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ON RECOGNITION BY SPECTRUM OF SYMMETRIC GROUPS

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ABSTRACT. The spectrum of a group is the set of its element orders. A finite group G is said to be recognizable by spectrum if every finite group with the same spectrum is isomorphic to G. We prove that if $n \in \{15, 16, 18, 21, 27\}$ then symmetric groups Sym_n are recognizable by spectrum.

Keywords: finite group, simple group, symmetric group, spectrum of a group, recognizability by spectrum.

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of its order, $\omega(G)$ be the spectrum of G, i.e. the set of its element orders. The Gruenberg-Kegel graph, or the prime graph, $GK(G)$ is defined as follows. The vertex set of the graph is $\pi(G)$. Two distinct primes p and q of $\pi(G)$ seen as verticies of the graph $GK(G)$, are connected by an edge if and only if $pq \in \omega(G)$. A group G is said to be recognizable by spectrum (shortly, recognizable) if for every finite group L the equality $\omega(L) = \omega(G)$ implies that $L \simeq G$. Two groups are said to be isospectral if they have the same spectra. Denote the symmetric group of degree n by Sym_n.

It was proved in [1, 2, 3, 4] that if $n \in \{2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 14\}$ then the group Sym_n is recognizable. It was shown in [5] that Sym_p is recognizable where p is a prime and $p > 13$, there were also obtained strong constraints on a group with the same spectrum as Sym_{p+1}. It was shown in [6] that Sym_n is recognizable if $n \notin \{2, 3, 4, 5, 6, 8, 10, 15, 16, 18, 21, 27, 33, 35, 39, 45\}$, there it was also proved

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that if Sym_{16} is recognizable then the groups $Sym_{33}, Sym_{35}, Sym_{39}, Sym_{45}$ are recognizable too.

In this paper we prove recognizability of the symmetric groups

$$Sym_n, \quad n \in \{15, 16, 18, 21, 27\}.$$

Theorem 1. *The group Sym_n, where $n \in \{15, 16, 18, 21, 27\}$, is recognizable.*

Corollary 1. *The group Sym_n, where $n \in \{33, 35, 39, 45\}$, is recognizable.*

Corollary 2. *The recognizability problem for Sym_n, $n \neq 10$, is solved.*

2. Preliminaries

Lemma 1 ([7, Lemma 2.2]). *Let $S = P_1 \times ... \times P_r$, where P_i are isomorphic non-Abelian simple groups. Then $Aut(S) \simeq (Aut(P_1) \times ... \times Aut(P_r)).Sym_r$.*

Lemma 2 ([8, Theorem 3.1]). *Given a Frobenius group G with kernel A and complement B, we have*

(a) A is nilpotent;

(b) every Sylow p-subgroup of B is a cyclic group for an odd prime p, and a cyclic or generalized quaternion group for $p = 2$.

Lemma 3 ([9, Proposition 1]). *Let G be a finite group, $t(G) \geq 3$, and let K be the maximal normal soluble subgroup of G. Then for every subset ρ of primes in $\pi(G)$ such that $|\rho| \geq 3$ and all primes in ρ are pairwise nonadjacent in $GK(G)$, the intersection $\pi(K) \cap \rho$ contains at most one number. In particular, G is insoluble.*

Lemma 4 ([10, Lemma 3.6]). *Let s and p be distinct primes, a group H be a semidirect product of a normal p-subgroup T and a cyclic subgroup $C = \langle g \rangle$ of order s, and let $[T, g] \neq 1$. Suppose that H acts faithfully on a vector space V of positive characteristic t not equal to p. If the minimal polynomial of g on V does not equal $x^s - 1$, then*

(i) $C_T(g) \neq 1$;

(ii) T is non-Abelian;

(iii) $p = 2$ and $s = 2^{2^\delta} + 1$ is a Fermat prime.

Lemma 5 ([11, Lemma 14]). *Any odd element from $\pi(Out(P))$ where P is any simple group, either belongs to the spectrum of P or does not exceed $m/2$, where $m = max_{p \in \pi(P)} p$.*

Lemma 6 ([5, Lemma 6]). *Let H be a finite group and let V be a proper normal subgroup of H such that H/V is isomorphic to Alt_m. Then $\omega(H) \nsubseteq \omega(Sym_m)$ provided that $m \geq 6$ and $m \neq 8$.*

Lemma 7 ([5]). *Recognizability of the symmetric group of degree $r + 1$, where $r \geq 17$ is prime, amounts to the following: for every proper covering $G = N.A$ of an arbitrary finite group N by a group A isomorphic to Sym_r or Alt_r, the inequality $\omega(G) \neq \omega(Sym_{r+1})$ holds.*

Lemma 8 ([6, Theorem 2]). *If Sym_{16} is recognizable then the groups*

$$Sym_{33},\ Sym_{35},\ Sym_{39},\ Sym_{45}$$

are recognizable too.

Lemma 9 ([12, Lemma 1]). *If a Frobenius group FC with kernal F and cyclic complement $C = \langle c \rangle$ of order n acts faithfully on a vector space V of nonzero characteristic p coprime with the order of F then the natural semidirect product VC contains an element of order $p \cdot n$.*

3. Proof of Main Theorem for Sym_{15}

Proposition 1. *The group Sym_{15} is recognizable.*

Let $\omega = \omega(G) = \omega(Sym_{15})$, K be the maximal normal soluble subgroup of G, $S = Soc(G/K) \simeq S_1 \times ... \times S_n$, where $S_i, 1 \leq i \leq n$ are non-Abelian simple groups. Obviously, the prime divisors of $|S|$ are not greater than 13. Using the classification of finite simple groups it is not hard to obtain the full list of all finite simple groups L with the property $\pi(L) \subseteq \{2, 3, 5, 7, 11, 13\}$ (see [13]).

