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PURE AND FINITELY PRESENTABLE MODULES, DUALITY HOMOMORPHISMS AND THE COHERENCE PROPERTY OF A RING

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ABSTRACT. The homological properties of pure modules are considered, showing, in particular, that for coherent rings the pure modules occupy roughly the same position with respect to injective modules as the flat with respect to projective (for arbitrary rings). The duality homomorphisms $\mathrm{Tor}_p(A^*, F) \to \mathrm{Ext}^p(F, A)^*$ are examined in situations where they are not isomorphisms; dependence of the structure of these homomorphisms on the finite presentability or the purity of the modules F and A, as well as on the coherence of the base ring, is studied. Characterizations of pure and flat modules in terms of duality, and characterizations of coherence, semihereditariness and noetherianness in terms of duality, purity and finite presentability are given.

Bibliography: 21 titles.

A submodule $A \subset B$ is called *pure in* B if the mapping $M \otimes A \to M \otimes B$ is monomorphic for every right module M. A module A will be called *pure* if A is a pure submodule of any module containing it. Such modules are often called absolutely pure, and other terms are also in use; see §1. In particular, obviously, all injective modules are pure, and the two notions are equivalent if the base ring is noetherian.

The properties of pure modules and the position they occupy depend strongly on whether the base ring is coherent; and the requirement of coherence plays the role, from the homological point of view, of the distinctive condition of compactness. In the category of modules over a coherent ring, the pure modules occupy roughly the same position with respect to injective modules as the flat with respect to projective (over an arbitrary ring). Many characterizations of pure modules are connected with finitely presentable modules, there being even a certain duality between them (e.g., Lemma 1.4 and Proposition 1.11, Propositions 2.4 and 2.5, et al.), but a full duality is impossible: for example, the requirement that purity be inherited under passage to quotient modules reduces to the requirement of semihereditariness of the ring, while the requirement that finite presentability be inherited (by submodules of finite type) is precisely the requirement of coherence for the ring.

In §1 we examine the mutually related properties of pure and finitely presentable modules. Many of these are already known (for some we give brief proofs). Pure modules are used to characterize the weak homological dimension of coherent rings in terms of the right derived functors, in the second argument, of the functor $\otimes$.

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In §2 we examine the duality homomorphisms in situations where they are not isomorphisms, and study the dependence of their structure on the finite presentability or the purity of the modules involved, as well as on the requirement of coherence of the base ring. In terms of duality we give related characterizations of pure and flat modules.

The results of §2 play an essential role in §3. Most of the facts collected into this third section can be regarded as results of a negative character for modules over noncoherent rings, since they show that various properties or assertions connected with purity, finite presentability and duality, that are easily provable or self-evident in the category of modules over a coherent ring, are lost as soon as we pass to the noncoherent case.

All modules will be assumed to be left modules, except as otherwise evident from the text. Ring properties like coherence and hereditariness will be assumed to be satisfied on the left.

§1. Pure and finitely presentable modules

LEMMA 1.1. *A module A is pure if and only if it is pure in its injective hull* $\bar{A}$ (see also Proposition 1.7 in [6]).

Indeed, any inclusion $A \subset B$ determines inclusions $A \subset \bar{A} \subset \bar{B}$ (see [5], Chapter III, §11); hence, from the fact that $\bar{A}$ is a direct summand in $\bar{B}$ it follows that A is a pure submodule in $\bar{B}$ and in B, since $A \subset B \subset \bar{B}$.

LEMMA 1.2. *A submodule $A \subset B$ is pure in B if and only if, for any free module P of finite rank and any finitely generated submodule $\Phi \subset P$, every homomorphism $\Phi \to A$ that extends to a homomorphism $P \to B$ has an extension $P \to A$. Similarly, a module A is pure if and only if every mapping $\Phi \to A$ extends to a mapping $P \to A$ (for any pair $\Phi \subset P$ of the designated type).*

The first half of the lemma is a rephrasing of the characterization of pure submodules in terms of "relations" (see [1], Chapter 1, Exercise 24 of §2; see also [6], Proposition 1.45, and also [19]). The second half follows from the first by Lemma 1.1 and the injectivity of $\bar{A}$.

COROLLARY 1.3. *A direct product $\Pi_\lambda A_\lambda$ is a pure module if and only if all the A_λ are pure.*

LEMMA 1.4. *A module A is pure if and only if* $\operatorname{Ext}^1(F, A) = 0$ *for every finitely presentable module F.*

To prove this, it suffices to write out the exact sequence of derived functors of the functor $\operatorname{Hom}(\ , A)$ corresponding to the exact triple $0 \to \Phi \to P \to F \to 0$, where Φ and P are finitely generated and P is free, and to use Lemma 1.2.

REMARK 1.5. Modules A for which $\operatorname{Ext}^1(F, A)$ vanishes for every finitely presentable F (or equivalently, that satisfy the condition in the second half of Lemma 1.2) are sometimes called *FP-injective* (see [20], and also a similar definition in [10]). The equivalence of these requirements to the purity of A is noted in [18], [20] and elsewhere (see also [6], Propositions 1.30, 1.31 and 1.45). We show below (Proposition 1.11) that the requirement on F of finite presentability cannot be replaced by the requirement of finite generation.

Lemma 1.4 implies

COROLLARY 1.6. *If in the exact triple $0 \to A \to B \to C \to 0$ the modules A and C are pure, so is the module B.*

LEMMA 1.7. *The union of a directed system of submodules A_λ of a given module (or the direct limit of injections) is pure if all the A_λ are pure* (see [20], [18], [14]).

Indeed, suppose $A = \bigcup_\lambda A_\lambda = \varinjlim_\lambda A_\lambda$, and let $A \subset B$ be any inclusion. From the directedness of the system it follows that any element of $M \otimes A$ can be written as an element of some $M \otimes A_\lambda$. Hence the assertion follows from the purity of A_λ and the inclusions $A_\lambda \subset A \subset B$.

