V. I. Danchenko, Estimates of derivatives of simplest fractions and other questions, *Sbornik: Mathematics*, 2006, Volume 197, Issue 4, 505–524

DOI: 10.1070/SM2006v197n04ABEH003768

Sbornik: *Mathematics* **197**:4 505–524

Matematicheskiĭ Sbornik **197**:4 33–52
DOI 10.1070/SM2006v197n04ABEH003768

Estimates of derivatives of simplest fractions and other questions

V. I. Danchenko

Abstract. The approximation properties of simplest fractions (s.f.'s), that is, of the logarithmic derivatives of complex polynomials, have recently become a subject of intensive research. These properties of s.f.'s prove to have many similarities with those of polynomials. For instance, one has for them analogues of Mergelyan's and Jackson's classical results on uniform polynomial approximation. In connection with approximation by s.f.'s estimates of the Markov–Bernstein kind for derivatives of s.f.'s on various subsets of the complex plane arouse interest. Such estimates are obtained in this paper on circles, straight lines and their intervals, and some applications of these estimates are indicated. Several other questions relating to approximation properties of s.f.'s are also considered.

Bibliography: 28 titles.

§ 1. Introduction

By a *simplest fraction* (*s.f.*) *of degree* n, $n \geqslant 1$, of the complex variable $z \in \mathbb{C}$ we mean a rational function of the following form:

$$\rho(z) = \rho_n(z) = \sum_{k=1}^{n} \frac{1}{z - z_k}\,, \tag{1}$$

that is, the logarithmic derivative of a complex variable (some of the points $z_k \in \mathbb{C}$ can be equal). The approximation properties of s.f.'s have recently become an object of intensive study (see [1]–[8]). For this reason one is interested in estimates of the Markov–Bernstein kind for s.f.'s on various subsets K of the complex plane $\mathbb{C}$, that is, in estimates of the following form:

$$|\rho'(z)| \leqslant A(z, K, n, \|\rho\|_K), \qquad \|\rho\|_K = \sup\{|\rho(t)| : t \in K\}, \quad z \in K, \tag{2}$$

where A is a positive quantity that is finite at each point $z \in K$ and depends only on the indicated parameters (but is independent of the location of the poles of the s.f. ρ). In what follows we consider only sets K of the simplest form: circles, straight lines, and straight line intervals. As is known, in the class of rational functions of general form (of arbitrary fixed degree n) there can be no estimates of the form (2) at any point of a set K of the above form (see, for instance, [9]). Moreover, there exist

This research was carried out with the support of the Russian Foundation for Basic Research (grant nos. 04-01-00717, 05-01-00962) and the Programme of Support of Leading Scientific Schools of RF (grant no. NSh-1892.2003.1).

AMS 2000 *Mathematics Subject Classification*. Primary 41A20, 41A17.

no estimates either in the class of rational functions representable as a difference of s.f.'s. One example here is the s.f. $f_{a,b}(z) = (z-a)^{-1} - (z-b)^{-1}$, where K does not separate the points a and b, $a \neq b$. Then $f'_{a,b}(z) = -f_{a,b}((z-a)^{-1} + (z-b)^{-1})$ and for each $z \in K$ one can clearly find $a \to b \to z$ such that both $\|f_{a,b}\|_K \to 0$ and $|f'_{a,b}(z)| \to \infty$. For an s.f. (1) there exist estimates of the form (2), and in the case of bounded sets K they are similar to inequalities of Bernstein type for derivatives of polynomials (see §§ 6, 7). Moreover, one can find estimates of s.f.'s also on unbounded sets. For instance, it is shown in [10] that for $x \in \mathbb{R} = (-\infty, \infty)$, for an s.f. of the form (1) with $\operatorname{Im} z_k > 0$, $k = 1, \dots, n$, we have the inequalities

$$|\operatorname{Re} \rho'(x)| + |\operatorname{Im} \rho'(x)| \leqslant 2 \operatorname{Im} \rho(x)(|\operatorname{Re} \rho(x)| + \|\operatorname{Re} \rho\|_{\mathbb{R}}) \leqslant 4\|\rho\|^2_{\mathbb{R}},$$

and for each s.f. ρ of the first degree the first inequality here becomes an equality for some real x. Other precise inequalities of this type hold on $\mathbb{R}$ and other sets of the above-mentioned form. One can find the proofs of the main results in §§ 4–7. In § 8 we consider some additional properties of s.f.'s. Some of our results here were published in the Proceedings of Conferences [6]–[8].

§ 2. Auxiliary results

Let G be a simply connected domain in the extended complex plane $\overline{\mathbb{C}}$, with boundary γ that is a piecewise analytic curve in $\overline{\mathbb{C}}$, that is, it consists of finitely many regular Jordan analytic arcs γ_m. (Each arc γ_m is the image on the Riemann sphere of the interval $[0, 1]$ under a locally conformal map.) Here γ is not necessarily a simple curve: it can be a two-sided bounded or unbounded cut. We shall call points $z \in \gamma$ distinct from ∞ and the end-points of the arcs γ_m *regularity points of* γ. We denote by $w = w(z)$ a fixed univalent conformal map of the domain G onto the unit disc $D : |w| < 1$. We fix n and some points $z_k \in G$, $k = 1, \dots, n$, distinct from ∞ (points with distinct indices are not necessarily distinct). In what follows we often write w in place of $w(z)$ for $z \in \overline{G}$ and v in place of $w(\zeta)$ for $\zeta \in \gamma$. For $w_k = w(z_k)$ we set

$$B(z) = \prod_{k=1}^{n} \frac{w - w_k}{1 - w\overline{w_k}}\,, \quad \tau(\zeta) = \frac{w'(\zeta)}{w(\zeta)}\,, \quad \mu(\zeta) = \frac{B'(\zeta)}{\tau(\zeta)B(\zeta)} = \sum_{k=1}^{n} \frac{1 - |w_k|^2}{|v - w_k|^2} > 0, \tag{3}$$

where $\zeta \in \gamma$ is a regularity point of the curve γ (at such a point ζ the quantity $\tau(\zeta)$ is well defined, finite, and non-zero). For fixed real φ we consider the fraction

$$f(z, \varphi) = \frac{1}{B(z) - e^{i\varphi}}\,,$$

where we choose φ such that all the roots $\zeta_k \in \gamma$, $k = 1, \dots, n$, of the equation $B(\zeta) = e^{i\varphi}$ are regularity points of γ. In view of the definition of μ, we can write down the expansion of $f(z, \varphi)$ into elementary fractions (with respect to the variable $w = w(z)$ the function $f(z, \varphi)$ is a rational function):

$$f(z, \varphi) = a + \sum_{k=1}^{n} \frac{1}{B(\zeta_k)} \, \frac{w'(\zeta_k)B(\zeta_k)}{v_k B'(\zeta_k)} \, \frac{v_k}{w - v_k} = a + e^{-i\varphi} \sum_{k=1}^{n} \frac{1}{\mu(\zeta_k)} \, \frac{v_k}{w - v_k}\,,$$

where a is a finite constant and $v_k = w(\zeta_k)$, $|v_k| = 1$ for all k. We now calculate the z-derivatives of both sides of this equality and after a simple transformation taking account of (3) obtain

$$\frac{B'(z)}{B(z)}\,\frac{w}{w'(z)}\,\frac{B(z)e^{i\varphi}}{(B(z)-e^{i\varphi})^2} = \sum_{k=1}^{n}\frac{1}{\mu(\zeta_k)}\,\frac{v_k w}{(w-v_k)^2}\,, \qquad w = w(z).$$

Since for real x, y we have

$$\frac{e^{ix}e^{iy}}{(e^{ix}-e^{iy})^2} = -\frac{1}{4}\operatorname{cosec}^2\frac{x-y}{2}\,, \tag{4}$$

substituting in the above identity $z = \zeta \in \gamma$, $B(\zeta) = e^{i\beta}$, $v = w(\zeta) = e^{i\alpha}$, and $v_k = w(\zeta_k) = e^{i\alpha_k}$ we obtain

$$\mu(\zeta) = \sin^2\frac{\beta-\varphi}{2}\sum_{k=1}^{n}\frac{1}{\mu(\zeta_k)}\operatorname{cosec}^2\frac{\alpha-\alpha_k}{2}\,. \tag{5}$$

We shall require one further identity. Consider the expansion in simplest fractions (with respect to $w = w(z)$):

$$\frac{1}{w}\,\frac{e^{i\varphi}}{B(z)-e^{i\varphi}} = \frac{1}{w}\,\frac{e^{i\varphi}}{B_0-e^{i\varphi}} + \sum_{k=1}^{n}\frac{1}{\mu(\zeta_k)}\,\frac{1}{w-v_k}\,, \qquad B_0 = (-1)^n\prod_{k=1}^{n} w_k,$$

after which, multiplying both sides by w, setting $w \to \infty$ and performing simple transformations we obtain

$$\sum_{k=1}^{n}\frac{1}{\mu(\zeta_k)} = \frac{e^{i\varphi}}{(-1)^n\prod_{k=1}^{n}1/\overline{w_k}-e^{i\varphi}} - \frac{e^{i\varphi}}{B_0-e^{i\varphi}} = \frac{1-|B_0|^2}{|B_0-e^{i\varphi}|^2}\,. \tag{6}$$

§ 3. Main lemma

Consider the functions

$$f_1(z) = \frac{P(w)}{\prod_{k=1}^{n}(w-w_k)}\,, \quad f_2(z) = \frac{Q(w)}{\prod_{k=1}^{n}(1-w\overline{w_k})}\,, \qquad f(z) = f_1(z)+f_2(z), \tag{7}$$

where $P(w)$ and $Q(w)$ are arbitrary polynomials of degree at most n and where, as before, $w = w(z)$, $w_k = w(z_k)$, $z_k \in G$, $|w_k| < 1$ (points with distinct indices can be the same).

