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On the homotopy finiteness of DG categories

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Abstract. This paper gives a short overview of results related to homotopy finiteness of DG categories. A general plan is explained for proving homotopy finiteness of derived categories of coherent sheaves and coherent matrix factorizations on separated schemes of finite type over a field of characteristic zero.

Bibliography: 39 titles.

Keywords: derived categories, differential graded categories, homotopy finiteness, Verdier localization, resolution of singularities.

Contents

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1. Introduction

In this paper we give an overview of results on homotopy finiteness of differential graded (DG) categories and on smooth categorical compactifications.

According to one of the approaches to non-commutative algebraic geometry, a non-commutative space is a triangulated DG (or A_∞-) category which admits a single generator ([19], [27]).

By a theorem of Bondal and Van den Bergh ([5], Theorem 3.1.1) and by results of Keller [17], for any separated scheme X of finite type over a field k there is a DG k-algebra A, defined up to a Morita equivalence, such that

$$D(\mathrm{QCoh}(X)) \simeq D(A).$$

Here $D(A)$ is the derived category of (right) DG A-modules. This equivalence identifies the full subcategories of perfect complexes (which are exactly the compact objects): $D_{\mathrm{perf}}(X) \simeq D_{\mathrm{perf}}(A)$. Let us denote by $\mathrm{Perf}(X)$ the DG enhancement (see [3]) of the triangulated category $D_{\mathrm{perf}}(X)$. The DG category $\mathrm{Perf}(X)$ is treated as a non-commutative space associated with X.

It is known (see [27], Proposition 3.30, and [22], Proposition 3.13) that a scheme X is smooth (respectively, proper) if and only if the DG category $\mathrm{Perf}(X)$ is smooth (respectively, proper). Thus, these basic geometric properties of X are reflected by the DG category $\mathrm{Perf}(X)$. The notions of smoothness and properness for DG categories are recalled in § 2.

The situation is quite different for the DG category $D^b_{\mathrm{coh}}(X) \supset \mathrm{Perf}(X)$ — an enhancement of the derived category of coherent sheaves. Namely, the following theorem has been proven by Lunts.

Theorem 1.1 ([22], Theorem 6.3). *Let X be a separated scheme of finite type over a perfect field* k. *Then the DG category $D^b_{\mathrm{coh}}(X)$ is smooth.*

This is quite surprising: the scheme X can have arbitrary singularities and even be non-reduced, but the DG category $D^b_{\mathrm{coh}}(X)$ is always smooth.

The class of smooth DG categories contains some 'large' examples. For example, the field of rational functions $\mathrm{k}(x_1, \dots, x_n)$ is a smooth DG algebra. It is natural to try to impose some conditions on a smooth DG category so that it is 'finitely presented' in an appropriate sense.

Toën and Vaquié [36] introduced the class of so-called homotopically finitely presented (homotopically finite) DG categories.

Definition 1.2 [36]. 1) A DG algebra A is homotopically finite if in the homotopy category of DG algebras A there is a retract of a free graded algebra $\mathrm{k}\langle x_1, \dots, x_n\rangle$ with differential satisfying the condition

$$dx_i \in \mathrm{k}\langle x_1, \dots, x_{i-1}\rangle, \qquad 1 \leqslant i \leqslant n.$$

2) A small DG category is homotopically finite if it is Morita equivalent to a smooth DG algebra which is homotopically finite.

See § 2 for a more detailed discussion, in particular for the notion of a retract. The homotopically finite DG categories play the same role in the Morita homotopy category of DG categories as the perfect $\mathscr{A}$-modules play in the derived category of all $\mathscr{A}$-modules (where $\mathscr{A}$ is some small DG category). Also, their analogue in the homotopy category of CW complexes is the category of so-called finitely dominated spaces, which are homotopy retracts of finite CW complexes (see [38], [39], and [37]).

The basic facts about homotopically finite DG categories are recalled in § 2. Here we mention that if a DG category is hfp, then it is smooth. On the other hand, if a DG category is smooth and proper, then it is homotopically finite.

The main result of [8] is the following theorem.

Theorem 1.3 ([8], Theorem 1.4). *Let Y be a separated scheme of finite type over a field* k *of characteristic zero. Then the DG category $D^b_{\mathrm{coh}}(Y)$ is homotopically finite.*

The statement of this theorem was previously conjectured by Kontsevich at the conference at the University of Miami in 2010.[1]

In [8] a similar result is also proved for coherent matrix factorizations [11]. For any regular function W on Y we have a $\mathbb{Z}/2$-graded DG category $D^{\mathrm{abs}}_{\mathrm{coh}}(X,W)$ — an enhancement of the absolute derived category of coherent matrix factorizations of W.

