j}}{a_n} = \left(\frac{q}{n^{\alpha}}\right)^{j} \left(1 + \sum_{k=1}^{\infty} \frac{b^{(j)}_k}{n^k}\right). \tag{1.12}$$

Condition (1.12) defines the rate of decrease of the Fourier coefficients of f and, in combination with (1.11), in a certain sense describes the regularity of their decrease.[3] In particular, it follows from (1.11) and (1.12) that f is the real part of

[3] We point out that the sequence $\{b^{(j)}_k\}_{k=1}^{\infty}$ is determined by j alone and is independent of n, and c is an absolute constant, which is the same for all j.

the restriction to $\mathbb{R}$ of the entire function

$$\widehat{f}(z) = c_0 + 2\sum_{n=1}^{\infty} c_n e^{inz}.$$

In some special cases verification of (1.11) and (1.12) can be simplified. In fact, let $T(\alpha, q)$, $\alpha \in \mathbb{N}$, $q \in \mathbb{R}$, be the set of functions f representable in the form (1.8) for which $a_n \neq 0$, $n \in \mathbb{N}$, and a sequence of real numbers $\{b_k\}_{k=1}^{\infty}$ exists such that for $n \geqslant n_0$,

$$\frac{a_{n+1}}{a_n} = \frac{q}{n^\alpha}\left(1 + \sum_{k=1}^{\infty} \frac{b_k}{n^k}\right), \qquad |b_k| \leqslant M^k, \quad M = \text{const}, \quad k = 0, 1, 2, \dots .$$

By Theorem 5.1 in [20], $T(\alpha, q) \subset T_2^\alpha(q)$. Taking this into account we see, for example, that the function

$$f_h(x) = e^{h\cos x}\cos(h\sin x) = \sum_{n=0}^{\infty} \frac{h^n}{n!}\cos nx, \qquad h \in \mathbb{R}, \tag{1.13}$$

is in $T_2^1(h)$.

We say that g belongs to $T_\beta^\alpha(q, d)$ if $g(x) = f(dx)$, where $f \in T_\beta^\alpha(q)$, $d \in \mathbb{N}$. For example,

$$g_h(x) = \operatorname{ch}(h\cos x)\cos(h\sin x) = \sum_{n=0}^{\infty} \frac{h^{2n}}{(2n)!}\cos 2nx \tag{1.14}$$

is in the class $T_2^2(h^2/4; 2)$. Using the definition of trigonometric Padé approximants it is easy to show that if $g \in T_\beta^\alpha(q, d)$ and $g(x) = f(dx)$, then $\pi^t_{dn+i,\, dm+j}(x; g) = \pi^t_{n,m}(dx; f)$ for $0 \leqslant i + j \leqslant d - 1$, and therefore

$$g(x) - \pi^t_{dn+i,\, dm+j}(x; g) = f(dx) - \pi^t_{n,m}(dx; f). \tag{1.15}$$

The following theorem is one of the central results of our paper (see § 3).

Theorem 1. *Let f be a function in $T_\beta^\alpha(q)$, $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$. If*

$$\lim_{n\to\infty} \frac{(m(n))^{2+\beta}}{n} = 0,$$

then for all $x \in \mathbb{R}$, uniformly for $0 \leqslant m \leqslant m(n)$,

$$f(x) - \pi^t_{n,m}(x; f) = (-1)^m m!\, a_{n+1}\left(\frac{\alpha q}{n^{\alpha+1}}\right)^m \operatorname{Re}\{e^{i(n+m+1)x}(1 + o(1))\},$$
$$\mathrm{R}^t_{n,m}(f) \sim \|f - \pi^t_{n,m}(\,\cdot\,; f)\| \sim m!\, |a_{n+1}|\left(\frac{\alpha|q|}{n^{\alpha+1}}\right)^m$$

as $n \to \infty$.

In § 4 we give applications of Theorem 1 and some other results in § 3 to problems dealing with finding the sharp constants of rational approximation (in this connection see Braess [21], Gonchar and Rakhmanov [22], Stahl [23], [24] and Aptekarev [25]). In particular, we prove a rational analogue of Bernstein's theorem (see [26], paper 3, Ch. 5, § 54) which establishes the asymptotics of decrease of the best algebraic polynomial approximations for functions representable by a Fourier-Chebyshev series with regularly decreasing coefficients.

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§ 2. Asymptotic behaviour of Hadamard-type determinants

Consider m power series which converge in the disc $D_\rho = \{z : |z| < \rho\}$ and have the following form:

$$f_{m-i}(z) = \sum_{k=0}^{\infty} a_k^{(m-i)} z^k, \qquad i = 1, \dots, m, \tag{2.1}$$

where

$$\begin{cases} a_k^{(i)} = 0, & 1 \leqslant i \leqslant m-1,\ k < \alpha i, \\ a_{\alpha i}^{(i)} = 1, & 1 \leqslant i \leqslant m-1, \\ a_0^{(0)} = 1,\ a_k^{(0)} = 0, & k \geqslant 1. \end{cases} \tag{2.2}$$

We shall assume that the other coefficients of the power series (2.1) satisfy the following conditions:

$$|a_k^{(i)}| \leqslant (ci^\beta)^{k-\alpha i}, \qquad c = \text{const}, \quad k > \alpha i, \quad 1 \leqslant i \leqslant m-1. \tag{2.3}$$

For $\overrightarrow{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_m)$, $\lambda_j \in D_\rho$, $j = 1, \dots, m$, consider the determinants

$$\Delta_m(\overrightarrow{\lambda}) := |f_{m-i}(\lambda_j)|_{i,j=1}^m.$$

The proof of the following result can be found in [20].

Proposition 3. *If* $c_1, c_2 > 0$, $\beta \geqslant 1$,

$$\frac{1}{c_1 n + c_2 m} \leqslant \lambda_j \leqslant \frac{1}{c_1 n - c_2 m}, \qquad j = 1, \dots, m, \tag{2.4}$$

and condition (1.10) *holds, then uniformly for* $0 \leqslant m \leqslant m(n)$,

$$\Delta_m(\overrightarrow{\lambda}) \sim W_m(\overrightarrow{\lambda^\alpha})$$

as $n \to \infty$, *where*

$$W_m(\overrightarrow{\lambda}) = |\lambda_j^{m-i}|_{i,j=1}^m = \prod_{1 \leqslant i < j \leqslant m} (\lambda_i - \lambda_j)$$

is the Vandermonde determinant of order m *of the variables* $\lambda_1, \lambda_2, \dots, \lambda_m$ *and* $\overrightarrow{\lambda^\alpha} = (\lambda_1^\alpha, \lambda_2^\alpha, \dots, \lambda_m^\alpha)$.

Remark 1. Proposition 3 holds uniformly for all $k = 1, 2, 3, \dots$ if we have

$$\frac{1}{c_1(n+k)+c_2 m} \leqslant \lambda_1 \leqslant \frac{1}{c_1(n+k)-c_2 m},$$

and the other λ_j satisfy the same conditions (2.4) as before.

