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Strong asymptotics of multiply orthogonal polynomials for Nikishin systems

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Abstract. Strong asymptotic formulae for the Hermite–Padé polynomials for systems of Markov-type functions in the Nikishin class are obtained. The proof is based on the properties of certain rational functions on Riemann surfaces associated with the supports of the measures generating the Nikishin system.

Bibliography: 23 titles.

Introduction

0.1. Definition and statement of the problem. Let

$$\sigma := \{\sigma_\alpha(x)\}_{\alpha=1}^p \qquad (0.1_1)$$

be a collection of positive Borel measures with supports $\operatorname{supp}\sigma_\alpha$ lying in some intervals $E := \{E_\alpha := [a_\alpha, b_\alpha]\}_{\alpha=1}^p$ of the real axis,

$$\operatorname{supp}\sigma_\alpha \subseteq E_\alpha, \qquad \alpha = 1, \dots, p, \qquad (0.1_2)$$

such that

$$E_\alpha \cap E_{\alpha-1} = \varnothing, \qquad \alpha = 2, \dots, p. \qquad (0.1_3)$$

Nikishin [1] has considered a system of measures $\{\mu_\alpha(x)\}_{\alpha=1}^p$ generated by the collection (σ, E) in accordance with the following recursive formulae:

$$\text{(N)}: \quad \begin{aligned} &d\mu_1(x) := d\sigma_1(x), \\ &d\mu_2(x) := d\langle\sigma_1, \sigma_2\rangle(x) := \biggl(\int_{E_2} \frac{d\sigma_2(t)}{x-t}\biggr) d\sigma_1(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &d\mu_\alpha(x) := d\langle\sigma_1, \sigma_2, \dots, \sigma_\alpha\rangle := d\langle\sigma_1, \langle\sigma_2, \dots, \sigma_\alpha\rangle\rangle, \quad \alpha = 2, \dots, p \end{aligned} \qquad (0.2)$$

(the notation $\langle\sigma_1, \dots, \sigma_\alpha\rangle$ has been proposed recently in [2]).

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The measures making up the *Nikishin system* (0.1), (0.2), are concentrated on the same interval E_1. In spite of this, they possess a certain 'independence' property, which suffices for the existence of a *unique* polynomial defined by the system of orthogonality relations

$$\begin{aligned} &(*) \qquad \int Q_{\mathbf{n}}(x)x^{\nu}\,d\mu_{\alpha}(x)=0, \qquad \nu=0,\dots,n_{\alpha}, \quad \alpha=1,\dots,p, \\ &(**) \qquad \deg Q_{\mathbf{n}}(x)\leqslant|\mathbf{n}|:=\sum_{\alpha=1}^{p}n_{\alpha} \end{aligned} \tag{0.3}$$

(for the first proof of the uniqueness $Q_{\mathbf{n}}$ in the case $p=2$ and $\mathbf{n}=(n,n)$, $(n+1,n)$ see [1]; at present, the uniqueness has been established for multi-indices $\mathbf{n}$ such that $n_{\alpha}\leqslant n_{\alpha-1}+1$, $\alpha=2,3,\dots,p$; see, for instance, [2]).

Polynomials $Q_{\mathbf{n}}$ indexed by *vector-valued subscripts* $\mathbf{n}=(n_1,\dots,n_p)$ and satisfying (0.3) are called *multiply orthogonal polynomials with respect to the systems of measures* $\mu=(\mu_1,\dots,\mu_p)$. If $p=1$, then relations (0.3) define ordinary orthogonal polynomials.

Another well-known system of measures μ for which multiply orthogonal polynomials are also uniquely defined is the so-called *Angelesco system*

$$\text{(A)}: \ \{\mu_{\alpha}(x)\}_{\alpha=1}^{p}, \quad \operatorname{supp}\mu_{\alpha}\subseteq E_{\alpha}, \quad E_{\alpha}\cap E_{\beta}=\varnothing, \ \alpha\neq\beta, \ \alpha,\beta=1,\dots,p. \tag{0.4}$$

Multiply orthogonal polynomials with respect to the system (0.4) were considered for the first time in [3] (and, 60 years later, re-discovered by Nikishin [4]). A system of measures generalizing both Angelesco and Nikishin systems on the basis of the concept of 'tree graph' has recently been proposed by Gonchar and his colleagues in [2].

Note that the restrictions on the measures participating in the definitions of the Angelesco and the Nikishin systems are of a 'general' nature and concern only the 'geometry' of the supports of the measures generating these systems. Therefore the corresponding polynomials (0.3) belong to the so-called classes of *general polynomials of multiple orthogonality* (in contrast to polynomials orthogonal with respect to *special weights* and generalizing classical orthogonal polynomials: Jacobi polynomials, Hermite polynomials, and so on; see [5]).

Multiply orthogonal polynomials are the common denominators of the Hermite–Padé rational approximants

$$\pi_{\mathbf{n}}(z):=\left(\frac{P^{(1)}}{Q_{\mathbf{n}}},\dots,\frac{P^{(p)}}{Q_{\mathbf{n}}}\right) \tag{0.5}$$

for systems of Markov functions

$$\widehat{\mu}_{\alpha}(z):=\int\frac{d\mu_{\alpha}(x)}{z-x}, \qquad \alpha=1,\dots,p; \tag{0.6}$$

they are also called *Hermite–Padé polynomials* for that reason. Indeed, it follows from the definition of the Hermite–Padé approximations (see, for instance, [6]) that

$$Q_{\mathbf{n}}(z)\widehat{\mu}_{\alpha}(z)-P^{(\alpha)}(z)=O\left(\frac{1}{z^{n_{\alpha}+1}}\right), \qquad \alpha=1,\dots,p, \tag{0.5$'$}$$

which is equivalent to (0.3).

Multiply orthogonal polynomials are important not only in the approximations of analytic vector-valued functions by rational ones, but also in other domains of mathematics: number theory, the spectral calculus of non-symmetric operators, and the theory of special functions.

In his cornerstone paper [7] Nuttall carried out an investigation of many *special* cases of Hermite–Padé polynomials and formulated on this basis conjectures on the *strong* asymptotic behaviour of *general* multiply orthogonal polynomials. (We point out also Kalyagin's paper [8], in which the author proved strong asymptotic formulae for some special multiply orthogonal polynomials.) A *strong asymptotic formula* for a sequence of polynomials $\{Q_n(z)\}_{n=0}^{\infty}$, $\deg Q_n = n$, as $n \to \infty$ is a formula of the following form:

$$Q_n(z) \sim \Phi^n(z)\bigl(F(z) + o(1)\bigr),$$

where the *leading term* $\Phi(z)$ is an analytic function with first-order pole at infinity (which usually depends on the geometry of the supports of the orthogonality measures) and $F(z)$ (the *Szegő function*) is some function analytic in a neighbourhood of infinity (which is usually defined in terms of the densities of the orthogonality measures). The first results on the strong asymptotic behaviour of general polynomials defined by orthogonality relations were the classical theorems of Bernstein (on the strong asymptotic behaviour of general polynomials orthogonal on an interval, see [9]) and Szegő (for polynomials orthogonal on a circle, see [10]).

This author [11] proved strong asymptotic formulae for general multiply orthogonal polynomials in the Angelesco class. (The leading term of the formula and the convergence of the Hermite–Padé approximants (0.5) for the collection of Markov functions (0.6) generated by the Angelesco system (0.4) has been studied by Gonchar and Rakhmanov in their fundamental paper [12].)

In the present paper we prove a result on the strong asymptotic behaviour of general multiply orthogonal polynomials in the Nikishin class. We point out that the issues of uniqueness and convergence, and also the properties of the leading term in the asymptotic formula for the Hermite–Padé approximants for the Nikishin system have already been thoroughly studied by several authors (see [1], [2], [13]–[18]).

In the next subsection of the introduction we consider a system of polynomials orthogonal with respect to '*variable*' weights (depending on polynomials in the system). These polynomials are crucially important for the solution of the problem under consideration. The formulation of the theorem on the strong asymptotic behaviour proved in this paper (Theorem 1) involves such polynomials. One consequence of this theorem is strong asymptotic formulae for the multiply orthogonal polynomials for Nikishin systems (Theorem 1′). In conclusion we discuss briefly the main points of the proof, which we present in the following sections.

The author wishes to express his gratitude to D. N. Tulyakov and H. Stahl for useful discussions he had with them when he was writing this paper.

0.2. System of polynomials orthogonal with respect to variable weights. A central role in the study of general multiply orthogonal polynomials is played by the associated system of polynomials $\mathbf{q} := \{q_{\alpha,\mathbf{n}}\}_{\alpha=1}^{p}$ satisfying the usual orthogonality relations, but with respect to variable weights depending on the polynomials themselves in this system:

$$\int_{E_\alpha} q_{\alpha,\mathbf{n}}(x)x^\nu H_\alpha(\mathbf{q};x)\,d\sigma_\alpha(x)=0,\quad \nu=0,\dots,\deg q_{\alpha,\mathbf{n}}-1,\ \ \alpha=1,\dots,p. \quad (0.7)$$

Such a system is very easy to produce in the Angelesco case: $q_{\alpha,\mathbf{n}}$ is a polynomial whose zeros are those zeros of the polynomial $Q_\mathbf{n}$ lying in E_α:

$$Q_\mathbf{n}(x)=:\prod_{\alpha=1}^{p} q_{\alpha,\mathbf{n}}(x),\qquad \deg q_{\alpha,\mathbf{n}}=n_\alpha,\qquad \operatorname{Zer}[q_{\alpha,\mathbf{n}}]\subset E_\alpha. \quad (0.8)$$

For polynomials $q_{\alpha,\mathbf{n}}$ in (0.8) relations of multiple orthogonality (0.3), (0.4) give one the system of usual orthogonality relations (0.7) with respect to the variable weights

$$H_\alpha(\mathbf{q};x)\,d\sigma_\alpha(x):=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{p} q_{\beta,\mathbf{n}}(x)\,d\mu_\alpha(x).$$

In the Nikishin case a similar system of polynomials $q_{\alpha,\mathbf{n}}$, $\alpha=1,\dots,p$, is much more difficult to construct. We present this construction now. We have the following result.

Assertion (see, for instance, [2] and [13]). *For vector-valued indices*

$$\mathbf{n}:\quad n_\alpha\leqslant n_{\alpha-1}+1,\ \ \alpha=2,\dots,p,$$

and Borel measures (0.1) *there exists a unique system of polynomials*

$$\mathbf{q}:=\big(q_{1,\mathbf{n}}(x),\dots,q_{p,\mathbf{n}}(x)\big),\qquad \deg q_{\alpha,\mathbf{n}}=\sum_{\beta=\alpha}^{p} n_\beta,$$

satisfying the system of orthogonality relations (0.7), *where the 'variable' weight of constant sign on E_α is as follows*:

$$H_\alpha(\mathbf{q};x):=\frac{h_{\alpha,\mathbf{n}}(x)}{q_{\alpha-1,\mathbf{n}}(x)\,q_{\alpha+1,\mathbf{n}}(x)}\,,\qquad q_{0,\mathbf{n}}\equiv q_{p+1,\mathbf{n}}\equiv 1, \quad (0.9)$$

and the functions $h_{\alpha,\mathbf{n}}$ are recursively defined:

$$h_1(x)=1,\qquad h_{\alpha,\mathbf{n}}(x):=\int_{E_{\alpha-1}}\frac{q^2_{\alpha-1,\mathbf{n}}(t)}{x-t}\frac{h_{\alpha-1,\mathbf{n}}(t)\,d\sigma_{\alpha-1}(x)}{q_{\alpha-2,\mathbf{n}}(t)q_{\alpha,\mathbf{n}}(t)}\,,\quad \alpha=2,\dots,p.$$

The multiply orthogonal polynomial $Q_\mathbf{n}(x)$ for the Nikishin system (0.1), (0.2), *is connected with the $\{q_{\alpha,\mathbf{n}}\}$ as follows*:

$$Q_\mathbf{n}(x)=q_{1,\mathbf{n}}(x). \quad (0.10)$$

It is the asymptotic behaviour of the polynomials $\{q_{\alpha,\mathbf{n}}(x)\}$ that we shall be studying in this paper. We restrict ourselves to the discussion of *diagonal* sequences of vector-valued indices

$$\mathbf{n} = (n, n, \dots, n),$$

therefore we shall denote the vector-valued index $\mathbf{n}$ by n in what follows.

We now discuss the normalization of the polynomials $\{q_{\alpha,n}\}_{\alpha=1}^{p}$. Besides polynomials with leading coefficient one,

$$q_{\alpha,n} := z^{n(p-\alpha+1)} + \cdots, \qquad m_{\alpha,n} := \int_{E_\alpha} q_{\alpha,n}^2(x) \left| \frac{h_{\alpha,n}(x)}{q_{\alpha+1,n}(x)\, q_{\alpha-1,n}(x)} \right| d\sigma_\alpha(x),$$

we shall also consider the polynomials

$$\tilde{q}_{\alpha,n} := k_{\alpha,n} q_{\alpha,n}$$

with the following normalization:

$$1 = \int_{E_\alpha} \tilde{q}_{\alpha,n}^2(x) \left| \frac{\tilde{h}_{\alpha,n}(x)}{q_{\alpha-1,n}(x)\, q_{\alpha+1,n}(x)} \right| d\sigma_\alpha(x), \tag{0.11}$$

where the integral component of the variable weight is normalized as follows:

$$\tilde{h}_{\alpha,n} = l_{\alpha,n} h_{\alpha,n},$$

$$\tilde{h}_{1,n} := 1, \qquad \tilde{h}_{\alpha,n}(x) := \int_{E_{\alpha-1}} \frac{\tilde{q}_{\alpha-1}^2(t)}{x-t} \, \frac{\tilde{h}_{\alpha-1,n}(t)}{q_{\alpha-1,n}(t)\, q_{\alpha,n}(t)} \, d\sigma_{\alpha-1}(t), \tag{0.12}$$

$\alpha = 2, \dots, m$. In that case it is easy to see that

$$l_{1,n} \equiv 1, \qquad l_{\alpha,n} = \prod_{\nu=1}^{\alpha-1} k_{\nu,n}^2,$$

$$k_{1,n} = \frac{1}{m_{1,n}}, \qquad k_{\alpha,n} = \frac{m_{\alpha-1,n}}{m_{\alpha,n}}.$$

We now observe that a weak asymptotic formula is known for the integrand in (0.11). This enables us to simplify considerably the statement of the problem. As was proved by López Lagomasino (see [19], Theorem 9 and also [20]), for polynomials $\{t_n(x)\}_{n=0}^{\infty}$ that are orthonormal on an interval $E = [a, b]$ with respect to the 'variable' weight

$$\frac{d\sigma(x)}{|T_{2n}(x)|},$$

where $d\sigma(x)$ is a measure such that

$$\sigma'(x) > 0 \quad \text{a.e. on } E,$$

$\{T_{2n}(x)\}_{n=0}^{\infty}$ is an arbitrary sequence of polynomials such that

$$T_{2n}(x) := \prod_{\nu=1}^{k}(x - x_{\nu,2n}), \qquad k \leqslant 2n, \quad \{x_{\nu,2n}\} \subset \widetilde{E} \subset \mathbb{R},$$

$\widetilde{E}$ is a compact set, and $E \cap \widetilde{E} = \varnothing$, we have the relation

$$\frac{t_n^2(x)\,d\sigma(x)}{|T_{2n}(x)|} \xrightarrow[n\to\infty]{*} \frac{1}{\pi}\,\frac{dx}{\sqrt{(x-a)(b-x)}} =: d\lambda_{0,E}. \tag{0.13}$$

(We present here a weak version of the statement of Theorem 9 in [19], which is sufficient for applications to Nikishin systems.) We note also that by repeating the proof of asymptotic formula (0.13) in [19] one can show that if one multiplies the variable weight by a continuous coefficient $h_n(x) > 0$, $x \in [a,b]$, such that

$$h_n(x) \rightrightarrows h(x) \in C[a,b] \quad \text{as} \quad n \to \infty,$$

then the orthonormal polynomials $\{\tilde{t}_n(x)\}_{n=0}^{\infty}$ with respect to

$$\frac{h_n(x)\,d\sigma(x)}{|T_{2n}(x)|}$$

also satisfy the relation

$$\frac{\tilde{t}_n^2(x) h_n(x)\,d\sigma(x)}{|T_{2n}(x)|} \xrightarrow[n\to\infty]{*} d\lambda_{0,E}(x). \tag{0.13$'$}$$

