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INEQUALITIES OF BERNSTEIN TYPE FOR DERIVATIVES OF RATIONAL FUNCTIONS, AND INVERSE THEOREMS OF RATIONAL APPROXIMATION

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ABSTRACT. Let H_p be the Hardy space of functions f that are analytic in the disk $|z| < 1$ and let $J^\alpha f$ be the derivative of f of order α in the sense of Weyl. It is shown, for example, that if r is a rational function of degree $n \geqslant 1$ with all its poles in the domain $|z| > 1$, then $\|J^\alpha r\|_{H_\sigma} \leqslant cn^\alpha \|r\|_{H_p}$, where $p \in (1, \infty]$, $\alpha > 0$, $\sigma = (\alpha + p^{-1})^{-1}$ and $c > 0$ depends only on α and p.

Bibliography: 32 titles.

Let X be a quasinormed space of functions that are analytic in the disk $|z| < 1$, and let $R_n(f, X)$ ($f \in X$, $n = 1, 2, \dots$) be the best approximation to f in X by rational fractions of degree at most $n - 1$. Dolzhenko [17] showed that if $f \in H_\infty$ and $\sum R_n(f, H_\infty) < \infty$ then f belongs to the Hardy-Sobolev space H_1^1. Under the same conditions on f, Peller [13] showed that f belongs to the Hardy-Besov space B_1^1. Since $B_1^2 \subsetneqq H_1^1$, Peller's result is stronger than Dolzhenko's. Nevertheless (see [17]) both of these inverse theorems on rational approximation are best possible in the following sense. For every nonincreasing sequence of numbers a_n ($n = 1, 2, \dots$) that satisfies the condition $\sum a_n = +\infty$, there exists an $f_* \in H_\infty$ such that $R_n(f_*, H_\infty) = O(a_n)$ and $f_* \notin H_1^1$, and consequently $f_* \notin B_1^1$. These results are generalized in the present paper. In particular, we obtain the best possible sufficient conditions on the rate of decrease of $R_n(f, H_p)$ ($1 < p \leqslant \infty$) that guarantee that f belongs to the Hardy-Sobolev space H_σ^α or the Hardy-Besov space B_σ^α ($\alpha > 0$, $\sigma = (\alpha + p^{-1})^{-1}$). In addition, in contrast to [13], [27] and [28], we prove the implication $\sum (R_n(f, \mathrm{BMOA}))^{1/\alpha} < \infty \Rightarrow f \in B_{1/\alpha}^\alpha$ (first obtained by Peller [13] for $0 < \alpha \leqslant 1$ and then generalized to the case $\alpha > 1$ in [27], [28] and [31]) without making use of the connection of $R_n(f, \mathrm{BMOA})$ with Hankel operators. The method for solving these problems uses inequalities of Bernstein type, obtained here, for derivatives of rational functions.

The main results of this paper were presented without proof in [29]–[32].

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§1. On some spaces of functions analytic in a disk

Let S be a rectifiable curve in the complex plane. We denote by $L_p(S)$, $p \in (0, \infty]$, the set of functions f, measurable on S, for which $\|f\|_{p,S} \leqslant \infty$, where we set

$$\|f\|_{p,S} = \left(\int_S |f(z)|^p |dz|\right)^{1/p}, \qquad p \neq \infty,$$

$$\|f\|_{\infty,S} = \operatorname*{ess\,sup}_{z \in S} |f(z)|, \qquad p = \infty.$$

We denote by T, D_+, and D_-, respectively, the circle $|z| = 1$, the disk $|z| < 1$ and the domain $|z| > 1$; by $A(D_\pm)$ we denote the set of functions that are analytic in $D_\pm$. We denote by H_p, $0 < p \leqslant \infty$, the Hardy space [1] of functions in $A(D_+)$ for which the quasinorm

$$\|f\|_{H_p} = \lim_{\rho \to 1-0} \|f(\cdot\rho)\|_p$$

is finite, where we write for short $\|g\|_p = \|g\|_{p,T}$ for $g \in L_p(T)$. The indicated limit exists because of the monotonicity of $\|f(\cdot\rho)\|_p$ with respect to ρ ([1], p. 273). If $f \in H_p$ and $z \in T$, we denote by $f(z)$ the nontangential limit of $f(\zeta)$ as $\zeta \to z$ ([1], p. 276). It is known that $\|f\|_{H_p} = \|f\|_p$.

Let $f \in A(D_+)$ and let $\hat{f}(k)$ $(k = 0, 1, \dots)$ be the Taylor coefficients of f. If $\alpha \geqslant 0$, the following functions in $A(D_+)$,

$$f^{(\alpha)}(z) = \sum_{k=[\alpha]}^{\infty} \frac{\Gamma(k - [\alpha] + 1 + \alpha)}{\Gamma(k - [\alpha] + 1)} \hat{f}(k) z^{k-[\alpha]},$$

$$J^\alpha f(z) = \sum_{k=0}^{\infty} (k+1)^\alpha \hat{f}(k) z^k,$$

where Γ is Euler's gamma function and $[\alpha]$ is the integral part of α, are called the derivatives of f in the Riemann-Liouville and the Weyl senses, respectively. Evidently, if $\alpha = l$ is a positive integer, $f^{(l)}(z)$ is the ordinary derivative, and $J^l f(z) = [(d/dz)z]^l f(z)$. The function $J^\alpha f$ will also be considered for $\alpha < 0$. In this case it is called the integral of f of order $-\alpha$ in the sense of Weyl. It is easy to establish (see also [2]) that when $\alpha > 0$

$$f^{(\alpha)}(z) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{|\zeta| = \rho} f(\zeta)\left(1 - \frac{z}{\zeta}\right)^{-1-\alpha} \zeta^{-1-[\alpha]} d\zeta, \qquad |z| < \rho < 1, \tag{1}$$

where the branch of $(1 - \eta)^{-1-\alpha}$ is chosen so that $(1 - \eta)^{-1-\alpha} > 0$ for $\eta \in (-\infty, 1)$. We denote by H_p^α $(\alpha \in (-\infty, \infty)$, $p \in (0, \infty])$ the Hardy-Sobolev space, i.e. the set of $f \in A(D_+)$ with finite quasinorm $\|f\|_{H_p^\alpha} = \|J^\alpha f\|_{H_p}$. We denote by $B_{p,q}^\alpha$ $(\alpha \in (-\infty, \infty)$, $p \in (0, \infty]$, $q \in (0, \infty])$ the Hardy-Besov space, i.e. the set of $f \in A(D_+)$ with finite quasinorm

$$\|f\|_{B_{p,q}^\alpha}^{(\beta)} = \left(\int_0^1 (1-\rho)^{(\beta-\alpha)q-1} \|J^\beta f(\cdot\rho)\|_{H_p}^q d\rho\right)^{1/q}, \qquad q \neq \infty,$$

$$\|f\|_{B_{p,\infty}^\alpha}^{(\beta)} = \sup_{0<\rho<1} (1-\rho)^{\beta-\alpha} \|J^\beta f(\cdot\rho)\|_{H_p}, \qquad q = \infty.$$

Here β is arbitrary, $\beta > \alpha$. The space $B^\alpha_{p,q}$ is independent of β [2] and the quasinorms for different values of β are equivalent. In this connection, we call the quasinorm with $\beta = \alpha + 1$ *fundamental*, and denote it by $\|f\|_{B^\alpha_{p,q}}$. We abbreviate $B^\alpha_{p,q}$ to B^α_p.

Unlike $J^\alpha f$, the derivative $f^{(\alpha)}$ does not have the semigroup property. In fact, the equality $f^{(\alpha_1+\alpha_2)} = (f^{(\alpha_1)})^{(\alpha_2)}$ is satisfied for every f only in the case when α_1 and α_2 are integers. Lemma 1.1, proved below, lets one avoid this inconvenience.

DEFINITION. Let W be a quasinormed space of elements of $A(D_+)$. A sequence $\{\lambda_k\}_0^\infty$ is called a *multiplier* in W if, for each $f \in W$, we have $\|g\|_W \leqslant c\|f\|_W$, where $g(z) = \sum_0^\infty \lambda_k \hat{f}(k) z^k$, with $c > 0$ and independent of f.

