L. A. Skornyakov, More on quasi-Frobenius rings, *Mathematics of the USSR-Sbornik*, 1973, Volume 21, Issue 4, 511–522

DOI: 10.1070/SM1973v021n04ABEH002032

Mat. Sbornik
Tom 92 (134) (1974), No. 4

Math. USSR Sbornik
Vol. 21 (1974), No. 4

MORE ON QUASI-FROBENIUS RINGS

UDC 519.48

L. A. SKORNJAKOV*

Abstract. Let R be a ring and J its Jacobson radical. Let us set $J^1 = J$, $J^\alpha = JJ^{\alpha-1}$, and $J^\alpha = \bigcap_{\beta<\alpha} J^\beta$ if α is a limit ordinal. We call a ring an annihilating ring if the left (right) annihilator of the right (left) annihilator of an arbitrary left (right) ideal I is I itself. We prove that a ring R is quasi-Frobenius if and only if it is a left self-injective annihilating ring and $J^\alpha = 0$ for some transfinite α.

Bibliography: 15 items.

Up to the present time numerous criteria have been obtained for a ring to be quasi-Frobenius (see, e.g., [2], [3], [8]–[10], [12], [13]). The original definition of a quasi-Frobenius ring includes annihilator conditions, whereas most of the criteria include self-injectivity. Moreover, some sort of chain conditions are imposed. In this note we determine how the annihilator conditions and self-injectivity may be combined. Furthermore, a certain restriction is imposed on the radical. An example is given which shows the restriction to be a necessary one.

All rings under consideration are assumed to be associative with identity element. If R is a ring and H is a subset of R, we set

$$l(H) = \{x \mid x \in R;\ xh = 0 \quad \text{for all} \quad h \in H\}$$

and

$$r(H) = \{x \mid x \in R,\ hx = 0 \quad \text{for all} \quad h \in H\}.$$

We shall say that a ring is *left-annihilating* if $lr(I) = I$ for every left ideal I. A subset H of R is called *right-balanced* if for every mapping f of H into R for which each relation $\Sigma a_i h_i = 0$, $a_i \in R$, $h_i \in H$, implies $\Sigma a_i h_i f(h_i) = 0$, the intersection $\bigcap_{h \in H}(f(h) + r(h))$ is nonempty. A ring is called *right-balanced* if all its subsets are right-balanced. The properties of being *right-annihilating* and *left-balanced* are defined analogously. A ring is called annihilating if it is both right- and left-annihilating.

AMS (MOS) subject classifications (1970). Primary 16A36, 16A34, 16A52.

**Editor's note*: The present translation incorporates corrections made by the author.

If I is an ideal of R, we set $I^1 = I$, $I^\alpha = II^{\alpha-1}$ and $I^\alpha = \bigcap_{\beta<\alpha} I^\beta$ if α is a limit ordinal. If $I^\alpha = 0$ for some transfinite ordinal α, then I is called *transfinitely nilpotent.* It is easy to check that nothing is changed if instead of $I^\alpha = II^{\alpha-1}$ we set $I^\alpha = I^{\alpha-1}I$.

Main Theorem. *The following properties of a ring R with a transfinitely nilpotent Jacobson radical are equivalent*:

(1) *R is quasi-Frobenius.*

(2) *R is left and right self-injective, while $r(L)$ and $l(I)$ are nonzero for every left ideal L and right ideal I distinct from R.*

(3) *R is a left self-injective annihilating ring.*

(4) *R is a right-balanced annihilating ring.*

Note. This theorem generalizes a result of T. Kato ([13], p. 493, Theorem 10), who proved that an annihilating two-sided self-injective ring with nilpotent Jacobson radical is quasi-Frobenius.

Let us begin by proving some lemmas.

Lemma 1. *If I is a right-balanced left ideal of a ring R and $rl(a) = aR$ for each $a \in R$, then for each homomorphism ϕ: $I \to R$ there exists an element $s \in R$ such that $x\phi = xs$ for all $x \in I$.*

Proof. Since $l(x) \subseteq l(x\phi)$ for all $x \in I$, we have

$$x\varphi \in rl(x\varphi) \subseteq rl(x) = xR,$$

i.e. $x\phi = xf(x)$, where $f(x) \in R$. If $\Sigma a_i x_i = 0$, $a_i \in R$, $x_i \in I$, then

$$\sum a_i x_i f(x_i) = \sum a_i (x_i \varphi) = \left(\sum a_i x_i\right) \varphi = 0.$$

Consequently, there exists an $s \in \bigcap_{x\in I} (f(x) + r(x))$. Hence $xs = xf(x) = x\phi$ for all $x \in I$.