Theorem 1.4 ([8], Theorem 1.5). *Let Y be a separated scheme of finite type over a field* k *of characteristic zero, and let W be a regular function on Y. Then the $\mathbb{Z}/2$-graded DG category $D^{\mathrm{abs}}_{\mathrm{coh}}(Y,W)$ is homotopically finite.*

There is a particularly nice class of homotopically finite DG categories: those which admit a so-called smooth categorical compactification. We recall that a DG quasi-functor $F\colon \mathscr{A} \to \mathscr{B}$ between small DG categories is a DG bimodule $N_F \in \mathscr{A}\text{-}\mathrm{Mod}\text{-}\mathscr{B}$, such that for each object $X \in \mathscr{A}$ the DG $\mathscr{B}$-module $N_F(X,-)$ is quasi-isomorphic to a representable module.

Definition 1.5. A smooth categorical compactification of a DG category $\mathscr{A}$ is a DG quasi-functor $F\colon \mathscr{C} \to \mathscr{A}$, where the DG category $\mathscr{C}$ is smooth and proper, the extension of scalars functor $F^*\colon \mathrm{Perf}(\mathscr{C}) \to \mathrm{Perf}(\mathscr{A})$ is a Verdier localization (up to direct summands), and its kernel is generated by a single object.

The motivation for the term 'smooth categorical compactification' is as follows. Suppose that Y is smooth, and $\overline{Y} \supset Y$ is a usual (algebro-geometric) smooth compactification. Then the restriction functor $D^b_{\mathrm{coh}}(\overline{Y}) \to D^b_{\mathrm{coh}}(Y)$ is a smooth categorical compactification.

One can show that the existence of a smooth categorical compactification implies homotopy finiteness (see Corollary 2.9). In the recent paper [9] the author gives examples of homotopically finite DG categories which do not admit a smooth categorical compactification.

In [8] the following theorem is proved, which is stronger than Theorems 1.3 and 1.4.

Theorem 1.6 ([8], Theorem 1.8). *Let Y be a separated scheme of finite type over a field* k *of characteristic zero. Then*:

[1] Workshop on homological mirror symmetry and related topics (University of Miami, 2010), Discussion session.

1) *the DG category* $D^b_{\mathrm{coh}}(Y)$ *has a smooth categorical compactification of the form* $D^b_{\mathrm{coh}}(\widetilde{Y}) \to D^b_{\mathrm{coh}}(Y)$, *where* $\widetilde{Y}$ *is a smooth and proper variety*;

2) *for any regular function* $W \in \mathscr{O}(Y)$ *the* $D(\mathbb{Z}/2\text{-})G$ *category* $D^{\mathrm{abs}}(Y,W)$ *has a* $\mathbb{Z}/2$*-graded smooth categorical compactification* $C_W \to D^{\mathrm{abs}}(X,W)$, *with a semi-orthogonal decomposition* $C_W = \langle D^{\mathrm{abs}}(V_1,W_1),\dots,D^{\mathrm{abs}}(V_m,W_m)\rangle$, *where each* V_i *is a* k*-smooth variety and the morphisms* $W_i\colon V_i \to \mathbb{A}^1_{\mathrm{k}}$ *are proper.*

The general idea of the proof of Theorem 1.6 is motivated by the following conjecture of Bondal and Orlov.

Conjecture 1.7 [4]. *Let* Y *be a variety with rational singularities, and let* $f\colon X \to Y$ *be a resolution of singularities* (*recall that rationality of singularities means that* $\mathbf{R}f_*\mathscr{O}_X \cong \mathscr{O}_Y$). *Then the functor* $\mathbf{R}f_*\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ *is a localization.*

If we are able to prove Conjecture 1.7 and also to show that the kernel of the localization is generated by a single object, then choosing any smooth compactification $\overline{X}$ of X, we get a smooth categorical compactification $D^b_{\mathrm{coh}}(\overline{X})$ of $D^b_{\mathrm{coh}}(Y)$. Unfortunately, we have not been able to prove Conjecture 1.7 in general, but the technique developed in [8] allows us to prove it in a certain class of cases.