LEMMA 1.1. *Let $\alpha, \beta > 0$. Then the sequences $\lambda_k = \Gamma(k+\alpha+\beta)[\Gamma(k+\alpha)(k+1)^\beta]^{-1}$ and $\mu_k = \lambda_k^{-1}$ $(k = 0,1,2,\dots)$ are multipliers in the spaces H^γ_p and $B^\gamma_{p,q}$.*

PROOF. It follows from the definitions of H^γ_p and $B^\gamma_{p,q}$ that we may restrict our attention to $H^0_p = H_p$. Let m be the smallest integer such that $m \geqslant p^{-1} + 1$. From the asymptotic series for the gamma function ([3], p. 339) we obtain

$$\lambda_k = b_0 + (k+1)^{-1} b_1 + (k+1)^{-2} b_2 + \cdots + (k+1)^{-m} b_m + (k+1)^{-m-1} d_k,$$

where $b_0, b_1, \dots, b_m$ are numbers depending only on α and β, and $\{d_k\}_0^\infty$ is a bounded sequence. Consequently, if $f \in H_p$ and $g(z) = \sum_0^\infty \lambda_k \hat{f}(k) z^k$, then

$$g(z) = \sum_{j=0}^{m} b_j J^{-j} f(z) + \sum_{k=0}^{\infty} \hat{f}(k)(k+1)^{-m-1} d_k z^k \overset{\text{def}}{=} \sum_{j=0}^{m} b_j f_j(z) + \psi(z). \tag{2}$$

Moreover, we have (see [4], p. 142) $\|f_j\|_{H_p} \leqslant c_1(j)\|f\|_{H_p}$ and (see [2])

$$|\hat{f}(k)| \leqslant c_2(p)(k+1)^{1/p} \qquad (k = 0,1,2,\dots).(^1)$$

Consequently $|\hat{\psi}(k)| \leqslant c_3(p)\|f\|_{H_p}(k+1)^{-2}$ and $\|\psi\|_{H_p} \leqslant c_4(p)\|f\|_{H_p}$. Thus we obtain $\|g\|_{H_p} \leqslant c_5(p)\|f\|_{H_p}$ from (2). We can show in a similar way that the sequence $\{\mu_k\}_0^\infty$ is a multiplier in H_p. This completes the proof of Lemma 1.1.

Let X and Y be quasinormed spaces. By an embedding $X \subset Y$ we shall always understand a continuous embedding, i.e. if $f \in X$ then $f \in Y$ and $\|f\|_Y \leqslant c\|f\|_X$, where $c > 0$ is independent of f.

Lemma 1.1 lets us extend various embedding theorems that were proved for the Riemann-Liouville derivative to the Weyl derivative. For example, we have ([4], p. 142)

$$H^{\alpha_0}_{p_0} \subset H^{\alpha_1}_{p_1} \qquad \left(0 < p_0 \leqslant p_1 < \infty,\ p_0^{-1} - p_1^{-1} = \alpha_0 - \alpha_1\right). \tag{3}$$

There are the following embeddings between the spaces $B^\alpha_{p,q}$ [2]:

$$B^{\alpha_1}_{p_1,q_1} \subset B^{\alpha_0}_{p_0,q_0} \qquad (\alpha_1 > \alpha_0,\ p_1 \geqslant p_0), \tag{4}$$

$$B^{\alpha}_{p,q_1} \subset B^{\alpha}_{p,q_0} \qquad (q_1 < q_0), \tag{5}$$

$$B^{\alpha_1}_{p_1,q} \subset B^{\alpha_0}_{p_0,q} \qquad \left(\alpha_1 - \alpha_0 = p_1^{-1} - p_0^{-1} > 0,\ p_0 \neq \infty\right). \tag{6}$$

The following two embeddings [2] reflect the connection between H^α_p and B^α_p:

$$H^\alpha_p \subset B^\alpha_p \quad (2 \leqslant p < \infty), \tag{7}$$

$$B^\alpha_p \subset H^\alpha_p \quad (0 < p \leqslant 2). \tag{8}$$

$(^1)$ By $c(\alpha,\beta,\dots)$, $c_1(\alpha,\beta\dots)$, $c_2(\alpha,\beta,\dots),\dots$ we denote positive numbers, different in different places, depending only on $\alpha,\beta,\dots$.

We denote by BMOA the space of analytic functions of bounded mean oscillation [5], i.e. $f \in \mathrm{BMOA}$ if there exists $g \in L_\infty(T)$ such that

$$f(z) = \frac{1}{2\pi i}\int_T \frac{g(\zeta)\,d\zeta}{\zeta - z}, \qquad z \in D_+. \tag{9}$$

The norm in BMOA is defined as follows:

$$\|f\|_{\mathrm{BMOA}} = \inf\|g\|_\infty,$$

where the lower bound is taken over all $g \in L_\infty(T)$ for which (9) holds. Evidently

$$H_\infty \subset \mathrm{BMOA} \subset H_p \qquad (0 < p < \infty). \tag{10}$$

§2. Inequalities of Bernstein type for the derivatives of rational functions

Surveys of inequalities for the derivatives of rational functions are given by Gonchar [6] and Rusak [7]. Here we present only the inequalities that are directly related to the subject of the present paper. The first result in this direction was obtained by Dolzhenko [8], who showed that a rational function r of degree $n \geqslant 1$ with poles only in D_- satisfies

$$\|r\|_{H_1^1} \leqslant c_1 n \|r\|_{H_\infty}, \tag{11}$$

$$\|r\|_{B_2^{1/2}} \leqslant c_2 n^{1/2} \|r\|_{H_\infty}. \tag{12}$$

For any $s \in \mathbf{N}$([2]) and $p \in (0, \infty]$ the following generalization of (11) follows from the results of Sevast′yanov [9]:

$$\|r\|_{H^s_{\sigma-\varepsilon}} \leqslant c_3(s, p, \varepsilon) n^s \|r\|_{H_p} \qquad \left(\sigma = \left(s + p^{-1}\right)^{-1},\ \varepsilon \in (0, \sigma)\right). \tag{13}$$

As was observed in [9], one cannot take $\varepsilon = 0$ in the preceding inequality if $1/p \in \mathbf{N}$. To see this, it is enough to consider the function $r(z) = (1 + \delta - z)^{-1}$ as $\delta \to +0$. We showed in [10] that for $p = \infty$ and any $s \in \mathbf{N}$ we can take $\varepsilon = 0$ in (13). Inequality (12) was generalized by Danchenko [12], who showed that

$$\|r\|_{B^\alpha_{t,q}} \leqslant c_3(\alpha, t, q, n) \|r\|_{H_p} \qquad \left(\alpha \in (0, 1],\ p \in (1, \infty],\ t \leqslant \left(\alpha + p^{-1}\right)^{-1},\ q > 0\right). \tag{14}$$

Another generalization follows from a result of Peller [13] on best rational approximations for the class $B^\alpha_{1/\alpha}$, $\alpha \in (0, 1]$, in the space BMOA. This is

$$\|r\|_{B^\alpha_{1/\alpha}} < c_4(\alpha) n^\alpha \|r\|_{\mathrm{BMOA}} \qquad (0 < \alpha \leqslant 1).$$

Our Theorem 2.1 (below) generalizes and strengthens the results quoted above.

THEOREM 2.1. *Let r be a rational function of degree $n \geqslant 1$ with all its poles in D_-; let $\alpha > 0$, $p \in (1, \infty]$, and $\sigma = (\alpha + p^{-1})^{-1}$. Then*

$$\|r\|_{H^\alpha_\sigma} \leqslant c_1(\alpha, p) n^\alpha \|r\|_{H_p}, \tag{15}$$

$$\|r\|_{B^\alpha_\sigma} \leqslant c_2(\alpha, p) n^\alpha \|r\|_{H_p}, \tag{16}$$

$$\|r\|_{H^\alpha_{1/\alpha}} \leqslant c_3(\alpha) n^\alpha \|r\|_{\mathrm{BMOA}}, \tag{17}$$

$$\|r\|_{B^\alpha_{1/\alpha}} \leqslant c_4(\alpha) n^\alpha \|r\|_{\mathrm{BMOA}}. \tag{18}$$

([2]) $\mathbf{N}$ is the set of positive integers.

REMARKS. 1) The extremal exponent $\sigma = (\alpha + p^{-1})^{-1}$ also occurs in the following inequality of Brudnyĭ [22]:

$$\| \Delta_n^k r \|_{\sigma,[0,1-kh]} \leqslant c(\alpha, p)(nh)^{\alpha} \| r \|_{p,[0,1]},$$

where r is a rational function of degree n with poles outside $[0,1]$, $\alpha > 0$, $k = [\alpha] + 1$, $\Delta_n^k r$ is the kth finite difference of r with step $0 < h < 1/k$, and $1 \leqslant p \leqslant \infty$.

2) From the embeddings (7) and (8) we obtain that (15) and (16) are equivalent for $\sigma = 2$. For $\sigma > 2$, (15) is stronger, but (16) is stronger for $\sigma < 2$. A similar statement holds for (17) and (18).