Lemma 2. *A left self-injective ring is right-balanced.*

Proof. Let H be a subset of a left self-injective ring R. Let us consider the mapping f mentioned in the definition of a balanced ring and the left ideal $I = RH$. Let us define a homomorphism ϕ: $I \to R$ by setting

$$\left(\sum a_i h_i\right) \varphi = \sum a_i h_i f(h_i) \quad (a_i \in R,\ h_i \in H),$$

and let us pick an $s \in R$ such that $x\phi = xs$ for all $x \in I$. Then $h(s - f(h)) = h\phi - hf(h) = 0$ for all $h \in H$, so that $s \in f(h) + r(h)$ for all $h \in H$.

A system of nonzero submodules of a left (right) module is called independent if each submodule of the system has zero intersection with the sum of the remaining ones. By the *Goldie dimension* of a module we mean the smallest cardinal number which is greater than or equal to the cardinality of every independent system of submodules. The least cardinal number which is greater than or equal to the Goldie dimensions of all factor modules of a module will be called its *thickness*. We shall denote the Goldie dimension and the thickness of a module A by dim A and thick A, respectively. We note that both the Goldie dimension and the thickness of a completely reducible module coincide with the cardinality of the set of all its irreducible summands ([1], Chapter IV, § 1).

Lemma 3. *If A, B and C are left R-modules, $A \subseteq B \subseteq C$, if the factor module B/A is completely reducible, and if* $\dim (B/A) \geq \aleph_0$, *then*

$$\text{thick } (C/A) \leqslant \dim (B/A) + \text{thick } (C/B).$$

Proof. Let us consider the direct sum

$$\dot{\sum_{\alpha\in\Omega}} (H_\alpha/D) \subseteq C/D,$$

where $A \subseteq D \subsetneqq H_\alpha$. Let us set $W = B \bigcap (\Sigma_{\alpha\in\Omega} H_\alpha)$. The module W/A is completely reducible. Since $W/D \cong (W/A)/(D/A)$, the module W/D also turns out to be completely reducible ([1], Chapter IV, § 1, Corollary 3). Here we have

$$\dim (W/D) \leqslant \dim (W/A) \leqslant \dim (B/A)$$

([1], Chapter IV, § 1). Each irreducible summand of W/D belongs to the finite sum

$$(H_{\alpha_1}/D) + \ldots + (H_{\alpha_m}/D).$$

Consequently, $W \subseteq \Sigma_{\alpha\in\Omega'} H_\alpha$, where $\Omega' \subseteq \Omega$ and

$$\text{Card}\,\Omega' = \dim (W/D) \leqslant \dim (B/A).$$

If $\gamma \notin \Omega'$ and $h \in H_\gamma \bigcap \Sigma_{\gamma\neq\beta\notin\Omega'} (H_\beta + B)$, then $h = \Sigma h_\beta + b$, $h_\beta \in H_\beta$, $b \in B$. Hence

$$b \in \Big(\sum_{\beta\notin\Omega'} H_\beta\Big) \cap B \subseteq \Big(\sum_{\beta\in\Omega'} H_\beta\Big) \cap W \subseteq \Big(\sum_{\beta\notin\Omega'} H_\beta\Big) \cap \Big(\sum_{\alpha\in\Omega'} H_\alpha\Big) \subseteq D.$$

Thus

$$h \in H_\gamma \cap \Big(\sum_{\gamma\neq\beta\in\Omega'} H_\beta\Big) \subseteq D$$

and, consequently,

$$H_\gamma \cap \sum_{\gamma\neq\beta\in\Omega'} (H_\beta + B) \subseteq D.$$

But then

$$(H_\gamma + B) \cap \sum_{\gamma\neq\beta\notin\Omega'} (H_\beta + B) \subseteq B + D,$$

i.e., the sum

$$S = \sum_{\beta \notin \Omega'} ((H_\beta + B + D)/(B + D))$$

is direct. Since $C/(B + D) \cong (C/B)/((B + D)/B)$, we have

$$\text{Card}\,(\Omega \setminus \Omega') \leqslant \text{thick}\,(C/B),$$

whence

$$\text{Card}\,\Omega = \text{Card}\,\Omega' + \text{Card}\,(\Omega \setminus \Omega') \leqslant \dim\,(B/A) + \text{thick}\,(C/B).$$

If $M \subseteq \mathfrak{S}$, where $\mathfrak{S}$ is a subset of a ring R, then by a *right* $(M, \mathfrak{S})$-*hyperplane* we mean the set of elements of the form $\Sigma x_i a_i + \Sigma y_j b_j$, where $a_i, b_j \in R$, $x_i \in M$, $y_j \in \mathfrak{S} \setminus M$ and $\Sigma a_i = 0$. It is clear that a right $(M, \mathfrak{S})$-hyperplane is a right ideal of R.