Theorem 1.8 ([8], Theorem 1.10). *Suppose that* Y *has rational singularities,* $Z \subset Y$ *is a closed smooth subscheme, and the blow-up* $X = \mathrm{Bl}_Z Y$ *is smooth, so that* $f\colon X \to Y$ *is a resolution of singularities. Let us denote by* $T = f^{-1}(Z)$ *the exceptional divisor, by* $p\colon T \to Z$ *the induced morphism, and by* $j\colon T \to X$ *the embedding. Suppose that* $\mathbf{R}f_* I_T^n = I_Z^n$ *for* $n \geqslant 1$. *Then the functor* $\mathbf{R}f_*\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ *is a localization, and its kernel is generated by the subcategory* $j_*((p^*D^b_{\mathrm{coh}}(Z))^{\perp})$.

A more general version of this result is Theorem 7.20. It can be applied, for example, when $Y \subset \mathbb{A}^m$ is a cone over some projective embedding of a smooth Fano variety, and $Z = \{0\} \subset Y$ is the origin.

For an arbitrary scheme Y (separated, of finite type) the idea is to use the so-called categorical resolution constructed by Kuznetsov and Lunts [21]. It plays the same role as the derived category of the resolution of rational singularities, and it exists for any separated scheme of finite type. Surprisingly, in this framework we are able to prove the analogue of Conjecture 1.7, which allows us to prove Theorem 1.6. We note that even if Y has rational singularities, we still use the categorical resolution to obtain the smooth categorical compactification.

This overview is organized as follows.

In § 2 we discuss the notions of homotopy finiteness of DG categories and smooth categorical compactifications, and also formulate some basic results related to these notions.

In § 3 we recall the Neeman criterion for a functor to be a localization, and we introduce homological epimorphisms of DG categories (in the terminology of [14] and [28]), which generalize localizations.

In § 4 we recall the notion of gluing DG categories via a bimodule.

In § 5 we recall the notions of coderived category and absolute derived category. We formulate basic results about them for locally Noetherian Abelian categories, in particular, the statement about compact generation.

Section 6 is devoted to specific convenient enhancements for derived categories of coherent sheaves and absolute derived categories of coherent matrix factorizations. For these enhancements, we have natural DG direct image functors (not just quasi-functors) for a proper morphism.

In § 6.3 we introduce the category of nice ringed spaces. Its objects are pairs $(X, \mathscr{A}_X)$, where X is a separated Noetherian scheme and $\mathscr{A}_X$ is a coherent sheaf of $\mathscr{O}_X$-algebras satisfying a certain additional condition. We discuss the (co)derived categories of (quasi-)coherent sheaves on nice ringed spaces, and functors between them.

Section 7 is devoted to the proof of Theorem 1.6 (see Theorem 7.23).

2. Homotopy theory of DG algebras and DG categories

For an introduction to DG categories, we refer the reader to [17]. Our basic reference for model categories is [15]. The references for model structures on DG algebras and DG categories are [34] and [35]. The notion of homotopy finiteness is taken from [36]. The references for DG quotients are [6] and [18].

We fix some base field k. We will consider either $\mathbb{Z}$-graded or $\mathbb{Z}/2$-graded DG categories. The latter can be treated as DG categories over $\mathrm{k}[u^{\pm 1}]$, where u has degree 2. These two cases are parallel for our discussion. If we do not specify the grading, then we mean that everything holds in both frameworks. We write $\otimes$ for $\otimes_{\mathrm{k}}$. Also, for a homogeneous element v of a graded vector space V, we denote by $|v|$ its grading.

All DG modules are assumed to be right modules unless otherwise stated. Given a small DG category $\mathscr{A}$, we denote by $\operatorname{Mod-}\mathscr{A}$ the DG category of right DG modules (it is denoted by $\operatorname{Dif}\mathscr{A}$ in [17], § 1.2). We denote by $\mathscr{A}\text{-}\operatorname{Mod} = \operatorname{Mod-}\mathscr{A}^{\mathrm{op}}$ the DG category of left $\mathscr{A}$-modules. We have a fully faithful Yoneda embedding functor $\mathscr{A} \to \operatorname{Mod-}\mathscr{A}$. For any DG category T (not necessarily small) the k-linear category $H^0(T)$ has the same objects as T, and the morphisms are given by

$$H^0(T)(X, Y) = H^0(T(X, Y)).$$

It is shown in [17], Lemma 2.2, that the category $H^0(\operatorname{Mod-}\mathscr{A})$ is naturally triangulated. The derived category $D(\mathscr{A})$ is defined to be the Verdier quotient of $H^0(\operatorname{Mod-}\mathscr{A})$ by the full triangulated subcategory of acyclic DG modules.