Indeed, in this case

$$\frac{c_1(n+k)+c_2 m}{c_1(n+k)-c_2 m} \leqslant \frac{c_1 n+c_2 m}{c_1 n-c_2 m},$$

and under the assumptions in this remark, Theorem 2.1 in [20] also holds.

We consider two square matrices of order $m+1$:

$$\widetilde{A}(k) = \begin{bmatrix} 2c_{n+m+k} & c_{n+m+k-1}+c_{n+m+k+1} & \dots & c_{n+k}+c_{n+2m+k} \\ 2c_{n+1} & c_n + c_{n+2} & \dots & c_{n-m+1}+c_{n+m+1} \\ \dots & \dots & \dots & \dots \\ 2c_{n+m} & c_{n+m-1}+c_{n+m+1} & \dots & c_n + c_{n+2m} \end{bmatrix},$$

$$A(y) = \begin{bmatrix} 2 & y^{-1}+y & \dots & y^{-m}+y^m \\ 2c_{n+1} & c_n + c_{n+2} & \dots & c_{n-m+1}+c_{n+m+1} \\ \dots & \dots & \dots & \dots \\ 2c_{n+m} & c_{n+m-1}+c_{n+m+1} & \dots & c_n + c_{n+2m} \end{bmatrix},$$

and two square matrices of order m:

$$A_0 = \begin{bmatrix} c_{n+2}+c_n & c_{n+3}+c_{n-1} & \dots & c_{n+m+1}+c_{n-m+1} \\ c_{n+3}+c_{n+1} & c_{n+4}+c_n & \dots & c_{n+m+2}+c_{n-m+2} \\ \dots & \dots & \dots & \dots \\ c_{n+m+1}+c_{n+m-1} & c_{n+m+2}+c_{n+m-2} & \dots & c_{n+2m}+c_n \end{bmatrix},$$

$$B = \begin{bmatrix} c_{n+2}-c_n & c_{n+3}-c_{n-1} & \dots & c_{n+m+1}-c_{n-m+1} \\ c_{n+3}-c_{n+1} & c_{n+4}-c_n & \dots & c_{n+m+2}-c_{n-m+2} \\ \dots & \dots & \dots & \dots \\ c_{n+m+1}-c_{n+m-1} & c_{n+m+2}-c_{n+m-2} & \dots & c_{n+2m}-c_n \end{bmatrix}.$$

Here and throughout, $c_k = 2a_k$, where the a_k are the Fourier coefficients of the series (1.8).

Lemma 1. *Let f be a function represented by a series* (1.8). *Then*

$$\Delta_{n,m} = \det A_0 \cdot \det B, \tag{2.5}$$

$$\widetilde{c}_{n+m+k} = \frac{1}{2}\det \widetilde{A}(k) \cdot \det B, \tag{2.6}$$

$$q_m^t(x) = \frac{1}{2}\det A(y) \cdot \det B, \qquad y = e^{ix}. \tag{2.7}$$

Proof. Since $c_j = c_{-j}$, we can replace c_{-j} by c_j in all the rows of the determinant $\Delta_{n,m}$ starting from the $(m+1)$th. Then

$$\Delta_{n,m} = \begin{vmatrix} c_{n+2m} & \cdots & c_{n+m+1} & c_{n+m-1} & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n+m+1} & \cdots & c_{n+2} & c_n & \cdots & c_{n-m+1} \\ --- & --- & --- & --- & --- & --- \\ c_{n-m+1} & \cdots & c_n & c_{n+2} & \cdots & c_{n+m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & \cdots & c_{n+m-1} & c_{n+m+1} & \cdots & c_{n+2m} \end{vmatrix}.$$

Now we transpose the $(m+1)$st row and the last one, the $(m+2)$nd and the next before last and so on, and in the resulting matrix we transpose the $(m+1)$st column and the last one, the $(m+2)$nd column and the next before last and so on. After these transformations we obtain

$$\Delta_{n,m} = \begin{vmatrix} c_{n+2m} & \cdots & c_{n+m+1} & c_n & \cdots & c_{n+m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n+m+1} & \cdots & c_{n+2} & c_{n-m+1} & \cdots & c_n \\ --- & --- & --- & --- & --- & --- \\ c_n & \cdots & c_{n+m-1} & c_{n+2m} & \cdots & c_{n+m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n-m+1} & \cdots & c_n & c_{n+m+1} & \cdots & c_{n+2} \end{vmatrix} = \begin{vmatrix} P & Q \\ Q & P \end{vmatrix},$$

where P and Q are matrices of order m located in the first m columns and first m rows and in the first m rows and last m columns, respectively. We transform the above matrix by adding the $(m+1)$st column to the first, the $(m+2)$nd column to the second and so on up to the mth column to which we add the $2m$th. After that we subtract the first row from the $(m+1)$st, the second row from the $(m+2)$nd and so on; finally, from the last row we subtract the mth. Then it follows from the properties of determinants that

$$\Delta_{n,m} = \begin{vmatrix} P+Q & Q \\ 0 & P-Q \end{vmatrix} = \det(P+Q) \cdot \det(P-Q),$$

where

$$\det(P+Q) = \det A_0, \qquad \det(P-Q) = \det B.$$

The proof of (2.5) is complete.

Now we prove (2.6). In $d_{n,m,n+m+k}$ we replace the coefficients c_{-j} by c_j starting from the $(m+2)$nd row. Then

$$\widetilde{c}_{n+m+k} = \begin{vmatrix} c_{n+2m} & \cdots & c_{n+m+1} & c_{n+m} & c_{n+m-1} & \cdots & c_n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n+m+1} & \cdots & c_{n+2} & c_{n+1} & c_n & \cdots & c_{n-m+1} \\ c_{n+2m+k} & \cdots & c_{n+m+k+1} & c_{n+m+k} & c_{n+m+k-1} & \cdots & c_{n+k} \\ c_{n-m+1} & \cdots & c_n & c_{n+1} & c_{n+2} & \cdots & c_{n+m+1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_n & \cdots & c_{n+m-1} & c_{n+m} & c_{n+m+1} & \cdots & c_{n+2m} \end{vmatrix}.$$

After elementary transformations of rows and columns, similarly to above we obtain

$$\widetilde{c}_{n+m+k} = \begin{vmatrix} X & Y \\ 0 & Z \end{vmatrix} = \det X \cdot \det Z,$$

where

$$\det X = \frac{1}{2}\det \widetilde{A}(k), \qquad \det Z = \det B.$$

The proof of equality (2.6) is complete, the proof of (2.7) is similar, and these complete the proof of Lemma 1.

Lemma 2. *Let n_i, $i = 1, \ldots, m$, be distinct positive integers and let*

$$\gamma_1 n - \gamma_2 m \leqslant n_i \leqslant \gamma_1 n + \gamma_2 m, \qquad i = 1, \ldots, m,$$

where γ_1 and γ_2 are positive constants independent of m and n. If a function f can be represented in the form (1.8) and $f \in T_\beta^\alpha(q)$, then uniformly for $0 \leqslant m \leqslant m(n)$,

$$\begin{vmatrix} c_{n_1} & c_{n_2} & \cdots & c_{n_m} \\ c_{n_1+1} & c_{n_2+1} & \cdots & c_{n_m+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n_1+m-1} & c_{n_2+m-1} & \cdots & c_{n_m+m-1} \end{vmatrix} \sim q^{m(m-1)/2} \prod_{i=1}^m c_{n_i} \prod_{1\leqslant i<j\leqslant m} (n_j^{-\alpha} - n_i^{-\alpha}) \tag{2.8}$$

as $n \to \infty$.