Lemma 4. *Let $\mathfrak{S}$ be an infinite subset of R. If for each nonempty subset $M \subseteq \mathfrak{S}$ the left annihilator L of $\mathfrak{S}$ is different from the left annihilator of the right $(M, \mathfrak{S})$-hyperplane, then*

$$\dim\,(R/L) > \text{Card}\,\mathfrak{S}.$$

Proof. As is well known ([5], § 14, Example F), there exists an independent system Φ of subsets(1) of $\mathfrak{S}$ such that $\text{Card}\,\Phi > \text{Card}\,\mathfrak{S}$. If $\varnothing \neq M \subseteq \mathfrak{S}$, we let $S(M)$ denote the right $(M, \mathfrak{S})$-hyperplane. Repeating word for word the arguments of the proof of the lemma in [6], we see that

$$S(M_0) + \bigcap_{i=1}^{n} S(M_i) = \sum_{x \in \mathfrak{S}} xR$$

for any choice of distinct subsets $M_0, M_1, \ldots, M_n \in \Phi$.(2) Hence

$$l(S(M_0)) \cap \sum_{i=1}^{n} l(S(M_i)) \subseteq l(S(M_0)) \cap l(\bigcap_{i=1}^{n} S(M_i))$$

$$= l(S(M_0) + \bigcap_{i=1}^{n} S(M_i)) = L.$$

This proves that the sum $\Sigma_{M \in \Phi}(l(S(M))/L)$ is direct. The inequality $l(S(M))/L \neq 0$ is included in the hypothesis.

Lemma 5. *Let R be a right-annihilating ring with $1 = e_1 + \cdots + e_n$, where $e_i^2 =$*

(1) We recall that a set X of elements of a Boolean algebra $\mathfrak{B}$ is called independent (see [5], § 14) if $x_1^{\epsilon_1} \wedge \cdots \wedge x_n^{\epsilon_n} \neq 0$ for every choice of distinct elements x_i of X and numbers $\epsilon_i = \pm 1$ (here $x^1 = x$ and x^{-1} is the complement of x). It is easy to verify that this implies the impossibility of expressing any element of X in terms of the rest using the operations of union, intersection and complementation a finite number of times.

(2) As has been noted by W. Stephenson, the main result of [6] is valid only for local rings. In fact, in the proof of a lemma in that paper it is assumed that (in the terminology of the present note) the left $(\varnothing, \mathfrak{S})$- and $(M, \mathfrak{S})$-hyperplanes are distinct for any choice of a nonempty subset M of the independent system Φ. However, this assertion has been proved only for local rings.

$e_i \neq 0$ and $e_i e_j = 0$ for $i \neq j$ and the $e_i R e_i$ are local rings, H and I are right ideals of R with

$$H/I = \dot{\sum_{\alpha\in\Omega}} (H_\alpha/I) \quad \text{and} \quad \operatorname{Card}\Omega = t \geqslant \aleph_0.$$

Then there exist a number i and a left ideal L of R such that

$$l(He_i) \subseteq L \quad \text{and} \quad \dim (R/L) > t.$$

Proof. By choosing from each H_α one element not belonging to I, we form a subset $\mathfrak{D}$ of R. Of course, Card $\mathfrak{D} = t$. In $\mathfrak{D}$ let us consider relations θ_i $(i = 1, \ldots, n)$, defined as follows: $x\theta_i y$ if and only if $xe_i - ye_i \in I$. It is clear that the θ_i are equivalence relations. If $x\theta_i y$ for all i, then $x - y \in I$, so that $x = y$. Therefore $\mathfrak{D}$ may be considered as a subset of the direct product $(\mathfrak{D}/\theta_1)\times\cdots\times(\mathfrak{D}/\theta_n)$. This makes it possible to assume that Card$(\mathfrak{D}/\theta_1) = t$. It is clear that at most one equivalence class of θ_1 contains elements which fall into I after right multiplication by e_1. Choosing one element from each of the remaining classes and multiplying on the right by e_1, we obtain a system $\mathfrak{S}$ of elements of R. Of course, Card $\mathfrak{S} = t$ and $I \cap \mathfrak{S} = \emptyset$. If x_0, x_1, $\ldots$, x_m are distinct elements of $\mathfrak{S}$, then

$$x_0 R \cap \sum_{i=1}^{m} x_i R \subseteq H_{\alpha_0} \cap \sum_{i=1}^{m} H_{\alpha_i} \subseteq I.$$

We see that for any nonempty subset $M \subseteq \mathfrak{S}$ the right ideal $\mathfrak{S}R$ is different from the right $(M, \mathfrak{S})$-hyperplane. In fact, if this were not so, then for every $x_0 \in M$ we would have

$$x_0 = x_0 a + \sum x b_x + \sum y c_y,$$

where a, b_x, $c_y \in R$, $x_0 \neq x \in M$, $y \in \mathfrak{S}\backslash M$ and $a + \Sigma b_x = 0$. Since $ue_1 = u$ for all $u \in \mathfrak{S}$, we may suppose that a, b_x, $c_y \in e_1 R e_1$. Let $\bar{J}$ be the Jacobson radical of $e_1 R e_1$. If $a \in \bar{J}$, then $(e_1 - a)d = e_1$ for some $d \in e_1 R e_1$ (see [4], §3.7, Proposition 1). Hence

$$x_0 = x_0 e_1 = \sum x b_x d + \sum y c_y d \in x_0 R \cap \sum_{x_0 \neq x \in \mathfrak{S}} xR \subseteq I,$$

which contradicts the construction of $\mathfrak{S}$. If, however, $a \notin \bar{J}$, then for some $x \in M\backslash x_0$ we have $b_x \notin \bar{J}$. Since b_x is invertible in $e_1 R e_1$, as above there arises a contradiction to the condition that $x \notin I$. Because R is right-annihilating, this result permits an application of Lemma 4. It remains to set $L = l(\mathfrak{S})$ and turn our attention to the validity of the inclusion $l(He_1) \subseteq L$, which follows from $\mathfrak{S} \subseteq \mathfrak{D}e_1 \subseteq He_1$.