It is also convenient to define the category $Z^0(T)$ for any DG category T, similarly to $H^0(T)$. Here for a complex $\mathscr{K}^\bullet$ of vector spaces we denote the vector space of closed elements of degree zero by $Z^0(\mathscr{K}^\bullet)$.

By results in [17], § 3, the full subcategory $H^0(\operatorname{Acycl}(\mathscr{A}))$ of acyclic DG modules in $H^0(\operatorname{Mod-}\mathscr{A})$ is both left and right admissible. Recall that an $\mathscr{A}$-module M is said to be *h-projective* (respectively, *h-injective*) if it is in the left (respectively, right) orthogonal to $H^0(\operatorname{Acycl}(\mathscr{A}))$. We denote by $\operatorname{h-proj}(\mathscr{A}) \subset \operatorname{Mod-}\mathscr{A}$ the full DG subcategory of h-projective $\mathscr{A}$-modules. In particular, we have an equivalence $D(\mathscr{A}) \simeq H^0(\operatorname{h-proj}(\mathscr{A}))$. This allows us to define the left derived functor $\mathbf{L}F\colon D(\mathscr{A}) \to D(\mathscr{B})$ of any exact functor $F\colon H^0(\operatorname{Mod-}\mathscr{A}) \to H^0(\operatorname{Mod-}\mathscr{B})$ to be the composition

$$D(\mathscr{A}) \xrightarrow{\sim} H^0(\operatorname{h-proj}(\mathscr{A})) \xrightarrow{F} H^0(\operatorname{Mod-}\mathscr{B}) \to D(\mathscr{B}).$$

The tensor product bifunctor

$$-\otimes_{\mathscr{A}} -\colon \ \mathrm{Mod}\text{-}\mathscr{A} \otimes \mathscr{A}\text{-}\mathrm{Mod} \to \mathrm{Mod}\text{-}\mathrm{k}$$

is given by

$$M \otimes_{\mathscr{A}} N = \operatorname{Coker}\biggl(\bigoplus_{X,Y\in\mathscr{A}} M(Y)\otimes \mathscr{A}(X,Y)\otimes N(Y) \xrightarrow{\nu} \bigoplus_{X\in\mathscr{A}} M(X)\otimes N(X)\biggr),$$

where $\nu(m\otimes f\otimes n) = mf\otimes n - m\otimes fn$.

Given small DG categories $\mathscr{A}$ and $\mathscr{B}$, we denote by $\mathscr{A}\text{-}\mathrm{Mod}\text{-}\mathscr{B}$ the DG category $\mathrm{Mod}\text{-}(\mathscr{A}^{\mathrm{op}}\otimes\mathscr{B})$ of $\mathscr{A}$-$\mathscr{B}$-bimodules. Then an $\mathscr{A}$-$\mathscr{B}$-bimodule N defines a DG functor

$$-\otimes_{\mathscr{A}} N\colon \ \mathrm{Mod}\text{-}\mathscr{A} \to \mathrm{Mod}\text{-}\mathscr{B},$$

given by

$$(M\otimes_{\mathscr{A}} N)(X) = M\otimes_{\mathscr{A}} N(-,X).$$

This DG functor induces an exact functor

$$-\otimes_{\mathscr{A}} N\colon H^0(\mathrm{Mod}\text{-}\mathscr{A}) \to H^0(\mathrm{Mod}\text{-}\mathscr{A}).$$

We denote by $-\overset{\mathbf{L}}{\otimes}_{\mathscr{A}} N\colon D(\mathscr{A})\to D(\mathscr{B})$ the left derived functor.

Similarly, for an $\mathscr{A}$-$\mathscr{B}$-bimodule M and a $\mathscr{B}$-$\mathscr{C}$-bimodule N, their tensor product $M\otimes_{\mathscr{B}} N\in \mathscr{A}\text{-}\mathrm{Mod}\text{-}\mathscr{C}$ is given by

$$(M\otimes_{\mathscr{B}} N)(X,Y) = M(X,-)\otimes_{\mathscr{B}} N(-,Y).$$

Deriving the resulting bifunctor on either side gives the same bi-exact bifunctor

$$D(\mathscr{A}^{\mathrm{op}}\otimes\mathscr{B})\times D(\mathscr{B}^{\mathrm{op}}\otimes\mathscr{C}) \to D(\mathscr{A}^{\mathrm{op}}\otimes\mathscr{C}).$$

