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FINITE GROUPS WITH FORMATIONAL SUBNORMAL PRIMARY SUBGROUPS OF BOUNDED EXPONENT

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Abstract. Let $\mathfrak{U}_k$ be the class of all supersoluble groups in which exponents are not divided by the $(k+1)$-th powers of primes. We investigate the classes $\mathrm{w}\mathfrak{U}_k$ and $\mathrm{v}\mathfrak{U}_k$ that contain all finite groups in which every Sylow and, respectively, every cyclic primary subgroup is $\mathfrak{U}_k$-subnormal. We prove that $\mathrm{w}\mathfrak{U}_k$ and $\mathrm{v}\mathfrak{U}_k$ are subgroup-closed saturated formations and obtain the characterizations of these formations.

Keywords: finite group, primary subgroup, subnormal subgroup.

1. Introduction

All groups in this paper are finite. A primary group is a group of prime power order. All fragments of the theory of group classes that we used correspond to [1].

Let $\mathfrak{F}$ be a non-empty formation. A subgroup H of a group G is called $\mathfrak{F}$-subnormal in G if either $G = H$ or there is a subgroup chain

$$H = H_0 \lessdot \ldots \lessdot H_i \lessdot H_{i+1} \lessdot \ldots \lessdot H_n = G \qquad (1)$$

such that $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{F}$ for all i, [1, Definition 6.1.2]. We write $X \lessdot Y$ if X is a maximal subgroup of a group Y, and $X_Y = \cap_{y\in Y} X^y$ is the core of a subgroup X in a group Y. If $\mathfrak{X}$ and $\mathfrak{Y}$ are formations and $\mathfrak{X} \subseteq \mathfrak{Y}$, then, clearly, every $\mathfrak{X}$-subnormal subgroup is $\mathfrak{Y}$-subnormal. If $\mathfrak{F}$ is a soluble formation (i. e. all groups in $\mathfrak{F}$ are soluble) and H is a soluble $\mathfrak{F}$-subnormal subgroup of a group G, then G is soluble, [2, Lemma 2.13].

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Let $\mathbb{P}$ be the set of all primes. If $|H_{i+1} : H_i| \in \mathbb{P}$ for every i in (1), then H is $\mathbb{P}$-subnormal in G, [3, Definition 1].

The class of groups with all Sylow subgroups (all cyclic primary subgroups) $\mathfrak{F}$-subnormal is denoted by $\mathrm{w}\mathfrak{F}$ ($\mathrm{v}\mathfrak{F}$, respectively). If $\mathfrak{F} = \mathfrak{U}$ is the formation of all supersoluble groups, then the class $\mathrm{w}\mathfrak{U}$ ($\mathrm{v}\mathfrak{U}$) coincides with the class of all groups in which every Sylow subgroup (every cyclic primary subgroup, respectively) is $\mathbb{P}$-subnormal, [6, lemma 1.12]. The classes $\mathrm{w}\mathfrak{U}$ and $\mathrm{v}\mathfrak{U}$ are quite well investigated [3]–[9]. In particular, these classes are subgroup-closed saturated formations, $\mathrm{w}\mathfrak{U} \subset \mathrm{v}\mathfrak{U}$ and every group from $\mathrm{v}\mathfrak{U}$ has a Sylow tower of supersoluble type. The inclusion $\mathrm{w}\mathfrak{U} \subset \mathrm{v}\mathfrak{U}$ is proper, every biprimary minimal non-supersoluble group with non-cyclic non-normal Sylow subgroup belongs to $\mathrm{v}\mathfrak{U} \setminus \mathrm{w}\mathfrak{U}$, see [9, Example 2, Example 3].

The exponent of a group G is the least common multiple of the orders of all elements of G and denoted by $\exp(G)$. The set of all positive integers is denoted by $\mathbb{N}$ and the set of all positive integers not divided by the $(k+1)$-th powers of primes for $k \in \mathbb{N}$ is denoted by $\mathbb{N}_k$. If $\mathfrak{X}$ is a formation, then $\mathfrak{X}_k$ is the class of all groups from $\mathfrak{X}$ with exponents from $\mathbb{N}_k$. It is clear that $\mathfrak{X}_k = \mathfrak{X} \cap \mathfrak{E}_k$, where $\mathfrak{E}$ is the formation of all finite groups.

Introduce the following classes of groups:

- $\mathfrak{U}_k$ is the class of all supersoluble groups with exponents from $\mathbb{N}_k$;
- $\mathrm{w}\mathfrak{U}_k = \mathrm{w}(\mathfrak{U}_k)$ is the class of all groups in which every Sylow subgroup is $\mathfrak{U}_k$-subnormal;
- $\mathrm{v}\mathfrak{U}_k = \mathrm{v}(\mathfrak{U}_k)$ is the class of all groups in which every cyclic primary subgroup is $\mathfrak{U}_k$-subnormal.

Since $\mathfrak{U}_k \subset \mathfrak{U}$, we have $\mathrm{w}\mathfrak{U}_k \subset \mathrm{w}\mathfrak{U}$ and $\mathrm{v}\mathfrak{U}_k \subset \mathrm{v}\mathfrak{U}$. Hence groups in $\mathrm{w}\mathfrak{U}_k$ and $\mathrm{v}\mathfrak{U}_k$ possess the properties of groups from $\mathrm{w}\mathfrak{U}$ and $\mathrm{v}\mathfrak{U}$, respectively. In particular, groups in $\mathrm{w}\mathfrak{U}_k$ and $\mathrm{v}\mathfrak{U}_k$ have Sylow towers of supersoluble type. In addition, $\mathrm{w}\mathfrak{U}_k \subset \mathrm{v}\mathfrak{U}_k$ (Lemma 10) and this inclusion is proper for every k (Example 4).

Although $\mathfrak{U}_k$ is a subgroup-closed non-saturated formation, $\mathrm{w}\mathfrak{U}_k$ and $\mathrm{v}\mathfrak{U}_k$ are subgroup-closed saturated formations (Proposition 1 and Proposition 2). The following theorems contain the characterizations of groups from these formations.

Theorem 1. *For a group G, the following statements are equivalent.*

(1) *Every Sylow subgroup of G is $\mathfrak{U}_k$-subnormal in G, i. e. $G \in \mathrm{w}\mathfrak{U}_k$.*

(2) *$G/\Phi(G) \in (\mathrm{w}\mathfrak{U}_k)_k$;*

(3) *$A/\Phi(A) \in \mathfrak{U}_k$ for every metanilpotent subgroup A of G;*

(4) *$B/\Phi(B) \in \mathfrak{U}_k$ for every biprimary subgroup B of G.*

Corollary 1. *If $G \in \mathrm{w}\mathfrak{U}_k$, then $G/F(G) \in \mathcal{A}_k$.*

Here $\mathcal{A}_k$ is the class of all groups with abelian Sylow subgroups of exponent from $\mathbb{N}_k$.

Corollary 2. *For a metanilpotent group G, the following statements are equivalent.*

(1) *Every Sylow subgroup of G is $\mathfrak{U}_k$-subnormal in G.*

(2) *$G/\Phi(G) \in \mathfrak{U}_k$.*

(3) *$G \in \mathfrak{U}$ and $G/F(G) \in \mathfrak{A}_k$.*

Here $\mathfrak{A}_k$ is the class of all abelian groups with exponents from $\mathbb{N}_k$.