A weak asymptotic formula for the integrand in (0.11) and, therefore, a uniform asymptotic formula for the normalized integral component (0.12) of the 'variable' weight for the polynomials $\{q_{\alpha,n}\}$ are now consequences of (0.13′):

$$\begin{gathered}
\tilde{q}_{\alpha,n}^2(x)\left|\frac{\tilde{h}_{\alpha,n}(x)}{q_{\alpha-1,n}(x)\,q_{\alpha+1,n}(x)}\right| d\sigma_\alpha(x) \xrightarrow[n\to\infty]{*} d\lambda_{0,E_\alpha}(x), \qquad \alpha = 1,\dots,p, \\
\tilde{h}_{\alpha,n}(z) \rightrightarrows \widehat{\lambda}_{0,E_{\alpha-1}}(z) := \int_{E_{\alpha-1}} \frac{d\lambda_{0,E_{\alpha-1}}(x)}{z-x}, \qquad \alpha = 2,\dots,p,
\end{gathered} \tag{0.14}$$

where the last limit holds uniformly on compact subsets K of $\overline{\mathbb{C}} \setminus E_{\alpha-1}$.[1]

0.3. Statement of the main result. In view of formula (0.10), which relates the multiple orthogonal polynomials Q_n for the Nikishin system and the collection of polynomials $\{q_{\alpha,n}\}$ orthogonal with respect to the variable weights (0.9), and taking into account the known asymptotic formula for the integral component of the variable weight (0.14), we have reduced the problem of a strong asymptotic

[1]Details of the proof of (0.13′) and (0.14) can be found in the recently published paper [21].

formula for the Q_n to the problem of the asymptotic behaviour of the polynomials $\{q_{\alpha,n}\}_{\alpha=1}^{p}$ satisfying a system of orthogonality relations:

$$\begin{gathered} q_{\alpha,n}(x) = x^{n(p-\alpha+1)} + \cdots, \\ \int_{E_\alpha} q_{\alpha,n}(x) x^{\nu} \frac{\tilde{h}_{\alpha,n}(x)\, d\sigma_\alpha(x)}{q_{\alpha-1,n}(x)\, q_{\alpha+1,n}(x)} = 0, \\ \nu = 0,\dots,n(p-\alpha+1)-1, \qquad \alpha = 1,\dots,p, \qquad q_{0,n} \equiv q_{p+1,n} \equiv 1, \end{gathered} \tag{0.15}$$

where the $\tilde{h}_{\alpha,n}$ are arbitrary positive continuous functions on the intervals E_α such that, as $n \to \infty$, we have

$$\tilde{h}_{\alpha,n}(x) \rightrightarrows h_\alpha(x) > 0, \qquad x \in E_\alpha. \tag{0.16}$$

Besides conditions (0.1_1)–(0.1_3) on the measures $\{d\sigma_\alpha(x)\}$ we also impose the Szegő condition, which is natural in problems relating to strong asymptotic behaviour. That is, we assume that these measures are absolutely continuous:

$$d\sigma_\alpha(x) = \rho_\alpha(x)\, dx, \qquad x \in E_\alpha, \tag{0.17}$$

and that

$$\int_{E_\alpha} \ln \rho_\alpha(x)\, d\lambda_{0,E_\alpha}(x) > -\infty, \qquad \alpha = 1,\dots,p. \tag{0.18}$$

We now introduce some standard functions necessary for the description of the strong asymptotic behaviour of the polynomials $\{q_{\alpha,n}\}$.

Let

$$\mathcal{R}(E) = \overline{\bigcup_{\alpha=0}^{p} \mathcal{R}_\alpha} \tag{0.19}$$

be the $(p+1)$-sheeted Riemann surface with quadratic branch points at the endpoints of the intervals $E_\alpha = [a_\alpha, b_\alpha]$, $\alpha = 1,\dots,p$, and with monodromy matrices

$$M_{a_\alpha} = M_{b_\alpha} := \mathcal{E}_{\alpha-1,\alpha}, \tag{0.20}$$

where $\mathcal{E}_{i,j}$ is the matrix interchanging the ith and the jth components of a vector. That is, $\mathcal{R}$ is formed by the consecutively 'glued' sheets

$$\mathcal{R}_0 := \overline{\mathbb{C}} \setminus E_1, \qquad \mathcal{R}_\alpha := \overline{\mathbb{C}} \setminus \{E_\alpha \cup E_{\alpha+1}\}, \quad \alpha = 1,\dots,p-1, \qquad \mathcal{R}_p = \overline{\mathbb{C}} \setminus E_p$$

where the upper and the lower banks of the slits on two neighbouring sheets are identified. Clearly, this Riemann surface has genus 0 (which can be verified by the Riemann–Hurwitz formula), therefore by fixing an arbitrary divisor (a set of zeros and poles) we can define a unique (up to a multiplicative constant) rational function on $\mathcal{R}$.

We define a rational function $\Psi(z)$ on $\mathcal{R}$ as a function with zero of order p at the point $\infty^{(0)} \in \mathcal{R}_0$ and first-order poles at the points $\infty^{(\alpha)} \in \mathcal{R}_\alpha$ that is regular (with respect to local variables) at other points of $\mathcal{R}$:

$$\Psi(z) := \begin{cases} \Psi(\infty^{(0)}) := \Psi_0(\infty) = \dfrac{1}{C_0 z^p} + \cdots, \\ \Psi(\infty^{(\alpha)}) := \Psi_\alpha(\infty) = \dfrac{z}{C_\alpha} + \cdots, \end{cases} \qquad z \in \mathcal{R}. \tag{0.21}$$

We choose the constant coefficient by imposing the following conditions:

$$\prod_{\alpha=1}^{p} C_\alpha = 1, \qquad C_1 > 0. \tag{0.22}$$

The resulting function Ψ depends on the lengths and the location of the intervals $\{E_\alpha\}$. We shall describe in terms of this function the leading term in the asymptotic formula for $\{q_{\alpha,n}\}$.

The function $\{f_\alpha(z)\}_{\alpha=1}^{p}$ describing the next term of the asymptotic formula (the analogue of the Szegő function) can be defined as the solution of the system of boundary-value problems

$$\begin{aligned} &(1) \quad f_\alpha, \frac{1}{f_\alpha} \in H_{2,\rho_\alpha}(\overline{\mathbb{C}} \setminus E_\alpha), \qquad f_\alpha(\infty) > 0, \quad \alpha = 1, \dots, p, \\ &(2) \quad |f_\alpha(x)|^2 \frac{h_\alpha(x)\rho_\alpha(x)}{|(f_{\alpha+1} f_{\alpha-1})(x)|} = \lambda'_{0,E_\alpha}, \quad x \in E_\alpha, \qquad f_0 \equiv f_{p+1} \equiv 1. \end{aligned} \tag{0.23}$$

We shall show below that the problem (0.23) has a unique solution under the Szegő condition (0.18).

In the present paper we establish the following result.

Theorem 1. *There exists a sequence of polynomials* $\{q_{\alpha,n}\}_{\alpha=1}^{p}$, $n \in \mathbb{N}$, *satisfying the system of orthogonality relations* (0.15) *with weight functions corresponding to conditions* (0.16)–(0.18) *such that, as* $n \to \infty$,

$$(1) \quad \left\| \frac{q_{\alpha,n}(x)}{|c_\alpha \Phi_\alpha(x)|^n} - \left\{ \left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|} \right)^n F_\alpha(x) + \overline{\left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|} \right)^n F_\alpha(x)} \right\} \right\|_{L^2_{\rho_\alpha}(E_\alpha)} = o(1), \tag{0.24$_1$}$$

where c_α *and* $\Phi_\alpha(z)$ *are defined in terms of the function* $\Psi(z)$ *as follows* (see (0.21)):

$$\Phi_\alpha(z) = \prod_{j=\alpha}^{p} \Psi(z^{(j)}), \qquad c_\alpha = \prod_{j=\alpha}^{p} C_j, \tag{0.25}$$

and the function F_α *is the solution of the boundary-value problem* (0.23); *namely,*

$$F_\alpha(z) = \frac{f_\alpha(z)}{f_\alpha(\infty)}, \qquad \alpha = 1, \dots, p; \tag{0.26}$$

$$(2) \quad \frac{q_{\alpha,n}(z)}{(c_\alpha \Phi_\alpha(z))^n} \rightrightarrows F_\alpha(z), \quad z \in K \Subset \overline{\mathbb{C}} \setminus E_\alpha, \tag{0.24$_2$}$$

uniformly on compact subsets K *of* $\overline{\mathbb{C}} \setminus E_\alpha$, $\alpha = 1, \dots, p$.

Remark 1. For the normalized polynomials $\tilde{q}_{\alpha,n}$ (see (0.11)) the theorem establishes the relation

$$\frac{q_{\alpha,n}}{\tilde{q}_{\alpha,n}} = \frac{c_\alpha^n}{f_\alpha(\infty)}\big(1 + o(1)\big), \qquad \alpha = 1, \dots, p.$$

Remark 2. By imposing the additional constraints that the weight functions be positive and smooth, for instance, $h_{\alpha,\rho_\alpha} \in C^{1+}(E_\alpha)$, one can prove by standard methods (see [22] and other papers) that asymptotic formula (0.24_1) holds also with respect to the uniform norm $C(E_\alpha)$. Note also that for $p = 1$ ($h_{1,n} \equiv \text{const}$) our theorem becomes the classical result of Bernstein–Szegő on the strong asymptotic formula for orthogonal polynomials with weight $\rho_1(x)$ on an interval E_1.

0.4. Uniqueness of q. Strong asymptotic formula for multiply orthogonal polynomials for Nikishin systems. In connection with the result of Theorem 1 we now discuss the uniqueness problem for the polynomials $\{q_{\alpha,\mathbf{n}}\}_{\alpha=1}^{p}$ that are defined for fixed $\mathbf{n} = (n_1, \dots, n_p)$ and arbitrary fixed measure $h_{\alpha,\mathbf{n}}\,d\sigma_\alpha$ by the system of orthogonality relations (0.15):

$$\begin{gathered} q_{\alpha,\mathbf{n}}(x) = x^{n_\alpha} + \cdots, \\ \int_{E_\alpha} q_{\alpha,\mathbf{n}}(x) x^\nu \frac{h_{\alpha,\mathbf{n}}(x)\,d\sigma_\alpha(x)}{(q_{\alpha-1,\mathbf{n}} q_{\alpha+1,\mathbf{n}})(x)} = 0, \\ \nu = 0, \dots, n_\alpha - 1, \qquad \alpha = 1, \dots, p, \qquad q_{0,\mathbf{n}} \equiv q_{p+1,\mathbf{n}} \equiv 1. \end{gathered}$$

The point is that, by Theorem 1, there exists (under certain limiting conditions on $h_{\alpha,n}$) a strong asymptotic formula for *some* sequence of solutions of the non-linear system (0.15). (The existence of $\{q_{\alpha,\mathbf{n}}\}_{\alpha=1}^{p}$, satisfying (0.15) is a consequence of Brouwer's fixed-point theorem.) Hence information on the uniqueness of $\{q_{\alpha,\mathbf{n}}\}_{\alpha=1}^{p}$ would render a more definite character to Theorem 1.

For $p = 2$ it is easy to show that the system (0.15) is uniquely soluble for all $\mathbf{n} = (n_1, n_2)$. In fact, were there two solutions (q_1, q_2) and $(\bar{q}_1, \bar{q}_2)$, then by orthogonality relations (0.15) the rational function $\dfrac{q_1}{q_2} - \dfrac{\bar{q}_1}{\bar{q}_2}$ would make at least n_1 changes of sign on E_1, and the function $\dfrac{q_2}{q_1} - \dfrac{\bar{q}_2}{\bar{q}_1}$ would make at least n_2 changes of sign on E_2. Thus, the polynomial $q_1\bar{q}_2 - q_2\bar{q}_1$ of degree $n_1 + n_2 - 1$ would have at least n_1 zeros on the interval E_1 and n_2 zeros on E_2, which would lead to a contradiction.

However, for $p \geqslant 3$ the uniqueness problem for $\mathbf{q}$ is much more complicated and, as shown by Tulyakov, the system (0.15) can have several solutions.

Example (Tulyakov)**.** Let $d\sigma_\alpha(x) = dx$ on the segments $E_1 \equiv E_3 \equiv [-150, -1]$ and $E_2 = [0, 120]$, and let

$$\begin{aligned} h_1 := h_3(x) &:= \begin{cases} 1, & x \in E_1^* = [-150, -149.990] \cup [-1.0220, -1], \\ 0, & x \in E_1 \setminus E_1^*, \end{cases} \\ h_2(x) &:= \begin{cases} 1, & x \in E_2^* = [0, 0.01333\ldots] \cup [119.333\ldots, 120], \\ 0, & x \in E_2 \setminus E_2^*. \end{cases} \end{aligned}$$

Then for each $\mathbf{n} = (1,1,1)$ there exist three collections of polynomials $\mathbf{q}^{(\beta)}$:

$$\begin{aligned} \mathbf{q}^{(1)} &= \{(x + 16.833\ldots); (x - 51.7158\ldots); (x + 16.833\ldots)\}, \\ \mathbf{q}^{(2)} &= \{(x + 4.163\ldots); (x - 6.4270\ldots); (x + 4.163\ldots)\}, \\ \mathbf{q}^{(3)} &= \{(x + 2.757\ldots); (x - 2.98\ldots); (x + 2.757\ldots)\}, \end{aligned}$$

satisfying the system of orthogonality relations (0.15).

The question of the uniqueness of solutions to (0.15) for large n or the existence of common strong asymptotic formulae for all solutions (in the case when the limiting conditions (0.16) on $h_{\alpha,n}$ are satisfied) is still open.

Nevertheless, if we do not treat the $h_{\alpha,n}$ as arbitrary weights satisfying only relation (0.16), but take account of their particular nature, as specified for fixed $\mathbf{n}$ in the case of Nikishin systems by relations (0.12), then the uniqueness of the collection of polynomials satisfying (0.15), (0.12) is a consequence of the uniqueness of the polynomial of multiple orthogonality $Q_{\mathbf{n}}$ for the Nikishin system. Thus, Theorem 1 gives us the following result.

Theorem 1′. *Let Q_n be the diagonal sequence ($\mathbf{n} = (n, \dots, n)$) of multiply orthogonal polynomials* (0.3) *for the Nikishin system* (0.2) *generated by absolutely continuous measures* (0.17) *in the Szegő class* (0.18). *Then the following strong asymptotic formulae hold for Q_n as $n \to \infty$:*

$$(1)\quad \left\| \frac{Q_n(x)}{|c_1\Phi_1(x)|^n} - \left\{ \left(\frac{\Phi_1(x)}{|\Phi_1(x)|}\right)^n F_1(x) + \overline{\left(\frac{\Phi_1(x)}{|\Phi_1(x)|}\right)^n F_1(x)} \right\} \right\|_{L^2_{\rho_1}(E_1)} = o(1),$$

$$(2)\quad \left\| \frac{Q_n(z)}{(c_1\Phi_1(z))^n} - F_1(z) \right\|_{C(K)} = o(1) \quad \text{for each } K \Subset \overline{\mathbb{C}} \setminus E_1,$$

where the functions Φ_1 and F_1 are defined by relations (0.25), (0.26).

0.5. Scheme of the proof of Theorem 1. Structure of the paper. The next two sections are devoted to the proof of Theorem 1. The proof repeats in its main points the scheme of the proof of the strong asymptotic formula for the Angelesco case proposed in [11].

The existence of a sequence of polynomials $\mathbf{q}$ satisfying the system of orthogonality relations (0.15) and having strong asymptotic behaviour (0.24_1), (0.24_2), will be a consequence of the existence of some sequence of polynomials $\mathbf{P}_n$ with strong asymptotic behaviour (0.24_1), (0.24_2) (but not necessarily satisfying the orthogonality relations). That is, Theorem 1 will be obtained as a consequence of the following weaker result.

Theorem 2. *Let $\{F_\alpha(z)\}_{\alpha=1}^p$ be a solution of the problem* (0.23), (0.26), *let $\{C_\alpha\}_{\alpha=1}^p$ be the constants, and $\{\Phi_\alpha(z)\}_{\alpha=1}^n$ the algebraic functions and defined by* (0.21), (0.22), (0.25). *Then there exists a sequence of polynomials $\mathbf{P}_n = \{P_\alpha\}_{\alpha=1}^p$, $P_\alpha(x) = x^{n_\alpha} + \cdots$, $n_\alpha = n(p - \alpha + 1)$, such that*

$$(1)\quad \left\| \frac{P_\alpha(x)}{|c_\alpha\Phi_\alpha(x)|^n} - \left\{ \left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n F_\alpha(x) + \overline{\left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n F_\alpha(x)} \right\} \right\|_{L^2_{\rho_\alpha}(E_\alpha)} = o(1),$$

$$(2)\quad \left\| \frac{P_\alpha(z)}{(c_\alpha\Phi_\alpha(z))^n} - F_\alpha(z) \right\|_{C(K)} = o(1) \quad \text{for each } K \Subset \overline{\mathbb{C}} \setminus E_1.$$

The reduction of Theorem 1 to Theorem 2 is discussed in § 1. There we regard solutions of systems of boundary-value problems (0.23) for analytic functions and solutions of systems of orthogonality relations (0.15) for polynomials as fixed points of certain non-linear maps $\mathbf{T}$ and $\mathbf{T}_n$. To establish Theorem 1 we shall show that there exist fixed points of $\mathbf{T}_n$ in a neighbourhood of a fixed point of $\mathbf{T}$ as $n \to \infty$.

