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# TCHEBYCHEFF RATIONAL APPROXIMATION IN THE DISK, ON THE CIRCLE, AND ON A CLOSED INTERVAL

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ABSTRACT. Suppose that the function $f$ is analytic in the disk $\{z\colon |z| < 1\}$ and continuous in its closure. Let $R_n(f)$ denote the best uniform approximation of $f$ by rational functions of degree at most $n$. In 1965 Dolzhenko established that if $\sum R_n(f) < \infty$, then $f'$ belongs to the Hardy space $H_1$. The following converse of this result is obtained here: if $f' \in H_1$, then $R_n(f) = O(1/n)$. In combination with results of Peller, Semmes, and the author, this estimate yields, in particular, a description of the set of functions $f$ with $[\sum(2^{k\alpha}R_{2^k}(f))^q]^{1/q} < \infty$, where $\alpha > 1$ and $0 < q \leq \infty$.

Bibliography: 38 titles.

Let $\Omega$ be a subset of the complex plane $\mathbf{C}$, and let $\overline{\Omega}$ be its closure. Denote by $C(\Omega)$ the set of continuous functions on $\Omega$, with the norm $\|f\|_{\infty,\Omega} = \sup\{|f(z)|\colon z \in \Omega\}$. If $\Omega$ is a domain, then $A(\Omega)$ is the set of functions analytic in $\Omega$. The set of rational functions of degree at most $n$ ($n \geq 0$) with poles only in $\mathbf{C}\backslash\Omega$ is denoted by $\mathscr{R}_n(\Omega)$. We introduce the best uniform approximation $R_n(f,\Omega) = \inf\{\|f - r\|_{\infty,\Omega}\colon r \in \mathscr{R}_n(\Omega)\}$ of $f$ by the set $\mathscr{R}_n(\Omega)$. We also introduce the notation $D_+ = \{z \in \mathbf{C}\colon |z| < 1\}$, $D_- = \mathbf{C}\backslash\overline{D}_+$, and $T = \{\xi \in \mathbf{C}\colon |\xi| = 1\}$.

Dolzhenko [1] showed that if $f \in C(T)$ and $\sum R_n(f,T) < \infty$, then $f$ is absolutely continuous on $T$. He also established that for $f$ absolutely continuous $R_n(f,T)$ can tend to zero arbitrarily slowly, i.e., the result in [1] does not admit a converse (see [2]). In [3] Dolzhenko considered the analogous problem for functions $f \in A(D_+) \cap C(\overline{D}_+)$. He showed that if $\sum R_n(f,\overline{D}_+) < \infty$, then $f'$ belongs to the Hardy space $H_1$. Thus, the following problem was posed: what can be said about the behavior of $R_n(f,\overline{D}_+)$ as $n \to \infty$ for functions $f \in A(D_+) \cap C(\overline{D}_+)$ such that $f' \in H_1$? In [4] the author established the estimate $R_n(f,\overline{D}_+) = O(\ln^3 n/n)$ for such functions. Later [5] the author succeeded in replacing $\ln^3 n$ by $\ln n$. We improve the method in [5] and get the following result: if $f \in A(D_+) \cap C(\overline{D}_+)$ and $f' \in H_1$, then([1])

$$R_n(f,\overline{D}_+) \leq cn^{-1}\|f'\|_{H_1} \qquad (n \geq 1). \tag{1}$$

The proof of (1) is based on the use of an atomic decomposition of the space $\operatorname{Re} H_1$ introduced by Coifman [6] (see also [7]), the use of rational operators of Jackson type

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([1])Here and below, $c, c_1, c_2, \ldots$ denote absolute positive constants, which are generally different in different places. Similarly, $c(\cdots), c_1(\cdots), c_2(\cdots), \ldots$ denote positive quantities depending only on the parameters indicated in the parentheses.

constructed by Rusak [8], [9], and the use of an inequality of the author which connects best rational approximations on $\overline{D}_+$ and on $T$ [10]. Results in [11] on best rational approximation in $H_1$ are used here to generalize inequality (1) to, for example, the Besov space $B^{\alpha}_{1/\alpha}$ ($\alpha \geq 1$) of functions in the disk $D_+$, on the circle $T$, and on the interval $[-1,1]$. The results are definitive improvements of the direct theorems on Tchebycheff rational approximation due to Brudnyĭ [12], Peller [13], and the author [11]. In combination with known inverse theorems of Peller [13], Semmes [14], and the author [15], the results in this paper give a description of the set of continuous functions whose best rational approximations tend to zero at the rate of a power.

The main results in this article were announced in [16] and presented at a session of the All-Union School on Function Theory and Approximations held in Saratov in February 1986.

## §1. Lemmas on simple functions

A real-valued function $\varphi$ defined on $\mathbf{R}$ will be called a *simple* function if it is absolutely continuous, $\|\varphi'\|_{\infty,\mathbf{R}} < \infty$, and there exists a (finite) interval $I(\varphi)$ such that $\operatorname{supp}\varphi \subset I(\varphi)$. Since the interval $I(\varphi)$, which we call a *support interval* for $\varphi$, is not uniquely determined, we shall assume below in speaking of simple functions that a certain specific support interval is given for them. The quantity $\mu(\varphi) = |I(\varphi)| \cdot \|\varphi'\|_{\infty,\mathbf{R}} < \infty$ is an important characteristic of a simple function, where $|I(\varphi)|$ is the length of $I(\varphi)$. The function identically equal to zero will also be called simple.

LEMMA 1.1. *Let $f(x) = \sum_{k=1}^{p} \varphi_k(x)$, where the $\varphi_k$ are simple functions such that $I(\varphi_1) \supset I(\varphi_2) \supset \cdots \supset I(\varphi_p)$. Let $\sum_{k=1}^{p} \mu(\varphi_k) = v$. Then for any $n \geq 6$ ($n \in \mathbf{N}$) there exist simple functions $\psi_1, \dots, \psi_q$ ($q \leq n$) satisfying the following conditions*:

a) $|f(x) - \sum_{j=1}^{q} \psi_j(x)| \leq c_1 v/n$ *for* $x \in \mathbf{R}$,

b) $f(x) - \sum_{j=1}^{q} \psi_j(x) = 0$ *for* $x \in I(\varphi_p) \cup [\mathbf{R} \backslash I(\varphi_1)]$,

c) $\sum_{j=1}^{q} \mu(\psi_j) \leq c_2 v$.

PROOF. The lemma is obvious for $p \leq 6$; therefore, we assume that $p > 6$. Without loss of generality it can also be assumed that $I(\varphi_k) = (-a_k, b_k)$, $k = 1, \dots, p$, where $a_k, b_k \geq 1$ and $a_p = b_p = 1$. For $y > 0$ we introduce the simple function $\Delta_y(x) = \max\{0, 1 - |x|/y\}$, for which $I(\Delta_y) = (-y, y)$ and $\mu(\Delta_y) = 2$. Let

$$y_k = \min\{a_k, b_k\}, \qquad \mathring{\varphi}_k(x) = \varphi_k(x) - \varphi_k(0)\Delta_{y_k}(x),$$

$$f_1(x) = \sum_{k=1}^{p} \varphi_k(0)\Delta_{y_k}(x), \qquad f_2(x) = \sum_{k=1}^{p} \mathring{\varphi}_k(x).$$

Fix some $m \in \mathbf{N}$. We show that there exist real numbers $h_1, \dots, h_{m_1}$ and positive numbers $y_1 = z_1 > z_2 > \cdots > z_{m_1} = y_p = 1$ ($m_1 \leq 2(m+1)$) such that:

a′) $|f_1(x) - \sum_{j=1}^{m_1} h_j \Delta_{z_j}(x)| \leq (c_3/m) \sum_{k=1}^{p} |\varphi_k(0)|$ for $x \in \mathbf{R}$,

b′) $f_1(x) - \sum_{j=1}^{m_1} h_j \Delta_{z_j}(x) = 0$ for $x \in [-1,1] \cup [\mathbf{R} \backslash (-y_1, y_1)]$,

c′) $\sum_{j=1}^{m_1} |h_j| \leq \sum_{k=1}^{p} |\varphi_k(0)|$.