Lemma 10. *The group S is a finite simple group.*

Proof. Let $\overline{G} = G/K, \widetilde{G} = \overline{G}/S$. Obviously, $\overline{G} \leq Aut(S)$ and $\widetilde{G} \leq Out(S)$. Suppose that $n > 1$. By Lemma 3 we may assume that there exists $p \in \{11, 13\}$ such that $p \notin \pi(K)$. Suppose that $|\widetilde{G}|$ is divisible by p. Then $\overline{G}$ contains an element g of order p that acts by conjugation on S and induces an outer automorphism. We have $Out(S) \simeq Out(P_1) \times ... \times Out(P_r)$, where the groups P_j are direct products of isomorphic S_i. For some j, therefore, $g \in Out(P_j)$. It follows by Lemma 1 that $g \in Out(S_i)$ or $S_i^g \neq S_i$. By [13], for all non-Abelian finite simple groups R with the property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13\}$ except for $R \simeq L_3(3)$, we have $\{5, 7\} \cap \pi(R) \neq \varnothing$. Assume that there exists $1 \leq i \leq n$ such that $S_i \not\simeq L_3(3)$, we can assume that $i = 1$. Suppose that $S_1^g = S_1$. By Lemma 5, g is not an outer automorphism of a group $S_j, j \in \{1, ..., n\}$. Hence $S_1 \leq C_{\overline{G}}(g)$ and so $\overline{G}$ has an element whose order is equal to pt, where $t \in \{5, 7\} \cap \pi(S_1)$, but $pt \notin \omega$. Thus $S_1 \neq S_1^g$. Let $x = hh^g h^{g^2} ... h^{g^{p-1}}, h \in S_1, |h| \in \{5, 7\} \cap \pi(S_1)$. It is easy to check that $x \in C_{\overline{G}}(g)$, $|x| = |h|$. Hence $\overline{G}$ contains an element y and $|y| = p|h|$, but $p|h| \notin \omega$ and so $S_i \simeq L_3(3)$ for all $1 \leq i \leq n$. We have $\{3, 13\} \subset \pi(L_3(3))$. The group S has an element of order 39, since $n > 1$, but $39 \notin \omega$.Thus $p \in \pi(S)$.

Suppose that there exists S_i such that $13 \in \pi(S_i)$. By [13], for all non-Abelian finite simple groups R with the property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13\}$, we have $\{3, 5\} \cap \pi(R) \neq \varnothing$. Let $g \in S_i, |g| = 13, h \in S_j, i \neq j, |h| \in \{3, 5\} \cap \pi(S_j)$. Then $|gh| = 13|h|$, but $13|h| \notin \omega$. Hence $11 \in \pi(S)$. It is easy to check that there exists $x \in S$ and $|x| = 11t$, where $t \in \{5, 7\} \cap \pi(S)$; a contradiction. Then $n = 1$. □

By Lemma 10, we may assume that S is a non-Abelian finite simple group and $\pi(S) \subseteq \{2, 3, 5, 7, 11, 13\}$.

Lemma 11. $11, 13 \in \pi(S)$.

Proof. Assume that $13 \notin \pi(S)$. It follows from Lemmas 3, 5 and [14] that $\{5, 7, 11\} \subseteq \pi(S)$, $\{5, 7, 11\} \cap \pi(|G|/|S|) = \varnothing$. By Lemmas 5 and 10 we have $13 \in \pi(K)$. Hence $35 \in \omega(S)$. From [13] and [14], it follows that $S \simeq Alt_{12}$. Note that S contains a subgroup T isomorphic to a Frobenius group with kernel of order 11 and complement of order 5. Let $P \in Syl_{13}(K), N = N_G(P)$. Since $N_G(P)/N_K(P) \simeq G/K$, $\{5, 11\} \cap \pi(K) = \varnothing$ and the Schur-Zassenhaus theorem, we see that there exists $\widetilde{T} \leq N$ such that $\widetilde{T} \simeq T$. Let $\overline{N} = N/\Phi(P)$ and $\overline{T}$ isomorphic to T. From

Lemma 4 it follows that $\overline{N}$ contains an element of order $13t$, where $t \in \{5, 11\}$, but $\omega(\overline{N}) \subseteq \omega$; a contradiction.

Assume that $11 \notin \pi(S)$. It follows from Lemma 3 that $\{5, 7, 13\} \subseteq \pi(S)$ and $\{5, 7, 13\} \cap \pi(|G|/|S|) = \varnothing$. Hence $35 \in \omega(S)$. By [13] and [14], there are no such groups. □

From [13] and Lemma 11 it follows that S is isomorphic to one of the groups $L_5(3)$, $L_6(3)$, Alt_{13}, Alt_{14}, Alt_{15}, Alt_{16}, Suz, Fi_{22}.

Lemma 12. $S \notin \{L_5(3), L_6(3), Alt_{16}, Fi_{22}\}$.

Proof. Note that $121 \in \omega(L_5(3))\backslash\omega \subseteq \omega(L_6(3))$, $16 \in \omega(Fi_{22})\backslash\omega$, $63 \in \omega(Alt_{16})\backslash\omega$. Hence $S \notin \{L_5(3), L_6(3), Alt_{16}, Fi_{22}\}$. □

Thus the group S is isomorphic to one of the groups Alt_{13}, Alt_{14}, Suz or Alt_{15}. Assume that $S \in \{Alt_{13}, Alt_{14}, Suz\}$.