Theorem 2. *For a group G, the following statements are equivalent.*

(1) *Every cyclic primary subgroup of G is $\mathfrak{U}_k$-subnormal in G, i. e. $G \in \mathrm{v}\mathfrak{U}_k$.*

(2) $G/\Phi(G) \in (\mathrm{v}\mathfrak{U}_k)_k$.
(3) $A/\Phi(A) \in \mathfrak{U}_k$ *for every subgroup* A *with nilpotent derived subgroup.*
(4) $B/\Phi(B) \in \mathfrak{U}_k$ *for every biprimary subgroup* B *with cyclic Sylow subgroup.*

Corollary 3. $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k = \mathfrak{N}^2 \cap \mathrm{w}\mathfrak{U}_k = \mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k = \mathfrak{N}\mathfrak{A} \cap \mathrm{v}\mathfrak{U}_k$. *In particular, every Sylow subgroup of a supersoluble group* G *is* $\mathfrak{U}_k$*-subnormal in* G *if and only if every cyclic primary subgroup of* G *is* $\mathfrak{U}_k$*-subnormal in* G.

Here $\mathfrak{N}^2$ is the class of all metanilpotent groups and $\mathfrak{N}\mathfrak{A}$ is the class of all groups with nilpotent derived subgroup. Both of these classes are subgroup-closed saturated formations.

2. Preliminaries

Throughout this paper, k denotes an positive integer. We write $H \leq G$ ($H < G$) if H is a (proper) subgroup of G. A subgroup H of G is non-trivial if $H \neq 1$ and $H \neq G$. By $\pi(k)$ we denote the set of all primes dividing k. For a group G, $\pi(G) = \pi(|G|)$, where $|G|$ is the order of G. If

$$|G| = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n},\ p_1 < p_2 < \dots < p_n,$$

and G has a normal series $G = G_0 \geq G_1 \geq \dots \geq G_{n-1} \geq G_n = 1$ such that G_{i-1}/G_i is isomorphic to a Sylow p_i-subgroup of G for every i, then we say that G has *a Sylow tower of supersoluble type*. It is easy to check that the class $\mathfrak{D}$ of all groups with Sylow tower of supersoluble type is a subgroup-closed saturated Fitting formation. The class $\mathcal{A}$ of all groups with abelian Sylow subgroups is a subgroup-closed formation, but it is not a saturated formation and it is not a Fitting formation.

The greatest common divisor (gcd) and the least common multiple (lcm) of integers a and b are denoted by (a, b) and $[a, b]$, respectively. We repeatedly use the following simplest properties of $\mathbb{N}_k$.

Lemma 1. (1) *If* $n \in \mathbb{N}_k$ *and* d *divides* n*, then* $d \in \mathbb{N}_k$ *and* $n/d \in \mathbb{N}_k$.
(2) *If* $a, b \in \mathbb{N}_k$*, then* $(a, b) \in \mathbb{N}_k$ *and* $[a, b] \in \mathbb{N}_k$.

Lemma 2. (1) $\pi(G) = \pi(\exp(G))$ *and* $\exp(G)$ *divides* $|G|$.
(2) *The exponent of* G *is equal to* lcm *of orders of primary elements of* G.
(3) *If* H *is a subgroup of* G *and* N *is a normal subgroup of* G*, then* $\exp(H)$ *and* $\exp(G/N)$ *divide* $\exp(G)$.
(4) *If* $G = G_1 \times G_2$*, where* $G_1 \leq G$ *and* $G_2 \leq G$*, then* $\exp(G) = [\exp(G_1), \exp(G_2)]$.

Proof. (1) For every $p \in \pi(G)$, there is an element of order p in G by Sylow's Theorem. Hence $\pi(G) = \pi(\exp(G))$. In view of Lagrange's Theorem, the order of every element of G divides $|G|$. Therefore $\exp(G)$ divides $|G|$ by Lemma 1 (2).

(2) Assume that $\pi(G) = \{p_1, p_2, \dots, p_m\}$, $p_1 < p_2 < \dots < p_m$, and $\exp(G) = p_1^{n_1} p_2^{n_2} \dots p_m^{n_m}$. For every $i = 1, 2, \dots, m$, there is an element x_i in G such that $|x_i| = p_i^{n_i} t_i$ and p_i does not divide t_i. It is clear that $x_i^{t_i}$ is a primary element of order $p_i^{n_i}$ and $[x_1^{t_1}, x_2^{t_2}, \dots, x_m^{t_m}] = \exp(G)$.

(3) This statement true in view of Lagrange's Theorem.

(4) According to Statement (3), $\exp(G_1)$ and $\exp(G_2)$ divide $\exp(G)$. Hence $[\exp(G_1), \exp(G_2)]$ divides $\exp(G)$. Since any element $g \in G$ can be represented as $g = g_1 g_2$, where $g_1 \in G_1$, $g_2 \in G_2$ and $|g| = [|g_1|, |g_2|]$, we conclude that $\exp(G)$ divides $[\exp(G_1), \exp(G_2)]$. Consequently, $[\exp(G_1), \exp(G_2)] = \exp(G)$. □

A class $\mathfrak{X}$ is saturated if $G \in \mathfrak{X}$ whenever $G/\Phi(G) \in \mathfrak{X}$. Here $\Phi(G)$ is the Frattini subgroup of a group G. If $H \in \mathfrak{X}$ whenever $H \leq G$ and $G \in \mathfrak{X}$, then $\mathfrak{X}$ is a subgroup-closed class.

Lemma 3. (1) *$\mathfrak{E}_k$ is a subgroup-closed formation.*

(2) *If $\mathfrak{X}$ is a (subgroup-closed) formation, then $\mathfrak{X}_k = \mathfrak{X} \cap \mathfrak{E}_k$ is a (subgroup-closed) formation.*

(3) *If $\mathfrak{X}$ and $\mathfrak{Y}$ are formations, then $(\mathfrak{X} \cap \mathfrak{Y})_k = \mathfrak{X}_k \cap \mathfrak{Y}_k$ and $(\mathfrak{X}\mathfrak{Y})_k \subset \mathfrak{X}_k\mathfrak{Y}_k$.*

Proof. (1) Assume that $G \in \mathfrak{E}_k$ and N is a normal subgroup of G. Then $\exp(G) \in \mathbb{N}_k$ and $\exp(G/N)$ divides $\exp(G)$ by Lemma 2 (3). Hence $\exp(G/N) \in \mathbb{N}_k$ by Lemma 1 (1), and $G/N \in \mathfrak{E}_k$. Consequently, $\mathfrak{E}_k$ is a homomorph.

Let N_1 and N_2 be normal subgroups of G and let $G/N_1, G/N_2 \in \mathfrak{E}_k$. By Remak's Lemma, $G/(N_1 \cap N_2)$ is isomorphic to a subgroup which is a subdirect product of the direct product $G/N_1 \times G/N_2$. Since $\exp(G/N_i) \in \mathbb{N}_k$ for $i = 1, 2$ and $\exp(G/N_1 \times G/N_2) = [\exp(G/N_1), \exp(G/N_2)]$ by Lemma 2 (4), we get $\exp(G/N_1 \times G/N_2) \in \mathbb{N}_k$ by Lemma 1 (2). Consequently, $\mathfrak{E}_k$ is a formation.

Assume that $G \in \mathfrak{E}_k$ and H is a subgroup of G. In that case, $\exp(G) \in \mathbb{N}_k$ and $\exp(H)$ divides $\exp(G)$. Consequently, $\exp(H) \in \mathbb{N}_k$ by Lemma 1 (1) and $H \in \mathfrak{E}_k$. Thus $\mathfrak{E}_k$ is a subgroup-closed formation.

(2) Since the intersection of (subgroup-closed) formations is a (subgroup-closed) formation and in view of Statement (1), Statement (2) is true.