We recall that a ring R is called a *left PF-ring* if every exact left R-module is a generator of the category of all left R-modules, and a *right S-ring* if $r(I) \neq 0$ for every left ideal I distinct from R.

Lemma 6. *If R is a left self-injective right S-ring with Jacobson radical J, then R is a left-annihilating left PF-ring, every simple left R-module is isomorphic to a* *left ideal of R, and the factor ring R/J is classically semisimple.*

Proof. By the results of T. Kato ([13], p. 490, Theorem 7), a ring R satisfying the conditions of the lemma turns out to be a left PF-ring. The classical semisimplicity of R/J was proved by Utumi ([15], p. 60, Theorem 3.4). Azumaya ([7], p. 703, Theorem 7) noted that every simple left module over a left PF-ring is isomorphic to a left ideal. Therefore, the fact that the ring is left-annihilating is a consequence of results due to Björk ([8], p. 65, Proposition 2.1).

Lemma 7. *Let R be a left self-injective left-annihilating ring, J its Jacobson radical, and $1 = e_1 + \cdots + e_n$, where $e_i^2 = e_i \neq 0$, the e_iRe_i are local rings, $e_ie_j = 0$ and $e_iRe_j \subseteq J$ for $i \neq j$. Let I and H be left ideals of R, $I \subseteq H$, $\mathfrak{S}$ an infinite subset of H, and*

$$H/I = \dot{\sum_{x \in \mathscr{E}}} R(x + I),$$

where the $R(x + I)$ are irreducible left R-modules. Then

$$\dim (r(J\mathscr{E})/r(\mathscr{E})) > \operatorname{Card} \mathscr{E}.$$

Proof. By Lemma 6, R is a left PF-ring and there exist isomorphisms

$$\chi_x : R(x + I) \to Rc_x',$$

where the c_x' lie in the left socle C of R. Of course, there exists a number i such that $c_x \in c_x' e_i \neq 0$. In addition, $c_x \in C$. Changing the numbering, if necessary, we have

$$\operatorname{Card} \mathscr{E}_1 = \operatorname{Card} \mathscr{E}, \quad \text{where} \quad \mathscr{E}_1 = \{x \mid x \in \mathscr{E},\ 0 \neq c_x \in Ce_1\}.$$

If $\sum_{x \in \mathfrak{S}} a_x x = 0$, where $a_x \in R$, then $a_x x \in I$ for every $x \in \mathfrak{S}$. Hence $a_x c_x' = 0$ and thus $a_x c_x = 0$. Now let Φ be an independent system of subsets of $\mathfrak{S}_1$ with Card $\Phi >$ Card $\mathfrak{S}_1$ ([5], § 14, Example F). By what was proved above, for each $M \in \Phi$ the condition

$$x\varphi_M = \begin{cases} c_x, & \text{if } x \in M, \\ 0, & \text{if } x \in \mathscr{E} \setminus M, \end{cases}$$

defines a homomorphism $\phi_M \colon R(\mathfrak{S}) \to Ce_1$ of left R-modules. By the fact that R is left self-injective, there is an element $s_M \in Re_1$ such that $x\phi_M = xs_M$ for all $x \in \mathfrak{S}$. By [15] (p. 62, Proposition 3.10) we have $J\mathfrak{S}s_M \subseteq JC = 0$, i.e. $s_M \in r(J\mathfrak{S})$. If

$$s_{M_1}a_{M_1} + \ldots + s_{M_k}a_{M_k} \in r(\mathscr{E}),$$

where $M_i \in \Phi$ and $a_{M_i} \in R$, then, choosing an element

$$x \in M_i \setminus (M_1 \cup \ldots \cup M_{i-1} \cup M_{i+1} \cup \ldots \cup M_k),$$

we obtain

$$c_x a_{M_i} = xs_{M_i}a_{M_i} = x(s_{M_1}a_{M_1} + \ldots + s_{M_k}a_{M_k}) = 0.$$