The commutativity of the functor Ext with finite direct sums implies:

COROLLARY 1.8 (see [14] and [20]). *A direct sum $\Sigma_\lambda A_\lambda$ is pure if and only if all the A_λ are.*

Since injectivity of direct summands implies injectivity of direct sums only for noetherian rings [7], by Lemma 1.2 we have

COROLLARY 1.9 (see [18]). *The notion of purity for a module is equivalent to the notion of injectivity if and only if the ring is noetherian.*

In particular, if pure submodules always separate out as direct summands, then the ring is noetherian (cf. [8]).

In the general case, a limit of pure modules need not be pure. We note the following result.

PROPOSITION 1.10 (see [20]). *Each of the following requirements is equivalent to the requirement of coherence for the ring*:

(a) *For every finitely presentable module F, the natural transformation*

$$\varinjlim \operatorname{Ext}^*(F, A_\lambda) \to \operatorname{Ext}^*(F, \varinjlim A_\lambda)$$

is an isomorphism.

(b) *A direct limit $A = \varinjlim A_\lambda$ of pure modules is pure.*

The following assertion supplements Lemma 1.4. The proof, actually, presents a method for constructing pure modules.

PROPOSITION 1.11. *A module F with finitely many generators is finitely presentable if and only if* $\operatorname{Ext}^1(F, A) = 0$ *for every pure A.*

It suffices to show that if F has no finite presentation, one can exhibit a pure module A such that $\operatorname{Ext}^1(F, A) \neq 0$. Let $0 \to G \to P \to F \to 0$ be an exact triple in which P is a free module of finite rank, and let $\{g_\lambda\}$ be an (infinite) system of generators in G of minimal cardinality, indexed over all ordinals $\lambda < \omega$, where ω is the first ordinal of this cardinality. Let G_λ be the submodule of G generated by all g_ν, $\nu < \lambda$. Let i_1 be the natural inclusion of G_1 in the injective hull $A_1 = \overline{G_1}$. For A_2, take the direct sum $A_1 + \overline{G_2'}$, where $G_2' = G_2/G_1$, and for i_2 take the monomorphism which is equal to the sum of some extension i_1': $G_2 \to A_1$ of the mapping i_1 and of the natural mapping $G_2 \to G_2' \subset \overline{G_2'} \subset A_2$. If the mappings i_ν: $G_\nu \to A_\nu$ have been constructed for all $\nu < \lambda$, and if λ is not a limit ordinal, then i_λ: $G_\lambda \to A_\lambda$ is constructed from $i_{\lambda-1}$ and G_λ in the same way as i_2 from i_1 and G_2. Since the restriction of each i_ν to G_μ, for a lower index μ, coincides with i_μ, when λ is a limit ordinal we can take for i_λ the natural mapping of G_λ into $A_\lambda = \bigcup_{\nu<\lambda} A_\nu$. We have thus constructed mappings i_λ: $G_\lambda \to A_\lambda$ for all $\lambda < \omega$. By Lemma 1.7, the module $A = \bigcup_\lambda A_\lambda$ is pure. On account of the exact sequence for the functor Ext corresponding to the above exact triple of modules, it suffices to show that the mapping $i\colon G \to A$, $i = \varinjlim i_\lambda$, has no extension $P \to A$. If it did, the fact that P is finitely generated would imply that $i(G)$ is contained in some A_λ, $\lambda < \omega$; and this is impossible, since by construction of i_λ this would mean that $G_{\nu+1}/G_\nu = 0$ for $\nu \geqslant \lambda$, contradicting the minimality of the system of generators $\{g_\lambda\}$.

Let Q be the group of rational numbers reduced modulo 1. For any left module A, the group $A^* = \mathrm{Hom}(A, Q)$ is a right module. We shall frequently make use of the natural duality homomorphisms

$$\rho\colon \mathrm{Ext}^p(M, A^*) \to \mathrm{Tor}_p(M, A)^* \quad \text{and} \quad \sigma\colon \mathrm{Tor}_p(A^*, F) \to \mathrm{Ext}^p(F, A)^*,$$

of which the first is always an isomorphism (see [3], Proposition 5.1 in Chapter VI). As for σ, it is an isomorphism for every finitely generated F, provided the ring is noetherian (see [3], Proposition 5.3 in Chapter VI).

LEMMA 1.12. *The natural transformation σ is an isomorphism if F is finitely presentable and the base ring is coherent.*

Indeed, in this case F has a projective resolution consisting of finitely generated modules, and the lemma follows from the Remark to Proposition 5.3 of Chapter VI in [3].

LEMMA 1.13. *Each of the following conditions is equivalent to the purity of the module A*: a) *for every inclusion $A \subset B$ the mapping $A^* \otimes A \to A^* \otimes B$ is monomorphic*; b) *the mapping $B^* \to A^*$ is a split epimorphism* (in terms of pure submodules part b) is proved in [8]).

Indeed, the condition that $A^* \otimes A \to A^* \otimes B$ be monomorphic is equivalent to the condition that $(A^* \otimes B)^* \to (A^* \otimes A)^*$ be epimorphic, or, in view of the duality ρ, to the condition that $\mathrm{Hom}(A^*, B^*) \to \mathrm{Hom}(A^*, A^*)$ be epimorphic, i.e., that $B^* \to A^*$ be a split epimorphism. If in the functor $\mathrm{Hom}(A^*, \ )$ we replace the module A^* by an arbitrary right module M and run through the reverse argument, we obtain that $M \otimes A \to M \otimes B$ is a monomorphism.

COROLLARY 1.14. *If A^* is flat, then A is pure.*

PROPOSITION 1.15. *If the ring is coherent, then a module A is pure if and only if A^* is flat.*

Indeed, by Lemma 1.12 the natural transformation σ is an isomorphism when F is finitely presentable. Hence, by Lemma 1.4, the purity of A is equivalent to the condition that $\mathrm{Tor}_1(A^*, F) = 0$ for all finitely presentable F. It remains only to observe that every module is the direct limit of finitely presentable ones, and that the functor Tor commutes with direct limits.