(3) Let $G \in (\mathfrak{X} \cap \mathfrak{Y})_k$. In that case, $G \in (\mathfrak{X} \cap \mathfrak{Y})$ and $\exp(G) \in \mathbb{N}_k$. Hence $G \in \mathfrak{X}_k$ и $G \in \mathfrak{Y}_k$. It follows that $G \in \mathfrak{X}_k \cap \mathfrak{Y}_k$ and $(\mathfrak{X} \cap \mathfrak{Y})_k \subseteq \mathfrak{X}_k \cap \mathfrak{Y}_k$. Now assume that $G \in \mathfrak{X}_k \cap \mathfrak{Y}_k$. Then $G \in \mathfrak{X}_k \subseteq \mathfrak{X}$ and $G \in \mathfrak{Y}_k \subseteq \mathfrak{Y}$, $\exp(G) \in \mathbb{N}_k$. Therefore $G \in (\mathfrak{X} \cap \mathfrak{Y})_k$ and $(\mathfrak{X} \cap \mathfrak{Y})_k = \mathfrak{X}_k \cap \mathfrak{Y}_k$.

Let $G \in (\mathfrak{X}\mathfrak{Y})_k$. In that case, $G \in \mathfrak{X}\mathfrak{Y}$ и $\exp(G) \in \mathbb{N}_k$. Since $G \in \mathfrak{X}\mathfrak{Y}$, we get $G^{\mathfrak{Y}} \in \mathfrak{X}$. From $\exp(G) \in \mathbb{N}_k$ it follows that $\exp(G^{\mathfrak{Y}}) \in \mathbb{N}_k$ and $\exp(G/G^{\mathfrak{Y}}) \in \mathbb{N}_k$. Hence $G^{\mathfrak{Y}} \in \mathfrak{X}_k$, $G/G^{\mathfrak{Y}} \in \mathfrak{Y}_k$ and $G^{\mathfrak{Y}_k} \leq G^{\mathfrak{Y}}$. But $\mathfrak{Y}_k \subseteq \mathfrak{Y}$. Therefore $G^{\mathfrak{Y}} \leq G^{\mathfrak{Y}_k}$. Consequently, $G^{\mathfrak{Y}_k} = G^{\mathfrak{Y}}$ and $G \in \mathfrak{X}_k\mathfrak{Y}_k$. Thus, $(\mathfrak{X}\mathfrak{Y})_k \subseteq \mathfrak{X}_k\mathfrak{Y}_k$. □

Example 1. Note that the reverse inclusion in Lemma 3 (3) does not hold, an example is $D_8 \in \mathfrak{N}_1\mathfrak{N}_1 \setminus (\mathfrak{N}\mathfrak{N})_1$. Here D_8 is the dihedral group of order 8.

Note that $\mathrm{w}\mathfrak{U}_k$ and $(\mathrm{w}\mathfrak{U})_k$ are distinct classes: $\mathrm{w}\mathfrak{U}_k = \mathrm{w}(\mathfrak{U}_k)$ is the class of all groups in which every Sylow subgroup is $\mathfrak{U}_k$-subnormal; the class $(\mathrm{w}\mathfrak{U})_k = \mathrm{w}\mathfrak{U} \cap \mathfrak{E}_k$ consists of all groups with $\mathfrak{U}$-subnormal Sylow subgroups and exponent that is not divided by the $(k+1)$-th powers of primes.

Similarly, $\mathrm{v}\mathfrak{U}_k$ and $(\mathrm{v}\mathfrak{U})_k$ are also distinct classes: $\mathrm{v}\mathfrak{U}_k = \mathrm{v}(\mathfrak{U}_k)$ is the class of all groups in which every primary cyclic subgroup is $\mathfrak{U}_k$-subnormal; $(\mathrm{v}\mathfrak{U})_k = \mathrm{v}\mathfrak{U} \cap \mathfrak{E}_k$ consists of all groups with $\mathfrak{U}$-subnormal primary cyclic subgroups and exponent that is not divided by the $(k+1)$-th powers of primes.

Lemma 4. (1) *$(\mathrm{w}\mathfrak{U})_k$ and $(\mathrm{v}\mathfrak{U})_k$ are subgroup-closed formations for any k.*

(2) *$(\mathrm{w}\mathfrak{U})_k \subset \mathrm{w}\mathfrak{U}_k$ and $(\mathrm{v}\mathfrak{U})_k \subset \mathrm{v}\mathfrak{U}_k$ for any k.*

Proof. (1) Since $\mathrm{w}\mathfrak{U}$ and $\mathrm{v}\mathfrak{U}$ are subgroup-closed saturated formations, $(\mathrm{w}\mathfrak{U})_k = \mathrm{w}\mathfrak{U} \cap \mathfrak{E}_k$ and $(\mathrm{v}\mathfrak{U})_k = \mathrm{v}\mathfrak{U} \cap \mathfrak{E}_k$, we deduce that $(\mathrm{w}\mathfrak{U})_k$ and $(\mathrm{v}\mathfrak{U})_k$ are subgroup-closed formations by Lemma 3 (2).

(2) Let $G \in (\mathrm{w}\mathfrak{U})_k$. In that case, $\exp(G) \in \mathbb{N}_k$ and every Sylow subgroup of G is $\mathfrak{U}$-subnormal in G. Assume that R is a Sylow subgroup of G. By hypothesis, there

is a subgroup chain

$$R = H_0 \lessdot \ldots \lessdot H_i \lessdot H_{i+1} \lessdot \ldots \lessdot H_n = G$$

such that $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{U}$ for every i. By Lemma 1, $\exp\left(H_{i+1}/(H_i)_{H_{i+1}}\right) \in \mathbb{N}_k$. Hence $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{U}_k$ for every i. Thus R is $\mathfrak{U}_k$-subnormal in G and $G \in \mathrm{w}\mathfrak{U}_k$.

Let $G \in (\mathrm{v}\mathfrak{U})_k$. Then $\exp(G) \in \mathbb{N}_k$ and every cyclic primary subgroup of G is $\mathfrak{U}$-subnormal in G. Assume that A is a cyclic primary subgroup of G. By hypothesis, there is a subgroup chain $A = H_0 \lessdot \ldots \lessdot H_i \lessdot H_{i+1} \lessdot \ldots \lessdot H_n = G$ such that $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{U}$ for every i. Since $\exp\left(H_{i+1}/(H_i)_{H_{i+1}}\right) \in \mathbb{N}_k$, we get $H_{i+1}/(H_i)_{H_{i+1}} \in \mathfrak{U}_k$ for every i. Therefore A is $\mathfrak{U}_k$-subnormal in G and $G \in \mathrm{v}\mathfrak{U}_k$. □

Example 2. In Lemma 4 (2), the inclusion is proper. In the non-cyclic group $G = C_3 \rtimes C_{2^{k+1}} = \langle a, b \mid a^3 = b^{2^{k+1}} = 1,\ a^b = a^2\rangle$, a Sylow 3-subgroup C_3 is normal. Therefore C_3 is $\mathfrak{U}_1$-subnormal in G. A Sylow 2-subgroup $C_{2^{k+1}}$ is also $\mathfrak{U}_1$-subnormal in G, since

$$(C_{2^{k+1}})_G \cong C_{2^k}, \quad G/(C_{2^{k+1}})_G \cong S_3 \in \mathfrak{U}_1.$$

Thus, $G \in \mathrm{w}\mathfrak{U}_1 \subset \mathrm{w}\mathfrak{U}_k$, but $G \notin (\mathrm{w}\mathfrak{U})_k$ in view of $\exp(G) = 3 \cdot 2^{k+1}$.

Remind the properties of $\mathfrak{F}$-subnormal subgroups that we use.