In the proof of Theorem 2.1 we shall consistently use the following notation. Let $a_1, \dots, a_n$ belong to D_+. We set

$$B(z) = \prod_{k=0}^{n} \frac{z - a_k}{1 - \bar{a}_k z} \qquad (a_0 = 0),$$

$$Q(z, \zeta) = \frac{B(z) - B(\zeta)}{z - \zeta},$$

$$\lambda(z, \beta) = \sum_{k=0}^{n} \left(\frac{1 - |a_k|}{|z - a_k|} \right)^{\beta} \frac{1}{|z - a_k|} \qquad (\beta > 0).$$

LEMMA 2.1. *If $z \in T$ and $l \in \mathbf{N}$ then*(3)

$$\frac{(2l-1)!}{2\pi} \int_T |Q(z,\zeta)|^{2l} |d\zeta| = z^l \sum_{j=1}^{l} C_{2l}^{l-j} (-1)^{l-j} B^{-j}(z) \left[B^j(z) z^{l-1} \right]^{(2l-1)}.$$

PROOF. For z and $\zeta \in T$ we have $|d\zeta| = d\zeta / i\zeta$ and

$$|Q(\zeta, z)|^2 = Q(\zeta, z) \overline{Q(\zeta, z)} = \frac{\zeta z}{B(\zeta) B(z)} Q^2(\zeta, z).$$

Consequently

$$\frac{(2l-1)!}{2\pi} \int_T |Q(\zeta, z)|^{2l} |d\zeta| = z^l B^{-l}(z) I_l(z), \tag{19}$$

where

$$I_l(z) = \frac{(2l-1)!}{2\pi i} \int_T Q^{2l}(\zeta, z) B^{-l}(\zeta) \zeta^{l-1} d\zeta.$$

Since $I_l(z)$ is continuous in $D_+ \cup T$, it is enough to calculate it for $z \in D_+$. Thus we have

$$I_l(z) = \sum_{j=-l}^{l} C_{2l}^{l-j} (-B(z))^{l-j} I_{l,j}(z) \qquad (z \in D_+), \tag{20}$$

where

$$I_{l,j}(z) = \frac{(2l-1)!}{2\pi i} \int_T \frac{B^j(\zeta) \zeta^{l-1}}{(\zeta - z)^{2l}} d\zeta.$$

By Cauchy's formula we obtain

$$I_{l,j}(z) = \left[B^j(z) z^{l-j} \right]^{(2l-1)} \qquad (j = 1, \dots, l). \tag{21}$$

(3) There is a similar assertion in [7] (pp. 115 and 132) for the real axis, with $l = 1$ and 2.

If $-l \leqslant j \leqslant 0$, the point $\zeta = \infty$ is a zero of order at least 2 of the function

$$B^j(\zeta)\zeta^{l-1}(\zeta - z)^{-2l}.$$

Therefore $I_{l,j}(z) = 0$ $(j = -l, \dots, 0)$. By using (19)–(21), we obtain the conclusion of Lemma 2.1.

LEMMA 2.2. *For all $z \in T$ and $s \in \mathbf{N}$*

$$|B^{(s)}(z)| \leqslant 2^s s! \lambda^s(z, 1/s).$$

PROOF. We set $b_k(z) = (z - a_k)(1 - \bar{a}_k z)^{-1}$. Then

$$|B^{(s)}(z)| = \sum \frac{s!}{j_0! j_1! \cdots j_n!} b_0^{(j_0)}(z) b_1^{(j_1)}(z) \cdots b_n^{(j_n)}(z), \tag{22}$$

where the summation is over all collections $j_0, j_1, \dots, j_n$ of nonnegative numbers satisfying the condition $j_0 + j_1 + \cdots + j_n = s$. It is evident that for every $z \in T$, $0 \leqslant k \leqslant n$ and $1 \leqslant j \leqslant s$ we have

$$|b_k^{(j)}(z)| \leqslant 2 j! \left(\left| \frac{1 - |a_k|}{z - a_k} \right|^{1/s} \frac{1}{|z - a_k|} \right)^j. \tag{23}$$

Lemma 2.2 follows from (22) and (23).

LEMMA 2.3. *If $z \in T$ and $\alpha > 0$, then*

$$\|Q(\cdot, z)\|_{1+\alpha} \leqslant c(\alpha) \lambda^{\alpha/(\alpha+1)}(z, 1/(\alpha + 2)).$$

PROOF. It follows immediately from Lemmas 2.1 and 2.2 that

$$\int_T |Q(\zeta, z)|^{2l} |d\zeta| \leqslant c(l) \lambda^{2l-1}\left(z, \frac{1}{2l - 1}\right) \qquad (z \in T,\ l \in \mathbf{N}). \tag{24}$$

Let m be the smallest odd number such that $m > \alpha$. We introduce $p = (m + 1)(\alpha + 1)^{-1}$, $q = (m + 1)(m - \alpha)^{-1}$ and $S(z) = \{\zeta \in T\colon |\arg\zeta - \arg z| \leqslant \lambda^{-1}(z, m^{-1})\}$. From (24) and Hölder's inequality, we obtain

$$\int_{S(z)} |Q(\zeta, z)|^{1+\alpha} |d\zeta| \leqslant \|1\|_{q, S(z)} \|Q^{1+\alpha}(\cdot, z)\|_{p, S(z)} \leqslant c_1(m) \lambda^\alpha\left(z, \frac{1}{m}\right). \tag{25}$$

On the other hand,

$$\int_{T \setminus S(z)} |Q(\zeta, z)|^{1+\alpha} |d\zeta| \leqslant 2^{1+\alpha} \int_{T \setminus S(z)} |\zeta - z|^{-1-\alpha} |d\zeta| \leqslant c_2(\alpha) \lambda^\alpha\left(z, \frac{1}{m}\right). \tag{26}$$

Since $\lambda(z, \beta)$ does not increase in β for fixed $z \in T$, Lemma 2.3 follows from (25) and (26).

LEMMA 2.4. *If $f \in L_p(T)$, $p \in (1, \infty]$, $\alpha > 0$ and*

$$g(z) = \int_T |Q(\zeta, z)|^{1+\alpha} |f(\zeta)|\, |d\zeta|,$$

then $\|g\|_\sigma \leqslant c(\alpha, p) n^\alpha \|f\|_p$, where $\sigma = (\alpha + p^{-1})^{-1}$.

PROOF. For $p = \infty$ the necessary inequality follows from Lemma 2.3 and the relation

$$\int_T \lambda(z, \beta) |dz| \leqslant c_1(\beta) n \qquad (\beta > 0). \tag{27}$$

Now let $p \in (1,\infty)$ and $\alpha = 1 - p^{-1}$. Then $\sigma = 1$ and consequently, by Lemma 2.3, Hölder's inequality, and (27),

$$\|g\|_1 \leqslant \int_T \|Q^{1+\alpha}(\cdot, z)\|_1 |f(\zeta)|\,|d\zeta| \leqslant c_2(\alpha)\int_T \lambda^\alpha\Big(\zeta, \frac{1}{\alpha+2}\Big)|f(\zeta)|\,|d\zeta|$$
$$\leqslant c_2(\alpha)\|\lambda\|_1^\alpha \|f\|_p \leqslant c_3(\alpha) n^\alpha \|f\|_p.$$

Therefore Lemma 2.4 is established in the case under consideration. Now let α be arbitrary. Choose positive numbers γ, τ, l, and s satisfying the conditions $l \in (1,p)$, $l^{-1} + s^{-1} = 1$, $\gamma + \tau = \alpha$ and $l\tau = 1 - l/p$. Then, by Hölder's inequality,

$$|g(z)| \leqslant \|Q^{s\gamma+1}(\cdot, z)\|_1^{1/s} \|Q^{l\tau+1}(\cdot, z) f'(\cdot)\|_1^{1/l} \stackrel{\text{def}}{=} \varphi(z)\psi(z) \tag{28}$$

for every $z \in T$. From Lemma 2.3 and (27) we have

$$\|\varphi\|_{1/\gamma} \leqslant c_4(s,\gamma) n^\gamma. \tag{29}$$

Using the fact that the lemma has already been established for $\alpha = 1 - p^{-1}$ (in this case $l\tau = 1 - (p/l)^{-1}$), we obtain

$$\|\psi\|_l \leqslant c_5(l,p) n^{1/l - 1/p} \|f\|_p. \tag{30}$$

Thus we obtain the conclusion of Lemma 2.4 in the case $p \in (1,\infty)$ and $\alpha > 0$ from (28)–(30) and Hölder's inequality.

