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ON DECAY OF A SOLUTION OF THE FIRST MIXED PROBLEM FOR THE LINEARIZED SYSTEM OF NAVIER-STOKES EQUATIONS IN A DOMAIN WITH NONCOMPACT BOUNDARY

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ABSTRACT. A. K. Gushchin, V. I. Ushakov, A. F. Tedeev, and other authors have investigated how stabilization rate of solutions of mixed problems for parabolic equations of second and higher orders depends on the geometry of an unbounded domain. Here an analogous problem is considered for the linearized system of Navier-Stokes equations in a domain with noncompact boundary in three-dimensional space. Estimates are obtained for the rate of decay of a solution as $t \to \infty$, in terms of a simple geometric characteristic of the unbounded domain. These estimates coincide in form with the corresponding estimates of a solution of the first mixed problem for a parabolic equation.

Bibliography: 21 titles.

Let Ω be an unbounded domain in $\mathbb{R}^3$ with a noncompact boundary. For the velocity vector $\mathbf{u}(t, x) = (u_1, u_2, u_3)$ and for the pressure $p(t, x)$ we consider the following problem in the domain $D = (0, \infty) \times \Omega$:

$$\mathbf{u}_t - \Delta\mathbf{u} = \nabla p, \qquad \operatorname{div}_x \mathbf{u} = 0, \tag{1}$$

$$\mathbf{u}|_{x\in\partial\Omega} = 0, \qquad \mathbf{u}|_{t=0} = \boldsymbol{\varphi}(x). \tag{2}$$

The author knows only one paper containing a study of stabilization of a solution of the mixed problem for the system (1): the article [5] by Rusanov. In it he established that if Ω is the exterior of a disk in the plane and if the initial function φ is bounded, then the solution of the problem (1), (2) satisfies the estimate $|\mathbf{u}(t, x)| \le c(x)/\ln t$ for sufficiently large values of the time.

There are many articles investigating the asymptotics of solutions of the Cauchy problem or the mixed problem in a half-space for a system of equations of the form

$$\mathbf{v}_t - [\mathbf{v}, \boldsymbol{\omega}] - \nu\Delta\mathbf{v} - \beta\nu\nabla \operatorname{div}\mathbf{v} + \nabla p = 0,$$
$$\alpha^2\frac{\partial p}{\partial t} + \operatorname{div}\mathbf{u} = 0,$$
$$v(0, x) = \boldsymbol{\varphi}(x), \qquad \boldsymbol{\omega} = (0, 0, \omega),$$

for different values of the constants α, β, and ν (see, for example, [4], [18]–[21]), and the bibliography in [21]). Due to the presence of the Coriolis term, this system differs strongly from (1). For example, in [18] it was established that in the case $\alpha = \beta = 0$ the solution of the Cauchy problem decays at the rate $t^{-5/2}$ uniformly over the whole space. But in the case of the system (1) the solution of the Cauchy problem coincides with the solution of the Cauchy problem for the heat equation, and thus it decays like $t^{-3/2}$ uniformly over the whole space.

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Many articles study the stabilization of solutions of the Cauchy problem for the nonlinear Navier-Stokes equations (see, for example, [6]–[10]). Heywood [17] obtained the estimate $O(t^{-1/2})$ for the uniform decay of a solution of the mixed problem for the system of Navier-Stokes equations in an arbitrary unbounded domain with smooth boundary. It is natural to expect that the actual rate of decay depends strongly on the geometry of the unbounded domain.

The purpose of the present article is to single out a geometric characteristic of a domain, that would determine the rate of decay for the problem (1), (2).

An analogous problem was considered in [1]–[3], [15], and [16] for the case of a parabolic equation.

We assume that the initial function φ has bounded support, and we consider a generalized solution of the problem (1), (2). The boundary of the domain is assumed to be Lipschitz everywhere in this article. In Theorem 2 the boundary has smoothness C^3 in the following sense. There exists a number r such that for each point $x \in \partial\Omega$ the ball neighborhood of radius r about x has a local Cartesian coordinate system in which the boundary of the domain can be represented locally by the equation $y_3 = F(y_1, y_2)$. Here the derivatives of F through order three are bounded by a constant independent of x.

In this paper we establish estimates of the rate of decay in terms of a rather simple characteristic of the domain—the function $l(r)$ in the condition (A) below.

Let $B_r = \{|x| < r\}$ be the ball of radius r. Denote by Ω_ρ^r, $0 \le \rho < r$, the set $\Omega \cap B_r \setminus B_\rho$. We say that Ω satisfies condition (A) if:

(A) There exist positive numbers α_1, r_1 and a continuous monotone nondecreasing function $l(r)$, $r \ge r_1$, with

$$\lim_{r\to\infty} l(r)/r = 0, \tag{3}$$

such that for all $r \ge r_1$ and $\rho \in [\alpha_1 l(r), r]$

$$\int_{\Omega_{r-\rho}^r} \psi^2\,dx \le l^2(r) \int_{\Omega_{r-\rho}^r} |\nabla\psi|^2\,dx, \qquad \psi \in \overset{\circ}{W}{}_2^1(\Omega). \tag{4}$$

In §3 it will be shown that condition (A) is satisfied, for example, by convex domains of revolution of the form

$$\Omega = \{x : x_1^2 + x_2^2 < f^2(x_3),\ x_3 > 0\}, \tag{5}$$

determined by a continuous function $f(r)$, $r \ge 0$, such that

$$\lim_{r\to\infty} f(r)/r = 0. \tag{6}$$

Here the function $3f(r)$ can be taken as $l(r)$.

Another condition on the domain Ω is due to the following circumstance. As is known, many results in the qualitative theory of equations of parabolic or elliptic type are obtained by substituting a test function of the form $\eta(x)u$ with suitable "cutoff" function $\eta(x)$ into the integral identity for a generalized solution. In the case of the system (1) such test functions are not admissible, since the vector $\eta(x)\mathbf{u}$ is no longer solenoidal. In this article we use test functions of the form $\operatorname{curl}(\eta(x)\mathbf{w})$, where $\mathbf{w}$ is a vector such that $\operatorname{curl}\mathbf{w} = \mathbf{u}$. The next condition ensures the existence of a suitable vector $\mathbf{w}$.

Denote by $\dot{\mathbf{J}}(\Omega)$ the set of smooth solenoidal vectors with compact support in Ω. We say that a domain satisfies condition (B) if it satisfies condition (A) and, moreover,

(B) There exist numbers $k > 0$ $\varepsilon \in (0, 1/2)$, $r_2 > r_1$ and $\alpha_2 \ge \alpha_1/\varepsilon$ such that for all $r \ge r_2$ and $\rho \in [\alpha_1 l(2r), r]$ there is a linear operator $\mathscr{R}$ mapping

$\dot{\mathbf{J}}(\Omega)$ to $\mathbf{W}_2^1(\Omega_{r+\rho/2}^{r+\rho/2+\varepsilon\rho})$ and satisfying the following requirements: if $\mathbf{w} = \mathscr{R}\mathbf{v}$, then $\mathbf{w}|_{\partial\Omega} = 0$, $\operatorname{curl}\mathbf{w} = \mathbf{v}$, and

(7) $$\int_{\Omega_{r+\rho/2}^{r+\rho/2+\varepsilon\rho}} |\nabla\mathbf{w}|^2\,dx \le k\int_{\Omega_r^{r+\rho}} \mathbf{v}^2\,dx.$$

It will be assumed everywhere except in §3 that the domain satisfies condition (B).

Let $r(t)$ be the function inverse to the monotone increasing continuous function $rl(r)$, $r \ge r_1$. It obviously satisfies the equalities

(8) $$\frac{t}{l^2(r(t))} = \frac{r(t)}{l(r(t))} = \frac{r^2(t)}{t}.$$

Theorem 1. *For any $R_0 > 0$ there exist positive numbers T and κ such that the solution of the problem* (1), (2) *satisfies for all $t \ge T$ the estimate*

(9) $$\|\mathbf{u}(t, x)\|_{\mathbf{W}_2^1(\Omega)} \le 3t^{1/2}\exp(-\kappa r^2(t)/t)\|\boldsymbol{\varphi}\|_{\mathbf{W}_2^1(\Omega)}$$

for any initial function $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ with $\operatorname{supp}\boldsymbol{\varphi} \subseteq B_{R_0}$ *and* $\operatorname{div}\boldsymbol{\varphi} = 0$.

The next theorem gives an estimate for the solution in the L_∞-norm, as well as an estimate of the pressure.

Theorem 2. *Let Ω be a domain with boundary of class C^3. Then for any $R_0 > 0$ there are positive numbers T, κ, and c such that the solution of the problem* (1), (2) *satisfies for all $t \ge T$ the estimates*

(10) $$|\mathbf{u}(t, x)| \le ct^{1/2}\exp(-\kappa r^2(t)/t)\|\boldsymbol{\varphi}\|_{\mathbf{W}_2^3(\Omega)}, \qquad x \in \Omega,$$

(11) $$\|\nabla p(t, x)\|_{\mathbf{L}_2(\Omega)} \le ct^{1/2}\exp(-\kappa r^2(t)/t)\|\boldsymbol{\varphi}\|_{\mathbf{W}_2^3(\Omega)},$$

for any initial function $\boldsymbol{\varphi} \in \mathbf{W}_2^3(\Omega)\cap\overset{\circ}{\mathbf{W}}{}_2^3(\Omega)$ such that $\operatorname{supp}\boldsymbol{\varphi} \subseteq B_{R_0}, \dots, \operatorname{div}\boldsymbol{\varphi} = 0$, *and* $\Delta\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$.