Proof. Let C be the determinant in (2.8). Then using elementary transformations we obtain

$$\begin{aligned} C &= (-1)^{m(m-1)/2} \begin{vmatrix} c_{n_1+m-1} & c_{n_2+m-1} & \cdots & c_{n_m+m-1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n_1+1} & c_{n_2+1} & \cdots & c_{n_m+1} \\ c_{n_1} & c_{n_2} & \cdots & c_{n_m} \end{vmatrix} \\ &= (-1)^{m(m-1)/2} c_{n_1} c_{n_2} \cdots c_{n_m} \cdot C', \end{aligned} \tag{2.9}$$

where

$$C' = \begin{vmatrix} \dfrac{c_{n_1+m-1}}{c_{n_1}} & \dfrac{c_{n_2+m-1}}{c_{n_2}} & \cdots & \dfrac{c_{n_m+m-1}}{c_{n_m}} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{c_{n_1+1}}{c_{n_1}} & \dfrac{c_{n_2+1}}{c_{n_2}} & \cdots & \dfrac{c_{n_m+1}}{c_{n_m}} \\ 1 & 1 & \cdots & 1 \end{vmatrix}.$$

We can write (1.12) as

$$\frac{c_{n+j}}{c_n} = q^j \sum_{k=0}^\infty b_k^j \frac{1}{n^{k+\alpha j}}. \tag{2.10}$$

Now defining some of the coefficients in (2.1) using (2.2) and defining the other coefficients by

$$a_k^{(i)} = b_{k-\alpha i}^{(i)}, \qquad k > \alpha i, \quad 1 \leqslant i \leqslant m-1,$$

it follows from (1.11) that (2.3) holds in this case. Taking account of (2.10) and the choice of the coefficients in (2.1) we have

$$C' = q^{m(m-1)/2} |f_{m-i}(\lambda_j)|_{i,j=1}^{m},$$

where $\lambda_j = 1/n_j$, $j = 1, \ldots, m$. By Proposition 3,

$$C' \sim q^{m(m-1)/2} \prod_{1 \leqslant i < j \leqslant m} (\lambda_i^\alpha - \lambda_j^\alpha).$$

Finally, by (2.9) we obtain that, as $n \to \infty$,

$$\begin{aligned} C &\sim (-q)^{m(m-1)/2} \prod_{i=1}^{m} c_{n_i} \prod_{1 \leqslant i < j \leqslant m} (n_i^{-\alpha} - n_j^{-\alpha}) \\ &= q^{m(m-1)/2} \prod_{i=1}^{m} c_{n_i} \prod_{1 \leqslant i < j \leqslant m} (n_j^{-\alpha} - n_i^{-\alpha}). \end{aligned}$$

The proof of Lemma 2 is complete.

Let

$$A_1 := \begin{vmatrix} c_{n+1} & c_n + c_{n+2} & \ldots & c_{n-m+1} + c_{n+m+1} \\ c_{n+2} & c_{n+1} + c_{n+3} & \ldots & c_{n-m+2} + c_{n+m+2} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+m+1} & c_{n+m} + c_{n+m+2} & \ldots & c_{n+1} + c_{n+2m+1} \end{vmatrix}.$$

Lemma 3. *If f is a function representable in the form (1.8) and $f \in T_\beta^\alpha(q)$, then uniformly for $0 \leqslant m \leqslant m(n)$,*

$$A_1 \sim A^0 := \begin{vmatrix} c_{n+1} & c_n & \ldots & c_{n-m+1} \\ c_{n+2} & c_{n+1} & \ldots & c_{n-m+2} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+m+1} & c_{n+m} & \ldots & c_{n+1} \end{vmatrix} \tag{2.11}$$

as $n \to \infty$.

Proof. We represent A_1 as the sum of the 2^m determinants obtained by decomposing the columns:

$$A_1 = \sum_{n_1, n_2, \ldots, n_m} \begin{vmatrix} c_{n+1} & c_{n_1} & \ldots & c_{n_m} \\ c_{n+2} & c_{n_1+1} & \ldots & c_{n_m+1} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n+m+1} & c_{n_1+m} & \ldots & c_{n_m+m} \end{vmatrix}, \tag{2.12}$$

where n_i can be equal to $n+1-i$ or $n+1+i$, $i = 1, \ldots, m$. By $A^j_{v_1, v_2, \ldots, v_j}$, $1 \leqslant v_1 < v_2 < \cdots < v_j \leqslant m$, $j = 1, \ldots, m$, we denote the determinant in (2.12) in which $n_{v_1} = n + 1 + v_1$, $n_{v_2} = n + 1 + v_2$, $\ldots$, $n_{v_j} = n + 1 + v_j$, and for the other values we have $n_i = n + 1 - i$, $i = 1, \ldots, m$, $i \neq v_1, v_2, \ldots, v_j$. Then (2.12) can be written as follows:

$$A_1 = A^0 + \sum_{v_1} A^1_{v_1} + \sum_{v_1<v_2} A^2_{v_1,v_2} + \cdots + \sum_{v_1<\cdots<v_j} A^j_{v_1,\ldots,v_j} + \cdots + A^m_{1,2,\ldots,m}$$
$$= A^0 + S^1 + S^2 + \cdots + S^j + \cdots + S^m = A^0\biggl(1 + \sum_{j=1}^{m} \frac{S^j}{A^0}\biggr), \tag{2.13}$$

where the sum S^j contains precisely $C^j_m = m!/(j!\,(m-j)!)$ terms. Now we set $\lambda_i^+ = (n+1+i)^{-\alpha}$ and $\lambda_i^- = (n+1-i)^{-\alpha}$. It follows from Lemma 2 that

$$A^0 \sim q^{m(m+1)/2} \prod_{i=0}^{m} c_{n+1-i} \prod_{0\leqslant i<j\leqslant m} (\lambda_j^- - \lambda_i^-),$$

$$A^j_{v_1,v_2,\ldots,v_j} \sim q^{m(m+1)/2} \prod_{p=1}^{j} \frac{c_{n+1+v_p}}{c_{n+1-v_p}} \prod_{i=0}^{m} c_{n+1-i} \prod_{0\leqslant i<j\leqslant m} (\lambda_j^- - \lambda_i^-)$$
$$\times \prod_{k=1}^{j} \biggl\{ \prod_{\substack{i=0\\ i\neq v_1,v_2,\ldots,v_{k-1}}}^{v_k-1} \frac{\lambda^+_{v_k} - \lambda^-_i}{\lambda^-_{v_k} - \lambda^-_i} \prod_{\substack{i=v_k+1\\ i\neq v_{k+1},v_{k+2},\ldots,v_j}}^{m} \frac{\lambda^-_i - \lambda^+_{v_k}}{\lambda^-_i - \lambda^-_{v_k}} \biggr\} \prod_{k=2}^{j} \prod_{i=k}^{j} \frac{\lambda^+_{v_j} - \lambda^+_{v_{k-1}}}{\lambda^-_{v_j} - \lambda^-_{v_{k-1}}} \tag{2.14}$$

as $n \to \infty$.

