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SOME SIMPLE GROUPS WHICH ARE DETERMINED BY THEIR CHARACTER DEGREE GRAPHS

S. HEYDARI, N. AHANJIDEH

Abstract. Let G be a finite group, and let $\rho(G)$ be the set of prime divisors of the irreducible character degrees of G. The character degree graph of G, denoted by $\Delta(G)$, is a graph with vertex set $\rho(G)$ and two vertices a and b are adjacent in $\Delta(G)$, if ab divides some irreducible character degree of G. In this paper, we are going to show that some simple groups are uniquely determined by their orders and character degree graphs. As a consequence of this paper, we conclude that M_{12} is not determined uniquely by its order and its character degree graph.

Keywords: Character degree, minimal normal subgroup, Sylow subgroup.

1. Introduction

Throughout this paper, we suppose that all groups are finite and G is a group. We denote by $\mathrm{cd}(G)$, the set of irreducible character degrees of G forgetting multiplicities and also, the set of irreducible character degrees of G counting multiplicities is denoted by $X_1(G)$. The set of prime divisors of $|G|$ forgetting multiplicities is shown by $\pi(G)$. The simple group G is called a simple K_n-group if $|\pi(G)| = n$. There are some characterization of groups according to their irreducible characters. For example, authors in [6, 17] characterized some simple K_4-groups and Mathieu groups according to their orders and some irreducible character degrees. Also, in

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[1, 7], it was proved that some extensions of $L_2(p^n)$ are uniquely determined by their X_1. The character degree graph of G, which is shown by $\Delta(G)$, is a graph with the vertex set $\rho(G)$ and two vertices a and b are adjacent in $\Delta(G)$, if there is some $f \in \mathrm{cd}(G)$ such that $ab \mid f$. Many researchers try to know the properties of $\Delta(G)$. For example, in [14, 15], it was shown that for every group G, the diameter of $\Delta(G)$ is at most 3. Also, White in [16] showed that if G is a simple group, then $\Delta(G)$ is connected unless $G \cong L_2(q)$. In [10], Khosravi and et al. introduced a new characterization of finite groups based on the character degree graph as if G has the same order and the character degree graph as that of a certain group M, then $G \cong M$. Khosravi and et al., in [10], proved that the simple groups of orders less than 6000 are uniquely determined by their character degree graphs and orders and they in [11, 12], showed that $L_2(p)$, $L_2(p^2)$ and some simple groups are determined by their character degree graphs and orders. In this paper, we prove the following:

Theorem 1. *Let G be a finite group, and let $M \in \{M_{11}, M_{22}, M_{23}\}$. Then $G \cong M$ if and only if $\Delta(G) = \Delta(M)$ and $|G| = |M|$. Also, $\Delta(G) = \Delta(M_{12})$ and $|G| = |M_{12}|$ if and only if $G \cong M_{12}$ or $G \cong A_4 \times M_{11}$.*

Throughout this paper, we use the following notations: Let H be a subgroup of G. If H is characteristic in G, then we write H ch G. The set of all p-Sylow subgroups of G is shown by $\mathrm{Syl}_p(G)$. Let b be integer, a be prime and n be natural. If $a^n \mid b$ and $a^{n+1} \nmid b$, then we write $|b|_a = a^n$. If $\chi = \sum_{i=1}^{N} n_i\chi_i$, where for every $1 \le i \le N$, $\chi_i \in \mathrm{Irr}(G)$, then those χ_i with $n_i > 0$ are called irreducible constituents of χ.

In the following, we bring some lemmas, which are used in the proof of Theorem 1:

Lemma 1. [8, Theorem 6.2 and Corollary 11.29] *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. Let θ be an irreducible constituent of χ_N and suppose that $\theta_1 = \theta, ..., \theta_t$ are the distinct conjugates of θ in G. Then $\chi_N = e\sum_{i=1}^{t} \theta_i$, where $e = [\chi_N, \theta]$. Also, $\chi(1)/\theta(1) \mid [G : N]$.*

Lemma 2. (Ito's theorem) [8, Theorem 6.15] *Let G be a finite group, and let A be a normal abelian subgroup of G. Then $\chi(1) \mid [G : A]$, for all $\chi \in \mathrm{Irr}(G)$.*

Lemma 3. [17] *Let G be a non-solvable group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$.*

Lemma 4. [17] *Let G be a finite solvable group of order ${p_1}^{a_1}{p_2}^{a_2}...{p_n}^{a_n}$, where $p_1, p_2, ..., p_n$ are distinct primes. If $kp_n + 1 \nmid {p_i}^{a_i}$ for each $i \le n-1$ and $k > 0$, then the p_n-Sylow subgroup of G is normal in it.*

Lemma 5. (i) [4] *If G is a simple K_3-group, then G is isomorphic to one of the following groups: A_5, A_6, $L_2(7)$, $L_2(8)$, $L_2(17)$, $L_3(3)$, $U_3(3)$ or $U_4(2)$.*
(ii) [1, 13] *If G is a simple K_4-group, then G is isomorphic to one of the following groups:*