A convex domain of revolution of the form (5) with function f of class $C^2(0, \infty)$ satisfies condition (B) (see §3) if

(12) $$\lim_{r\to\infty} f(r) = \infty,$$

(13) $$|f(r)f''(r)| \le \alpha_0, \qquad r \ge r_0.$$

The inequalities (9)–(11) are valid for such domains, with the function $r(t)$, $t > 0$, inverse to the monotone function $rf(r)$, $r > 0$. In particular, if $f(r) = r^\alpha$, $\alpha \in (0, 1)$, then $r^2(t)/t = t^{(1-\alpha)/(1+\alpha)}$, and the increasing factor $t^{1/2}$ in these inequalities is not essential. Thus, there are positive constants T_1 and κ_1 such that the solution of the problem (1), (2) in a domain of the form

$$\Omega = \{x\colon x_1^2 + x_2^2 < x_3^{2\alpha},\ x_3 > 0\}, \qquad \alpha \in (0, 1),$$

with compactly supported initial function satisfies for all $t \ge T_1$ the estimates

$$\|\mathbf{u}(t, x)\|_{\mathbf{W}_2^1(\Omega)} \le \exp(-\kappa_1 t^{(1-\alpha)/(1+\alpha)})\|\boldsymbol{\varphi}\|_{\mathbf{W}_2^1(\Omega)},$$

$$|\mathbf{u}(t, x)| \le \exp(-\kappa_1 t^{(1-\alpha)/(1+\alpha)})\|\boldsymbol{\varphi}\|_{\mathbf{W}_2^3(\Omega)}, \qquad x \in \Omega,$$

$$\|\nabla p(t, x)\|_{\mathbf{L}_2(\Omega)} \le \exp(-\kappa_1 t^{1-\alpha)/(1+\alpha)})\|\boldsymbol{\varphi}\|_{\mathbf{W}_2^3(\Omega)}.$$

If a domain of the form (5) is equipped with a "handle", then it becomes not simply connected, but condition (B) still holds. But if an infinite "cylinder" of constant radius or an expanding cylinder is deleted from a domain of the form (5), then the author does not know whether the remaining domain satisfies condition (B).

§1

In this section we prove the basic Lemma 1.

Denote by $\overset{\circ}{\mathbf{J}}{}^n(\Omega)$ the closure of the set $\dot{\mathbf{J}}(\Omega)$ in the space $\mathbf{W}_2^n(\Omega) \equiv (W_2^n(\Omega))^3$ and by $\overset{\circ}{\mathbf{J}}(\Omega)$ the closure of the same set in the space $\mathbf{L}_2(\Omega)$.

Let $D^T = (0, T) \times \Omega$, $D_0 = (0, \infty) \times \Omega$, and $D = R \times \Omega$ be cylindrical domains. Denote by $\dot{\mathbf{J}}(D)$ the set of smooth compactly supported vector fields $\mathbf{v}(t, x) = (v_1, v_2, v_3)$ on D satisfying the condition $\operatorname{div}_x \mathbf{v} = 0$. Its closure in the space $\mathbf{W}_2^1(D^T)$ is denoted by $\overset{\circ}{\mathbf{J}}{}^1(D^T)$.

The existence and uniqueness of generalized solutions of problem (1), (2) have been thoroughly investigated (see, for example, [11], [13]). But our generalized solution is taken from a smaller space; therefore, for completeness of the exposition we give a proof of existence by Galërkin's method and in passing establish the properties of a generalized solution that are needed in what follows.

Assume that the initial function $\varphi(x)$ is an element of $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$ and has bounded support, and let R_0 be the radius of a ball about the origin containing the support.

A *generalized solution of the problem* (1), (2) in D^T, $T > 0$, is defined to be a function $\mathbf{u}(t, x) \in \overset{\circ}{\mathbf{J}}{}^1(D^T)$ satisfying (2) and the identity

$$\int_{D^T} (\mathbf{u}_t\mathbf{v} + \nabla\mathbf{u}\nabla\mathbf{v})\, dx\, dt = 0 \tag{14}$$

for all $\mathbf{v} \in \dot{\mathbf{J}}(D)$. The function $\mathbf{u}(t, x)$ is called a generalized solution of the problem (1), (2) in D if it is a generalized solution of this problem in D^T for all $T > 0$.

We prove the existence of a generalized solution.

In the Hilbert space $\overset{\circ}{\mathbf{J}}{}^2(\Omega)$ we take a fundamental system of functions $\mathbf{a}^k(x) \in \dot{\mathbf{J}}(\Omega)$, $k = 1, 2, \dots$, that is orthogonal in $\mathbf{L}_2(\Omega)$. Assume first that φ is in $\dot{\mathbf{J}}(\Omega)$. Then there is a sequence of functions of the form

$$\mathbf{b}^n(x) = \sum_{i=1}^{n} C_{in}^0 \mathbf{a}^i(x)$$

that converges to φ in the norm of the space $\mathbf{W}_2^2(\Omega)$.

We look for an approximate solution $\mathbf{v}^n$ of the problem (1), (2) in the form

$$\mathbf{v}^n(x) = \sum_{i=1}^{n} C_{in}(t)\mathbf{a}^i(x),$$

where the functions C_{in} are found from the conditions

$$C_{in}(0) = C_{in}^0, \qquad i = 1, 2, \dots, n, \tag{15}$$

and the equations

$$(\mathbf{v}_t^n, \mathbf{a}^i) + (\nabla\mathbf{v}^n, \nabla\mathbf{a}^i) = 0, \qquad i = 1, 2, \dots, n, \tag{16}$$

in which $(\mathbf{u}, \mathbf{v})$ denotes the inner product in $\mathbf{L}_2(\Omega)$. The equations (16) are a system of ordinary differential equations with constant coefficients. This system with the conditions (15) is uniquely solvable on the interval $[0, T]$.

We multiply each of the equations (16) by the corresponding function $C_{in}(t)$ and add the results from 1 to n, obtaining the equality

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{v}^n\|^2 + \|\nabla \mathbf{v}^n\|^2 = 0,$$

in which $\|\mathbf{v}\|$ denotes the norm in $\mathbf{L}_2(\Omega)$. Integration with respect to t gives

$$\|\mathbf{v}^n(t)\|^2 + 2\int_0^t \|\nabla \mathbf{v}^n\|^2\,dt = \|\mathbf{b}^n\|^2. \tag{17}$$

We multiply (16) by the functions $C'_{in}(t)$. Analogous transformations give us

$$2\int_0^t \|\mathbf{v}_t^n(t)\|^2\,dt + \|\nabla \mathbf{v}^n(t)\|^2 = \|\nabla \mathbf{b}^n\|^2. \tag{18}$$

Finally, we differentiate (16) with respect to t, multiply by the corresponding functions $C'_{in}(t)$, and sum the result over i. After integration with respect to t we get

$$\|\mathbf{v}_t^n(t)\|^2 + 2\int_0^t \|\nabla \mathbf{v}_t^n\|^2\,dt = \|\mathbf{v}_t^n(0)\|^2. \tag{19}$$

We transform (16) by integrating by parts: $(\mathbf{v}_t^n - \Delta\mathbf{v}^n, \mathbf{a}^i) = 0$, $i = 1, 2, \dots, n$. It follows that $\mathbf{v}_t^n(0) = P_n\Delta\mathbf{v}^n(0) = P_n\Delta\mathbf{b}^n$, where P_n denotes the projection of $\mathbf{L}_2(\Omega)$ onto the linear span of the elements $\mathbf{a}^1, \dots, \mathbf{a}^n$. Since $\mathbf{b}^n$ tends to $\boldsymbol{\varphi}$ in the norm of $\mathbf{W}_2^2(\Omega)$, it follows from (17)–(19) that the sequence $\mathbf{v}^n$ is uniformly bounded in the completion $\mathbf{H}$ of $\dot{\mathbf{J}}(D)$ in the norm

$$\|\mathbf{u}\|_{\mathbf{H}}^2 = \int_{D^T} (\mathbf{u}^2 + \mathbf{u}_t^2 + \mathbf{u}_x^2 + \mathbf{u}_{tx}^2)\,dx\,dt.$$

A ball in a Hilbert space is weakly compact; therefore, we can assume by re-indexing if necessary that $\mathbf{v}^n \to \mathbf{u} \in \mathbf{H}$. It is easy to see that $\mathbf{u}$ is an element of $C([0, T]; \overset{\circ}{\mathbf{J}}{}^1(\Omega))$. We prove that this function satisfies the identity (14). To do this we integrate (16) with respect to t and, using the weak convergence, pass to the limit as $n \to \infty$. We get

$$\int_0^T [(\mathbf{u}_t, \mathbf{a}^i) + (\nabla\mathbf{u}, \nabla\mathbf{a}^i)]\,dt = 0.$$

Multiplying the last equality by the continuous functions $h_i(t)$ and adding over i, we establish the identity (14) for functions of the form $\mathbf{v} = \sum_{i=1}^n h_{in}(t)\mathbf{a}^i(x)$. Such functions are dense in the space $C([0, T]; \overset{\circ}{\mathbf{J}}{}^1(\Omega))$ (see [11], Russian p. 200). Consequently, in (14) we can substitute the functions $\mathbf{u}$, $\mathbf{u}_t$ in place of $\mathbf{v}$. After simple transformations we obtain

$$\int_\Omega \mathbf{u}^2(t, x)\,dx + 2\int_0^t\int_\Omega |\nabla\mathbf{u}|^2\,dx\,d\tau = \int_\Omega \boldsymbol{\varphi}^2(x)\,dx, \tag{20}$$

$$2\int_0^t\int_\Omega \mathbf{u}_t^2\,dx\,d\tau + \int_\Omega |\nabla\mathbf{u}(t, x)|^2\,dx = \int_\Omega |\nabla\boldsymbol{\varphi}(x)|^2\,dx. \tag{21}$$

Thus, we have proved the existence of a generalized solution for an initial function in $\dot{\mathbf{J}}(\Omega)$. The solution belongs to the space $\mathbf{H}$ and satisfies (20), (21).