If $e_1 a_{M_i} e_1 \notin J$, then $e_1 = e_1 a_{M_i} e_1 b$ for some $b \in e_1 R e_1$ ([4], §3.7, Proposition 1). Hence

$$c_x = c_x e_1 = c_x e_1 a_{M_i} e_1 b = c_x a_{M_i} e_1 b = 0,$$

which is impossible. If however, $e_1 a_{M_i} e_1 \in J$, then

$$s_{M_i} a_{M_i} = s_{M_i} e_1 a_{M_i} = s_{M_i} e_1 a_{M_i} e_1 + s_{M_i} e_1 a_{M_i} (1 - e_1) \in s_{M_i} J \subseteq r(\mathscr{E}),$$

since

$$\mathscr{E} s_{M_i} J \subseteq CJ = 0.$$

Thus the submodules $(s_M + r(\mathfrak{S}))R$, where $M \in \Phi$, form an independent system of submodules of the factor module $r(J\mathfrak{S})/r(\mathfrak{S})$. Since $\operatorname{Card}\{s_M \mid M \in \Phi\} > \operatorname{Card} \mathfrak{S}$, the lemma is proved.

Lemma 8. *If J is the Jacobson radical of a ring H, I and H are left ideals, and $J^m H = I + J^{m+1} H$ for some m, then $J^m H r(I) \subseteq \bigcap_\alpha J^\alpha$.*

Proof. We shall prove that $J^m H r(I) \subseteq J^\alpha$ for all α. For $\alpha \leq m$ this is obvious. Suppose it is true for all $\beta < \alpha$. If α is a limit ordinal, then

$$J^m H r(I) \subseteq \bigcap_{\beta < \alpha} J^\beta = J^\alpha.$$

However, if $\alpha - 1$ exists, then

$$J^m H r(I) = (I + J^{m+1} H) r(I) \subseteq J J^m H r(I) \subseteq J J^{\alpha - 1} = J^\alpha.$$

Lemma 9. *If R is an annihilating ring with a transfinitely nilpotent Jacobson radical J and with the properties listed in the statement of Lemma 7, H is a left ideal of R, and $\dim(H/JH) \geq \aleph_0$, then*

$$\operatorname{thick}(R/JH) > \dim(H/JH).$$

Proof. By [1] (Chapter III, §6, Theorem 2), Lemmas 6 and 7 imply the existence of a subset $\mathfrak{S} \subseteq H$ such that

$$H = R\mathscr{E} + JH \quad \text{and} \quad \dim(r(J\mathscr{E})/r(\mathscr{E})) > \operatorname{Card} \mathscr{E} = \dim(H/JH) \geqslant \aleph_0.$$

But then, according to Lemma 5, for a suitable idempotent e_i and a left ideal L we have

$$J\mathscr{E} = lr(J\mathscr{E}) \subseteq l(r(J\mathscr{E}) e_i) \subseteq L$$

and

$$\dim(R/L) \geqslant \dim(r(J\mathscr{E})/r(\mathscr{E})) > \dim(H/JH).$$

Since $JH = J\mathfrak{S} + J^2H$, by taking Lemma 8 into account, we obtain $JHr(J\mathfrak{S}) \subseteq \bigcap_\alpha J^\alpha = 0$, whence $JH \subseteq lr(J\mathfrak{S}) = J\mathfrak{S} \subseteq L$, and consequently

$$\text{thick}\,(R/JH) \geqslant \text{thick}\,(R/L) \geqslant \dim\,(R/L) > \dim\,(H/JH).$$

Lemma 10. *If R is a self-injective left annihilating right S-ring with Jacobson radical J and the left R-modules R/J^n are Noetherian for $n = 1, 2, \ldots$, then all submodules of the left R-module R/J^ω are countably generated. However, if, in addition, R/J^ω contains a submodule not admitting a finite system of generators and R has the properties listed in the hypotheses of Lemma 7, then there exists a left ideal I of R such that $J^\omega \subseteq I$ and $\dim(I/JI) = \aleph_0$.*

Proof. Let H be a left ideal of R containing J^ω. Let us set $H_n = H \cap J^n$ ($n = 0, 1, 2, \ldots$). Of course, $H = D_0 + H_1$, where D_0 is a finitely generated left ideal of R. Here, if $H = H_1$, we may assume that $D_0 = 0$. Let us assume that we have found finitely-generated ideals $D_0, D_1, \ldots, D_{n-1}$ with the following properties:

a) $H_i = (D_0 + D_1 + \ldots + D_{i-1}) \cap J^i + D_i + H_{i+1}$;
b) $(D_0 + D_1 + \ldots + D_{i-1}) \cap D_i \subseteq H_{i+1}$;
c) if $D_i \subseteq H_{i+1}$, then $D_i = 0$.