Lemma 13. $11, 13 \notin \pi(K)$.

Proof. Suppose that $\pi(K) \cap \{11, 13\} \neq \varnothing$. Let $p \in \pi(K) \cap \{11, 13\}$, $H = O_{p'}(K)$. There exists a normal p-subgroup T in a group G/H. Since $5p \notin \omega(G)$, we have a group have a Frobenius group TM with kernal T and complement $M \in Syl_5(G/H)$. From Lemma 2 it follows that M is cyclic. But $N \in Syl_5(S)$ is elementary Abelian group of order 25 and $N \leq M/(M \cap (K/H))$; a contradiction. □

Lemma 14. $5, 7 \notin \pi(K)$.

Proof. Suppose that $\pi(K) \cap \{5, 7\} \neq \varnothing$. Let $p \in \pi(K) \cap \{5, 7\}$, H be a Hall $\{3, 5, 7\}$-subgroup of K. Since $N_G(H)/N_K(H) \simeq G/K$ and $\omega(N_K(H)) \subseteq \omega$, we may assume that $H \lhd G$. Since $13t \notin \omega$ for $t \in \{3, 5, 7\}$, Lemma 2 implies that H is nilpotent. Let $\widetilde{G} = G/O_2(K)$, $\widetilde{K} = K/O_2(K)$, $T \in Syl_2(\widetilde{K})$. Assume that exists $g \in \widetilde{G}$, $|g| = 13$ and g acts on T nontrivially. From Lemma 4, it follows that in $\widetilde{G}$ there is a element of order $13p$, but $13p \notin \omega$. Hence if $g \in N_{\widetilde{G}}(T)$, $|g| = 13$, then $g \in C_{\widetilde{G}}(T)$. The group S is generated by elements of order 13. Thus $T.S$ is a central extension of T with S. Therefore $\widetilde{G}/\widetilde{H}$ contains a subgroup isomorphic to one of the groups Alt_{13}, $2.Alt_{13}$, Suz, $2.Suz$. From the tables of 5 and 7-modular characters of Alt_{13}, $2.Alt_{13}$, Suz, and $2.Suz$ (see [14]), it follows that G has an element of order $11p$, but $11p \notin \omega(G)$; a contradiction. □

Lemma 15. $2, 3 \in \pi(K)$.

Proof. Since $13 \cdot 2 \in \omega(G) \setminus \omega(Aut(S))$ and $13 \notin \pi(K)$, we have $2 \in \pi(K)$. Since $7 \cdot 5 \cdot 3 \notin \omega(Aut(S))$ and $\{5, 7\} \cap \pi(K) = \varnothing$, we have $3 \in \pi(K)$. □

Lemma 16. $S \notin \{Alt_{13}, Alt_{14}, Suz\}$.

Proof. By Lemmas 13, 14 and 15, $\pi(K) = \{2, 3\}$. Put $R_0 = 1, R_1 = O_2(G), R_2 = O_{2,3}(G), R_3 = O_{2,3,2}(G)$, and so forth. For some n, we have $R_n = K$ for the first time, and it is obvious that $n \geq 2$. Put $\overline{G} = G/R_{n-2}$ and $\overline{K} = K/R_{n-2}$. Then $\overline{K}$ is a group in which the Sylow p-subgroup for $p = 2$ or 3 is normal. Suppose that $p = 2$. Then $\widetilde{G} = G/R_{n-1}$ possesses a nontrivial normal 3-subgroup $\widetilde{K} = K/R_{n-1}$. Note that $\widetilde{G}/\widetilde{K}$ contains a subgroup T isomorphic to one of the groups Alt_{13}, Suz. Since $39 \notin \omega$, the action of T on $\widetilde{K}$ by conjugations is faithful. The table of 3-modular characters of Suz (see [14]) implies that $C_{\overline{K}}(g) \neq 1, |g| = 13$. Hence $T \simeq Alt_{13}$. The table of 3-modular characters of Alt_{13} (see [14]) implies that every chief factor of G lying in $\widetilde{K}$ is a 12-dimensional irreducible representation over a field of characteristic 3, in which the dimension of the space of fixed points of elements of order 11 is equal to 2. Since there is a complement to $\widetilde{K}$ in $\widetilde{G}$ (see [15]), it follows that Alt_{13} acts on $P = R_{n-1}/R_{n-2}$. It is clear from the table of 2-modular characters of Alt_{13} (see [14]) that $C_P(x) \neq 1$ for an element $x \in Alt_{13}$ of order 11. Thus $C_{\overline{K}}(x)$ is an extension of a nontrivial 2-group by a 3-group of rank at least 2, and thus it contains an element of order 6. By the choice of x we deduce that G contains an element of order 66; thus $p = 3$. In this case $T = R_{n-1}/R_{n-2}$ is a 3-group which contains its centralizer in $\widetilde{K} = K/R_{n-1}$. Assume that there exists $g \in \widetilde{G}, |g| = 13$, and g acts on $\widetilde{K}$ nontrivially. From Lemma 4, it follows that $39 \in \omega(\overline{G})$, but $39 \notin \omega$. The group S is generated by 13-elements. Thus the group $\widetilde{G}$ contains a subgroup isomorphic to $\widetilde{K} \times S$ or $H \times (2.S)$, for some group H. Let us show that in the second case $\widetilde{K}$ is of order 2. Since G contains no elements of order $4 \cdot 13$, it follows that $\widetilde{K}$ is of period 2. If $\widetilde{K}$ is noncyclic then $C_T(\widetilde{y}) \neq 1$ for some $\widetilde{y}$ in $\widetilde{K}$. As above, an element of $\widetilde{G}$ of order 11 centralizes in $C_T(\widetilde{y})$ some nontrivial element, and consequently G contains an element of order 66; a contradiction. Put $N = 2.S$ if $\widetilde{G} = 2.S$, and $N = S$ if $\widetilde{G} = \widetilde{K} \times S$. In each case, since $\overline{G}$ contains no elements of order $8 \cdot 7$, while G must, it follows that $R_{n-2} \neq 1$. The table of 3-modular characters (see [14]) implies that N acts trivially on $\overline{K}$. Furthermore, as in the case $p = 2$, we deduce that for elements x of N of order 11 the group $C_{R_{n-1}/R_{n-3}}(x)$ contains an element of order 22. Thus G contains an element of order 66; this is a contradiction. □