Suppose now that $\boldsymbol{\varphi} \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$ and $\boldsymbol{\varphi}^n \in \dot{\mathbf{J}}(\Omega)$ is a sequence of functions convergent to $\boldsymbol{\varphi}$ in the norm of $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$. Let $\mathbf{u}^n$ be the solutions constructed above for the problem (1), (2) with the initial functions φ^n. Applying (20) and (21) to the difference $\mathbf{u}^n - \mathbf{u}^m$ and using the fact that φ^n is a Cauchy sequence in $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$, we establish the convergence $\mathbf{u}^n \to \mathbf{u}$ in $\overset{\circ}{\mathbf{J}}{}^1(D^T)$. By passing to the limit in the corresponding identities for $\mathbf{u}^n$ we prove that $\mathbf{u}$ satisfies (14), (20), and (21), i.e., is a generalized solution of the problem (1), (2) and belongs to the space $C([0, T]; \overset{\circ}{\mathbf{J}}{}^1(\Omega))$.

It is easy to see that (20) is valid for an arbitrary generalized solution; therefore, it is unique.

It will be assumed without loss of generality that R_0 is not less than the constant r_2 in condition (B). We fix $R > R_0$ and introduce the notation $\mu = l^2(2R)$,

$$H(t, r) = e^{-t/\mu}\left\{\int_{\Omega_r^\infty} |\nabla \mathbf{u}(t, x)|^2\, dx + \int_0^t \int_{\Omega_r^\infty} \mathbf{u}_t^2\, dx\, d\tau\right\}, \tag{22}$$

where $\mathbf{u}(t, x)$ is a solution of the problem (1), (2) with initial function $\varphi \in \dot{\mathbf{J}}(\Omega)$, $\operatorname{supp}\varphi \in B_{R_0}$.

Let r and ρ be arbitrary numbers satisfying the relations $R_0 \le r < r + \rho \le R$, and $\rho \in [\alpha_2 l(2R), r]$.

Lemma 1. *There exists a number β dependent only on the ε, k, and α_2 in condition* (B) *such that for all $t > 0$ the function $H(t, r)$ satisfies the inequality*

$$H(t, r+\rho) \le \frac{\beta}{\rho^2}\left(\mu H(t, r) + \int_0^t H(\tau, r)\, d\tau\right). \tag{23}$$

Proof. Let $\xi(r)$ be a smooth function equal to 0 for $r < 1/2$, to 1 for $r > 1/2+\varepsilon$, and monotone in the remaining interval. Let $\eta(x) \equiv \eta(x, r, \rho, \varepsilon) = \xi((|x| - r)/\rho)$. Obviously, there is a constant α, depending only on ε, such that

$$|\nabla \eta(x)|^2 + \sum_{i,j=1}^{3} \left|\frac{\partial^2 \eta^2(x)}{\partial x_i \partial x_j}\right| \le \frac{\alpha}{\rho^2}, \qquad x \in \mathbb{R}^3, \quad r \ge 1, \quad \rho > 0. \tag{24}$$

The function η is equal to 0 in the ball B_r, and $r \ge R_0$, therefore, $\eta\varphi \equiv 0$. The gradient $\nabla\eta$ is nonzero only in the shell $B_{r+\rho/2+\varepsilon\rho} \setminus B_{r+\rho/2}$.

We verify that the inequality (4) is applicable to the domain $\Omega_{r+\rho/2}^{r+\rho/2+\varepsilon\rho}$. Indeed, the numbers $r' = r + \rho/2 + \varepsilon\rho$ and $\rho' = \varepsilon\rho$ satisfy the requirements in condition (A): $r' \ge R_0 \ge r_2 \ge r_1$; $\rho' \ge \varepsilon\alpha_2 l(2R) \ge \alpha_1 l(r')$.

By condition (B), the vector $\mathbf{w}(t, x) = \mathscr{R}\mathbf{v}(t, x)$ is defined for every $\mathbf{v} \in \dot{\mathbf{J}}(D)$. By (7) and (4), $\mathbf{w}$ is an element of $C([0, \infty); \mathbf{W}_2^1(\Omega_{r+\rho/2}^{r+\rho/2+\varepsilon\rho}))$. Let us consider the function

$$\mathbf{\Phi}(t, x) = \begin{cases} 0, & |x| < r + \rho/2, \\ \mathbf{v}(t, x), & x \in \Omega_{r+\rho/2+\varepsilon\rho}^\infty, \\ \operatorname{curl}(\eta^2 \mathbf{w}), & x \in \Omega_{r+\rho/2}^{r+\rho/2+\varepsilon\rho}. \end{cases}$$

Obviously, $\operatorname{div}\mathbf{\Phi} = 0$ and $\mathbf{\Phi}(t, x) = \eta^2\mathbf{v} + \nabla(\eta^2) \times \mathbf{w}$, since $\operatorname{curl}\mathbf{w} = \mathbf{v}$. The support of the function $\mathbf{\Phi}(t, x)$ is bounded for each t, and, by condition (B), $\mathbf{\Phi}(t, x)$ is an element of $\overset{\circ}{\mathbf{W}}{}_2^1(\Omega)$ for all $t \in \mathbb{R}$. Since Ω has a Lipschitz boundary, this leads to the conclusion that $\mathbf{\Phi}(t, x) \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$ for each $t \in \mathbb{R}$ (see, for example, [13], Chapter I, Theorem 1.6). Thus, $\mathbf{\Phi}(t, x)$ is an element of $C(\mathbb{R}; \overset{\circ}{\mathbf{J}}{}^1(\Omega))$.

Substituting the test function $\mathbf{\Phi}(t, x)$ in (14), we have

$$\int_{D^T} \mathbf{u}_t(\eta^2\mathbf{v} + \nabla(\eta^2) \times \mathbf{w})\, dx\, dt + \int_{D^T} \nabla\mathbf{u}(\eta^2\nabla\mathbf{v} + \nabla(\eta^2)\mathbf{v} + \nabla(\nabla\eta^2 \times \mathbf{w}))\, dx\, dt = 0. \tag{25}$$

We get an upper estimate of certain terms in the last expression.

Using the inequality $l^2(r') \leq \mu$, along with the relations (24), (4), and (7), we can write

$$\begin{aligned}\int_{D^T} |(\nabla(\eta^2)\times \mathbf{w})\mathbf{u}_t|\,dx\,dt &\leq \int_{D^T}\left(\frac{\eta^2}{4}\mathbf{u}_t^2 + 4|\nabla\eta|^2\mathbf{w}^2\right)dx\,dt \\ &\leq \int_{D^T}\frac{\eta^2}{4}\mathbf{u}_t^2\,dx\,dt + \frac{4\alpha l^2(r')}{\rho^2}\int_0^t\int_{\Omega^{r'}_{r'-\rho'}}|\nabla\mathbf{w}|^2\,dx\,dt \\ &\leq \frac{1}{4}\int_{D^T}\eta^2\mathbf{u}_t^2\,dx\,dt + \frac{4\alpha\mu k}{\rho^2}\int_0^t\int_{\Omega_r^{r+\rho}}\mathbf{v}^2\,dx\,dt.\end{aligned}$$

Further

$$\begin{aligned}\int_{D^T}|\nabla\mathbf{u}(\nabla(\eta^2)\times\mathbf{v})\,dx\,dt &\leq \int_{D^T}\left(\frac{\eta^2}{4}\mathbf{v}^2 + 4|\nabla\eta|^2|\nabla\mathbf{u}|^2\right)dx\,dt \\ &\leq \frac{1}{4}\int_{D^T}\eta^2\mathbf{v}^2\,dx\,dt + \frac{4\alpha}{\rho^2}\int_0^t\int_{\Omega_r^{r+\rho}}|\nabla\mathbf{u}|^2\,dx\,dt.\end{aligned}$$

Finally,

$$\begin{aligned}\mathbf{I} &\equiv \int_{D^T}|\nabla\mathbf{u}\nabla(\nabla(\eta^2)\times\mathbf{w})|\,dx\,dt \\ &\leq \int_{D^T}|\nabla\mathbf{u}|\left(2\eta|\nabla\eta|\,|\nabla\mathbf{w}| + |\mathbf{w}|\sum_{i,j=1}^{3}\left|\frac{\partial^2\eta^2}{\partial x_i\,\partial x_j}\right|\right)dx\,dt \\ &\leq \int_{D^T}\left(\frac{\eta^2}{2\mu}|\nabla\mathbf{u}|^2 + 2\mu|\nabla\eta|^2|\nabla\mathbf{w}|^2\right)dx\,dt + \frac{\alpha}{\rho^2}\int_0^t\int_{\Omega^{r'}_{r'-\rho'}}(\mathbf{w}^2 + |\nabla\mathbf{u}|^2)\,dx\,dt.\end{aligned}$$