Lemma 5. *Let $\mathfrak{F}$ be a formation, let H and K be subgroups of G and let N be a normal subgroup of G. The following statement hold.*

(1) *If K is $\mathfrak{F}$-subnormal in H and H is $\mathfrak{F}$-subnormal in G, then K is $\mathfrak{F}$-subnormal in G* [1, 6.1.6(1)].

(2) *If K/N is $\mathfrak{F}$-subnormal in G/N, then K is $\mathfrak{F}$-subnormal in G* [1, 6.1.6(2)].

(3) *If H is $\mathfrak{F}$-subnormal in G, then HN/N is $\mathfrak{F}$-subnormal in G/N* [1, 6.1.6(3)].

(4) *If $\mathfrak{F}$ is a subgroup-closed formation and H is $\mathfrak{F}$-subnormal in G, then $H \cap K$ is $\mathfrak{F}$-subnormal in K* [1, 6.1.7(2)].

(5) *If $\mathfrak{F}$ is a subgroup-closed formation and H and K are $\mathfrak{F}$-subnormal in G, then $H \cap K$ is $\mathfrak{F}$-subnormal in G* [1, 6.1.7(3)].

Lemma 6. *If $\mathfrak{F}$ is a subgroup-closed formation and H is $\mathfrak{F}$-subnormal in G, then $H^{\mathfrak{F}}$ is subnormal in G.*

Proof. Use induction on $|G|$. If $H = G$, then $H^{\mathfrak{F}} = G^{\mathfrak{F}}$ is normal in G. Let H be a proper subgroup of G. In that case, there is a maximal subgroup M of G such that M contains H and $G^{\mathfrak{F}}$. By induction, $H^{\mathfrak{F}}$ is subnormal in M. Since $H^{\mathfrak{F}} \leq G^{\mathfrak{F}} \leq M$, we deduce that $H^{\mathfrak{F}}$ is subnormal in $G^{\mathfrak{F}}$. But $G^{\mathfrak{F}}$ is normal in G. Therefore $H^{\mathfrak{F}}$ is subnormal in G. □

Lemma 7. *If H is a subnormal subgroup of a soluble group G, then H is $\mathfrak{U}_1$-subnormal in G.*

Proof. Assume that H is a subnormal subgroup of a soluble group G. In that case, there is a composition series such that

$$1 = H_0 \leq H_1 \leq \ldots \leq H_j = H \leq H_{j+1} \leq \ldots \leq H_m = G.$$

Since G is soluble, we get $|H_{j+1}/(H_j)_{H_{j+1}}| = |H_{j+1}/H_j| \in \mathbb{P}$ and $H_{j+1}/H_j \in \mathfrak{U}_k$. Therefore H is $\mathfrak{U}_k$-subnormal in G. □

Example 3. In the Frobenius group $F_5 = C_5 \rtimes C_4$, a Sylow subgroup C_4 is $\mathfrak{U}$-subnormal, but C_4 is not $\mathfrak{U}_1$-subnormal and not subnormal.

We repeatedly use the following lemma.

Lemma 8. *If H is a non-normal subgroup of a soluble group G and $|G:H| = r \in \mathbb{P}$, then $G/H_G \cong C_r \rtimes C_t$, where t divides $r-1$. In particular, G/H_G is supersoluble.*

Proof. According to $|G:H| \in \mathbb{P}$, we deduce that H is a maximal subgroup of G and $\overline{G} = G/H_G$ is a soluble primitive group. Therefore $\overline{G} = \overline{N} \rtimes \overline{H}$, where $\overline{N} = F(\overline{G}) = C_{\overline{G}}(\overline{N})$ is the unique minimal normal subgroup of $\overline{G}$, $\overline{H} = H/H_G$ is a maximal subgroup of $\overline{G}$. Hence

$$|\overline{N}| = |\overline{G}:\overline{H}| = |G:H| = r,\ \overline{N} \cong C_r,\ N_{\overline{G}}(\overline{N})/C_{\overline{G}}(\overline{N}) = \overline{G}/\overline{N} \cong \overline{H}$$

and $\overline{H}$ is isomorphic to a subgroup of the automorphism group of $\overline{N}$. Therefore $\overline{H} \cong C_t$ and t divides $r-1$. Thus, $G/H_G \cong C_r \rtimes C_t$, in particular, G/H_G is supersoluble. □

3. Groups with $\mathfrak{U}_k$-subnormal Sylow subgroups

We repeatedly use the following properties of groups with $\mathfrak{U}$-subnormal Sylow subgroups.

Lemma 9. (1) *A group $G \in \mathrm{w}\mathfrak{U}$ if and only if every metanilpotent subgroup of G is supersoluble,* [6, Theorem 2.6 (2)]. *In particular, $\mathfrak{U} = \mathrm{w}\mathfrak{U} \cap \mathfrak{N}^2$.*

(2) *A group $G \in \mathrm{w}\mathfrak{U}$ if and only if every biprimary subgroup of G is supersoluble,* [4, Theorem B (1)], [9, Theorem 1 (2)].

(3) *If $G \in \mathrm{w}\mathfrak{U}$, then G has a Sylow tower of supersoluble type and every Sylow subgroup of $G/F(G)$ is abelian,* [3, Proposition 2.8; Theorem 2.13 (3)].

(4) *Every minimal non-supersoluble subgroup of G is threeprimary if and only if $G \in \mathrm{w}\mathfrak{U}$,* [9, Corollary 1 (2)].

Proposition 1. $\mathrm{w}\mathfrak{U}_k$ *is a subgroup-closed saturated formation.*

Proof. By Lemma 3 (2), $\mathfrak{U}_k$ is a subgroup-closed formation. Therefore $\mathrm{w}\mathfrak{U}_k$ is a subgroup-closed formation by [10, Theorem 3.1 (5)].

Now we prove that $\mathrm{w}\mathfrak{U}_k$ is a saturated formation. Assume the contrary and let G be a group of least order such that $G/\Phi(G) \in \mathrm{w}\mathfrak{U}_k$ and $G \notin \mathrm{w}\mathfrak{U}_k$.

Assume that $N \neq 1$ is a normal subgroup of G and $\Phi(G/N) = K/N$. Since

$$\Phi(G)N/N = (\cap_{M \lessdot G} M)\, N/N \leq (\cap_{N \leq H \lessdot G} H)\,/N = \Phi(G/N) = K/N,$$

we get $\Phi(G)N \leq K$. Since

$$G/K \cong (G/\Phi(G))/(K/\Phi(G)),\ G/\Phi(G) \in \mathrm{w}\mathfrak{U}_k$$

and $\mathrm{w}\mathfrak{U}_k$ is a homomorph, we have $G/K \in \mathrm{w}\mathfrak{U}_k$. Hence

$$(G/N)/(\Phi(G/N)) = (G/N)/(K/N) \cong G/K \in \mathrm{w}\mathfrak{U}_k.$$

Since $|G/N| < |G|$, we get $G/N \in \mathrm{w}\mathfrak{U}_k$. Thus $G/N \in \mathrm{w}\mathfrak{U}_k$ for every non-identity normal subgroup N of G. Since $\mathrm{w}\mathfrak{U}_k$ is a formation, G has the unique minimal normal subgroup.

Since G has a Sylow tower of supersoluble type, a Sylow r-subgroup R of G is normal in G for $r = \max \pi(G)$. It is clear that $R = F(G)$ and $O_p(G) = 1$ for all $p \in \pi(G) \setminus \{r\}$. In view of Lemma 7, R is $\mathfrak{U}_k$-subnormal in G.