Lemma 1. *At each regularity point ζ of γ,*

$$\begin{aligned}\sin\frac{\beta-\varphi}{2}&\bigl(f_1'(\zeta)e^{i(\beta-\varphi)/2}+f_2'(\zeta)e^{-i(\beta-\varphi)/2}\bigr)\\ &= \frac{\tau(\zeta)}{2i}f(\zeta)\mu(\zeta) - \frac{\tau(\zeta)}{2i}\sin^2\frac{\beta-\varphi}{2}\sum_{k=1}^{n}\frac{f(\zeta_k)}{\mu(\zeta_k)}\operatorname{cosec}^2\frac{\alpha-\alpha_k}{2}\\ &= \frac{\tau(\zeta)}{2i}\sin^2\frac{\beta-\varphi}{2}\sum_{k=1}^{n}\frac{f(\zeta)-f(\zeta_k)}{\mu(\zeta_k)}\operatorname{cosec}^2\frac{\alpha-\alpha_k}{2}\,,\end{aligned} \tag{8}$$

where the real parameters α, β, *are defined by the relations*

$$v = w(\zeta) = e^{i\alpha}, \qquad B(\zeta) = e^{i\beta},$$

and φ *is an arbitrary real number such that all the roots* ζ_k, $k = 1, \dots, n$, *of the equation* $B(z) = e^{i\varphi}$, $z \in \gamma$, *are regularity points of the curve* γ, *and* $\beta - \varphi \neq 2\pi l$ *for integer* l, $w(\zeta_k) = e^{i\alpha_k}$.

Proof. For fixed $\zeta \in \gamma$ and $\varphi \neq \beta + 2\pi l$ consider the function

$$F(z) = F(z, \varphi) = \frac{f_1(z)B(z) + f_2(z)e^{i\varphi}}{B(z) - e^{i\varphi}} .$$

This function is rational in the variable $w = w(z)$, with poles only at the points $v_k = w(\zeta_k) = e^{i\alpha_k}$. Hence, in view of the definition (3), we obtain

$$F(z) = A + \sum_{k=1}^{n} \frac{w'(\zeta_k)e^{i\varphi}}{v_k B'(\zeta_k)} \frac{v_k f(\zeta_k)}{w - v_k} = A + \sum_{k=1}^{n} \frac{v_k f(\zeta_k)}{\mu(\zeta_k)(w - v_k)} ,$$

where A is a finite constant. Consequently, for $z = \zeta \in \gamma$ and $v = w(\zeta) = e^{i\alpha}$ (see also (4)),

$$F'(\zeta) = -\tau(\zeta) \sum_{k=1}^{n} \frac{f(\zeta_k)}{\mu(\zeta_k)} \frac{v v_k}{(v - v_k)^2} = \frac{\tau(\zeta)}{4} \sum_{k=1}^{n} \frac{f(\zeta_k)}{\mu(\zeta_k)} \operatorname{cosec}^2 \frac{\alpha - \alpha_k}{2} . \qquad (9)$$

On the other hand,

$$F'(z) = \frac{f_1'(z)B(z)}{B(z) - e^{i\varphi}} + \frac{f_2'(z)e^{i\varphi}}{B(z) - e^{i\varphi}} - \frac{f(z)e^{i\varphi}B'(z)}{(B(z) - e^{i\varphi})^2} .$$

It is easy to verify that for $z = \zeta \in \gamma$ and $B(\zeta) = e^{i\beta}$ we have the equalities

$$\frac{B(\zeta)}{B(\zeta) - e^{i\varphi}} = \frac{1}{2}\left(1 - i \operatorname{ctg} \frac{\beta - \varphi}{2}\right), \qquad \frac{e^{i\varphi}}{B(\zeta) - e^{i\varphi}} = -\frac{1}{2}\left(1 + i \operatorname{ctg} \frac{\beta - \varphi}{2}\right),$$

$$\frac{e^{i\varphi}B'(\zeta)}{(B(\zeta) - e^{i\varphi})^2} = \frac{B'(\zeta)}{B(\zeta)} \frac{e^{i\varphi}B(\zeta)}{(B(\zeta) - e^{i\varphi})^2} = -\frac{1}{4}\mu(\zeta)\tau(\zeta) \operatorname{cosec}^2 \frac{\beta - \varphi}{2} .$$

Hence we can write the above expression for $F'(z)$ as follows:

$$2 \sin^2 \frac{\beta - \varphi}{2} F'(\zeta) = i \sin \frac{\varphi - \beta}{2} \left(f_1'(\zeta)e^{i(\beta - \varphi)/2} + f_2'(\zeta)e^{-i(\beta - \varphi)/2}\right) + \frac{1}{2} f(\zeta)\mu(\zeta)\tau(\zeta).$$

Comparing this equality with (9) we obtain the first equality in (8), which, in view of (5), yields the second equality in (8). The proof of Lemma 1 is complete.

Remark. Methods similar to the ones used in Lemma 1 and based on various interpolation identities, were used by Bernstein [11], Szegő [12], Akhiezer, Levin [13], [14], Videnskiĭ [15], Rusak [16], Pekarskiĭ [17], and many other authors. This approach was widely used for the derivation of precise inequalities of Markov–Bernstein type for derivatives of rational, algebraic, and entire functions, in the analysis of extremal properties of Chebyshëv–Markov fractions, and in other questions.

§ 4. Consequences of Lemma 1

4.1. For ζ and β fixed as in the lemma we set $t = t(\varphi) = (\beta - \varphi)/2$. Since the equalities in (8) hold for all admissible t described in Lemma 1 and therefore, by continuity, also for all $t \neq \pi l$, the following result is a consequence of (5) and the first equality in (8).

Theorem 1. *At each regularity point ζ of the curve γ the functions (7) have the following estimate for all $t \in \mathbb{R}$:*

$$|\sin t|\,|f_1'(\zeta)e^{it} + f_2'(\zeta)e^{-it}| \leqslant \frac{|\tau(\zeta)|}{2}(|f(\zeta)| + \|f\|_\gamma)\mu(\zeta). \tag{10}$$

4.2. Setting $t = t(\varphi) = (\beta - \varphi)/2$ we isolate the real and imaginary parts of the expressions $f_1'(\zeta) = u_1 + iv_1$, $f_2'(\zeta) = u_2 + iv_2$, and

$$W(\zeta, t) := \sin t\big(f_1'(\zeta)e^{it} + f_2'(\zeta)e^{-it}\big) = U(\zeta, t) + iV(\zeta, t),$$

so that

$$\begin{aligned} 2U(\zeta,t) &= (v_1 - v_2)\cos(2t) + (u_1 + u_2)\sin(2t) + v_2 - v_1, \\ 2V(\zeta,t) &= (u_2 - u_1)\cos(2t) + (v_2 + v_1)\sin(2t) + u_1 - u_2, \\ 2|W(\zeta,t)|^2 &= -C\cos(4t) - B\sin(4t) - A\cos(2t) + 2B\sin(2t) + A + C, \end{aligned}$$

where $A = (u_1 - u_2)^2 + (v_1 - v_2)^2$, $B = u_1v_2 - u_2v_1$, $C = u_1u_2 + v_1v_2$.

Theorem 1a. *Let $\zeta \in \gamma$ be a regularity point of the curve γ. Then the quantities $W_1(\zeta), \dots, W_8(\zeta)$ defined below have the estimate $2^{-1}|\tau(\zeta)|(|f(\zeta)| + \|f\|_\gamma)\mu(\zeta)$:*

$$W_1(\zeta) = \frac{1}{2}\Big(\sqrt{(v_1 - v_2)^2 + (u_1 + u_2)^2} + |v_1 - v_2|\Big),$$

$$W_2(\zeta) = \frac{1}{2}\Big(\sqrt{(u_1 - u_2)^2 + (v_1 + v_2)^2} + |u_1 - u_2|\Big),$$

$$W_3(\zeta) = \sqrt{(u_1 - u_2)^2 + (v_1 - v_2)^2}, \qquad W_4(\zeta) = \frac{1}{\sqrt{2}}\sqrt{u_1^2 + v_1^2 + u_2^2 + v_2^2}\,,$$

$$W_5(\zeta) = \sqrt{|u_1v_2 - u_2v_1|}\,, \qquad W_6(\zeta) = \frac{1}{2}\sqrt{u_1^2 + v_1^2 + u_2^2 + v_2^2 + u_1u_2 + v_1v_2}\,,$$

$$W_7(\zeta) = \frac{1}{\sqrt{2}}\sqrt{\frac{k_2^2}{8k_1} + k_1 + A + C}\,, \qquad k_2 \leqslant 4k_1,$$

$$W_8(\zeta) = \frac{1}{\sqrt{2}}\sqrt{k_2 - k_1 + A + C}\,, \qquad k_2 \geqslant 4k_1,$$

where $k_1 = \sqrt{B^2 + C^2}$, $k_2 = \sqrt{A^2 + 4B^2}$. In particular, for all k_1 and k_2 the same estimate holds for $W_9(\zeta) = \sqrt{k_1 + A + C}/\sqrt{2}\,$.

Proof. The quantities $W_1(\zeta)$ and $W_2(\zeta)$ are equal to the maximum values (in the t-variable) of the quantities $|U(\zeta,t)|$ and $|V(\zeta,t)|$, respectively. The estimate of $W_3(\zeta)$ is a consequence of (10) and the equality $W_3(\zeta) = |W(\zeta, \pi/2)|$. The estimates of $W_{4,5,6}(\zeta)$ follow from (10), the equalities between $||W(\zeta,\pi/4)|^2 \pm |W(\zeta,-\pi/4)|^2|$

and $2W_4^2(\zeta)$ and $2W_5^2(\zeta)$, and the equality $2W_6^2(\zeta) = |W(\zeta,\pi/6)|^2 + |W(\zeta,-\pi/6)|^2$. Finally, in the estimates of $W_7(\zeta)$ and $W_8(\zeta)$ we have

$$2\max_t |W(\zeta,t)|^2 \geqslant \min_\alpha \max_\tau (k_1\cos(2\tau) + k_2\cos(\tau-\alpha) + A + C).$$

An easy analysis shows that the min max on the right-hand side is attained for $\alpha = \pi/2$, in which case the maximum on the right-hand side is easy to calculate: for the corresponding values of k_1, k_2 it is equal to $2W_7^2(\zeta)$ and $2W_8^2(\zeta)$.