Therefore $S \simeq Alt_{15}$. By Lemma 6 it follows that the subgroup K is trivial. Since $\omega(S) \neq \omega$ and $Aut(S) = Sym_{15}$, we see that $G \simeq Sym_{15}$. The proposition is proved.

4. Proof of Main Theorem for Sym_{16}

Proposition 2. *The group Sym_{16} is recognizable.*

Let $\omega = \omega(G) = \omega(Sym_{16})$, K be the maximal normal soluble subgroup of G, $S = Soc(G/K) \simeq S_1 \times ... \times S_n$, where $S_i, 1 \leq i \leq n$ are non-Abelian simple groups. Obviously, the prime divisors of $|S|$ are not greater than 13. Using the classification of finite simple groups it is not hard to obtain the full list of all finite simple groups L with the property $\pi(L) \subseteq \{2, 3, 5, 7, 11, 13\}$ (see [13]).

Lemma 17. $13 \notin \pi(K)$.

Proof. Let $\overline{G} = G/K, \widetilde{G} = \overline{G}/S$. Suppose that $13 \in \pi(K)$. Then, from Lemma 3 we have $\{7, 11\} \cap \pi(K) = \varnothing$. Let $p \in \{5, 7, 11\}$. Using Frattini argument we can obtain that in $G/O_{13'}(K)$ there exists a subgroup $T.P$ such that T is isomorphic to Sylow 13-subgroup of K and P is isomorphic to Sylow p-subgroup of G/K. By Lemma 2 it follows that P and Sylow p-subgroups of the group G/K are cyclic of order p. Suppose that $11 \in \pi(\widetilde{G})$. Let $g \in \overline{G}, |g| = 11$ and the image of g in $\widetilde{G}$ is not trivial. Since $11 \notin \pi(Out(S_i))$ for all $1 \leq i \leq n$, we have $S_i^g \neq S_i$ for some i. The order of any non-Abelian finite simple group R with property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13\}$ is divisible by 5, 7 or 13(see [13]). Suppose that $p \in \{5, 7\} \cap \pi(S_i)$. Then the Sylow p-subgroups of group $\overline{G}$ are non-cyclic. Hence $\{5, 7\} \cap \pi(S_i) = \varnothing$. From [13] it follows that $S_i \simeq L_3(3)$ and $13 \in \pi(S_i)$. In the same way as in proof of Lemma 10, we obtain that in $\overline{G}$ there is element of order $13 \cdot 11$, but $13 \cdot 11 \notin \omega$. Thus $11 \in \pi(S)$. It is easy to prove that $7 \in \pi(S)$. Since $77 \notin \omega$ it follows that there exists S_i such that $7, 11 \in \pi(S_i)$. From [13] and the fact that the Sylow $5, 7$ and 11-subgroups of S are cyclic, we see that $S_i \simeq M_{22}$ or $U_6(2)$. Since $\{5, 7, 11\} \subseteq \pi(S_i)$, we have $S \simeq S_i$. From [16] we have $R < L_2(11) < M_{22} < U_6(2)$, where R is a Frobenius group with kernel of order 11 and complement of order 5. Let T be a Hall $\{13, 5\}$-subgroup of K. Using the Frattini argument we obtain that G contains a section isomorphic to $T.R$. From Lemma 4 it follows that $65 \in \omega(T.R)$ or $143 \in \omega(T.R)$; a contradiction. □

Lemma 18. *The group S is a finite simple group.*

Proof. Let $\overline{G} = G/K, \widetilde{G} = \overline{G}/S$. Suppose that $n > 1$. From Lemma 17 we have $13 \in \pi(\overline{G})$. By Lemma 3, it follows that there exists $p \in \{7, 11\} \cap \pi(\overline{G})$. Suppose that $13 \in \pi(\widetilde{G})$. Then there exists $g \in \overline{G}$ such that $|g| = 13$ and g acts by conjugation on S and induces an outer automorphism. By [13], for all non-Abelian finite simple groups R with property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13\}$ except when $R \simeq L_3(3)$, we have $\{5, 7\} \cap \pi(R) \neq \varnothing$. Assume that there exists $1 \leq i \leq n$ such that $S_i \not\simeq L_3(3)$, we can assume that $i = 1$. Suppose that $S_1^g = S_1$. By Lemma 5, g is not an outer automorphism of a group $S_j, j \in \{1, ..., n\}$. Hence $S_1 \leq C_{\overline{G}}(g)$ and so $\overline{G}$ has an element of order pt, where $t \in \{5, 7\} \cap \pi(S_1\}$, but $pt \notin \omega$. Thus $S_1 \neq S_1^g$. Let $x = hh^g h^{g^2} ... h^{g^{p-1}}, h \in S_1, |h| \in \{5, 7\} \cap \pi(S_1)$. It is easy to check that $x \in C_{\overline{G}}(g)$, $|x| = |h|$. Hence $\overline{G}$ has an element x such that $|x| = p|h|$, but $p|h| \notin \omega$ and so $S_i \simeq L_3(3)$ for all $1 \leq i \leq n$. Since $p \notin \pi(L_3(3))$, it follows that $p \in \pi(\widetilde{G})$. It is easy to check that $13p \in \omega(\overline{G})$; a contradiction. Hence $13, p \in \pi(S_i)$. If $n > 1$ then $\{65, 91, 143\} \cap \omega(\overline{G}) \neq \varnothing$; a contradiction. □