PROOF OF (15) AND (17). Let the poles of the rational function r be located, counting multiplicities, at the points $1/\bar{a}_1, \dots, 1/\bar{a}_n$, where $a_1, \dots, a_n$ belong to D_+. Then the function $r(\zeta)B^{-k}(\zeta)(1 - z/\zeta)^{-1-\alpha}\zeta^{-1-[\alpha]}$ ($k \in \mathbf{N}$ and $z \in D_+$) is an analytic function of ζ in D_- and has a zero of order at least 2 at ∞. Consequently

$$\int_T r(\zeta) B^{-k}(\zeta)\Big(1 - \frac{z}{\zeta}\Big)^{-1-\alpha} \zeta^{-1-[\alpha]}\,d\zeta = 0.$$

Therefore if we expand the function $(1 - B(z)/B(\zeta))^{1+\alpha}$ ($z \in D_+$ and $\zeta \in T$) in a Taylor series in $B(z)/B(\zeta)$, we obtain from (1)

$$r^{(\alpha)}(z) = \frac{\Gamma(1+\alpha)}{2\pi i}\int_T r(\zeta)\Big(1 - \frac{B(z)}{B(\zeta)}\Big)^{1+\alpha}\Big(1 - \frac{z}{\zeta}\Big)^{-1-\alpha}\zeta^{-1-[\alpha]}\,d\zeta. \tag{31}$$

From (31) and Lemmas 1.1 and 2.4 we obtain (15). To prove (17) it is enough to observe that (31) remains valid if we replace $r(\zeta)$ by $r(\zeta) + h(1/\zeta)$ on the right, where $h \in H_1$ and $h(0) = 0$.

LEMMA 2.5. *Let r be a rational function of degree $n \geqslant 1$ with all its poles in D_-, $\beta > 0$ and $p \in (1, \infty]$.*

1) *There are continuous functions $\lambda(\varphi)$ and $h(\varphi)$ of period 2π that satisfy the conditions*

$$\|\lambda\|_{p,[0,2\pi]} \leqslant c(\beta, p)\|r\|_{H_p} \quad \textit{and} \quad \lambda(\varphi) \geqslant 0,$$
$$\|h\|_{1,[0,2\pi]} \leqslant n \quad \textit{and} \quad h(\varphi) \geqslant 1,$$
$$|J^\beta r((1-x)e^{i\varphi})| \leqslant \lambda(\varphi)\big(\min(x^{-1}, h(\varphi))\big)^\beta, \qquad x \in (0,1),\ \varphi \in [0, 2\pi].$$

2) *There is a continuous function $g(\varphi)$ of period 2π that satisfies the conditions*

$$\|g\|_{1,[0,2\pi]} \leqslant c(\beta) n \quad \textit{and} \quad g(\varphi) > 1,$$
$$|J^\beta r((1-x)e^{i\varphi})| \leqslant \|r\|_{\mathrm{BMOA}}\big(\min(x^{-1}, g(\varphi))\big)^\beta, \qquad x \in (0,1),\ \varphi \in [0, 2\pi].$$

PROOF. It is evident that for $x \in (0,1)$ and $\varphi \in [0, 2\pi]$

$$|J^\beta r((1-x)e^{i\varphi})| \leqslant \max_{\rho\in[0,1]} |J^\beta r(\rho e^{i\varphi})| \overset{\text{def}}{=} G(\varphi). \tag{32}$$

From Lemma 1.1 we find that there is a function f, analytic in $D_+ \cup T$, such that $J^\beta r = f^{(\beta)}$ and $\|f\|_{H_p} \leqslant c_1(p)\|r\|_{H_p}$. Consequently we find from (1) that

$$J^\beta r(z) = \frac{\Gamma(1+\alpha)}{2\pi i}\int_{S(z)} f(\zeta)\left(1 - \frac{z}{\zeta}\right)^{-1-\beta} \zeta^{-1-[\beta]}\, d\zeta \qquad (z \in D_+\setminus\{0\}),$$

where $S(z)$ is the convex curve formed by the circle $|\zeta| = \frac{1}{2}$ and the tangents to it from the point $z/|z|$. Hence we obtain

$$|J^\beta r((1-x)e^{i\varphi})| \leqslant c_2(\beta)F(\varphi)x^{-\beta}, \qquad F(\varphi) = \max_{\zeta\in S(z)} |f(\zeta)|, \tag{33}$$

where $z = (1-x)e^{i\varphi}$. Let us show that the functions

$$h(\varphi) = c_3(\beta, p)n^{1-\beta\gamma}\left[\|r\|_{H_p}^{-1}G(\varphi)\right]^\gamma + 1 \qquad \left(\gamma = (\beta + p^{-1})^{-1}\right),$$

$$\lambda(\varphi) = F(\varphi) + c_4(\beta, p)n^{-\beta(1-\beta\gamma)}\|r\|_{H_p}^{\beta\gamma}G^{1-\beta\gamma}(\varphi)$$

satisfy the requirements of the lemma for suitable choices of the constants $c_3(\beta, p)$ and $c_4(\beta, p)$. In fact, from (15) together with Theorem (7.36) of [1], p. 278, we obtain

$$\|G\|_{\gamma,[0,2\pi]} \leqslant c_5(\beta, p)n^\beta\|r\|_{H_p},$$

$$\|F\|_{p,[0,2\pi]} \leqslant c_6(p)\|r\|_{H_p}.$$

Using (32) and (33), we obtain assertion 1) of Lemma 2.5.

For the proof of assertion 2) we observe that

$$J^\beta r(z) = \frac{\Gamma(1+\beta)}{2\pi i}\int_T [f(\zeta) + s(1/\zeta)]\left(1 - \frac{z}{\zeta}\right)^{-1-\beta}\zeta^{-1-[\beta]}\, d\zeta \qquad (z \in D_+),$$

where $s \in H_1$ and $s(0) = 0$. Consequently, instead of (33) we must use the inequality

$$|J^\beta r((1-x)e^{i\varphi})| \leqslant c_7(\beta)\|r\|_{\mathrm{BMOA}}x^{-\beta}.$$

To obtain the analog of (32) we have to use (17). Everything else is obtained as in the proof of assertion 1) for $p = \infty$.

PROOF OF (16) AND (18). Let h and λ be the functions from Lemma 2.5 corresponding to $\beta = \alpha + 1$. Then we obtain (16) from Lemma 2.5:

$$\|r\|_{B_\sigma^\alpha}^\sigma \leqslant \int_0^{2\pi}\lambda^\sigma(\varphi)\left(\int_0^{1/h(\varphi)} h^{\sigma(\alpha+1)}x^{\sigma-1}\,dx + \int_{1/h(\varphi)}^1 x^{-\alpha\sigma-1}\,dx\right)d\varphi$$

$$\leqslant c_1(\alpha, p)\int_0^{2\pi}\lambda^\sigma(\varphi)h^{\alpha\sigma}(\varphi)\,d\varphi \leqslant c_2(\alpha, p)n^{\alpha\sigma}\|r\|_{H_p}^\sigma.$$

Here in obtaining the last inequality we have also applied Hölder's inequality. Similarly we obtain (18) from Lemma 2.5.

COROLLARY 2.1 (compare (13)). *Let* $\alpha > 0$, $p \in (1, \infty]$, $\sigma = (\alpha + p^{-1})^{-1}$, $s \in (0, \infty]$, $q \in (0, \infty]$ *and*

$$A_n(\alpha, p, s, q) = \sup\left(\|r_n\|_{B_{s,q}^\alpha}\|r_n\|_{H_p}^{-1}\right),$$

where the upper bound is taken over all rational functions $r_n \not\equiv 0$ of degree at most n ($n \geqslant 1$). Then(4)

$$A_n(\alpha, p, \sigma, q) \asymp n^{\alpha} \qquad (q \geqslant \sigma), \tag{34}$$

$$A_n(\alpha, p, \sigma, q) \asymp n^{q^{-1}-p^{-1}} \qquad (q < \sigma), \tag{35}$$

$$A_n(\alpha, p, s, q) = +\infty \qquad (s > \sigma,\ q \in (0, \infty]), \tag{36}$$

$$A_n(\alpha, p, s, q) \asymp n^{\alpha} \qquad (s < \sigma,\ q \in (0, \infty]). \tag{37}$$

PROOF. The upper inequality in (34) follows from (16) and (5). To obtain the lower inequality in (34) it is enough to consider the function $r_n(z) = z^n$. The upper inequality in (35) follows from (16) and (6). To obtain the lower inequality we consider the function

$$r_n(z) = \sum_{k=0}^{n-1} \left[(1+\varepsilon)e^{2\pi i k/n} - z\right]^{-1}$$

for sufficiently small $\varepsilon > 0$. We immediately verify (36) by the example of the function $r_1(z) = (1 + \varepsilon - z)^{-1}$ as $\varepsilon \to +0$. To obtain the lower inequality in (37) we consider the function $r_n(z) = z^n$. To obtain the upper inequality in (37) we use Lemma 2.5. Let h and λ be the functions of Lemma 2.5 corresponding to $\beta = s^{-1} - p^{-1} > \alpha$. Then (with corresponding changes for $q = \infty$) we have

$$\|r\|^{(\beta)}_{B^{\alpha}_{s,q}} \leqslant 2^{1/q+1/s}\left(\int_{1/n}^{1}\left(\int_0^{2\pi}\left(\lambda(\varphi)x^{-\beta}\right)^s d\varphi\right)^{q/s} x^{q(\beta-\alpha)-1}\,dx\right)^{1/q}$$
$$+2^{1/q+1/s}\left(\int_0^{1/n}\left(\int_0^{2\pi}\left(\lambda(\varphi)h^{\beta}(\varphi)\right)^2 d\varphi\right)^{q/s} x^{q(\beta-\alpha)-1}\,dx\right)^{1/q}$$
$$\leqslant c_2(\alpha, p, q, s)n^{\alpha}\|r_n\|_{H_p}.$$

Corollary 2.1 is proved.