Section 2 is devoted to the study of certain properties of rational functions on the Riemann surface $\mathcal{R}$ (see (0.19)), in terms of which we construct a sequence of polynomials required in Theorem 2.

At the end of the paper we present an appendix devoted to an extremal property of orthogonal polynomials which is useful for reductions similar to the transition from Theorem 2 to Theorem 1.

§ 1. Properties of fixed points of the maps T and $\mathbf{T}_n$. Proof of Theorem 1

1.1. Definition of the map T. First, we describe spaces that are necessary for the definition of the maps $\mathbf{T}$ and $\mathbf{T}_n$. We shall denote by $H_{2,\rho}$ the Hilbert space of vector-valued analytic functions with components in the Hardy space H_{2,ρ_α}:

$$H_{2,\rho}(\Omega) \ni \mathbf{f} = (f_1, \dots, f_p): \quad f_\alpha \in H_{2,\rho_\alpha}(\Omega_\alpha), \;\; \Omega_\alpha = \overline{\mathbb{C}} \setminus E_\alpha, \;\; \alpha = 1, \dots, p,$$

and with norm

$$\|\mathbf{f}\|^2_{H_{2,\rho}} := \max_{\alpha=1,\dots,p} \|f_\alpha\|^2_{L^2_{\rho_\alpha}(\Gamma(\Omega_\alpha))} := \max_\alpha \oint_{\Gamma(\Omega_\alpha)} |f_\alpha(\xi)|^2 \rho_\alpha(\xi)\,|d\xi|.$$

Further, let $H(\Omega)$ be the locally convex space of vector-valued analytic functions

$$H(\Omega) \ni \mathbf{f} = (f_1, \dots, f_p): \quad f_\alpha \in H(\Omega_\alpha), \;\; \alpha = 1, \dots, p,$$

with family of norms

$$\|\mathbf{f}\|_{H(\Omega)} := \bigl\{\|\mathbf{f}\|^{(K)}_{H(\Omega)}\bigr\}_K: \quad \|\mathbf{f}\|^{(K)}_{H(\Omega)} = \max_{\alpha=1,\dots,p} \{\max_{z\in K_\alpha} |f_\alpha(z)|\} \quad \text{for } K = \{K_\alpha\} \Subset \Omega,$$

where the K_α are arbitrary compact subsets of Ω_α, $\alpha = 1, \dots, p$. Fixing some norm in $H(\Omega)$ we obtain a normed (non-complete) space, which we denote by $H_K(\Omega)$. Choosing a compact set K with interior points $\mathring{K} : K = \mathring{K} \cup \Gamma(\mathring{K})$, such that

$$K = \{K_\alpha\}, \qquad K_\alpha: \quad E_{\alpha-1} \cup E_{\alpha+1} \subset \mathring{K}_\alpha \subset K_\alpha \Subset \Omega_\alpha,$$
$$\alpha = 1, \dots, p, \qquad E_{-1} = E_{p+1} = \varnothing,$$

we obtain a particular case of the space $H_K(\Omega)$, which we denote by

$$H_{\mathbf{E}} \ni \mathbf{f} = (f_1, \dots, f_p); \qquad \|\mathbf{f}\|_{H_{\mathbf{E}}} = \max_{\alpha=1,\dots,p} \Bigl\{\max_{z\in K_\alpha} |f_\alpha(z)|\Bigr\}.$$

Finally, we consider the Banach space of continuous vector-valued functions

$$C_{\mathbf{E}} \ni \mathbf{f} = (f_1, \dots, f_p): \quad f_\alpha \in C\bigl(E_{\alpha-1} \cup E_{\alpha+1}\bigr), \;\; \alpha = 1, \dots, p, \;\; E_{-1} = E_{p+1} = \varnothing,$$

with norm

$$\|\mathbf{f}\|_{C_{\mathbf{E}}} := \max_{\alpha=1,\dots,p} \bigl\{\|f_\alpha\|_{C(E_{\alpha-1}\cup E_{\alpha+1})}\bigr\}. \tag{1.1}$$

Note the following chain of embeddings for the spaces so introduced (regarded as sets of functions):

$$H_{2,\rho} \subset H_{\mathbf{E}} \subset C_{\mathbf{E}}.$$

We shall denote the cones in these spaces containing the elements with components symmetric in $\mathbb{C}$ (with respect to the real axis) and non-negative on $\mathbb{R}$ (within their domains of definition) by

$$\widehat{H}_{2,\rho}, \quad \widehat{H}_{\mathbf{E}}, \quad \widehat{C}_{\mathbf{E}};$$

we also denote the cones containing the elements with components symmetric (with respect to the real axis) and non-vanishing (and positive on $\mathbb{R}$) by

$$\widehat{H}^+_{2,\rho}, \quad \widehat{H}^+_{\mathbf{E}}, \quad \widehat{C}^+_{\mathbf{E}}.$$

In the cone $\widehat{C}^+_{\mathbf{E}}$, besides the norm (1.1) we shall also consider the metric

$$d(\mathbf{f},\mathbf{g}) := \max_{\alpha=1,\dots,p} \bigl\{ d_{C(E_{\alpha-1}\cup E_{\alpha+1})}(f_\alpha, g_\alpha) \bigr\}, \tag{1.2}$$

where

$$d_{C(X)}(f,g) = \max_{x\in X} \left| \ln \frac{f(x)}{g(x)} \right|, \qquad f(x), g(x) > 0, \quad x \in X.$$

In addition, for the components of the vector-valued functions in question lying in the Szegő class (1.18) we shall use the metric

$$d_{L(X)}(f,g) = \int_X \left| \ln \frac{f(x)}{g(x)} \right| dx.$$

We now make two observations relating to the norm introduced above and the metric in the cone $\widehat{C}^+_{\mathbf{E}}$.

Remark 1.1. The set $\widehat{C}^+_{\mathbf{E}}$ is open in the norm topology, however $\widehat{C}^+_{\mathbf{E}}$ is complete with respect to the metric d. For taking the logarithms of functions in the cone transforms $\widehat{C}^+$ into C in one-to-one fashion and the induced metric coincides with the one corresponding to the norm in C.

Remark 1.2. The topologies in $\widehat{C}^+_{\mathbf{E}}$ corresponding to the norm and the metric are locally consistent. That is, if $\mathbf{g}$ lies in an r-neighbourhood of $\mathbf{f} \in \widehat{C}_{\mathbf{E}}$ such that

$$\|\mathbf{f}-\mathbf{g}\|_{C_{\mathbf{E}}} \leqslant \theta r, \qquad \theta < 1, \qquad r(\mathbf{f}) := \left\| \left(\frac{1}{f_1}, \dots, \frac{1}{f_2} \right) \right\|^{-1}_{C_{\mathbf{E}}}, \tag{1.3}$$

then

$$\ln\left(1 + \frac{\|\mathbf{f}-\mathbf{g}\|}{\|\mathbf{f}\|}\right) \leqslant d(\mathbf{f},\mathbf{g}) \leqslant \ln\left(1 + \frac{\|\mathbf{f}-\mathbf{g}\|}{r - \|\mathbf{f}-\mathbf{g}\|}\right).$$

We now define a map

$$\mathbf{T}\colon \widehat{C}^+_{\mathbf{E}} \to \widehat{H}^+_{2,\mathbf{w}},$$

associating with a function $\mathbf{f} \in \widehat{H}^+_{\mathbf{E}}$ the vector-valued function $\mathbf{T}\mathbf{f} = (Tf_1, \dots, Tf_p)$ in $\widehat{H}^+_{2,\mathbf{w}}$ such that its components solve the following boundary-value problem:

$$\begin{gathered} Tf_\alpha \in \widehat{H}^+_{2,w_\alpha}(\Omega_\alpha), \qquad |Tf_\alpha(x)|^2 = \frac{f_{\alpha-1}(x) f_{\alpha+1}(x)}{w_\alpha(x)}, \\ x \in E_\alpha, \qquad \alpha = 1, \dots, p, \qquad f_0 \equiv f_{p+1} \equiv 1, \end{gathered} \tag{1.4}$$

where $w_\alpha(x)$ is a fixed integrable function on E_α satisfying Szegő condition (0.18).

1.2. Properties of the map T. Unique solubility of the system of boundary-value problems for analytic functions. The map $\mathbf{T}$ is well defined in the cone $\widehat{H}^+(\Omega)$, and its values can be found by the solution of the Dirichlet problem for a harmonic function $\ln|Tf_\alpha|$ in Ω_α with boundary values integrable on $\Gamma(\Omega_\alpha)$ and equal to $\frac{1}{2}\ln(f_{\alpha+1}f_{\alpha-1}/w)$ and the subsequent solution of the problem of finding the harmonically conjugate function $\widetilde{\ln|Tf_\alpha|}$:

$$Tf_\alpha = \exp\bigl\{\ln|Tf_\alpha| + i\,\widetilde{\ln|Tf_\alpha|}\bigr\}.$$

The properties of solutions of the Dirichlet problem show (in view of Remark 1.2) that $\mathbf{T}$ is continuous, that is,

$$\|\mathbf{f}^{(n)}-\mathbf{f}\|_{C_{\mathbf{E}}} = o(1) \Rightarrow \|\mathbf{Tf}^{(n)}-\mathbf{Tf}\|_{H_{2,\rho}} = o(1) \Rightarrow \|\mathbf{Tf}^{(n)}-\mathbf{Tf}\|_{C_{\mathbf{E}}} = o(1). \quad (1.5)$$

Proposition 1.1. *The map* $\mathbf{T}$ *is a contraction with respect to the metric* d, *that is,*

$$d(\mathbf{Tf}^{(1)},\mathbf{Tf}^{(2)}) \leqslant \gamma\, d(\mathbf{f}^{(1)},\mathbf{f}^{(2)}), \quad (1.6)$$

where the constant $\gamma<1$ *depends only on the location of the intervals* E_α *satisfying condition* (0.1_3), $\alpha=1,\dots,p$.

Proof. For $\mathbf{f}\in\widehat{H}^+_{2,\rho}$ let

$$\psi := (\psi_1,\dots,\psi_p) := \bigl(\ln|f_1|,\dots,\ln|f_2|\bigr) \in \mathbf{h}(\Omega) := \bigotimes_{\alpha=1}^{p}\bigl\{\operatorname{Harm}(\Omega_\alpha)\cup L(E_\alpha)\bigr\},$$

where the components of ψ are harmonic functions in Ω with integrable boundary values.

The map $\mathbf{T}\colon \widehat{H}^+_{2,\rho}\to\widehat{H}^+_{2,\rho}$ induces the map

$$t\colon \mathbf{h}(\Omega)\to\mathbf{h}(\Omega),$$

so that

$$t\psi := \bigl(\ln|Tf_1|,\dots,\ln|Tf_p|\bigr),$$

and by the definition (1.4) of $\mathbf{T}$ we obtain

$$t\psi := \frac{1}{2}P\psi + \beta,$$

where β is the vector made up of harmonic functions

$$\beta_\alpha(z)\in\operatorname{Harm}(\Omega_\alpha), \qquad \beta_\alpha(x)=\ln w_\alpha(x) \quad \text{a.e. on } E_\alpha, \qquad \alpha=1,\dots,p,$$

and P is the linear operator

$$P := \begin{pmatrix} 0 & P_{1,2} & 0 & \dots & 0 \\ P_{2,1} & 0 & P_{2,3} & \dots & \dots \\ 0 & P_{3,2} & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & P_{p-1,p} \\ 0 & \dots & \dots & P_{p,p-1} & 0 \end{pmatrix}$$

such that

$$P_{i,j}\colon \operatorname{Harm}(\Omega_j) \to \operatorname{Harm}(\Omega_i)$$

is the following map:

$$P_{ij}\psi_j \in \operatorname{Harm}(\Omega_i), \qquad P_{ij}\psi_j(x) = \psi_j(x) \quad \text{a.e. on } E_i,$$

(that is, the operator P_{ij} associates with the values on the interval $E_i \subset \Omega_j$ of a function ψ_j harmonic in Ω_j the function $P_{ij}\psi_j$ harmonic in $\Omega_i = \overline{\mathbb{C}} \setminus E_i$.

The definition of the map t shows that, similarly to $\mathbf{T}$, it can be considered in a broader space, namely,

$$t\colon C_{\mathbf{E}} \to \mathbf{h}(\Omega).$$

We note also that the d-metric (1.2) in the cone $\widehat{C}_{\mathbf{E}}^{+}$ is transformed into the metric induced by the norm (1.1) in $C_{\mathbf{E}}$:

$$d(\mathbf{f}^{(1)}, \mathbf{f}^{(2)}) = \|\psi^{(1)} - \psi^{(2)}\|_{C_{\mathbf{E}}}.$$

Hence

$$\begin{aligned} d(\mathbf{T}\mathbf{f}^{(1)}, \mathbf{T}\mathbf{f}^{(2)}) &= \|t\psi^{(1)} - t\psi^{(2)}\|_{C_{\mathbf{E}}} = \frac{1}{2}\big\|P(\psi^{(1)} - \psi^{(2)})\big\| \\ &\leqslant \frac{1}{2}\|P\|\,\|\psi^{(1)} - \psi^{(2)}\| = \frac{\|P\|}{2}\, d(\mathbf{f}^{(1)}, \mathbf{f}^{(2)}). \end{aligned}$$

It remains to observe that, as follows from the definition (1.1) of the norm $\|\cdot\|_{C_{\mathbf{E}}}$ and the maximum principle for harmonic functions,

$$\begin{aligned} \|P\psi\| &= \max\bigl\{\|P_{1,2}\psi_2\|_{C(E_2)}; \|P_{2,1}\psi_1 + P_{2,3}\psi_3\|_{C(E_1\cup E_3)}; \dots; \|P_{p,p-1}\psi_{p-1}\|_{C(E_{p-1})}\bigr\} \\ &< \max\bigl\{\|\psi_2\|_{C(E_1)}; 2\max\{\|\psi_1\|_{C(E_2)}, \|\psi_3\|_{C(E_2)}\}; \dots; \|\psi_{p-1}\|_{C(E_p)}\bigr\} \leqslant 2\|\psi\|. \end{aligned}$$

Thus, there exists a constant $\delta > 0$ dependent only on the geometry of the set $E = \{E_\alpha\}_{\alpha=1}^{p}$ such that

$$\|P\| \leqslant 2 - \delta.$$

Setting $\gamma = 1 - \delta/2$ we obtain the assertion of Proposition 1.1.

The cone $\widehat{C}_{\mathbf{E}}^{+}$ is complete with respect to the metric d (see Remark 1.1), therefore Proposition 1.1 yields the following result.

Corollary 1.1. *The map* $\mathbf{T}$ *has a unique fixed point* $\mathbf{f}^{(\infty)} \in \widehat{H}_{2,\rho}^{+}$:

$$\mathbf{f}^{(\infty)} = \mathbf{T}\mathbf{f}^{(\infty)}.$$

Further, the unique fixed point of the map $\mathbf{T}$ *with*

$$w_\alpha = \frac{h_\alpha \rho_\alpha}{\lambda'_{0,E_\alpha}}, \qquad \alpha = 1, \dots, p \tag{1.7}$$

(*see* (1.4)) *is the unique solution of the system of boundary-value problems* (0.23).

We now make two observations which can be useful in what follows.

Remark 1.3. We now specify the domain of the map $\mathbf{T}$. In fact, it is not the whole of $\widehat{H}^+_{2,\rho}$, but the subset of *outer* functions $\widehat{H}^+_{2,\rho}$ (that is, the functions that can be recovered by the boundary values of their moduli by means of the solution of the Dirichlet problem). Thus, for each $\mathbf{w}$ with integrable components satisfying the Szegő conditions the fixed point of $\mathbf{T}$ lies in this subset $\widehat{H}^+_{2,\rho}$. Moreover, the converse is also obvious: *each outer function in* $\widehat{H}^+_{2,\rho}$ *can be represented as the solution of the system of boundary-value problems* (0.23) *with some integrable functions* h_α *satisfying the Szegő condition.* This concretization is not very essential, though, because the outer functions are dense in $\widehat{H}^+_{2,\rho}$, which is easy to verify in the scalar case of $\widehat{H}^+_{2,\rho}(D)$, where D is the unit disc. In that case each function $f(z) \in \widehat{H}^+_{2,\rho}(D)$ can be approximated by outer functions $f_r(z) = f(rz)$, $r < 1$, as $r \to 1$. In fact,

$$\|f - f_r\|^2 = \|f_r\|^2 - \|f\|^2 + 2\langle f, f - f_r\rangle \to 0,$$

where the difference of the first two terms on the right-hand side approaches zero by the definition of $H_{2,\rho}(D)$, and the last term converges to zero by Fatou's theorem (on the existence of non-tangential limit values). The verification in $\widehat{H}^+_{2,\rho}(\Omega)$ can be carried out in a similar way.