From [13], Lemmas 17 and 3 it follows that S is isomorphic to one of the groups $L_2(13)$, $L_2(27)$, $G_2(3)$, ${}^3D_4(2)$, $Sz(8)$, $L_2(64)$, $U_4(5)$, $L_3(9)$, $S_6(3)$, $O_7(3)$, $O_8^+(3)$, $G_2(4)$, $S_4(8)$, $L_5(3)$, $L_6(3)$, Alt_{13}, Alt_{14}, Alt_{15}, Alt_{16}, Suz, Fi_{22}.

Lemma 19. $S \notin \{L_2(64), U_4(5), L_5(3), L_6(3), L_3(9), S_4(8)\}$.

Proof. Note that $65 \in \omega(L_2(64)) \setminus \omega, 52 \in \omega(U_4(5)) \setminus \omega, 121 \in \omega(L_5(3)) \setminus \omega \subseteq \omega(L_6(3)), 91 \in \omega(L_3(9)) \setminus \omega, 65 \in \omega(S_4(8)) \setminus \omega$; a contradiction. □

Lemma 20. $S \notin \Omega = \{L_2(13)$, $L_2(27)$, $G_2(3)$, ${}^3D_4(2)$, $Sz(8)$, $S_6(3)$, $O_7(3)$, $O_8^+(3)$, $G_2(4)$, Alt_{13}, Alt_{14}, $Alt_{15}\}$.

Proof. Groups from Ω have no elements of order 55 (see [14]), it follows that $\{5, 11\} \cap \pi(K) \neq \varnothing$. From [16] we have that in the groups $G_2(3), O_7(3), O_8^+(3), G_2(4)$ there exists a subgroup isomorphic to $L_2(13)$, in the group $S_6(3)$ there exists a subgroup isomorphic to $L_2(27)$, in the groups Alt_{14}, Alt_{15} there exists a subgroup isomorphic Alt_{13}. Thus to prove the Lemma, it suffices to prove that $\omega(K.L) \setminus \omega(G) \neq \varnothing$ where $L \in \{L_2(13), L_2(27),{}^3 D_4(2), Sz(8), Alt_{13}\}$, there exists an element g and $|g| \notin \omega$.

Let $p \in \pi(K) \cap \{11, 5\}$, $P \in Syl_p(K)$. Without loss of generality it can be assumed that $P \lhd G$ and $C_K(P) \leq P$. Suppose that in G/P there exists an element g of order 13 and $K/P \not\leq C_{G/P}(g)$. From Lemma 4 it follows that G contains element of order $13p$, but $13p \notin \omega$; a contradiction. Since for all elements $x \in G/P$ of order 13 we have that x acts trivially on K/P and has no fixed point on P. Since S is a simple group, we see that all elements of order 13 generated S. Therefore, $(K/P).S$

is a central extension of K/P with S. Note that $(K/P).S$ contains a subgroup S or the Schur multiplier of S.

Suppose that $S \in \{L_2(27),^3D_4(2), Sz(8)\}$. From the tables of characters of S and the Schur multiplier it follows that G has an element of order $13p$, but $13p \notin \omega(G)$; contradiction.

Suppose that $S \simeq L_2(13)$. Since $11 \notin \pi(S)$, we can assume that $p = 11$. From the tables of characters of S and the Schur multiplier it follows that G has an element of order $13 \cdot 11$ or $7 \cdot 11$; contradiction.

Therefore, $S \simeq Alt_{13}$. From the tables of 5 and 11-modular characters of Alt_{13} and $2.Alt_{13}$ (see [14]) it follows that the element of order 13 acts with no fixed points only on the 12-dimensional permutation module, but in this case centralizes of an element of order 18 is nontrivial and hence $18p \in \omega$; a contradiction. □

Therefore, $S \simeq Alt_{16}$. By Lemma 6 it follows that the subgroup K is trivial. Hence $\omega(S) \neq \omega$ and $Aut(S) = Sym_{16}$ we see that $G \simeq Sym_{16}$. The proposition is proved.

5. Proof of Main Theorem for Sym_{18}

Proposition 3. *The group Sym_{18} is recognizable.*

From Lemma 7 it follows that if $\omega(G) = \omega(Sym_{18})$ where $G \not\simeq Sym_{18}$, then $G \simeq K.Alt_{17}$ or $K.Sym_{17}$ where K is a soluble group. Since $17t \notin \omega$, for all $t \in \pi(K)$, using Lemma 2 we can see that K is nilpotent. Since $77 \notin \omega(Sym_{17})$ we obtain $\{7, 11\} \cap \pi(K) \neq \varnothing$. Let $p \in \{7, 11\} \cap \pi(K), P \in Syl_p(K)$. We can assume that $K \simeq P$. From the tables of 7 and 11-modular characters of Alt_{14} (see [14]) it follows that G has an element g of order pt, $t \in \{7, 11\} \setminus \{p\}$. Note that $R.Alt_6 \leq C_G(g^p)$ where R is a p-group. From the tables of 7 and 11-modular characters of Alt_6 (see [14]) it follows that $C_G(g)$ has an element of order $3t$. Hence $3 \cdot 7 \cdot 11 \in \omega(G)$; a contradiction. Therefore, $G \simeq Sym_{18}$. The proposition is proved.