Using Lagrange's finite difference theorem we obtain

$$\biggl| \prod_{i=0}^{v_1-1} \frac{\lambda^+_{v_1} - \lambda^-_i}{\lambda^-_{v_1} - \lambda^-_i} \biggr| \leqslant \prod_{i=0}^{v_1-1} \frac{\frac{\alpha}{(n+1-i)^{\alpha-1}}\bigl(\frac{1}{n+1-i} - \frac{1}{n+1+v_1}\bigr)}{\frac{\alpha}{(n+1-i)^{\alpha-1}}\bigl(\frac{1}{n+1-v_1} - \frac{1}{n+1-i}\bigr)} = \prod_{i=0}^{v_1-1} \frac{v_1+i}{v_1-i} \, \frac{n+1-v_1}{n+1+v_1}$$
$$\leqslant \prod_{i=0}^{v_1-1} \frac{v_1+i}{v_1-i} = C^{v_1}_{2v_1-1} \leqslant v_1^{v_1}.$$

(We have used the fact that $C^r_{m+r-1} \leqslant m^r$.) Similarly, for $k = 2, 3, \ldots, j$,

$$\biggl| \prod_{\substack{i=0\\ i\neq v_1,v_2,\ldots,v_{k-1}}}^{v_k-1} \frac{\lambda^+_{v_k} - \lambda^-_i}{\lambda^-_{v_k} - \lambda^-_i} \biggr| \leqslant v_k^{v_k}.$$

Next, using Lagrange's theorem we obtain

$$\prod_{\substack{i=v_1+1\\ i\neq v_2,v_3,\ldots,v_j}}^{m} \frac{\lambda^-_i - \lambda^+_{v_1}}{\lambda^-_i - \lambda^-_{v_1}} \leqslant \prod_{i=v_1+1}^{m} \frac{\lambda^-_i - \lambda^+_{v_1}}{\lambda^-_i - \lambda^-_{v_1}} \leqslant C^{2v_1}_{m+v_1} \biggl(\frac{n+1-v_1}{n+1-m}\biggr)^{(\alpha-1)(m-v_1)}$$
$$\leqslant m^{2v_1} \biggl(\frac{n+1}{n+1-m}\biggr)^{(\alpha-1)m}.$$

Similarly, for $k = 2, 3, \ldots, j$,

$$\prod_{\substack{i=v_k+1\\ i\neq v_{k+1},v_{k+2},\ldots,v_j}}^{m} \frac{\lambda^-_i - \lambda^+_{v_k}}{\lambda^-_i - \lambda^-_{v_k}} \leqslant m^{2v_k} \biggl(\frac{n+1}{n+1-m}\biggr)^{(\alpha-1)m}.$$

Since for all $1 \leqslant v_p < v_k \leqslant m$ we have

$$\left|\frac{\lambda^+_{v_k} - \lambda^+_{v_p}}{\lambda^-_{v_k} - \lambda^-_{v_p}}\right| \leqslant \left(\frac{n+1-v_p}{n+1+v_p}\right)^{\alpha-1} \frac{(n+1-v_k)(n+1-v_p)}{(n+1+v_k)(n+1+v_p)} \leqslant 1,$$

the last factor on the right-hand side of (2.14) has absolute value at most 1.

Finally, in view of (1.12), for $n \geqslant n_0$ we obtain

$$\prod_{p=1}^{j} \frac{c_{n+1+v_p}}{c_{n+1-v_p}} \leqslant \prod_{p=1}^{j} \left(\frac{2q}{(n+1-v_p)^\alpha}\right)^{2v_p} \leqslant \left(\frac{2q}{(n+1-m)^\alpha}\right)^{2\sum_{k=1}^{j} v_k}.$$

It follows from these inequalities and (2.14) that

$$\begin{aligned}\left|\frac{A^j_{v_1,v_2,\dots,v_j}}{A^0}\right| &\leqslant \prod_{p=1}^{j} (v_p m^2)^{v_p} \cdot \left(\frac{n+1}{n+1-m}\right)^{jm(\alpha-1)} \left(\frac{2q}{(n+1-m)^\alpha}\right)^{2\sum_{k=1}^{j} v_k} \\ &\leqslant \left(\frac{n+1}{n+1-m}\right)^{jm(\alpha-1)} \left(\frac{2qm^{3/2}}{(n+1-m)^\alpha}\right)^{2\sum_{k=1}^{j} v_k}.\end{aligned}$$

Note that $S^j \leqslant C^j_m \max_{v_1,v_2,\dots,v_j} |A^j_{v_1,v_2,\dots,v_j}|$ and $C^j_m \leqslant m^j$. Hence

$$\begin{aligned}\sum_{j=1}^{m} \frac{S^j}{A^0} &\leqslant \left(\frac{n+1}{n+1-m}\right)^{m^2(\alpha-1)} \sum_{j=1}^{m} m^j \left(\frac{2qm^{3/2}}{(n+1-m)^\alpha}\right)^{2j} \\ &\leqslant \left(\frac{n+1}{n+1-m}\right)^{m^2(\alpha-1)} \sum_{j=1}^{m} \left(\frac{2qm^2}{(n+1-m)^\alpha}\right)^{2j}. \end{aligned} \tag{2.15}$$

In view of the constraints (1.10), uniformly for $0 \leqslant m \leqslant m(n)$,

$$\lim_{n\to\infty} \left(\frac{n+1}{n+1-m}\right)^{m^2(\alpha-1)} = 1.$$

Hence from (2.15) we finally obtain

$$A_1 = A^0\left(1 + \sum_{j=1}^{m} \frac{S^j}{S^0}\right) = A^0\left[1 + O\left(\frac{m^4}{(n+1-m)^{2\alpha}}\right)\right]$$

as $n \to \infty$. The proof of Lemma 3 is complete.

Corollary 1. *If a function f can be represented in the form (1.8) and $f \in T^\alpha_\beta(q)$, then for $0 \leqslant m \leqslant m(n)$ and $n > n_0$,* $\det A_0 \neq 0$, $\det B \neq 0$ *and* $\Delta_{n,m} \neq 0$.

This follows directly from an analysis of the proof of Lemma 3, equality (2.5) and Lemma 2.

Lemma 4. *If f is a function representable in the form (1.8) and $f \in T_\beta^\alpha(q)$, then uniformly for $0 \leqslant m \leqslant m(n)$,*

$$\det \widetilde{A}(1) \sim 2(-1)^m q^{m(m+1)/2} \prod_{i=0}^{m} c_{n+1-i} \prod_{0\leqslant i<j\leqslant m} (\lambda_j^- - \lambda_i^-), \tag{2.16}$$

$$\det \widetilde{A}(k) \sim 2q^{m(m+1)/2} c_{n+k} \prod_{i=1}^{m} c_{n+1-i} \prod_{1\leqslant i<j\leqslant m} (\lambda_j^- - \lambda_i^-) \times \prod_{j=1}^{m} \{(n+k)^{-\alpha} - (n+1-j)^{-\alpha}\}, \tag{2.17}$$

$$\det A(y) \sim 2q^{m(m-1)/2} \prod_{i=1}^{m} c_{n+1-i} \prod_{1\leqslant i<j\leqslant m} (\lambda_j^- - \lambda_i^-) \tag{2.18}$$

as $n \to \infty$, where $y = e^{ix}$.