4.3. Consider the domain $G = \mathbb{C}^+$ that is the open upper half-plane, and let $\{z_1,\dots,z_n\} \subset \mathbb{C}^+$ be a fixed set of points. Let

$$w(z) = \frac{z-i}{z+i};$$

then by the definition of (3), for real x we obtain

$$\tau(x) = \frac{2i}{x^2+1}\,, \qquad \mu(x) = (x^2+1)\sum_{k=1}^n \frac{\operatorname{Im} z_k}{|x-z_k|^2}\,, \qquad x\in\mathbb{R}.$$

Consider the rational functions

$$R_1(z) = \frac{P(z)}{\prod_{k=1}^n (z-z_k)}\,, \quad R_2(z) = \frac{Q(z)}{\prod_{k=1}^n (z-\overline{z_k})}\,, \qquad R(z) = R_1(z) + R_2(z),$$

where $P(z)$ and $Q(z)$ are arbitrary polynomials of degree at most n. The following result is a consequence of Theorems 1 and 1a (for $f_{1,2} = R_{1,2}$).

Theorem 2. *Let $x\in\mathbb{R}$, $u_1 = \operatorname{Re} R_1'(x)$, $v_1 = \operatorname{Im} R_1'(x)$, $u_2 = \operatorname{Re} R_2'(x)$, and $v_2 = \operatorname{Im} R_2'(x)$. Then the quantities $W_1(x),\dots,W_9(x)$ listed in Theorem 1a have the following estimate*:

$$W_m(x) \leqslant \frac{1}{2}(|R(x)| + \|R\|_{\mathbb{R}})\mu_1(x), \qquad \mu_1(x) := \sum_{k=1}^n \frac{2\operatorname{Im} z_k}{|x-z_k|^2}\,, \quad m=1,\dots,9.$$

A similar estimate of $|W(x,t)|$ holds for each real t.

The estimate of $W_3(x)$ and Rusak's result in [16] yield the inequalities

$$|R_1'(x) \pm R_2'(x)| \leqslant \|R\|_{\mathbb{R}}\mu_1(x). \tag{11}$$

Remark. In [16] Rusak obtained inequality (11) with '+' sign and pointed out that it was extremal, that is, became an equality for some rational function $R = R_1 + R_2$. See also inequalities (14). In [17], Theorem 3.1, Pekarskiĭ obtained inequalities of the form (11) for either component $R_{1,2}(z)$:

$$|R_1'(x)| \leqslant \|R\|_{\mathbb{R}} \sum \frac{2\operatorname{Im} t_k}{|x-t_k|^2}\,, \qquad |R_2'(x)| \leqslant \|R\|_{\mathbb{R}} \sum \frac{2|\operatorname{Im} t_m'|}{|x-t_m'|^2}\,,$$

where the sums are taken over all the poles $t_k\in\mathbb{C}^+$ and $t_m'\in\mathbb{C}^-$ of the functions R_1 and R_2. These inequalities produce both inequalities in (11) for an arbitrary mutual positioning of the poles of the rational functions R_1 and R_2, but with coefficient 2 on the right-hand side.

We point out that inequalities (11) have been generalized to various metrics and more general domains in [17]–[21] (where the authors obtain estimates that are sharp in order, but not extremal). For instance, estimates of type (11) have been obtained for domains G with rectifiable boundaries γ of bounded linear density [20] (that is, the length of the part of γ lying in an arbitrary disc of radius r is at most $A(\gamma)r$).

In the case of the upper half-plane one can refine the estimate of W_1 and W_2 since the coefficient $i\tau(\zeta)$ in (8) is real. Then, comparing the real (imaginary) parts in (8) we obtain

$$\begin{aligned}\bigl|R_1'(x)+\overline{R_2'(x)}\,\bigr|+\bigl|\mathrm{Im}\bigl(R_1'(x)+\overline{R_2'(x)}\,\bigr)\bigr| &\leqslant \bigl(|\mathrm{Re}\,R(x)|+\|\mathrm{Re}\,R\|_{\mathbb{R}}\bigr)\mu_1(x),\\ \bigl|R_1'(x)-\overline{R_2'(x)}\,\bigr|+\bigl|\mathrm{Re}\bigl(R_1'(x)-\overline{R_2'(x)}\,\bigr)\bigr| &\leqslant \bigl(|\mathrm{Im}\,R(x)|+\|\mathrm{Im}\,R\|_{\mathbb{R}}\bigr)\mu_1(x),\end{aligned}$$

so that for $R_2=0$ we have the inequalities

$$|R_1'(x)|+|\mathrm{Im}\,R_1'(x)| \leqslant (|\mathrm{Re}\,R_1(x)|+\|\mathrm{Re}\,R_1\|_{\mathbb{R}})\mu_1(x), \tag{12}$$
$$|R_1'(x)|+|\mathrm{Re}\,R_1'(x)| \leqslant (|\mathrm{Im}\,R_1(x)|+\|\mathrm{Im}\,R_1\|_{\mathbb{R}})\mu_1(x). \tag{13}$$

We point out that (12) becomes an identity for each fraction $R_1(z)=1/(z-z_0)$ with single pole $z_0\in\mathbb{C}^+$. The estimates (12) and (13) complement the following result of Rusak (see [16], Theorem 1):

$$|R_1'(x)| \leqslant \|\mathrm{Re}\,R_1\|_{\mathbb{R}}\mu_1(x), \qquad |R_1'(x)| \leqslant \|\mathrm{Im}\,R_1\|_{\mathbb{R}}\mu_1(x). \tag{14}$$

4.4. Let $G=g_r=\{z:\ |z|>r\}$ be the exterior of a disc of some positive radius r, and let $\{z_1,\dots,z_n\}\subset g_r$ be a fixed point set. Let $w(z)=r/z$. Then by the definition (3) we obtain

$$\tau(\zeta)=-\frac{1}{\zeta}\,, \quad \mu_2(\zeta):=\mu(\zeta)=\sum_{k=1}^{n}\frac{|z_k|^2-r^2}{|\zeta-z_k|^2}\,, \qquad |\zeta|=r.$$

For rational functions

$$R_1(z)=\frac{P(z)}{\prod_{k=1}^{n}(z-z_k)}\,, \quad R_2(z)=\frac{Q(z)}{\prod_{k=1}^{n}(r^2-z\overline{z_k})}\,, \qquad R(z)=R_1(z)+R_2(z),$$

where $P(z)$ and $Q(z)$ are arbitrary polynomials of degrees at most n, the following result is a consequence of Theorems 1 and 1a (for $f_{1,2}=R_{1,2}$).

Theorem 3. *Let* $|\zeta|{=}r$, $u_1{=}\mathrm{Re}\,R_1'(\zeta)$, $v_1{=}\mathrm{Im}\,R_1'(\zeta)$, $u_2{=}\mathrm{Re}\,R_2'(\zeta)$, $v_2{=}\mathrm{Im}\,R_2'(\zeta)$. *Then the quantities* $W_1(\zeta),\dots,W_9(\zeta)$ *listed in Theorem 1a have the following estimates*:

$$W_m(\zeta) \leqslant (2r)^{-1}(|R(\zeta)|+\|R\|_{\gamma_r})\mu_2(\zeta), \qquad m=1,\dots,9.$$

where γ_r *is the boundary of the domain* g_r. *A similar estimate of* $|W(\zeta,t)|$ *holds for each real* t.

The estimate for $W_3(x)$ and above-mentioned Rusak's inequality in [16] yield the following relations:

$$r|R_1'(\zeta) \pm R_2'(\zeta)| \leqslant \|R\|_{\gamma_r}\mu_2(\zeta). \tag{15}$$

We point out that Pekarskiĭ [17] obtained similar estimates for the components $R_{1,2}$ (see our remark to Theorem 2).

We can complement Theorem 3 as follows. Let $\theta \in (0,1]$, $M > 0$. We say that a function $R(z)$ belongs to the class $\operatorname{Lip}(M,\theta,\gamma_r)$ on the circle $\gamma_r = \{\zeta : |\zeta| = r\}$ if

$$|R(re^{it_1}) - R(re^{it_2})| \leqslant Mr^\theta \left|\sin\frac{t_1 - t_2}{2}\right|^\theta.$$

Theorem 3a. *The following estimates of a rational function of the above-indicated form $R = R_1 + R_2$ belonging to the class (M,θ,γ_r) hold for $m = 1,\dots,9$ and $\zeta \in \gamma_r$:*

$$W_m(\zeta) \leqslant \frac{M}{2r^{1-\theta}}(1 + \mu_2(\zeta))^{1-\theta/2}. \tag{16}$$

Proof. We denote the right-hand side of (8) (the last expression) by $A(\zeta,\varphi)$. Then in view of Hölder's inequality ($1/p = 1 - \theta/2$, $1/q = \theta/2$) and equalities (5)and (6), one obtains

$$\begin{aligned}
|A(\zeta,\varphi)| &\leqslant \frac{M}{2r^{1-\theta}} \sin^2\frac{\beta-\varphi}{2} \sum_{k=1}^n \frac{1}{\mu_2(\zeta_k)} \left|\operatorname{cosec}\frac{\alpha-\alpha_k}{2}\right|^{2-\theta} \\
&\leqslant \frac{M}{2r^{1-\theta}} \left(\sin^2\frac{\beta-\varphi}{2} \sum_{k=1}^n \frac{1}{\mu_2(\zeta_k)} \operatorname{cosec}^2\frac{\alpha-\alpha_k}{2}\right)^{1-\theta/2} \left(\sum_{k=1}^n \frac{1}{\mu_2(\zeta_k)}\right)^{\theta/2} \\
&\leqslant \frac{M}{2r^{1-\theta}} \mu_2^{1-\theta/2}(\zeta) \left(\frac{1+|B|}{1-|B|}\right)^{\theta/2}, \qquad B = (-1)^n \prod_{k=1}^n \frac{r}{z_k},
\end{aligned} \tag{17}$$

for $\zeta = re^{-i\alpha}$, $\zeta_k = re^{-i\alpha_k}$, where φ is arbitrary. Since φ can be arbitrary, the estimates of the quantities $W_m(\zeta)$, $m = 1,\dots,9$, proceed in the same way. If in place of R we now consider a variation of it, namely, the new rational function

$$\widetilde{R}(z) = R(z) + (z - z_0)^{-1}$$

with sufficiently large $|z_0|$, then setting $z_0 \to \infty$, by the estimates (17) for the corresponding quantities $\widetilde{W}_m(\zeta)$ one obtains (16).