REMARK 1.16. It will be shown below that the coherence requirement for the ring in the preceding proposition is essential. A similar result for injective modules over a noetherian ring is obtained in [11] and [8] (see also §3). Proposition 1.15 is the analogue of the following assertion (see, e.g., [4]): a module A is flat if and only if the module A^* is injective. We note (see also [9]) that the purity of A^* implies the injectivity of A^*: the fact that $\mathrm{Ext}^1(M, A^*) = 0$ for all finitely presentable right modules M (Lemma 1.4) implies, by the duality ρ, that $\mathrm{Tor}_1(M, A) = 0$, i.e., that A is flat.

COROLLARY 1.17. *If the ring is coherent, then a module A is pure if and only if* $\mathrm{Ext}^p(F, A) = 0$ *for* $p \geqslant 1$ *and all finitely presentable F.*

Thus, $\mathrm{Ext}^p(F, N)$ can be calculated by means of pure resolutions of N, provided the module F is finitely presentable.

Indeed, for the flat module A^* we have $\mathrm{Tor}_p(A^*, F) = 0$, $p \geqslant 1$ (see [2]), so that it suffices to use the isomorphism σ.

COROLLARY 1.18. *In the case of a coherent ring, if in the exact triple* $0 \to A \to B \to C \to 0$ *the modules A and B are pure, so is the module C* (cf. question 10 on p. 134 of [6] (question 7 on p. 127 of the translation)).

To prove this, it suffices to consider the exact sequence of the functor Ext corresponding to the given triple, and to use Corollary 1.17.

Below (§3) it will be shown that the assumption of coherence in these last two assertions cannot be omitted.

PROPOSITION 1.19 (see [20]). *If the ring R is coherent, then its weak dimension* $w \dim R$ *is equal to the smallest n such that* $\mathrm{Ext}^{n+1}(F, N) = 0$ *for all N and all finitely presentable F.*

Indeed, the inequality $n \geqslant w \dim R$ is ensured by the isomorphisms ρ (because every module is the direct limit of finitely presentable ones). The reverse inequality is ensured by the isomorphisms σ.

REMARK 1.20. In a standard fashion (see [3], Chapter VI, §2), the n in question, and therefore also $w \dim R$, can be characterized by the fact [20] that if in the exact sequence $0 \to N \to A^0 \to \cdots \to A^n \to 0$ the modules A^i are pure (for example, injective) for $i < n$ and for any N, then also A^n is pure. Similar characterizations of weak dimension and of related dimensional invariants of rings and modules, as well as relations between them, are given in [9], [8], [15], [20], [17] et al.

It is known [12] that for the tensor product, regarded as a functor of two arguments, the right derived functors $R^q \otimes$ are equal to zero for $q \geqslant 1$ if R is a commutative integral domain. The following result shows that the derived functors in the second argument are in general different from zero and, when the ring is coherent, characterize its weak dimension (or the usual homological dimension if the ring is noetherian).

PROPOSITION 1.21. *If the ring is coherent, then the condition* $w \dim R \leqslant n$ *is equivalent, for $n > 1$, to the condition that the functor* $\mathrm{Tor}^{n-1}(M, N)$, *regarded as a right derived functor of $M \otimes N$ in the argument N, is equal to zero; and for $n = 1$, to the condition that* $\mathrm{Tor}^0(M, N)$ *is the image of the mapping $M \otimes N \to M \otimes Y^0$ corresponding to the imbedding of N into an injective module Y^0 (in both cases we have automatically* $\mathrm{Tor}^q(M, N) = 0$ *for $q \geqslant n$).*

Let $0 \to N \to Y^0 \to \cdots \to Y^{n-1} \to \cdots$ be an injective resolution of N. If $w \dim R \leqslant n$, then, in keeping with Remark 1.20, the kernel $Z^n \subset Y^n$ of the mapping in the resolution is a pure module; and by Corollary 1.18, so are the kernels $Z^q \subset Y^q$ for $q \geqslant n$. From the exact sequence

$$M \otimes Z^{q-1} \xrightarrow{\alpha} M \otimes Y^{q-1} \xrightarrow{\beta} M \otimes Z^q \to 0$$

and from the purity of Z^q it follows that the kernel of $M \otimes Y^{q-1} \to M \otimes Y^q$ coincides with the image of α, which, in turn, is equal to the image of $M \otimes Y^{q-2} \to M \otimes Y^{q-1}$

for $q > 1$, so that $\operatorname{Tor}^{q-1}(M, N) = 0$, or to the image of $M \otimes N \to M \otimes Y^0$ for $q = 1$, so that $\operatorname{Tor}^0(M, N)$ is of the form indicated.

Conversely, the fact that $\operatorname{Tor}^{n-1}(M, N)$ is as described in the proposition means that the kernel of $M \otimes Y^{n-1} \to M \otimes Y^n$ coincides (because of the right exactness of the tensor product) with the image of the homomorphism α, i.e., with the kernel of β. But this means that the mapping $M \otimes Z^n \to M \otimes Y^n$ is monomorphic. Since the module M was arbitrary, and the module Y^n injective, it follows from Lemma 1.1 that Z^n is pure, and therefore $w \dim R \leqslant n$ by Remark 1.20.

It is evident from the proof that for coherent R the pure modules are acyclic with respect to the functor $\operatorname{Tor}^q(M, \ )$, so that to compute $\operatorname{Tor}^q(M, N)$ we can use pure resolutions of N.

For $n = 1$, the preceding results can be stated, without the assumption of coherence for the ring, in the following form.