Proof. As in the proof of Lemma 3, we can demonstrate that, as $n \to \infty$,

$$\det \widetilde{A}(k) \sim 2(-1)^m \begin{vmatrix} c_{n+1} & c_n & \cdots & c_{n-m+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+m} & c_{n+m-1} & \cdots & c_n \\ c_{n+m+k} & c_{n+m+k-1} & \cdots & c_{n+k} \end{vmatrix}.$$

Hence it is easy to show by elementary transformations that

$$\det \widetilde{A}(k) \sim 2(-1)^m \begin{vmatrix} c_{n+k} & c_n & \cdots & c_{n-m+1} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+m+k-1} & c_{n+m-1} & \cdots & c_n \\ c_{n+m+k} & c_{n+m} & \cdots & c_{n+1} \end{vmatrix}.$$

Taking account of Remark 1 we apply Lemma 2 to the determinant on the right-hand side to obtain (2.17). Now (2.16) follows from (2.17) for $k = 1$.

Next we proceed to prove (2.18). We expand $\det A(y)$ along the first row and then use Lemma 3. We see that, as $n \to \infty$,

$$\det A(y) \sim 2 \begin{vmatrix} c_n & c_{n-1} & \cdots & c_{n-m+1} \\ c_{n+1} & c_n & \cdots & c_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+m-1} & c_{n+m-2} & \cdots & c_n \end{vmatrix} - 2(y^{-1}+y) \begin{vmatrix} c_{n+1} & c_{n-1} & \cdots & c_{n-m+1} \\ c_{n+2} & c_n & \cdots & c_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+m} & c_{n+m-2} & \cdots & c_n \end{vmatrix} + \cdots + 2(-1)^m (y^{-m}+y^m) \begin{vmatrix} c_{n+1} & c_n & \cdots & c_{n-m+2} \\ c_{n+2} & c_{n+1} & \cdots & c_{n-m+3} \\ \cdots & \cdots & \cdots & \cdots \\ c_{n+m} & c_{n+m-1} & \cdots & c_{n+1} \end{vmatrix}.$$

If we bear in mind that $y=e^{ix}$ and use Lemma 2, as in the proof of Lemma 3 we can show that

$$\det A(y) \sim 2\begin{vmatrix} c_n & c_{n-1} & \dots & c_{n-m+1} \\ c_{n+1} & c_n & \dots & c_{n-m+2} \\ \dots & \dots & \dots & \dots \\ c_{n+m-1} & c_{n+m-2} & \dots & c_n \end{vmatrix}$$

$$\sim 2q^{m(m-1)/2}\prod_{i=1}^{m} c_{n+1-i}\prod_{1\leqslant i<j\leqslant m}\{(n+1-j)^{-\alpha}-(n+1-i)^{-\alpha}\}$$

as $n\to\infty$. The proof of Lemma 4 is complete.

§ 3. Trigonometric Padé approximants to functions in the class $T^{\alpha}_{\beta}(q)$

We will now prove our central results.

Theorem 2. *Let f be a function represented by a Fourier series (1.8) such that $f\in T^{\alpha}_{\beta}(q)$, $\alpha\in\mathbb{N}$, $\beta\geqslant 1$, $q\in\mathbb{R}$. If $\lim_{n\to\infty}(m(n))^{2+\beta}/n=0$, then for all $x\in\mathbb{R}$, uniformly for $0\leqslant m\leqslant m(n)$,*

$$f(x)-\pi^t_{n,m}(x;f)=(-1)^m m!\,a_{n+1}\biggl(\frac{\alpha q}{n^{\alpha+1}}\biggr)^m \operatorname{Re}\{e^{i(n+m+1)x}(1+o(1))\} \tag{3.1}$$

as $n\to\infty$.

Proof. In view of (1.9), for all $x\in\mathbb{R}$ we have

$$\begin{aligned} f(x)-\pi^t_{n,m}(x;f) &= \sum_{k=1}^{\infty}\frac{\widetilde{c}_{n+m+k}}{q^t_m(x)}(e^{i(n+m+k)x}+e^{-i(n+m+k)x}) = 2\operatorname{Re}\sum_{k=1}^{\infty}\frac{\widetilde{c}_{n+m+k}}{q^t_m(x)}e^{i(n+m+k)x} \\ &= 2\operatorname{Re}\biggl\{\frac{\widetilde{c}_{n+m+1}}{q^t_m(x)}e^{i(n+m+1)x}\biggl(1+\sum_{k=2}^{\infty}\frac{\widetilde{c}_{n+m+k}}{\widetilde{c}_{n+m+1}}e^{i(k-1)x}\biggr)\biggr\}. \end{aligned} \tag{3.2}$$

By Lemmas 1 and 4, as $n\to\infty$,

$$\begin{aligned} \frac{\widetilde{c}_{n+m+1}}{q^t_m(x)} &= \frac{\det\widetilde{A}(1)}{\det A(y)} \sim (-q)^m c_{n+1}\prod_{j=1}^{m}\{(n+1-j)^{-\alpha}-(n+1)^{-\alpha}\} \\ &\sim (-q)^m m!\,c_{n+1}\biggl(\frac{\alpha}{n^{\alpha+1}}\biggr)^m = (-1)^m m!\,c_{n+1}\biggl(\frac{\alpha q}{n^{\alpha+1}}\biggr)^m \end{aligned} \tag{3.3}$$

and

$$\frac{\widetilde{c}_{n+m+k}}{\widetilde{c}_{n+m+1}}=\frac{\det\widetilde{A}(k)}{\det\widetilde{A}(1)}\sim\frac{c_{n+k}}{c_{n+1}}\prod_{j=1}^{m}\frac{(n+1-j)^{-\alpha}-(n+k)^{-\alpha}}{(n+1-j)^{-\alpha}-(n+1)^{-\alpha}}.$$

Using the last equivalence, repeating the arguments in the proof of inequality (4.4) in [20] word for word, for $n\geqslant n_0$ we obtain

$$\sum_{k=2}^{\infty}\biggl|\frac{\widetilde{c}_{n+m+k}}{\widetilde{c}_{n+m+1}}\biggr|\leqslant c\sum_{k=2}^{\infty}\biggl(\frac{2|q|(m+1)}{n^{\alpha}}\biggr)^{k-1}=O\biggl(\frac{m}{n^{\alpha}}\biggr). \tag{3.4}$$

Since $2c_{n+1} = a_{n+1}$, relations (3.2)–(3.4) yield the asymptotic equality (3.1), which completes the proof of Theorem 2.