(1) *A_7, A_8, A_9, A_{10}, M_{11}, M_{12}, J_2, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^3D_4(2)$, ${}^2F_4(2)'$;*
(2) *$L_2(q)$, where q is a prime power such that $q(q^2-1) = (2, q-1)2^{\alpha_1}3^{\alpha_2}v^{\alpha_3}r^{\alpha_4}$, with $v, r > 3$ distinct prime numbers and for $1 \le i \le 4$, $\alpha_i \in \mathbb{N}$.*

(iii) [9] *If G is a simple K_5-group, then G is isomorphic to one of the following groups:*
$L_2(q)$, *where* $|\pi(q^2-1)| = 4$, $L_3(q)$, *where* $|\pi((q^2-1)(q^3-1))| = 4$, $U_3(q)$, *where* $|\pi((q^2-1)(q^3+1))| = 4$, $O_5(q)$, *where* $|\pi(q^4-1)| = 4$, $Sz(q)$, *where* $q = 2^{2k+1}$ *and* $|\pi((q-1)(q^2+1))| = 4$, $R(q)$, *where q is an odd power of* 3 *and* $|\pi((q^2-1)(q^2-q+1))| = 4$ *or one of the following simple groups:*
$L_4(4)$, $L_4(5)$, $L_4(7)$, $L_5(2)$, $L_5(3)$, $L_6(2)$, $O_7(3)$, A_{11}, A_{12}, $O_9(2)$, $S_6(3)$, $S_8(2)$, $U_4(4)$, $U_4(5)$, $U_4(7)$, $U_4(9)$, $U_5(3)$, $U_6(2)$, $O_8^+(3)$, $O_8^-(2)$, M_{22}, J_3, HS, He, McL, ${}^3D_4(3)$, $G_2(4)$, $G_2(5)$, $G_2(7)$, $G_2(9)$.

(iv) [9] *If G is a simple K_6-group, then G is isomorphic to one of the following groups:*
$L_2(q)$, *where* $|\pi(q^2-1)| = 5$, $L_3(q)$, *where* $|\pi((q^2-1)(q^3-1))| = 5$, $L_4(q)$, *where* $|\pi((q^2-1)(q^3-1)(q^4-1))| = 5$, $U_3(q)$, *where* $|\pi((q^2-1)(q^3+1))| = 5$, $U_4(q)$, *where* $|\pi((q^2-1)(q^3+1)(q^4-1))| = 5$, $O_5(q)$ *where* $|\pi(q^4-1)| = 5$, $G_2(q)$, *where* $|\pi(q^6-1)| = 5$, $Sz(2^{2m+1})$, *where* $|\pi(2^{2m+1}-1)(2^{4m+2}+1))| = 5$, $R(3^{2m+1})$, *where* $|\pi((3^{2m+1}-1)(3^{6m+3}+1))| = 5$ *or one of the following groups:*

$$A_{13}, A_{14}, A_{15}, A_{16}, M_{23}, M_{24}, J_1, Suz, Ru, Co_2, Co_3, Fi_{22}, HN, L_5(7), L_6(3), L_7(2), \\ O_7(4), O_7(5), O_7(7), O_9(3), S_6(4), S_6(5), S_6(7), S_8(3), U_5(4), U_5(5), \\ U_5(9), U_6(3), U_7(2), F_4(2), O_8^+(4), O_8^+(5), O_8^+(7), \\ O_{10}^+(2), O_8^-(3), O_{10}^-(2), {}^3D_4(4), {}^3D_4(5).$$

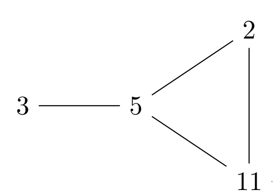
Lemma 6. *For $n \in \{3,4,5,6\}$, let G be a finite K_n-group. If there is not any finite simple group L in Lemma 5 such that $\pi(L) \subseteq \pi(G)$, then G is solvable.*

Proof. It follows immediately from Lemmas 3 and 5. □

2. Proof of the main Theorem.

Proof. First, note that for the irreducible character degrees of the finite groups, we refer the reader to [2]. It is obvious that if $G \cong M$, then $\Delta(G) = \Delta(M)$ and $|G| = |M|$. Thus in the following, assume that $\Delta(G) = \Delta(M)$ and $|G| = |M|$. We continue the proof in the following cases:

i. Let $M = M_{11}$. Then $|G| = |M_{11}| = 2^4.3^2.5.11$ and $\Delta(G) = \Delta(M_{11})$ is as follows:



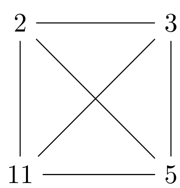
Therefore there exists $\chi \in \mathrm{Irr}(G)$ such that $5.11 \mid \chi(1)$. Now, we claim that G is non-solvable. On the contrary, suppose that G is solvable. Then since for every natural number k, $11k+1 \nmid 2^4, 3^2, 5$, Lemma 4 shows that $P \trianglelefteq G$, where P is a 11-Sylow subgroup of G. But since $|P| = 11$, P is abelian so, Ito's theorem forces $\chi(1) \mid [G:P] = 2^4.3^2.5$ and hence, $5.11 \mid [G:P] = 2^4.3^2.5$, which is impossible. Thus G is non-solvable. Therefore Lemma 3 shows that there is a normal series

$1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic non-abelian simple groups and $|G/K| \mid |\mathrm{Out}(K/H)|$. Now, considering $|G|$ and the order of the non-abelian simple K_3 or K_4-groups mentioned in Lemma 5 (i,ii) implies that

$$K/H \cong A_5, A_6, L_2(11) \text{ or } M_{11}.$$

Let $K/H \cong A_5$. Then since $|\mathrm{Out}(K/H)| = |\mathrm{Out}(A_5)| = 2$, $|H| = 2.3.11$ or $2^2.3.11$. Hence, Lemma 6 guarantees that H is solvable and the same argument as used in the proof of the non-solvability of G leads us to get a contradiction. Also, the same reasoning as above rules out $K/H \cong A_6$. Suppose that $K/H \cong L_2(11)$. Then $|H| = 2.3$ or $2^2.3$. Assume that $\theta \in \mathrm{Irr}(H)$ such that $[\chi_H, \theta] \neq 0$. Then Lemma 1 implies that $\chi(1) = et\theta(1)$, where $t = [G : I_G(\theta)]$. Since $\theta(1) \mid |H|$, $5, 11 \nmid \theta(1)$ and hence, $5.11 \mid et$. On the other hand, $C_G(H) \subseteq I_G(\theta)$. Thus $t \mid [G : C_G(H)]$. Since $G/C_G(H) \hookrightarrow \mathrm{Aut}(H)$, $t \mid |\mathrm{Aut}(H)|$. Now, by GAP [3], we can see that 5 and 11 dose not divide the orders of the automorphism groups of the finite groups of orders 6 and 12. Therefore $5, 11 \nmid t$ and so, $5.11 \mid e$. It follows that $[\chi_H, \chi_H] = e^2 t \geq (5.11)^2 > [G : H]$, which is a contradiction. These contradictions show that $K/H \cong M_{11}$ and hence, $H = 1$ and $G = K \cong M_{11}$.

ii. Let $M = M_{12}$. Then $|G| = |M_{12}| = 2^6.3^3.5.11$ and $\Delta(G) = \Delta(M_{12})$ is as follows:



Thus there are $\chi, \beta, \alpha \in \mathrm{Irr}(G)$ such that $5.11 \mid \chi(1)$, $3.11 \mid \beta(1)$ and $2.11 \mid \alpha(1)$. Now, we claim that G is non-solvable. On the contrary, suppose that G is solvable. Then Lemma 4 and an easy calculation show that $P \trianglelefteq G$, where $P \in \mathrm{Syl}_{11}(G)$, which is a contradiction by considering Ito's theorem and χ. Hence, G is non-solvable.

Let S be a minimal normal solvable subgroup of G. Then S is a r-elementary abelian group. Now, applying Ito's theorem to S and χ forces $r = 2$ or 3. Suppose that N is a maximal normal $\{2, 3\}$-subgroup of G and let L/N be a minimal normal subgroup of G/N such that $L/N \leq C_G(N)N/N$. Suppose that L/N is solvable. Then for some $t \in \pi(G)$, L/N is a t-elementary abelian group. Now, our assumption on N implies that $t \neq 2, 3$. Hence, $t = 5$ or 11. Since N and L/N are solvable, we conclude that L is solvable. If $11 \mid |L|$, then the same argument as used in the proof of the non-solvability of G leads us to get a contradiction. Thus $t = 5$. Since $|G|_5 = 5$ and $5 \mid |L|$, $5 \nmid |G/L|$. Hence, considering $|G|$ shows that $\pi(G/L) \subseteq \{2, 3, 11\}$. Therefore Lemma 6 guarantees that G/L is solvable. But this is a contradiction, because G is non-solvable and L is solvable. Hence, L/N is non-solvable and so, it is a direct product of isomorphic non-abelian simple groups. Now, considering $|G|$ and Lemma 5(i,ii) shows that

$$L/N \cong A_5, A_6, L_2(11), M_{11} \text{ or } M_{12}. \tag{1}$$

Let C/N be a minimal normal subgroup of G/N such that $C/N \leq C_{G/N}(L/N)$. Then $C/N = 1$ or applying the same reasoning as used for L/N shows that C/N

is isomorphic to one of the groups in 1. Assume that $C/N \neq 1$.