Indeed, suppose first that all $\varphi_k(0)$ are nonnegative. Then $f_1(x)$ is even, downwards convex on $[0,\infty)$, linear on $[0,1]$, and equal to zero on $[y_1,\infty)$. Take numbers $z_1, \dots, z_{m+1}$ ($m_1 = m+1$) so that the variation of $f_1(x)$ on each of the intervals $[z_{j+1}, z_j]$ ($j = 1, \dots, m$) does not exceed $(1/m)\sum_{k=1}^{p} \varphi_k(0)$. It is geometrically easy to find nonnegative numbers $h_1, \dots, h_{m+1}$ such that conditions a′)–c′) hold. Obviously, in this case we can set $c_3 = 1$ in a′). In the general case we introduce the functions

$$f_1^+(x) = \sum_{\varphi_k(0) \geq 0} \varphi_k(0)\Delta_{y_k}(x), \qquad f_1^-(x) = \sum_{\varphi_k(0) < 0} (-\varphi_k(0))\Delta_{y_k}(x).$$

Then $f_1(x) = f_1^+(x) - f_1^-(x)$, and it is necessary to consider each of the functions $f_1^+(x)$ and $f_1^-(x)$ separately. Conditions a′)–c′) hold with $m_1 = 2(m+1)$ and $c_3 = 2$.

We proceed to consider the function $f_2(x)$. Let us define a continuous function $\xi^+(x)$ on $\mathbf{R}$ by setting $\xi^+(\pm 2^j) = [1+(-1)^j]/2$ $(j = 0,1,2,\dots)$ and letting $\xi^+(x)$ be linear on the intervals $(-1,1), (2^j, 2^{j+1})$, and $(-2^{j+1}, -2^j)$ $(j = 0,1,2,\dots)$. Define $\xi^-(x) = 1 - \xi^+(x)$. Then $f_2(x) = f_2^+(x) + f_2^-(x)$, where

$$f_2^\pm(x) = \sum_{k=1}^{p} \mathring{\varphi}_k(x)\xi^\pm(x).$$

It follows from the condition $\mathring{\varphi}_k(0) = 0$ that the functions $\mathring{\varphi}_k(x)\xi^+(x)$ and $\mathring{\varphi}_k(x)\xi^-(x)$ decompose into sums of the simple functions $\psi_{k,i}^+(x)$ $(i = 1,\dots,s^+)$ and $\psi_{k,i}^-(x)$ $(i = 1,\dots,s^-)$, respectively. Here the intervals into which $(-a_1, b_1)$ is divided by the zeros of $\xi^+(x)$ and $\xi^-(x)$ are support intervals for $\psi_{k,i}^+(x)$ and $\psi_{k,i}^-(x)$, respectively. It can be assumed that each system of functions $\{\psi_{k,i}^+\}_{i=1}^{s^+}$ and $\{\psi_{k,i}^-\}_{i=1}^{s^-}$ for a fixed value of $i$ has one and the same support interval. It is easy to get that

$$\sum_{i=1}^{s^\pm} \mu(\psi_{k,i}^\pm) \le c_4\mu(\varphi_k), \qquad k = 1,\dots,p. \tag{2}$$

We introduce the simple functions

$$\psi_i^\pm(x) = \sum_{k=1}^{p} \psi_{k,i}^\pm(x), \qquad i = 1,\dots,s^\pm.$$

It is not hard to see that

$$f_2^\pm(x) = \sum_{i=1}^{s^\pm} \psi_i^\pm(x).$$

For example, consider the function $f_2^+(x)$. It can be assumed that the $\psi_i^+$ are indexed so that $I(\psi_1^+) = (-2,2)$ and $\mu(\psi_2^+) \ge \mu(\psi_3^+) \ge \cdots \ge \mu(\psi_{s^+}^+)$. We get from (2) that $\sum_{i=1}^{s^+} \mu(\psi_i^+) \le c_4 v$, and hence

$$\mu(\psi_i^+) \le c_4 v/(i-1), \qquad i = 2,\dots,s^+. \tag{3}$$

For the $m \in \mathbf{N}$ chosen earlier we set $m_2^+ = \min\{m, s^+\}$. Using inequality (3) and the fact that the intervals $I(\psi_i^+)$ $(i = 1,\dots,s^+)$ are disjoint, we get that

a″) $|f_2^+(x) - \sum_1^{m_2^+} \psi_i^+(x)| \le c_5 v/m$ for $x \in \mathbf{R}$,

b″) $f_2^+(x) - \sum_1^{m_2^+} \psi_i^+(x) = 0$ for $x \in [-1,1] \cup (\mathbf{R}\backslash(-a_1, b_1)]$,

c″) $\sum_1^{m_2^+} \mu(\psi_i^+) \le c_6 v$.

Obviously, analogous relations hold also for the function $f_2^-(x)$. Thus, Lemma 1.1 follows from a′)–c′) and a″)–c″).

LEMMA 1.2. *Suppose that the conditions of Lemma 1 hold, $p \ge 2$, and $1 \le k_1 < k_2 < \cdots < k_d \le p-1$ are positive integers. Then for any $n \in \mathbf{N}$ there exist simple functions $\psi_1,\dots,\psi_q$ $(q \le n + 6(d+1))$ satisfying the following conditions:*

a) $|f(x) - \sum_1^q \psi_j(x)| \le c_1 v/n$ *for* $x \in \mathbf{R}$,

b) $f(x) - \sum_1^q \psi_j(x) = 0$ *for* $x \in I(\varphi_p) \cup [\bigcup_{i=1}^d (I(\varphi_{k_i})\backslash I(\varphi_{k_i+1}))] \cup [\mathbf{R}\backslash I(\varphi_1)]$,

c) $\sum_1^q \mu(\psi_j) \le c_2 v$.

PROOF. We let $k_0 = 0$ and $k_{d+1} = p$, and, assuming that $v \neq 0$ (the lemma is obvious for $v = 0$), we introduce for $i = 1, \dots, d+1$ the following objects:(2)

$$f_i(x) = \sum_{k=k_{i-1}+1}^{k_i} \varphi_k(x), \quad v_i = \sum_{k=k_{i-1}+1}^{k_i} \mu(\varphi_k), \quad n_i = [v_i n/v] + 1.$$

According to Lemma 1.1, for any $i = 1, \dots, d+1$ there exist simple functions $\psi_{i,j}$ ($j = 1, \dots, q_i$, $q_i \leq n_i + 5$) satisfying the following conditions:

$\mathrm{a}_i$) $|f_i(x) - \sum_{j=1}^{q_i} \psi_{i,j}(x)| \leq c_1 v_i/n_i \leq c_1 v/n$ for $x \in \mathbf{R}$,

$\mathrm{b}_i$) $f_i(x) - \sum_{j=1}^{q_i} \psi_{i,j}(x) = 0$ for $x \in I(\varphi_{k_i}) \cup [\mathbf{R} \backslash I(\varphi_{k_{i-1}+1})]$,

$\mathrm{c}_i$) $\sum_{j=1}^{q_i} \mu(\psi_{i,j}) \leq c_2 v_i$.

It follows from $\mathrm{a}_i$)–$\mathrm{c}_i$) that $\psi_{i,j}(x)$ ($i = 1, \dots, d+1$, $j = 1, \dots, q_i$) are the desired functions. There are $q = \sum_1^{d+1} q_i \leq n + 6(d+1)$ of them. Lemma 1.2 is proved.