PROPOSITION 1.22. *Each of the following is equivalent to the condition that the ring be semihereditary* (see also §3): a) *the quotient modules of pure* (*injective*) *modules are pure* (see [18], [10]); b) *the functor* $\operatorname{Tor}^0(M, N)$ *is as described in Proposition* 1.21 (*in which case automatically* $\operatorname{Tor}^q = 0$ *for* $q > 0$); c) $\operatorname{Ext}^2(F, N) = 0$ *for every* N *and every finitely presentable* F.

Indeed, if R is semihereditary, it is coherent, so that a) and b) follow from Remark 1.20 and Proposition 1.21. As in the second part of the proof of Proposition 1.21, condition b) implies that the quotient modules of injective modules are pure. If F is an ideal of finite type, $B \to C$ an epimorphism with B injective, and $F \to C$ an arbitrary homomorphism, then the latter extends, since C is pure, to a mapping $R \to C$, which can be covered by a homomorphism $R \to B$. There is thus obtained a covering $F \to B$ of the original homomorphism $F \to C$, so that F is projective (see [3], Chapter I, §5) and R semihereditary. Part c) is an obvious consequence of Proposition 6.2 of Chapter I in [3].

REMARK 1.23. If R is a commutative integral domain, then by Proposition 1.2 of Chapter VII in [3] and Lemma 1.2, every pure module is divisible (hence, if it is not injective, it must necessarily have torsion, by Proposition 1.3 of [3]). Purity is equivalent to divisibility only when the ring is Prüfer (see [18]). In this case, $\operatorname{Tor}^0(M, N) = M' \otimes N$, where M' is the quotient of M by its torsion submodule (this is easily seen by considering a pure resolution of length 1 of the module N and applying Exercise 5 of Chapter VII in [3]).

§2. The duality homomorphisms

We examine now in detail the homomorphisms

$$\sigma\colon \operatorname{Tor}_k(N^*, F) \to \operatorname{Ext}^k(F, N)^*$$

in the lowest dimensions k for modules F with finitely many generators, without the assumption that the ring is noetherian or coherent.

LEMMA 2.1. *If F is finitely presentable, then the mapping* $N^* \otimes F \to \operatorname{Hom}(F, N)^*$ *is an isomorphism.*

This follows from Proposition 5.2 of Chapter VI in [3] and the right exactness of the tensor product (see also [1], Chapter I, Exercise 14 of §2).

PROPOSITION 2.2. *Every injective resolution Y of a module N, $0 \to N \to Y^0 \to Y^1 \to \cdots$, determines a homological spectral sequence converging to* $\mathrm{Tor}(N^*, F)$, *with second term* $E^2_{pq} = H_p(\mathrm{Tor}_q(Y^*, F))$. *Furthermore,* $E^2_{p0} = \mathrm{Ext}^p(F, N)^*$ *if F is finitely presentable.*

To prove this consider the double complex obtained by tensor multiplication by F of a projective resolution of the chain complex ([3], Chapter XVII) $0 \leftarrow N^* \leftarrow Y^{0*} \leftarrow Y^{1*} \leftarrow \cdots$ (which for convenience can be thought of as consisting of columns over N^*, Y^{0*}, $Y^{1*}, \ldots$). Exactness of the rows of the resolution, together with the projectivity of the modules, yields, when we take homology on rows, a projective resolution of N^* in the column with index 0 and zeros in the remaining positions; therefore, taking homology on columns, we have $\mathrm{Tor}(N^*, F)$ as the homology of the double complex. The second spectral sequence is of the form $E^1_{p,q} = \mathrm{Tor}_q(Y^{p*}, F)$, taking homology on columns, and $E^2_{p,q} = H_p(\mathrm{Tor}_q(Y^*, F))$, taking homology on rows. For F finitely presentable, we find that

$$E^2_{p0} = H_p(Y^* \otimes F) = H_p(\mathrm{Hom}(F, Y)^*) = H^p(\mathrm{Hom}(F, Y))^* = \mathrm{Ext}^p(F, N)^*,$$

where the first equality follows from Lemma 2.1; the second, from the commutativity of the homology functor with the exact functor $X \to X^*$; and the third, from the definition of the functor Ext.

The spectral sequence gives the standard result:

COROLLARY 2.3. *For finitely presentable F, we have the exact sequence*

$$\mathrm{Tor}_2(N^*, F) \to \mathrm{Ext}^2(F, N)^* \to H_0(\mathrm{Tor}_1(Y^*, F)) \to \mathrm{Tor}_1(N^*, F) \to \mathrm{Ext}^1(F, N)^* \to 0.$$

PROPOSITION 2.4. *The mapping* $\mathrm{Tor}_1(N^*, F) \to \mathrm{Ext}^1(F, N)^*$ *for a given F with finitely many generators is epimorphic (for all N) if and only if F is finitely presentable.*

In view of the preceding corollary, it suffices to show that the mapping is not epimorphic if F does not have a finite presentation. Take for N a pure module A such that $\mathrm{Ext}^1(F, A) \neq 0$ (see Proposition 1.11). As previously mentioned, any module F can be represented as $\varinjlim F_\lambda$, where the F_λ are finitely presentable. Using the fact that

$$\varinjlim \mathrm{Tor}_1(A^*, F_\lambda) = \mathrm{Tor}_1(A^*, F),$$

we conclude from Lemma 1.4 and the commutative diagram

$$\begin{array}{ccc} \mathrm{Tor}_1(A^*, F_\lambda) & \to & \mathrm{Tor}_1(A^*, F) \\ \downarrow & & \downarrow \\ \mathrm{Ext}^1(F_\lambda, A)^* & \to & \mathrm{Ext}^1(F, A)^* \end{array}$$

that the mapping in question has zero image.