Now, considering the orders of the groups mentioned in 1 shows that $5 \mid |L/N|$. Thus since $L/N \cap C/N = 1$ and $|G|_5 = 5$, $5 \nmid |C/N|$ and so, C/N is not isomorphic to any groups in 1, which is a contradiction. Thus $C/N = 1$ so, $C_{G/N}(L/N) = 1$ and hence,

$$G/N \hookrightarrow \mathrm{Aut}(L/N).$$

If $L/N \cong A_5$ or A_6, then $11 \nmid |\mathrm{Aut}(L/N)|$, which is a contradiction, because $11 \mid |G/N|$.

Let $L/N \cong L_2(11)$. Then since $\mathrm{Aut}(L_2(11)) = PGL_2(11)$, $G/N \cong L_2(11)$ or $PGL_2(11)$. On the other hand, $L/N \leq C_G(N)N/N \leq G/N$. It follows that $C_G(N)N \cong G$ or $C_G(N)N \cong L$. Thus considering $|G|$, $|L|$ and $|N|$ shows that $5, 11 \mid |C_G(N)|$ and so, $5, 11 \nmid |G/C_G(N)|$. Let $\theta \in \mathrm{Irr}(N)$ such that $[\chi_N, \theta] \neq 0$. Then Lemma 1 shows that $\chi(1) = es\theta(1)$, where $s = [G : I_G(\theta)]$. Now, we can see that $5, 11 \nmid \theta(1)$, because $\theta(1) \mid |N|$. Moreover, since $C_G(N) \leq I_G(\theta)$, the fact that $5, 11 \nmid |G/C_G(N)|$ implies that $5, 11 \nmid s = [G : I_G(\theta)]$ and hence, $5.11 \mid e$. Thus we obtain $[\chi_N, \chi_N] = e^2 s \geq (11.5)^2 > [G : N]$, which is a contradiction.

Let $L/N \cong M_{11}$. Then since $G/N \hookrightarrow \mathrm{Aut}(L/N)$ and $\mathrm{Aut}(M_{11}) = M_{11}$, we conclude that $G/N \cong L/N \cong M_{11}$ and so, $|N| = 12$. Now, by GAP, we can see that $\pi(\mathrm{Aut}(N)) \subseteq \{2, 3\}$. Therefore $\pi(G/C_G(N)) \subseteq \{2, 3\}$, because $G/C_G(N) \hookrightarrow \mathrm{Aut}(N)$. Hence, $C_G(N)$ is non-solvable. Also, $5, 11 \mid |C_G(N)|$. On the other hand, $C_G(N)N/N \leq G/N \cong M_{11}$. Thus $C_G(N)/C_G(N) \cap N \cong C_G(N)N/N \cong M_{11}$. Let $C_G(N) = (C_G(N))'$. Then since $C_G(N) \cap N \leq Z(C_G(N))$ and $\mathrm{Mult}(M_{11}) = 1$, we deduce that $C_G(N) \cong (C_G(N) \cap N) \times M_{11}$. Now, since $C_G(N) \cap N = Z(N)$ is abelian, we conclude that $\mathrm{cd}(C_G(N)) = \mathrm{cd}(M_{11})$. Also, by GAP, we get $|C_G(N) \cap N| = |Z(N)| \in \{1, 2, 12\}$.

Let $|Z(N)| = 2$. Then $|C_G(N)| = 2^5.3^2.5.11$. Let $\gamma \in \mathrm{Irr}(C_G(N))$ such that $[\beta_{C_G(N)}, \gamma] \neq 0$. Then Lemma 1 implies that $\beta(1) = es\gamma(1)$, where $s = [G : I_G(\gamma)]$ and also, $11 \mid \gamma(1)$. Now, if $3 \mid \gamma(1)$, then 3.11 divides some irreducible character degree of M_{11}, which is a contradiction. Thus $3 \mid e$ or $3 \mid s$. Let $3 \mid e$. Then $e^2 s \geq 3^2 > [G : C_G(N)] = 6$, which is a contradiction. Hence, $3 \mid s$. So, $C_G(N)$ has at least 3 irreducible characters of the same degrees such that 11 divides them. Now, since $Z(N)$ has two irreducible characters whose degrees are 1, we deduce that M_{11} has at least two irreducible characters of the same degrees such that 11 divides them. But since $X_1(M_{11}) = \{1, 10, 10, 10, 11, 16, 16, 44, 45, 55\}$, we get a contradiction.

Let $|Z(N)| = |C_G(N) \cap N| = 1$. Then $C_G(N)/C_G(N) \cap N = C_G(N) \cong M_{11}$. Thus $N \times C_G(N) \cong N \times M_{11} \trianglelefteq G$. Now, since $|N \times M_{11}| = |G|$, $N \times M_{11} \cong G$. Let $\beta_1 \in \mathrm{Irr}(N)$ and $\beta_2 \in \mathrm{Irr}(M_{11})$ such that $\beta = \beta_1 \times \beta_2$. Then since $3.11 \mid \beta(1)$ and $11 \nmid |N|$, $11 \mid \beta_2(1)$. Also, since in $\Delta(M_{11})$, 11 is not adjacent to 3, we conclude that $3 \mid \beta_1(1)$. Thus by GAP, we can see that $\mathrm{cd}(N) = \{1, 3\}$ and also, $N \cong A_4$. Hence, $G \cong A_4 \times M_{11}$, as desired.