LEMMA 1.3. *Suppose that $f(x) = \sum_1^p \varphi_k(x)$, where the $\varphi_k$ are simple functions such that any two intervals $I(\varphi_k)$ and $I(\varphi_{k'})$ with $k \neq k'$ are either disjoint or imbedded one in the other. Let $\sum_1^p \mu(\varphi_k) = v$. Then for any $n \in \mathbf{N}$ there exist simple functions $\psi_1, \dots, \psi_q$ ($q \leq n$) satisfying the following conditions*:

$$\left| f(x) - \sum_{j=1}^{q} \psi_j(x) \right| \leq \frac{c_1 v}{n} \quad \textit{for } x \in \mathbf{R}, \tag{4}$$

$$\sum_{j=1}^{q} \mu(\psi_j) \leq c_2 v. \tag{5}$$

PROOF. We introduce the function

$$\theta(x) = \sum_{k=1}^{p} \frac{\mu(\varphi_k)}{|I(\varphi_k)|} \chi_k(x),$$

where $\chi_k(x)$ is the characteristic function of the interval $I(\varphi_k)$. We fix some $m \in \mathbf{N}$ ($m \geq 2$) and denote by $(\xi_0, \xi_m)$ the smallest interval containing $\bigcup_1^p I(\varphi_k)$. Note that $\theta(x) \geq 0$ for all $x \in \mathbf{R}$, $\theta(x) \equiv 0$ for $x \in \mathbf{R} \backslash (\xi_0, \xi_m)$, and $\int_{\mathbf{R}} \theta(x)dx = v$. Therefore, there are points $\xi_0 < \xi_1 < \dots < \xi_{m-1} < \xi_m$ such that $\int_{\xi_i}^{\xi_{i+1}} \theta(x)dx = v/m$ ($i = 0, \dots, m-1$). We partition the set $\{1, \dots, p\}$ into subsets $G_s$ ($s = 1, \dots, m$):

$$G_1 = \{k \colon \xi_1 \in I(\varphi_k)\}, \qquad G_s = \{k \colon \xi_s \in I(\varphi_k)\} \backslash \bigcup_{i=1}^{s-1} G_i \quad (s = 2, \dots, m-1),$$

$$G_m = \{1, 2, \dots, p\} \backslash \bigcup_{s=1}^{m-1} G_s.$$

We also introduce the functions

$$f_s(x) = \sum_{k \in G_s} \varphi_k(x). \tag{6}$$

If some set $G_s$ is empty, then the corresponding function $f_s(x)$ is taken identically equal to zero. It is not hard to see that

$$\|f_m\|_{\infty, \mathbf{R}} \leq v/m. \tag{7}$$

(2)$[a]$ is the integer part of a number $a$.

Assume first that $G_j \neq \varnothing$ ($j = 1, \ldots, m-1$). The simple functions $\varphi_k$ in the decomposition of $f_s$ according to (6) will be denoted by $\Gamma_{s,i}$ ($i = 1, \ldots, p_s$). As follows from the construction of the sets $G_s$ and the conditions of the lemma, we can assume that $I(\Gamma_{s,1}) \supset I(\Gamma_{s,2}) \supset \cdots \supset I(\Gamma_{s,p_s})$. Let $I_s = I(\Gamma_{s,1})$. For each $s = 1, \ldots, m-1$ we define a set $E_s \subset I_s$ as follows. Let $E_s = \varnothing$ if $I_s$ does not contain any interval $I_{s'}$ other than itself. But if such an $I_{s'}$ exists,(3) then define $E_s$ as the union of all the intervals $I_{s^*} \subsetneqq I_s$ satisfying the condition that there is no interval $I_{s''}$ with $I_{s^*} \subsetneqq I_{s''} \subsetneqq I_s$. Obviously, if $E_s \neq \varnothing$, then $E_s$ is the union of certain disjoint intervals $I(\varphi_k)$. The number of such intervals $I(\varphi_k)$ is denoted by $d_s$. In the case $E_s = \varnothing$ we set $d_2 = 0$. It is not hard to see that

$$d_1 + d_2 + \cdots + d_{m-1} \leq m. \tag{8}$$

Assume that $v \neq 0$ (the lemma is obvious for $v = 0$), and let

$$m_s = [mv_s/v] + 1, \qquad v_s = \sum_{i=1}^{p_s} \mu(\Gamma_{s,i}).$$

According to Lemma 1.2, there exist simple functions $\psi_{s,j}(x)$ ($j = 1, \ldots, q_s$, $q_s \leq m_s + 6(d_s + 1)$) satisfying the following conditions:

a$_s$) $|f_s(x) - \sum_{j=1}^{q_s} \psi_{s,j}(x)| \leq c_1 v_s/m_s \leq c_1 v/m$ for $x \in \mathbf{R}$,

b$_s$) $f_s(x) - \sum_{j=1}^{q_s} \psi_{s,j}(x) = 0$ for $x \in E_s \cup (\mathbf{R} \backslash I_s)$,

c$_s$) $\sum_{j=1}^{q_s} \mu(\psi_{s,j}) \leq c_2 \sum_{i=1}^{p_s} \mu(\Gamma_{s,i}) = c_2 v_s$.

From condition b$_s$) and the way of constructing the sets $E_s$ and the functions $f_s$ we get that for a fixed $x \in \mathbf{R}$ at most one of the numbers $f_s(x) - \sum_{j=1}^{q_s} \psi_{s,j}(x)$ ($s = 1, \ldots, m-1$) is nonzero. Therefore, by condition a$_s$),

$$\left| \sum_{s=1}^{m-1} f_s(x) - \sum_{s=1}^{m-1} \sum_{j=1}^{q_s} \psi_{s,f}(x) \right| = \max_{s=1,\ldots,m-1} \left| f_s(x) - \sum_{j=1}^{q_s} \psi_{s,j}(x) \right| \leq \frac{c_1 v}{m} \quad \text{for } x \in \mathbf{R}. \tag{9}$$

By (7),

$$\left\| f - \sum_{s=1}^{m-1} \sum_{j=1}^{q_s} \psi_{s,j} \right\|_{\infty,\mathbf{R}} \leq \frac{c_3 v}{m}. \tag{10}$$

Also from condition c$_s$) we get

$$\sum_{s=1}^{m-1} \sum_{j=1}^{q_s} \mu(\psi_{s,j}) \leq c_2 \sum_{s=1}^{m-1} v_s \leq c_2 v. \tag{11}$$

By (8), the total number of terms in the double sum in (10) is

$$\sum_{s=1}^{m-1} q_s \leq \sum_{s=1}^{m-1} [m_s + 6(d_s + 1)] \leq 14m. \tag{12}$$

If some $G_s = \varnothing$ ($1 \leq s \leq m-1$), then in the above arguments we should consider only the functions $f_s$ with $G_s \neq \varnothing$. But if all the $G_s$ are empty for $s = 1, \ldots, m-1$, then (10) -(12) clearly hold, for example, for all $q_s = 1$ and for the simple functions $\psi_{s,1}$ equal identically to zero. Thus, relations (10)–(12) always hold, and they imply Lemma 1.3.

(3)By assumption, only these two cases are possible.

## §2. Lemmas on rational operators of Jackson type

Let $z_k$ $(k = 1, \dots, n)$ be points in the upper half-plane $\Pi = \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}$, i.e., $z_k = \alpha_k + i\beta_k$, where $-\infty < \alpha_k < \infty$ and $\beta_k > 0$. We define the Blaschke product

$$b(x) = \prod_{k=1}^{n} \frac{x - z_k}{x - \bar{z}_k}$$

and the rational kernel of Jackson type

$$g(x,t) = \left| \frac{b(t) - b(x)}{t - x} \right|^4 \qquad (x, t \in \mathbf{R}).$$

Let $g(x) = \int_{\mathbf{R}} g(x,t)dt$. Following Rusak [8], [9], we define a rational operator of Jackson type for a function $f$ integrable on $\mathbf{R}$ with respect to the measure $(1+t^2)^{-2}dt$:

$$\mathscr{D}_n(x, f) = \frac{1}{g(x)} \int_{\mathbf{R}} f(t) g(x,t) dt. \tag{13}$$

Actually, Rusak considered integration with respect to $(1+t^2)dt$ instead of integration with respect to $dt$ in (13). Obviously, this does not affect the following important properties, where were established in [8] and [9]:

a) *$\mathscr{D}_n(x, f)$ is a linear operator, and*

$$\mathscr{D}_n(x, 1) \equiv 1. \tag{14}$$

b) *$\mathscr{D}_n(x, f)$ is a rational function of degree at most* $4n - 4$.