§ 5. Estimates for derivatives of an s.f. on a straight line

5.1. Let $f_n(z) = \rho_n(z) + R(z)$ be a rational function such that ρ_n is an s.f. of the form (1) with set of poles $P_n = \{z_1,\dots,z_n\}$ lying in the open upper half plane $\mathbb{C}^+$, where R is an arbitrary rational function with poles in the open lower half plane $\mathbb{C}^-$, $R(\infty) \neq \infty$. We now prove an auxiliary result on the separation of the singularities of an s.f.

Lemma 2. *The following estimate holds*:

$$\|\rho_n\|_{\mathbb{R}} \leqslant (1 + \varepsilon_n)\ln n \cdot \|f_n\|_{\mathbb{R}}, \qquad n \geqslant 2, \tag{18}$$

where the positive ε_n *approach zero as* $n \to \infty$ *(the* ε_n *depend only* n*). The estimate is sharp in order in the following sense: there exists a sequence of functions of the above-indicated form* $\widetilde{f}_n(z) = \widetilde{\rho}_n(z) + \widetilde{R}_n(z)$ *such that* $\|\widetilde{\rho}_n\|_{\mathbb{R}} \geqslant 25^{-1} \ln n \cdot \|\widetilde{f}_n\|_{\mathbb{R}}$ *for integer* $n \geqslant 100$.

Recall for comparison that for arbitrary rational functions ρ_n of degree at most n in a similar problem of the separation of singularities, in place of $\ln n$ one has the coefficient n, which is the precise order (see, for instance, [17], [21]–[23]).

Proof. We set $f = f_n$, $\rho = \rho_n$ and shall assume that $\|f\| = \|f\|_{\mathbb{R}} = 1$ (we can always achieve this after a transformation of the form $f(z/\|f\|)/\|f\|$ preserving the form of the s.f.). Then, as shown in [10], Theorem 1, for the distance $\operatorname{dist}(P_n, \mathbb{R})$ of the set $P_n = \{z_1, \dots, z_n\}$ from the axis $\mathbb{R}$ we have $\operatorname{dist}(P_n, \mathbb{R}) \geqslant B \ln\ln n \cdot \ln^{-1} n$ for $n \geqslant 3$ (with some positive absolute constant B; this estimate is independent of the particular form of $R(z)$; one can assume that R is an arbitrary function in the Hardy class $H^\infty(\mathbb{C}^+)$).

In addition, it is shown in Lemma 1 of [10] that on the line $\operatorname{Im} z = -h$, $h \in (0, n)$, we have the inequality $|\rho(x - ih)| \leqslant (1/2)\ln(2en/h)$ for real x. Assume that $\delta > 0$ and let $h = n^{-1-\delta}$. Then $|\rho(x - ih)| \leqslant (1 + \delta/2)\ln(en)$, and therefore

$$
\begin{aligned}
|\rho(x)| &\leqslant |\rho(x - ih)| + |\rho(x) - \rho(x - ih)| \\
&\leqslant \left(1 + \frac{\delta}{2}\right)\ln(en) + \frac{1}{n^{1+\delta}} \sum_{k=1}^{n} \frac{1}{|x - z_k|^2} \leqslant \left(1 + \frac{\delta}{2}\right)\ln(en) + \frac{1}{n^\delta}\, \frac{\ln^2 n}{B^2 \ln^2 \ln n}\,.
\end{aligned}
$$

Selecting a sequence $\delta = \delta_n > 0$ convergent to zero sufficiently slowly so that the second term in the above majorant approaches zero as $n \to \infty$ we obtain inequality (18).

Example 1. For a corroboration of the sharpness of the order we can take for example the s.f. $\widetilde{\rho}_n$ in [10], § 4.1:

$$
\widetilde{\rho}(z) = \widetilde{\rho}_n(z) = \sum_{k=1}^{n} \frac{1}{z - ki} + \frac{b'(z)}{b(z) - a}\,, \qquad b(z) = b_n(z) = \prod_{k=1}^{n} \frac{z + ki}{z - ki}\,,
$$

where $a = a(n) = (-1)^n \ln(n+1)$, and we set $\widetilde{f}_n(z) = \widetilde{\rho}(z) + \widetilde{R}_n(z)$, $\widetilde{R}_n(z) = \overline{\widetilde{\rho}(\overline{z})}$. It is easy to see that $\widetilde{f}_n$ has the form indicated in Lemma 2 and for $n \geqslant 100$ has the following properties (for more detailed computations the reader can consult § 4.1 of [10]):

(a) the distance between its poles and the axis $\mathbb{R}$ is less that $2^{-1} \ln\ln n \cdot \ln^{-1} n$ (we do not require this property here);

(b) for $x \in \mathbb{R}$ one has

$$
|\widetilde{\rho}(0)| \geqslant \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{\ln n - 1}\left|\frac{b'(0)}{b(0)}\right| = \frac{\ln n - 3}{\ln n - 1} \sum_{k=1}^{n} \frac{1}{k} > \frac{2}{5} \ln n, \qquad n \geqslant 100,
$$

and, in addition,

$$
|\widetilde{f}_n(x)| = 2|\operatorname{Re}\widetilde{\rho}(x)| \leqslant \sum_{k=1}^{n} \frac{2|x|}{x^2 + k^2} + \frac{2}{\ln(n+1) - 1}\, \frac{|b'(x)|}{|b(x)|} \leqslant 10, \qquad x \in \mathbb{R}.
$$

The proof of Lemma 2 is complete.

5.2. We now proceed to estimates of the derivatives of an s.f. on the real axis $\mathbb{R}$.

Theorem 4. *Let $\rho_n(z)$ be an s.f. of the form (1) with set of poles $\{z_1, \dots, z_n\}$ lying in $\mathbb{C}^+$, and let R be a rational function with simple poles from the set $\{\overline{z}_1, \dots, \overline{z}_n\}$ such that $R(\infty) \neq \infty$. Then the inequalities*

$$|\rho_n'(x) \pm R'(x)| \leqslant 2\|\rho_n + R\|_{\mathbb{R}} \operatorname{Im} \rho_n(x) \tag{19}$$

hold. Moreover,

$$|\rho_n'(x)| \leqslant (|\rho_n(x)| + \|\rho_n\|_{\mathbb{R}}) \operatorname{Im} \rho_n(x), \tag{20}$$

$$|\operatorname{Im} \rho_n'(x)| \leqslant (|\operatorname{Re} \rho_n(x)| + \|\operatorname{Re} \rho_n\|_{\mathbb{R}}) \operatorname{Im} \rho_n(x), \tag{21}$$

$$|\operatorname{Re} \rho_n'(x)| \leqslant (|\operatorname{Im} \rho_n(x)| + \|\operatorname{Im} \rho_n\|_{\mathbb{R}}) \operatorname{Im} \rho_n(x), \tag{22}$$

$$|\rho_n'(x)| + |\operatorname{Im} \rho_n'(x)| \leqslant 2(|\operatorname{Re} \rho_n(x)| + \|\operatorname{Re} \rho_n\|_{\mathbb{R}}) \operatorname{Im} \rho_n(x), \tag{23}$$

$$|\rho_n'(x)| + |\operatorname{Re} \rho_n'(x)| \leqslant 2(|\operatorname{Im} \rho_n(x)| + \|\operatorname{Im} \rho_n\|_{\mathbb{R}}) \operatorname{Im} \rho_n(x). \tag{24}$$

Here relations (21) and (23) become equalities (the first relation at some point $x \in \mathbb{R}$ and the second on the whole of $\mathbb{R}$) for each s.f. $\rho(x)$ of the first degree.

Proof. Inequalities (19) follow from (11), (20), and the estimate for W_3 in Theorem 2 (for $R_1(z) = \rho_n(z)$, $R_2(z) = 0$). The same estimate yields inequalities (21) and (22) for $\overline{R_2(\overline{z})} = R_1(z) = \rho_n(z)$ and $-\overline{R_2(\overline{z})} = R_1(z) = \rho_n(z)$, respectively. Inequalities (23) and (24) follow from (12) and (13).

Theorem 4a. *Under the assumptions of Theorem 4,*

$$|\rho_n'(x) \pm R'(x)| \leqslant (2 + \varepsilon_n) \ln n \cdot \|\rho_n + R\|_{\mathbb{R}}^2, \qquad n \geqslant 2, \tag{25}$$

where the positive ε_n approach zero as $n \to \infty$ (the ε_n depend only on n). The estimate is sharp in order in the following sense: for each integer $n \geqslant 100$ there exists an s.f. $\widetilde{f}_n(z) = \widetilde{\rho}_n(z) + \widetilde{R}_n(z)$ of the form indicated in Theorem 4 such that $\|\widetilde{f}_n'\|_{\mathbb{R}} \geqslant 50^{-1} \ln n \cdot \|\widetilde{f}_n\|_{\mathbb{R}}^2$.