Remark 1.4. We point out the following method of approximation of the solution of the system of boundary-value problems (0.23). If $\mathbf{w}$ and $\mathbf{w}^{(m)}$ have integrable components satisfying the Szegő condition, then it follows from the relations

$$\|w_\alpha - w_\alpha^{(m)}\|_{L(E_\alpha)} = o(1), \quad d_{L(E_\alpha)}(w_\alpha, w_\alpha^{(m)}) = o(1), \qquad \alpha = 1, \dots, p,$$

that

$$\|\mathbf{f}^{(m)} - \mathbf{f}\|_{H_2,\mathbf{w}} = o(1).$$

(See [10], [22] in the scalar case; the proof in the vector case repeats the arguments in [11].)

1.3. Definition and properties of the map $\mathbf{T}_{\boldsymbol{n}}$. Let $H^n(\Omega)$ be the n-dimensional subspaces of $H_{2,\rho}(\Omega)$ with elements

$$H^n \ni \left(\frac{P_{1,n}(z)}{\Phi_1(z)^n}, \dots, \frac{P_{p,n}(z)}{\Phi_p^n(z)}\right), \tag{1.8}$$

where the $P_{\alpha,n} \in \mathcal{P}_{n(p-\alpha+1)}$ are polynomials, $\deg P_{\alpha,n} \leqslant n_\alpha = n(p - \alpha + 1)$, and the functions $\Phi_\alpha(z)$ are defined in (0.25), $\alpha = 1, \dots, p$.

Accordingly, $\widehat{H}^{+n}(\Omega)$ is the cone $\widehat{H}^{+n} \subset H^n$ of elements whose αth components may vanish only on E_α, $\alpha = 1, \dots, p$. Note that

$$\widehat{H}^{+n} \subset \widehat{H}^+_{2,\rho} \subset \widehat{H}^+_{\mathbf{E}} \subset \widehat{C}^+_{\mathbf{E}}.$$

In view of Remark 1.3, Theorem 2 formulated in § 0.5 (on the existence of a sequence of polynomials with fixed asymptotic behaviour) can be stated as follows.

Theorem 2′. *The family of cones* $\{\widehat{H}^{+n}\}_{n=0}^{\infty}$ *is dense in* $\widehat{H}^{+}_{2,\rho}$ *in the norms of* $H(\Omega)$. *That is, for each* $\mathbf{f}\in\widehat{H}^{+}_{2,\rho}$ *there exist* $\mathbf{f}^{(n)}\in\widehat{H}^{+n}$ *such that*

$$\|\mathbf{f}^{(n)}-\mathbf{f}\|_{H(\Omega)}=o(1);$$

moreover,

$$\left\|\frac{\Phi_\alpha^n}{|\Phi_\alpha|^n}f_\alpha^{(n)}-\left\{\frac{\Phi_\alpha^n}{|\Phi_\alpha|^n}f_\alpha+\overline{\frac{\Phi_\alpha^n}{|\Phi_\alpha|^n}f_\alpha}\right\}\right\|_{L_{2,\rho_\alpha}(E_\alpha)}=o(1),\qquad \alpha=1,\dots,p.$$

We consider now the map

$$\mathbf{T}_n\colon \widehat{C}^{+}_{\mathbf{E}}\to\widehat{H}^{+n}$$

associating with $\mathbf{f}\in\widehat{C}^{+}_{\mathbf{E}}$ the vector-valued function $\mathbf{T}_n\mathbf{f}=(T_nf_1,\dots,T_nf_p)$ in $\widehat{H}^{+n}$ with components satisfying the following system of orthogonality relations:

$$\begin{gathered}\int_{E_\alpha}(T_nf_\alpha)(x)\frac{x^\nu}{\overline{\Phi_\alpha^n(x)}}\,\frac{\tilde{h}_{\alpha,n}(x)\rho_\alpha(x)\,dx}{(f_{\alpha-1}f_{\alpha+1})(x)}=0,\\ \nu=0,\dots,n_\alpha-1,\quad \alpha=1,\dots,p,\qquad f_0\equiv f_{p+1}\equiv 1.\end{gathered}\tag{1.9}$$

Considering its restriction to functions $\mathbf{f}=\left\{\dfrac{P_{\alpha,n}}{\Phi_\alpha^n}\right\}_{\alpha=1}^{p}\in\widehat{H}^{+n}$, the map

$$\mathbf{f}\to\mathbf{T}_n\mathbf{f}=\left\{\frac{\widetilde{T}_nP_{\alpha,n}}{\Phi_\alpha^n}\right\}_{\alpha=1}^{p}\in\widehat{H}^{+n},$$

and using the following property of the functions $\{\Phi_\alpha\}_{\alpha=1}^{p}$:

$$\frac{|\Phi_\alpha(x)|^2}{(\Phi_{\alpha-1}\Phi_{\alpha+1})(x)}=1,\qquad x\in E_\alpha,\quad \alpha=1,\dots,p,\qquad \Phi_0\equiv\Phi_{p+1}\equiv 1,\tag{1.10}$$

which is a consequence of the definition (0.25) (see § 2 for greater detail), we obtain the following representation for the map $\widetilde{\mathbf{T}}_n$ induced on the cone $\widehat{\mathcal{P}}^{+}_{n}$ of polynomials $\{P_{\alpha,n}\}_{\alpha=1}^{p}$:

$$\begin{gathered}\widetilde{\mathbf{T}}_n\colon\widehat{\mathcal{P}}^{+}_{n}\to\widehat{\mathcal{P}}^{+}_{n},\\ \int_{E_\alpha}(\widetilde{T}_nP_{\alpha,n})(x)x^\nu\frac{(\tilde{h}_{\alpha,n}\rho_\alpha)(x)\,dx}{(P_{\alpha-1,n}P_{\alpha+1,n})(x)}=0,\\ \nu=0,\dots,n_\alpha-1,\quad \alpha=1,\dots,p,\qquad P_{0,n}\equiv P_{p+1,n}\equiv 1.\end{gathered}\tag{1.9′}$$

For fixed n the maps $\widetilde{\mathbf{T}}_n$ and $\mathbf{T}_n$ are continuous (which follows from the continuous dependence of the coefficients of an orthogonal polynomial on the moments of the orthogonality measure).

We consider now the restriction of $\widetilde{\mathbf{T}}_n$ to a set $\widetilde{\mathcal{P}}_n$ such that

$$\widetilde{\mathbf{T}}_n[\widehat{\mathcal{P}}_n^+] \subset \widetilde{\mathcal{P}}_n \subset \widehat{\mathcal{P}}_n^+;$$

namely, the elements $x \in \widetilde{\mathcal{P}}_n$ can be represented in the form $x = \bigcup_\alpha \{x_j^{(\alpha)}\}_{j=1}^{n_\alpha}$, where $\{x_j^{(\alpha)}\}$ is the zero set of the polynomial $P_{\alpha,n}$ and has the following properties:

$$\{x_j^{(\alpha)}\}_{j=1}^{n_\alpha} \subset E_\alpha, \qquad x_1^{(\alpha)} < x_2^{(\alpha)} < \dots < x_{n_\alpha}^{(\alpha)}, \quad \alpha = 1, \dots, p.$$

We introduce in $\mathbb{R}^{\sum_{\alpha=1}^p n_\alpha}$ coordinates such that

$$x + y = \bigcup_\alpha \Bigl\{(x_j^{(\alpha)} + y_j^{(\alpha)})\Bigr\}_{j=1}^{n_\alpha},$$

and we see that $\widetilde{\mathcal{P}}_n$ is a convex closed bounded subset of a finite-dimensional space, which is mapped into itself by the continuous map $\widetilde{\mathbf{T}}_n$. Hence $\widetilde{\mathbf{T}}_n$ and, therefore, also $\mathbf{T}_n$ have a fixed point for each n by Brouwer's theorem. *The fixed point of* $\widetilde{\mathbf{T}}_n$ *is a vector* $\mathbf{q}$ *with polynomial components satisfying the system of orthogonality relations* (1.9′) (cf. (0.15)).

The closeness of the fixed points of $\mathbf{T}$ and $\widetilde{\mathbf{T}}_n$ as $n \to \infty$ is the subject of our study.

If

$$\|\tilde{h}_{\alpha,n} - h_\alpha\|_{C(E_\alpha)} = o(1), \tag{1.11}$$

where $\tilde{h}_{\alpha,n}$ is as in (1.9) and h_α is as in (1.4), (1.7), then the density of the family of cones $\{\widehat{H}^{+n}\}$ in $\widehat{H}_{2,\rho}^+$ (Theorem 2′) ensures the following closeness properties of $\mathbf{T}_n$ and $\mathbf{T}$ as $n \to \infty$. The first of these properties can also be called 'the asymptotic behaviour of polynomials with fixed variable weight' or the local (pointwise) closeness of $\mathbf{T}_n$ and $\mathbf{T}$ and reads as follows.

Proposition 1.2. *Assume that* (1.11) *holds for maps* $\mathbf{T}$ *and* $\mathbf{T}_n$*. If*

$$\|\mathbf{f}^{(n)} - \mathbf{f}\|_{C(\mathbf{E})} = o(1) \tag{1.12}$$

then

$$\|\mathbf{T}_n \mathbf{f}^{(n)} - \mathbf{T}\mathbf{f}\|_{H(\Omega)} = o(1); \tag{1.13}$$

moreover,

$$\left\| \frac{\Phi_\alpha^n}{|\Phi_\alpha|^n} T_n f_\alpha^{(n)} - \left\{ \frac{\Phi_\alpha^n}{|\Phi_\alpha|^n} T f_\alpha + \overline{\frac{\Phi_\alpha^n}{|\Phi_\alpha|^n} T f_\alpha} \right\} \right\|_{L_{2,\rho_j}(E_j)} = o(1), \quad \alpha = 1, \dots, p. \tag{1.14}$$

Proof. First we deduce (1.14) from (1.12) and then obtain (1.13) from (1.14).

We prove (1.14). Let

$$\widehat{f}_\alpha := \bigl\{e^{in\theta_\alpha} f_\alpha + \overline{e^{in\theta_\alpha} f_\alpha}\bigr\},$$

where

$$\theta_\alpha(x) = \arg \Phi_\alpha(x), \qquad x \in E_\alpha, \quad \alpha = 1, \dots, p.$$

We consider a sequence $\mathbf{g}^{(n)} \in \widehat{H}^{+n}$, which exists by Theorem 2′ such that

$$(1) \quad \|\mathbf{g}^{(n)} - \mathbf{Tf}\|_{H(\Omega)} = o(1), \tag{1.15}$$

$$(2) \quad \|e^{in\theta_\alpha} g_\alpha^{(n)} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}(E_\alpha)} = o(1), \quad \alpha = 1, \dots, p. \tag{1.16}$$

We apply to the left-hand side of (1.14) the triangle inequality:

$$\|e^{in\theta_\alpha} T_n f_\alpha^{(n)} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}} \leqslant \|e^{in\theta_\alpha} T_n f_\alpha^{(n)} - \widehat{Tf_\alpha^{(n)}}\,\|_{L_{2,\rho_\alpha}} + \|\widehat{Tf_\alpha^{(n)}} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}}. \tag{1.17}$$

In view of (1.11) and (1.12), the first term on the right-hand side satisfies for large n the relation

$$\begin{aligned} \|e^{in\theta_\alpha} T_n f_\alpha^{(n)} - \widehat{Tf_\alpha^{(n)}}\,\|_{L_{2,\rho_\alpha}} &\asymp \int_{E_\alpha} \left|e^{in\theta_\alpha(x)} T_n f_\alpha^{(n)}(x) - \widehat{Tf_\alpha^{(n)}(x)}\right|^2 \left(\frac{\tilde{h}_{\alpha,n}\rho_\alpha}{f_{\alpha-1}^{(n)} f_{\alpha+1}^{(n)}}\right)(x)\,dx \\ &= \int_{E_\alpha} \left|\frac{P_{\alpha,n}(x)}{|\Phi_\alpha(x)|^n} - \widehat{Tf_\alpha^{(n)}}(x)\right|^2 \left(\frac{\tilde{h}_{\alpha,n}\rho_\alpha}{f_{\alpha-1}^{(n)} f_{\alpha+1}^{(n)}}\right)(x)\,dx. \end{aligned} \tag{1.18}$$

The polynomial $P_{\alpha,n}$ is orthogonal with respect to the weight

$$\frac{h_{\alpha,n}\rho_\alpha}{\Phi_{\alpha-1}^n f_{\alpha-1}^{(n)} \Phi_{\alpha+1}^n f_{\alpha+1}^{(n)}}\,,$$

and Tf_α is the Szegő function corresponding to the weight

$$\frac{h_{\alpha,n}\rho_\alpha}{f_{\alpha-1}^{(n)} f_{\alpha+1}^{(n)}}\,,$$

therefore it follows from the extremality of orthogonal polynomials, the reproducing property of the Szegő function, and (1.10) that the value of the integral on the right-hand side of (1.18) can only increase after the replacement of $P_{\alpha,n}$ by another polynomial of the same degree and the same leading coefficient (see § 3 for greater detail).

Thus, replacing the functions $T_n f_\alpha^{(n)}$ on the left-hand side of (1.18) by the functions $g_\alpha^{(n)}$ satisfying (1.16) we obtain

$$\begin{aligned} \|e^{in\theta_\alpha} T_n f_\alpha^{(n)} - \widehat{Tf_\alpha^{(n)}}\,\|_{L_{2,\rho_\alpha}} &\lesssim \|e^{in\theta_\alpha} g_\alpha^{(n)} - \widehat{Tf_\alpha^{(n)}}\,\|_{L_{2,\rho_\alpha}} \\ &\leqslant \|e^{in\theta_\alpha} g_\alpha^{(n)} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}} + \|\widehat{Tf_\alpha^{(n)}} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}}. \end{aligned}$$

Returning to (1.17), for sufficiently large n we obtain the relation

$$\|e^{in\theta_\alpha}T_n f_\alpha^{(n)} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho}(E_\alpha)} \lesssim \|e^{in\theta_\alpha}g_\alpha^{(n)} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}} + 2\|\widehat{Tf_\alpha^{(n)}} - \widehat{Tf_\alpha}\,\|_{L_{2,\rho_\alpha}},$$

where the first term on the right-hand side approaches zero by (1.16), and the convergence to zero of the second term is ensured by (1.12) and the continuity of $\mathbf{T}$ as expressed by (1.5) (see Remarks 1.4 and 1.2; one can also find details in the proof of relation (5.10) in [11]).

All this establishes (1.14).

To prove (1.13) we observe that, in view of (1.14),

$$\|g^{(n)} - \mathbf{T}_n f^{(n)}\|_{H_{2,\rho}} = o(1)$$

and therefore, by Cauchy's integral formula,

$$\|g^{(n)} - \mathbf{T}_n f^{(n)}\|_{H(\Omega)} = o(1),$$

which yields (1.13) in view of (1.15).

The proof of Proposition 1.2 is complete.

One consequence of the above Proposition and the compactness principle for analytic functions is the global ('uniform') closeness of $\mathbf{T}$ and $\mathbf{T}_n$ expressed by the following result.

Proposition 1.3. *Let $\mathbf{T}$ and $\mathbf{T}_n$ be maps such that* (1.11) *holds. Then*

$$\|\mathbf{T}_n\mathbf{f} - \mathbf{T}\mathbf{f}\|_{C_\mathbf{E}} \rightrightarrows 0$$

as $n \to \infty$ uniformly for $\mathbf{f} \in H_\mathbf{E}$ such that $\|\mathbf{f}\|_{H_\mathbf{E}} \leqslant C$ for each $C > 0$.