LEMMA 2.1. *For any* $x \in \mathbf{R}$

$$g(x) \geq 8\pi \left[ \sum_{k=1}^{n} \frac{\beta_k}{(x - \alpha_k)^2 + \beta_k^2} \right]^3, \qquad g(x) \geq 8\pi \sum_{k=1}^{n} \frac{\beta_k}{[(x-\alpha_k)^2 + \beta_k^2]^2}.$$

PROOF. We compute $g(x)$ in a way completely analogous to that in Rusak's paper [8] (see also [9], pp. 132–136). As a result,

$$g(x) = \frac{\pi i}{3} [b^{-2}(x)(b^2(x))''' - 4b^{-1}(x)b'''(x)].$$

Consequently,

$$g(x) = \frac{8\pi}{3} \left\{ 4 \left[ \sum_{k=1}^{n} \frac{\beta_k}{(x-\alpha_k)^2 + \beta_k^2} \right]^3 - 4 \sum_{k=1}^{n} \frac{\beta_k^3}{[(x-\alpha_k)^2 + \beta_k^2]^3} \right.$$
$$\left. + 3 \sum_{k=1}^{n} \frac{\beta_k}{[(x-\alpha_k)^2 + \beta_k^2]^2} \right\}.$$

This proves Lemma 2.1.

LEMMA 2.2. *Suppose that* $-\infty < \alpha < \infty$, $\beta > 0$, *and* $\varphi$ *is a simple function such that* $I(\varphi) = (\alpha - \beta, \alpha + \beta)$ *and* $\mu(\varphi) \leq 1$. *Then for any* $x \in \mathbf{R}$

$$|\varphi(x) - \mathscr{D}_n(x, \varphi)| \leq \frac{43}{\sqrt[3]{g(x)}} \frac{\beta}{(x-\alpha)^2 + \beta^2} + \frac{16}{g(x)} \frac{\beta}{[(x-\alpha)^2 + \beta^2]^2}.$$

PROOF. Suppose that $\mathscr{E}_x = \{t \in \mathbf{R}\colon |t-x| \le 1/\sqrt[3]{g(x)}\}$ and $\mathscr{E}'_x = \mathbf{R}\backslash\mathscr{E}_x$. By (14),

$$|\varphi(x) - \mathscr{D}_n(x,\varphi)| \le \frac{1}{g(x)}\int_{\mathbf{R}} |\varphi(x)-\varphi(t)|g(x,t)dt = \frac{1}{g(x)}\left[\int_{\mathscr{E}_x} + \int_{\mathscr{E}'_x}\right].$$

If $t \in \mathscr{E}_x$, then $|\varphi(x)-\varphi(t)| \le 1/2\beta\sqrt[3]{g(x)}$, and

$$\int_{\mathscr{E}_x} \le \frac{1}{2\beta\sqrt[3]{g(x)}}\int_{\mathscr{E}_x} g(x,t)dt \le \frac{1}{2\beta}\sqrt[3]{g^2(x)}.$$

But if $t \in \mathscr{E}'_x$, then $|\varphi(x)-\varphi(t)| \le |x-t|/2\beta$, and

$$\int_{\mathscr{E}'_x} \le \frac{1}{2\beta}\int_{\mathscr{E}'_x} |x-t|g(x,t)dt \le \frac{16}{\beta}\int_{1/\sqrt[3]{g(x)}}^{\infty} \frac{dy}{y^3} = \frac{8}{\beta}\sqrt[3]{g^2(x)}.$$

Combining these estimates, we find that

$$|\varphi(x) - \mathscr{D}_n(x,\varphi)| \le \frac{8+1/2}{\beta\sqrt[3]{g(x)}} \qquad (x \in \mathbf{R}). \tag{15}$$

Under the condition that $x \notin [\alpha - 2\beta, \alpha + 2\beta]$ the estimate (15) can be refined as follows:

$$\begin{aligned} |\varphi(x) - \mathscr{D}_n(x,\varphi)| &= |\mathscr{D}_n(x,\varphi)| \le \frac{1}{2g(x)}\int_{\alpha-\beta}^{\alpha+\beta} g(x,t)dt \\ &\le \frac{8}{g(x)}\int_{\alpha-\beta}^{\alpha+\beta}\frac{dt}{(t-x)^4} \le \frac{16}{g(x)}\frac{\beta}{[(x-\alpha)^2+\beta^2]^2}. \end{aligned} \tag{16}$$

Lemma 2.2 follows from (15) and (16).

LEMMA 2.3. *Suppose that $\varphi_1, \dots, \varphi_n$ are simple functions, and let $f = \varphi_1 + \dots + \varphi_n$ and $\mu_k = \mu(\varphi_k)$. Then the half-plane $\Pi$ contains at most $2n$ numbers $z_1, \dots, z_m$ such that the operator $\mathscr{D}_m(x,\cdot)$ determined by them satisfies the relation*

$$|f(x) - \mathscr{D}_m(x,f)| \le \frac{c}{n}\sum_{k=1}^{n}\mu_k \qquad (x \in \mathbf{R}).$$

PROOF. Let $I(\varphi_k) = (\alpha_k - \beta_k, \alpha_k + \beta_k)$, where $-\infty < \alpha_k < \infty$ and $\beta_k > 0$. Since the operator $\mathscr{D}_n(x,\cdot)$ is linear, it can be assumed that $\sum_1^n \mu_k = 1$. The parameters $z_{k,j}$ of the desired operator $\mathscr{D}_m(x,\cdot)$ are determined as follows: $z_{k,j} = \alpha_k + i\beta_k$, where $k = 1, \dots, n$ and $j = 1, \dots, [n\mu_k]+1$. Obviously, there are $m = \sum_1^n([n\mu_k]+1) \le 2n$ such numbers $z_{k,j}$. Setting $\delta_k(x) = [(x-\alpha_k)^2 + \beta_k^2]^{-1}$, we get from Lemma 2.2 that for any $x \in \mathbf{R}$

$$|f(x) - \mathscr{D}_m(x,f)| \le \frac{43}{\sqrt[3]{g(x)}}\sum_{k=1}^{n}\mu_k\beta_k\delta_k(x) + \frac{16}{g(x)}\sum_{k=1}^{n}\mu_k\beta_k\delta_k^2(x).$$

On the basis of Lemma 2.1 we conclude that for any $x \in \mathbf{R}$

$$\sqrt[3]{g(x)} \ge \sqrt[3]{8\pi n}\sum_{k=1}^{n}\mu_k\beta_k\delta_k(x), \qquad g(x) \ge 8\pi n\sum_{k=1}^{n}\mu_k\beta_k\delta_k^2(x).$$

Lemma 2.3 is proved.

REMARK. It can be shown similarly that under the conditions of Lemma 2.3 the estimate

$$|f(x) - \mathscr{F}_m(x,f)| \leq \frac{c\ln(n+1)}{n} \sum_{k=1}^{n} \mu_k \qquad (x \in \mathbf{R})$$

holds for the Fejér-type operator $\mathscr{F}_n(x,f)$ also introduced by Rusak in [8] and [9], and determined by the kernel $|b(t)-b(x)|^2/|t-x|^2$.