Proof. Inequality (25) follows by (19) and Lemma 2. For an example demonstrating the sharpness one can take the s.f. $\widetilde{\rho} = \widetilde{\rho}_n$ of Example 1 and set $\widetilde{R}_n(z) = \overline{\widetilde{\rho}(\overline{z})}$. Then

$$\widetilde{\rho}\,'(x) = \frac{b}{b-a}\left(\frac{b'}{b}\right)^2 + \frac{b}{b-a}\left(\frac{b'}{b}\right)' - \left(\frac{b'}{b}\right)^2\left(\frac{b}{b-a}\right)^2 - \sum_{k=1}^{n} \frac{1}{(x-ki)^2}\,,$$

where $a = a(n) = (-1)^n \ln(n+1)$, $b = b_n(x)$. We observe that here for $x = 0$ the absolute value of the first term on the right-hand side is greater that $4 \ln n$, the second term vanishes, and the sum of the absolute values of the last two terms is bounded by an absolute constant $A \leqslant 12$ (we bear in mind that the set of poles of the s.f. $\widetilde{\rho}_n$ is symmetric relative to the imaginary axis). Hence we obtain the lower bound

$$\frac{|\widetilde{f}_n'(0)|}{2} = |\widetilde{\rho}\,'(0)| \geqslant 4 \ln n - A > \ln n, \qquad n \geqslant 100,$$

so that in view of the estimate $\|\widetilde{f}_n\|_{\mathbb{R}} \leqslant 10$ (see Example 1), we see that the estimate (25) is sharp in order.

5.3. In the case of an arbitrary mutual positioning of the poles of ρ_n and R (having the form indicated in the beginning of § 5.1) there exists no estimate of the kind of (19), (25). This is a consequence of the following well-known fact: there exist no estimates of the Bernstein kind for derivatives of rational functions of general form (see, for instance, Dolzhenko [9]). However, if $R = \widetilde{\rho}$ is also an s.f., then such an estimate is possible.

Theorem 4b. *Let ρ and $\widetilde{\rho}$ be s.f.'s such that the set of poles of the first fraction lies in $\mathbb{C}^+$ and that of the second lies in $\mathbb{C}^-$. Then*

$$|\rho'(x)| + |\widetilde{\rho}'(x)| \leqslant A\|\rho + \widetilde{\rho}\|_{\mathbb{R}}^2 (\ln^2(en) + \ln^2(e\widetilde{n})), \tag{26}$$

where n and $\widetilde{n}$ are the degrees of the s.f.'s ρ and $\widetilde{\rho}$, respectively, and A is a positive absolute constant.

Inequality (26) is an immediate consequence of Lemma 2 and the estimate (23) applied separately to each of the fractions ρ and $\widetilde{\rho}$.

§ 6. Estimates of derivatives of an s.f. on the circle

6.1. Let $f_n(z) = \rho_n(z) + R(z)$ be a rational function, where ρ_n is an s.f. of the form (1) with set of poles $P_n = \{z_1, \dots, z_n\}$ in the domain $g_r = \{z : |z| > r\}$, $r > 0$, where R is an arbitrary rational function with poles inside the circle $\gamma_r = \{z : |z| = r\}$ such that $R(\infty) = 0$. We shall prove an analogue of Lemma 2 for circles. We set $\rho = \rho_n$, $M = \|f_n\|_{\gamma_r}$.

Lemma 3. *The following estimate holds for $n \geqslant n_0(rM)$:*

$$\|\rho_n\|_{\gamma_r} \leqslant 3 \ln n \cdot M + \frac{1}{nr}\,. \tag{27}$$

Here one can set $n_0(x) = 10^3(x^2 + 1)$.

Proof. We shall use in the proof the following inequality from Theorem 6 in [10] for the distance $\operatorname{dist}(P_n, \gamma_r)$ between the set $P_n = \{z_1, \dots, z_n\}$ and the circle γ_r:

$$\operatorname{dist}(P_n, \gamma_r) \geqslant \frac{1}{2}\,\frac{r}{n+1}\biggl(\ln \frac{n+1}{1 + 2Mr\ln(3n)} - 2\biggr), \tag{28}$$

provided that $n \geqslant 4(r^2M^2 + 1)$. Hence it is easy to see that $\operatorname{dist}(P_n, \gamma_r) \geqslant r/n$ for $n \geqslant n_0(rM) = 10^3(r^2M^2 + 1)$.

In the case of $R(\infty) = 0$ it follows by Cauchy's integral formula that $|\rho(0)| \leqslant M$ and $|\rho'(z)| \leqslant Mr/(r^2 - |z|^2)$, where $|z| < r$. Integrating the last inequality over the radial interval $L_n = [0, a_n]$, $a_n = r(1 - n^{-4})e^{it}$ for some fixed $t \in \mathbb{R}$ we obtain

$$|\rho(z)| \leqslant |\rho(0)| + \frac{M}{2}\ln\frac{r + |z|}{r - |z|}\,, \quad z \in L_n, \qquad |\rho(a_n)| \leqslant M + 2M\ln(2n).$$

Hence taking account of the inequality $\operatorname{dist}(P_n, \gamma_r) > r/n$, $n \geqslant n_0$, we obtain

$$|\rho(e^{it})| \leqslant |\rho(a_n)| + |\rho(e^{it}) - \rho(a_n)| \leqslant M(1 + 2\ln(2n)) + \frac{1}{nr}\,,$$

which proves (27) since t can be arbitrary.

Remark. The estimate (28) is independent of the form of the function $R(z)$; we can assume that R is an arbitrary function in the Hardy class $H^\infty(g_r)$ such that $R(\infty) = 0$ (see [10]). It is easy to see from (28) that for all positive integers n we have

$$\operatorname{dist}(P_n, \gamma_r) \geqslant a(M, r)\frac{\ln n}{n}, \tag{29}$$

where the positive quantity $a(M, r)$ depends only on $M = \|f_n\|_{\gamma_r}$ and r.

Inequalities of the type of (28), (29), and therefore ones similar to (27) hold also if one replaces g_r by the exterior $G(\gamma)$ of a Jordan curve γ with the generalized Lyapunov property:

$$\operatorname{dist}(P_n, \gamma) \geqslant a_1(M, \gamma)\frac{\ln n}{n}, \qquad \|\rho_n\|_\gamma \leqslant a_2(M, \gamma)\ln n, \tag{30}$$

where the positive quantities $a_{1,2}(M, \gamma)$ depend only on M and γ. Recall that by the *generalized Lyapunov property* of a smooth curve one means that the modulus of continuity $\omega(r)$ of the argument of its tangent satisfies the Dini condition as a function of arc-length s: $\displaystyle\int_0 \frac{\omega(s)}{s}\,ds < \infty$.

In fact, let $z = \psi(w)$ be a conformal univalent map of the exterior of the unit disc $\{g_1 : |w| > 1\}$ onto $G(\gamma)$ such that $\psi(\infty) = \infty$. By a result of Warshawski [24],

$$0 < A_1(\gamma) \leqslant |\psi'(w)| \leqslant A_2(\gamma) < \infty, \qquad w \in g_1. \tag{31}$$

We observe that

$$\psi'(w)f_n(\psi(w)) = \psi'(w)R(\psi(w)) + \sum_{k=1}^{n}\frac{\psi'(w)}{\psi(w) - z_k} = F(w) + \sum_{k=1}^{n}\frac{1}{w - w_k},$$

where $z_k = \psi(w_k)$, and $F(w)$ is a function in the class $H^\infty(g_1)$ such that $F(\infty) = 0$. It follows from (29) and (31) that $|w_k| - 1 \geqslant a_3(M, \gamma)n^{-1}\ln n$ for all k. Hence (31) yields the first inequality in (30), while the proof of the second is perfectly similar to the proof of Lemma 2.

6.2. The following results hold.

Theorem 5. *For $r > 0$ let $\rho_n(z)$ be an s.f. of the form (1) with set of poles $\{z_1, \dots, z_n\}$ lying in the exterior of the circle γ_r, and let R be a rational function with simple poles belonging to the set $\{r^2/\overline{z_1}, \dots, r^2/\overline{z_n}\}$ such that $R(\infty) \neq \infty$. Then the two inequalities*

$$\|\rho_n' \pm R'\|_{\gamma_r} \leqslant \|f_n\|_{\gamma_r}(nr^{-1} + 2\|\rho_n\|_{\gamma_r}), \qquad f_n = \rho_n + R \tag{32}$$

hold.

This result is a consequence of (15) and the equality

$$\mu_2(\zeta) = \operatorname{Re}\biggl(\sum_{k=1}^{n}\frac{z_k + \zeta}{z_k - \zeta}\biggr) = n - 2\operatorname{Re}(\zeta\rho_n(\zeta)).$$

Theorem 5a. *Under the assumptions of Theorem 5, for $R(\infty)=0$ one has*

$$\|\rho_n' \pm R'\|_{\gamma_r} \leqslant n\|f_n\|_{\gamma_r}(r^{-1}+\varepsilon_n), \tag{33}$$

where $\varepsilon_n = 6n^{-1}(r^{-1}+\ln n\cdot\|f_n\|_{\gamma_r})$ and $n \geqslant n_0(r\|f_n\|_{\gamma_r})$, and the quantity n_0 is defined in Lemma 3. The estimate (33) is sharp in order in the following sense: for an arbitrary integer $n \geqslant 2$ and arbitrary positive r there exists an s.f. $\widetilde{\rho}_n$ of degree n with poles in g_r such that $\|\widetilde{\rho}_n\|_{\gamma_r} \leqslant 1$ and $\|\widetilde{\rho}_n'\|_{\gamma_r} \geqslant r^{-1}(n-1)$.

Proof. Inequality (33) is a consequence of (27) and (32). For an example of an s.f. one can take $\widetilde{\rho}_n = nz^{n-1}(z^n - Ar^n)^{-1}$ with $A = 1+n/r$. Then calculations yield

$$\|\widetilde{\rho}_n\|_{\gamma_r} \leqslant \frac{n}{r(A-1)} = 1, \qquad |\widetilde{\rho}_n'(r)| = \frac{n}{r^2}\,\frac{(n-1)A+1}{(A-1)^2} > \frac{n-1}{r}\,,$$

which proves the second part of Theorem 5a concerning the sharpness of the estimate.