6. Proof of Main Theorem for Sym_{21}

Proposition 4. *The group Sym_{21} is recognizable.*

Let $\omega = \omega(G) = \omega(Sym_{21})$, K be the maximal normal soluble subgroup of G, $S = Soc(G/K) \simeq S_1 \times ... \times S_n$, where $S_i, 1 \leq i \leq n$ are non-Abelian simple groups. Obviously, the prime divisors of $|S|$ are not greater then 19. Using the classification of finite simple groups it is not hard to obtain a full list of all finite simple groups L with $\pi(L) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19\}$ (see [13]).

Lemma 21. *The group S is a finite simple group.*

Proof. Let $\overline{G} = G/K, \widetilde{G} = \overline{G}/S$. Obviously $\overline{G} \leq Aut(S)$ and $\widetilde{G} \leq Out(S)$. Suppose that $n > 1$. By Lemma 3 we may assume that there exists $p \in \{17, 19\}$ and $p \notin \pi(K)$. Suppose that $|\widetilde{G}|$ is divisible by p. Then $\overline{G}$ contains an element g of order p that acts by conjugation on S and induces an outer automorphism. By Lemma 5, g is not an outer automorphism of a group $S_i, 1 \leq i \leq n$. By [13], for all non-Abelian finite simple groups R with property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19\}$ except when $R \simeq L_2(17)$, we have $\{5, 7, 13\} \cap \pi(R) \neq \varnothing$. Assume that there exists $1 \leq i \leq n$ such that $S_i \not\simeq L_2(17)$, we can assume that $i = 1$. Suppose that $S_1^g = S_1$. Hence $S_1 \leq C_{\overline{G}}(g)$ and so $\overline{G}$ has an element whose order is equal to pt, where

$t \in \{5,7,13\} \cap \pi(S_1)$, but $pt \notin \omega$. Thus $S_1 \neq S_1^g$. Let $x = hh^g h^{g^2} ... h^{g^{p-1}}, h \in S_1, |h| \in \{5,7,13\} \cap \pi(S_1)$. It is easy to check that $x \in C_{\overline{G}}(g)$, $|x| = |h|$. Hence $\overline{G}$ has an element x such that $|x| = p|h|$, but $p|h| \notin \omega$ and so $S_i \simeq L_2(17)$ for all $1 \leq i \leq n$. We have $\{9,17\} \subset \omega(L_2(17))$. The group S has an element of order $9 \cdot 17$ since $n > 1$, but $9 \cdot 17 \notin \omega$.

Thus $p \in \pi(S)$. Without loss of generality it can be assumed that $p \in \pi(S_1)$. It is easy to see that there exists $x \in S$ and $|x| = pt$, where $t \in \{5,7,9,13\} \cap \omega(S_2)$; a contradiction. Then $n = 1$. □

Lemma 22. $19 \in \pi(S)$.

Proof. Assume that $19 \notin \pi(S)$. Then $\{5,7,11,13,17\} \subset \pi(S)$ and

$$\{7,13\} \cap \pi(|G|/|S|) = \varnothing.$$

Hence $7 \cdot 13 \in \omega(S)$. From [13] and [14] it follows that there are no such groups. □

Lemma 23. $13, 17 \in \pi(S)$.

Proof. Suppose that $17 \notin \pi(S)$. Then $\{11,13,19\} \subset \pi(S)$. From [13] it follows that there are no such groups.

Suppose that $13 \notin \pi(S)$. Then $\{11,17,19\} \subset \pi(S)$. From [13] and Lemmas 22 and 23 it follows that there are no such groups. □

From [13] it follows that S is isomorphic to one of the groups

$$Alt_n, 19 \leq n \leq 22, {}^2E_6(2).$$

Lemma 24. $S \not\simeq Alt_{22}$.

Proof. Note that $57 \in \omega(Alt_{22})$ but ω has no such elements; contradiction. □

Lemma 25. $S \not\simeq {}^2E_6(2)$.

Proof. Group ${}^2E_6(2)$ have no elements of order 91 (see [14]), it follows that $\{7,13\} \cap \pi(K) \neq \varnothing$. From [16] we have that in the group ${}^2E_6(2)$ there exists a subgroup T isomorphic to $O_8^-(2)$.

Let $p \in \pi(K) \cap \{7,13\}$, $P \in Syl_p(K)$. Without loss of generality it can be assumed that $P \lhd G$ and $C_K(P) \leq P$. Suppose that in G/P there exists an element g of order 17 and $K/P \not\leq C_{G/P}(g)$. From Lemma 4 it follows that G contains element of order $17p$, but $17p \notin \omega$; a contradiction. Hence for all elements $x \in G/P$ of order 17 we have that x acts trivially on K/P and has no fixed point on P. Since T is a simple group, we see that all elements of order 17 generated T. Therefore, $(K/P).T$ is a central extension of K/P with T. Note that $(K/P).T$ contains a subgroup T or the Schur multiplier of T. From the tables of p-modular characters of T and the Schur multiplier (see [14]), it follows that G has an element of order $17p$, but $17p \notin \omega(G)$; contradiction. □

Lemma 26. $S \notin \{Alt_{19}, Alt_{20}\}$.