Since

$$H_n/H_{n+1} = H_n/(H_n \cap J^{n+1}) \cong (H_n + J^{n+1})/J^{n+1} \subseteq J^n/J^{n+1},$$

it follows from Lemma 6 and properties of completely reducible modules ([1], Chapter III, §6, Theorem 2; Chapter IV, §1, Corollaries 1 and 3) that

$$H_n/H_{n+1} = ((D_0 + D_1 + \ldots + D_{n-1}) \cap J^n + H_{n+1})/H_{n+1} \oplus (D_n + H_{n+1})/H_{n+1}$$

for some left ideal $D_n \subseteq H_n$. If $D_n \subseteq H_{n+1}$, we set $D_n = 0$. Since the left R-module R/J^{n+1} is Noetherian, the left ideal D_n can be taken to be finitely-generated. It is obvious that properties a)–c) are valid. Let us set $H' = J^\omega + D_0 + D_1 + \cdots$. It is clear that $H = H' + H_n$ for each n. Since

$$r(H_n) = r(lr(H) \cap lr(J^n)) = rl(r(H) + r(J^n)) = r(H) + r(J^n),$$

we have

$$r(H) = r(H') \cap r(H_n) = r(H') \cap (r(H) + r(J^n)) = r(H) + r(H') \cap r(J^n),$$

i.e. $r(H') \cap r(J^n) \subseteq r(H)$ for all n. Hence

$$r(H') = r(H') \cap r(J^\omega) = r(H') \cap rl(\bigcup_{n=1}^{\infty} r(J^n))$$

$$= r(H') \cap (\bigcup_{n=1}^{\infty} r(J^n)) = \bigcup_{n=1}^{\infty} (r(H') \cap r(J^n)) \subseteq r(H) \subseteq r(H')$$

and consequently

$$H' = lr(H') = lr(H) = H.$$

To prove the first assertion we have only to observe that H'/J^{ω} is a countably-generated left R-module. We now assume that the left R-module H/J^{ω} is not finitely generated. If $D_i \subseteq H_{i+1}$ for all $i > p$, then, by construction, $D_i = 0$ for $i > p$. Hence

$$H = H' = J^{\omega} + D_0 + D_1 + \ldots + D_p,$$

which implies that H/J^{ω} is finitely generated. Thus there exist arbitrarily large values of i for which $D_i \nsubseteq H_{i+1}$. But then among the idempotents mentioned in Lemma 7 there is an e_k such that $e_k D_i \nsubseteq H_{i+1}$ for $i = n_1, n_2, \ldots$, where $n_1 < n_2 < \cdots$. In each set $e_k D_{n_j}$ choose an x_j not belonging to H_{nj+1}, and set $I = J^{\omega} + Rx_1 + Rx_2 + \cdots$.

Suppose that $ax_j \in Rx_1 + \cdots + Rx_{j-1}$. By b), $ax_j \in H_{n_j+1}$. If $ae_k \notin J$, then $e_k a e_k \notin J$ since $e_l a e_k \in J$ for $l \neq k$. By the properties of local rings ([4], § 3.7, Proposition 1), for some $b \in e_k R e_k$ we have

$$x_j = e_k x_j = b e_k a e_k x_j = b e_k a x_j \in H_{n_j+1},$$

contrary to construction. Thus, $ax_j = ae_k x_j \in JI$, i.e. the system $\{R(x_i + JI) | i = 1, 2, \ldots\}$ is an independent system of submodules of I/JI, q.e.d.

Lemma 11. *An annihilating ring R with a transfinitely nilpotent Jacobson radical J and which satisfies the conditions of Lemma 7 is left-Noetherian.*

Proof. Let us assume that $\dim(J^n/J^{n+1}) \geq \aleph_0$ for some n. Of course, we may assume that $\dim(J^i/J^{i+1}) < \aleph_0$ for $i = 1, \ldots, n-1$. By Lemma 9,

$$\text{thick}\,(R/J^{n+1}) > \dim\,(J^n/J^{n+1}).$$

On the other hand, a simple induction using Lemma 3 permits us to establish that thick $(R/J^n) < \aleph_0$. Again applying Lemma 3, we obtain

$$\dim\,(J^n/J^{n+1}) < \text{thick}\,(R/J^{n+1}) \leqslant \dim\,(J^n/J^{n+1}) + \text{thick}\,(R/J^n) = \dim\,(J^n/J^{n+1}),$$

which, of course, is impossible. Thus, all the factor modules J^n/J^{n+1} are Noetherian, which implies that all the left R-modules R/J^n are Noetherian (see [4], § 1.4, Proposition 6). If the left R-module R/J^{ω} is not Noetherian, then, according to Lemma 10, we have $\dim(I/JI) = \aleph_0$ for some left ideal I of R which contains J^{ω}. Since $(JI + J^{\omega})/JI \subseteq I/JI$, the module $(JI + J^{\omega})/JI$ is completely reducible, and by Lemmas 3 and 9, we have

$$\aleph_0 = \dim\,(I/JI) < \text{thick}\,(R/JI) \leqslant \dim\,((JI + J^{\omega})/JI) + \text{thick}((R/JI + J^{\omega})).$$