We denote the full DG subcategory of *semi-free finitely generated* modules by $\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A})\subset \mathrm{Mod}\text{-}\mathscr{A}$. That is, a module M is in $\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A})$ if it has a finite filtration by DG submodules such that all the subquotients are isomorphic to shifts of representable DG modules. In particular, all representable $\mathscr{A}$-modules are in $\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A})$. In fact, $\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A})\subset \text{h-proj}(\mathscr{A})$, and the category $H^0(\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A}))$ is identified with the full triangulated subcategory of $D(\mathscr{A})$ generated by representable modules via shifts and cones (note that the category $H^0(\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A}))$ is not necessarily Karoubi-closed). We recall that a DG category $\mathscr{A}$ is said to be weakly (respectively, strongly) pre-triangulated if the Yoneda functor $\mathscr{A}\to\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A})$ is a quasi-equivalence (respectively, a DG equivalence). In particular, for a weakly pre-triangulated DG category, the category $H^0(\mathscr{A})$ is triangulated. By definition [3], an enhancement of a triangulated category $\mathscr{T}$ is a weakly pre-triangulated DG category $\mathscr{A}$, together with an exact equivalence $H^0(\mathscr{A})\simeq T$.

We recall that the triangulated subcategory $D_{\mathrm{perf}}(\mathscr{A})\subset D(\mathscr{A})$ is defined to be the Karoubi completion of $H^0(\mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A}))$ inside $D(\mathscr{A})$. In fact, the triangulated category $D(\mathscr{A})$ is compactly generated (see [17], § 4.2) and the subcategory $D(\mathscr{A})^c$

of compact objects coincides with $D_{\mathrm{perf}}(\mathscr{A})$ (see [32], [25], Lemma 2.2, and [17], Theorem 5.3).

We denote by $\mathrm{dgalg}_{\mathrm{k}}$ the category of DG algebras over k. By [16] it has a model structure, with weak equivalences being quasi-isomorphisms and fibrations being surjections. This model category is finitely generated in the terminology of [15]. Its finite-cell objects are as follows.

Definition 2.1 [36]. A finite-cell DG algebra B is a DG algebra which is isomorphic as a graded algebra to a free algebra of finite type:

$$B^{\mathrm{gr}} \cong \mathrm{k}\langle x_1, \dots, x_n\rangle,$$

and moreover,

$$dx_i \in \mathrm{k}\langle x_1, \dots, x_{i-1}\rangle, \qquad 1 \leqslant i \leqslant n.$$

We recall that for a category $\mathscr{C}$, an object $X \in \mathscr{C}$ is called a *retract* of an object $Y \in \mathscr{C}$ if there exist morphisms $f\colon X \to Y$ and $g\colon Y \to X$ such that $gf = \mathrm{id}_X$.

The following definition is due to Toën and Vaquié [36]. It makes sense for all finitely generated model categories.

Definition 2.2 [36]. A DG algebra A is homotopically finitely presented (homotopically finite) if in the homotopy category $\mathrm{Ho}(\mathrm{dgalg}_{\mathrm{k}})$ the object A is a retract of some finite-cell DG algebra B.

We recall the notions of smoothness and properness.

Definition 2.3 [19]. 1) A DG algebra A is smooth over k if the diagonal A-A-bimodule is perfect:

$$A \in D_{\mathrm{perf}}(A \otimes A^{\mathrm{op}}).$$

2) A DG algebra A is proper over k if $A \in D_{\mathrm{perf}}(\mathrm{k})$, or in other words, the total cohomology of A is finite-dimensional.

We have the following implications, which were proved in [36].

Theorem 2.4 [36]. 1) *If a DG algebra is homotopically finite over* k, *then it is smooth.*

2) *If a DG algebra is smooth and proper over* k, *then it is homotopically finite.*

3) *If DG algebras A and A' are Morita equivalent and A is hfp, then so is A'.*

Part 3) of the above theorem implies that we can talk about homotopy finiteness of small DG categories. We can also consider the category of small DG categories $\mathrm{dgcat}_{\mathrm{k}}$, and define weak equivalences as Morita equivalences. Tabuada [35] has constructed the corresponding model structure, which is again finitely generated. We denote by $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ the corresponding homotopy category.

By [34] there is another model structure on $\mathrm{dgcat}_{\mathrm{k}}$, with weak equivalences being quasi-equivalences. We denote by $\mathrm{Ho}(\mathrm{dgcat}_{\mathrm{k}})$ the corresponding homotopy category.

Definition 2.5. A DG category $\mathscr{B}$ is called a finite-cell category if:

i) $\mathscr{B}$ has a finite number of objects;

ii) the graded category $\mathscr{B}^{\mathrm{gr}}$ is freely generated by a finite number of morphisms $f_1, \dots, f_n$;

iii) $df_i \in \mathrm{k}\langle f_1, \dots f_{i-1}\rangle$, $1 \leqslant i \leqslant n$.