## §3. Approximation in the disk

We introduce the function spaces needed in what follows. Our main object of investigation is the approximation space $R^{\alpha}_{\infty,q}(\Omega)$ ($\alpha > 0$, $0 < q \leq \infty$) of functions $f \in C(\Omega)$ with finite quasinorm

$$\|f\|_{R^{\alpha}_{\infty,q}(\Omega)} = \|f\|_{\infty,\Omega} + \left[\sum_{k=0}^{\infty} (2^{k\alpha} R_{2^k}(f,\Omega))^q\right]^{1/q} \qquad (q \neq \infty), \tag{17}$$

$$\|f\|_{R^{\alpha}_{\infty,\infty}(\Omega)} = \|f\|_{\infty,\Omega} + \sup_{k\geq 0} 2^{k\alpha} R_{2^k}(f,\Omega) \qquad (q = \infty). \tag{18}$$

Let $S$ be a locally rectifiable curve in $\mathbf{C}$, and let $0 < p \leq \infty$. Denote by $L_p(S)$ the Lebesgue space of measurable functions $f$ on $S$ with

$$\|f\|_{p,S} = \left(\int_S |f(\xi)|^p |d\xi|\right)^{1/p} < \infty \qquad (0 < p < \infty),$$

$$\|f\|_{\infty,S} = \operatorname*{ess\,sup}_{\xi \in S} |f(\xi)| < \infty \qquad (p = \infty).$$

The Hardy space $H_p = H_p(D_+)$ is defined as the set of $f \in A(D_+)$ with

$$\|f\|_{H_p} = \lim_{\rho \to 1-0} \|f(\cdot\rho)\|_{p,T} < \infty. \tag{19}$$

The limit in (19) exists because $\|f(\cdot\rho)\|_{p,T}$ is monotone in $\rho$ (see [17], p. 77). For the definition of the Hardy space $H_p(D_-)$ in $D_-$ one should consider the functions $f \in A(D_-)$ vanishing at infinity and let $\rho \to 1+0$ instead of $\rho \to 1-0$. It is known (see [17] and [18]) that the functions $f \in H_p$ ($H_p(D_-)$) have nontangential boundary values $f(\xi)$ for almost all $\xi \in T$. Let $f^{(\alpha)}$ ($\alpha > 0$, $f \in A(D_+)$) denote the $\alpha$th derivative of $f$ in the Riemann-Liouville sense (see [19] and [15]). The Hardy-Sobolev space $H^{\alpha}_p$ ($\alpha > 0$, $0 < p \leq \infty$) is defined as the set of $f \in A(D_+)$ such that

$$\|f\|_{H^{\alpha}_p} = \|f\|_{H_p} + \|f^{(\alpha)}\|_{H_p} < \infty. \tag{20}$$

Let $S$ be the circle $T$ or the interval $[-1,1]$, and let $f \in L_p(S)$. Denote by $\omega_{p,k}(\cdot, f)$ the $k$th modulus of smoothness of $f$ in $L_p(S)$. The Besov space $B^{\alpha}_p(S)$ ($\alpha > 0$, $0 < p < \infty$) is defined as the set of functions $f \in L_p(S)$ with

$$\|f\|_{B^{\alpha}_p(S)} = \|f\|_{p,S} + \left[\int_0^1 \left(\frac{\omega_{k,p}(t,f)}{t^{\alpha}}\right)^p \frac{dt}{t}\right]^{1/p} < \infty, \tag{21}$$

where $k = [\alpha] + 1$. The Hardy-Besov space $B^{\alpha}_p(D_+)$ ($B^{\alpha}_p(D_-)$) is defined as the set of $f \in H_p(D_+)$ ($f \in H_p(D_-)$) such that the boundary function $f(\xi)$ belongs to $B^{\alpha}_p(T)$. The spaces $R^{\alpha}_{\infty,q}(D_+)$ and $B^{\alpha}_p(D_+)$ will sometimes be denoted by $R^{\alpha}_{\infty,q}$ and $B^{\alpha}_p$ for brevity. We note that in [11] and [15] we used another equivalent definition of $H^{\alpha}_p$ and $B^{\alpha}_p$ (see [19] and [22] for more details on this). We have the imbeddings([4])

$$B^{\alpha}_p \subset H^{\alpha}_p \quad (p \leq 2), \qquad B^{\alpha}_p \supset H^{\alpha}_q \quad (p \geq 2), \tag{22}$$

and both imbeddings are strict (see [19] and [22]) for $p \neq 2$.

([4])Only continuous imbeddings are considered in this paper.

If $f \in H^{\alpha}_{1/\alpha}$ or $f \in B^{\alpha}_{1/\alpha}$ and $\alpha \geq 1$, then the boundary function $f(\xi)$ is continuous on $T$, i.e., the spaces $H^{\alpha}_{1/\alpha}$ and $B^{\alpha}_{1/\alpha}$ are imbedded in $C(\overline{D}_+)$ for $\alpha \geq 1$. If $f \in B^{\alpha}_{1/\alpha}(S)$, where $\alpha \geq 1$ and $S$ is the circle $T$ or the interval $[-1,1]$, then $f$ coincides almost everywhere on $S$ with some function in $C(S)$. Thus, we again have that $B^{\alpha}_{1/\alpha}(S) \subset C(S)$ for $\alpha \geq 1$. In the case $\alpha \in (0,1)$ the spaces $H^{\alpha}_{1/\alpha}, B^{\alpha}_{1/\alpha}$, and $B^{\alpha}_{1/\alpha}(S)$ contain essentially unbounded functions. Therefore, Theorems 3.2 and 4.1 (see below), our main results, do not hold for $\alpha < 1$.

To prove the inequalities (1) we also need the Hardy space $H_1(\Pi)$ in the half-plane $\Pi = \{z \in \mathbf{C}\colon \operatorname{Im} z > 0\}$. It is defined [18] as the set of $f \in A(\Pi)$ such that

$$\|f\|_{H_1(\Pi)} = \sup_{y>0} \|f(\cdot + iy)\|_{1,\mathbf{R}} < \infty.$$

The functions $f \in H_1(\Pi)$ have nontangential boundary values $f(x)$ for almost all $x \in \mathbf{R}$.

A real-valued function $a \in L_\infty(\mathbf{R})$ is called an *atom* (see [6] and [7]) if there exists a (finite) interval $J(a)$ such that $\operatorname{supp} a \subset J(a)$, $\|a\|_{\infty,\mathbf{R}} \leq 1/|J(a)|$, and, moreover, $\int_{\mathbf{R}} a(x)dx = 0$. If $a(x)$ is an atom, then $\varphi(x) = \int_{-\infty}^{x} a(t)dt$ is a simple function (see §1) for which $I(\varphi) = J(a)$ and $\mu(\varphi) \leq 1$.

LEMMA 3.1. *Suppose that $g \in H_1(\Pi)$ and $g \not\equiv 0$. Then there exist a sequence (finite or infinite) of atoms $a_1, a_2, \ldots$ and a sequence of positive numbers $\lambda_1, \lambda_2, \ldots$ such that:*

a) $\operatorname{Re} g(x) = \sum_k \lambda_k a_k(x)$ *for almost all* $x \in \mathbf{R}$;

b) $\sum_k \lambda_k \leq c\|g\|_{H_1(\Pi)}$;

c) *for any $k$ and $k'$ ($k \neq k'$) the intervals $J(a_k)$ and $J(a_{k'})$ are either disjoint or imbedded one in the other.*

Lemma 3.1 was obtained by Coifman in [6], where, however, the condition c) is missing in its formulation. This condition is not hard to see from the proof, which is constructive. There is a simpler proof in [7].

LEMMA 3.2. *If $f \in A(D_+) \cap C(\overline{D}_+)$, then*

$$R_n(f,T) \leq R_n(f,\overline{D}_+) \leq 2R_n(f,T), \qquad n \geq 0.$$

The first inequality is obvious, and the second was obtained in [10].

THEOREM 3.1. *If $f \in H^1_1$, then*

$$R_n(f,\overline{D}_+) \leq (c/n)\|f'\|_{H_1}, \qquad n \geq 1.$$

PROOF. We introduce the auxiliary function $g(\eta) = f[\Gamma(\eta)]$ ($\eta \in \overline{\Pi}$), where $z = \Gamma(\eta) = (1+i\eta)/(\eta+i)$ is a linear fractional mapping of the half-plane $\overline{\Pi}$ onto the disk $\overline{D}_+$. It is easy to show that $g' \in H_1(\Pi)$ and $\|g'\|_{H_1(\Pi)} = \|f'\|_{H_1(D_+)}$. Therefore, Lemmas 1.3, 2.3, and 3.1 imply that for any $m \in \mathbf{N}$ there exist $r_j \in \mathscr{R}_m(\mathbf{R})$ ($j = 1,2$) such that

$$\|\operatorname{Re} g - r_1\|_{\infty,\mathbf{R}} \leq c_1 m^{-1}\|f'\|_{H_1}, \tag{23}$$

$$\|\operatorname{Im} g - r_2\|_{\infty,\mathbf{R}} \leq c_2 m^{-1}\|f'\|_{H_1}. \tag{24}$$

Making the inverse substitution $\eta = \Gamma^{-1}(z) = (1-iz)/(z-i)$, we get Theorem 3.1 from (23) and (24) and Lemma 3.2.