Let Q be a Sylow q-subgroup of G for $q \neq r$. Since $G/\Phi(G) \in \mathrm{w}\mathfrak{U}_k$, we deduce that $Q\Phi(G)/\Phi(G)$ is $\mathfrak{U}_k$-subnormal in $G/\Phi(G)$. By Lemma 6,

$$(Q\Phi(G)/\Phi(G))^{\mathfrak{U}_k} = Q^{\mathfrak{U}_k}\Phi(G)/\Phi(G)$$

is subnormal in $G/\Phi(G)$. Consequently,

$$Q^{\mathfrak{U}_k}\Phi(G)/\Phi(G) \leq F(G/\Phi(G)) = F(G)/\Phi(G),\ Q^{\mathfrak{U}_k} = 1.$$

Therefore exponents of all Sylow r'-subgroup of G belong to $\mathbb{N}_k$. Since QR/R is a Sylow q-subgroup of $G/R \in \mathrm{w}\mathfrak{U}_k$, QR/R is $\mathfrak{U}_k$-subnormal in G/R. According to Lemma 5 (2), QR is $\mathfrak{U}_k$-subnormal in G. In view of $QR \leq G \in \mathrm{w}\mathfrak{U}$, we have Q is $\mathfrak{U}$-subnormal in QR. Therefore there is a subgroup chain

$$Q = M_0 \lessdot M_1 \lessdot \ldots \lessdot M_i \lessdot M_{i+1} \lessdot \ldots \lessdot M_n = QR$$

such that $|M_{i+1} : M_i| \in \mathbb{P}$ for every i. Denote $M_i = A$ and $M_{i+1} = B$. Clearly, $|B : A| = r$. In view of Lemma 8, $B/A_B \cong C_r \rtimes C_t$, where t divides $r-1$. Since $\exp(Q) \in \mathbb{N}_k$, we deduce that $\exp(B/A_B) \in \mathbb{N}_k$ and $B/A_B \in \mathfrak{U}_k$. Hence Q is $\mathfrak{U}_k$-subnormal in QR. Consequently, Q is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (1). Thus all Sylow subgroups of G are $\mathfrak{U}_k$-subnormal in G and $G \in \mathrm{w}\mathfrak{U}_k$. □

Proof of Theorem 1. (1) ⇒ (2): Assume that every Sylow subgroup of G is $\mathfrak{U}_k$-subnormal in G, i.e. $G \in \mathrm{w}\mathfrak{U}_k$. Use induction on $|G|$ to prove $G/\Phi(G) \in (\mathrm{w}\mathfrak{U}_k)_k$. Suppose that there is a maximal subgroup M of G such that $M_G = 1$. In that case, G is a primitive group, $\Phi(G) = 1$, $G = F(G) \rtimes M$, where $F(G)$ is the unique minimal normal subgroup of G. Since G has a Sylow tower of supersoluble type, a Sylow r-subgroup R is normal in G for $r = \max \pi(G)$. Hence $R = F(G)$ and R is an elementary abelian r-subgroup. If Q is a Sylow q-subgroup of G for $q \neq r$, Q is $\mathfrak{U}_k$-subnormal in G and $Q^{\mathfrak{U}_k}$ is subnormal in G by Lemma 6. Therefore $Q^{\mathfrak{U}_k} \leq F(G) = R$ in view of [11, Theorem 2.2]. Consequently, $Q^{\mathfrak{U}_k} = 1$ and the exponent of every Sylow r'-subgroup of G belongs to $\mathbb{N}_k$. Thus all Sylow subgroups of G have exponents from $\mathbb{N}_k$ and $G \in (\mathrm{w}\mathfrak{U}_k)_k$ by Lemma 2 (2).

Now assume that $M_G \neq 1$ for every maximal subgroup M of G. Since $G/M_G \in \mathrm{w}\mathfrak{U}_k$, by induction,

$$(G/M_G)/\Phi(G/M_G) \in (\mathrm{w}\mathfrak{U}_k)_k.$$

But G/M_G is a primitive group, hence $\Phi(G/M_G) = 1$ and $G/M_G \in (\mathrm{w}\mathfrak{U}_k)_k$ for every maximal subgroup M of G. Since $\Phi(G) = \bigcap_{M \lessdot G} M_G$ and $(\mathrm{w}\mathfrak{U}_k)_k$ is a formation, we get $G/\Phi(G) \in (\mathrm{w}\mathfrak{U}_k)_k$.

(1) ⇐ (2): Let $G/\Phi(G) \in (\mathrm{w}\mathfrak{U}_k)_k$. Since $(\mathrm{w}\mathfrak{U}_k)_k \subseteq \mathrm{w}\mathfrak{U}_k$ and $\mathrm{w}\mathfrak{U}_k$ is a saturated formation in view of Proposition 1, we get $G \in \mathrm{w}\mathfrak{U}_k$.

Thus, (1) ⇔ (2) is proved.

(1) ⇒ (3): Assume that $G \in \mathrm{w}\mathfrak{U}_k$ and A is a metanilpotent subgroup of G. In that case, $G \in \mathrm{w}\mathfrak{U}$, and by Lemma 9 (1), $A \in \mathfrak{U}$. Since $\mathrm{w}\mathfrak{U}_k$ is a subgroup-closed formation in view of Proposition 1, we get $A \in \mathrm{w}\mathfrak{U}_k$. According proved Statement (1) ⇒ (2), $A/\Phi(A) \in (\mathrm{w}\mathfrak{U}_k)_k$. Consequently, $A/\Phi(A) \in \mathfrak{U} \cap (\mathrm{w}\mathfrak{U}_k)_k \subseteq \mathfrak{U}_k$.

(1) ⇐ (3): Let $A/\Phi(A) \in \mathfrak{U}_k$ for every metanilpotent subgroup A of G. Since $\mathfrak{U}_k \subseteq \mathfrak{U}$, every metanilpotent subgroup A of G is supersoluble. In view of Lemma 9 (1), $G \in \mathrm{w}\mathfrak{U}$. Choose G of least order such that $G \in \mathrm{w}\mathfrak{U} \setminus \mathrm{w}\mathfrak{U}_k$. Since $G \in \mathrm{w}\mathfrak{U}$, a Sylow r-subgroup R of G is normal in G for $r = \max \pi(G)$. In view of Lemma 7, R is $\mathfrak{U}_k$-subnormal in G. Assume that Q is a Sylow q-subgroup of G for $q \neq r$. In that case, $R \rtimes Q$ is metanilpotent and $R \rtimes Q/\Phi(R \rtimes Q) \in \mathfrak{U}_k \subseteq \mathrm{w}\mathfrak{U}_k$ by the

choice of G. Since $\mathrm{w}\mathfrak{U}_k$ is a saturated formation by Proposition 1, we get $R \rtimes Q \in \mathrm{w}\mathfrak{U}_k$. Hence QR is a proper subgroup of G and Q is $\mathfrak{U}_k$-subnormal in QR. Let U_1/R be a metanilpotent subgroup of G/R. Since $(|U_1/R|, |R|) = 1$, by the Schur-Zassenhaus Theorem, there is a subgroup U such that $U_1 = R \rtimes U$ and $U_1/R \cong U$ is metanilpotent. By the choice of G, $U/\Phi(U) \in \mathfrak{U}_k$. Hence

$$(U_1/R)/\Phi(U_1/R) \cong U/\Phi(U) \in \mathfrak{U}_k.$$

Thus G/R satisfies Statement (3) and $G/R \in \mathrm{w}\mathfrak{U}_k$ by the choice of G. Hence a Sylow subgroup QR/R is $\mathfrak{U}_k$-subnormal in G/R. According to Lemma 5 (2), QR is $\mathfrak{U}_k$-subnormal in G, and Q is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (1). Thus all Sylow subgroups of G is $\mathfrak{U}_k$-subnormal in G and $G \in \mathrm{w}\mathfrak{U}_k$.