Homotopically finite DG categories are defined in the same way.

Definition 2.6. A small DG category $\mathscr{A}$ is homotopically finite if in the homotopy category $\mathrm{Ho}_M(\mathrm{dgcat}_{\mathrm{k}})$ the object $\mathscr{A}$ is a retract of a finite-cell DG category.

By [36], Corollary 2.12, a DG category is homotopically finite if and only if it is Morita equivalent to a homotopically finite DG algebra.

Remark 2.7. The notions of smoothness and properness make sense for all small DG categories, and statements 1) and 2) of Theorem 2.4 are also true for small DG categories.

The following result holds.

Proposition 2.8 ([8], Proposition 2.8). *Let $\mathscr{C}$ be a small DG category which is hfp, and let $E \in \mathrm{Ob}(\mathscr{C})$ be an object. Then the DG quotient $\mathscr{C}/E$ is also homotopically finite.*

In the Introduction we defined the notion of a smooth categorical compactification (Definition 1.5). We have the following corollary.

Corollary 2.9. *Assume that a small DG category $\mathscr{A}$ has a smooth categorical compactification (see Definition 1.5). Then $\mathscr{A}$ is homotopically finite.*

Proof. Indeed, this follows directly from Proposition 2.8 and Theorem 2.4. □

3. Homological epimorphisms and localizations

We recall the following result of Neeman on the localizations of compactly generated triangulated categories. If $\mathscr{T}$ is a compactly generated triangulated category, then $\mathscr{T}^c \subset \mathscr{T}$ denotes the full triangulated subcategory of compact objects.

Theorem 3.1 [26]. *Let $\mathscr{T}$ and $\mathscr{S}$ be compactly generated triangulated categories, and let $F\colon \mathscr{T} \to \mathscr{S}$ be an exact functor commuting with small direct sums and preserving compact objects. The following are equivalent:*

(i) *the induced functor $F^c\colon \mathscr{T}^c \to \mathscr{S}^c$ is a localization up to direct summands (that is, it is a localization onto its image, and the Karoubi completion of the image coincides with $\mathscr{S}^c$);*

(ii) *the functor $F\colon \mathscr{T} \to \mathscr{S}$ is a localization, and its kernel is generated (as a localizing subcategory) by its intersection with $\mathscr{T}^c$.*

We will restrict ourselves to triangulated categories with a DG enhancement. We need to specify our notation.

Notational convention. For a DG functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ between small DG categories we denote by $\Phi_*\colon D(\mathscr{B}) \to D(\mathscr{A})$ the restriction of scalars functor. Its left adjoint, the extension of scalars, is denoted by $\Phi^*\colon D(\mathscr{A}) \to D(\mathscr{B})$. We denote by $I_{\mathscr{A}} \in \mathscr{A}\text{-}\mathrm{Mod}\text{-}\mathscr{A}$ the diagonal bimodule given by $I_{\mathscr{A}}(X,Y) = \mathscr{A}(Y,X)$. When it does not lead to confusion, we also denote this bimodule by $\mathscr{A}$, as well as its various restrictions of scalars. For example, the extension of scalars functor above can be written as

$$\Phi^*(-) = - \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \mathscr{B}.$$

The following notion of a homological epimorphism is a straightforward generalization of the corresponding notions from [14] (the case of associative rings) and [28] (the case of DG algebras).

Definition 3.2. A DG functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ between small DG categories is a homological epimorphism if the extension of scalars functor

$$\Phi^*\colon D(\mathscr{A}) \to D(\mathscr{B})$$

is a localization.

Remark 3.3. An exact functor $F\colon \mathscr{T} \to \mathscr{S}$ between (not necessarily small) triangulated categories is a localization if the induced functor $\overline{F}\colon \mathscr{T}/\ker(F) \to \mathscr{S}$ is an equivalence, which is in general hard to verify (for example, Conjecture 1.7 is a statement of this kind). However, if we assume moreover that the functor F has a left (respectively, right) adjoint G, then the condition on F to be a localization is equivalent to the condition on G to be fully faithful.

The property of a functor to be a homological epimorphism has several reformulations.