The sharpness of Theorem 3.1 can be judged by the example of the function $\varphi_n(z) = z^{n+1}/2\pi(n+1)$, for which (see [9], p. 167) $\|\varphi_n'\|_{H_1} = 1$ and $R_n(\varphi_n,\overline{D}_+) = 1/2\pi(n+1)$.

We introduce the following best approximation for a function $f \in H_1$:

$$\overline{R}_n(f,H_1) = \inf\{\|f - r'\|_{H_1} : r \in \mathscr{R}_n(\overline{D}_+)\}.$$

LEMMA 3.3. *If* $f \in H_1^1$, *then*

$$R_n(f, \overline{D}_+) \le (c/n)\overline{R}_{n/2}(f', H_1), \qquad n \ge 2.$$

PROOF. Let $r_* \in \mathscr{R}_{n/2}(\overline{D}_+)$ be such that $\|f' - r_*'\|_{H_1} = \overline{R}_{n/2}(f', H_1)$. Then by Theorem 3.1

$$\begin{aligned} R_n(f, \overline{D}_+) &= R_n[r_* + (f - r_*)] \le R_{n/2}(r_*, \overline{D}_+) + R_{n/2}(f - r_*, \overline{D}_+) \\ &\le (c/n)\|f' - r_*'\|_{H_1} = (c/n)\overline{R}_{n/2}(f', H_1). \end{aligned}$$

Lemma 3.3 is proved.

COROLLARY 3.1. *If* $f \in H_1^1$, *then* $R_n(f, \overline{D}_+) = o(1/n)$.

The proof follows directly from Lemma 3.3 and Jackson's theorem.

THEOREM 3.2. *The following imbeddings are valid:*

$$R^1_{\infty,1} \subset H^1_1 \subset R^1_{\infty,\infty}, \tag{25}$$

$$R^\alpha_{\infty,1/\alpha} \subset H^\alpha_{1/\alpha} \subset R^\alpha_{\infty,2} \qquad (\alpha > 1), \tag{26}$$

$$R^1_{\infty,1} \subset B^1_1 \subset R^1_{\infty,\infty}, \tag{27}$$

$$R^\alpha_{\infty,1/\alpha} = B^\alpha_{1/\alpha} \qquad (\alpha > 1). \tag{28}$$

PROOF. The left-hand imbeddings in (25)–(27), as well as the imbedding "$\subset$" in (28), are known ([3], [13]–[15], and [20]). The right-hand imbeddings in (25) and (27) follow directly from Theorem 3.1 and (22). We get the right-hand imbedding (26). From Lemma 3.3,

$$R_{2^k}(f, \overline{D}_+) \le c2^{-k} R_{2^{k-1}}(f', H_1) \qquad (k \ge 1) \tag{29}$$

for a function $f \in H^\alpha_{1/\alpha}$ ($\alpha > 1$). Since $f' \in H^{\alpha-1}_{1/\alpha}$, we get from Theorem 4.1 in [11] that

$$\|f'\|_{H_1} + \left[\sum_{k=0}^{\infty} (2^{(\alpha-1)k}\overline{R}_{2^k}(f', H_1))^2\right]^{1/2} \le c(\alpha)\|f'\|_{H^{\alpha-1}_{1/\alpha}}. \tag{30}$$

The right-hand imbedding in (26) follows from (29) and (30). The imbedding "$\supset$" in (28) is proved similarly. This proves Theorem 3.2.

For $s \in \mathbf{N}$ and $\beta > 0$ we introduce the function

$$\varphi_{s,\beta}(z) = \left(\ln_{(s)} \frac{a}{1-z}\right)^{-\beta},$$

where $\ln_{(1)} x = \ln(x)$ and $\ln_{(s)} x = \ln(\ln_{(s-1)} x)$ for $s \ge 2$, the principal branch is taken for all logarithms, and the positive number $a$ is chosen so that $\varphi_{s,\beta}(z)$ is continuous in $\overline{D}_+$. For sufficiently large $n$ we have the relations(5)

$$R_n(\varphi_{1,\beta}, [0,1]) \asymp 1/n^{1+\beta}, \tag{31}$$

$$R_n(\varphi_{s,\beta}, [0,1]) \asymp 1/n(\ln_{(s-1)} n)^\beta \qquad (s \ge 2). \tag{32}$$

The equivalences (31) and (32) were obtained for $s = 2$ in [23]. The case $s > 2$ is handled similarly. We mention that the first nontrivial upper and lower estimates for $R_n(\varphi_{s,\beta}, [0,1])$ were obtained by Gonchar in [24] and [25]. See also Bulanov's paper [26] about a lower estimate of $R_n(\varphi_{1,\beta}, [0,1])$. It is shown in Example 3.1 that (31) and (32) are preserved if $[0,1]$ is replaced by $T$ or by $\overline{D}_+$.

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(5) $a_n \asymp b_n \Leftrightarrow a_n = O(b_n) \& b_n = O(a_n)$.

EXAMPLE 3.1. *For sufficiently large n*

$$R_n(\varphi_{1,\beta},\overline{D}_+) \asymp R_n(\varphi_{1,\beta},T) \asymp 1/n^{1+\beta}, \tag{33}$$

$$R_n(\varphi_{s,\beta},\overline{D}_+) \asymp R_n(\varphi_{s,\beta},T) \asymp 1/n(\ln_{(s-1)} n)^{\beta} \qquad (s \geq 2). \tag{34}$$

PROOF. It follows from the lower estimate in (31) and (32) and from Lemma 3.2 that it suffices for us to get an upper estimate for $R_n(\varphi_{s,\beta},\overline{D}_+)$ $(s \geq 1)$. With this goal we introduce $\varphi_{s,\beta,n}(z) = \varphi_{s,\beta}((1-e^{-n})z)$ and choose some $k \in \mathbf{N}$ such that $k > 1+\beta$. From the right-hand imbeddings in (25) and (26) we have

$$R_n(\varphi_{s,\beta},\overline{D}_+) \leq c_1 n^{-1}\|\varphi'_{s,\beta} - \varphi'_{s,\beta,n}\|_{H_1} + c_2(k)n^{-k}\|\varphi^{(k)}_{s,\beta,n}\|_{H_{1/k}}. \tag{35}$$

The necessary upper estimate follows from (35). The relations (33) and (34) are proved.

Since $\varphi_{s,\beta} \in B_1^1$ for any $s$ and $\beta$, (34) and (22) imply that the right-hand imbeddings in (25) and (27) are sharp in the sense that $R^1_{\infty,\infty}$ cannot be replaced by $R^1_{\infty,q}$ for any $q < \infty$. The impossibility of an analogous improvement in the left-hand imbeddings in (25) and (27) follows from results of Dolzhenko [3]. Examples in [11] and [15] give us that the imbeddings (26) also cannot be improved.

For Lebesgue measurable sets $\mathscr{E} \subset D$ we define the measure

$$\mu(\mathscr{E}) = \iint_{\mathscr{E}} (1-|z|)^{-2}\,dx\,dy \qquad (z = x+iy).$$

Denote by $L_{p,q}(D_+,\mu)$ the Lorentz space of $\mu$-measurable functions in $D_+$ (see [27], §5.3).

COROLLARY 3.2. *If* $\alpha > 1$, $k > \alpha$ $(k \in \mathbf{N})$, *and* $0 < q \leq \infty$, *then*

$$f \in R^\alpha_{\infty,q} \Leftrightarrow f^{(k)}(z)(1-|z|)^k \in L_{1/\alpha,q}(D_+,\mu).$$

*In particular,*

$$R_n(f,\overline{D}_+) = O(n^{-\alpha})$$
$$\Leftrightarrow \mu\{z \in D_+ : |f^{(k)}(z)|(1-|z|)^k > t\} = O(t^{-1/\alpha}) \quad \text{as } t \to +0.$$

The proof is based on (28) and is analogous to that of Corollary 4.2 in [11].