Statement (1) $\Leftrightarrow$ (3) is proved.

(1) $\Rightarrow$ (4): Assume that $G \in \mathrm{w}\mathfrak{U}_k$ and B is a biprimary subgroup of G. In that case, $G \in \mathrm{w}\mathfrak{U}$, and by Lemma 9 (2), B is supersoluble. Since $\mathrm{w}\mathfrak{U}_k$ is a subgroup-closed formation by Proposition 1, we have $B \in \mathrm{w}\mathfrak{U}_k$. By proved Statement (1) $\Rightarrow$ (2), $B/\Phi(B) \in (\mathrm{w}\mathfrak{U}_k)_k$. Consequently, $B/\Phi(B) \in \mathfrak{U} \cap (\mathrm{w}\mathfrak{U}_k)_k \subseteq \mathfrak{U}_k$.

(1) $\Leftarrow$ (4): Let G be a group of least order such that $B/\Phi(B) \in \mathfrak{U}_k$ for every biprimary subgroup B of G and $G \notin \mathrm{w}\mathfrak{U}_k$. In that case, G has a Sylow q-subgroup Q for a prime $q \in \pi(G)$ that is not $\mathfrak{U}_k$-subnormal in G. Since $\mathfrak{U}_k \subseteq \mathfrak{U}$, every biprimary subgroup of G is supersoluble. By Lemma 9 (2), $G \in \mathrm{w}\mathfrak{U}$, in particular, G has a Sylow tower of supersoluble type. Consequently, for $r = \max \pi(G)$, a Sylow r-subgroup R of G is normal in G. In view of Lemma 7, R is $\mathfrak{U}_k$-subnormal in G and $r > q$. By the choice of G, $QR/\Phi(QR) \in \mathfrak{U}_k \subseteq \mathrm{w}\mathfrak{U}_k$. Hence $QR \in \mathrm{w}\mathfrak{U}_k$ by Proposition 1, in particular, Q is $\mathfrak{U}_k$-subnormal in QR and $QR < G$. Assume that H/R is a biprimary subgroup of G/R. By the Schur-Zassenhaus Theorem, there is a biprimary subgroup B of H such that $H = R \rtimes B$, $H/R \cong B$. By the choice of G, $B/\Phi(B) \in \mathfrak{U}_k$. Therefore

$$(H/R)/\Phi(H/R) \cong B/\Phi(B) \in \mathfrak{U}_k.$$

By induction, $G/R \in \mathrm{w}\mathfrak{U}_k$, hence QR/R is $\mathfrak{U}_k$-subnormal in G/R. It follows that QR is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (2), and Q is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (1), a contradiction.

Statement (1) $\Leftrightarrow$ (4) is proved. □

Proof of Corollary 1. Since $G \in \mathrm{w}\mathfrak{U}_k \subset \mathrm{w}\mathfrak{U}$, we get $G/F(G) \in \mathcal{A}$ by Lemma 9 (3). In view of theorem 1 ((1) $\Rightarrow$ (2)) $G/\Phi(G) \in (\mathrm{w}\mathfrak{U}_k)_k$. Therefore

$$G/F(G) \cong (G/\Phi(G))/(F(G)/\Phi(G)) \in \mathcal{A} \cap (\mathrm{w}\mathfrak{U}_k)_k \subseteq \mathcal{A}_k. \qquad \square$$

Proof of Corollary 2. (1) $\Leftrightarrow$ (2): If $G \in \mathfrak{N}^2$ and every Sylow subgroup of G is $\mathfrak{U}_k$-subnormal in G, then $G/\Phi(G) \in \mathfrak{U}_k$ by Theorem 1 ((1) $\Rightarrow$ (3)). Conversely, if $G/\Phi(G) \in \mathfrak{U}_k$, then $G \in \mathrm{w}\mathfrak{U}_k$ by Theorem 1 ((1) $\Leftarrow$ (2)).

(1) $\Leftrightarrow$ (3): If $G \in \mathfrak{N}^2 \cap \mathrm{w}\mathfrak{U}_k$, then $G/\Phi(G) \in \mathfrak{U}_k$ by proved Statement (1) $\Rightarrow$ (2). Since $G/F(G)$ is abelian, we get $G/F(G) \cong (G/\Phi(G))/(F(G)/\Phi(G)) \in \mathfrak{A} \cap \mathfrak{U}_k = \mathfrak{A}_k$. Conversely, let $G/F(G) \in \mathfrak{A}_k$ and let $G \in \mathfrak{U}$. Use induction on $|G|$ to prove that every Sylow subgroup of G is $\mathfrak{U}_k$-subnormal in G. Assume that P is a Sylow p-subgroup and N is a minimal normal subgroup of G such that $|N| = r$ and $r = \max \pi(G)$. By induction, PN/N is $\mathfrak{U}_k$-subnormal in G/N. Hence PN is $\mathfrak{U}_k$-subnormal in G and $p < r$. Since

$$F(G) \leq C_G(N), \ PN/C_{PN}(N) = PN/(PN \cap C_G(N)) \cong PC_G(N)/C_G(N) \leq$$

$$\le G/C_G(N) \cong (G/F(G))/(C_G(N)/F(G)) \in \mathfrak{A}_k,$$

and $G/C_G(N)$ is cyclic, we deduce that $PN/C_{PN}(N)$ is cyclic and $|PN/C_{PN}(N)| = p^t \le p^k$. Next,

$$C_{PN}(N) = P_1 \times N,\ P_1 = P_{PN} \le P \lessdot PN,\ PN/P_1 \cong C_r \rtimes C_{p^t} \in \mathfrak{U}_k,$$

therefore P $\mathfrak{U}_k$-subnormal in PN. By Lemma 5 (1), P is $\mathfrak{U}_k$-subnormal in G. □

4. Groups with $\mathfrak{U}_k$-subnormal cyclic primary subgroups

Groups with $\mathfrak{U}$-subnormal cyclic primary subgroups were first considered in [4]. The class of such groups was later denoted by $\mathrm{v}\mathfrak{U}$. In Introduction, we indicate that $\mathrm{w}\mathfrak{U} \subset \mathrm{v}\mathfrak{U}$ and this inclusion is proper.

Lemma 10. $\mathrm{w}\mathfrak{U}_k \subset \mathrm{v}\mathfrak{U}_k$.

Proof. Let $G \in \mathrm{w}\mathfrak{U}_k$. Then every Sylow subgroup of G is $\mathfrak{U}_k$-subnormal in G. In view of Lemma 7, every p-subgroup is $\mathfrak{U}_1$-subnormal in a Sylow p-subgroup. Hence every primary subgroup of G is $\mathfrak{U}_k$-subnormal in G and $G \in \mathrm{v}\mathfrak{U}_k$. □

Example 4. In $GL(3,7)$, there is a non-abelian subgroup Q of order 3^3 and exponent 3 that acts irreducibly on an elementary abelian group P of order 7^3 [12]. The semidirect product $G = P \rtimes Q$ is a minimal non-supersoluble group and $G \in \mathrm{v}\mathfrak{U}$ according to [9, Corollary 2 (2)]. It corresponds to the group from [13, Theorem 9 (Type 10)]. Since $\exp(G) = 3 \cdot 7$, we have $G \in \mathrm{v}\mathfrak{U}_1$. Biprimary groups in $\mathrm{w}\mathfrak{U}$ are supersoluble, therefore $G \notin \mathrm{w}\mathfrak{U}$, and $G \notin \mathrm{w}\mathfrak{U}_1$. Clearly, $G \in \mathrm{v}\mathfrak{U}_k \setminus \mathrm{w}\mathfrak{U}_k$ for any k.