Again using (4) and (7), we have

$$\mathbf{I} \leq \frac{1}{2\mu}\int_{D^T}\eta^2|\nabla\mathbf{u}|^2\,dx\,dt + \frac{\alpha}{\rho^2}\int_0^t\int_{\Omega_r^{r+\rho}}(|\nabla\mathbf{u}|^2 + 3\mu k\mathbf{v}^2)\,dx\,dt.$$

Using these estimates, from (25) we deduce the inequality

$$\begin{aligned}\int_{D^T}\eta^2(\mathbf{u}_t\mathbf{v} + \nabla\mathbf{u}\nabla\mathbf{v})\,dx\,dt &\leq \frac{1}{2\mu}\int_{D^T}\eta^2|\nabla\mathbf{u}|^2\,dx\,dt + \int_{D^T}\frac{\eta^2}{4}(\mathbf{u}_t^2 + \mathbf{v}^2)\,dx\,dt \\ &\quad + \frac{7\alpha}{\rho^2}\int_0^t\int_{\Omega_r^{r+\rho}}(|\nabla\mathbf{u}|^2 + \mu k\mathbf{v}^2)\,dx\,dt.\end{aligned}$$

Now let us replace $\mathbf{v}$ by a sequence of functions $\mathbf{u}^n \in \dot{\mathbf{J}}(D)$ convergent to $\mathbf{u}_t$ in $\mathbf{W}_2^{0,1}(D^T)$. Passing to the limit as $n\to\infty$ and using simple transformations, we get

$$\begin{aligned}&\int_{D^T}\eta^2\mathbf{u}_t^2\,dx\,dt + \int_\Omega\eta^2|\nabla\mathbf{u}(T,x)|^2\,dx \\ &\leq \int_\Omega\eta^2|\nabla\boldsymbol{\varphi}(x)|^2\,dx + \frac{1}{\mu}\int_{D^T}\eta^2|\nabla\mathbf{u}|^2\,dx\,dt \\ &\quad + \frac{14\alpha}{\rho^2}\int_0^T\int_{\Omega_r^{r+\rho}}(|\nabla\mathbf{u}|^2 + \mu k\mathbf{u}_t^2)\,dx\,dt.\end{aligned} \tag{26}$$

The first term on the right-hand side of (26) is equal to zero, because $\eta\nabla\varphi \equiv 0$.

Let $\beta = \max(14\alpha, 14\alpha k, \alpha_2^2)$. For the functions

$$z(t) = \int_{D^T} \eta^2 |\nabla \mathbf{u}|^2 \, dx\, dt,$$

$$h(t) = \frac{\beta}{\rho^2} \int_0^t \int_{\Omega_r^\infty} (|\nabla \mathbf{u}|^2 + \mu \mathbf{u}_t^2) \, dx\, dt$$

we get the inequality $\dot{z} \le h(t) + z/\mu$ from (26). This inequality gives us that $z(T) \le \int_0^T \exp((T-t)/\mu) h(t)\, dt$. Substituting this in the right-hand side of (26) and taking into account that $\eta = 1$ for $|x| > r + \rho$, we get

$$\begin{aligned} e^{T/\mu} H(T, r+\rho) \le & \frac{\beta}{\rho^2} \int_0^T \int_{\Omega_r^\infty} (|\nabla \mathbf{u}|^2 + \mu \mathbf{u}_t^2) \, dx\, dt \\ & + \frac{\beta}{\mu\rho^2} \int_0^T e^{(T-t)/\mu} \int_0^t \int_{\Omega_r^\infty} (|\nabla \mathbf{u}|^2 + \mu \mathbf{u}_t^2) \, dx\, d\tau\, dt. \end{aligned} \tag{27}$$

We transform one of the integrals by parts:

$$\begin{aligned} &\frac{1}{\mu} \int_0^T e^{(T-t)/\mu} \int_0^t \int_{\Omega_r^\infty} |\nabla \mathbf{u}|^2 \, dx\, d\tau\, dt \\ &= -e^{(T-t)/\mu} \int_0^t \int_{\Omega_r^\infty} |\nabla \mathbf{u}|^2 \, dx\, d\tau \bigg|_0^T + \int_0^t e^{(T-t)/\mu} \int_{\Omega_r^\infty} |\nabla \mathbf{u}(t, x)|^2 \, dx\, dt. \end{aligned}$$

Combining this with (27), we obtain

$$e^{T/\mu} H(T, r+\rho) \le \frac{\beta\mu}{\rho^2} \int_0^T \int_{\Omega_r^\infty} \mathbf{u}_t^2 \, dx\, dt + \frac{\beta}{\rho^2} e^{T/\mu} \int_0^T H(t, r)\, dt.$$

Multiplication of the last inequality by $\exp(-T/\mu)$ completes the proof of Lemma 1.

§2

In this section we prove Theorems 1 and 2.

Proposition. *There exist positive numbers γ and R_1, dependent on Ω, R_0, and the constants in condition* (B), *such that the generalized solution of the problem* (1), (2) *satisfies for all $R \ge R_1$ and $T > 0$ the estimate*

$$\int_0^T \int_{\Omega_R^\infty} \mathbf{u}_t^2(t, x)\, dx\, dt \le \exp\left(1 + \frac{2T}{l^2(2R)} - \frac{\gamma R}{l(2R)}\right) \int_\Omega |\nabla \varphi(x)|^2 \, dx \tag{28}$$

for all $\varphi \in \dot{\mathbf{J}}^1(\Omega)$, $\operatorname{supp} \varphi \in B_{R_0}$.

Proof. By the condition (3), it is possible to choose a number $R_1 \ge 2R_0$ such that $2(2\beta e)^{1/2} l(2R) \le R/2$ for all $R \ge R_1$.

Fix an $R \ge R_1$ and an $r \in [R_0, R]$. We prove by induction on n that for all ρ and n with $r + n\rho \le R$ and $\rho \in [\alpha_2 l(2R), r]$ we have the relation

$$H(t, r+n\rho) \le A \left(\frac{\beta}{\rho^2}\right)^n \sum_{i=0}^n \frac{t^{n-i} \mu^i \binom{i}{n}}{(n-i)!}, \qquad t > 0, \tag{29}$$

where

$$A = \int_\Omega |\nabla \varphi|^2 \, dx.$$

For $n = 0$ it is a consequence of (22) and (21). Further, if (29) is valid for some n, then, replacing r by $r + n\rho$ in (23) and using (29), we get

$$H(t, r + (n+1)\rho) \le A\left(\frac{\beta}{\rho^2}\right)^{n+1}\left[\sum_{m=0}^{n}\frac{t^{n-m}\mu^{m+1}\binom{m}{n}}{(n-m)!} + \sum_{i=0}^{n}\frac{t^{n+1-i}\mu^i\binom{n}{i}}{(n+1-i)!}\right].$$

Replacing m by $i - 1$ in the first sum, we have

$$H(t, r + (n+1)\rho) \le A\left(\frac{\beta}{\rho^2}\right)^{n+1}\sum_{i=1}^{n+1}\frac{t^{n+1-i}\mu^i\left(\binom{n}{i-n} + \binom{n}{i}\right)}{(n+1-i)!}.$$

Here $\binom{n}{-1} = \binom{n}{n+1} = 0$. This concludes the proof of the inequality (29), because $\binom{n+1}{i} = \binom{n}{i} + \binom{n}{i-1}$.

As is known, $x^i/i! < \exp(x)$, $i = 0, 1, \dots$, for positive x. Therefore,

$$t^{n-i}\mu^i/(n-i)! \le \mu^n e^{t/\mu}$$

for $i = 0, \dots, n$. Using the equality $\sum_{i=0}^n \binom{n}{i} = 2^n$, we now get from (29) and (22)

$$\int_0^t \int_{\Omega_{r+n\rho}^\infty} \mathbf{u}_t^2\, dx\, d\tau \le A\left(\frac{2\beta\mu}{\rho^2}\right)^n e^{2t/\mu}. \tag{30}$$

Let $\gamma = (8\beta e)^{-1/2}$, $n = [\gamma R\mu^{-1/2}]$. If $n = 0$, i.e., $1 > \gamma R\mu^{-1/2} = \gamma R/l(2R)$, then (28) is a consequence of (21). But if $n \ge 1$, then let $r = R/2$ and $\rho = R/2n$ and verify the condition $\rho \in [\alpha_2 l(2R), R/2]$. By the choice of the number β, $\rho = R/2n \ge (2\beta e\mu)^{1/2} \ge \alpha_2\mu^{1/2} = \alpha_2 l(2R)$. Further, by the choice of the number R_1, $\rho = R/2n < 2(2\beta e\mu)^{1/2} \le R/2$. Thus, the inequality (30) can be used. Here

$$\left[\frac{2\beta\mu}{\rho^2}\right]^n = \left[\frac{\beta\mu n^2}{R^2}\right] \le e^{-n} \le \exp\left[1 - \frac{\gamma R}{l(2R)}\right].$$

This proves the inequality (28) of the proposition.