Assume that $|Z(N)| = |C_G(N) \cap N| = 12 = |N|$. Then $N \leq C_G(N)$ and so, $C_G(N) \cong N \times M_{11}$. Hence, $C_G(N) = G$, because $|C_G(N)| = |N \times M_{11}| = |G|$. Thus $N \leq Z(G)$ and so, $\mathrm{cd}(G) = \mathrm{cd}(M_{11})$, which is a contradiction.

Now, we suppose that $(C_G(N))' < C_G(N)$. Since $C_G(N)$ is non-solvable, for some natural number n, $C_G^{(n)}(N) = C_G^{(n+1)}(N)$. Also, $C_G^{(n)}(N)N/N \cong C_G^{(n)}(N)/C_G^{(n)}(N) \cap N \cong M_{11}$. Now, $C_G^{(n)}(N) \cap N \leq C_{C_G^{(n)}(N)}(C_G^{(n)}(N) \cap N)$, because $C_G^{(n)}(N) \cap$

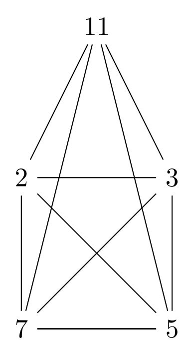
N is abelian. Since $C_G^{(n)}(N)/C_{C_G^{(n)}(N)}(C_G^{(n)}(N) \cap N) \le \mathrm{Aut}(C_G^{(n)}(N) \cap N)$ and $C_G^{(n)}(N) \cap N \le N$, we conclude that $5, 11 \mid |C_{C_G^{(n)}(N)}(C_G^{(n)}(N) \cap N)|$. Thus $C_{C_G^{(n)}(N)}(C_G^{(n)}(N) \cap N)/C_G^{(n)}(N) \cap N \cong C_G^{(n)}(N)/C_G^{(n)}(N) \cap N \cong M_{11}$. Therefore $C_{C_G^{(n)}(N)}(C_G^{(n)}(N) \cap N) = C_G^{(n)}(N)$ and so, $C_G^{(n)}(N) \cap N \le Z(C_G^{(n)}(N))$. Hence, since $C_G^{(n)}(N) = C_G^{(n+1)}(N)$ and $\mathrm{Mult}(M_{11}) = 1$, we deduce that $C_G^{(n)}(N) \cong (C_G^{(n)}(N) \cap N) \times M_{11}$. On the other hand, $|C_G(N)N/N| = |C_G(N)/C_G(N) \cap N| = |C_G^{(n)}(N)/C_G^{(n)}(N) \cap N| = |M_{11}|$. Thus $C_G^{(n)}(N) \cap N < C_G(N) \cap N$, because $C_G^{(n)}(N) < C_G(N)$. Hence, since $|C_G(N) \cap N| = |Z(N)| \in \{1, 2, 12\}$, $|C_G^{(n)}(N) \cap N| = 1$ or $C_G^{(n)}(N) \cap N$ is an abelian group of order $2, 3$, 4 or 6. Therefore $|C_G^{(n)}(N)| = 2^4.3^2.5.11, 2^5.3^2.5.11$, $2^4.3^3.5.11, 2^6.3^2.5.11$ or $2^5.3^3.5.11$ and also, $\mathrm{cd}(C_G^{(n)}(N)) = \mathrm{cd}(M_{11})$.

Now, if $|C_G^{(n)}(N)| = 2^4.3^2.5.11$, then the same argument as used when $(C_G(N))' = C_G(N)$ shows that $G \cong A_4 \times M_{11}$, as claimed.

If $|C_G^{(n)}(N)| = 2^5.3^2.5.11$, then the same argument as used when $(C_G(N))' = C_G(N)$, leads us to get a contradiction. Let $|C_G^{(n)}(N)| = 2^4.3^3.5.11$ or $2^5.3^3.5.11$ and let $\theta \in \mathrm{Irr}(C_G^{(n)}(N))$ such that $[\beta_{C_G^{(n)}(N)}, \theta] \ne 0$. Then Lemma 1 implies that $3.11 \mid \theta(1)$, which is a contradiction, because $\theta(1) \in \mathrm{cd}(M_{11})$. Also, when $|C_G^{(n)}(N)| = 2^6.3^2.5.11$, since $C_G^{(n)}(N) < C_G(N)$, considering $|G|$ shows that $C_G(N) = G$ and so, $N \le Z(G)$. Thus $G \cong N \times M_{11}$, because $G/N \cong M_{11}$ and $\mathrm{Mult}(M_{11}) = 1$. Hence, $\mathrm{cd}(G) = \mathrm{cd}(M_{11})$, which is a contradiction.