The same argument gives a similar result for pure modules (using Lemma 1.4):

PROPOSITION 2.5. *The homomorphism* $\mathrm{Tor}_1(A^*, N) \to \mathrm{Ext}^1(N, A)^*$, *for fixed A, has zero image for all N if and only if A is pure.*

LEMMA 2.6. *The mapping* $N^* \otimes F \to \mathrm{Hom}(F, N)^*$ *is always epimorphic if the module F has finitely many generators.*

To prove this, represent F as a direct limit $\varinjlim F_\lambda$, where the F_λ are finitely presentable and the projections $F_\lambda \to F_\mu$ and $F_\lambda \to F$ are epimorphisms. Then the groups $\mathrm{Hom}(F_\lambda, N)$ are included one in the other, $\mathrm{Hom}(F_\lambda, N) \supset \mathrm{Hom}(F_\mu, N)$, filtering down to the intersection $\mathrm{Hom}(F, N)$. Since the group Q is injective, we have epimorphisms $\mathrm{Hom}(F_\lambda, N)^* \to \mathrm{Hom}(F_\mu, N)^*$. If we call the elements of these groups functionals, then each such epimorphism is the restriction of the functionals to $\mathrm{Hom}(F_\mu, N) \subset \mathrm{Hom}(F_\lambda, N)$. Every element of $\mathrm{Hom}(F, N)^*$ is represented by the germ of a functional in $\mathrm{Hom}(F_\lambda, N)^* = N^* \otimes F_\lambda$ (Lemma 2.1), and consequently by an element of $N^* \otimes F = \varinjlim N^* \otimes F_\lambda = \varinjlim \mathrm{Hom}(F_\lambda, N)^*$. This proves the lemma.

PROPOSITION 2.7. *If F has finitely many generators, then the mapping $N^* \otimes F \to \mathrm{Hom}(F, N)^*$ is an isomorphism for all N only when F is finitely presentable.*

In proving this, we shall use the terminology of the preceding lemma. By Lemmas 2.1 and 2.6, it suffices to show that if F has no finite presentation, then we can select N so that the mapping in question has a nonzero kernel, i.e., so that in some $\mathrm{Hom}(F_\lambda, N)^*$ there exists a nontrivial germ whose restriction to $\mathrm{Hom}(F, N)$ is equal to zero (we call the germ of a functional on $\mathrm{Hom}(F_\lambda, N)$ nontrivial if it itself and all its restrictions to the $\mathrm{Hom}(F_\mu, N) \subset \mathrm{Hom}(F_\lambda, N)$ are different from zero). Take a fixed λ_0, and denote the module F_{λ_0} by F_0. We shall consider only those $\lambda \geqslant \lambda_0$. Let N be the direct product $\prod_\lambda F_\lambda$ of the F_λ. For each ν, denote by i_ν the canonical inclusion $F_\nu \subset \prod_\lambda F_\lambda$. Since the inclusion $\mathrm{Hom}(F, N) \subset \mathrm{Hom}(F_\nu, N)$ is induced by a mapping of F_ν onto F with nonzero kernel, the element $i_\nu \in \mathrm{Hom}(F_\nu, N)$ cannot belong to $\mathrm{Hom}(F, N) = \bigcap_\lambda \mathrm{Hom}(F_\lambda, N)$. On the other hand, we have $i_\nu \in \mathrm{Hom}(F_0, N)$, because of the mapping $F_0 \to F_\nu$. Regard the group $\mathrm{Hom}(F_0, N)$ as the product $\prod_\lambda \mathrm{Hom}(F_0, F_\lambda)$. It contains in a natural fashion the direct sum $\sum_\lambda \mathrm{Hom}(F_0, F_\lambda)$ as a subgroup, and obviously every i_ν belongs to this subgroup. Consider for each λ a homomorphism of $\mathrm{Hom}(F_0, F_\lambda)$ into Q that is different from zero on i_λ but equal to zero on the subgroup $\mathrm{Hom}(F, F_\lambda)$ (it suffices to construct a mapping of the quotient $\mathrm{Hom}(F_0, F_\lambda)/\mathrm{Hom}(F, F_\lambda)$ into Q which is different from zero on the image of i_λ). These homomorphisms determine a mapping into Q of the direct sum $\sum_\lambda \mathrm{Hom}(F_0, F_\lambda) \subset \mathrm{Hom}(F_0, N)$ which is equal to zero on $\sum_\lambda \mathrm{Hom}(F, F_\lambda)$. If we factor out this subgroup, imbed the quotient group into the quotient

$$\prod_\lambda \mathrm{Hom}(F_0, F_\lambda) / \prod_\lambda \mathrm{Hom}(F, F_\lambda),$$

and extend the mapping to the whole of this quotient (which is possible because Q is injective), we obtain finally a mapping of $\mathrm{Hom}(F_0, N) = \prod_\lambda \mathrm{Hom}(F_0, F_\lambda)$, i.e., a functional on $\mathrm{Hom}(F_0, N)$. Clearly, the restriction of this functional to $\mathrm{Hom}(F, N)$ is trivial. But the restriction to any subgroup of the form $\mathrm{Hom}(F_\nu, N)$ is not, since such a subgroup is of the form $\prod_\lambda \mathrm{Hom}(F_\nu, F_\lambda)$ and contains the element i_ν, on which the functional is different from zero by construction. This proves the proposition.

REMARK 2.8. In the above construction, we could obviously have taken N to be the product $\prod_\lambda \bar{F}_\lambda$. Thus, the mapping in Proposition 2.7 need not be an isomorphism even for injective N (in particular, for pure modules there can be no analogue of Proposition 2.7). If F does not have finitely many generators, then, in addition, the mapping is not epimorphic.

Let F be a module of finite type but without finite presentation, and $0 \to G \to P \to F \to 0$ an exact sequence in which P is a free module with finitely many generators. Then $F = \varinjlim F_\lambda$, where $F_\lambda = P/G_\lambda$ and the G_λ are all the submodules of finite type in G. Denote by D_λ the quotient module G/G_λ.