§ 7. Estimates of derivatives of s.f. on an interval

Assume that all the poles of an s.f. $\rho(z)=\rho_n(z)$ of the form (1) lie outside an interval $[-a,a]$, $a>0$. We shall assume in addition that $\rho(x)$, $x\in\mathbb{R}$, takes only real values, so that the set of poles z_k is symmetric relative to the real axis. Let $\|\rho\|^*_{[-a,a]} = \max_{x\in[-a,a]}|\sqrt{a^2-x^2}\,\rho(x)|$.

Theorem 6. *Let $\rho(x)=\rho_n(x)$ be a real-valued s.f. Then for $n \geqslant n_0(\|\rho\|^*_{[-a,a]})$ (the value of n_0 was defined in Lemma 3),*

$$(a^2-x^2)|\rho'(x)| \leqslant |x\rho(x)| + n\|\rho\|^*_{[-a,a]}(1+\varepsilon_n), \tag{34}$$

where $\varepsilon_n = 6n^{-1}\ln n\cdot(1+\|\rho\|^*_{[-a,a]})$. The estimate (34) is sharp in order in the following sense: for all $a>0$ and $n\in\mathbb{N}$ there exists an s.f. $\widetilde{\rho}(x)=\widetilde{\rho}_n(x)$ of the above-indicated form such that

$$\|\widetilde{\rho}\|^*_{[-a,a]} \leqslant 1, \qquad \max_{x\in[-a,a]}|(a^2-x^2)\widetilde{\rho}\,'(x) - x\widetilde{\rho}(x)| \geqslant n. \tag{35}$$

Proof. It is sufficient to prove (34) for $a=1$. In fact, for $a>0$ and $x=at$, $t\in[-1,1]$, $x\in[-a,a]$, we can define an s.f. ρ_0 by the equality $\rho_0(t)=a\rho(at)$. Then $(1-t^2)|\rho_0'(t)| = (a^2-x^2)|\rho'(x)|$, $t\rho_0(t)=x\rho(x)$, $\sqrt{1-t^2}\,\rho_0(t)=\sqrt{a^2-x^2}\,\rho(x)$. Thus, if (34) holds for ρ_0 with $a=1$, then it holds also for $\rho(x)$ with arbitrary $a>0$.

Making the change of variable $z=(w+1/w)/2$, $z_k=(w_k+1/w_k)/2$, where we assume for definiteness that $|w_k|>1$, we verify directly that

$$\rho(z) = \frac{2w}{w^2-1}\,F(w), \qquad F(w) = \sum_{k=1}^{n}\biggl(\frac{1}{ww_k-1}+\frac{w}{w-w_k}\biggr), \tag{36}$$

$$(z^2-1)\rho'(z)+z\rho(z) = wF'(w), \qquad z=\frac{w+1/w}{2}; \tag{37}$$

here each $z \neq \pm 1$ corresponds to two distinct values of w with product 1. By (36) and the equality $\sqrt{1-x^2} = |v^2-1|/|2v|$, which holds, for $x \in [-1,1]$ and $w = v = e^{it}$, $t \in \mathbb{R}$,

$$|F(v)| = \sqrt{1-x^2}\,|\rho(x)| \leqslant \|\rho\|^*_{[-1,1]}, \qquad x = \frac{v+1/v}{2}\,.$$

We observe that the set of points w_k is symmetric relative to the real axis, therefore the set of poles of $F(w)$ is symmetric relative to the unit circle. Hence by (37), (15), and the last inequality we obtain

$$(1-x^2)|\rho'(x)| \leqslant |x\rho(x)| + |F'(v)| \leqslant |x\rho(x)| + \|F\|_{\gamma_1}\mu_2(v) \leqslant |x\rho(x)| + \|\rho\|^*_{[-1,1]}\mu_2(v),$$
$$\mu_2(v) = \sum_{k=1}^{n} \frac{|w_k|^2-1}{|v-w_k|^2}\,, \tag{38}$$

for $x = (v+1/v)/2$. We now take into account the inequality

$$\mu_2(v) = n - 2\,\mathrm{Re}(v\mathscr{R}(v)) \leqslant n + 2\|\mathscr{R}\|_{\gamma_1}, \qquad \mathscr{R}(v) = \sum_{k=1}^{n} \frac{1}{v-w_k}\,.$$

By Lemma 3, $|\mathscr{R}(v)| \leqslant 3\ln n \cdot \|F\|_{\gamma_1} + 1/n \leqslant 3\ln n \cdot (1 + \|\rho\|^*_{[-1,1]})$, therefore $\|\mu_2\|_{\gamma_1} \leqslant n(1+\varepsilon_n)$ with $\varepsilon_n = 6n^{-1}\ln n \cdot (1+\|\rho\|^*_{[-1,1]})$ for $n \geqslant n_0(\|F\|_{\gamma_1})$, which in combination with (38) proves inequality (34).

Example 2. It is sufficient to present an example substantiating the sharpness of the estimate (34) with $a = 1$. Let $n \in \mathbb{N}$ and let w_k, $k = 1,\dots,n$, be the roots of the equation $w^n - A = 0$ for some $A > 1$. Then from (36) we obtain

$$\widetilde{\rho}(z) = \widetilde{\rho}_n(z) = \frac{2w}{w^2-1}\widetilde{F}(w) = \frac{2nw}{w^2-1}\,\frac{A(w^{2n}-1)}{(w^nA-1)(w^n-A)}\,. \tag{39}$$

Let A be a solution of the equation $2An(A-1)^{-2} = 1$. Then for $w = v$, $|w| = |v| = Z1$, $x \in (-1,1)$, taking account of the equality $\sqrt{1-x^2} = |v^2-1|/2$ we obtain

$$|\widetilde{\rho}(x)| \leqslant \frac{2An}{(A-1)^2}\min\left\{\frac{1}{\sqrt{1-x^2}}\,, n\right\}, \qquad \|\widetilde{\rho}\|^*_{\gamma_1} \leqslant \frac{2An}{(A-1)^2} = 1. \tag{40}$$

Simple calculations for $|v| = 1$ yield (see (37))

$$(1-x^2)\widetilde{\rho}\,'(x) - x\widetilde{\rho}(x) = -v\widetilde{F}'(v) = 2An^2\,\mathrm{Re}\left(\frac{v^n}{(v^n-A)^2}\right).$$

At the points v such that $v^n = 1$, the last expression is equal to $2An^2(A-1)^{-2} = n$; comparing this with (40) we obtain (35) (for $a = 1$).

We point out that from (39), using the substitution $w = e^{it}$, $t = \cos x$, one obtains a representation of the s.f. $\widetilde{\rho}(x)$ in terms of the Chebyshëv polynomials $T_n(x) = \cos n\cos^{-1}x$:

$$\widetilde{\rho}(x) = 2n\frac{T_{n-1}(x) - T_{n+1}(x)}{2T_n(x) - T_{n-2}(x) - T_{n+2}(x) + (A+1/A)(T_2(x)-1)}\,, \qquad n \geqslant 2$$

(a reducible fraction).

§ 8. Several additional properties of s.f.'s and their generalizations

8.1. Zolotarev and Chebyshëv problems for s.f.'s. An *analogue of the Zolotarev problem for s.f.'s* can be stated as follows. Let $\delta \in (0, 1/2)$, $\Delta_1 = [-1+\delta, -\delta]$, $\Delta_2 = [\delta, 1-\delta]$. For an s.f. ρ of the form (1) we set $m_\delta(\rho) = \min\{|\rho(x)| : x \in \Delta_1\}$ and

$$\lambda_n(\delta) = \sup\left\{\frac{m_\delta(\rho)}{\|\rho\|_{\Delta_2}} : \deg \rho \leqslant n\right\}, \qquad n = 1, 2, \dots, \tag{41}$$

where one takes the sup over all the s.f.'s (1) of degree at most n. One must find the precise growth order of the quantities $\lambda_n(\delta)$.

As Example 2 shows, for each fixed δ, $\lambda_n(\delta)$ grows more rapidly than each power n^α, $\alpha > 1$. In fact, let $A = n^\alpha$. The s.f. $\widetilde{\rho}(x)$ in Example 2 has the following properties. Its poles lie on the ellipse

$$z = \frac{1}{2}\left(a + \frac{1}{a}\right)\cos t + \frac{i}{2}\left(a - \frac{1}{a}\right)\sin t, \qquad t \in [0, 2\pi], \quad a = a_n = \sqrt[n]{A}\,. \tag{42}$$

Hence they are located in the $((a - 1/a)/2)$-neighbourhood of the interval $[-1, 1]$, and moreover,

$$\frac{1}{2}\left(a - \frac{1}{a}\right) \leqslant b_n = 2\alpha\frac{\ln n}{n}$$

provided that $b_n \leqslant 2$ (the last relation holds, for instance, if $n \geqslant e^\alpha$). In addition (see (40)), $\sqrt{1-x^2}\,|\widetilde{\rho}(x)| \leqslant 3n^{1-\alpha}$ for $x \in [-1, 1]$ and $n \geqslant n_0(\alpha)$.

Then the s.f. $\rho_0(x) = 2\widetilde{\rho}(2x - 1)$ has similar properties with respect to the interval $[0, 1]$: all its poles lie in the $(\alpha(\ln n)/n)$-neighbourhood of $[0, 1]$ and, in addition, $\|\rho_0\|_{\Delta_2} \leqslant 6n^{1-\alpha}(\delta(1-\delta))^{-1/2}$. It follows by the first property that for sufficiently large n the function $|\rho_0(x)|$ increases on $[-1, -\delta]$ and attains its minimum at $x = -1$; moreover, $|\rho_0(-1)| > n/2$. Thus, for each $\delta \in (0, 1/2)$ and $n \geqslant n_1(\alpha) \geqslant n_0(\alpha)$ we have $\lambda_n(\delta) \geqslant 12^{-1}(\delta(1-\delta))^{1/2} n^\alpha$.