Proof. We must show that for each $\varepsilon > 0$ there exists N such that for all $n > N$ and $\mathbf{f}$, $\|\mathbf{f}\|_{H_\mathbf{E}} \leqslant C$, we have the inequality

$$\|\mathbf{T}_n\mathbf{f} - \mathbf{T}\mathbf{f}\|_{C_\mathbf{E}} \leqslant \varepsilon.$$

Assume the contrary, that is, let the above assertion fail. Then there exist $\varepsilon > 0$ and an infinite sequence of indices Λ and functions $\mathbf{f}^{(n)}, n \in \Lambda$, $\|\mathbf{f}^{(n)}\|_{H_\mathbf{E}} \leqslant C$ such that

$$\|\mathbf{T}_n\mathbf{f}^{(n)} - \mathbf{T}\mathbf{f}^{(n)}\|_{C_\mathbf{E}} > \varepsilon \qquad \text{for } n \in \Lambda. \tag{1.19}$$

However, $\|\mathbf{f}^{(n)}\|_{H_\mathbf{E}} \leqslant C$, therefore by Montel's theorem the sequence $\{\mathbf{f}^{(n)}\}_{n\in\Lambda}$ is a compact family on $\mathbf{E}$, so that there exist $\Lambda' \subseteq \Lambda$ and $\mathbf{f} \in C(\mathbf{E})$ such that

$$\|\mathbf{f}^{(n)} - \mathbf{f}\|_{C_\mathbf{E}} \to 0, \qquad n \in \Lambda'.$$

Hence, for indices in the subsequence Λ' we have

$$\|\mathbf{T}_n\mathbf{f}^{(n)} - \mathbf{T}\mathbf{f}^{(n)}\|_{C_\mathbf{E}} \leqslant \bigl(\|\mathbf{T}_n\mathbf{f}^{(n)} - \mathbf{T}\mathbf{f}\|_{C_\mathbf{E}} + \|\mathbf{T}\mathbf{f}^{(n)} - \mathbf{T}\mathbf{f}\|_{C_\mathbf{E}}\bigr) \to 0,$$

where the first term on the right-hand side approaches zero by Proposition 1.2 and the second approaches zero by the continuity of $\mathbf{T}$ (see (1.5)). This is a contradiction with (1.19).

The proof of Proposition 1.3 is complete.

1.4. Asymptotic closeness of fixed points of the maps $\mathbf{T}_n$ and T. Proof of Theorem 1. The contraction property of the map **T** with respect to the d-metric (Proposition 1.1) and the closeness of the maps **T** and $\mathbf{T}_n$ for large n (Proposition 1.3), which is a consequence of the density of the cones $\widehat{H}^{+n}$ in $\widehat{H}^+_{2,\rho}$ (Theorem 2′), enable one to verify that each neighbourhood of a fixed point of **T** contains fixed points of $\mathbf{T}_n$ for large n. This is, in essence, a result equivalent to Theorem 1 stated in the introduction.

Theorem 1″. *Let* **T** *and* $\mathbf{T}_n$ *be maps satisfying* (1.11). *Then there exists a sequence* $\{\mathbf{f}^{(n)}\}$ *of fixed points of* $\mathbf{T}_n$,

$$\mathbf{f}^{(n)} = \mathbf{T}_n \mathbf{f}^{(n)}, \qquad \mathbf{f}^{(n)} = \left(\frac{q_{1,n}}{\Phi_1^n}, \dots, \frac{q_{p,n}}{\Phi_p^n}\right), \tag{1.20}$$

such that

$$\|\mathbf{f}^{(n)} - \mathbf{f}^{(\infty)}\|_{C_{\mathbf{E}}} \to 0 \tag{1.21}$$

as $n \to \infty$, *where* $\mathbf{f}^{(\infty)}$ *is a fixed point of* **T**.

Proof. We fix $\theta < 1$. Let ω be a closed neighbourhood of $\mathbf{f}^{(\infty)}$ in the $C_{\mathbf{E}}$-norm such that

$$\|\mathbf{f}^{(\infty)} - g\|_{C(\mathbf{E})} \leqslant \theta\, r(\mathbf{f}^{(\infty)}) \quad \text{for each } g \in \omega, \tag{1.22}$$

where $r(\mathbf{f}^{(\infty)})$ is as in Remark 1.2; see (1.3). We consider a family ω_ε of closed neighbourhoods of $\mathbf{f}^{(\infty)}$ in the d-metric such that

$$d(\mathbf{f}^{(\infty)}, g) \leqslant \varepsilon \quad \text{for each } g \in \omega_\varepsilon, \quad 0 < \varepsilon \leqslant \varepsilon_0,$$

where ε_0 is chosen from the condition

$$\omega_{\varepsilon_0} \subset \omega \tag{1.23}$$

($\mathbf{f}^\infty$ belongs to the cone of continuous positive functions $\widehat{C}^+_{\mathbf{E}}$, therefore there exists such an ε_0).

Now let

$$\omega_{\varepsilon,n} := \omega_\varepsilon \cap \widehat{H}^{+n}.$$

The set $\omega_{\varepsilon,n}$ is a closed and bounded (and therefore compact) subset of a finite-dimensional space. By Theorem 2′ the set $\omega_{\varepsilon,n}$ is non-empty. Moreover, it is easy to verify that $\omega_{\varepsilon,n}$ is convex. (For $\omega_{\varepsilon,n}$ consists of vector-valued functions of the form (1.8) whose components have graphs lying in a 'tube' with axis along the graph of $\mathbf{f}^\infty$. Elements $t\mathbf{g}^{(1)} + (1-t)\mathbf{g}^{(2)}$, where $\mathbf{g}^{(1)}, \mathbf{g}^{(2)} \in \omega_{\varepsilon,n}$ and $t \in [0,1]$, retain the form (1.8), and a pointwise inspection shows that their graphs stay within the tube.)

We claim that for each $\varepsilon \in (0, \varepsilon_0]$ there exists N_ε such that for each $n > N_\varepsilon$ we have

$$\mathbf{T}_n[\omega_{\varepsilon,n}] \subset \omega_{\varepsilon,n}. \tag{1.24}$$

In fact, the elements $g \in \omega_{\varepsilon,n}$ are bounded in $\widehat{H}^+_{\mathbf{E}}$, therefore it follows from Proposition 1.3, in view of the consistency of the norm and the metric in $\omega_{\varepsilon,n}$ (as shown by (1.23) and (1.22); see Remark 1.2), that there exists N_ε such that for all $n > N_\varepsilon$ and $g \in \omega_{\varepsilon,n}$ we have

$$d(\mathbf{T}_n g, \mathbf{T} g) < (1-\gamma)\varepsilon,$$

where γ was defined in (1.6). Now, in view of Proposition 1.1 (see (1.6)), we obtain

$$d(\mathbf{f}^\infty, \mathbf{T}_n g) \leqslant d(\mathbf{f}^\infty, \mathbf{T} g) + d(\mathbf{T} g, \mathbf{T}_n g) < \gamma\, d(\mathbf{f}^\infty, g) + (1-\gamma)\varepsilon < \varepsilon,$$

which proves (1.24). Hence,Theorem 1$''$ follows by Brouwer's fixed point theorem.

A *verification of Theorem* 1 in the introduction can now be carried out by the application of Proposition 1.2 to the sequence of fixed points (1.20), (1.21).

It remains to note that if, in the definition of the map $\mathbf{T}_n\colon \mathbf{f} \to \mathbf{T}_n\mathbf{f}$ (see (1.9)), one defines the weight function $\{\tilde h_{\alpha,n}\}_{\alpha=1}^p$ by formula (0.12), as for Nikishin systems:

$$\tilde h_{\alpha,n}(\mathbf{f};x) := \int_{E_\alpha} \frac{f^2_{\alpha-1}(t)}{x-t}\,\frac{\tilde h_{\alpha-1,n}(\mathbf{f};t)}{f_{\alpha-2}(t) f_\alpha(t)}\, d\sigma_{\alpha-1}(t), \quad \alpha = 2,\dots,p; \quad \tilde h_{1,n} := 1, \tag{1.25}$$

then limit relations (0.14) ensure (1.11) and we see that the map (1.9), (1.25) satisfies the assumptions of Theorem 1$''$. Taking into account the uniqueness of the Hermite–Padé polynomials Q_n for the Nikishin system (see the introduction) we see that the map $\mathbf{T}_n$ so defined has a unique fixed point. Thus (provided that Theorem 2 holds), we *have verified* Theorem 1$'$.

§ 2. Properties of some rational functions on $\mathfrak{R}$. Proof of Theorem 2

2.1. Properties of the functions $\Psi(z)$ and $\Phi(z)$. In this subsection we study the properties of the function Ψ governing the leading terms of the asymptotic formula, the functions $\{\Phi_\alpha\}_{\alpha=1}^p$ (see (0.25)). We defined the function $\Psi(z)$ in the introduction as a rational function on the Riemann surface $\mathfrak{R}$ (see (0.19), (0.20)) with divisor (0.21).

First we point out some general properties of rational functions with divisor (0.21) on $(p+1)$-sheeted Riemann surfaces with quadratic branch points at the end-points of the intervals $\{E_\alpha\}_{\alpha=1}^p$, that is, properties that are independent of the monodromy group of the Riemann surface (0.20).

(1) It is easy to see that these algebraic functions satisfy the equation

$$\Psi^{p+1} + r_1(z)\Psi^p + r_2(z)\Psi^{p-1} + \dots + r_p(z)\Psi + r_0 = 0 \tag{2.1}$$

with polynomial coefficients $r_k(z)$, where

$$\deg r_k = k, \quad k = 0,1,\dots,p, \qquad r_0 = (-1)^p.$$

Note that although the coefficients $\{r_k\}_{k=1}^p$ are completely defined by the divisor and the monodromy group, their actual calculation by these data is a non-trivial problem.

(2) We are interested in the subset I of the complex plane $\overline{\mathbb{C}}$ on which distinct branches of the function $\Psi(z)$ — the functions

$$\Psi_l(z) := \Psi(z^{(l)}), \qquad z \in \overline{\mathbb{C}}, \quad z^{(l)} \in \overline{\mathfrak{R}}_l, \quad l = 0,1,\dots,p,$$

— have the same absolute value. That is,

$$I := \left\{ z : \left| \frac{\Psi_{l_1}}{\Psi_{l_2}} \right| = 1;\ l_1 \neq l_2;\ l_1, l_2 = 0, 1, \dots, p \right\}.$$

The set I can be described as the union of the trajectories of all the roots $z(\nu)$ of the equation

$$J(\nu, z) := \prod_{\substack{l_1, l_2 = 0 \\ l_1 > l_2}}^{p+1} \left(\nu - \left(\frac{\Psi_{l_1}}{\Psi_{l_2}} + \frac{\Psi_{l_2}}{\Psi_{l_1}} \right) \right) = 0 \tag{2.2}$$

for $\nu \in [-2, 2]$. The coefficients of the equation with respect to the algebraic function $\nu(z)$ are symmetric functions of $\{\Psi_l\}_{l=0}^{p}$, therefore expressing them in terms of the elementary symmetric functions of $\{\Psi_l\}_{l=0}^{p}$, which are the polynomials $\{r_k\}_{k=0}^{p}$ in (2.1), we see that these coefficients are also polynomials in z:

$$J(\nu, z) = \nu^{m_p} + S_{m_p - 1}(z)\nu^{m_p - 1} + \dots = \sum_{k=0}^{m_p} S_k(z)\nu^k, \quad m_p = \frac{p(p+1)}{2}, \tag{2.3}$$

and in view of what we know about the degree of r_m, we obtain

$$\max_{k=0,\dots,m_p} \deg S_k = p(p+1). \tag{2.4}$$

Hence the algebraic function $z(\nu)$ has $p(p+1)$ branches and the subset I (of the complex z-plane) is the union of $p(p+1)$ trajectories $\{I_j\}_{j=1}^{p(p+1)}$ starting at the points $z(2)$ and arriving at $z(-2)$.

We consider now the set of starting points $\{z_j\}$ and the set of terminal points $\{d_j\}$ (both treated as geometric sets, with no account of multiplicities). That is,

$$\{z_j\} : \quad J(2, z_j) = 0; \qquad \{d_j\} : \quad J(-2, d_j) = 0.$$

The polynomial $J(2, z)$ is the discriminant of the algebraic function (2.1), and the values of distinct branches of the function $\Psi(z)$ coincide at its roots:

$$\Psi_k(z_j) = \Psi_l(z_j) \quad \text{for some } k, l, \quad k \neq l.$$

The number of roots of $J(2, z)$ counted with multiplicities is $p(p+1)$, of which $2p$ are simple roots, which are the branch points of $\Psi(z)$ at the end-points of the intervals $E_\alpha = [a_\alpha, b_\alpha]$, and the rest are of a larger (moreover, even) multiplicity, for $\Psi(z)$ does not branch at these points. That is,

$$\{z_j\} = \{a_\alpha, b_\alpha\}_{\alpha=1}^{p} \cup \{c_j\}_{j=1}^{u_p},$$

where

$$u_p \leqslant \frac{(p+1)p - 2p}{2} = \frac{p(p-1)}{2}. \tag{2.5}$$

The polynomial $J(-2,z)$ is the square of a symmetric function of $\{\Psi_l\}_{l=0}^{p}$ (see the definition (2.2)), therefore it has roots of even multiplicity,

$$\{d_j\}_{j=1}^{v_p}: \quad \Psi_k(d_j) = -\Psi_l(d_j) \quad \text{for some } k,l, \quad k\neq l,$$

and we have

$$v_p \leqslant \frac{p(p+1)}{2}. \tag{2.6}$$

Thus, the set I is the union of $p(p+1)$ trajectories (which may have common points, but no common segments). Of these, $2p$ start one at a time at the end-points of the E_α, $\alpha=1,\dots,p$ (and run along E_α until they terminate at some point d_j or, more generally, until they run against an oncoming trajectory along E_α, after which both trajectories turn into the complex plane as two conjugate curves). The remaining trajectories start at the points c_j (and either go opposite ways along the real axis or run into the complex plane as conjugate curves). All trajectories end in pairs at the points d_j.

We point out again that all the above concerns arbitrary $(p+1)$-sheeted Riemann surfaces with quadratic branch points at a_α, b_α, $\alpha=1,\dots,p$, and we take no account of the special features of the monodromy group (0.20).

We proceed now to the particular Riemann surface (0.19) with monodromy group (0.20) associated with the Nikishin system. In this case the geometry of I is trivial (in contrast to the Riemann surfaces associated with the Angelesco system or with systems of Markov functions on intersecting intervals; see [11], [18]).

Proposition 2.1. *We have the relation*

$$I = \bigcup_{\alpha=1}^{p} E_\alpha.$$

Proof. By the definition (0.21),

$$\Psi_0(z) \in \widehat{H}^+(\mathbb{C}\setminus E_1), \qquad \Psi_0(z)\big|_\infty = \frac{1}{C_0 z^p} + \cdots,$$

therefore it follows from the argument principle that

$$\frac{1}{2\pi} \mathop{\triangle}_{E_1} \operatorname{Arg}\Psi_0 = p,$$

that is, Ψ_0 takes purely real values at $p+1$ points in E_1 at least (where the end-points a_1 and b_1 are taken into account) and the limit values of Ψ_0 from both above and below are purely imaginary at p points in E_1 at least. Thus, since $\Psi(z)$ is symmetric with respect to the real axis:

$$\Psi_0(x) = \overline{\Psi_1(x)}, \qquad x \in E_1,$$

at least $p-1$ points in $\{c_j\}$ (see (2.5)) and p points in $\{d_j\}$ (see (2.6)) must be interior points of E_1:

$$\begin{aligned} c_{j_k} \in E_1: &\quad \Psi_0(c_{j_k}) = \Psi_1(c_{j_k}), \qquad k=1,\dots,k_1, \quad k_1 \geqslant p-1,\\ d_{j_s} \in E_1: &\quad \Psi_0(d_{j_s}) = -\Psi_1(d_{j_s}), \quad s=1,\dots,s_1, \quad s_1 \geqslant p. \end{aligned}$$

We also point out that

$$\frac{1}{2\pi}\mathop{\triangle}_{E_1}\operatorname{Arg}\Psi_1 = -p. \tag{2.7}$$

Further, by the definition of $\Psi(z)$, see (0.21), we obtain

$$\Psi_1(z)\in\widehat{H}^+\big(\mathbb{C}\setminus\{E_1\cup E_2\}\big), \qquad \Psi_1(z)\big|_\infty = \frac{z}{C_1}+\cdots.$$

Hence, by the argument principle,

$$\frac{1}{2\pi}\mathop{\triangle}_{E_1\cup E_2}\operatorname{Arg}\Psi_1 = -1,$$

therefore, in view of (2.7),

$$\frac{1}{2\pi}\mathop{\triangle}_{E_2}\operatorname{Arg}\Psi_1 = p-1,$$

which requires E_2 to contain in its interior at least $p-2$ points in $\{c_j\}$ and $p-1$ points in $\{d_j\}$. We also point out the equality

$$\frac{1}{2\pi}\mathop{\triangle}_{E_2}\operatorname{Arg}\Psi_2 = -(p-1).$$

Thus, moving along α from 1 to m we obtain

$$\begin{aligned} &c_{j_k^{(\alpha)}}\in E_\alpha: \quad \Psi_\alpha\big(c_{j_k^{(\alpha)}}\big)=\Psi_{\alpha-1}\big(c_{j_k^{(\alpha)}}\big), \quad k=1,\dots,k_\alpha, \quad k_\alpha\geqslant p-\alpha,\\ &d_{j_s^{(\alpha)}}\in E_\alpha: \quad \Psi_\alpha\big(d_{j_s^{(\alpha)}}\big)=\Psi_{\alpha-1}\big(d_{j_s^{(\alpha)}}\big), \quad s=1,\dots,s_\alpha, \quad s_\alpha\geqslant p-\alpha+1, \end{aligned} \tag{2.8}$$

therefore the total numbers of points $\{c_j\}$ and $\{d_j\}$ have lower bounds:

$$u_p=\sum_{\alpha=1}^{p}k_\alpha\geqslant\frac{p(p-1)}{2}, \qquad v_p=\sum_{\alpha=1}^{p}s_\alpha\geqslant\frac{(p+1)p}{2}.$$

Thus, in view of the general upper bound (2.5), (2.6), we obtain

$$\begin{aligned} u_p&=\frac{p(p-1)}{2} \quad\Rightarrow\quad k_\alpha=p-\alpha,\\ v_p&=\frac{(p+1)p}{2} \quad\Rightarrow\quad s_\alpha=p-\alpha+1, \end{aligned} \qquad \alpha=1,\dots,p.$$

Thus, we have proved that the sets of points $\{c_j\}$ and $\{d_j\}$ lie in the interior of the intervals $\{E_\alpha\}_1^p$ each containing $k_\alpha=p-\alpha$ and $s_\alpha=p-\alpha+1$ points, respectively. That is, I is the union of $p(p+1)$ trajectories (2.2), which start (for $\nu=2$) in the amount of $2(p-\alpha)+2$ from $p-\alpha+2$ points in each interval E_α, $\alpha=1,\dots,m$ (one trajectory from each of the end-points a_α and b_α and pairs of trajectories from the points $c_{j_k^{(\alpha)}}\in E_\alpha$, see (2.8)), and which terminate (for $\nu=-2$) at $p-\alpha+1$

points of *the same* (‘*launching*’) interval E_α (two at each point $d_{j_s^{(\alpha)}} \in E_\alpha$); at these trajectories we have $|\Psi_\alpha| = |\Psi_{\alpha-1}|$.