Theorem 3. *Let* $g \in T^{\alpha}_{\beta}(q;d)$, $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$, *and let* $g(x) = f(dx)$, *where* f *is representable by the Fourier series* (1.8). *If* $\lim_{n\to\infty}(m(n))^{2+\beta}/n = 0$, *then for all* $x \in \mathbb{R}$, *uniformly for* $0 \leqslant m \leqslant m(n)$,

$$g(x) - \pi^{t}_{dn+i,\, dm+j}(x;g) = (-1)^m m!\, a_{n+1}\biggl(\frac{\alpha q}{n^{\alpha+1}}\biggr)^m \operatorname{Re}\{e^{i(n+m+1)dx}(1+o(1))\}$$

as $n \to \infty$, *where* $0 \leqslant i + j \leqslant d-1$.

The result of Theorem 3 follows from equalities (1.15) and (3.1).

Now we show that the trigonometric Padé approximants $\pi^t_{n,m}(\,\cdot\,;f)$ converge to f in the class under consideration at a rate asymptotically equal to the highest possible.

Theorem 4. *Let* f *be a function representable by the Fourier series* (1.8) *and let* $f \in T^{\alpha}_{\beta}(q)$, $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$. *If* $\lim_{n\to\infty}(m(n))^{2+\beta}/n = 0$, *then uniformly for* $0 \leqslant m \leqslant m(n)$,

$$\mathrm{R}^t_{n,m}(f) \sim \|f - \pi^t_{n,m}(\,\cdot\,;f)\| \sim m!\,|a_{n+1}|\biggl(\frac{\alpha|q|}{n^{\alpha+1}}\biggr)^m$$

as $n \to \infty$.

Proof. Let

$$\varphi(x) = (-1)^m m!\, a_{n+1}\biggl(\frac{\alpha q}{n^{\alpha+1}}\biggr)^m e^{i(n+m+1)x}.$$

For sufficiently large n the difference $f(x) - \pi^t_{n,m}(x;f)$ has the same sign as $\operatorname{Re}\varphi(x)$. As x ranges over $[0,2\pi)$, $(n+m+1)x$ ranges over $[0, 2\pi(n+m+1))$, therefore there exist $2(n+m+1)$ real numbers x_j, $j = 1,2,\dots,2(n+m+1)$, such that

$$0 \leqslant x_1 < x_2 < \dots < x_{2(n+m+1)} < 2\pi, \qquad \varphi(x_j) = (-1)^{m+j} m!\, a_{n+1}\biggl(\frac{\alpha q}{n^{\alpha+1}}\biggr)^m.$$

Then at the points x_j the difference $f(x) - \pi^t_{n,m}(x;f)$ takes values with alternating signs. Hence by the analogue of the de la Vallée Poussin theorem for trigonometric rational functions (see, for instance, [1], [27])

$$\mathrm{R}^t_{n,m}(f) \geqslant \min_{1\leqslant j\leqslant 2(n+m+1)} |f(x_j) - \pi^t_{n,m}(x_j;f)| \geqslant m!\,|a_{n+1}|\biggl(\frac{\alpha|q|}{n^{\alpha+1}}\biggr)^m (1 - |o(1)|).$$

On the other hand

$$\mathrm{R}^t_{n,m}(f) \leqslant \max_{x\in\mathbb{R}} |f(x) - \pi^t_{n,m}(x;f)| \leqslant m!\,|a_{n+1}|\biggl(\frac{\alpha|q|}{n^{\alpha+1}}\biggr)^m (1 + |o(1)|).$$

The proof of Theorem 4 is complete.

Remark 2. Theorem 1 is a consequence of Theorems 2 and 4.

The following result has a similar proof.

Theorem 5. *Let $g \in T^{\alpha}_{\beta}(q;d)$, $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$, and let $g(x) = f(dx)$, where f is representable by the Fourier series* (1.8)*. If $\lim_{n\to\infty}(m(n))^{2+\beta}/n = 0$, then uniformly for $0 \leqslant m \leqslant m(n)$,*

$$\mathrm{R}^{t}_{dn+i,\,dm+j}(g) \sim \|g - \pi^{t}_{dn,dm}(\,\cdot\,;g)\| \sim m!\,|a_{n+1}|\biggl(\frac{\alpha|q|}{n^{\alpha+1}}\biggr)^{m}$$

as $n \to \infty$, where $0 \leqslant i + j \leqslant d - 1$.

Corollary 2. *Let $m(n) = o(n^{1/4})$. Then uniformly for $0 \leqslant m \leqslant m(n)$,*

$$\mathrm{R}^{t}_{n,m}(f_h) \sim \|f_h - \pi^{t}_{n,m}(\,\cdot\,;f_h)\| \sim \frac{m!\,h^{m}}{n^{2m}}\mathrm{R}^{t}_{n,0}(f_h) \sim \frac{m!\,h^{n+m+1}}{n^{2m}(n+1)!},$$

$$\mathrm{R}^{t}_{2n+i,\,2m+j}(g_h) \sim \|g_h - \pi^{t}_{2n+i,\,2m+j}(\,\cdot\,;g_h)\| \sim \frac{m!\,h^{2m}}{(2n^3)^{m}}\mathrm{R}^{t}_{2n,0}(g_h)$$
$$\sim \frac{m!\,h^{2(n+m+1)}}{2^{m}n^{3m}(2(n+1))!}$$

as $n \to \infty$, where $0 \leqslant i + j \leqslant 1$ and the functions f_h and g_h are defined by (1.13) *and* (1.14).

In [20], in connection with a conjecture due to Dzyadyk (see [28]), it was shown that if f is a defined by a power series then the approximation properties of Padé approximants to f get worse in comparison with Taylor polynomials as the series get more lacunary. Analyzing the results of Corollary 2 while bearing in mind that (1.14) is obtained from (1.13) by deleting the terms with odd indices it is easy to see that a similar phenomenon also occurs in the periodic case. Indeed,

$$\frac{\|f_h - \pi^{t}_{2n+1,\,2m}(\,\cdot\,;f_h)\|}{\|f_h - \pi^{t}_{2(n+m)+1,\,0}(\,\cdot\,;f_h)\|} \sim \frac{(2m)!\,h^{2(n+m+1)}}{(2n)^{4m}(2n+2)!}\,\frac{(2n+2m+2)!}{h^{2(n+m+1)}}$$
$$= (2m)!\,\frac{(2n+2m+2)!}{(2n)^{4m}(2n+2)!} \sim (2m)!\biggl(\frac{1}{2n}\biggr)^{2m} \sim \sqrt{4\pi m}\biggl(\frac{1}{e}\,\frac{m}{n}\biggr)^{2m}$$

as $n \to \infty$, whereas

$$\frac{\|g_h - \pi^{t}_{2n+1,\,2m}(\,\cdot\,;g_h)\|}{\|g_h - \pi^{t}_{2(n+m)+1,\,0}(\,\cdot\,;g_h)\|} \sim \frac{m!\,h^{2(n+m+1)}}{(2n^3)^{m}(2n+2)!}\,\frac{(2n+2m+2)!}{h^{2(n+m+1)}}$$
$$\sim m!\,(4n^2)^{m}\frac{1}{(2n^3)^{m}} \sim m!\biggl(\frac{2}{n}\biggr)^{m} \sim \sqrt{2\pi m}\biggl(\frac{2}{e}\,\frac{m}{n}\biggr)^{m}.$$