These show that $L/N \cong M_{12}$ and so, $G = L \cong M_{12}$.

iii. Let $M = M_{22}$. Then $|G| = |M_{22}| = 2^7.3^2.5.7.11$ and $\Delta(G) = \Delta(M_{22})$ is as follows:



Hence, there exist $\chi, \beta \in \mathrm{Irr}(G)$ such that $7.11 \mid \chi(1)$ and $5.11 \mid \beta(1)$. If G is solvable, then Lemma 4 and an easy calculation imply that a 11-Sylow subgroup of G is normal in it, which contradicts Ito's theorem. Thus G is non-solvable.

Assume that S is a minimal normal solvable subgroup of G. Then for some $r \in \pi(G)$, S is a r-elementary abelian group. Now, applying Ito's theorem to S and χ leads us to see that $r = 2$ or 3.

Assume that N is a maximal normal $\{2, 3\}$-subgroup of G and suppose that L/N is a minimal normal subgroup of G/N such that $L/N \le C_G(N)N/N$. Then we claim that L/N is non-solvable. On the contrary, suppose that L/N is solvable. Then for some $t \in \pi(G)$, L/N is a t-elementary abelian group. Now, our assumption on N and the fact that $|L/N| \mid |G|$ show that $t = 5, 7$ or 11. Since L/N and N are solvable, we conclude that L is solvable. Thus G/L is non-solvable, because G is non-solvable. Hence, considering Lemma 3 shows that

$$3 \mid |G/L| \text{ and } |G/L|_2 \geq 2^2. \tag{2}$$

Now, suppose that N is abelian. Then $L/N \leq C_G(N)/N$. It follows that there is a t-subgroup Q of G such that $L = Q \times N \trianglelefteq G$ and so, $Q \trianglelefteq G$, which is a contradiction by considering Ito's theorem and β and χ. Thus in the following, we assume that N is non-abelian.

If $t = 11$, then $11 \mid |L|$. Suppose that $P \in \mathrm{Syl}_{11}(L)$. Then P ch $L \trianglelefteq G$. Since $|P| = 11$, P is abelian. Hence, Ito's theorem shows that $\chi(1) \mid [G : P]$, which is impossible. Hence, $t = 5$ or 7.

Suppose that $t = 5$. If $|L|_2 = |N|_2 \leq 8$, then P ch $L \trianglelefteq G$, where $P \in \mathrm{Syl}_5(L)$. Now, considering Ito's theorem and β leads us to get a contradiction. Thus $|L|_2 \geq 16$. Hence, $|G/L| \mid 2^3.3^2.7.11$. Also, 2 implies that $2^2.3.7.11 \mid |G/L|$. Now, since G/L is non-solvable, considering Lemmas 3 and 5 shows that G/L has a normal series $1 \trianglelefteq H/L \trianglelefteq K/L \trianglelefteq G/L$ such that $\frac{K/L}{H/L} \cong L_2(7)$ or $L_2(8)$. Thus $11 \mid |H/L|$ and $|H/L| \mid 3.11$ so, H/L and consequently, H is solvable and $11 \mid |H|$. Let $P \in \mathrm{Syl}_{11}(H)$. Then P ch $H \trianglelefteq G$. Now, applying Ito's theorem to P and χ leads us to get a contradiction.

Now, suppose that $t = 7$. If $|L|_2 = |N|_2 \leq 2^2$, then a 7-Sylow subgroup of L is normal in L. It follows that G has a normal abelian 7-Sylow subgroup. But considering Ito's theorem and χ leads us to get a contradiction. Hence, $|N|_2 \geq 8$ and so, according to 2, we conclude that $|N| \in \{8, 3.8, 16, 3.16, 32, 3.32\}$.

Suppose that $|N| \in \{8, 3.8, 16, 32\}$. Then by GAP, we can see that $7 \nmid |\mathrm{Aut}(N)|$. Assume that $\gamma \in \mathrm{Irr}(L)$ such that $[\chi_L, \gamma] \neq 0$. Then Lemma 1 implies that $7 \mid \gamma(1)$. Let $\mu \in \mathrm{Irr}(N)$ such that $[\gamma_N, \mu] \neq 0$. Then Lemma 1 shows that $\gamma(1) = es\mu(1)$, where $s = [L : I_L(\mu)]$. Now, $7 \nmid \mu(1)$, because $7 \nmid |N|$ and so, $7 \mid es$. On the other hand, $L/C_L(N) \hookrightarrow \mathrm{Aut}(N)$. Now, since $7 \nmid |\mathrm{Aut}(N)|$, $7 \nmid |L/C_L(N)|$. Hence, $7 \nmid s = [L : I_L(\mu)]$, because $C_L(N) \subseteq I_L(\mu)$. It follows that $7 \mid e$ and so, $e^2 s \geq 7^2 > [L : N] = 7$, which is a contradiction. Assume that $|N| \in \{3.16, 3.32\}$. If $7 \nmid |\mathrm{Aut}(N)|$, then the same reasoning as above leads us to get a contradiction. Thus $7 \mid |\mathrm{Aut}(N)|$. Now, by GAP, we can see that if $|N| = 3.16$, then $|Z(N)| = 8$ and if $|N| = 3.32$, then $|Z(N)| = 16$ or 8. Since $Z(N)$ ch $N \trianglelefteq L$, $Z(N) \trianglelefteq L$. Suppose that when $|N| = 3.32$, $|Z(N)| = 16$. Then $|L/Z(N)| = 2.3.7$. Now, by replacing N with $Z(N)$ in the above argument, we get a contradiction. Now, assume that $|N| = 3.32$ and $|Z(N)| = 8$ and suppose that $P \in \mathrm{Syl}_2(N)$. Then by GAP, $P \trianglelefteq N$ hence, P ch $N \trianglelefteq L$ and so, $P \trianglelefteq L$. Thus $|L/P| = 3.7$. Now, the same reasoning as above leads us to get a contradiction.