On the other hand, well-known properties of completely reducible modules ([1], Chapter IV, § 1) imply that

$$\dim\,((JI + J^{\omega})/JI) \leqslant \dim\,(I/JI) = \aleph_0.$$

Consequently

$$\aleph_0 < \text{thick}\,(R/(JI + J^{\omega})) \leqslant \text{thick}\,(R/J^{\omega}),$$

which contradicts Lemma 10. The fact that R/J^{ω} is Noetherian implies that the chain

$$J^{\omega} = lr(J^{\omega}) \subseteq l(Jr(J^{\omega})) \subseteq l(J^2r(J^{\omega})) \subseteq \cdots$$

must stabilize. But then $J^m r(J^{\omega}) = J^{m+1}r(J^{\omega})$ for some m, and by Lemma 8

$$J^m r(J^{\omega}) = J^m r(J^{\omega}) r(0) \subseteq \bigcap_{\alpha} J^{\alpha} = 0.$$

Hence $J^m \subseteq lr(J^{\omega}) = J^{\omega} \subseteq J^{m+1} \subseteq J^m$, and consequently $J^m = J^{m+1} \subseteq \bigcap_{\alpha} J^{\alpha} = 0$. Thus $R = R/J^m$ and is Noetherian by what we have proved earlier.

Lemma 12. *If the Jacobson radical J of R is transfinitely nilpotent and $e^2 = e \in R$, then eJe is transfinitely nilpotent.*

Proof. Suppose $(eJe)^{\beta} = (eJe)^{\beta+1}$. It is clear that $(eJe)^{\beta} \subseteq J^1$. If $(eJe)^{\beta} \subseteq J^{\alpha-1}$, then

$$(eJe)^{\beta} = (eJe)(eJe)^{\beta} \subseteq JJ^{\alpha-1} = J^{\alpha}.$$

Consequently $(eJe)^{\beta} \subseteq \bigcap_{\alpha} J^{\alpha} = 0$.

Proof of the Main Theorem. The validity of the implication (1)$\Rightarrow$(2) is well known ([3], Theorem 58.6). The implication (2)$\Rightarrow$(3) is a consequence of Lemma 6 and its right-hand analog. The equivalence of properties (3) and (4) follows from Lemmas 1 and 2. Finally, suppose R satisfies condition (3). Since a left-annihilating ring is obviously a right S-ring, Lemma 6 implies the classical semisimplicity of the factor ring R/J, where J is the Jacobson radical of R. Taking into account the Wedderburn-Artin theorem and other well-known results ([1], Chapter III, §3; §8, Propositions 5 and 1; §7, Corollary), as well as the Faith-Utumi theorem on lifting idempotents ([11], p. 174, Theorem 4.5), we obtain

$$1 = (e_{11} + \cdots + e_{1p_1}) + \cdots + (e_{m1} + \cdots + e_{mp_m}),$$

where $e_{ij}^2 = e_{ij}$, $e_{ij}e_{kl} = 0$ for $(i, j) \neq (k, l)$, $e_{ij}Re_{kl} \subseteq J$ for $i \neq k$, the $e_{ij}Re_{ij}$ are local rings, and the left R-modules Re_{ij} and Re_{ik} are isomorphic. Moreover, there exist elements $u_{ij} \in e_{i1}Re_{ij}$ and $v_{ij} \in e_{ij}Re_{i1}$ such that $e_{i1} = u_{ij}v_{ij}$ and $e_{ij} = v_{ij}u_{ij}$ ([1], Chapter III, §7, Proposition 4). Let us set $e = e_{11} + e_{21} + \cdots + e_{m1}$, $\overline{R} = eRe$ and $\overline{J} = eJe$. It is known that $\overline{J}$ is the Jacobson radical of $\overline{R}$ ([1], Chapter III, §7, Proposition 1).

We shall prove several auxiliary propositions.

(a) *The rings $e_{i1}Re_{i1}$ and $e_{ij}Re_{ij}$ are isomorphic.*

To prove this it is sufficient to note that the mapping ϕ defined by $\phi(x) = v_{ij}xu_{ij}$ for all $x \in e_{i1}Re_{i1}$ is the desired isomorphism.

(b) *If $x \in R$, L is a left ideal of R, and $eLx = 0$, then $Lx = 0$.*

In fact, $e_{ij}Lx = v_{ij}eu_{ij}Lx \subseteq v_{ij}eLx = 0$, whence

$$Lx = \sum_{i,j} e_{ij}Lx = 0.$$

Similarly, we may prove

(c) *If* $x \in R$, I *is a right ideal of* R, *and* $xIe = 0$, *then* $xI = 0$.

Let $\bar{l}(\ )$ and $\bar{r}(\ )$ have the same meaning in $\bar{R}$ as $l(\ )$ and $r(\ )$ have in R. It is easily verified that $\bar{r}(\Delta) = er(\Delta)e$ and $\bar{l}(\Delta) = el(\Delta)e$ for every subset $\Delta \subseteq \bar{R}$.

(d) $F\bar{l}(\bar{I}) = \bar{I}$ *for every right ideal* $\bar{I}$ *of* $\bar{R}$.