Let the rational function r of degree $n + m$ have no poles on T, but n poles in D_+ and m in D_-. Then $r(z) = r_+(z) + r_-(1/z)$, where r_+ and r_- are rational functions of respective degrees n and m with all their poles in D_-. It is easy to obtain the following corollary of Theorem 4.1.

COROLLARY 2.2. *If $\alpha > 0$, $p \in (1, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$ then*

$$\|r_+\|_{B^{\alpha}_{\sigma}} \leqslant c(\alpha, p)n^{\alpha}\|r\|_P, \qquad \|r_-\|_{B^{\alpha}_{\sigma}} \leqslant c(\alpha, p)m^{\alpha}\|r\|_P.$$

In conclusion, we remark that it would be interesting to extend Theorem 2.1 to the Smirnov spaces E_p. Some special results in this direction were obtained in [11], [14] and [15].

§3. Inverse theorems on rational approximation

Let $f \in H_p$ and $n \geqslant 1$. Let $R_n(f, H_p)$ denote the best approximation to f in H_p by rational fractions of degree at most $n - 1$. Following [16], we introduce the approximation space $R^{\alpha}_{p,q}$ ($\alpha > 0$, $p \in (0, \infty]$, $q \in (0, \infty]$) of functions $f \in H_p$ with finite quasinorm

$$\|f\|_{R^{\alpha}_{p,q}} = \|f\|_{H_p} + \left(\sum_{k=0}^{\infty}\left(2^{k\alpha}R_{2^k}(f, H_p)\right)^q\right)^{1/q}, \qquad q \neq \infty,$$

$$\|f\|_{R^{\alpha}_{p,\infty}} = \|f\|_{H_p} + \sup_{k=0,1,\ldots} 2^{k\alpha}R_{2^k}(f, H_p).$$

(4) The symbol $a_n \asymp b_n$ means that there are constants $c_1, c_2 > 0$ such that $c_1 b_n \leqslant a_n \leqslant c_2 b_n$, $n = 1, 2, \ldots$.

We denote by $R_n(f, \mathrm{BMOA})$ the best approximation to f in BMOA by rational fractions of degree at most $n-1$, and the corresponding approximation space by $R^\alpha_{*,q}$.

THEOREM 3.1. *Let* $\alpha > 0$, $p \in (1, \infty]$ *and* $\sigma = (\alpha + p^{-1})^{-1}$. *Then*

$$R^\alpha_{p,\sigma} \subset B^\alpha_\sigma, \tag{38}$$

$$R^\alpha_{p,\min(2,\sigma)} \subset H^\alpha_\sigma, \tag{39}$$

$$R^\alpha_{*,1/\alpha} \subset B^\alpha_{1/\alpha}, \tag{40}$$

$$R^\alpha_{*,\min(2,1/\alpha)} \subset H^\alpha_{1/\alpha}. \tag{41}$$

REMARK. Some special cases of the embeddings (38)–(41) were obtained earlier by Dolzhenko [11], [17], Danchenko [12], Peller [13], and the author [10]. These special cases are corollaries of inequalities of Bernstein type for derivatives of rational functions (see the survey in §2). An exception is (40), obtained earlier by Peller [13] for $0 < \alpha \leqslant 1$ and then generalized to the case $\alpha > 1$ simultaneously and independently by Peller [27], Semmes [28] and the author [31]. As we noted in the Introduction, our proof differed from those of Peller and Semmes by not using the connection between best rational approximations in BMOA and Hankel operators. Peller (see [13] and [27]) also obtained the inverse of the embedding (40). Embeddings (38), (39), and (41) admit partial inverses [31], and if $1 < p < \infty$ embedding (38) also admits a complete inverse (see [32]). Proofs of these results will be given in another paper.

For the proof of Theorem 3.1 we need the following Lemmas 3.1 and 3.2.

LEMMA 3.1 ([1], p. 20). *Let* $f(x)$ *be a nonnegative function defined for* $x > 0$, *and let* $r > 1$ *and* $s < r - 1$. *If* $f^r(x)x^s$ *is integrable on* $(0, \infty)$, *then*

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^r x^s\,dx \leqslant \left(\frac{r}{r-s-1}\right)^r \int_0^\infty f^r(x)x^s\,dx.$$

LEMMA 3.2. *Let* $\{\lambda_k\}_{-\infty}^\infty$ *and* $\{h_k\}_{-\infty}^\infty$ *be sequences of nonnegative numbers satisfying the conditions*

$$\frac{h_{k+1}}{h_k} \geqslant q \quad (k = 0, \pm 1, \pm 2, \dots), \qquad \sum_{k=-\infty}^{\infty} \left(h_k^{l-m}\lambda_k\right)^r < \infty,$$

where $l > m > 0$, $r > 1$ *and* $q > 1$. *If*

$$\psi(x) = \sum_{k=-\infty}^{\infty} \lambda_k\left(\min(h_k, x^{-1})\right)^l \qquad (x \in [0, \infty)),$$

then

$$\int_0^\infty \psi^r(x)x^{mr-1}\,dx \leqslant c(l, m, q, r)\sum_{k=-\infty}^{\infty}\left(h_k^{l-m}\lambda_k\right)^r.$$

PROOF. We define a function $\varphi(y)$ on $(0, \infty)$ in the following way. If j is a positive integer and $y \in (q^{j-1}, q^j]$ then $\varphi(y)$ equals $\lambda_k q^{-j}$ if $h_k \in (q^{j-1}, q^j]$ and equals 0 when no h_k belongs to $(q^{j-1}, q^j]$. Since $h_{k+1}/h_k \geqslant q$ for every k, the interval $(q^{j-1}, q^j]$ contains at most one h_k and consequently $\varphi(y)$ is well defined. It is easy to verify the inequality

$$\psi(x) \leqslant c_1(q)\int_0^{1/x} \varphi(y)y^l\,dy + \frac{c_2(q)}{x^l}\int_0^x \varphi\left(\frac{1}{y}\right)\frac{dy}{y^2} \qquad (x > 0).$$

Making an appropriate change of variable in the improper integral, we find from Lemma 3.1 that

$$\int_0^\infty \psi^r(x)x^{mr-1}\,dx \leqslant c_3(r,q)\int_0^\infty \left(\frac{1}{x}\int_0^x \varphi(y)y^l dy\right)^r x^{r(l-m-1)}\,dx$$
$$+c_4(r,q)\int_0^\infty \left(\frac{1}{x}\int_0^x \varphi\left(\frac{1}{y}\right)\frac{dy}{y^2}\right)^r x^{mr-1}\,dx$$
$$\leqslant c_5(r,m,q,l)\int_0^\infty \varphi^r(x)x^{r(l-m-1)-1}\,dx.$$

By the definition of $\varphi(x)$ we obtain

$$\int_0^\infty \varphi^r(x)x^{r(l-m-1)-1}\,dx \leqslant c_6(r,m,q,l)\sum_{k=-\infty}^{\infty}\left(h_k^{l-m}\lambda_k\right)^r.$$

Thus the conclusion of Lemma 3.2 follows from the preceding two inequalities.

The proof of Theorem 3.1 is divided into five cases:

1. *Embedding* (3.1) *for* $\sigma \leqslant 1$. Following Bernstein's classical method, we represent a function $f \in R^\alpha_{p,\sigma}$ in the form

$$f(z) = a_0 + \sum_{k=0}^{\infty} u_k(z) \qquad (z \in D_+), \tag{42}$$

where u_k is a rational function of degree at most 2^{k+1}, with all its poles in D_-, that satisfies

$$\|u_k\|_{H_p} \leqslant 3R_{2^k}(f, H_p), \tag{43}$$

and a_0 is a constant such that $|a_0| \leqslant 2\|f\|_{H_p}$.