Thus, for g_h the advantage of trigonometric Padé approximants over Fourier sums is $(2en/m)^{m}/\sqrt{2}$ times less than for f_h. If the function $P_h \in T^{4}_{2}(h^4/256;4)$ is representable as

$$P_h(x) = \sum_{k=0}^{\infty}\frac{h^{4k}}{(4k)!}\cos 4kx,$$

from Theorem 5, for $0 \leqslant i+j \leqslant 3$ we obtain

$$\mathrm{R}^t_{4n+i,\,4m+j}(P_h) \sim \|P_h - \pi^t_{4n,4m}(\,\cdot\,;P_h)\| \sim m!\,\frac{h^{4(n+m+1)}}{(64n^5)^m(4n+4)!}$$
$$\sim \frac{m!\,h^{4m}}{(64n^5)^m}\mathrm{R}^t_{4n,0}(P_h).$$

It is now easy to show that, as $n\to\infty$, in the case of P_h this advantage is $(4en/m)^m/\sqrt{2}$ times less than for f_h.

So far we have only considered functions f representable as Fourier series (1.8). This case is interesting for applications (see § 4), but in fact all the above results also hold for functions f representable as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx, \tag{3.5}$$

where we impose the same conditions on the Fourier coefficients as before. In particular, if

$$\widetilde{f}_h(x) = e^{h\cos x}\sin(h\sin x) = \sum_{n=1}^{\infty}\frac{h^n}{n!}\sin nx,$$
$$\widetilde{g}_h(x) = \sinh(h\cos x)\sin(h\sin x) = \sum_{n=0}^{\infty}\frac{h^{2n}}{(2n)!}\sin 2nx,$$

then we have the following result.

Corollary 3. *Let $m(n) = o(n^{1/4})$. Then uniformly for $0\leqslant m\leqslant m(n)$,*

$$\mathrm{R}^t_{n,m}(\widetilde{f}_h) \sim \|\widetilde{f}_h - \pi^t_{n,m}(\,\cdot\,;\widetilde{f}_h)\| \sim \frac{m!\,h^m}{n^{2m}}\mathrm{R}^t_{n,0}(\widetilde{f}_h) \sim \frac{m!\,h^{n+m+1}}{n^{2m}(n+1)!},$$
$$\mathrm{R}^t_{2n+i,\,2m+j}(\widetilde{g}_h) \sim \|\widetilde{g}_h - \pi^t_{2n+i,\,2m+j}(\,\cdot\,;\widetilde{g}_h)\| \sim \frac{m!\,h^{2m}}{(2n^3)^m}\mathrm{R}^t_{2n,0}(\widetilde{g}_h)$$
$$\sim \frac{m!\,h^{2(n+m+1)}}{2^m n^{3m}(2(n+1))!}$$

as $n\to\infty$, where $0\leqslant i+j\leqslant 1$.

For Fourier series of the form (3.5) the corresponding theorems have similar proofs, with only slight technical modifications; we will not discuss them here.

§ 4. Some applications

Let $\mathscr{R}_{n,m}$ be the set of algebraic rational functions $r(x) = p_n(x)/q_m(x)$, where p_n and q_m are real algebraic polynomials, $\deg p_n \leqslant n$, $\deg q_m \leqslant m$. For $f \in C[-1,1]$, that is, for a continuous real function f on the interval $[-1,1]$ we consider its best uniform algebraic rational approximations

$$\mathrm{R}_{n,m}(f) = \mathrm{R}_{n,m}(f;[-1,1]) := \inf\{\|f-r\| : r\in\mathscr{R}_{n,m}\},$$

where $\|g\| = \max\{|g(x)| : x \in [-1,1]\}$. With each $f \in C[-1,1]$ we associate a function $\psi(x) = f(\cos x)$. As in the polynomial case (see [8], Ch. 5) we can prove that for all n and m,

$$\mathrm{R}_{n,m}(f;[-1,1]) = \mathrm{R}^t_{n,m}(\psi). \tag{4.1}$$

Suppose that ψ is representable by a Fourier series:

$$\psi(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos kx. \tag{4.2}$$

Then f can be expanded in the corresponding Fourier series in Chebyshev polynomials $T_n(x) = \cos(n\cos^{-1} x)$:

$$f(x) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k T_k(x). \tag{4.3}$$

Furthermore, if f is the restriction to $[-1,1]$ of an entire function, then by Bernstein's theorem (see [29], Ch. 2, § 2) there exists a sequence $\{n_k\}_{k=1}^{\infty} \subset \mathbb{N}$ for which

$$\mathrm{R}_{n_k,0}(f) \sim |A_{n_k+1}| \quad \text{as } k \to \infty. \tag{4.4}$$

Moreover, for "most usual functions the coefficients decrease so regularly that formula (4.4) holds for all n or, in any case, for those n of the same parity" (see [29], Ch. 2, § 2). Regularity conditions were stated by Bernstein in the following form (see [26], paper 3, Ch. 5, § 54).

Proposition 4. *If f is represented by a series (4.3) and*

$$\lim_{n\to\infty} \frac{|A_{n+1}| + |A_{n+2}| + \cdots}{A_n} = 0, \tag{4.5}$$

then as $n \to \infty$,

$$\mathrm{R}_{n,0}(f;[-1,1]) \sim \|f - S_n(\,\cdot\,;f)\| \sim |A_{n+1}|,$$

where $S_n(x;f) = A_0/2 + \sum_{k=1}^{n} A_k T_k(x)$ is a partial sum of the series (4.3).

Using Proposition 4 it is fairly easy (see [26]) to establish the decreasing asymptotics of the $\mathrm{R}_{n,0}(f;[-1,1])$ as $n \to \infty$ for most elementary functions ($\exp x$, $\cos x$, $\sin x$, $\sinh x$, $\cosh x$ and others). For $m \geqslant 1$ finding the asymptotic behaviour of the $\mathrm{R}_{n,m}(f)$ is much more difficult. Classical examples when this problem can be solved are well known (for instance, see the survey [30]):

$$\mathrm{R}_{n,m}(e^x;[-1,1]) \sim \frac{n!\,m!}{2^{n+m}(n+m)!\,(n+m+1)!} \quad \text{as } n+m \to \infty \qquad \text{(Braess [21]);}$$

$$\mathrm{R}_{n,n}(|x|;[-1,1]) \sim 8e^{-\pi\sqrt{n}} \quad \text{as } n \to \infty \qquad \text{(Stahl [23]);}$$

$$\mathrm{R}_{n,n}(x^{\alpha};[0,1]) \sim 4^{1+\alpha}|\sin \pi\alpha| e^{-2\pi\sqrt{\alpha n}} \quad \text{as } n \to \infty \qquad \text{(Stahl [24]);}$$

$$\lim_{n\to\infty} \mathrm{R}^{1/n}_{n,n}(e^{-x};[0,+\infty]) = v \qquad \text{(Gonchar and Rakhmanov [22]);}$$

$$\mathrm{R}_{n,n}(e^{-x};[0,+\infty]) \sim 2v^{n+1/2} \quad \text{as } n \to \infty \qquad \text{(Aptekarev [25]);}$$

here $v = 0.10765\ldots$ is the Halphen constant. Note also Gonchar's paper [31]; it follows from the results there that if F is an analytic function on $[-1, 1]$, then

$$\varlimsup_{n\to\infty} R_{n,m}^{1/n}(f) = \frac{1}{l_m},$$

where l_m is the sum of the half-axes of the largest ellipse with foci at ± 1 such that f extends into its interior as a meromorphic function with at most m poles.