LEMMA 2.9. *If F is a module of finite type but without finite presentation, and N an injective module, then the kernel of the mapping $N^* \otimes F \to \mathrm{Hom}(F, N)^*$ is the group* $\varinjlim \mathrm{Hom}(D_\lambda, N)^*$.

Indeed, we have for each λ the exact triple $0 \to D_\lambda \to F_\lambda \to F \to 0$, which for injective N gives an exact sequence

$$0 \to \mathrm{Hom}(D_\lambda, N)^* \to \mathrm{Hom}(F_\lambda, N)^* \to \mathrm{Hom}(F, N)^* \to 0.$$

Hence our assertion follows from the exactness of the functor $\varinjlim$ and from the equality $\varinjlim \mathrm{Hom}(F_\lambda, N)^* = N^* \otimes F$ (see the proof of Lemma 2.6).

PROPOSITION 2.10. *If the ring R is coherent and the module F finitely generated, then for any N the duality homomorphisms σ are contained in an exact sequence*

$$\ldots \xrightarrow{\sigma} \mathrm{Ext}^{n+1}(F, N)^* \xrightarrow{\delta} \varinjlim \mathrm{Ext}^n(D_\lambda, N)^* \to \mathrm{Tor}_n(N^*, F) \xrightarrow{\sigma} \mathrm{Ext}^n(F, N)^*$$
$$\ldots \xrightarrow{\delta} \varinjlim \mathrm{Hom}(D_\lambda, N)^* \to N^* \otimes F \xrightarrow{\sigma} \mathrm{Hom}(F, N)^* \to 0.$$

To prove this, let Y be an injective resolution of N. Then by Lemma 2.9 we have an exact triple of chain complexes

$$0 \to \varinjlim \mathrm{Hom}(D_\lambda, Y)^* \to Y^* \otimes F \to \mathrm{Hom}(F, Y)^* \to 0.$$

The exact sequence of Proposition 2.10 is the corresponding homology sequence. Indeed, the homology of the complex $\mathrm{Hom}(F, Y)^*$ is obviously $\mathrm{Ext}(F, N)^*$; and the homology of $Y^* \otimes F$ is $\mathrm{Tor}(N^*, F)$, since by Proposition 1.15 the sequence Y^* that constitutes a resolution of N^* consists of flat modules. Finally,

$$H_p(\varinjlim \mathrm{Hom}(D_\lambda, Y)^*) = \varinjlim H_p(\mathrm{Hom}(D_\lambda, Y)^*) = \varinjlim H^p(\mathrm{Hom}(D_\lambda, Y))^*$$
$$= \varinjlim \mathrm{Ext}^p(D_\lambda, N)^*$$

(the first of these equalities follows from the commutativity of the homology functor H_p with the exact functor $\varinjlim$; the second, from the commutativity of H_p with the exact functor $X \to X^*$; the third, from the definition of Ext).

Observe that, since $\varinjlim D_\lambda = 0$, we have for any N the equality

$$\varprojlim \mathrm{Hom}(D_\lambda, N) = \bigcap \mathrm{Hom}(D_\lambda, N) = 0.$$

Observe also that the homomorphism δ in the exact sequence above has another description: it is the composite of the isomorphism

$$\mathrm{Ext}^{n+1}(F, N)^* = \mathrm{Ext}^n(G, N)^*$$

and the mapping

$$\mathrm{Ext}^n(G, N)^* \to \varinjlim \mathrm{Ext}^n(D_\lambda, N)^*,$$

induced by the homomorphisms $G \to D_\lambda$.

We conclude this section by giving mutually related characterizations, in terms of duality, of pure and flat modules.

PROPOSITION 2.11. *A module A is pure if and only if the mapping*

$$\mathrm{Tor}_1(A^*, \bar{A}/A) \to \mathrm{Ext}^1(\bar{A}/A, A)^*$$

is the zero mapping.

Indeed, if A is pure, our assertion follows from Proposition 2.5. To prove the converse, consider the following diagram, corresponding to the exact sequence $0 \to A \to \bar{A} \to \bar{A}/A \to 0$:

$$\begin{array}{ccccccc} & \mathrm{Tor}_1(A^*, \bar{A}/A) & \xrightarrow{\alpha} & A^* \otimes A & \xrightarrow{\beta} & A^* \otimes \bar{A} \\ & \sigma\downarrow & & \sigma\downarrow & & \downarrow \\ 0 \to & \mathrm{Ext}^1(\bar{A}/A, A)^* & \to & \mathrm{Hom}(A, A)^* & \to & \mathrm{Hom}(\bar{A}, A)^* \end{array}$$

The mapping in the middle column is given by the formula

$$\sigma(f \otimes a) g = f(g(a)),$$

where $g \in \mathrm{Hom}(A, A)$, $a \in A$ and $f \in A^*$ (see [3], Chapter VI, §5), and is therefore monomorphic: it suffices to take for g the identity isomorphism, and for a any element of A on which f is different from zero. Thus, if the mapping in the left-hand column is zero, then so is the homomorphism α, so that β is a monomorphism. Using the same argument as in Lemma 1.13, we see that the mapping $M \otimes A \to M \otimes \bar{A}$ is monomorphic for every right module M. Hence A is pure, by Lemma 1.1.

REMARK 2.12. It follows from Proposition 1.15 that for a coherent ring the criterion of Proposition 2.11 reduces to the condition $\mathrm{Tor}_1(A^*, \bar{A}/A) = 0$. Furthermore, the injective hull $\bar{A}$ in Proposition 2.11 can be replaced by any injective module B containing A (since $\bar{A}$ is contained in B as a direct summand).

Now suppose that in the exact triple $0 \to G \to P \to F \to 0$ the module P is projective. The following assertion supplements the characterization of flat modules given by Chase (see [7], Proposition 2.2).

PROPOSITION 2.13. *The module F is flat if and only if* $\mathrm{Tor}_1(G^*, F) = 0$.