An *analogue of the Chebyshëv problem for s.f.'s* can be stated as follows. Find an s.f. of degree n having the least deviation from zero on $[-1, 1]$ in the norm $\|\cdot\|^*_{[-1,1]}$ among all the s.f.'s ρ_n of the form (1) with distance $d(\rho_n)$ from the set of poles to the interval $[-1, 1]$ not exceeding 1. Example 2 shows that for $A = 2^n$ all the poles of $\widetilde{\rho}_n(x)$ lie in the $\frac{3}{4}$-neighbourhood of the interval $[-1, 1]$, and moreover, $\|\widetilde{\rho}\|^*_{[-1,1]} \leqslant 2^{n+1}n(2^n - 1)^{-2} \asymp n2^{-n+1}$. Is this the precise order? Some lower bounds for the least deviation in question under the assumption $d(\rho_n) \leqslant 1$ were obtained in [25]. It is shown there, for instance, that if $\|\rho_n\|_{[-1,1]} \leqslant b^{-n-1}$ for some $b > 2$, then all the poles of the s.f. ρ_n lie outside the ellipse of the form (42) with $a = b/2$.

8.2. Approximation properties of s.f.'s and polynomials have much in common (see [1]–[4]). For instance, one has an analogue of Mergelyan's theorem on uniform approximation by simplest fractions of complex-valued functions $f(z)$ on compact subsets E of the complex plane.

Theorem [1], [2]. *A function $f(z)$ that is continuous on a compact subset E of $\mathbb{C}$ with connected complement and analytic at its interior points can be uniformly approximated on E to an arbitrary accuracy by simplest fractions.*

We say that a compact subset E of $\mathbb{C}$ *is of class* $\mathscr{A}$ if it separates no points in the plane and one can connect two arbitrary points of it by a rectifiable curve of length at most A lying in E; here $A = A(E)$ is a finite quantity. Let $\mathscr{R}_n = \mathscr{R}_n(f, E)$ and $\mathscr{E}_n = \mathscr{E}_n(f, E)$ be the smallest uniform deviations on E of the function f from the sets of s.f.'s and polynomials of degree at most n, respectively. As shown in [1] and [2], if $E \in \mathscr{A}$, then $\mathscr{R}_{[n \ln(1/\mathscr{E}_n)]} < C\mathscr{E}_n$, $C = C(f, E)$, $n \geqslant n_0(f, E)$. Let f be a complex-valued function satisfying the assumptions of Mergelyan's theorem on E. For fixed $b \in E \in \mathscr{A}$ we set $\alpha(f; z) = \displaystyle\int_b^z f(t)\,dt$, where the integral is taken over a rectifiable curve in E joining b to $z \in E$. Kosukhin [4] has shown that the deviations $\mathscr{R}_{n+1}(f, E)$ and $\mathscr{E}_n(fe^{\alpha(f;\,\cdot\,)}, E)$ are weakly equivalent on the above-defined class $\mathscr{A}$ of compact sets E: $\mathscr{R}_{n+1}(f, E) \asymp \mathscr{E}_n(fe^{\alpha(f;\,\cdot\,)}, E)$.

Consider now special fractions of the following form:

$$\theta(z) = \frac{\rho_{1,n_1}(z) - \rho_{2,n_2}(z)}{\rho_{3,n_3}(z) - \rho_{4,n_4}(z)}\,, \tag{43}$$

where the $\rho_{s,n_s}(z)$, $s = 1, \dots, 4$, are s.f.'s of the form (1) and of degree at most n_s. This slightly more complicated form of fractions results in significantly stronger approximation properties. We have the following result.

Theorem 7 [6]. *Let E be an arbitrary compact set, $R(z)$ a rational function of degree $n \geqslant 1$, and $r = \|R\|_E < \infty$. Then for $p \geqslant 5r$ there exists a fraction of the form* (43) *with degrees n_1, n_2, n_3, n_4 not exceeding pn such that $\|\theta - R\|_E \leqslant 2e^r r^{p+1}/p!$.*

For a positive integer p we set

$$q_p(z) = \sum_{k=0}^{p} \frac{1}{k!}\, R^k(z), \qquad \delta_p(z) = R'(z) - \frac{q_p'(z)}{q_p(z)}\,.$$

Lemma 4 [2]**.** *For $z \in E$ and $p \geqslant 5r$ one has $p!\,|\delta_p(z)| \leqslant |R'(z)| 2e^r r^p$.*

In fact, $\delta_p(z) = R'(z)R^p(z)/(p!\,q_p(z))$; furthermore, for $z \in E$ and $p \geqslant 5r$ we have

$$|q_p(z)| \geqslant e^{-r} - \sum_{k=p+1}^{\infty} \frac{|R(z)|^k}{k!} \geqslant e^{-r} - \frac{r^{p+1}}{(p+1)!}\left(1 + \frac{re^r}{p+1}\right) \geqslant \frac{e^{-r}}{2}\,, \qquad r = \|R\|_E.$$

Proof of Theorem 7. Let $\theta(z) = (q_p'(z)/q_p(z))/(R'(z)/R(z))$. Then taking account of Lemma 4 we obtain

$$R(z) = \theta(z) + \delta_p(z)\frac{R(z)}{R'(z)}\,, \qquad |\delta_p(z)|\left|\frac{R(z)}{R'(z)}\right| \leqslant 2e^r\,\frac{r^{p+1}}{p!}\,.$$

It follows from Theorem 7 that if for some function f its best uniform approximations $R_n = R_n(f, E)$ on E by rational functions of degree at most n decrease to zero as $n \to \infty$, then for its best uniform approximations $\Theta_n = \Theta_n(f, E)$ by fractions of the form (43) of degree at most n we have the estimate $\Theta_{[n \ln(1/R_n)]} \leqslant C(f) R_n$, $n \geqslant n_0(f)$. For instance, for the best uniform approximations $\Theta_n(|x|, [-1, 1])$

of $|x|$ on the interval $[-1,1]$ we obtain $\Theta_n(|x|,[-1,1]) \leqslant C\exp(-\sqrt[3]{n}\,)$. Recall for comparison that the corresponding rational approximations by polynomials have order $1/n$ (Bernstein), and the ones by rational functions of general form have order $\exp(-\pi\sqrt{n}\,)$ (Newman [26], Bulanov [27], Vyacheslavov [28]).

8.3. We now present an application of approximation by s.f.'s to numerical differentiation of analytic functions. The corresponding results are a joint work [7]. Let $z_0 \in D$ (the unit disc) and let s and n be fixed positive integers. For integer $p \geqslant 1$ we set

$$\alpha(z) = -\frac{1}{s}\,\frac{1}{(z-z_0)^s}\,, \qquad A = A_s(z_0) = \frac{1}{s}\,\frac{1}{(1-|z_0|)^s}\,, \qquad q(z) = \sum_{k=0}^{p} \frac{\alpha^k(z)}{k!}\,.$$

Then for $z \in \gamma = \{z : |z| = 1\}$ and $p \geqslant 5A$, by Lemma 4 we obtain

$$\frac{1}{(z-z_0)^{s+1}} - \frac{q'(z)}{q(z)} = \frac{1}{p!\,s^p}\,\frac{(-1)^p}{(z-z_0)^{sp+s+1}}\,\frac{1}{q(z)}\,, \qquad |q(z)| \geqslant \frac{e^{-A}}{2}\,.$$

We point out that it follows, in particular, by a similar estimate for $q(z)$ with $\alpha(z) = z$ that all the roots z_m of the equation $\sum_{k=0}^{p} z^k/k! = 0$ satisfy the inequality $|z_m| \geqslant p/5$, $m = 1,\dots,p$. It is also easy to show (on the basis of Rouché's theorem) that $|z_m| \leqslant 2p$.

Thus,

$$\int_\gamma \biggl|\frac{1}{(z-z_0)^{s+1}} - \frac{q'(z)}{q(z)}\biggr|\,|dz| \leqslant \frac{2e^{A+p}}{(ps)^p} \int_\gamma \frac{|dz|}{|z-z_0|^{sp+s+1}} \leqslant 4\pi e^{A+p} As\biggl(\frac{A}{p}\biggr)^p. \tag{44}$$

The estimate of the last integral I proceeds as follows (we set $\beta = (sp+s+1)/2 \geqslant 1$ and $b = |z_0|$):

$$\begin{aligned} I &= 2\int_0^\pi \frac{d\varphi}{(1-2b\cos\varphi + b^2)^\beta} = \int_0^\infty \frac{4(1+t^2)^{\beta-1}}{((1-b)^2(1+t^2)+4bt^2)^\beta}\,dt \\ &= \frac{4}{(1-b)^{2(\beta-1)}} \int_0^\infty \frac{((1-b)^2+(1-b)^2t^2)^{\beta-1}}{((1-b)^2+(1+b)^2t^2)^\beta}\,dt \\ &\leqslant \frac{4}{(1-b)^{2(\beta-1)}} \int_0^\infty \frac{((1-b)^2+(1+b^2)t^2)^{\beta-1}}{((1-b)^2+(1+b)^2t^2)^\beta}\,dt = \frac{2\pi}{(1-b)^{2\beta-1}(1+b)}\,. \end{aligned}$$

Next, we have $q'(z)/q(z) = \sum_{k=1}^{ps}(z-\zeta_k)^{-1} - ps(z-z_0)^{-1}$, where the $\zeta_k = z_0 + \tau_k^{-1}$ are the zeros of $q(z)$ and the τ_k are the roots of the equation (in t)

$$\sum_{m=0}^{p} \frac{(-1)^m}{m!}\biggl(\frac{t^s}{s}\biggr)^m = 0. \tag{45}$$

Let f be a holomorphic function in D. If all the points ζ_k lie in D, then for sufficiently small positive ε we have

$$\frac{1}{s!}\,f^{(s)}(z_0) + psf(z_0) - \sum_{k=1}^{ps} f(\zeta_k) = \frac{1}{2\pi i}\int_{|z|=1-\varepsilon} f(z)\biggl(\frac{1}{(z-z_0)^{s+1}} - \frac{q'(z)}{q(z)}\biggr)\,dz.$$

Hence (44) yields the following result.