Corollary. *A generalized solution of the problem* (1), (2) *satisfies the estimate*

$$\int_{\Omega_R^\infty} \mathbf{u}^2(t, x)\, dx < 3tA\exp\left[\frac{2t}{l^2(2R)} - \frac{\gamma R}{l(2R)}\right] \tag{31}$$

for all $t > 0$ *and* $R \ge R_1$. *The constants* R_1 *and* γ *and the condition on* $\boldsymbol{\varphi}$ *are the same as in the proposition.*

Proof. The condition on the initial function and the Newton-Leibniz formula enable us to write the inequalities

$$|u_i(t, x)| \le \int_0^t \left|\frac{\partial u_i(\tau, x)}{\partial t}\right| d\tau, \qquad i = 1, 2, 3,$$

which are valid for a.e. $x \in \Omega_{R_0}^\infty$. We square them and use the Cauchy-Schwarz-Bunyakovskiĭ inequality

$$u_i^2(t, x) \le t\int_0^t \left[\frac{\partial u_i}{\partial t}\right]^2 d\tau, \qquad i = 1, 2, 3.$$

We sum the last relation over i and integrate with respect to $x \in \Omega_R^\infty$, $R \ge R_0$. This gives

$$\int_{\Omega_R^\infty} \mathbf{u}^2(t, x)\, dx \le t\int_0^t \int_{\Omega_R^\infty} \mathbf{u}_t^2\, dx\, d\tau.$$

Now (31) is a simple consequence of (28).

Proof of Theorem 1. It suffices to carry out the proof for initial functions in the space $\dot{\mathbf{J}}(\Omega)$. Let R_1 and γ be the numbers in the proposition. Since $r(t)$ tends to infinity as $t \to \infty$, there is a number $T > 1$ such that $r(4t/\gamma) \geq 2R_1$ for all $t \geq T$.

Fix $t \geq T$ and let $R = r(8t/\gamma)/2$. From (8),

$$\frac{2t}{l^2(2R)} = \frac{\gamma R}{2l(2R)} = \frac{\gamma^2 R^2}{8t}. \tag{32}$$

Consequently, the estimate (31) can be represented in the form

$$\int_{\Omega_R^\infty} \mathbf{u}^2(\tau, x)\, dx \leq \delta \equiv 3AT \exp\left(-\frac{\gamma^2 R^2}{8t}\right) \tag{33}$$

for all $\tau \in [0, t]$.

We write the inequality (4) for the vector $\mathbf{u}$ with $\tau \geq 0$

$$\frac{1}{l^2(R)} \int_{\Omega_0^R} \mathbf{u}^2(\tau, x)\, dx \leq \int_{\Omega_0^R} |\nabla \mathbf{u}(\tau, x)|^2\, dx \leq \int_\Omega |\nabla \mathbf{u}(\tau, x)|^2\, dx.$$

It follows from the identity (20) that the function $E(\tau) \equiv \int_\Omega \mathbf{u}^2(\tau, x)\, dx$ is absolutely continuous. Using (33), we deduce for it the differential inequality

$$\begin{aligned}\frac{1}{l^2(R)}(E(\tau) - \delta) &\leq \frac{1}{l^2(R)} \int_{\Omega_0^R} \mathbf{u}^2(\tau, x)\, dx \\ &\leq \int_\Omega |\nabla \mathbf{u}(\tau, x)|^2\, dx = -\frac{1}{2}\frac{\partial}{\partial t} E(\tau), \qquad \tau \in [0, t].\end{aligned}$$

Solving it for the monotone nonincreasing function $E(\tau)$, we get the estimate

$$E(t) \leq \delta + E(0) \exp(-2t/l^2(R)).$$

The monotonicity of the function l in (32) gives us the inequality $2t/l^2(R) \geq \gamma^2 R^2/8t$. Replacing δ by its value, we have

$$E(t) \leq 3t \exp(-\gamma^2 R^2/8t)(A + E(0)) = 3t \exp(-\gamma^2 R^2/8t) \|\boldsymbol{\varphi}\|^2_{\mathbf{W}_2^1(\Omega)}. \tag{34}$$

To estimate the integral

$$F(t) = \int_\Omega |\nabla \mathbf{u}(t, x)|^2\, dx$$

we rewrite the identity (20) in the form

$$\int_\Omega \mathbf{u}^2(2t, x)\, dx + 2\int_t^{2t} \int_\Omega |\nabla \mathbf{u}|^2\, dx\, d\tau = \int_\Omega \mathbf{u}^2(t, x)\, dx.$$

Then the inequalities

$$F(2t) \leq \frac{1}{t} \int_t^{2t} \int_\Omega |\nabla \mathbf{u}|^2\, dx\, d\tau \leq \frac{1}{t} \int_\Omega \mathbf{u}^2(t, x)\, dx$$

hold for the monotone nonincreasing function $F(t)$ (see [21]). Combining the latter with (34) and replacing t by $t/2$, we get

$$F(t) \leq 3 \exp(-\gamma^2 r^2(4t/\gamma)/16t) \|\boldsymbol{\varphi}\|^2_{\mathbf{W}_2^1(\Omega)}.$$

The inequality (9) with $\kappa = \gamma^2/32$ is a consequence of the last inequality, (34), and the inequality $r(4t/\gamma) \geq r(t)$, which follows from the monotonicity of r for $\gamma \leq 4$. Theorem 1 is proved.

Proof of Theorem 2. By the hypothesis of the theorem, the function $\psi(x) = \Delta\varphi$ is an element of $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$ (see [13], Chapter I, Theorem 1.6), and it is clear that $\operatorname{supp}\psi \subseteq B_{R_0}$. By Theorem 1, the solution $\mathbf{w}(t, x)$ of the problem (1), (2) with the initial function ψ satisfies the estimate

$$(35) \qquad \|\mathbf{w}(t, x)\|_{\mathbf{W}_2^1(\Omega)} \leq 3t^{1/2}\exp(-\kappa r^2(t)/t)\|\psi\|_{\mathbf{W}_2^1(\Omega)}.$$

We show that the function $\mathbf{u}(t, x) = \varphi(x) + \int_0^t \mathbf{w}(\tau, x)\,d\tau$ is the generalized solution of the problem (1), (2) with initial function φ.

By the definition of a generalized solution, $\mathbf{w}$ satisfies for any $\mathbf{V}$ in $\dot{\mathbf{J}}(D^T)$ the identity

$$\int_{D^T}(\mathbf{w}_t\mathbf{V} + \nabla\mathbf{w}\nabla\mathbf{V})\,dx\,d\tau = 0.$$

By substituting

$$\mathbf{V}(t, x) = \int_t^T \mathbf{v}(\tau, x)\,d\tau, \qquad \mathbf{v} \in \dot{\mathbf{J}}(D),$$

and transforming the integrals by parts it is not hard to get the identity (14), which is what was to be shown.

By using the continuity of the functions $\mathbf{u}$ and $\mathbf{w}$ in the norm of $\overset{\circ}{\mathbf{J}}{}^1(\Omega)$, it is not hard to get from (14) with the test function $\xi_n(t)\mathbf{v}(x)$ (ξ_n is a δ-shaped sequence) that

$$\int_\Omega(\mathbf{w}\mathbf{v} + \nabla\mathbf{u}\nabla\mathbf{v})\,dx = 0, \qquad t > 0,$$

for all $\mathbf{v} \in \overset{\circ}{\mathbf{J}}{}^1(\Omega)$. We write the estimate (6) in [17] in the form

$$(36) \qquad \|\mathbf{u}(t)\|_{\mathbf{W}_2^2(K)} \leq C(\|\mathbf{w}(t)\|_{\mathbf{L}_2(\Omega)} + \|\nabla\mathbf{u}(t)\|_{\mathbf{L}_2(\Omega)})$$

with a constant C depending only on Ω and the measure of the compact set $K \subset \Omega$. It is easy to verify (see, for example, [11], Russian p. 57) that $\Delta\mathbf{u} - \mathbf{w}$ is the gradient of some function p satisfying the estimate

$$(37) \qquad \|\nabla p(t)\|_{\mathbf{L}_2(\Omega)} \leq \|\mathbf{w}(t)\|_{\mathbf{L}_2(\Omega)} + \|\mathbf{u}(t)\|_{\mathbf{W}_2^2(K)}.$$

Since $\psi = \nabla\varphi$, we get the inequality (11) of Theorem 2 from (36), (37), (35), and (9).

It follows from the smoothness of the boundary of Ω that there is a sufficiently small cone K that can touch any point of Ω from the inside. The functions $\mathbf{u}$ and $\mathbf{w}$ belong to $C([0, \infty]; \overset{\circ}{\mathbf{J}}{}^1(\Omega))$. Therefore, by (36) and the Sobolev inequality

$$|\mathbf{u}(t, x)| \leq C_1\|\mathbf{u}(t)\|_{\mathbf{W}_2^2(K_x)}, \qquad x \in \Omega,$$

$\mathbf{u}(t, x)$ is continuous for $t > 0$ and $x \in \Omega$ and satisfies the estimate

$$|\mathbf{u}(t, x)| \leq C_2(\|\mathbf{w}(t)\|_{\mathbf{L}_2(\Omega)} + \|\nabla\mathbf{u}(t)\|_{\mathbf{L}_2(\Omega)}), \qquad t > 0, \quad x \in \Omega.$$

The inequality (10) now follows from (35) and (9). Theorem 2 is proved.

§3

In this section prove that the conditions (A) and (B) hold for domains of revolution of the form (5) if f satisfies (6), (12), and (13).

We first derive an inequality.