These contradictions show that L/N is non-solvable. So, it is a direct product of isomorphic non-abelian simple groups. Now, considering $|G|$ and Lemma 5 shows that

$$L/N \cong A_5, A_6, A_7, A_8, L_3(4), L_2(7), L_2(8), L_2(11), M_{11} \text{ or } M_{22}. \tag{3}$$

Let C/N be a minimal normal subgroup of G/N such that $C/N \leq C_{G/N}(L/N)$. Then $C/N = 1$ or applying the same argument as above implies that C/N is isomorphic to one of the groups in 3. Assume that $C/N \neq 1$.

Suppose that $L/N \cong A_5$. Then considering $|G/N|$ and the fact that $L/N \cap C/N = 1$ shows that $C/N \cong L_2(7)$. Set $D/N := L/N \times C/N$. Then $D/N \trianglelefteq G/N$. Now, we claim that $C_{G/N}(D/N) = 1$. On the contrary, suppose that $C_{G/N}(D/N) \neq 1$ and assume that R/N is a minimal normal subgroup of G/N such that $R/N \leq C_{G/N}(D/N)$. Then the same argument as used for L/N forces R/N to be isomorphic to one of the groups in 3. But since $R/N \cap D/N = 1$, considering $|D/N|$ shows that $|R/N| \mid 2^2.11$ and so, R/N is solvable, which is a contradiction. This contradiction shows that $C_{G/N}(D/N) = 1$ and so, $G/N \hookrightarrow \mathrm{Aut}(D/N) = \mathrm{Aut}(A_5 \times L_2(7)) = S_5 \times PGL(2,7)$. Now, since $11 \nmid |S_5 \times PGL(2,7)|$, we deduce that $11 \nmid |G/N|$, which is a contradiction.

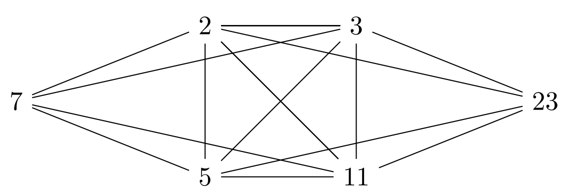
Also, if $L/N \cong L_2(7)$, then we can see that $C/N \cong A_5$ or $L_2(11)$ and so, $A_5 \times L_2(7) \trianglelefteq G/N$ or $L_2(7) \times L_2(11) \trianglelefteq G/N$. If $L/N \times C/N \cong A_5 \times L_2(7)$, then the same argument as the previous case leads us to get a contradiction. Thus $L/N \times C/N \cong L_2(7) \times L_2(11)$. Now, considering $|G|$ shows that $|N| \mid 2^2$. Let $\iota \in \mathrm{Irr}(C)$ such that $[\beta_C, \iota] \neq 0$. Then Lemma 1 shows that $5.11 \mid \iota(1)$. If $N = 1$, then $C \cong L_2(11) \trianglelefteq G$. But considering $\iota(1)$ leads us to get a contradiction, because $\mathrm{cd}(C) = \mathrm{cd}(L_2(11)) = \{1, 11, 10, 12, 5\}$. Thus $N \neq 1$ and so, $|N| = 4$ or 2. Let $\vartheta \in \mathrm{Irr}(N)$ such that $[\iota_C, \vartheta] \neq 0$. Then by Lemma 1, we obtain $\iota(1) = et\vartheta(1)$, where $t = [C : I_C(\vartheta)]$. Since N is abelian, $\vartheta(1) = 1$ and so, $5.11 \mid et$. On the other hand, $t \mid |\mathrm{Aut}(N)|$. Now, since $5, 11 \nmid |\mathrm{Aut}(N)|$, $5, 11 \nmid t$ and hence, $5.11 \mid e$. Therefore $[\iota_N, \iota_N] = e^2 t \geq (5.11)^2 > [C : N] = |L_2(11)|$, which is a contradiction.

Assume that $L/N \cong L_2(11)$. Then we can see that $C/N \cong L_2(7)$. Now, the same argument as used in the previous case leads us to get a contradiction.