Let us compare the degree of best rational approximation in $C(\overline{D}_+)$ and the space BMOA of analytic functions of bounded mean oscillation in $D_+$ (see [13], [18], and [20]). By definition, an $f \in A(D_+)$ belongs to BMOA if it is representable as an integral of Cauchy type with bounded density:

$$f(z) = \mathscr{K}^+ g(z) = \frac{1}{2\pi i}\int_T \frac{g(\xi)}{\xi - z}\,d\xi, \tag{36}$$

where $g \in L_\infty(T)$ and $z \in D_+$. Here we set $\|f\|_{\mathrm{BMOA}} = \inf \|g\|_{\infty,T}$, where the infimum runs over all $g$ such that (36) holds. Let $R_n(f,\mathrm{BMOA})$ denote the best approximation of $f$ in BMOA by the set $\mathscr{R}_n(\overline{D}_+)$, and by $R^\alpha_{*,q}$ the approximation space determined by (17) and (18) when $\|f\|_{\infty,\Omega}$ is replaced by $\|f\|_{\mathrm{BMOA}}$ and $R_n(f,\Omega)$ is replaced by $R_n(f,\mathrm{BMOA})$. Obviously, for $f \in A(D_+) \cap C(\overline{D}_+)$

$$R_n(f,\mathrm{BMOA}) \leq R_n(f,\overline{D}_+). \tag{37}$$

Therefore, (28) implies the imbedding $B^\alpha_{1/\alpha} \subset R^\alpha_{*,1/\alpha}$ for $\alpha > 1$. By using real interpolation this imbedding can be generalized to $\alpha \leq 1$. The necessary interpolation theorems are in [27] and [28]. Thus, we have obtained a new proof of a result of Peller [13], [20]: $B^\alpha_{1/\alpha} \subset R^\alpha_{*,1/\alpha}$ $(\alpha > 0)$. The reverse imbedding also holds (see [20], [13], [14], and [15]).

We show that inequality (37) can be reversed for "sufficiently smooth functions".

COROLLARY 3.3. *If $f \in \mathrm{BMOA}$ and*

$$\sum_{k=0}^{\infty} R_k(f, \mathrm{BMOA}) < \infty, \tag{38}$$

*then $f \in C(\overline{D}_+)$, and for any $n \geq 1$*

$$R_n(f, \overline{D}_+) \leq \frac{c}{n} \sum_{k \geq n/2} R_k(f, \mathrm{BMOA}). \tag{39}$$

The proof is by the standard method with use of Theorem 3.1 and the inequality (see [20] and [13]) $\|r'\|_{H_1} \leq cn\|r\|_{\mathrm{BMOA}}$, where $r \in \mathscr{R}_n(\overline{D}_+)$ and $n \geq 1$.

The sharpness of inequality (39) can be judged by the example of the function $\varphi_{s,\beta}(z)$, for which we obtained (33) and (34) earlier. For sufficiently large $n$ we also have the equivalences

$$R_n(\varphi_{1,\beta}, \mathrm{BMOA}) \asymp 1/n^{1+\beta}, \tag{40}$$

$$R_n(\varphi_{s,\beta}, \mathrm{BMOA}) \asymp 1/n \ln_{(1)} n \cdots \ln_{(s-2)} (\ln_{(s-1)} n)^{1+\beta} \qquad (s \geq 2). \tag{41}$$

The assertions (40) and (41) were obtained in [11] for $s = 2$, and the case $s > 2$ is handled similarly.

REMARKS. 1. We can show that if (38) does not hold, then $f \notin C(\overline{D}_+)$ in general.

2. Relation (39), in combination with a result of Peller in [13] and [20] ($B^{\alpha}_{1/\alpha} \subset R^{\alpha}_{*,1/\alpha}$, $\alpha > 0$), also leads to the imbedding $B^{\alpha}_{1/\alpha} \subset R_{\infty,1/\alpha}$ ($\alpha > 1$) in (38).

## §4. Approximation on the circle and on a closed interval

Together with the integral $\mathscr{K}^+g(z)$ defined by (36), we also introduce for a $g \in L_1(T)$ the integral $\mathscr{K}^-g(z)$ obtained from (36) by replacing $z \in D_+$ by $z \in D_-$. For $\xi \in T$ let $\mathscr{K}^{\pm}g(\xi)$ denote the nontangential boundary values of $\mathscr{K}^{\pm}g(z)$. As is known [17], $g(\xi) = \mathscr{K}^+g(\xi) + \mathscr{K}^-g(\xi)$ for almost all $\xi \in T$. We also define the conjugate function

$$\tilde{g}(\xi) = -\frac{1}{\pi} \int_T \frac{g(\eta)}{\eta - \xi} d\eta \qquad (\xi \in T),$$

where the integral is understood in the sense of the Cauchy principal value. It is known [17] that $\tilde{g}(\xi)$ exists almost everywhere on $T$ and

$$\tilde{g}(\xi) = -i\mathscr{K}^+g(\xi) + i\mathscr{K}^-g(\xi) + i\hat{g}(0),$$

where

$$\hat{g}(0) = (1/2\pi) \int_T g(\xi)|d\xi|.$$

LEMMA 4.1 ([13], [20]). *The operators $\mathscr{K}^+, \mathscr{K}^-$, and $\tilde{}$ act continuously from $B^{\alpha}_{1/\alpha}(T)$ ($\alpha > 0$) into $B^{\alpha}_{1/\alpha}(D_+), B^{\alpha}_{1/\alpha}(D_-)$, and $B^{\alpha}_{1/\alpha}(T)$, respectively.*

LEMMA 4.2 ([13], [29]). *If $f(x) \in B^{\alpha}_{1/\alpha}[-1, 1]$ ($\alpha > 0$), then $g(\xi) = f[(\xi + \bar{\xi})/2] \in B^{\alpha}_{1/\alpha}(T)$, and this mapping is continuous.*

THEOREM 4.1. *Suppose that $S$ is the circle $T$ or the interval $[-1, 1]$. Then*

$$R^1_{\infty,1}(S) \subset B^1_1(S) \subset R^1_{\infty,\infty}(S), \tag{42}$$

$$R^{\alpha}_{\infty,1/\alpha}(S) = B^{\alpha}_{1/\alpha}(S) \qquad (\alpha > 1). \tag{43}$$

PROOF. The left-hand imbedding in (42) is known, as is the imbedding "$\subset$" in (43) (see [13]–[15], [20], and [29]). The right-hand imbedding in (42) and the imbedding "$\supset$" in (43) follow from Theorem 3.2 and Lemmas 4.1 and 4.2. Theorem 4.1 is proved.

THEOREM 4.2. *If the functions $f$ and $\tilde{f}$ are absolutely continuous on $T$, then for any $n \geq 1$*

$$R_n(f,T) \geq cn^{-1}(\|f'\|_{1,T} + \|(\tilde{f})'\|_{1,T}).$$

The proof follows immediately from Theorem 3.1 and the properties given above for an integral of Cauchy type and for conjugate functions.

REMARK. Sevast'yanov [35] pointed out the advisability of studying best rational approximations of the functions in Theorem 4.2. As is clear from Lemma 3.2, Theorems 3.1 and 4.2 are equivalent.

COROLLARY 4.1. *Suppose that $f \in C(T)$ and that the function $f(e^{ix})$ is even and convex on $[0, 2\pi]$. Then $\tilde{f} \in C(T)$, and for any $n \geq 1$*

$$R_n(f,T) \leq cn^{-1}\|f'\|_{1,T}, \tag{44}$$

$$R_n(\tilde{f},T) \leq cn^{-1}\|f'\|_{1,T}. \tag{45}$$

PROOF. We introduce the function $\Delta_t(x) = \max\{0, 1 - |x|/t\}$, where $x \in [-\pi, \pi]$ and $t > 0$. It is not hard to show (see also [30], Chapter II, §1), that there exists a nondecreasing function $h(t)$ on $[0, \pi]$ satisfying the conditions $h(\pi) - h(0) = \frac{1}{2}\|f\|_{1,T}$ and

$$f(e^{ix}) = f(-1) + \int_0^\pi \Delta_t(x)dh(t) \qquad (x \in [\pi, \pi]).$$

We get (44) and (45) from Theorem 4.2 (or Theorem 4.1 and Lemma 4.1) and the last relation.