LEMMA 2.14. *For every module N, the following conditions are equivalent:*

a) *The mapping* $\mathrm{Tor}_1(N^*, F) \to \mathrm{Ext}^1(F, N)^*$ *has zero image.*

b) *For any finitely generated submodule $G_\lambda \subset G$, every mapping $G_\lambda \to N$ that has an extension $G \to N$ extends to a mapping $P \to N$.*

To obtain Proposition 2.13 from this lemma, put $N = G$ and apply the lemma to the inclusions $G_\lambda \subset G$, obtaining mappings $j_\lambda \colon P \to G$ that are stationary on G_λ. Since P is the direct summand of a free module, the mappings j_λ can be selected so that the modules $G'_\lambda = \mathrm{Im}\, j_\lambda$ are finitely generated. If we put $\lambda < \mu$ only when $G'_\lambda \subset G_\mu$, we obtain a direct spectrum consisting of modules $P_\lambda = P$ and mappings $P_\lambda \to P_\mu$ given by $\theta^\lambda_\mu = 1 - j_\mu$. Since F is the limit of this spectrum (see [21], Proposition 2), F is flat.

To prove the lemma, consider the diagram

$$\begin{array}{ccccccc} & \mathrm{Tor}_1(N^*, F) & \to & N^* \otimes G & \to & N^* \otimes P \\ & \downarrow & & \downarrow\sigma & & \downarrow \\ 0 \to & \mathrm{Ext}^1(F, N)^* & \to & \mathrm{Hom}(G, N)^* & \xrightarrow[\alpha]{} & \mathrm{Hom}(P, N)^*. \end{array}$$

The mapping in the right-hand column is a monomorphism (this is obvious for free P, and consequently also for projective, since the projective are direct summands of free). Hence, as is easily seen, condition a) of the lemma is equivalent to the condition

Ker $\alpha \cap \operatorname{Im} \sigma = 0$. If G_λ is any finitely generated submodule of G, then the fact that the mapping in the left-hand column of the commutative square

$$\begin{array}{ccc} N^* \otimes G_\lambda & \to & N^* \otimes G \\ \downarrow & & \downarrow \\ \operatorname{Hom}(G_\lambda, N)^* & \underset{\beta_\lambda}{\to} & \operatorname{Hom}(G, N)^* \end{array}$$

is epimorphic (Lemma 2.6) implies that $\operatorname{Im} \beta_\lambda \subset \operatorname{Im} \sigma$. Therefore, since $G = \varinjlim G_\lambda$, the condition Ker $\alpha \cap \operatorname{Im} \sigma = 0$ is equivalent to the condition that Ker $\alpha \cap \operatorname{Im} \beta_\lambda = 0$ for all finitely generated submodules $G_\lambda \subset G$. It is easily verified that Ker α consists of precisely those functionals on $\operatorname{Hom}(G, N)$ (in the terminology of the proof of Lemma 2.6) that vanish on the homomorphisms $h' \in \operatorname{Hom}(G, N)$ that extend to P. Similarly, $\operatorname{Im} \beta_\lambda$ consists of precisely those functionals in $\operatorname{Hom}(G, N)^*$ that vanish on the homomorphisms $h'' \in \operatorname{Hom}(G, N)$ whose restrictions to G_λ are zero. Therefore, if the homomorphisms of type h' and h'' generate the group $\operatorname{Hom}(G, N)$, then Ker $\alpha \cap \operatorname{Im} \beta_\lambda = 0$. Conversely, if the homomorphisms of type h' and h'' do not generate $\operatorname{Hom}(G, N)$, then, factoring out from $\operatorname{Hom}(G, N)$ the subgroup generated by all such h' and h'', we can construct a nontrivial functional on $\operatorname{Hom}(G, N)$ that vanishes both on homomorphisms of type h' and on homomorphisms of type h'', i.e., a nonzero element of Ker $\alpha \cap \operatorname{Im} \beta_\lambda$. Thus, for any given finitely generated submodule $G_\lambda \subset G$, the condition Ker $\alpha \cap \operatorname{Im} \beta_\lambda = 0$ is equivalent to the condition that $\operatorname{Hom}(G, N)$ be generated by the homomorphisms of the two types. Now suppose $f \in \operatorname{Hom}(G_\lambda, N)$, and let $h \in \operatorname{Hom}(G, N)$ be an extension of f. If $h = h' + h''$, then $h' = h - h''$ coincides with f on G_λ and has an extension to P; i.e., f extends to a homomorphism $P \to N$. Conversely, suppose $h \in \operatorname{Hom}(G, N)$, and let $f \in \operatorname{Hom}(G_\lambda, N)$ be the restriction of h. If f extends to a mapping $P \to N$, and h' is the restriction of this extension to G, then the mapping $h'' = h - h'$ is equal to zero on G_λ; i.e., $\operatorname{Hom}(G, N)$ is generated by the homomorphisms of the two types. This proves the lemma.

§3. Characterizations of coherence and semihereditariness

The impact of coherence of the base ring on many properties of pure modules and on certain concomitant characterizations of finitely presentable modules has already been made apparent in several cases above. This connection is more completely reflected in the following proposition.