Suppose that $L/N \cong A_6$, A_7, A_8, $L_2(8)$, $L_3(4)$ or M_{11}. Then considering $|G/N|$ and the fact that $L/N \cap C/N = 1$ shows that $3 \nmid |C/N|$ and so, C/N is not isomorphic to any groups mentioned in 3, which is a contradiction.

These contradictions imply that $C_{G/N}(L/N) = 1$ and hence, $G/N \hookrightarrow \mathrm{Aut}(L/N)$. Now, if $L/N \cong A_5, A_6, A_7, A_8, L_3(4), L_2(7), L_2(8), L_2(11)$ or M_{11}, then considering $|\mathrm{Aut}(L/N)|$ shows that 7 or 11 dose not divide $|\mathrm{Aut}(L/N)|$. But this is a contradiction, because $7, 11 \mid |G/N|$. Therefore $L/N \cong M_{22}$ and so, $G \cong M_{22}$.

iv. Let $M = M_{23}$. Then $|G| = |M_{23}| = 2^7.3^2.5.7.11.23$ and $\Delta(G) = \Delta(M_{23})$ is as follows:



Thus there are $\chi, \beta, \alpha \in \mathrm{Irr}(G)$ such that $11.23 \mid \chi(1)$, $5.11 \mid \beta(1)$ and $7.11 \mid \alpha(1)$. Now, if G is solvable, then Lemma 4 and an easy calculation show that a 23-Sylow subgroup P of G is normal in G. But applying Ito's theorem to P and χ leads us to get a contradiction. Thus G is non-solvable. Hence, considering Lemmas 3 and 5 and $|G|$ shows that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that

$$K/H \cong A_5, A_6, A_7, A_8, L_3(4), L_2(7), L_2(8), L_2(11), M_{11}, M_{22} \text{ or } M_{23}.$$

Let $\theta, \eta, \lambda \in \mathrm{Irr}(H)$ such that $[\chi_H, \theta] \neq 0$, $[\beta_H, \eta] \neq 0$ and $[\alpha_H, \lambda] \neq 0$.

First, suppose that $K/H \cong A_5$. Then $|G/K| \mid |\mathrm{Out}(A_5)| = 2$. Hence, $|H| =$

$2^4.3.7.11.23$ or $2^5.3.7.11.23$. Thus Lemma 1 implies that $11.23 \mid \theta(1)$ and $7.11 \mid \lambda(1)$. If H is solvable, then the same reasoning as used in the proof of the non-solvability of G leads us to get a contradiction. Thus H is non-solvable. Therefore considering Lemmas 3 and 5 and $|H|$ shows that H has a normal series $1 \trianglelefteq N \trianglelefteq R \trianglelefteq H$ such that $R/N \cong L_2(7)$ or $L_2(23)$. Let $R/N \cong L_2(7)$. Then $11.23 \mid |N|$ and $|N| \mid 2^2.11.23$. Thus Lemma 6 shows that N is solvable and the same argument as proving the non-solvability of H leads us to get a contradiction. Hence, $R/N \cong L_2(23)$ and so, $7 \mid |N|$ and $|N| \mid 2^2.7$. Suppose that $P \in \mathrm{Syl}_7(N)$. Then P ch $N \trianglelefteq H$. But applying Ito's theorem to P and λ leads us to get a contradiction.

Let $K/H \cong L_2(7)$. Then $|H| = 2^4.3.5.11.23$ or $2^3.3.5.11.23$ and by Lemma 1, we have $11.23 \mid \theta(1)$ and $5.11 \mid \eta(1)$. Also, the same reasoning as used in the proof of the non-solvability of G leads us to see that H is non-solvable. So, there is a normal series $1 \trianglelefteq N \trianglelefteq R \trianglelefteq H$ such that $R/N \cong A_5, L_2(11)$ or $L_2(23)$. Let $R/N \cong L_2(23)$. Then $5 \mid |N|$ and $|N| \mid 2.5$. Let $P \in \mathrm{Syl}_5(R)$. Then we can check at once $P \trianglelefteq G$. But since P is abelian, applying Ito's theorem to P and η leads us to get a contradiction. Suppose that $R/N \cong A_5$ or $L_2(11)$. Then an easy calculation shows that N is solvable and $23 \mid |N|$. Let $Q \in \mathrm{Syl}_{23}(N)$. Then Q ch $N \trianglelefteq H$ and so, $Q \trianglelefteq H$. But considering Ito's theorem and θ leads us to get a contradiction.

Also, the same argument as the above cases rules out $K/H \cong L_2(11)$.

If $K/H \cong L_2(8), A_6, A_7, A_8, L_3(4), M_{11}$ or M_{22}, then we can see that $23 \mid |H|$ and H is solvable and the same argument as used in the above cases leads us to get a contradiction.

Thus $K/H \cong M_{23}$ and hence, $G \cong M_{23}$. □

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