Let S be a convex surface of revolution of class C^2 with axis of revolution Ox_3, $O \in S$. Let φ be the polar angle, and t the geodesic distance from a point of S to the point O, $t \in [0, \alpha]$. Denote by $r(\theta)$ the radius of the circle $C_\theta = \{t = \theta\}$ and by $R(\theta)$ the integral of this function, $R' = r$, $R(0) = 0$. The monotonicity of $r(t)$ implies the inequality

$$R(t) \leq tr(t), \qquad t \in [0, \alpha].$$

We prove that for any function $g \in C^1(S)$

$$\int_S g^2\, dS \leq 2\alpha^2 \int_S |\nabla_S g|^2\, dS + 2\alpha \int_{C_\alpha} g_2\, dl. \tag{38}$$

Here $|\nabla_S g|$ can be understood as the usual gradient of an extension of g to a neighborhood of S such that the extension is constant along the normal to S.

For a fixed φ we write the Newton-Leibniz formula for g:

$$g(\theta) = -\int_\theta^\alpha g_\theta(t)\, dt + g(\alpha),$$

from which

$$g^2(\theta) \leq 2\alpha \int_\theta^\alpha g_\theta^2(t)\, dt + 2g^2(\alpha).$$

Consequently,

$$\begin{aligned}\int_0^\alpha g^2(\theta) r(\theta)\, d\theta &\leq 2\alpha \int_0^\alpha r(\theta) \int_\theta^\alpha g_\theta^2\, dt\, d\theta + 2R(\alpha) g^2(\alpha)\\ &= 2\alpha \int_0^\alpha R(\theta) g_\theta^2\, d\theta + 2R(\alpha) g^2(\alpha)\\ &\leq 2\alpha \left(\alpha \int_0^\alpha r(\theta) g_\theta^2\, d\theta + r(\alpha) g^2(\alpha)\right).\end{aligned}$$

Integrating the latter with respect to φ on $(0, 2\pi)$, we get (38).

If Ω is a domain of the form (5) and $S(r) = \Omega \cap \{|x| = r\}$, then it follows from the monotonicity of f that $\alpha \leq \pi f(r)/2$. Therefore, by (38), the inequality

$$\int_{S(t)} g^2\, dS \leq 9 f^2(r) \int_{S(t)} |\nabla g|^2\, dS, \qquad t \in (0, r],$$

holds for any smooth function with compact support in Ω. Integrating it with respect to $t \in [r - \rho, r]$, we conclude that (4) holds for $r > 0$, $\rho \in (0, r]$, and $l(r) = 3f(r)$.

Theorem 3. *Suppose that a function $f(r)$ of class $C^2(0, \infty)$ satisfies the conditions (6), (12), and (13), and Ω is a convex domain of the form (5). Then there exist numbers k and r_2 such that for all $r \geq r_2$ and $\rho \in [6f(2r), r]$ there is a linear operator $\mathscr{R}$ mapping $\mathbf{\dot{J}}(\Omega)$ to $\mathbf{W}_2^1(U)$, $U = \Omega_{r+\rho/2}^{r+2\rho/3}$, and satisfying the following requirements: if $\mathbf{w} = \mathscr{R}\mathbf{v}$, then $\mathbf{w}|_{\partial\Omega} = 0$, $\operatorname{curl} \mathbf{w} = \mathbf{v}$, and*

$$\int_U |\nabla \mathbf{w}|^2\, dx \leq k \int_{\Omega_r^{r+\rho}} \mathbf{v}^2\, dx.$$

Remark. In the process of proving the theorem we construct a function $\hat{\mathbf{v}}$, then $\hat{\mathbf{w}}$ and $\mathbf{w}$. It is easy to see that the correspondences $\mathbf{v} \mapsto \hat{\mathbf{v}}$, $\hat{\mathbf{v}} \mapsto \hat{\mathbf{w}}$, and $\hat{\mathbf{w}} \mapsto \mathbf{w}$ determine certain linear mappings, and in what follows we shall not make special mention of this.

We consider some auxiliary domains.

Let $t_1 = r + f(r)$ and $t_2 = r + \rho - f(r)$. In view of the convexity of Ω and (6) there is a number $r_2 \geq r_0$ such that

$$f(2r) \leq 2f(r), \quad r > 0; \qquad f'(r) \leq 1, \quad r \geq r_2. \tag{39}$$

Let $B = \Omega \cap \{t_1 < x_3 < t_2\}$. If the numbers r and ρ satisfy the conditions of Theorem 3, then in view of (13) the domain B can be supplemented to form a convex domain of revolution V with $B \subset V \subset \Omega_r^{r+\rho}$ whose boundary ∂V is of class C^2 and has radius of curvature $r_\gamma \geq af(r)$ for any normal section of the surface ∂V at an arbitrary point of it. Here and in what follows in this section the letter a with indices will denote various positive constants depending only on the α_0 in (13).

The condition on the radius of curvature enables us to introduce coordinates (τ, ω) in the layer F of thickness $a_1 f(r)$ contiguous to ∂V, where ω stands for the coordinates on ∂V, and τ is the distance from a point in V to the boundary.

For simplicity we assume that $S_0 \equiv \partial\Omega \cap \partial V = \partial\Omega \cap \{t_1 \leq x_3 \leq t_2\}$. The domain V without U is broken up into two connected components Q_1 and Q_2. The parts $S_1 = \partial Q_1 \cap \{x_3 < t_1\}$ and $S_2 = \partial Q_2 \cap \{x_3 > t_2\}$ of their boundaries are said to be *spherical*.

Fix an arbitrary vector $\mathbf{v} \in \dot{\mathbf{J}}(\Omega)$. We construct a vector $\hat{\mathbf{v}} \in \overset{\circ}{\mathbf{J}}(V)$ such that

$$\hat{\mathbf{v}}(x) = \mathbf{v}(x), \qquad x \in U, \tag{40}$$

and

$$\int_V \hat{\mathbf{v}}^2\,dx = a_2 \int_V \mathbf{v}^2\,dx. \tag{41}$$

We consider on Q_1 harmonic functions g^i $(i = 1, 2)$ satisfying Neumann conditions on the boundary:

$$\left.\frac{\partial g^i}{\partial n}\right|_{S_1} = (\mathbf{v}, n), \qquad \left.\frac{\partial g^i}{\partial n}\right|_{2Q_i \setminus S_1} = 0, \qquad \int_{Q_i} g^i\,dx = 0, \qquad i = 1, 2.$$

The solvability of the Neumann problems follows easily from the fact that the vector $\mathbf{v}$ is compactly supported and solenoidal. Let

$$\hat{\mathbf{v}} = \begin{cases} \mathbf{v} - \nabla g^i, & x \in Q_i, \\ \mathbf{v}, & x \in U. \end{cases}$$

Since $(\hat{\mathbf{v}}, n)|_{\partial V} = 0$ and, as is easily verified, $\operatorname{div}\hat{\mathbf{v}} = 0$ in V in the sense of generalized functions, it follows that $\hat{\mathbf{v}} \in \overset{\circ}{\mathbf{J}}(V)$ (see [13], Chapter I, Theorem 1.4).

We prove the inequality (41). Assume that $\varphi(t)$ is a continuous function equal to 1 for $r < r + \rho/3$ and to 0 for $r > r + \rho/2$, and linear in the remaining interval. Obviously, $|\varphi'| \leq 6/\rho$. Since $\rho/3 \geq 2f(2r)$, it follows that $S_1 \subset B_{r+\rho/3}$. By the fact that $\mathbf{v}$ has compact support and by the Neumann conditions,

$$\begin{aligned} \int_{Q_1} |\nabla g^1|^2\,dx &= \int_{S_1} g^1 \frac{\partial g^1}{\partial n}\,dS = \int_{\partial Q_1} g^1 \varphi(r)(\mathbf{v}, n)\,dS \\ &= \int_{Q_1} \sum_{i=1}^3 v_i \frac{\partial}{\partial x_i}(g^1\varphi(r))\,dx \\ &\leq \left[2\int_{Q_1} \mathbf{v}^2\,dx \left(\int_{Q_1} |\nabla g^1|^2\,dx + \frac{36}{\rho^2}\int_{Q_1} |g^1|^2\,dx\right)\right]^{1/2}. \end{aligned} \tag{42}$$

By the Poincaré inequality, for the function g^1

$$\frac{36}{\rho^2}\int_{Q_1} |g^1|^2\,dx \leq \frac{a_3}{\rho^2}(\operatorname{diam} Q_1)^2 \int_{Q_1} |\nabla g^1|^2\,dx \leq a_4 \int_{Q_1} |\nabla g^1|^2\,dx.$$

The inequality

$$\int_{Q_1} |\nabla g^1|^2\,dx \leq 2(1 + a_4)^2 \int_{Q_1} \mathbf{v}^2\,dx$$

now follows from (42). The latter, together with the analogous inequality for g^2, proves (41).

The following assertion is taken from (12) (p. 20).