Theorem 8. *Let f be a holomorphic function in the unit disc D; $\|f\|=\|f\|_D<\infty$. Then at points $z_0\in D$, for positive integers s and $p>5A=5s^{-1}(1-|z_0|)^{-s}$ one has*

$$\frac{1}{s!}\,f^{(s)}(z_0)\approx -psf(z_0)+\sum_{k=1}^{ps}f(\zeta_k)$$

with the following error bound:

$$\left|\frac{1}{s!}\,f^{(s)}(z_0)+psf(z_0)-\sum_{k=1}^{ps}f(\zeta_k)\right|\leqslant 2\|f\|e^{p+A}As\left(\frac{A}{p}\right)^{p},$$

where the $\zeta_k=z_0+\tau_k^{-1}\in D$ and the τ_k are the roots of equation (45), *and where*

$$\left(\frac{ps}{5}\right)^{1/s}\leqslant|\tau_k|\leqslant(2ps)^{1/s}.$$

8.4. We now present another application of the estimates (14). Consider the fraction

$$R(z)=\sum_{k=1}^{n}A_k(z-a_k)^{-1}, \tag{46}$$

where the A_k are arbitrary quantities and $a_k\in\mathbb{C}^+$. Then for real positive λ,

$$V:=\sum_{k=1}^{n}A_ke^{i\lambda a_k}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}e^{i\lambda x}R(x)\,dx=\frac{1}{2\pi\lambda}\int_{-\infty}^{\infty}e^{i\lambda x}R'(x)\,dx.$$

Using (14) we now obtain

$$|V|\leqslant\frac{\|\operatorname{Re}R\|_{\mathbb{R}}}{2\pi\lambda}\int_{-\infty}^{\infty}\mu_1(x)\,dx=\frac{n}{\pi\lambda}\,\|\operatorname{Re}R\|_{\mathbb{R}}. \tag{47}$$

One can also obtain the same estimates with $\operatorname{Re}R(x)$ replaced by $\operatorname{Im}R(x)$.

We use inequality (47) for the estimate of the rate of decrease as $x\to+\infty$ of solutions $v(x)$ of the equation $v^{(n)}+c_{n-1}v^{(n-1)}+\dots+c_0v=0$ with constant coefficients c_k, provided that the roots z_k, $k=1,\dots,n$, of the characteristic polynomial $P(z)=z^n+c_{n-1}z^{n-1}+\dots+c_0$ lie in the left half-plane. For compactness we assume that all the roots z_k are simple. For fixed quantities $A_1,\dots,A_n$ we consider the fractions

$$R_0(z)=R_0(\{A_k\},z)=\sum_{k=1}^{n}\frac{A_k}{z-z_k}\,,\qquad r_0(y)=\operatorname{Im}R_0(iy),$$

and set $\|r_0\|=\max\{|r_0(y)|:-\infty<y<\infty\}$.

Theorem 9. *The estimate $|v(x)|\leqslant n\|r_0\|x^{-1}$ holds for $x>0$ for solutions of the form $v(x)=\sum_{k=1}^{n}A_ke^{xz_k}$, $\operatorname{Re}z_k<0$. A similar estimate with $r_0(y)=\operatorname{Re}R_0(iy)$ in place of $r_0(y)=\operatorname{Re}R_0(iy)$ also holds.*

In fact, it is sufficient to consider in (46) the fraction $R(z)$ with poles $a_k = -iz_k$ lying in $\mathbb{C}^+$. Since $\operatorname{Re} R(x) = -r_0(x)$, the required bounds are consequences of (47).

Bibliography

[1] V. I. Danchenko and D. Ya. Danchenko, "Uniform approximation by logarithmic derivatives of polynomials", *Function theory, its applications, and related questions. In celebration of the* 130*th birthday of D. F. Egorov* (Kazan', September 1999), Kazan' Mathematical Society, Kazan' 1999, pp. 74–77. (Russian)

[2] V. I. Danchenko and D. Ya. Danchenko, "Approximation by simple fractions", *Mat. Zametki* **70**:4 (2001), 553–559; English transl. in *Math. Notes* **70**:4 (2001), 502–507.

[3] E. P. Dolzhenko, "Simple fractions", *Function theory, its applications, and related questions* (Kazan', June–July 2001), DAS, Kazan' 2001, pp. 90–94.

[4] O. N. Kosukhin, "Approximation properties of most simple fractions", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **2001**:4, 54–59; English transl. in *Moscow Univ. Math. Bull.* **56**:4 (2001), 36-40.

[5] P. A. Borodin and O. N. Kosukhin, "Approximation by simplest fractions on the real axis", *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* **2005**:1, 3–9; English transl. in *Moscow Univ. Math. Bull.* **60**:1 (2005), 1–6.

[6] V. I. Danchenko and D. Ya. Danchenko, "Approximation by fractions of special form", *Modern methods of function theory and related problems. Voronezh Winter Mathematical School* (*Voronezh, January–February*, 2003), Voronezh State University 2003, pp. 86–87. (Russian)

[7] V. I. Danchenko and D. Ya. Danchenko, "Numerical differentiation of analytic functions", *Contemporary problems of function theory and their applications*, 12th Saratov Winter School. Abstracts of lectures (Saratov, January–February 2004), Educational and Research Centre "Kolledzh", Saratov 2004, pp. 66–67. (Russian)

[8] V. I. Danchenko, "Estimates of derivatives of simplest fractions on a circle and a line interval", *Trudy Mat. Tsentra im. Lobachevskogo* **9** (2003), 84–88. (Russian)

[9] E. P. Dolzhenko, "Estimates of derivatives of rational functions", *Izv. Akad. Nauk SSSR Ser. Mat.* **27**:1 (1963), 9–28. (Russian)

[10] V. I. Danchenko, "Estimates of the distances from the poles of logarithmic derivatives of polynomials to lines and circles", *Mat. Sb.* **185**:8 (1994), 63–80; English transl. in *Russian Acad. Sci. Sb. Math.* **82**:2 (1995), 425–440.

[11] S. N. Bernstein, *Extremal properties of polynomials and best approximation of continuous functions of one real variable*, Part 1, ONTI, Leningrad 1937. (Russian)

[12] G. Szegő, "Über einen Satz des Herrn Serge Bernstein", *Schriften Königsberg*, 1928, no. 5, 59–70.

[13] N. I. Akhiezer, *Theory of approximation*, Fizmatlit, Moscow 1965; English transl. of 2nd ed., Frederick Ungar, New York 1956.

[14] N. I. Akhiezer and B. Ya. Levin, "Generalization of S. N. Bernstein's inequality to derivatives of entire functions", *Studies in modern problems of theory of functions of a complex variable*, GIFML, Moscow 1960, pp. 111–165. (Russian)

[15] V. S. Videnskiĭ, "Some estimates of derivatives of rational fractions", *Izv. Akad. Nauk SSSR Ser. Mat.* **26**:3 (1962), 415–426. (Russian)

[16] V. N. Rusak, *Rational functions as approximation machinery*, Belorussia State University, Minsk 1979. (Russian)

[17] A. A. Pekarskiĭ, "Estimates of the derivative of a Cauchy-type integral with meromorphic density and their applications", *Mat. Zametki* **31**:3 (1982), 389–402; English transl. in *Math. Notes* **31**:3 (1982), 199-206.

[18] A. A. Pekarskiĭ, "Estimates for higher derivatives of rational functions and their applications", *Izv. Akad. Nauk BSSR Ser. Fiz.-Mat. Nauk* **1980**:5, 21–28. (Russian)

[19] A. A. Pekarskiĭ, "Estimates of the derivatives of rational functions in $L_p[-1,1]$", *Mat. Zametki* **39**:3 (1986), 388–394; English transl. in *Math Notes* **39**:3 (1986), 212–216.

[20] V. I. Danchenko, "Several integral estimates of the derivatives of rational functions on sets of finite density", *Mat. Sb.* **187**:10 (1996), 33–52; English transl. in *Sb. Math.* **187**:10 (1996), 1443-1463.

[21] V. I. Danchenko, "On rational components of meromorphic functions and their derivatives", *Anal. Math.* **16**:4 (1990), 241–255.

[22] L. D. Grigoryan, "Estimates of the norm of the holomorphic components of functions meromorphic in domains with a smooth boundary", *Mat. Sb.* **100(142)**:1 (1976), 156–164; English transl. in *Math. USSR-Sb.* **29** (1978), 139-146.

[23] V. I. Danchenko, "On separation of singularities of meromorphic functions", *Mat. Sb.* **125(167)**:2 (1984), 181–198; English transl. in *Math. USSR-Sb.* **53**:1 (1986), 183–201.

[24] S. E. Warschawski, "On differentiability at the boundary in conformal mapping", *Proc. Amer. Math. Soc.* **12**:4 (1961), 614–620.

[25] D. Ya. Danchenko, "Some question of approximation and interpolation by rational functions. Application to equations of elliptic type", *Kandidat Thesis*, Vladimir State Pedagogical University, Vladimir 2001. (Russian)

[26] D. J. Newman, "Rational approximation to $|x|$", *Michigan Math. J.* **11**:1 (1964), 11–14.

[27] A. P. Bulanov, "Asymptotic behavior of the least deviations of the function $\operatorname{sign} x$ from rational functions", *Mat. Sb.* **96(138)**:2 (1975), 171–188; English transl. in *Math. USSR-Sb.* **25**:2 (1975), 159–176.

[28] N. S. Vyacheslavov, "On the uniform approximation of $|x|$ by rational functions", *Dokl. Akad. Nauk SSSR* **220**:3 (1975), 512–515; English transl. in *Soviet Math. Dokl.* **16** (1975), 100-104.

V. I. Danchenko
Vladimir State University
E-mail: danch@vpti.vladimir.ru

Received 23/AUG/05
Translated by IPS(DoM)