We repeatedly use the following properties of groups with $\mathfrak{U}$-subnormal primary cyclic subgroups.

Lemma 11. (1) *A group $G \in \mathrm{v}\mathfrak{U}$ if and only if every subgroup of G with nilpotent derived subgroup is supersoluble,* [6, Theorem 2.6 (1)],[9, Theorem 2 (1)]. *In particular, $\mathfrak{U} = \mathrm{v}\mathfrak{U} \cap \mathfrak{N}\mathfrak{A}$.*

(2) *A group $G \in \mathrm{v}\mathfrak{U}$ if and only if every biprimary subgroup of G with cyclic Sylow subgroup is supersoluble,* [4, Theorem B (3)],[9, Theorem 2 (2)].

(3) *The quotient group $H/H^{\mathfrak{U}}$ is non-cyclic for every minimal non-supersoluble subgroup H of G if and only if $G \in \mathrm{v}\mathfrak{U}$,* [9, Corollary 2 (2)].

(4) $\mathrm{w}\mathfrak{U} = \mathrm{v}\mathfrak{U} \cap \mathfrak{N}\mathcal{A}$ *and every group of* $\mathrm{v}\mathfrak{U}$ *has a Sylow tower of supersoluble type,* [9, Theorem 3 (1)].

Proposition 2. $\mathrm{v}\mathfrak{U}_k$ *is a subgroup-closed saturated formation.*

Proof. By Lemma 3 (2), $\mathfrak{U}_k$ is a subgroup-closed formation. Therefore $\mathrm{v}\mathfrak{U}_k$ is a subgroup-closed formation by [7, Theorem A (3)].

Now we prove that $\mathrm{v}\mathfrak{U}_k$ is a saturated formation. Assume the contrary and let G be a group of least order such that $G/\Phi(G) \in \mathrm{v}\mathfrak{U}_k$ and $G \notin \mathrm{v}\mathfrak{U}_k$. By analogy with the proof of Proposition 1, we can easily prove that G has the unique minimal normal subgroup. Since G has a Sylow tower of supersoluble type, a Sylow r-subgroup R is normal in G for $r = \max \pi(G)$. It is clear that $R = F(G)$ and $O_p(G) = 1$ for all $p \in \pi(G) \setminus \{r\}$.

Let A be a cyclic q-subgroup for a prime $q \in \pi(G)$. If $q = r$, then A is $\mathfrak{U}_k$-subnormal in G in view of Lemma 7. Analogously, if $A \le \Phi(G)$, then A is $\mathfrak{U}_k$-subnormal in G by Lemma 7. Assume that $q \ne r$ and A is not contained in $\Phi(G)$.

Since $G/\Phi(G) \in \mathrm{v}\mathfrak{U}_k$, we deduce that $A\Phi(G)/\Phi(G)$ is $\mathfrak{U}_k$-subnormal in $G/\Phi(G)$. By Lemma 6,

$$(A\Phi(G)/\Phi(G))^{\mathfrak{U}_k} = A^{\mathfrak{U}_k}\Phi(G)/\Phi(G)$$

is subnormal in $G/\Phi(G)$. Consequently,

$$A^{\mathfrak{U}_k}\Phi(G)/\Phi(G) \le F(G/\Phi(G)) = F(G)/\Phi(G) = R/\Phi(G),\ A^{\mathfrak{U}_k} = 1.$$

Therefore $A \in \mathfrak{U}_k$. Since AR/R is a cyclic q-subgroup of $G/R \in \mathrm{v}\mathfrak{U}_k$, we deduce that AR/R is $\mathfrak{U}_k$-subnormal in G/R. By Lemma 5 (2), AR is $\mathfrak{U}_k$-subnormal in G. Since $AR \le G \in \mathrm{v}\mathfrak{U}$, we get A is $\mathfrak{U}$-subnormal in AR. Hence there is a subgroup chain

$$A = M_0 \lessdot M_1 \lessdot \ldots \lessdot M_i \lessdot M_{i+1} \lessdot \ldots \lessdot M_n = AR$$

such that $|M_{i+1} : M_i| \in \mathbb{P}$ for every i. Denote $M_i = H$ and $M_{i+1} = K$. Clearly, $|K : H| = r$. It follows that $K/H_K \cong C_r \rtimes C_t$, where t divides $r-1$ in view of Lemma 8. Since $\exp(A) \in \mathbb{N}_k$, we have $\exp(K/H_K) \in \mathbb{N}_k$ and $K/H_K \in \mathfrak{U}_k$. Hence A is $\mathfrak{U}_k$-subnormal in AR. Consequently, A is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (1). Thus all primary cyclic subgroups of G are $\mathfrak{U}_k$-subnormal in G and $G \in \mathrm{v}\mathfrak{U}_k$. □

Proof of Theorem 2. (1) ⇒ (2): Let $G \in \mathrm{v}\mathfrak{U}_k$. Use induction on $|G|$ to prove $G/\Phi(G) \in (\mathrm{v}\mathfrak{U}_k)_k$. Suppose that there is a maximal subgroup M of G such that $M_G = 1$. In that case, G is a primitive group, $\Phi(G) = 1$, $G = F(G) \rtimes M$, where $F(G)$ is the unique minimal normal subgroup of G. In view of Lemma 11 (1), a Sylow r-subgroup R is normal in G for $r = \max \pi(G)$. Hence $R = F(G)$ and R is an elementary abelian r-subgroup.

Let A be a cyclic q-subgroup for a prime $q \in \pi(G)$, $q \ne r$. In that case, A is $\mathfrak{U}_k$-subnormal in G, and by Lemma 6, $A^{\mathfrak{U}_k}$ is subnormal in G. Hence $A^{\mathfrak{U}_k} \le F(G) = R$ by [11, Theorem 2.2]. Consequently, $A^{\mathfrak{U}_k} = 1$ and the exponent of every primary cyclic r'-subgroup belongs to $\mathbb{N}_k$. Thus all primary cyclic subgroups of G have exponents from $\mathbb{N}_k$ and $G \in (\mathrm{v}\mathfrak{U}_k)_k$ by Lemma 2 (2).

Now assume that $M_G \ne 1$ for every maximal subgroup M of G. Since $G/M_G \in \mathrm{v}\mathfrak{U}_k$, we get $(G/M_G)/\Phi(G/M_G)) \in (\mathrm{v}\mathfrak{U}_k)_k$ by induction. But G/M_G is a primitive group, therefore $\Phi(G/M_G) = 1$ and $G/M_G \in (\mathrm{v}\mathfrak{U}_k)_k$ for every maximal subgroup M of G. Since $\Phi(G) = \bigcap_{M \lessdot G} M_G$ and $(\mathrm{v}\mathfrak{U}_k)_k$ is a formation, we conclude that $G/\Phi(G) \in (\mathrm{v}\mathfrak{U}_k)_k$.

(1) ⇐ (2): Let $G/\Phi(G) \in (\mathrm{v}\mathfrak{U}_k)_k$. Since $(\mathrm{v}\mathfrak{U}_k)_k \subseteq \mathrm{v}\mathfrak{U}_k$ and $\mathrm{v}\mathfrak{U}_k$ is a saturated formation by Proposition 2, we get $G \in \mathrm{v}\mathfrak{U}_k$.

Statement (1) ⇔ (2) is proved.