THEOREM 3.1. *Each of the following assertions is valid precisely when the ring R is coherent*:

a) *The natural transformations*

$$\sigma : \operatorname{Tor}_k(N^*, F) \to \operatorname{Ext}^k(F, N)^*$$

are isomorphisms for any finitely presentable F, any module N, and any $k \geqslant 0$ (we can restrict ourselves to the case $k = 1$).

b) *For every module F with finitely many generators, and every N, the transformations σ are contained in an exact sequence like that in Proposition* 2.10.([1])

c) *A module A is pure if and only if the module A^* is flat.*

([1])In a) and b), we can use homomorphisms

$$\sigma : \operatorname{Tor}_k(\operatorname{Hom}(N, J), F) \to \operatorname{Hom}(\operatorname{Ext}^k(F, N), J)$$

with J injective, where we have in mind all allowable situations in the sense of [3].

d) *A module A is pure if and only if* $\operatorname{Ext}^k(F, A) = 0$ *for* $k \geqslant 1$ *and for all finitely presentable F (we can, however, restrict ourselves to the values* $k = 1, 2$).

e) *A module F with finitely many generators is finitely presentable if and only if* $\operatorname{Ext}^k(F, A) = 0$ *for* $k \geqslant 1$ *and for all pure modules A (we can, however, restrict ourselves to the values* $k = 1, 2$).

f) *If in the exact triple* $0 \to A \to B \to C \to 0$ *the modules A and B are pure, so is the module C.*

That all these assertions hold when the ring is coherent has been proved above: a) follows from Lemma 1.12; b), from Proposition 2.10; c), from Proposition 1.15; d) and f), from Corollaries 1.17 and 1.18; and e), from Proposition 1.11 and Corollary 1.17.

Assuming now that R is not coherent, we shall disprove each of the assertions in the theorem. Let Φ be an ideal of finite type in R that has no finite presentation (as a module). For the exact sequence $0 \to \Phi \to R \to F \to 0$, consider the diagram

$$\begin{array}{ccccccc} 0 \to & \operatorname{Tor}_1(N^*, F) & \to & N^* \otimes \Phi & \to & N^* \otimes R \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & \operatorname{Ext}^1(F, N)^* & \to & \operatorname{Hom}(\Phi, N)^* & \to & \operatorname{Hom}(R, N)^*. \end{array}$$

Select the module N so that (see Proposition 2.7) the mapping in the middle column has nontrivial kernel. Since in the right-hand column we have an isomorphism, this kernel is contained in $\operatorname{Tor}_1(N^*, F)$ (the rows of the diagram are exact) and coincides with the kernel of the mapping in the left-hand column. Since F is finitely presentable, this disproves assertion a). Furthermore, by Remark 2.8, N can be chosen to be an injective module A, and since for this choice $\operatorname{Tor}_1(A^*, F) \neq 0$, i.e., A^* is not flat, this disproves assertion c). This also disproves assertion b): for the A in question, we should have in the exact sequence of Proposition 2.10 that $\operatorname{Ext}^1(F, A) = \operatorname{Ext}^1(D_\lambda, A) = 0$, contradicting the fact that A^* is not flat.

For the same Φ, we can find, by Proposition 1.11, a pure A such that $\operatorname{Ext}^1(\Phi, A) \neq 0$. Since $\operatorname{Ext}^2(F, A) = \operatorname{Ext}^1(\Phi, A)$, this disproves assertion d). It also disproves e). Finally, selecting a finitely presentable F and a pure A such that $\operatorname{Ext}^2(F, A) \neq 0$, consider a triple $0 \to A \to B \to C \to 0$ in which B is injective. Then $\operatorname{Ext}^1(F, C) = \operatorname{Ext}^2(F, A)$, so that, by Lemma 1.4, C is not pure. This disproves f). The theorem is now proved.

Parts d) and f) of the theorem explain why a dimensional invariant similar to injective dimension but defined by means of pure resolutions [20] (and having a connection with weak dimension) can be properly constructed only when the ring is coherent (cf. [9]).

Corollary 1.9, part c) of Theorem 3.1, and Proposition 2.7 yield the following result (see Remark 1.16).

THEOREM 3.2. *Each of the following conditions is equivalent to the ring being noetherian*:

a) *A module A is injective if and only if the module* A^* *is flat.*

b) *The duality homomorphisms* σ *are isomorphisms for any module F with finitely many generators.*

Let A_1 and A_2 be submodules of a given module, and $A_1 + A_2$ their usual sum (as submodules). Part f) of Theorem 3.1 can be formulated in the following way.

LEMMA 3.3. *Coherence of the ring R is equivalent to the condition that whenever the modules* A_1, A_2 *and* $A_1 \cap A_2$ *are pure, so is the sum* $A_1 + A_2$.

Indeed, if R is coherent, the assertion of the lemma follows from assertion f), since $A_1 + A_2$ is the quotient of the direct sum $A_1 \otimes A_2$ by a submodule isomorphic to $A_1 \cap A_2$. If R is not coherent, select a pure module A such that $\mathrm{Ext}^2(F, A) \neq 0$ for some finitely presentable F, imbed it in an injective module B, and consider the exact triple $0 \to A \to B \oplus B \to C \to 0$, where the first mapping is the diagonal imbedding of A into the direct sum. The argument at the end of the proof of Theorem 3.1 shows that C is not pure, even though C is the sum of two submodules isomorphic to B.

THEOREM 3.4. *Each of the following assertions is valid precisely when the ring R is semihereditary*:

a) *The sum of two pure modules (as submodules of a third module) is pure.*

b) *The set of all submodules of an arbitrary module that are themselves pure modules has a greatest element (containing all the others).*

The characterizations in Proposition 1.22 and Theorem 3.4 of semihereditary rings in terms of pure modules are analogues of characterizations of hereditary rings in terms of injective modules (see [16]); the analogue of part b) of Theorem 3.4 is valid for hereditary rings provided R is noetherian.

If R is semihereditary, assertion a) is a consequence of Proposition 1.22 (the sum of the modules is a quotient of their direct sum), and assertion b) follows from a) and Lemma 1.7 (it is proved in [18]; see also [13]). That b) implies a) is obvious. We show now that a) implies that R is semihereditary. By Proposition 1.22, it suffices to show that the module A/H is pure whenever A is. If $B = A \oplus A$ and H' is the image of H in B under the diagonal imbedding, then A/H is isomorphic to the quotient of the module B/H' by a submodule isomorphic to A (argument as in [16]). Since B/H' is the sum of submodules isomorphic to A, it is itself pure. Furthermore, a) implies (Lemma 3.3) that R is coherent. The purity of A/H is therefore a consequence of Corollary 1.18.

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