Taking account of the restriction $\sigma \leqslant 1$, we find from (16) and (43) that

$$\|f\|^\sigma_{B^\alpha_\sigma} \leqslant \|a_0\|^\sigma_{B^\alpha_\sigma} + \sum_{k=0}^{\infty}\|u_k\|^\sigma_{B^\alpha_\sigma} \leqslant c(\alpha,p)\|f\|^\sigma_{R^\alpha_{p,\sigma}}.$$

2. *Embedding* (38) *for* $\sigma > 1$. We again use (42) and (43), and also suppose that all $u_k \not\equiv 0$. Let λ_k and h_k be the continuous functions of period 2π from Lemma 2.5 for u_k and $\beta = \alpha + 1$. We set

$$h_k^*(\varphi) = 2^{k/2}h_0(\varphi) + 2^{(k-1)/2}h_1(\varphi) + \cdots + h_k(\varphi).$$

Then, for every φ, we have

$$\frac{h^*_{k+1}(\varphi)}{h^*_k(\varphi)} \geqslant \sqrt{2} \quad (\varphi \in [0,2\pi]), \qquad \|h^*_k\|_{1,[0,2\pi]} \leqslant 2^{k+2}, \tag{44}$$

$$\|\lambda_k\|_{p,[0,2\pi]} \leqslant c_1(\alpha,p)R_{2^k}(f,H_p), \tag{45}$$

$$\left|J^{\alpha+1}u_k((1-x)e^{i\varphi})\right| \leqslant \lambda(\varphi)\left(\min(x^{-1}, h^*_k(\varphi))\right)^{\alpha+1} \qquad (x \in (0,1)).$$

Therefore we find from Lemma 3.2 that for every $\varphi \in [0,2\pi]$

$$\begin{aligned}&\int_0^1 \left|J^{\alpha+1}f((1-x)e^{i\varphi})\right|^\sigma x^{\sigma-1}\,dx \\ &\quad\leqslant c_2(\alpha,p)|a_0|^\sigma + c_3(\alpha,p)\sum_{k=0}^{\infty}\left[\lambda_k(\varphi)\left(h^*_k(\varphi)\right)^\alpha\right]^\sigma.\end{aligned} \tag{46}$$

From Hölder's inequality and (44) and (45) we obtain

$$\int_0^{2\pi} \left[\lambda_k(\varphi)\left(h_k^*(\varphi)\right)^\alpha\right]^\sigma d\varphi \leqslant c_4(\alpha, p)\left(2^{k\alpha} R_{2^k}(f, H_p)\right)^\sigma .$$

Thus the required embedding follows from (46). If some $u_k \equiv 0$ in (42), we have to make evident modifications in the proof.

3. *Embedding* (39) *for* $\sigma \in (0, 2]$. This follows from (38) and (8).

4. *Embedding* (39) *for* $\sigma > 2$. This is proved just like (38) for $\sigma > 1$. Here, along with Lemmas 2.5 and 3.2, we also have to use the Littlewood-Paley theorem ([4], p. 214) according to which

$$\|f\|_{H_\sigma^\alpha}^2 \leqslant c(\alpha, p)\left\| \int_0^1 \left|J^{\alpha+1} f((1-x)e^{i\varphi})\right|^2 x\,dx \right\|_{\sigma/2,[0,2\pi]} \qquad (\sigma > 2).$$

5. *Embeddings* (40) *and* (41). These are proved just like the embeddings (38) and (39) respectively.

Theorem 3.1 is proved.

§4. Embeddings (38)–(41) are best possible

THEOREM 4.1. *Let* $\alpha > 0$, $p \in (1, \infty]$, *and* $\sigma = (\alpha + p^{-1})^{-1}$.

1) *Corresponding to every sequence* $\{a_n\}_1^\infty$ *that is nonincreasing and tends to zero, and satisfies*

$$\sum_{k=0}^{\infty} \left(2^{k\alpha} a_{2^k}\right)^{\min(2,\sigma)} = +\infty, \tag{47}$$

there is an $f \in H_p$ *such that* $R_n(f, H_p) = O(a_n)$ *and* $f \notin H_\sigma^\alpha$.

2) *Corresponding to every sequence* $\{a_n\}_1^\infty$ *that is nonincreasing and tends to zero, and satisfies*

$$\sum_{k=0}^{\infty} \left(2^{k\alpha} a_{2^k}\right)^\sigma = +\infty, \tag{48}$$

there is an $f \in H_p$ *such that* $R_n(f, H_p) = O(a_n)$ *and* $f \notin B_\sigma^\alpha$.

Thus, embeddings (38) and (39) cannot be improved. It follows from (10) that, in the same sense, embeddings (40) and (41) also cannot be improved. Moreover, by a result of Peller [13], [27], there is actually equality in (40). In addition, since (38) admits an inverse for $1 < p < \infty$ (see §3), assertion 2) of Theorem 4.1 is of interest only when $p = \infty$. Since the proof is the same for all p, we take $p \in (1, \infty]$ for the sake of completeness of presentation. Assertion 1) for $\alpha = 1$ and $p = \infty$, and 2) for $\alpha = \frac{1}{2}$ and $p = \infty$ in Theorem 4.1, were obtained previously by Dolzhenko [17].

The proof of Theorem 4.1 is based on the following lemmas, 4.1 and 4.2.

LEMMA 4.1. *If* $\alpha > 0$, $0 < q < \infty$, *and if the sequence* $\{b_k\}_0^\infty$ *is nonincreasing and tends to zero, and the series*

$$\sum_{k=0}^{\infty} \left(2^{k\alpha} b_k\right)^q \tag{49}$$

diverges, then the series

$$\sum_{k=0}^{\infty} \left(2^{k\alpha}(b_k - b_{k+1})\right)^q \tag{50}$$

also diverges.

PROOF. Suppose that (50) converges. We show that in this case

$$b_k^q \leqslant c_1(\alpha, p) 2^{-\gamma q k} \sum_{j=k}^{\infty} 2^{j\gamma q} \beta_j^q \qquad \left(\gamma = \frac{\alpha}{2},\ \beta_j = b_j - b_{j+1}\right) \tag{51}$$

for all $k = 0, 1, \ldots$. In fact, since $b_k \downarrow 0$, then $b_k = \beta_k + \beta_{k+1} + \cdots$ and since (50) converges we have, for $q \leqslant 1$,

$$b_k^q \leqslant \sum_{j=k}^{\infty} \beta_j^q \leqslant 2^{-\gamma q k} \sum_{j=k}^{\infty} \left(2^{j\gamma} \beta_j\right)^q.$$

If $q > 1$, let $q' = q(q-1)^{-1}$, and from Hölder's inequality we obtain

$$b_k \leqslant \left(\sum_{j=k}^{\infty} 2^{-jq'\gamma}\right)^{1/q'} \left(\sum_{j=k}^{\infty} \left(2^{j\gamma}\beta_j\right)^q\right)^{1/q} \leqslant c_2(\alpha, q) 2^{-\gamma k} \left(\sum_{j=k}^{\infty} \left(2^{j\gamma}\beta_j\right)^q\right)^{1/q}.$$

Thus we obtain (51) from the preceding two relations. From (51) we obtain

$$\sum_{k=0}^{\infty} \left(2^{k\alpha} b_k\right)^q \leqslant c_3(\alpha, q) \sum_{k=0}^{\infty} 2^{k\gamma q} \sum_{j=k}^{\infty} \left(2^{i\gamma}\beta_j\right)^q$$

$$= c_3(\alpha, q) \sum_{j=0}^{\infty} \sum_{k=0}^{j} \left(2^{(j+k)\gamma}\beta_j\right)^q \leqslant c_4(\alpha, q) \sum_{j=0}^{\infty} \left(2^{j\alpha}(b_j - b_{j+1})\right)^q.$$

The last inequality contradicts the divergence of (49). This completes the proof of Lemma 4.1.

For use below, we introduce the notation

$$\beta_{n,j} = \pi/j \qquad \left(n \in \mathbf{N},\ 2^n - 1 \leqslant j \leqslant 2^{n+1} - 2\right),$$

$$G_{n,j} = \left\{z \colon 1 - 2^{-2n} \leqslant |z| < 1,\ \arg z = \beta_{n,j}\right\}, \qquad G_n = \bigcup_{j=2^n-1}^{2^{n+1}-2} G_{n,j}.$$

LEMMA 4.2. *Let $\alpha > 0$, $p \in (1, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$. Then for every $n \in \mathbf{N}$ there is a rational function φ_n of degree 2^n that satisfies the conditions*

1) $\|\varphi_n\|_{H_p} \leqslant c_1(\alpha, p)$,
2) $\|\varphi_n^{(\alpha)}\|_{\sigma, G_n} \geqslant 2^{n\alpha} c_2(\alpha, p)$,
3) $\|\varphi_n^{(\alpha)}\|_{\infty, G_m} \leqslant 1$ $(n \neq m)$.