(1) ⇒ (3): Assume that $G \in \mathrm{v}\mathfrak{U}_k$ and A is a subgroup of G with nilpotent derived subgroup. In that case, $G \in \mathrm{v}\mathfrak{U}$, and by Lemma 11 (1), $A \in \mathfrak{U}$. By proved Statement (1) ⇒ (2), $A/\Phi(A) \in (\mathrm{v}\mathfrak{U}_k)_k$. Consequently, $A/\Phi(A) \in \mathfrak{U} \cap (\mathrm{v}\mathfrak{U}_k)_k \subseteq \mathfrak{U}_k$.

(1) ⇐ (3): Let $A/\Phi(A) \in \mathfrak{U}_k$ for every subgroup A of G with nilpotent derived subgroup. Since $\mathfrak{U}_k \subseteq \mathfrak{U}$, every subgroup A of G with nilpotent derived subgroup is supersoluble. In view of Lemma 11 (1), $G \in \mathrm{v}\mathfrak{U}$. Choose a group G of least order such that $G \in \mathrm{v}\mathfrak{U} \setminus \mathrm{v}\mathfrak{U}_k$. Since $G \in \mathrm{v}\mathfrak{U}$, a Sylow r-subgroup R of G is normal in G for $r = \max \pi(G)$. In view of Lemma 7, every cyclic r-subgroup of G is $\mathfrak{U}_k$-subnormal in G. Let H be a cyclic q-subgroup of G for a prime $q \in \pi(G)$, $q \ne r$. The derived subgroup $(R \rtimes H)' \le R \in \mathfrak{N}$. Therefore by the choice of G, $R \rtimes H/\Phi(R \rtimes H) \in \mathfrak{U}_k \subseteq \mathrm{w}\mathfrak{U}_k$. By Proposition 2, we get $R \rtimes H \in \mathrm{w}\mathfrak{U}_k$. Hence HR is a proper subgroup of G and H is $\mathfrak{U}_k$-subnormal in HR.

Let U_1/R be a subgroup with nilpotent derived subgroup in G/R. Since $(|U_1/R|, |R|) = 1$, by the Schur-Zassenhaus theorem, there is a subgroup U such that $U_1 = R \rtimes U$ and $U_1/R \cong U$ has the derived subgroup. By the choice of G, $U/\Phi(U) \in \mathfrak{U}_k$. Hence

$$(U_1/R)/\Phi(U_1/R) \cong U/\Phi(U) \in \mathfrak{U}_k.$$

Thus G/R satisfies Statement (3) and $G/R \in \mathrm{v}\mathfrak{U}_k$ by the choice of G. Therefore HR is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (2), and H is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (1). Thus, $G \in \mathrm{v}\mathfrak{U}_k$.

Statement (1) $\Leftrightarrow$ (3) is proved.

(1) $\Rightarrow$ (4): Assume that $G \in \mathrm{v}\mathfrak{U}_k$ and B is a biprimary subgroup with cyclic Sylow subgroup in G. In that case, $G \in \mathrm{v}\mathfrak{U}$, and by Lemma 11 (2), B is supersoluble. Since $\mathrm{v}\mathfrak{U}_k$ is a subgroup-closed formation by Proposition 2, we get $B \in \mathrm{v}\mathfrak{U}_k$. According to proved Statement (1) $\Rightarrow$ (3), we have $B/\Phi(B) \in \mathfrak{U} \cap (\mathrm{w}\mathfrak{U}_k)_k \subseteq \mathfrak{U}_k$.

(1) $\Leftarrow$ (4): Let G be a group of least order such that $B/\Phi(B) \in \mathfrak{U}_k$ for every biprimary B with cyclic Sylow subgroup and $G \notin \mathrm{v}\mathfrak{U}_k$. In that case, G contains a cyclic q-subgroup H for a prime $q \in \pi(G)$ that is not $\mathfrak{U}_k$-subnormal in G. Since $\mathfrak{U}_k \subseteq \mathfrak{U}$, every biprimary subgroup with cyclic Sylow subgroup in G is supersoluble. By Lemma 11 (2), $G \in \mathrm{v}\mathfrak{U}$, in particular, G has a Sylow tower of supersoluble type. Consequently, a Sylow r-subgroup R of G is normal in G for $r = \max \pi(G)$. In view of Lemma 7, R is $\mathfrak{U}_k$-subnormal in G and $r > q$. By the choice of G, $HR/\Phi(HR) \in \mathfrak{U}_k \subseteq \mathrm{v}\mathfrak{U}_k$. Hence $HR \in \mathrm{v}\mathfrak{U}_k$ by Proposition 2. Consequently, HR is a proper subgroup of G and H is $\mathfrak{U}_k$-subnormal in HR. Let K_1/R be a biprimary subgroup with cyclic Sylow subgroup in G/R. By the Schur-Zassenhaus theorem, there is a biprimary subgroup K with cyclic Sylow subgroup in K_1 such that $K_1 = R \rtimes K$ and $K_1/R \cong K$. By the choice of G, $K/\Phi(K) \in \mathfrak{U}_k$. Therefore

$$(K_1/R)/\Phi(K_1/R) \cong K/\Phi(K) \in \mathfrak{U}_k.$$

By induction, $G/R \in \mathrm{v}\mathfrak{U}_k$. It follows that HR/R is $\mathfrak{U}_k$-subnormal in G/R. Hence HR is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (2), and H is $\mathfrak{U}_k$-subnormal in G by Lemma 5 (1), a contradiction.

Statement (1) $\Leftrightarrow$ (4) is proved. □

Proof of Corollary 3. Since every supersoluble group is metanilpotent, we have $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k \subseteq \mathfrak{N}^2 \cap \mathrm{w}\mathfrak{U}_k$. If $G \in \mathfrak{N}^2 \cap \mathrm{w}\mathfrak{U}_k$, then $G/\Phi(G) \in \mathfrak{U}_k$ by Theorem 1 ((1) $\Rightarrow$ (3)). Now $G \in \mathfrak{U}$ and $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k \supseteq \mathfrak{N}^2 \cap \mathrm{w}\mathfrak{U}_k$. Hence $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k = \mathfrak{N}^2 \cap \mathrm{w}\mathfrak{U}_k$.

Since the derived subgroup of a supersoluble group is nilpotent, we get $\mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k \subseteq \mathfrak{N}\mathfrak{A} \cap \mathrm{v}\mathfrak{U}_k$. If $G \in \mathfrak{N}\mathfrak{A} \cap \mathrm{v}\mathfrak{U}_k$, then $G/\Phi(G) \in \mathfrak{U}_k$ by Theorem 2 ((1) $\Rightarrow$ (3)). Now $G \in \mathfrak{U}$ and $\mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k \supseteq \mathfrak{N}\mathfrak{A} \cap \mathrm{v}\mathfrak{U}_k$. Hence $\mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k = \mathfrak{N}\mathfrak{A} \cap \mathrm{v}\mathfrak{U}_k$.

In view of Lemma 10, $\mathrm{w}\mathfrak{U}_k \subset \mathrm{v}\mathfrak{U}_k$. Therefore $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k \subseteq \mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k$.

Conversely, let $G \in \mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k$. By Theorem 2 ((1) $\Rightarrow$ (2)), $G/\Phi(G) \in \mathfrak{U} \cap (\mathrm{v}\mathfrak{U}_k)_k \subseteq \mathfrak{U}_k \subseteq \mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k$, and $G \in \mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k$ since $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k$ is a saturated formation.

Since $\mathfrak{U} \cap \mathrm{w}\mathfrak{U}_k = \mathfrak{U} \cap \mathrm{v}\mathfrak{U}_k$, it follows that every Sylow subgroup of a supersoluble group G is $\mathfrak{U}_k$-subnormal in G if and only if every cyclic primary subgroup of G is $\mathfrak{U}_k$-subnormal in G. □

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