We now emphasize two points.

First, *the trajectories starting at E_α do not intersect the real axis outside the interval E_α* because at each point of $\mathbb{R} \setminus E_\alpha$ one of the functions Ψ_α and $\Psi_{\alpha-1}$ must be real, therefore if such a trajectory intersects $\mathbb{R} \setminus E_\alpha$, then there emerges a new point e such that

$$\Psi_\alpha(e) = \Psi_{\alpha-1}(e) \quad \text{or} \quad \Psi_\alpha(e) = -\Psi_{\alpha-1}(e),$$

which is in contradiction with (2.8) and (2.4).

Second, *the trajectories starting from E_α cannot leave E_α for the complex plane* because the functions $z(\nu)$ are symmetric with respect to the real axis (see (2.3)), and the union of such trajectories and the symmetric (complex conjugate) trajectories (which necessarily exist in this case), which terminate at E_α, bound a domain in which the harmonic functions $|\Psi_\alpha(z)|$ and $|\Psi_{\alpha+1}|$ are the same by the maximum principle; this would lead to a contradiction.

Hence the trajectories forming I start at the points $a_\alpha, b_\alpha, c_{j_k^{(\alpha)}}$ in each interval E_α, where $k = 1, \dots, p-\alpha$ and $\alpha = 1, \dots, p$; go (for $\nu\colon 2 \to -2$) along the interval E_α (one trajectory from each of the points a_α, b_α and pairs of trajectories, going in the opposite directions, from the points $c_{j_k^{(\alpha)}}$) and terminate (two trajectories from the opposite directions at a point) at the points $d_{j_s^{(\alpha)}}$, $s = 1, \dots, p-\alpha+1$, which alternate with the $c_{j_k^{(\alpha)}}$ (see Fig. 1).

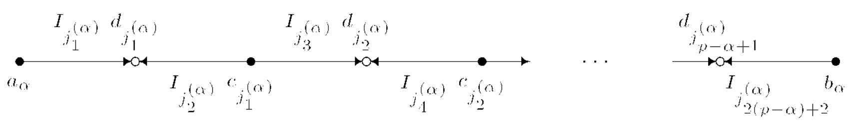

Figure 1

Thus, $I = \bigcup_{\alpha+1}^{p} E_\alpha$ and

$$|\Psi_{l-1}(x)| = |\Psi_l(x)|, \qquad x \in E_\alpha;$$

moreover, if $|l-k| > 1$, $l, k = 1, \dots, p$, then

$$|\Psi_k(z)| \neq |\Psi_l(z)| \quad \text{for each } z \in \mathbb{C}. \tag{2.9}$$

The proof of Proposition 2.1 is complete.

We now state one immediate (and important for what follows) consequence of this result.

Proposition 2.2. *For each point z in the complex plane,*

$$|\Psi_0(z)| \leqslant |\Psi_1(z)| \leqslant \dots \leqslant |\Psi_p(z)|, \tag{2.10}$$

and the equality between $|\Psi_{l-1}(z)|$ and $|\Psi_l(z)|$ holds only for $z \in E_l$.

Proof. By the definition (0.21) of $\Psi(z)$ we obtain

$$|\Psi_0(\infty)| < |\Psi_1(\infty)|,$$

therefore, by (2.9) (in view of the continuity of $|\Psi_l|$ in $\mathbb{C}$, $l = 0, \dots, p$),

$$\begin{aligned} |\Psi_0(z)| &< |\Psi_1(z)|, \qquad z \in \overline{\mathbb{C}} \setminus E_1, \\ |\Psi_0(x)| = |\Psi_1(x)| &< |\Psi_2(x)|, \qquad x \in E_1 \end{aligned} \tag{2.11$_1$}$$

(the last inequality holds because otherwise — in the case of the reverse inequality — it follows from (0.21) that there exists $\tilde{z} \in \overline{\mathbb{C}}$ such that $|\Psi_0(\tilde{z})| = |\Psi_2(\tilde{z})|$, which contradicts (2.9)). The last inequality in (2.11$_1$) means by continuity that

$$|\Psi_1(z)| < |\Psi_2(z)|, \qquad z \in \overline{\mathbb{C}} \setminus E_2, \tag{2.11$_2$}$$

$$|\Psi_1(x)| = |\Psi_2(x)| < |\Psi_3(x)|, \qquad x \in E_2 \tag{2.11$_3$}$$

(the reverse inequality in (2.11$_3$) would mean that $|\Psi_3(z)| < |\Psi_1(z)|$ for each $z \in \mathbb{C}$, in view of (2.9), which leads to a contradiction with (2.11$_3$) and with the equality $|\Psi_2(x)| = |\Psi_3(x)|$ for $x \in E_3$).

Repeating these arguments for $l = 1, \dots, p$ we obtain

$$|\Psi_{l-1}(z)| < |\Psi_l(z)|, \qquad z \in \mathbb{C} \setminus E_l.$$

The proof of Proposition 2.2 is complete.

To complete the subsection we point out several properties of the leading terms in the asymptotic formulae, the functions $\{\Phi_\alpha(z)\}_{\alpha=1}^p$ (see (0.25)), where

$$\Phi_\alpha(z) := \prod_{l=\alpha}^{p} \Psi_l(z), \qquad \alpha = 1, \dots, p.$$

It is an immediate consequence of the definition that

$$\begin{aligned} &(1) \quad \Phi_\alpha \in \widehat{H}^+(\mathbb{C} \setminus E_\alpha), \\ &(2) \quad \Phi_\alpha(z)\big|_{z=\infty} \simeq \frac{z^{p-\alpha+1}}{c_\alpha}\,, \quad c_\alpha = \prod_{l=\alpha}^{p} C_l, \\ &(3) \quad |\Phi_\alpha(x)|^2 \frac{1}{|\Phi_{\alpha-1}(x)\,\Phi_{\alpha+1}(x)|} = 1, \quad x \in E_\alpha, \quad \alpha = 1, \dots, p \end{aligned} \tag{2.12}$$

(as usual, we agree that $\Phi_{p+1} \equiv \Phi_0 \equiv 1$).

We now verify property (3) in (2.12). By the definition of $\{\Phi_\alpha\}$,

$$\frac{\Phi_\alpha}{\Phi_{\alpha-1}} = \frac{1}{\Psi_{\alpha-1}}\,,$$

so that for $\alpha = 2, 3, \dots, p$, in view of the equality

$$\Psi_\alpha(x) = \overline{\Psi_{\alpha-1}(x)}, \qquad x \in E_\alpha, \tag{2.13}$$

we have

$$\frac{|\Phi_\alpha(x)|^2}{\big|(\Phi_{\alpha-1}\Phi_{\alpha+1})(x)\big|} = \left|\frac{\Psi_\alpha(x)}{\Psi_{\alpha-1}(x)}\right| = 1, \qquad x \in E_\alpha, \quad \alpha = 2, \dots, p.$$

For $\alpha = 1$, in view of (2.13) and bearing in mind that $\prod_{l=0}^{p} \Psi_l \equiv 1$ (see (0.22)), we obtain

$$\frac{|\Phi_1(x)|^2}{|\Phi_2(x)|} = |\Psi_1(x)|^2 \prod_{l=2}^{p} |\Psi_l(x)| = \left|\prod_{l=0}^{p} \Psi_l(x)\right| = 1, \qquad x \in E_1.$$

2.2. Bernstein–Szegő polynomials for $\mathfrak{R}$. Proof of Theorem 2. In this subsection we prove Theorem 2 stated in the introduction (see § 0.5). We recall what we require to this end. For an arbitrary fixed collection of (non-negative, integrable and satisfying the Szegő condition on $\mathbf{E}$) weight functions $\mathbf{w} = \{w_\alpha\}_{\alpha=1}^{p}$ giving rise to a solution of the boundary-value problem

$$\begin{gathered} \mathbf{f} = \{f_\alpha\} \in \widehat{H}^{+}_{2,w}(\Omega), \\ \left(\frac{|f_\alpha|^2 w_\alpha}{|f_{\alpha+1} f_{\alpha-1}|}\right)(x) = 1, \quad x \in E_\alpha, \quad \alpha = 1, \dots, p, \qquad f_0 \equiv f_{p+1} \equiv 1, \end{gathered} \tag{2.14}$$

we must construct a sequence of polynomials

$$\mathbf{P}_n = \{P_\alpha\}_{\alpha=1}^{p}, \qquad P_\alpha(z) = z^{n_\alpha} + \cdots, \quad n_\alpha = n(p - \alpha + 1),$$

such that

$$\begin{aligned} &(1) \quad \left\| \frac{P_\alpha(x)}{|c_\alpha \Phi_\alpha(x)|^n} - \left\{ \left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n \frac{f_\alpha(x)}{f_\alpha(\infty)} + \overline{\left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n \frac{f_\alpha(x)}{f_\alpha(\infty)}} \right\} \right\|_{L^2_{w_\alpha}(E_\alpha)} = o(1), \\ &(2) \quad \left\| \frac{P_\alpha(z)}{(c_\alpha \Phi_\alpha(z))^n} - \frac{f_\alpha(z)}{f_\alpha(\infty)} \right\|_{H(\Omega_\alpha)} = o(1) \end{aligned} \tag{2.15}$$

as $n \to \infty$.

We construct the $\mathbf{P}_n$ in accordance with the following scheme.

(A) First, we construct a sequence of special weight functions $\{w_j^{(m)}\}_{j=1}^{p}$ approximating $\mathbf{w}$ in the integral norm and in the d_L-metric:

$$\|w_j^{(m)} - w\|_{L(E_j)} = o(1), \quad \left\|\ln \frac{w_j^{(m)}}{w}\right\|_{L(E_j)} = o(1), \qquad j = 1, \dots, p, \tag{2.16}$$

as $m \to \infty$. For these special weights we write down the solution of the boundary-value problem (2.14) explicitly in terms of some rational functions $\{f_j^{(m)}\}_{j=1}^p$ on $\mathcal{R}$. In view of Remark 1.4,

$$\|\mathbf{f}^{(m)} - \mathbf{f}\|_{H_{2,\mathbf{w}(\Omega)}} = o(1) \tag{2.16$'$}$$

as $m \to \infty$.

(B) Next, for arbitrary fixed m we construct a sequence of polynomials

$$\mathbf{P}_n^{(m)} = \{P_\alpha^{(m)}\}_{\alpha=1}^p, \qquad P_\alpha^{(m)}(z) = z^{n_\alpha} + \cdots, \quad n_\alpha = n(p-\alpha+1), \quad n \geqslant N_m,$$

such that

$$\begin{aligned}
&(1) \quad \left\| \frac{P_\alpha^{(m)}(x)}{|c_\alpha \Phi_\alpha(x)|^n} - \left\{ \left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n \frac{f_\alpha^{(m)}(x)}{f_\alpha^{(m)}(\infty)} \right. \right. \\
&\qquad\qquad \left.\left. + \overline{\left(\frac{\Phi_\alpha(x)}{|\Phi_\alpha(x)|}\right)^n \frac{f_\alpha^{(m)}(x)}{f_\alpha^{(m)}(\infty)}} \right\} \right\|_{C(E_\alpha)} = O(C_{E_\alpha}^n), \\
&(2) \quad \left\| \frac{P_\alpha^{(m)}(z)}{(c_\alpha \Phi_\alpha(z))^n} - \frac{f_\alpha^{(m)}(z)}{f_\alpha^{(m)}(\infty)} \right\|_{C(K)} = O(C_K^n)
\end{aligned} \tag{2.17}$$

as $n \to \infty$, where

$$C_{E_\alpha}, C_K < 1 \quad \text{for each } K \Subset \Omega_\alpha, \quad \alpha = 1, \dots, p.$$

For $p = 1$ the $P^{(m)}$ coincide with the polynomials introduced by Bernstein (in the case of an interval) and Szegő (in the circle case), in the proof of strong asymptotic formulae for ordinary orthogonal polynomials.

Finally, on accomplishing steps (A) and (B) we arrive at the following proof.

Proof of Theorem 2. We fix an arbitrary monotonic sequence $\{\varepsilon_k\}_{k=0}^\infty$ of positive numbers approaching zero:

$$\varepsilon_k > 0, \qquad \varepsilon_k \searrow 0 \quad \text{as } k \to \infty.$$

For each k we choose m_k such that the right-hand side of (2.16′) is less than $\varepsilon_k/2$. Next, for each m_k we choose N_k such that for each $n > N_k$ the right-hand sides of both asymptotic formulae in (2.17) are less than $\varepsilon_k/2$. The required sequence of polynomials $\mathbf{P}_n$ can now be defined as follows:

$$\mathbf{P}_n := \mathbf{P}_n^{(m_k)}, \qquad n \in [N_k, N_{k+1}], \quad k \in \mathbb{N};$$

it is obvious (from the triangle inequality) that asymptotic formulae (2.15) hold for $\mathbf{P}_n$.

The proof of Theorem 2 is complete.

It now remains to perform steps (A) and (B).

We fix arbitrary $\alpha \in \{1, \dots, p\}$ and discuss the construction of a polynomial $P_\alpha^{(m)}$ satisfying (2.17).

(A1) *Construction of* $\{w_j^{(m)}\}_{j=1}^p$. We shall use three distinct constructions for $j=\nu\in\{\alpha+1,\dots,p\}$, $j=k\in\{1,\dots,\alpha-1\}$, and for $j=\alpha$.

Let $\nu\in\{\alpha+1,\dots,p\}$. We set

$$w_\nu^{(m)}(x):=\frac{1}{t^{(\nu)}(x)}, \quad x\in E_\nu, \quad \text{where} \quad t^{(\nu)}(z)=C_{\nu,m_\nu}\prod_{\mu=1}^{m_\nu}(z-z_{\mu,m_\nu}). \quad (2.18_1)$$

Here $t^{(\nu)}(z)$ is a polynomial on $\overline{\mathbb{C}}$, that is, a rational function on the Riemann surface $\overline{\mathbb{C}}$ with pole of order m_ν at infinity and with m_ν zeros (counted with multiplicities) at some finite points. By Weierstrass's theorem we can choose $t^{(\nu)}$ such that $w_\nu^{(m)}$ approximates w_ν with the required accuracy (so that (2.16) holds).

Let $k\in\{1,\dots,\alpha-1\}$. We set

$$w_k^{(m)}(x):=\frac{1}{\tilde t_{k-1}^{(k)}(x)}, \qquad x\in E_k, \qquad (2.18_2)$$

where $\tilde t_{k-1}^{(k)}$ is the $(k-1)$th branch of the rational function $\tilde t^{(k)}\in\mathfrak{M}(r^{(k)})$ on the k-sheeted Riemann surface

$$r^{(k)}:=\overline{\bigcup_{l=0}^{k-1} r_l^{(k)}}, \qquad r_l^{(k)}:\equiv\mathcal{R}_l, \quad l=0,\dots,k-1.$$

The Riemann surface $r^{(k)}$ (with sheets denoted by $r_l^{(k)}$, $l=0,\dots,k-1$) is by definition made up of the first k sheets of $\mathcal{R}$ with punctured branch points at the projections of the end-points of the interval E_k onto the $(k-1)$th sheet of $\mathcal{R}_{k-1}$. The function $\tilde t^{(k)}$ is defined by its divisor as follows:

$$\tilde t^{(k)}\in\mathfrak{M}(r^{(k)}):\begin{cases}\tilde t^{(k)}(z)=\infty, & z\big|_{\infty^{(0)}}=c_{k,m_k}z^{m_k}+\cdots, \\ \tilde t^{(k)}(z)=0, & z\in\{\tilde z_{\mu,m_k}\}_{\mu=1}^{m_k},\end{cases} \qquad (2.19)$$

where the zeros $\{\tilde z_{\mu,m_k}\}_{\mu=1}^{m_k}$ of the 'polynomial' $\tilde t^{(k)}$ on the Riemann surface $r^{(k)}$ are chosen so that (by the Weierstrass–Lavrent'ev theorem) the 'polynomial' $\tilde t^{(k)}$ approximates the function $1/w_k(x)$ on the last sheet of the surface $r^{(k)}$, at the points of the projection of E_k onto $r_{k-1}^{(k)}$, with accuracy required for (2.16).