Lemma 2. *Suppose that* V *is a bounded convex domain in* $\mathbb{R}^3$ *with boundary of class* C^2. *Then for any vector* $\mathbf{v}$ *in* $\overset{\circ}{\mathbf{J}}(V)$ *there is a unique solenoidal vector* $\mathbf{w} \in \mathbf{W}_2^1(V)$, *with zero tangent component on the boundary, such that* $\operatorname{curl} \mathbf{w} = \mathbf{v}$. *Furthermore,*

$$\int_V |\nabla \mathbf{w}|^2 \, dx \leq \int_V \mathbf{v}^2 \, dx .$$

Let $\hat{\mathbf{w}}$ be the vector constructed for the function $\hat{\mathbf{v}}$ according to the lemma. In view of (41) it satisfies the inequality

$$\int_V |\nabla \hat{\mathbf{w}}|^2 \, dx = a_2 \int_V \mathbf{v}^2 \, dx . \tag{43}$$

For brevity we let

$$\mathbf{I} = \int_V \mathbf{v}^2(x) \, dx .$$

Theorem 3 will be proved if we construct a function $h \in W_2^2(U)$ satisfying on $S = \partial U \cap \partial \Omega$ the conditions

$$h|_S = 0, \qquad \left. \frac{\partial h}{\partial n} \right|_S = (\widehat{\mathbf{w}}, n) \tag{44}$$

and such that

$$\int_U \sum_{i,j=1}^{3} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)^2 dx \leq b\mathbf{I}, \tag{45}$$

with a constant b depending only on the α_0 in the condition (13). Indeed, it suffices to set $\mathbf{w} = \widehat{\mathbf{w}} - \nabla h$.

We remark that a direct application of general trace theorems gives us the inequality (45) with the constant b dependent on U, which is not good enough.

Let us consider the mapping $y(x)$ given by the formulas

$$y_i = \frac{x_i}{f(x_3)}, \quad i = 1, 2, \qquad y_3 = \frac{x_3 - r - \rho/3}{f(r)} .$$

It maps V diffeomorphically into some domain V'. Further, the image of U will lie in the cylinder

$$Q = \{y : y_1^2 + y_2^2 < 1 , \ 0 < y_3 < \rho/(3f(r))\} .$$

According to (39), the Jacobian of the mapping $y(x)$ satisfies the estimate

$$\frac{1}{4f^3(r)} \leq \left| \frac{Dy}{Dx} \right| = \frac{1}{f(r) f^2(x_3)} \leq \frac{1}{f^3(r)} . \tag{46}$$

The inequality

$$\int_V \widehat{\mathbf{w}}^2 \, dx \leq a_6 f^2(r) \mathbf{I} \tag{47}$$

will be proved below.

Let $\mathbf{u}(y)$ be a function such that $u(y(x)) \equiv \widehat{w}_1 x_1 + \widehat{w}_2 x_2$. Then by using (46), (47), and (43) it is not hard to get the estimate

$$\begin{aligned} \|u\|^2_{W_2^1(V')} &\le \frac{a_7}{f(r)} \int_V \left(\frac{u_2(x)}{f^2(r)} + |\nabla u(x)|^2 \right) dx \\ &\le a_8 f(r) \int_V \left(\frac{\widehat{\mathbf{w}}^2}{f^2(r)} + |\nabla \widehat{\mathbf{w}}|^2 \right) dx \le a_9 f(r) \mathbf{I}. \end{aligned} \tag{48}$$

We consider a smooth partition of unity $\{\varphi_i(t)\}$, $i \in \mathbb{Z}$, on the real axis such that the support of the function $\varphi_0(t)$ lies in the interval $[\frac{1}{4}, \frac{7}{4}]$ and $\varphi_i(t) = \varphi_0(t-i)$. Let $u_i(y) = u(y)\varphi_i(y_3)$, $i = -1, \dots, n = [\rho/(3f(r))]$.

We prove that for the functions $u_i(y)$ there exist functions $h_i(y)$ satisfying the inclusions $\operatorname{supp} h_i \subseteq \{y \colon y_1^2 + y_2^2 \le 1, i + \frac{1}{8} \le y_3 \le i + \frac{15}{8}\}$, the boundary conditions

$$h_i|_{r'=1} = 0, \qquad \left.\frac{\partial h_i}{\partial n}\right|_{r'=1} = u_i(y),$$

and the inequalities

$$\|h_i\|_{W_2^2(Q)} \le a_{10} \|u_i\|_{W_2^1(V')}, \qquad i = -1, \dots, n.$$

To do this it suffices to describe a way of constructing the function h_0. Let $C_t = Q \cap \{t < y_3 < 2 - t\}$ a cylinder, and let $\widehat{C}$ be a domain with smooth boundary such that $C_{1/8} \subset \widehat{C} \subset C_0$. According to Theorem 8.3 in the book [14] (Chapter 1, §8.2), there is a function g such that

$$g|_{r'=1} = 0, \qquad \left.\frac{\partial g}{\partial n}\right|_{r'=1} = u_0(y), \qquad \|g\|_{W_2^2(\widehat{C})} \le d \|u_0\|_{W_2^1(\widehat{C})},$$

where the constant d does not depend on the function u_0. It remains to set $h_0 = \eta g$, where η is a smooth cutoff function equal to 1 in $C_{1/4}$ and to 0 outside $C_{1/8}$. Denote by $h(y)$ the sum $\sum_{i=-1}^n h_i(y)$. Since each point y belongs to no more than two supports of the functions φ_i, it follows from the inequality (48) that

$$\begin{aligned} \|h\|_{W_2^2(Q)} &\le \sum_{i=-1}^{n} \|h_i\|_{W_2^2(Q)} \le a_{10} \sum_{i=-1}^{n} \|u_i\|_{W_2^1(V')} \\ &\le a_{11} \sum_{i=-1}^{n} \|u\|_{W_2^1(\operatorname{supp} \varphi_i)} \le 2a_{11} \|u\|_{W_2^1(V')} \le 2a_{11} (a_9 f(r) \mathbf{I})^{1/2}. \end{aligned}$$

By using (13), (39), and (46) it is not hard to prove the inequality

$$\int_U \sum_{i,j=1}^{3} \left(\frac{\partial^2 h}{\partial x_i \partial x_j} \right)^2 dx \le \frac{b_1}{f(r)} \int_Q \left(|\nabla_y h|^2 + \sum_{i,j=1}^{3} \left(\frac{\partial^2 h}{\partial y_i \partial y_j} \right)^2 \right) dy,$$

with a constant b_1 depending only on the α_0 in the condition (13). Combining the last two inequalities, we get (45).

It remains to verify that the function $h(y(x))$ satisfies the condition (44). We write the equalities

$$(\nabla h, (x_1, x_2, 0))|_S = (\nabla h, r)|_S = f(x_3) \left.\frac{\partial h}{\partial r}\right|_S = \left.\frac{\partial h}{\partial r'}\right|_{r'=1} = \sum_{i=-1}^{n} u_i = u = (\widehat{\mathbf{w}}, \mathbf{r}).$$

Since $h(y(x))$ is equal to 0 on S, this gives $\nabla h = \widehat{\mathbf{w}}$ on S. Theorem 3 is proved.

Proof of inequality (47). We denote by $\mathbf{q}$ the vector

$$\int_{\mathbf{R}^3} \frac{\hat{\mathbf{v}}(y)\, dy}{|x-y|}$$

and let $\mathbf{u}(x) = (4\pi)^{-1} \operatorname{curl} \mathbf{q}(x)$. Let us prove that $\operatorname{curl} \mathbf{u}(x) = \hat{\mathbf{v}}$. Since $\mathbf{q}(x)$ is a solution of the Poisson equation $\Delta \mathbf{q} = -4\pi \hat{\mathbf{v}}$, to do this it suffices to verify that the vector $\mathbf{q}$ is solenoidal. It is easy to see that for every $\mathbf{v} \in \overset{\circ}{\mathbf{J}}(V)$, extended by zero outside V, $\operatorname{div} \mathbf{v} = 0$ in the sense of generalized functions. Therefore, $\Delta \operatorname{div} \mathbf{q} = -4\pi \operatorname{div} \hat{\mathbf{v}} = 0$, and hence the decreasing harmonic function $\operatorname{div} \mathbf{q}$ is identically zero.

It follows from the equality $\operatorname{curl}(\mathbf{u} - \widehat{\mathbf{w}}) = 0$ that the vector $\mathbf{u} - \widehat{\mathbf{w}}$ is the gradient of a function $g(x)$ defined in V. Further, $\operatorname{div} \nabla g = \operatorname{div}(\mathbf{u} - \widehat{\mathbf{w}}) = 0$, therefore, g is a harmonic function. Since $\mathbf{u}, \widehat{\mathbf{w}} \in \mathbf{W}_2^1(V)$, it is not hard to prove that g belongs to $\mathbf{W}_2^2(V)$.

To prove (47) we establish corresponding estimates for the vectors $\mathbf{u}$ and ∇g.