PROOF. We set

$$\varphi_n(z) = 2^{-n/p} \delta^{1-1/p} \sum_{j=2^n-1}^{2^{n+1}-2} \varphi_{n,j}(z),$$

$$\varphi_{n,j}(z) = \left(z_{n,j} - z\right)^{-1}, \qquad z_{n,j} = (1+\delta) e^{i\beta_{n,j}} \quad (\delta > 0).$$

It is easily shown that

$$\lim_{\delta \to +0} \delta^{1-1/p} \|\varphi_{n,j}\|_{H_p} = \left(\sqrt{\pi}\, \Gamma\left(\frac{p-1}{2}\right) \Big/ \Gamma\left(\frac{p}{2}\right)\right)^{1/p}, \tag{52}$$

$$\lim_{\delta \to +0} \delta^{1-1/p} \left\|\varphi_{n,j}^{(\alpha)}\right\|_{\sigma, G_{n,j}} = \Gamma(1+\alpha)\left(\sigma - \frac{\sigma}{p}\right)^{1/\sigma}. \tag{53}$$

In (52) the right-hand side is to be taken to be 1 for $p = \infty$; to obtain (53) we need to use the equality

$$\varphi_{n,j}^{(\alpha)}(z) = \Gamma(1+\alpha)\left(1 - zz_{n,j}^{-1}\right)^{-1-\alpha} z_{n,j}^{-1-[\alpha]},$$

which follows from (1). The functions φ_n and $\varphi_n^{(\alpha)}$ tend uniformly to zero as $\delta \to +0$, outside an arbitrarily small neighborhood of G_n. Hence it follows from (52) and (53) that φ_n satisfies conditions 1)–3) for sufficiently small $\delta > 0$. This completes the proof of Lemma 4.2.

The proof of Theorem 4.1 is divided into four cases.

1) *Assertion* 1) *for* $\sigma \leqslant 2$. As the required function we take

$$f(z) = \sum_{k=1}^{\infty} p_k \varphi_k(z),$$

where $p_k = a_{2^{k+1}} - a_{2^{k+2}}$ and the φ_k are the rational fractions from Lemma 4.2. From condition 1) of Lemma 4.2 we obtain

$$R_{2^j}(f, H_p) \leqslant \left\| \sum_{k=j}^{\infty} p_k \varphi_k \right\|_{H_p} \leqslant c_1(\alpha, p) a_{2^{j+1}}$$

for every $n \in \mathbf{N}$, and consequently $R_n(f, H_p) = O(a_n)$ as $n \to \infty$. On the other hand, for arbitrary $n \in \mathbf{N}$ we have from conditions 2) and 3) of Lemma 4.2.

$$\left\|f^{(\alpha)}\right\|_{\sigma,G_n}^{\sigma} \geqslant 2^{-\sigma}\|p_n\varphi_n\|_{\sigma,G_n}^{\sigma} - \left\| \sum_{\substack{k=1 \\ k\neq n}}^{\infty} p_k\varphi_k \right\|_{\sigma,G_n}^{\sigma} \geqslant c_2(\alpha,p)\left[\left(2^{n\alpha}p_n\right)^{\sigma} - a_2^{\sigma}2^{-n}\right].$$

Setting $G = \bigcup_1^{\infty} G_n$, we obtain $\|f^{(\alpha)}\|_{\sigma,G} = +\infty$ from Lemma 4.1 and (47), and consequently, by Carleson's embedding theorem ([18], pp. 195–198), $f^{(\alpha)} \notin H_\sigma$. The proof of this part of the theorem is completed by applying Lemma 1.1.

2. *Assertion* 1 *for* $\sigma > 2$. As the required function we take

$$f(z) = \sum_{k=1}^{\infty} p_k z^{2^k} \qquad \left(p_k = a_{2^{k+1}} - a_{2^{k+2}}\right). \tag{54}$$

Evidently $R_n(f, H_p) = O(a_n)$. On the other hand, for every $\rho \in (0,1)$ we have, by Hölder's inequality and Parseval's equality,

$$\|J^\alpha f(\rho\,\cdot)\|_{\sigma,T} \geqslant (2\pi)^{1/\sigma - 1/2}\left(\sum_{k=1}^{\infty}\left(\rho_k 2^{k\alpha}\rho^{2^k}\right)^2\right)^{-1/2}.$$

Consequently, we obtain $f \notin H_\sigma^\alpha$ by letting $\rho \to 1-0$ and using Lemma 4.1 and (47).

3. *Assertion* 2) *for* $\sigma \leqslant 2$. This follows from assertion 1) and (8).

4. *Assertion* 2) *for* $\sigma > 2$. We show that the function (54) is the required function. In fact, let $\rho_n = 1 - 2^{-n}$, $n \in \mathbf{N}$, and $\rho \in [\rho_n, \rho_{n+1}]$. Then, by Hölder's inequality and Parseval's theorem,

$$\int_0^{2\pi} \left|J^{\alpha+1} f(\varphi e^{i\varphi})\right|^{\sigma} d\varphi \geqslant (2\pi)^{1-2/\sigma}\left(\int_0^{2\pi}\left|J^{\alpha+1}f(\rho e^{i\varphi})\right|^2 d\varphi\right)^{\sigma/2}$$
$$\geqslant c_3(\alpha,p)\left(2^{n(\alpha+1)}p_n\right)^{\sigma}.$$

By (48) we find from Lemma 4.1 that

$$\|f\|_{B_\sigma^\alpha}^\sigma \geqslant \sum_{n=1}^{\infty} \int_{\rho_n}^{\rho_{n+1}} d\rho \int_0^{2\pi} \left|J^{\alpha+1} f(\rho e^{i\varphi})\right|^\sigma (1-\rho)^{\sigma-1} d\varphi = +\infty.$$

This completes the proof of Theorem 4.1.

§5. Degree of rational approximation and smoothness of functions

We denote by $\omega_k(\delta, f)_p$ ($k \in \mathbf{N}$, $\delta \geqslant 0$, $f \in L_p(T)$) the kth order modulus of smoothness of f, i.e.

$$\omega_k(\delta, f)_p = \sup_{|h| \leqslant \delta} \left\| \sum_{\nu=0}^{k} (-1)^{k-\nu} C_k^\nu f(e^{i(\cdot+\nu h)}) \right\|_{p,[0,2\pi]}.$$

THEOREM 5.1. *Let $\alpha > 0$, $p \in (1, \infty]$ and $\sigma = (\alpha + p^{-1})^{-1}$, and let k be the smallest positive integer such that $k > \alpha$.*

1) *If $f \in H_p$ then for every $n \in \mathbf{N}$*

$$\sum_{m=0}^{n} \left(2^{m\alpha} \omega_k(2^{-m}, f)_\sigma\right)^\sigma \leqslant c(\alpha, p) \sum_{m=0}^{n} \left(2^{m\alpha} R_{2^m}(f, H_p)\right)^\sigma. \tag{55}$$

2) *If $f \in$ BMOA then for every $n \in \mathbf{N}$*

$$\sum_{m=0}^{n} \left(2^{m\alpha} \omega_k(2^{-m}, f)_{1/\alpha}\right)^{1/\alpha} \leqslant c(\alpha) \sum_{m=0}^{n} \left(2^{m\alpha} R_{2^m}(f, \mathrm{BMOA})\right)^{1/\alpha}. \tag{56}$$

COROLLARY 5.1. *If l is the smallest positive integer such that $l \geqslant \alpha$, then for every $\delta \in (0, \frac{1}{2}]$*

$$\omega_l(\delta, f)_\sigma \leqslant c(\alpha, p) \delta^\alpha \left[\sum_{0 \leqslant m \leqslant \log_2(1/\delta)} \left(2^{m\alpha} R_{2^m}(f, H_p)\right)^\sigma \right]^{1/\sigma}. \tag{57}$$

To obtain (57) we observe that for $\alpha \notin \mathbf{N}$ we have $l = k$ and it suffices to suppress the terms with $m = 0, 1, \ldots, n-1$ on the left-hand side of (55). However, if $\alpha \in \mathbf{N}$, then $l = k - 1$ and by Marchaud's inequality (see, for example, [19]) the left-hand side of (55) majorizes $c_1(\alpha, p)(2^{nl}\omega_l(2^{-n}, f)_\sigma)^\sigma$.

In view of Corollary 2.2, inequalities (55) and (57) remain valid if we suppose that $f \in L_p(T)$ and $R_{2^m}(f, H_p)$ is replaced by $R_{2^m}(f, L_p(T))$, the best approximation to f in $L_p(T)$ by rational fractions of degree $2^m - 1$.