Finally, let

$$w_\alpha^{(m)}(x):=\frac{[\psi_{\alpha-1}^{(\alpha)}(x)]^{M_\alpha}}{\tilde t_{\alpha-1}^{(\alpha)}(x)}, \qquad M_\alpha=\sum_{\nu=\alpha+1}^{p}m_\nu, \quad x\in E_\alpha,$$

where $\psi_{\alpha-1}^{(\alpha)}$ is the value on the last sheet of the standard rational function $\psi^{(\alpha)}$ on $r^{(\alpha)}$ defined as follows by its divisor:

$$\psi^{(\alpha)}(z)\in\mathfrak{M}(r^{(\alpha)}):\begin{cases}\psi^{(\alpha)}(z)=\infty, & z\in\{\infty^{(1)},\infty^{(2)},\dots,\infty^{(\alpha-1)}\}, \\ \psi^{(\alpha)}(z)=0, & z\in\{(\infty^{(0)})^{\alpha-1}\},\end{cases}$$

$$\prod_{l=0}^{\alpha-1}\psi_l^{(\alpha)}=1, \qquad \psi_l^{(\alpha)}(z)=\psi^{(\alpha)}(z^{(l)}), \qquad l=0,\dots,\alpha-1$$

(note that $\psi^{(p+1)} = \Psi$, see (0.21)), and $\tilde{t}^{(\alpha)}_{\alpha-1}$ is the $(\alpha-1)$th branch of the rational function $\tilde{t}^{(\alpha)}$ on $r^{(\alpha)}$ defined by its divisor similarly to (2.19), where the zeros of the 'polynomial' $\tilde{t}^{\alpha}$ on $r^{(\alpha)}$ are chosen so that $\tilde{t}^{(\alpha)}_{\alpha-1}$ approximates $[\psi^{(\alpha)}_{\alpha-1}]^{M_\alpha}/w_\alpha$ with accuracy required for (2.16).

We point out that the approximations of the weights w_j with $j \neq \alpha$ are carried out independently. At the same time, the approximation of w_α must be carried out after the approximation of the w_ν, $\nu = \alpha+1, \dots, p$, and the approximant $\tilde{t}^{(\alpha)}$ depends on the sum of the degrees of the approximants $t^{(\nu)}$. This, however, involves no complications.

We note also that $\psi^{(1)} \equiv 1$ for $\alpha = 1$ and the definition of $w^{(m)}_\alpha$ coincides with (2.18_1), while for $\alpha = p$ we have $M_\alpha = 0$ and the definition of $w^{(m)}_\alpha$ coincides with (2.18_2).

(A2) *Solution of the boundary-value problem for the functions* $\{w^{(m)}_j\}$. *The definition of* $\{f^{(m)}_j(z)\}^p_{j=1}$.

We construct a rational function $g(z)$ on $\mathcal{R}$ by specifying its zeros and poles as follows.

We place all the zeros of each polynomial $t^{(\nu)}$, $\nu = \alpha+1, \dots, p$, on *each* sheet $\mathcal{R}_l$, $l = 0, \dots, \nu-1$, that is,

$$z^{(l)}_{\mu,m_\nu} = \pi^{-1}_{\mathcal{R}_l}(z_{\mu,m_\nu}), \qquad \mu \in \{1, \dots, m_\nu\}, \quad l = 0, \dots, \nu-1, \quad \nu = \alpha+1, \dots, p.$$

All the zeros of each 'polynomial' $\tilde{t}^{(k)}$, $k = 1, \dots, \alpha$, are put on the *corresponding* sheets of $\mathcal{R}$, that is,

$$r^{(k)}_l \ni \tilde{z}_{\mu,m_k} \to \tilde{z}_{\mu,m_k} \in \mathcal{R}_l, \qquad \mu \in \{1, \dots, m_k\}, \quad l = 0, \dots, k-1, \quad k = 1, \dots, \alpha.$$

On the sheets $\mathcal{R}_l$, $l = \alpha, \dots, p-1$, we place poles at the points $(\infty^{(l)})$, and their multiplicity must be equal to the number of zeros on $\mathcal{R}_l$. All other poles (with multiplicities) required in order that $g(z)$ be a rational function on $\mathcal{R}$ are put at $\infty^{(0)}$.

Thus, $g(z)$ is defined by its divisor as follows:

$$g(z) \in \mathfrak{M}(\mathcal{R}) : \begin{cases} g(z) = \infty, \\ \quad z \in \big\{(\infty^{(0)})^{\sum_{k=1}^{\alpha} m_k + \alpha M_\alpha}, (\infty^{(\alpha)})^{\sum_{\nu=\alpha+1}^{p} m_\nu}, \\ \qquad (\infty^{(\alpha+1)})^{\sum_{\nu=\alpha+2}^{p} m_\nu}, \dots, (\infty^{(p-1)})^{m_p}\big\}, \\ g(z) = 0, \\ \quad z \in \big\{\{\{\tilde{z}_{\mu,m_k}\}^{m_k}_{\mu=1}, k = 1, \dots, \alpha\}; \\ \qquad \{\{z^{(l)}_{\mu,m_\nu}\}^{m_\nu}_{\mu=1},\ l = 0, \dots, \nu-1,\ \nu = \alpha+1, \dots, p\}\big\}. \end{cases} \tag{2.20}$$

We choose a normalization of $g(z)$ such that

$$\prod_{l=0}^{p} g_l = \prod_{k=1}^{\alpha} t^{(k)} \cdot \prod_{\nu=\alpha+1}^{p} (t^{(\nu)})^\nu, \tag{2.21}$$

where $t^{(k)}$ is a polynomial on $\mathbb{C}$:

$$t^{(k)} = \prod_{l=0}^{k-1} \tilde{t}_l^{(k)}.$$

In view of this normalization we point out one identity, which is important for what follows:

$$\prod_{k=1}^{\alpha} t^{(k)} = \prod_{k=1}^{\alpha} \prod_{l=0}^{k-1} \tilde{t}_l^{(k)} = \prod_{l=0}^{\alpha-1} \prod_{k=l+1}^{\alpha} \tilde{t}_l^{(k)}.$$

Besides the functions $g(z)$, $\tilde{t}^{(k)}$, and $t^{(\nu)}$ we shall require for the definition of $\{f_j^{(m)}(z)\}_{j=1}^p$ other rational functions on $r^{(\alpha)}$,

$$\widetilde{T}^{(\alpha)} \in \mathfrak{M}(r^{(\alpha)}) : \begin{cases} \widetilde{T}^{(\alpha)}(z) = \infty, \\ \qquad\qquad z \in \big\{(\infty^{(0)})^{\alpha M_\alpha}\big\}, \\ \widetilde{T}^{(\alpha)}(z) = 0, \\ \qquad\qquad z \in \big\{\{z_{\mu,m_\nu}^{(l)}\}_{\mu=1}^{m_\nu},\ \nu = \alpha+1, \dots, p,\ l = 0, \dots, \alpha-1\big\}. \end{cases}$$

In connection with $\widetilde{T}^{(\alpha)}$, we point out an additional identity, which is useful in what follows:

$$\widetilde{T}^{(\alpha)}(\psi^{(\alpha)})^{M_\alpha} = T^{(\alpha)} := \prod_{\nu=\alpha+1}^{p} t^{(\nu)}. \tag{2.22}$$

(Identity (2.22) becomes obvious once one continues the polynomial $T^{(\alpha)}$ on $\overline{\mathbb{C}}$ analytically to $r^{(\alpha)}$ by copying its values on each sheet $r_l^{(\alpha)}$, $l = 0, \dots, \alpha-1$.)

We now define a collection of functions $\{f_j^{(m)}\}$:

$$f_j^{(m)} := \begin{cases} g_p, & j = p, \\ g_p \displaystyle\prod_{l=j}^{p-1} \frac{g_l}{\prod_{\nu=l+1}^{p} t^{(\nu)}}, & j = \alpha, \dots, p-1, \\ g_p \Big(\displaystyle\prod_{l=\alpha}^{p-1} \frac{g_l}{\prod_{\nu=l+1}^{p} t^{(\nu)}}\Big) \displaystyle\prod_{l=j}^{\alpha-1} \frac{g_l}{\widetilde{T}_l^{(\alpha)} \prod_{k=l+1}^{\alpha} \tilde{t}_l^{(k)}}, & j = 1, \dots, \alpha-1. \end{cases} \tag{2.23}$$

We verify that $\{f_j^{(m)}\}_{j=1}^p$ is a solution of the boundary-value problem (2.14) with weights $\{w_j^{(m)}\}$.

First, we claim that $f_j^{(m)} \in \widehat{H}^+(\Omega_j)$. For the successive multiplication of branches of g in the definition of $f_j^{(m)}$ 'removes' the branch points from the last sheets of $\mathcal{R}$ (the Riemann surface of g) and the successive divisions by the polynomials $t^{(\nu)}$ and the corresponding branches of $\tilde{t}^{(k)}$ and $\widetilde{T}^{(\alpha)}$ 'remove' the zeros and poles from these sheets of $\mathcal{R}$. As a result, $f_j^{(m)}(z)$ is an analytic function in $\overline{\mathbb{C}}$ branching at the end-points of the intervals E_j and without zeros or poles on the principal sheet of its Riemann surface, that is, the function $f_j^{(m)}$ is single-valued, analytic, and non-vanishing in Ω.

Second, we verify that the functions $\{f_j^{(m)}\}_{j=1}^p$ satisfy boundary conditions (2.14). We use the fact that polynomials approximating real functions can be selected to have real coefficients; hence the algebraic functions g, $\tilde{t}^{(k)}$, and $\widetilde{T}^{(\alpha)}$ can also be considered symmetric with respect to the real axis. Hence

$$|g_l^2(x)| = \bigl|(g_l\, g_{l-1})(x)\bigr|, \qquad x \in E_l, \quad l = 1, \dots, p,$$

on E_l.

For $j = \alpha + 1, \dots, p$ we have

$$\frac{|f_p^{(m)}(x)|^2}{|f_{p-1}^{(m)}(x)|} = \left|\left(\frac{g_p^2 t^{(p)}}{g_p g_{p-1}}\right)(x)\right| = t^{(p)}(x) = \frac{1}{w_p^{(m)}(x)}\,, \quad x \in E_p,$$

$$\frac{|f_j^{(m)}(x)|^2}{|(f_{j-1}f_{j+1})(x)|} = \left|\left(\frac{g_j \prod_{\nu=j}^p t^{(\nu)}}{g_{j-1}\prod_{\nu=j+1}^p t^{(\nu)}}\right)(x)\right| = t^{(j)}(x) = \frac{1}{w_j^{(m)}(x)}\,, \quad x \in E_j,$$
$$j = \alpha + 1, \dots, p-1.$$

For $j = \alpha$, in view of identity (2.22), we obtain

$$\frac{|f_\alpha^{(m)}(x)|^2}{|(f_{\alpha-1}f_{\alpha+1})(x)|} = \left|\left(\frac{g_\alpha \bigl[\prod_{l=\alpha+1}^{p-1}\prod_{\nu=l+1}^p t^{(\nu)}\bigr]^2 \prod_{\nu=\alpha+1}^p t^{(\nu)} \widetilde{T}_{\alpha-1}^{(\alpha)} \tilde{t}_{\alpha-1}^{(\alpha)}}{g_{\alpha-1}\bigl[\prod_{l=\alpha}^{p-1}\prod_{\nu=l+1}^p t^{(\nu)}\bigr]^2}\right)(x)\right|$$
$$= \frac{\widetilde{T}_{\alpha-1}^{(\alpha)} \tilde{t}_{\alpha-1}^{(\alpha)}}{T^{(\alpha)}} = \frac{1}{w_\alpha^{(m)}(x)}\,, \qquad x \in E_\alpha.$$

For $j = 2, \dots, \alpha - 1$, by the symmetry of $\tilde{t}^{(k)}$ and $\widetilde{T}^{(\alpha)}$ relative to the real axis we obtain

$$\bigl|\widetilde{T}_j^{(\alpha)}(x)\bigr| = \bigl|\widetilde{T}_{j-1}^{(\alpha)}(x)\bigr|, \quad \bigl|\tilde{t}_j^{(k)}(x)\bigr| = \bigl|\tilde{t}_{j-1}^{(k)}(x)\bigr|, \qquad x \in E_j, \quad k = j+1, \dots, \alpha,$$

and therefore

$$\frac{|f_j^{(m)}(x)|^2}{|(f_{j-1}f_{j+1})(x)|} = \left|\left(\frac{g_j \widetilde{T}_{j-1}^{(\alpha)} \prod_{k=j}^\alpha \tilde{t}_{j-1}^{(k)}}{g_{j-1}\widetilde{T}_j^{(\alpha)} \prod_{k=j+1}^\alpha \tilde{t}_j^{(k)}}\right)(x)\right|$$
$$= \tilde{t}_{j-1}^{(j)}(x) = \frac{1}{w_j^{(m)}(x)}\,, \qquad x \in E_j, \quad j = 2, \dots, \alpha - 1.$$

Finally, for $j = 1$, taking into account, besides the symmetry of the functions g, $\tilde{t}^{(k)}$, and $\widetilde{T}^{(\alpha)}$, also their normalizations (see (2.21)), we obtain for $x \in E_1$ the equality

$$\frac{|f_1^{(m)}(x)|^2}{|f_2^{(m)}(x)|} = \left|\left(\frac{g_1^2 g_2 \cdots g_p \prod_{\nu=\alpha+1}^p (t^{(\nu)})^{\nu-\alpha} \prod_{l=2}^{\alpha-1} \widetilde{T}_l^{(\alpha)} \prod_{l=2}^{\alpha-1}\prod_{k=l+1}^\alpha \tilde{t}_l^{(k)}}{\bigl[\prod_{\nu=\alpha+1}^p (t^{(\nu)})^{\nu-\alpha}\bigr]^2 \bigl[\prod_{l=1}^{\alpha-1} \widetilde{T}_l^{(\alpha)}\bigr]^2 \bigl[\prod_{l=1}^{\alpha-1}\prod_{k=l+1}^\alpha \tilde{t}_l^{(k)}\bigr]^2}\right)(x)\right|$$
$$= \left|\left(\frac{\prod_{k=1}^\alpha \prod_{l=0}^{k-1} \tilde{t}_l^{(k)}}{\prod_{k=2}^\alpha (\tilde{t}_1^{(k)})^2 \prod_{l=2}^{\alpha-1}\prod_{k=l+1}^\alpha \tilde{t}_l^{(k)}}\right)(x)\right|$$
$$= \tilde{t}_0^{(1)} = \frac{1}{w_1^{(m)}(x)}\,, \qquad x \in E_1.$$

Thus, the resulting functions $\{f_j^{(m)}\}_{j=1}^p$ (see (2.23)) make up a solution of the boundary-value problem (2.14) with weight functions $\{w_j^{(m)}\}_{j=1}^p$ (see (2.18)), which, in their turn, approximate the collection $\{w_j\}_{j=1}^p$ of integrable functions satisfying the Szegő condition.

(B1) *Definition of the polynomials* $P_\alpha^{(m)}$, *their asymptotic behaviour.* We consider the following rational function on $\mathcal{R}$:

$$\chi(z) := g(z) \cdot \Psi^n(z),$$

where g and Ψ are defined above (see (2.20) and (0.21)). We set

$$S_\nu(\chi; z) := \sum_{0\leqslant i_0<i_1<\dots<i_\nu\leqslant p} \prod_{l=0}^{\nu} \chi_{i_l}(z), \qquad \nu = 0, 1, \dots, p. \tag{2.24}$$

The elementary symmetric functions $S_\nu(\chi)$ of the distinct branches of the algebraic function χ are single-valued in $\overline{\mathbb{C}}$, and since all poles of χ lie over the point ∞, the $S_\nu(z)$ are polynomials.