Lemma 3. *Let $r \geq r_2$ and $\rho \in [6f(2r), r]$. Then there is an absolute constant a_{12} such that for every vector $\mathbf{v} \in \dot{\mathbf{J}}(\Omega)$*

$$\int_{\partial V} \mathbf{u}^2 dS \leq a_{12} f(r) \mathbf{I}, \tag{49}$$

$$\int_V \mathbf{u}^2\, dx \leq a_{12} f^2(r) \mathbf{I}. \tag{50}$$

Proof of Lemma 3. Obviously,

$$|\mathbf{u}(x)| \leq \int_V \frac{|\hat{\mathbf{v}}(y)|\, dy}{|x-y|^2}. \tag{51}$$

Let $C_R = \{x \colon x_1^2 + x_2^2 < R^2\}$, a cylinder. Then for $\alpha \in (1, 3)$

$$\int_{C_R} \frac{dy}{|y|^\alpha} = b(\alpha) R^{3-\alpha}.$$

Since $V \subseteq C_{2f(r)}$, for all $x \in V$

$$\int_V \frac{dy}{|x-y|^2} \leq 4b(2) f(r).$$

By the Cauchy-Schwarz-Bunyakovskiĭ inequality, it follows from (51) that

$$\mathbf{u}^2(x) \leq \int_V \frac{\hat{\mathbf{v}}^2(y)\, dy}{|x-y|^2} \int_V \frac{dy}{|x-y|^2} \leq 4b(2) f(r) \int_V \frac{\hat{\mathbf{v}}^2(y)\, dy}{|x-y|^2}.$$

After integration with respect to x, we have

$$\begin{aligned} \int_V \mathbf{u}^2(x)\, dx &\leq 4b(2) f(r) \int_V \hat{\mathbf{v}}^2(y) \left(\int_V \frac{dx}{|x-y|^2} \right) dy \\ &\leq (4b(2) f(r))^2 \int_V \hat{\mathbf{v}}^2(y)\, dy. \end{aligned}$$

Now (50) follows from (41). Let us apply to (51) the Cauchy-Schwarz-Bunyakovskiĭ inequality in a different form:

$$\mathbf{u}^2(x) \leq \int_V \frac{\hat{\mathbf{v}}^2(y)\, dy}{|x-y|^{3/2}} \int_V \frac{dy}{|x-y|^{5/2}} \leq b(\tfrac{5}{2}) (4f(r))^{1/2} \int_V \frac{\hat{\mathbf{v}}^2(y)\, dy}{|x-y|^{3/2}}.$$

We integrate the last inequality over the boundary of V:

$$\int_{\partial V} \mathbf{u}^2(x)\, dS \leq b(\tfrac{5}{2}) (4f(r))^{1/2} \int_V \hat{\mathbf{v}}(y) \left(\int_{\partial V} \frac{dS_x}{|x-y|^{3/2}} \right) dy. \tag{52}$$

By using the convexity of V and the inclusion $V \subseteq C_{2f(r)}$ it is not hard to prove the inequality

$$\max_{y\in\mathbb{R}^3} \int_{\partial V} \frac{dS_x}{|x-y|^{3/2}} \le a_{13} f^{1/2}(r),$$

which together with (52) and (41) yields (49). Lemma 3 is proved.

Denote by $\nabla_S g$ the tangent component of the vector ∇g on the boundary ∂V. Since the vector $\widehat{\mathbf{w}}$ is collinear to the normal on ∂V, $\nabla_S g$ coincides with the tangent component $\mathbf{u}_S$ of the vector $\mathbf{u}$ on the boundary. Therefore, from (49),

$$\int_{\partial V} |\nabla_S g|^2\, dS \le a_{12} f(r)\mathbf{I}. \tag{53}$$

In view of (50) and the equality $\widehat{\mathbf{w}} = \mathbf{u} - \nabla g$, to prove (47) it suffices to establish the estimate

$$\int_V |\nabla g|^2\, dx \le a_{14} f^2(r)\mathbf{I}. \tag{54}$$

Since a harmonic function has minimum Dirichlet integral among functions with a fixed value on the boundary of a domain, it remains to construct some function $\hat{g}$ satisfying (54) and having the same boundary values as g, i.e.,

$$(\hat{g} - g)|_{\partial V} = 0.$$

On the surface ∂V the function g is an element of $W_2^1(\partial V)$. Consequently, for all $t \in [t_1, t_2]$ its trace is defined on the circle $\gamma_t = \partial V \cap \{x_3 = t\}$ and belongs to $L_2(\gamma_t)$.

We consider a continuous function $\psi(t)$, $t \in \mathbb{R}$, that is constant outside $[t_1, t_2]$ and such that

$$\psi(t) = \frac{1}{2\pi f(t)} \int_{\gamma(t)} g\, d\gamma, \qquad t \in [t_1, t_2].$$

It belongs to $W_2^1([t_1, t_2])$, and

$$\psi'(t) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial g}{dt}\, d\varphi \quad \text{for a.e. } t \in [t_1, t_2].$$

Here the partial derivative is computed in the coordinates (t, φ) on the surface $\partial\Omega$. By (39),

$$\left|\frac{\partial g}{\partial t}\right|^2 \le (1 + f'^2(t))|\nabla_S g|^2 \le 2|\nabla_S g|^2, \quad \text{a.e. } \omega \in S_0.$$

According to the Cauchy-Schwarz-Bunyakovskiĭ inequality,

$$\psi'^2(t) \le \frac{1}{\pi f(t)} \int_{\gamma_t} |\nabla_S g|^2\, d\gamma \quad \text{for a.e. } t \in [t_1, t_2].$$

Using (39) and (53), we can write

$$\begin{aligned} \int_V |\nabla\psi(x_3)|^2\, dx &= \int_V |\psi'|^2\, dx \le \int_{t_1}^{t_2} f(t) \int_{\gamma_t} |\nabla_S g|^2\, d\gamma\, dt \\ &\le 4f(r) \int_{S_0} |\nabla_S g|^2\, dS \le 4a_{12} f^2(r)\mathbf{I}. \end{aligned} \tag{55}$$

It is easy to see that the trace of the function $\psi(x_3)$ on ∂V is an element of $W_2^1(\partial V)$. Denote by h the trace on ∂V of the difference $g - \psi(x_3)$. We establish the estimate

$$\int_{\partial V} |\nabla_S h|^2\, dS \le a_{15} f(r)\mathbf{I}. \tag{56}$$

To do this we estimate the Dirichlet integral of the trace of ψ on ∂V:

$$\int_{\partial V} |\nabla \psi(x_3)|^2 \, dS = \int_{S_0} \psi'^2 \, dS \leq \int_{S_0} \left(\frac{1}{\pi f(t)} \int_{\gamma_t} |\nabla_S g|^2 \, d\gamma \right) dS \leq a_{16} \int_{S_0} |\nabla_S g|^2 \, dS .$$

Now (56) follows from (53).

We prove the inequality

$$\int_{\partial V} h^2 \, dS \leq a_{17} f^2(r) \int_{\partial V} |\nabla_S h|^2 \, dS . \tag{57}$$

Note that for any contour γ_t, $t \in [t_1, t_2]$,

$$\int_{\gamma_t} h \, d\gamma = 0 .$$

The radius of each circle γ_t does not exceed $2f(r)$; therefore, the Poincaré inequality gives us

$$\int_{\gamma_t} h^2 \, d\gamma \leq 4f^2(r) \int_{\gamma_t} |\nabla_S h|^2 \, d\gamma \quad \text{for a.e. } t \in [t_1, t_2] .$$

Multiplying the latter by the appropriate Jacobian and integrating with respect to t, we get

$$\int_{S_0} h^2 \, dS \leq 4f^2(r) \int_{S_0} |\nabla_S h|^2 \, dS . \tag{58}$$

To estimate the integral over S_1 we use an inequality following from (38) for the surfaces $S(t) = \partial V \cap \{x_3 < t\}$, $t \in [t_1, t_1 + f(r)]$:

$$\int_{S_1} h^2 \, dS \leq a_{18} \left(\alpha(t) \int_{\gamma_t} h^2 \, d\gamma + \alpha^2(t) \int_{S(t)} |\nabla_S h|^2 \, dS \right) ,$$

where $\alpha(t)$ is easily seen not to exceed $a_{19} f(r)$. Integrating it with respect to t in $[t_1, t_1 + f(r)]$ and using (58), we get

$$\int_{S_1} h^2 \, dS \leq a_{20} f^2(r) \int_{\partial V} |\nabla_S h|^2 \, dS .$$

The analogous inequality for S_2, together with (58), completes the proof of the estimate (57).

Recall that the radius of curvature r_γ of the normal section γ of the surface ∂V at any point ω is $\geq a f(r)$. Under parallel displacement of the surface by a quantity t along the inner normal the element $d\gamma$ is shortened by a factor of $r\gamma/(r\gamma - t)$. Assuming without loss of generality that $a \geq a_1$, we conclude that for $t < a_1 f(r)$ this ratio does not exceed 2. Consequently, if we regard $h(\omega)$ as a function on the layer F, then the inequality

$$|\nabla_S h| \,|_{t=t} \leq 2 |\nabla_S h| \,|_{t=0}$$

holds for the component of its gradient tangent to ∂V.

Using the inequality $J \leq 1$ for the Jacobian of the system of coordinates (t, ω), we can write the following estimates for the function $\hat{h} = h(\omega)\eta(t)$, where $\eta = 1 - t/a_1 f(r)$ for $t < a_1 f(r)$ and $\eta = 0$ for all other values of t:

$$\begin{aligned} \int_{\partial V} |\nabla \hat{h}|^2 \, dx &\leq \int_0^{a_1 f(r)} dt \int_{\partial V} \left(\left(\frac{\partial \hat{h}}{\partial t} \right)^2 + 4 |\nabla_S \hat{h}|^2 |_{t=0} \right) d\omega \\ &\leq a_1 f(r) \int_{\partial V} \left(\frac{1}{a_1^2 f^2(r)} h^2(\omega) + 4 |\nabla_S h|^2 \right) d\omega . \end{aligned}$$

We now conclude from (57) and (56) that the extension $\hat{h}$ of h from the boundary inside the domain satisfies an estimate of the type (47). Thus, in view of (55) the function $\hat{g} = \hat{h} + \psi(x_3)$ forms an extension of g from the boundary ∂V inside V and satisfies the estimate (47). All the more so, the harmonic function g satisfies this estimate.

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