Proof. Let $S \in \{Alt_{19}, Alt_{20}\}$, H be a Hall $2'$-subgroup of K. Since $13 \cdot 5 \cdot 3 \notin \omega(Aut(S))$, we see that H is not trivial. Without loss of generality it can be assumed that $H \lhd G$. Since $19p \notin \omega, p \in \pi(H)$, by Lemma 2 the subgroup H is nilpotent. Note that there exists $R < S$ such that R is isomorphic to a Frobenius group with kernel order 19 and complement order 9. Since $\pi(K/H) \subseteq \{2\}$, we see that R acts on H. If $\{3,13\} \cap \pi(H) \neq \varnothing$ then by Lemma 9 we obtain that $H.R$ has an element

x and $|x| \in \{57, 27, 117, 247\}$; a contradiction. Since $13 \cdot 5 \cdot 3, 11 \cdot 7 \cdot 3 \notin \omega(G/K)$ we see that $\pi(H) = \{5, 7\}$ or $\pi(H) = \{5, 11\}$. From the table of 5-modular characters of Alt_{13} and $2.Alt_{13}$ (see [14]) it follows that G has an element of order $11 \cdot 5 \cdot 7$; a contradiction. □

Therefore, $S \simeq Alt_{21}$. By Lemma 6 it follows that K is trivial. Since $\omega(S) \neq \omega$ and $Aut(S) = Sym_{21}$, we see that $G \simeq Sym_{21}$. The proposition is proved.

7. Proof of Main Theorem for Sym_{27}

Proposition 5. *The group Sym_{27} is recognizable.*

Let $\omega = \omega(G) = \omega(Sym_{27})$, K be the maximal normal soluble subgroup of G, $S = Soc(G/K) \simeq S_1 \times ... \times S_n$, where $S_i, 1 \leq i \leq n$ are non-Abelian simple groups. Obviously, the prime divisors of $|S|$ are not greater then 23. Using the classification of finite simple groups it is not hard to obtain a full list of all finite simple groups L with the property $\pi(L) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$ (see [13]).

Lemma 27. $23 \notin \pi(K)$.

Proof. Let $\overline{G} = G/K, \widetilde{G} = \overline{G}/S$. Suppose that $23 \in \pi(K)$. From Lemma 3 we have $\{11, 13, 17, 19\} \cap \pi(K) = \varnothing$. By Lemma 2 and the Frattini argument it follows that a Sylow p-subgroup of G/K is cyclic, for any $p \in \{5, 7, 11, 13, 17, 19\}$. Assume that $19 \in \pi(\widetilde{G})$. Let $g \in \overline{G}, |g| = 19$ and the image of g in $\widetilde{G}$ is not trivial. Since $19 \notin \pi(Out(S_i))$ for all $1 \leq i \leq n$, we obtain that there exists $1 \leq i \leq n$ such that $S_i^g \neq S_i$. By [13], for all non-Abelian finite simple groups R with the property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$, we have $\{5, 7, 11, 13, 17\} \cap \pi(R) \neq \varnothing$. Let $p \in \{5, 7, 11, 13, 17\} \cap \pi(S_i)$. Then a Sylow p-subgroup P of $\overline{G}$ is not cyclic; a contradiction. Thus $19 \in \pi(S)$. It is easy to see that $17 \in \pi(S)$. Since $19 \cdot 17 \notin \omega$ we obtain that there exists S_i such that $19, 17 \in \pi(S_i)$. We have that a Sylow t-subgroup of S_i must be cyclic for all $t \in \{5, 7, 11, 13, 17\} \cap \pi(S_i)$. By [13] and [14] it follows that there are no such groups. □

Lemma 28. *The group S is a finite simple group.*

Proof. Let $\overline{G} = G/K, \widetilde{G} = \overline{G}/S$. Suppose that $n > 1$. From Lemma 27 we have $23 \in \pi(\overline{G})$. Suppose that $23 \in \pi(\widetilde{G})$. Then there exists $g \in \overline{G}$ such that $|g| = 23$ and g acts by conjugation on S and induces an outer automorphism. It follows by Lemma 1 that $g \in Out(S_i)$ or $S_i^g \neq S_i$. By [13], for all non-Abelian finite simple groups R with the property $\pi(R) \subseteq \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$, we have $\{5, 7, 11, 13, 17\} \cap \pi(R) \neq \varnothing$. Suppose that there exists $1 \leq i \leq n$ such that $S_i^g = S_i$, we can assume that $i = 1$. By Lemma 5, g is not an outer automorphism of a group $S_j, j \in \{1, ..., n\}$. Hence $S_1 \leq C_{\overline{G}}(g)$ and so $\overline{G}$ has an element whose order is equal to $23t$, where $t \in \{5, 7, 11, 13, 17\} \cap \pi(S_1)$, but $23t \notin \omega$. Thus $S_1 \neq S_1^g$. Let $x = hh^g h^{g^2} ... h^{g^{p-1}}, h \in S_1, |h| \in \{5, 7, 11, 13, 17\} \cap \pi(S_1)$. It is easy to check that $x \in C_{\overline{G}}(g)$, $|x| = |h|$. Hence $\overline{G}$ has an element x and $|x| = 23|h|$, but $23|h| \notin \omega$; a contradiction. Hence $23 \in \pi(S_i)$. If $n > 1$ then $23t \in \omega, t \in \{5, 7, 11, 13, 17\} \cap \pi(S_j)$; contradiction. □

From [13] and Lemma 3 it follows that S is isomorphic to one of the groups Fi_{23}, Alt_{23}, Alt_{24}, Alt_{25}, Alt_{26}, Alt_{27}, Alt_{28}.