We set

$$\widetilde{P}_\alpha^{(m)} := \frac{S_{p-\alpha}}{\prod_{\nu=\alpha+1}^{p} (t^{(\nu)})^{\nu-\alpha}}\,. \tag{2.25}$$

We claim that, first, the rational function $\widetilde{P}_\alpha^{(m)}$ is a polynomial and, second, $\deg \widetilde{P}_\alpha^{(m)} = n(p-\alpha+1)$.

In fact, $S_{p-\alpha}$ is a polynomial and each term in (2.24) vanishes (with the corresponding multiplicities) at the zeros of the polynomial $\prod_{\nu=\alpha+1}^{p}(t^{(\nu)})^{\nu-\alpha}$ (by the definition of χ and g; see (2.20)). An estimate for the degree of the polynomial $\widetilde{P}_\alpha^{(m)}$ can be obtained by a consideration of the order of the pole at infinity for each term

$$\frac{\prod_{l=0}^{\nu} g_{i_l}\Psi_{i_l}^n}{\prod_{\nu=\alpha+1}^{p}(t^{(\nu)})^{\nu-\alpha}}\,, \qquad 0 \leqslant i_0 < i_1 < \dots < i_\nu \leqslant p, \tag{2.26}$$

in the definition (2.25), (2.24). For $\sum_{k=1}^{\alpha} m_k + \alpha \sum_{\nu=\alpha+1}^{p} m_k - pn \leqslant n$ an estimate for the degree of $\widetilde{P}_\alpha^{(m)}$ is provided by the order of the pole at infinity of the term

$$\left(\frac{\prod_{j=\alpha}^{p} g_j \Psi_j^n}{\prod_{\nu=\alpha+1}^{p}(t^{(\nu)})^{\nu-\alpha}}\right)(x)\bigg|_{z=\infty} = O(z^{n(p-\alpha+1)}).$$

Thus, for

$$n \geqslant \frac{\widetilde{M}_\alpha + \alpha M_\alpha}{(p+1)}\,, \qquad \widetilde{M}_\alpha = \sum_{k=1}^{\alpha} m_k, \quad M_\alpha = \sum_{\nu=\alpha+1}^{p} m_\nu,$$

the function defined in (2.25) is the polynomial $\widetilde{P}_\alpha^{(m)}$ of degree $n(p-\alpha+1)$.

It remains to verify that after normalization the polynomials have the asymptotic behaviour (2.17) as $n \to \infty$. Indeed, it follows from property (2.10) of the weights of Ψ (see Proposition 2.2) that the main contribution to the asymptotic behaviour as $n\to\infty$ in the domain Ω_α is provided by one term in (2.26):

$$\widetilde{P}_\alpha^{(m)}(z)=\frac{\prod_{j=\alpha}^p g_j(z)}{\prod_{\nu=\alpha+1}^p(t^{(\nu)}(z))^{\nu-\alpha}}\prod_{j=\alpha}^p\Psi_j^n(z)+o\biggl(\prod_{j=\alpha}^p\Psi_j^n(z)\biggr),\qquad z\in\Omega_\alpha,$$

where $o(\,\cdot\,)$ uniformly decreases with exponential rate on compact subsets of Ω_α. Normalizing this relation and taking account of the definitions of the functions Φ_α and $f_\alpha^{(m)}$ (see (0.25) and (2.23)) we see that the polynomials

$$P_\alpha^{(m)}(z):=\frac{\widetilde{P}_\alpha^{(m)}(z)}{\widetilde{P}_\alpha^{(m)}(\infty)}$$

satisfy asymptotic formula (2) in (2.17). A contribution to the asymptotic formula on the interval E_α is made by two terms in (2.26):

$$\widetilde{P}_\alpha^{(m)}(x)=\frac{\prod_{j=\alpha+1}^p g_j(x)\Psi_j^n(x)}{\prod_{\nu=\alpha+1}^p(t^{(\nu)}(x))^{\nu-\alpha}}\bigl[g_\alpha(x)\Psi_\alpha^n(x)+g_{\alpha-1}(x)\Psi_{\alpha-1}^n(x)+o\bigl(\Psi_\alpha^n(x)\bigr)\bigr],\qquad x\in E_\alpha,$$

therefore, in view of the symmetry of g and Ψ with respect to the real axis:

$$g_\alpha(x)=\overline{g_{\alpha-1}(x)},\quad \Psi_\alpha(x)=\overline{\Psi_{\alpha-1}(x)},\qquad x\in E_\alpha,$$

and also the definitions of Φ_α and $f_\alpha^{(m)}$, we obtain

$$\widetilde{P}_\alpha^{(m)}(x)=f_\alpha^{(m)}(x)\Phi_\alpha^n(x)+\overline{f_\alpha^{(m)}(x)\Phi_\alpha^n(x)}+o\bigl(\Phi_\alpha^n(x)\bigr),\qquad x\in E_\alpha,$$

which gives us after normalization asymptotic formula (1) in (2.17).

Thus, we have constructed polynomials satisfying (2.17) for arbitrary fixed α. The same construction can be carried out for the other α in the set $\{1,2,\dots,p\}$. This completes the proof of Theorems 2 and 1.

§ 3. Appendix: An extremal property of orthogonal polynomials

In this section we present an extremal property of polynomials defined by orthogonality relations. It is a useful tool in the proof of strong asymptotic formulae (see, for instance, [11]). We demonstrate it for ordinary orthogonal polynomials first, and for polynomials orthogonal with respect to a variable weight dependent on the index after that.

Let

$$q_n(x)=x^n+\cdots,\qquad \int_a^b q_n(x)x^\nu\rho(x)\,dx=0,\quad \nu=0,1,\dots,n-1,\qquad\text{(A.1)}$$

be an orthogonal polynomial on the interval $E=[a,b]$ with respect to a weight $\rho(x)$ satisfying the Szegő condition and such that

$$\int_a^b\ln\rho(x)\,\frac{dx}{\sqrt{(x-a)(b-x)}}>-\infty.$$

Let $F(x)$ be the *Szegő function* of the weight $\rho(x)$, which is the solution of the following boundary value problem:

$$F(x) := \frac{f(z)}{f(\infty)},$$

$$\begin{aligned} &(1) \quad f, \frac{1}{f} \in H_{2,\rho}(\overline{\mathbb{C}} \setminus E), \quad f(\infty) > 0, \\ &(2) \quad |f(x)|^2 \rho(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}, \quad x \in E = [a,b]. \end{aligned} \tag{A.2}$$

We denote the leading term of the asymptotic formula for the polynomials $q_n(z)$ by $\Phi(z)$:

$$\Phi(z) := z + \sqrt{z^2 - 1} \qquad \big(E = [-1,1]\big).$$

Recall that $\Phi(z)$ can be described in terms of the complex logarithmic potential

$$v_\lambda(z) := V_\lambda(z) + i\widetilde{V_\lambda(z)}, \quad \text{where} \quad V_\lambda(z) := \int_a^b \ln\frac{1}{|z-x|}\, d\lambda(x),$$

of the equilibrium measure $\lambda(x)$ of the interval E, which is characterized by the equilibrium property

$$V_\lambda(x) = \gamma, \qquad x \in E.$$

With this notation,

$$\Phi(z) = C(E)e^{-v_\lambda(z)}, \qquad C(E) := e^\gamma. \tag{A.3}$$

We also set

$$\mathcal{F}_n(x) := \Phi^n(x)F(x) + \overline{\Phi^n(x)F(x)}, \qquad x \in E. \tag{A.4}$$

We have the following result.

Proposition A.1. *Let q_n be the orthogonal polynomial* (A.1)*, and let C and $\mathcal{F}_n$ be defined by conditions* (A.2)–(A.4)*. Then, for arbitrary $n \in \mathbb{N}$ and each polynomial $P_n(x) = x^n + \cdots$,*

$$\left\| \frac{q_n(x)}{C^n} - \mathcal{F}_n(x) \right\|_{L_{2,\rho}(E)} \leqslant \left\| \frac{P_n(x)}{C^n} - \mathcal{F}_n(x) \right\|_{L_{2,\rho}(E)}.$$

Proof. We have

$$\begin{aligned} &\left\| \frac{q_n(x)}{C^n} - \mathcal{F}_n(x) \right\|^2_{L_{2,\rho}(E)} \\ &\quad = \frac{1}{C^n}\int_a^b |q_n(x)|^2\rho(x)\,dx - 2\operatorname{Re}\int_a^b \frac{q_n(x)}{C^n}\,\overline{\mathcal{F}_n(x)}\rho(x)\,dx + \int_a^b |\mathcal{F}_n|^2\rho(x)\,dx \\ &\quad =: I_1(q_n) + 2\operatorname{Re} I_2(q_n) + I_3. \end{aligned} \tag{A.5}$$

By the extremal property of orthogonal polynomials,

$$I_1(q_n) \leqslant I_1(P_n).$$

In view of the identity

$$\overline{\Phi(x)} = \frac{1}{\Phi(x)}$$

and the symmetry of Φ and F with respect to the real axis, the second integral in (A.5) can be represented as follows:

$$\begin{aligned} I_2(q_n) &= \int_a^b \frac{q_n(x)}{C^n\Phi^n(x)}\,\overline{F(x)}\rho(x)\,dx + \int_a^b \frac{q_n(x)}{C^n}\,\frac{F(x)}{\overline{\Phi^n(x)}}\rho(x)\,dx \\ &= \int_a^b \frac{q_n(x)}{C^n\Phi_+^n(x)}\,\overline{F_+(x)}\rho(x)\,dx + \int_a^b \frac{q_n(x)}{C^n\Phi_-^n(x)}\,\overline{F_-(x)}\rho(x)\,d(-x), \end{aligned}$$

where '+' indicates the boundary values of an analytic function when the point approaches the interval from above; correspondingly, '−' indicates the boundary values on the lower bank of E. Thus,

$$I_2(q_n) = \oint_E \frac{q_n(\xi)}{C^n\Phi^n(\xi)}\overline{F(\xi)}\rho(\xi)\,|d\xi|.$$

In view of the representing property of the Szegő function $F(z)$ expressed by the equality

$$H(\infty) = \frac{1}{\mu}\oint_E H(\xi)\overline{F(\xi)}\rho(\xi)\,|d\xi| \text{for } H(z) \in H_{2,\rho}(\overline{\mathbb{C}} \setminus E),$$

where

$$\mu = \oint_E |F(\xi)|^2\rho(\xi)\,|d\xi|$$

(see, for instance, [22], p. 165), and of the expansion

$$C^n\Phi^n(z)\big|_{z=\infty} = z^n + \cdots, \qquad \frac{q_n}{C^n\Phi^n}, \frac{P_n}{C^n\Phi^n} \in H_{2,\rho}(\overline{\mathbb{C}} \setminus E),$$

we obtain

$$I_2(q_n) = I_n(P_n) = \mu.$$

It remains to observe that the last integral I_3 in (A.5) is independent on q_n. The proof of Proposition A.1 is complete.

We consider now a similar extremal property of the polynomials $q_n(x) = x^n + \cdots$ orthogonal with respect to a variable weight $w_n(x)\rho(x)$:

$$\int_E q_n(x)x^\nu w_n(x)\rho(x)\,dx = 0, \qquad \nu = 0,\dots,n-1, \quad w_n \in C(E), \tag{A.6}$$

where the weight must satisfy the condition of the existence of limits

$$\begin{aligned} \big(w_n(x)\big)^{1/n} &\underset{n\to\infty}{\rightrightarrows} \varphi(x), \qquad \varphi(x)>0, \quad x\in E; \\ \frac{w_n(x)}{\varphi^n(x)} &\underset{n\to\infty}{\rightrightarrows} \psi(x), \end{aligned} \tag{A.7}$$

which must hold uniformly on E.

We now define analogues of the functions Φ and F in (A.3) and (A.2). Let λ_φ be the equilibrium measure (concentrated on the interval E) of the equilibrium problem for a potential in the exterior field $\ln\varphi(x)$. This measure is uniquely specified by the equilibrium condition on E:

$$V_{\lambda_\varphi} + \frac{1}{2}\ln\varphi \begin{cases} =\gamma, & x\in \operatorname{supp}\lambda_\varphi, \\ >\gamma, & x\in E\setminus \operatorname{supp}\lambda_\varphi. \end{cases} \tag{A.8}$$

By analogy with (A.3) we set

$$\Phi(x) := Ce^{-v_{\lambda_\varphi}(z)}, \qquad v_{\lambda_\varphi}(z) := V_{\lambda_\varphi}(z) + i\widetilde{V_{\lambda_\varphi}(z)}, \qquad C := e^{\gamma}.$$

Assume that the exterior field has the property that the interval E is the support of the equilibrium measure in the problem (A.8):

$$\operatorname{supp}\lambda_\varphi = E = [a,b], \tag{A.9}$$

and let $\psi(x)$ in (A.7) be a function satisfying the Szegő condition on E:

$$\int_a^b \ln\big(\psi(x)\rho(x)\big)\,\frac{dx}{\sqrt{(x-a)(b-x)}} > -\infty. \tag{A.10}$$

As before (see (A.2)), let $F(z)$ be the Szegő function for the weight $\psi\rho$ on E, that is,

$$F(z) := \frac{f(z)}{f(\infty)},$$

$$(1) \quad f, \frac{1}{f} \in H_{2,\psi\rho}(\overline{\mathbb{C}}\setminus E), \quad f(\infty)>0,$$

$$(2) \quad |f(x)|^2\psi(x)\rho(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}}\,, \quad x\in E;$$

we also set

$$\mathcal{F}_n(x) := \frac{\Phi^n(x)}{|\Phi^n(x)|}F(x) + \overline{\frac{\Phi^n(x)}{|\Phi^n(x)|}F(x)}.$$

Finally, we recall the Gonchar–Rakhmanov asymptotic formula (see [23]) for the nth root of the polynomials (A.6) and (A.7):

$$|q_n(z)|^{1/n} \rightrightarrows \exp\{-V_{\lambda_\varphi}(z)+\gamma\}, \qquad z\in K \Subset \overline{\mathbb{C}}\setminus E. \tag{A.11}$$

We have the following result.

Proposition A.2. *Let $q_n(x)$ be the polynomials* (A.6) *orthogonal on E with respect to variable weights $w_n\rho$, satisfying conditions* (A.7), (A.9), *and* (A.10). *Then the following relation holds for an arbitrary sequence of polynomials $P_n(x) = x^n + \cdots$:*

$$\left\| \frac{q_n(x)}{(C|\Phi(x)|)^n} - \mathcal{F}_n(x) \right\|_{L_{2,\psi\rho}(E)} \leqslant \left\| \frac{P_n}{(C|\Phi(x)|)^n} - \mathcal{F}_n(x) \right\|_{L_{2,\psi\rho}(E)} \cdot \bigl(1 + o(1)\bigr). \quad \text{(A.12)}$$

Proof. As before, we represent the left-hand side of (A.12) as a sum of three integrals:

$$\begin{aligned} &\left\| \frac{q_n}{(C|\Phi|)^n} - \mathcal{F}_n \right\|_{L_{2,\psi\rho}(E)} \\ &\qquad =: I_1(q_n) - 2\operatorname{Re} I_2(q_n) + I_3 \\ &\qquad = \int_a^b \left| \frac{q_n(x)}{C^n\Phi^n(x)} \right|^2 \psi(x)\rho(x)\,dx \\ &\qquad\qquad - 2\operatorname{Re} \int_a^b \frac{q_n(x)}{C^n\Phi^n(x)}\, \overline{\mathcal{F}_n(x)}\psi(x)\rho(x)\,dx + \int_a^b |\mathcal{F}_n(x)|^2\psi(x)\rho(x)\,dx. \end{aligned}$$

In view of (A.7), we obtain

$$I_1(q_n) = \int_a^b \frac{q_n^2(x)w_n(x)\rho(x)\,dx}{|C^2\Phi^2(x)\varphi(x)|^n}\bigl(1 + o(1)\bigr).$$

By equilibrium relation (A.8) the denominator of the integrand is equal to one, therefore

$$I_1(q_n) = \int_a^b q_n^2(x)w_n(x)\rho(x)\,dx\,\bigl(1 + o(1)\bigr).$$

By the extremal property of orthogonal polynomials (A.6) we obtain

$$I_1(q_n) \leqslant I_1(P_n).$$

Finally, as before,

$$\begin{aligned} I_2(q_n) &= \int_a^b \frac{q_n(x)}{|C\Phi(x)|^n} \left\{ \overline{\frac{\Phi^n(x)}{|\Phi(x)|^n}F(x)} + \frac{\Phi^n(x)}{|\Phi(x)|^n}F(x) \right\} \psi(x)\rho(x)\,dx \\ &= \oint_E \frac{q_n(\xi)}{C^n\Phi^n(\xi)}\, \overline{F(\xi)}\psi(\xi)\rho(x)\,|dx| \end{aligned}$$

and from the reproducing property of the Szegő function we see that

$$I_2(q_n) = I_2(P_n).$$

The proof of Proposition A.2 is complete.

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