Proposition 3.4 ([8], Proposition 3.4). *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a DG functor between small DG categories. The following are equivalent*:

(i) Φ *is a homological epimorphism*;

(ii) *the restriction of scalars functor $\Phi_*\colon D(\mathscr{B}) \to D(\mathscr{A})$ is fully faithful*;

(iii) *for any $X, Y \in \mathrm{Ob}(\mathscr{B})$ the natural (composition) morphism*

$$\mathscr{B}(\Phi(-), Y) \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \mathscr{B}(X, \Phi(-)) \to \mathscr{B}(X, Y)$$

is an isomorphism in $D(\mathrm{k})$;

(iv) *the natural morphism*

$$\mathscr{B} \overset{\mathbf{L}}{\otimes}_{\mathscr{A}} \mathscr{B} = (\Phi \otimes \Phi^{\mathrm{op}})^* I_{\mathscr{A}} \to I_{\mathscr{B}} \tag{3.1}$$

is an isomorphism in $D(\mathscr{B} \otimes \mathscr{B}^{\mathrm{op}})$.

Corollary 3.5. *If $\Phi\colon \mathscr{A} \to \mathscr{B}$ is a homological epimorphism and $\mathscr{A}$ is smooth, then $\mathscr{B}$ is also smooth.*

Proof. By the definition of smoothness, the bimodule $I_{\mathscr{A}} \in D(\mathscr{A} \otimes \mathscr{A}^{\mathrm{op}})$ is perfect. The extension of scalars functor always preserves perfect complexes. Hence, condition (iv) in Proposition 3.4 implies that $I_{\mathscr{B}} \in D(\mathscr{B} \otimes \mathscr{B}^{\mathrm{op}})$ is also perfect, and thus $\mathscr{B}$ is smooth. □

We note that the properties of being a homological epimorphism and of being quasi-fully-faithful are dual to each other (see [8], Proposition 3.6). We recall that a DG functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ is said to be quasi-fully-faithful if the morphisms

$$\Phi(X, Y)\colon \mathscr{A}(X, Y) \to \mathscr{B}(\Phi(X), \Phi(Y)), \qquad X, Y \in \mathscr{A},$$

are quasi-isomorphisms.

Definition 3.6. We call a DG functor $\Phi\colon \mathscr{A} \to \mathscr{B}$ between small DG categories a localization if the functor $\Phi^*\colon D_{\mathrm{perf}}(\mathscr{A}) \to D_{\mathrm{perf}}(\mathscr{B})$ is a Verdier localization up to direct summands.

Theorem 3.1 directly implies the following.

Corollary 3.7. *Let $\Phi\colon \mathscr{A} \to \mathscr{B}$ be a functor between small DG categories. The following are equivalent*:

(i) Φ *is a localization*;

(ii) Φ *is a homological epimorphism and the kernel of* $\Phi^*\colon D(\mathscr{A}) \to D(\mathscr{B})$ *is generated* (*as a localizing subcategory*) *by its intersection with* $D_{\mathrm{perf}}(\mathscr{A})$.

We finish this section by mentioning a situation when a homological epimorphism is automatically a localization.

Lemma 3.8. *For a commutative square*

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{F_1} & \mathscr{B} \\ {\scriptstyle G_1}\downarrow & & \downarrow{\scriptstyle G_2} \\ \mathscr{C} & \xrightarrow{F_2} & \mathscr{D} \end{array}$$

of DG functors, let F_1 and F_2 be quasi-fully-faithful, let G_1 be a localization, let G_2 be a homological epimorphism, and let the induced functor $\overline{G}_2\colon \mathscr{B}/F_1(\mathscr{A}) \to \mathscr{D}/F_2(\mathscr{C})$ be a Morita equivalence. Then G_2 is a localization. Moreover, there is an identification of subcategories

$$\ker(G_2^*\colon D_{\mathrm{perf}}(\mathscr{B}) \to D_{\mathrm{perf}}(\mathscr{D})) = F_1^*(\ker(G_1^*\colon D_{\mathrm{perf}}(\mathscr{A}) \to D_{\mathrm{perf}}(\mathscr{C}))).$$

Proof. By Corollary 3.7, we only need to show that $\ker(G_2^*\colon D(\mathscr{B}) \to D(\mathscr{D}))$ is identified with $F_1^*(\ker(G_1^*\colon D(\mathscr{A}) \to D(\mathscr{C})))$. Let

$$\mathrm{pr}_1\colon \mathscr{B} \to \mathscr{B}/F_1(\mathscr{A}) \quad \text{and} \quad \mathrm{pr}_2\colon \mathscr{D} \to \mathscr{D}/F_2(\mathscr{C})$$

denote the projection DG functors. We have semi-orthogonal decompositions

$$D(\mathscr{B}) = \langle \mathrm{pr}_{1*} D(\mathscr{B}/F_1(\mathscr{A})), F_1^* D(\mathscr{A})\rangle \text{ and } D(\mathscr{D}) = \langle \mathrm{pr}_{2*} D(\mathscr{D}/F_2(\mathscr{C})), F_2^* D(\mathscr{C})\rangle. \tag{3.2}$$