Lemma 29. $S \not\simeq Fi_{23}$.

Proof. Suppose that $S \simeq Fi_{23}$. Since $19 \notin \pi(Fi_{23})$, we obtain $19 \in \pi(K)$. From Lemma 3, it follows that $11, 23 \notin \pi(K)$. From [16] we obtain that in S there exists a Frobenius group with kernel order 23 and complement of order 11. By Lemma 4 we have that $19 \cdot 11 \in \omega$ or $19 \cdot 23 \in \omega$; a contradiction. □

Hence S contains a subgroup isomorphic to Alt_{23}.

Lemma 30. *The set $\pi(K)$ has no elements greater than 7. In particular $S \not\simeq Alt_{23}$.*

Proof. Since $11 \cdot 13 \notin \omega(Aut(Alt_{23}))$, we see that if $S \simeq Alt_{23}$ then $\{11, 13\} \cap \pi(K) \neq \varnothing$. Suppose that in $\pi(K)$ there is a number $p \in \{11, 13, 17, 19\}$. Let H be a Hall $\{2, 3\}'$-subgroup of K. We can assume that $H \lhd G$ and $C_K(H) \leq H$. Since $23t \notin \omega$, for any $t \in \pi(H)$, then using Lemma 2 we see that H is nilpotent. Suppose that there exists $g \in G/H$, $|g| = 23$ and $K/H \not\leq C_{G/H}(g)$. From Lemma 4 it follows that in $23p \in \omega$; a contradiction. Thus any element of order 23 of G/H acts trivially on K/H and has no fixed points on H. Since S is a simple group, it follows that S is generated by elements of order 23. Thus $(K/H).S$ is a central extension of K/H with S. Suppose that $p = 11$. Note that G/K contains Frobenius group with kernel of order 23 and complement of order 11. By Lemma 9 we see that $121 \in \omega$ or $253 \in \omega$; contradiction. Let $h \in G, |h| = 11$ and the image $\overline{h}$ of h in G/H is not trivial. Note that $C_{G/H}(\overline{h})$ contains a subgroup isomorphic to Alt_{10} or $2.Alt_{10}$. Since a Sylow 5-subgroup of Alt_{10} is elementary Abelian it follows that in $C_G(h)$ there exist elements of order $5p$. Thus in G there exists element of order $55p$; a contradiction. □

Hence S has a subgroup isomorphic to Alt_{24}.

Lemma 31. $5, 7 \notin \pi(K)$. *In particular $S \simeq Alt_{26}$ or $S \simeq Alt_{27}$.*

Proof. We have $19 \cdot 7 \notin \omega(Aut(Alt_{25})) \supseteq \omega(Aut(Alt_{24}))$. Thus if $S \simeq Alt_{24}$ or Alt_{25}, then $7 \in \pi(K)$. Suppose that $p \in \{5, 7\} \cap \pi(K) \neq \varnothing$. Let H be a Hall $\{2, 3\}'$-subgroup of K. We can assume that $H \lhd G$ and $C_K(H) \leq H$. Since $23t \notin \omega$ for any $t \in \pi(H)$, using Lemma 2 we see that H is nilpotent. Suppose that there exists $g \in G/H$, $|g| = 23$ and $K/H \not\leq C_{G/H}(g)$. From 4 it follows that $23p \in \omega$; a contradiction. Thus any element of order 23 of G/H acts trivially on K/H and has no fixed points on H. Since S is a simple group, it follows that S is generated by elements of order 23. Thus $(K/H).S$ is a central extension of K/H with S. In G/H there exists a subgroup isomorphic to Alt_{12} or $2.Alt_{12}$. From the table of 5 and 7-modular characters of Alt_{12}, $2.Alt_{13}$, Alt_8, and $2.Alt_8$ (see [14]) it follows that G has an element of order $66pr$, $r \in \{5, 7\} \setminus \{p\}$; a contradiction. □

Lemma 32. $S \simeq Alt_{27}$.

Proof. Suppose that $S \simeq Alt_{26}$. We have $3 \cdot 5 \cdot 19 \notin \omega(Out(Alt_{26}))$. Since $5, 7 \notin \pi(K)$, it follows that $3 \in \pi(K)$, and $3 \in \pi(C_K(g))$, $g \in G, |g| = 19$. Let $C = C_G(g)$. We can assume that a Sylow 3-subgroup P of $C \cap K$ is normal in C and $3 \notin \pi((C \cap K)/P)$. In C/P there exists a Frobenius group R with kernel of order 7 and complement of order 3. From 9 it follows that $9 \in \omega(C)$ or $21 \in \omega(C)$. Thus $9 \cdot 19 \in \omega$ or $21 \cdot 19 \in \omega$; a contradiction. □

Therefore, $S \simeq Alt_{27}$. By Lemma 6 it follows that the subgroup K is trivial. Hence $\omega(S) \neq \omega$ and $Aut(S) = Sym_{27}$, we see that $G \simeq Sym_{27}$. The proposition is proved.

8. Proof of Main Theorem and Corollaries

The theorem follows from Propositions 1–5. The corollary 1 follows from Proposition 2 and Lemma 8. The corollary 2 follows from Theorem and [1]–[6].

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