We remark that (44) is an obvious consequence of Theorem 5.1 in [31], while (45) is new. Relation (44) can be obtained similarly in the nonperiodic case (see [31] and [32]).

For $x \geq 0$ we introduce the function $\Phi_0(x) = x \ln x$ for $x \geq 1$, $\Phi_0(x) = 0$ for $x < 1$. Denote by $L^*_{\Phi_0}(T)$ the corresponding Orlicz space [33] of functions on $T$.

COROLLARY 4.2 [31]. *If $f$ is absolutely continuous on $T$ and $f' \in L^*_{\Phi_0}(T)$, then for any $n \geq 1$*

$$R_n(f,T) \leq cn^{-1}\|f'\|_{L^*_{\Phi_0}(T)}.$$

The proof follows immediately from Theorem 4.2 and the theorem of Zygmund on conjugate functions [18].

Grigoryan [34] proved that if

$$\sum_{k=0}^{\infty} R_k(f,T) < \infty, \tag{46}$$

then $\tilde{f} \in C(T)$. Sevast'yanov [35] constructed examples which imply that if (46) does not hold, then $\tilde{f} \notin C(T)$ in general. Theorem 4.3 below gives an analogue, for rational approximations, of an inequality of Stechkin [36] connecting best polynomial approximations of a function and its conjugate.

THEOREM 4.3. *Suppose that $f \in C(T)$ and condition (46) holds. Then for any $n \geq 2$*

$$R_n(\tilde{f},T) \leq \frac{c}{n} \sum_{k \geq n/2} R_k(f,T).$$

The proof is by a standard method with the use of Theorem 4.2 along with the inequalities $\|r'\|_{1,T} \leq cn\|r\|_{\infty,T}$ and $\|(\tilde{r})'\|_{1,T} \leq cn\|r\|_{\infty,T}$, where $r \in \mathscr{R}_n(T)$ and $n \geq 1$. The first of these inequalities is due to Dolzhenko [1], and the second to Rusak [9], [35].

The sharpness of Theorem 4.3 can be judged by the following example.

EXAMPLE 4.1. For any $s \in \mathbf{N}$ and $\beta > 0$ there exists an $f_{s,\beta} \in C(T)$ such that $\tilde{f}_{s,\beta} \in C(T)$ and, for sufficiently large $n$,

$$R_n(f_{1,\beta}, T) \asymp R_n(\tilde{f}_{1,\beta}, T) \asymp 1/n^{1+\beta}, \tag{47}$$

$$R_n(f_{s,\beta}, T) \asymp 1/n \ln_{(1)} n \cdots \ln_{(s-2)} n (\ln_{(s-1)} n)^{1+\beta} \qquad (s \geq 2), \tag{48}$$

$$R_n(\tilde{f}_{1,\beta}, T) \asymp 1/n(\ln_{(s-1)} n)^{\beta} \qquad (s \geq 2). \tag{49}$$

PROOF. The constructions are based on the functions $\varphi_{s,\beta}$ in §3. Let $f_{1,\beta}(\xi) = \varphi_{1,\beta}(\xi)$, $\xi \in T$. Since $\varphi_{1,\beta} \subset C(\overline{D}_+) \cap A(D_+)$, it follows that $\tilde{f}_{1,\beta} = -if_{1,\beta} + \text{const}$, and hence (47) follows from (33). In the case $s \geq 2$ we introduce functions $r_k \in \mathscr{R}_{2^k}(\overline{D}_+)$ which satisfy the conditions $\|\varphi_{s,\beta} - r_k\|_{\text{BMOA}} = R_{2^k}(\varphi_{s,\beta}, \text{BMOA})$. Fix some sufficiently large $k_0$. We have that $\varphi_{s,\beta} = u_1 + u_2 + u_3 + \cdots$, where $u_1 = r_{k_0}$, $u_2 = r_{k_0+1} - r_{k_0}$, $u_3 = r_{k_0+2} - r_{k_0+1}$ and so on. According to (41), $\|u_j\|_{\text{BMOA}} \leq c\lambda_s(2^{j+k_0})$, where $\lambda_s(n)$ is the right-hand side of (41). Since $u_j \in \mathscr{R}_{2^{j+k_0}}(\overline{D}_+)$, it follows [37] that there is a $v_j \in \mathscr{R}_{2^{j+k_0}}(\overline{D}_-)$ such that the function $w_j = u_j + v_j$ satisfies the condition

$$\|w_j\|_{\infty,T} = \|u_j\|_{\text{BMOA}} \leq c\lambda_s(2^{j+k_0}). \tag{50}$$

We show that $f_{s,\beta} = w_1 + w_2 + \cdots$ is the desired function. Indeed, (50) implies that $f_{s,\beta} \in C(T)$ and that the upper estimate in (48) is valid. To obtain the lower estimate in (48), note that $\mathscr{K}^+ f_{s,\beta} = \varphi_{s,\beta} + \text{const}$ and $R_n(f_{s,\beta}, T) \geq R_n(\varphi_{s,\beta}, \text{BMOA}) \geq c\lambda_s(n)$ by (41). Thus, (48) is proved. The upper estimate in (49) follows from Theorem 4.3 and the upper estimate in (48). It remains to get the lower estimate in (49). For this, note that $\tilde{f}_{s,\beta} = if_{s,\beta} - 2i\varphi_{s,\beta} + \text{const}$, and hence

$$R_n(\tilde{f}_{s,\beta}, T) \geq 2R_{2n}(\varphi_{s,\beta}, T) - R_n(f_{s,\beta}, T).$$

The lower estimate in (49) is obtained from the last inequality, relation (34), and the upper estimate in (48). This proves (47)–(49).

COROLLARY 4.3. *Suppose that $\alpha > 1$ and $0 < q \leq \infty$. Then the following conditions are equivalent:*

a) $f \in R^{\alpha}_{\infty,q}(T)$,

b) $\tilde{f} \in R^{\alpha}_{\infty,q}(T)$,

c) $\mathscr{K}^+ f \in R^{\alpha}_{\infty,q}(\overline{D}_+)$ *and* $\mathscr{K}^- f \in R^{\alpha}_{\infty,q}(\overline{D}_-)$.

Recall that a description of $R^{\alpha}_{\infty,q}(\overline{D}_+)$ is given in Corollary 3.2. The space $R^{\alpha}_{\infty,q}(\overline{D}_-)$ admits an analogous description. Thus, Corollaries 3.2 and 4.3 give a description of the space $R^{\alpha}_{\infty,q}(T)$.

Let $(\cdot,\cdot)_{\theta,q}$ be the Peetre interpolation functor [27]. One application of the results obtained is given in Corollary 4.4.

COROLLARY 4.4 (cf. [13], [20] and [28]). *Suppose that $S$ is the circle $T$ or the interval $[-1,1]$, $s > 1$, $0 < \theta < 1$, and $\alpha = \theta s > 1$. Then*

$$(C(S), B^s_{1/s}(S))_{\theta,1/\alpha} = B^{\alpha}_{1/\alpha}(S).$$

The proof follows immediately from Theorem 4.1 and the equalities (see [27], Chapter 7)

$$(C(S), R^s_{\infty,1/s}(S))_{\theta,1/\alpha} = R^{\alpha}_{\infty,1/\alpha}(S).$$

In conclusion we note that it is possible to give a "real" description of the spaces $R^{\alpha}_{\infty,q}(T)$ and $R^{\alpha}_{\infty,q}[-1,1]$ for $\alpha > 1$. This description is based on the idea of an atomic decomposition of the spaces $H^\alpha_{1/\alpha}$ and $B^\alpha_{1/\alpha}$. It is also possible to introduce a characteristic, analogous to moduli of smoothness in polynomial approximation, which connects smoothness of functions and the degree of Tchebycheff rational approximation. Such a characteristic was introduced for the spaces $L_p$ $(1 < p < \infty)$ in [4] and [16]. We propose to consider these questions separately. We remark also that by using the Faber transformation method ([13], [21], and [29]) it is possible to generalize Theorem 3.2 for Lipschitz domains [38].

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