In fact, if $x \in \bar{r}\bar{l}(\bar{I})$, then $x \in eRe$ and

$$el(\bar{I})x = el(\bar{I})ex = \bar{l}(\bar{I})x = 0.$$

By (b), it follows that $l(\bar{I})x = 0$, i.e.

$$x \in erl(\bar{I})e = e\bar{I}Re = e\bar{I}eRe \subseteq e\bar{I} = \bar{I}.$$

Thus $\bar{r}\bar{l}(\bar{I}) \subseteq \bar{I}$. The reverse inclusion is obvious.

Similarly, we may prove

(e) $\bar{l}\bar{r}(\bar{L}) = \bar{L}$ *for every left ideal* $\bar{L}$ *of* $\bar{R}$.

From (d), (e) and Björk's results ([8], p. 64, Theorem 1.1) we have

(f) $\bar{R}$ *is a left self-injective annihilating ring.*

From (f) and Lemma 12 it follows that $\bar{R}$ satisfies the hypotheses of Lemma 11. Consequently $\bar{R}$ is left Noetherian. By an Eilenberg-Nakayama theorem ([9], pp. 11–12, Theorem 18), $\bar{R}$ is quasi-Frobenius and, in particular, Artinian and right self-injective. By (a) the Jacobson radical of each of the rings $e_{ij}Re_{ij}$ is nilpotent ([1], Chapter III, § 1, Theorem 1; § 7, Proposition 1). By the results of Björk ([8], p. 64, Theorem 1.1; p. 72, Theorem 6.2), R is right self-injective and right and left complete (co-complete in Björk's terminology), after which we may cite a theorem of Koĭfman ([2], p. 58, Theorem 2.4).

Example. Let S be the ring of generalized power series studied by Levy ([14], p. 151). The factor ring $S/(x^{>1/2})$, like all proper factor rings of S, is self-injective. It is easily verified that every ideal of the ring contains a least ideal $(x^{1/2})/(x^{>1/2})$. By [15] (p. 60, Theorem 3.4), R is a PF-ring. The fact that it is annihilating follows, e.g., from Lemma 6 (besides, this is readily established by a straightforward calculation). Nevertheless, it is obvious that R is not quasi-Frobenius.

Remark. It is not yet clear whether it is impossible to restrict ourselves to a one-sided annihilator condition. Moreover, the question of whether there exist one-sided PF-rings remains open.

Received 7/JUNE/73

BIBLIOGRAPHY

1. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ., vol. 37, Amer. Math. Soc., Providence, R. I., 1956; Russian transl., IL, Moscow, 1961. MR 18, 373; 22 #11005.

2. L. A. Koĭfman, *Cogenerators of self-injective rings*, Algebra i Logika 9 (1970), 52–66 = Algebra and Logic 9 (1970), 32–40. MR 43 #7469.

3. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Appl. Math., vol. 11, Interscience, New York, 1962; Russian transl., "Nauka", Moscow, 1969. MR 26 #2519; 40 #1490.

4. J. Lambek, *Lectures on rings and modules*, Blaisdell, Waltham, Mass., 1966; Russian transl., "Mir", Moscow, 1971. MR 34 #5857.

5. R. Sikorski, *Boolean algebras*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 25, Springer-Verlag, New York, 1969; Russian transl., "Mir", Moscow, 1969. MR 39 # 4053; 40 #2581.

6. L. A. Skornjakov, *Modules with self-dual lattice of submodules*, Sibirsk. Mat. Ž. 1 (1960), 238–241. (Russian) MR 24 # A147.

7. G. Azumaya, *Completely faithful modules and self-injective rings*, Nagoya Math. J. 27 (1966), 697–708. MR 35 #4253.

8. J.-E. Björk, *Rings satisfying certain chain conditions*, J. Reine Angew. Math. 245 (1970), 63–73. MR 43 #3295.

9. S. Eilenberg and T. Nakayama, *On the dimension of modules and algebras*. II. *Frobenius algebras and quasi-Frobenius rings*, Nagoya Math. J. 9 (1955), 1–16. MR 17, 453; 1437; 19, 1431.

10. C. Faith, *Rings with ascending condition on annihilators*, Nagoya Math. J. 27 (1966), 179–191. MR 33 #1328.

11. C. Faith and Y. Utumi, *Quasi-injective modules and their endomorphism rings*, Arch. Math. 15 (1964), 166–172. MR 29 #3503.

12. C. Faith and E. A. Walker, *Direct-sum representations of injective modules*, J. Algebra 5 (1967), 203–221. MR 34 #7575.

13. T. Kato, *Self-injective rings*, Tôhoku Math. J. (2) 19 (1967), 485–495. MR 37 #247.

14. L. S. Levy, *Commutative rings whose homomorphic images are self-injective*, Pacific J. Math. 18 (1966), 149–153. MR 33 #2663.

15. Y. Utumi, *Self-injective rings*, J. Algebra 6 (1967), 56–64. MR 35 #219.

Translated by W. L. WERNER