Now we show that the theorems in § 3 enable us to solve similar problems for functions f representable by a Fourier-Chebyshev series (4.3), provided that the coefficients $\{A_k\}_{k=0}^{\infty}$ of the series decrease sufficiently regularly.

By a *Padé-Chebyshev approximant* to a function f defined by a series (4.3) we mean (see, for example, [6]) the rational fraction $\pi_{n,m}^{\mathrm{ch}}(x) = p_n^{\mathrm{ch}}(x)/q_m^{\mathrm{ch}}(x)$ in $\mathscr{R}_{n,m}$ whose numerator and denominator satisfy

$$q_m^{\mathrm{ch}}(x) f(x) - p_n^{\mathrm{ch}}(x) = \sum_{k=n+m+1}^{\infty} c_k T_k(x),$$

where the c_k are real numbers.

Clearly, $\pi_{n,0}^{\mathrm{ch}}(x; f) = S_n(x; f)$. For $m \geqslant 1$ the $\pi_{n,m}^{\mathrm{ch}}(x; f)$ are rational analogues of Fourier partial sums in Chebyshev polynomials. In this connection the next theorem can be viewed as a generalization of Proposition 4.

Theorem 6. *Let $f \in C[-1, 1]$ and assume that the corresponding function ψ can be represented in the form (4.2) and $\psi \in T_\beta^\alpha(q)$, $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$. If*

$$\lim_{n\to\infty} \frac{(m(n))^{2+\beta}}{n} = 0,$$

then uniformly for $0 \leqslant m \leqslant m(n)$,

$$\mathrm{R}_{n,m}(f; [-1, 1]) \sim \|f - \pi_{n,m}^{\mathrm{ch}}(\,\cdot\,; f)\| \sim m!\, |A_{n+1}| \biggl(\frac{\alpha |q|}{n^{\alpha+1}}\biggr)^m$$

as $n \to \infty$.

Proof. Applying Theorem 4 to ψ while taking account of (4.1) and the identity $\pi_{n,m}^t(\cos^{-1} x; \psi) = \pi_{n,m}^{\mathrm{ch}}(x; f)$ we obtain the required result.

In a similar way we can prove the following result.

Theorem 7. *Let $f \in C[-1, 1]$ and assume that the corresponding function ψ can be represented in the form (4.2) and $\psi \in T_\beta^\alpha(q; d)$, where $\alpha \in \mathbb{N}$, $\beta \geqslant 1$, $q \in \mathbb{R}$, $d \in \mathbb{N}$. If*

$$\lim_{n\to\infty} \frac{(m(n))^{2+\beta}}{n} = 0,$$

then uniformly for $0 \leqslant m \leqslant m(n)$,

$$\mathrm{R}_{dn+i,\,dm+j}(f; [-1, 1]) \sim \|f - \pi_{dn,dm}^{\mathrm{ch}}(\,\cdot\,; f)\| \sim m!\, |A_{n+1}| \biggl(\frac{\alpha |q|}{n^{\alpha+1}}\biggr)^m$$

as $n \to \infty$, where $0 \leqslant i + j \leqslant d - 1$.

Analyzing §§ 1–4, it is easy to see that we impose conditions of regularity on the sequence $(A_n)_{n=0}^{\infty}$ such that uniformly for $0 \leqslant m \leqslant m(n)$,

$$\lim_{n\to\infty} \frac{|D_{n,m,2}| + |D_{n,m,3}| + \cdots}{D_{n,m,1}} = 0, \tag{4.6}$$

where

$$D_{n,m,k} = \begin{vmatrix} A_{n+1} & A_n & \cdots & A_{n-m+1} \\ \hdotsfor{4} \\ A_{n+m} & A_{n+m-1} & \cdots & A_n \\ A_{n+m+k} & A_{n+m+k-1} & \cdots & A_{n+k} \end{vmatrix}.$$

By definition $D_{n,0,k} = A_{n+k}$, therefore conditions (4.6) for $m = 0$ coincide with Bernstein's conditions (4.5). The conclusions of Proposition 4 and Theorem 6 also coincide in this case.

Now we illustrate some applications of Theorems 6 and 7 using examples of trigonometric series considered before.

Corollary 4. *Let $m(n) = o(n^{1/4})$ and let*

$$F_h(x) = f_h(\cos^{-1} x) := e^{hx} \cos(h\sqrt{1-x^2}).$$

Then uniformly for $0 \leqslant m \leqslant m(n)$,

$$\mathrm{R}_{n,m}(F_h; [-1,1]) \sim \frac{m!\, h^{n+m+1}}{n^{2m}(n+1)!} \sim \frac{m!\, h^m}{n^{2m}} \mathrm{R}_{n,0}(F_h; [-1,1])$$

as $n \to \infty$.

Corollary 5. *Let $m(n) = o(n^{1/4})$ and let*

$$G_h(x) = g_h(\cos^{-1} x) := \cosh(hx) \cos(h\sqrt{1-x^2}).$$

Then uniformly for $0 \leqslant m \leqslant m(n)$,

$$\mathrm{R}_{2n+i,\,2m+j}(G_h; [-1,1]) \sim \frac{m!\, h^{2(n+m+1)}}{2^m n^{3m}(2(n+1))!} \sim \frac{m!\, h^{2m}}{2^m n^{3m}} \mathrm{R}_{2n,0}(G_h; [-1,1])$$

as $n \to \infty$, where $0 \leqslant i + j \leqslant 1$.

We prove Corollaries 4 and 5 under the assumption that $m(n) = o(n^{1/4})$. We can also show that they hold under the less restrictive condition $m(n) = o(n^{2/3})$. It is plausible that these results hold for $m(n) = o(n)$ and that this condition is best possible (see [19]).

In conclusion we point out that if f is one of the elementary functions $\exp x$, $\cos x$, $\sin x$, $\sinh x$ or $\cosh x$, then we cannot find the decreasing asymptotics of the $\mathrm{R}_{n,m}(f; [-1,1])$ as $m \to \infty$ and $n \to \infty$ using Theorems 6 and 7. Nevertheless there are grounds for optimism as regards our method of analysis. For instance, it is shown in [9] that for all the elementary functions f mentioned above, if m is fixed, the infinitesimals $\mathrm{R}_{n,m}(f; [-1,1])$ and $\|f - \pi^{\mathrm{ch}}_{n,m}(\,\cdot\,; f)\|$ are equivalent as $n \to \infty$.

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