An inequality of the type of (57) was obtained by Dolzhenko [20] for $\alpha = 1$ and $p = \infty$; by Sevast'yanov [21] for $\alpha \in (0, 1)$ and $p = \infty$; and finally by Brudnyĭ [22] for $\alpha \geqslant 1 - p^{-1}$, $p \in [1, \infty]$, and with k instead of l.

For the proof of Theorem 5.1 we require the following two lemmas.

LEMMA 5.1. *Let $p \in (0, \infty]$, $s = \min(1, p)$, $k \in \mathbf{N}$ and $f \in B_{p,s}^0$. Then for every $\delta \in (0, 1]$*

$$\omega_k(\delta, f)_p \leqslant c(k, p) \left(\int_{1-\delta}^{1} \|J^k f(\rho\,\cdot)\|_{H_p}^s (1-\rho)^{ks-1} d\rho \right)^{1/s}.$$

PROOF. For every $z \in D_+$ we have $f(z) = f_1(z) + f_2(z)$, where

$$f_1(z) = \sum_{\nu=0}^{k} C_k^\nu (-1)^\nu f\left(\left(1 - \frac{\nu}{k}\delta\right) z\right), \qquad f_2(z) = f(z) - f_1(z).$$

From Lemma 1.1 and a result of Storozhenko [23] we obtain, since $\|g(\rho\,\cdot)\|_p$ is nondecreasing with respect to ρ ($g \in H_p$),

$$\omega_k(\delta, f_2)_p \leqslant c_1(k, p) \delta^k \|J^k f_2\|_{H_p} \leqslant c_2(k, p) \delta^k \|J^k f(\cdot(1 - \delta/k))\|_{H_p}. \tag{58}$$

From the properties of finite differences ([24], p. 157) we have, for every $z \in D_+$,

$$\begin{aligned} |f_1(z)| &\leqslant \left| z^k \int_0^{\delta/k} dt_1 \int_0^{\delta/k} dt_2 \cdots \int_0^{\delta/k} f^{(k)}((1-(t_1+t_2+\cdots+t_k))z)\, dt_k \right| \\ &\leqslant \underset{\substack{t_1, t_2, \ldots, t_k > 0 \\ t_1+t_2+\cdots+t_k \leqslant \delta}}{\iint \cdots \int} |f^{(k)}((1-(t_1+t_2+\cdots+t_k))z)|\, dt_1\, dt_2 \cdots dt_k \\ &= \frac{1}{(k-1)!} \int_0^\delta |f^{(k)}((1-t)z)|^{k-1} dt. \end{aligned} \tag{59}$$

If $p \in [1, \infty]$ we find from (59) that

$$\|f_1\|_p \leqslant \frac{1}{(k-1)!} \int_0^\delta \|f^{(k)}(\cdot(1-t))\|_{H_p} t^{k-1}\, dt. \tag{60}$$

Therefore we obtain the necessary inequality for $p \in [1, \infty]$ from (58), (60), and Lemma 1.1. For $p \in (0, 1)$ we introduce

$$F(z) = \max_{0 \leqslant \tau \leqslant 1} |f^{(k)}(\tau z)|.$$

We find from (59) that

$$\begin{aligned} |f_1(z)|^p &\leqslant [(k-1)!]^{-p} \left(\sum_{m=0}^{\infty} \int_{2^{-(m+1)}\delta}^{2^{-m}\delta} F((1-t)z) t^{k-1}\, dt \right)^p \\ &\leqslant c_3(k, p) \int_{1-\delta}^{1} F^p(\rho z)(1-\rho)^{kp-1}\, d\rho. \end{aligned} \tag{61}$$

Using the fact that $\|F(\cdot\rho)\|_p \leqslant c_4(p)\|f^{(k)}(\cdot\rho)\|_p$ for every $\rho \in (0,1)$ ([1], p. 278), we obtain the conclusion of Lemma 5.1 for $p \in (0,1)$ from (58), (60) and Lemma 1.1. This completes the proof of Lemma 5.1.

LEMMA 5.2. *Let $\alpha > 0$, $p \in (0, \infty]$ and $q \in (0, \infty)$, and let k be the smallest positive integer such that $k > \alpha$. Then a function $f \in H_p$ belongs to class B_p^α if and only if*

$$\|f\|'_{B^\alpha_{p,q}} = \|f\|_{H_p} + \left(\sum_{m=1}^{\infty} (2^{m\alpha}\omega_k(2^{-m}, f)_p)^q \right)^{1/q} < \infty. \tag{62}$$

Here the quasinorm (62) *is equivalent to the quasinorm* $\|f\|_{B^\alpha_{p,q}}$.

REMARKS. 1) With a corresponding definition of $\|f\|'_{B^\alpha_{p,q}}$ the conclusion of the lemma remains valid for $q = \infty$.

2) The lemma is well known for $p \in [1, \infty]$ and $q \in [1, \infty]$ (see, for example, [25]).

3) For the proof of Theorem 5.1 we need only the necessity for $p = q$.

PROOF OF LEMMA 5.2. For $j \in \mathbf{N}$ we introduce $\mu_j = \|J^k f((1-2^{-j})\cdot)\|_{H_p}$. From Lemma 5.1 we obtain

$$\omega_k(2^{-m}, f)_p \leqslant c_1(k, p) \left(\sum_{j=m}^{\infty} (2^{-kj}\mu_j)^s \right)^{1/s}, \qquad s = \min(1, p).$$

As in the proof of Lemma 4.1, we obtain

$$(2^{m\alpha}\omega_k(2^{-m}, f)_p)^q \leqslant c_2(\alpha, p, q) 2^{\gamma m q} \sum_{j=m}^{\infty} (2^{-(k-\gamma)j}\mu_j)^q, \qquad \gamma = \alpha/2.$$

Consequently $\|f\|'_{B^\alpha_{p,q}} \leqslant c_3(\alpha, p, q)\|f\|_{B^\alpha_{p,q}}$, since $B^\alpha_{p,q} \subset H_p$ for $\alpha > 0$. The reverse inequality follows from a result of Storozhenko [26]:

$$\left\| f^{(k)}(\rho\,\cdot) \right\|_{H_p} \leqslant c_4(p,k)(1-\rho)^{-k}\omega_k(1-\rho,f)_p \qquad \left(\tfrac{1}{2} \leqslant \rho < 1\right)$$

and Lemma 1.1. This completes the proof of Lemma 5.2.

PROOF OF THEOREM 5.1. Let $f \in H_p$, $1 < p \leqslant \infty$, and let r_n be a rational function of degree $2^n - 1$ for which $\|f - r\|_{H_p} \leqslant 2R_{2^n}(f, H_p)$. From (38) and Lemma 5.2 we obtain

$$\|r_n\|'_{B^\alpha_\sigma} \leqslant c_1(\alpha,p)\|r_n\|_{R^\alpha_{p,\alpha}} \qquad \left(\alpha > 0,\ \sigma = \left(\alpha + p^{-1}\right)^{-1}\right). \tag{63}$$

Evidently, $R_{2^j}(r_n, f) = 0$ for $j \geqslant n$, and

$$R_{2^j}(r_n, H_p) = R_{2^j}\left(f - (f - r_n), H_p\right) \leqslant R_{2^j}(f, H_p) + \|f - r_n\|_{H_p} \leqslant 3R_{2^j}(f, H_p)$$

for $j = 0, 1, \ldots, n-1$. On the other hand, for every $j \in \mathbf{N}$,

$$\begin{aligned}\omega_k\left(2^{-j}, r_n\right)_\sigma = \omega_k\left(2^{-j}, f - (f - r_n)\right)_\sigma &\geqslant -2^{-1/\sigma}\omega_k\left(2^{-j}, f\right)_\sigma - \omega_k\left(2^{-j}, f - r_n\right) \\ &\geqslant 2^{-1/\sigma}\omega_k\left(2^{-j}, f\right)_\sigma - 2^{k+1}R_n(f, H_p).\end{aligned}$$

Consequently, from (63) we obtain

$$\sum_{m=1}^{n} \left(2^{m\alpha}\omega_k(2^{-m}, f)_\sigma\right)^\sigma \leqslant c_2(\alpha,p)\|f\|_p + c_3(\alpha,p)\sum_{m=0}^{n}\left(2^{m\alpha}R_{2^m}(f, H_p)\right)^\sigma.$$

Now if in the preceding inequality we replace $f(z)$ by $f(z) - f(0)$ and use the inequality

$$\|f(z) - f(0)\|_{H_p} \leqslant c_4(p)R_1(f, H_p),$$

we obtain (55). Inequality (56) is proved similarly. This completes the proof of Theorem 5.1.

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* *Editor's note*. See also his paper, *Trace ideal criteria for Hankel operators, and applications to Besov spaces*, Integral Equations Operator Theory **7** (1984), 241–281.