The functor $G_2^*\colon D(\mathscr{B}) \to D(\mathscr{D})$ is compatible with the semi-orthogonal decompositions (3.2), and it induces the functors $\overline{G}_2^*$ and G_1^* on the components. By our assumptions, the functor $\overline{G}_2^*$ is an equivalence. It follows that $\ker(G_2^*\colon D(\mathscr{B}) \to D(\mathscr{D}))$ is contained in $F_1^*(D(\mathscr{A}))$. □

4. Gluing of DG categories

First we recall the notion of gluing, following the notation of [27].

Definition 4.1. Let $\mathscr{A}$ and $\mathscr{B}$ be small DG categories, and let $M \in D(\mathscr{A} \otimes \mathscr{B}^{\mathrm{op}})$ be a bimodule.

1) Define the DG category $\mathscr{C} = \mathscr{A} \sqcup_M \mathscr{B}$ as follows. First, $\mathrm{Ob}(\mathscr{A} \sqcup_M \mathscr{B}) = \mathrm{Ob}(\mathscr{A}) \sqcup \mathrm{Ob}(\mathscr{B})$. The complexes of morphisms are defined by

$$\mathscr{C}(X,Y) = \begin{cases} \mathscr{A}(X,Y) & \text{for } X, Y \in \mathscr{A}; \\ \mathscr{B}(X,Y) & \text{for } X, Y \in \mathscr{B}; \\ M(X,Y) & \text{for } X \in \mathscr{A},\ Y \in \mathscr{B}; \\ 0 & \text{for } X \in \mathscr{B},\ Y \in \mathscr{A}. \end{cases}$$

The composition in $\mathscr{A} \sqcup_M \mathscr{B}$ is given by the compositions in $\mathscr{A}$ and $\mathscr{B}$ and by the bimodule structure on M.

2) The DG category $\mathscr{A} \oplus_M \mathscr{B}$ is defined as follows. Its objects are triples (X, Y, μ), where $X \in \mathrm{Ob}(\mathscr{A})$, $Y \in \mathrm{Ob}(\mathscr{B})$, and $\mu \in M^0(X,Y)$ is a cocycle of degree zero. The graded k-modules of morphisms are defined by

$$\mathrm{Hom}((X,Y,\mu),(X',Y',\mu')) = \mathscr{A}(X,X') \oplus \mathscr{B}(Y,Y') \oplus M(X,Y')[-1].$$

The differential is given by the formula

$$d(f_1, f_2, f_{12}) = \big(d(f_1), d(f_2), -d(f_{12}) - f_2\mu + \mu' f_1\big).$$

The composition is given by

$$(f_1, f_2, f_{12}) \circ (g_1, g_2, g_{12}) = (f_1 \circ g_1, f_2 \circ g_2, f_{12} g_1 + (-1)^{|f_2|} f_2 g_{12}).$$

These two versions of gluing are related to each other as follows.

Proposition 4.2. 1) *There is a natural fully faithful DG functor*

$$\Phi \colon \mathscr{A} \oplus_M \mathscr{B} \to \mathscr{S}\mathscr{F}_{\mathrm{fg}}(\mathscr{A} \sqcup_M \mathscr{B}), \qquad (X,Y,\mu) \mapsto \mathrm{Cone}(h_X \xrightarrow{\mu} h_Y).$$

Moreover, Φ *is a Morita equivalence.*

2) *If both* $\mathscr{A}$ *and* $\mathscr{B}$ *are weakly (respectively, strongly) pre-triangulated, then* Φ *is a quasi-equivalence (respectively, a DG equivalence).*

Proof. Straightforward checking. □

Remark 4.3. It follows from 1) in Proposition 4.2 that for any DG functor $F \colon \mathscr{A} \sqcup_M \mathscr{B} \to \mathscr{C}$, where $\mathscr{C}$ is strongly pre-triangulated, we have a natural DG functor

$$F' \colon \mathscr{A} \oplus_M \mathscr{B} \to \mathscr{C}, \quad \text{where } F'(X,Y,\mu) = \mathrm{Cone}(F(\mu) \colon F(X) \to F(Y))$$

(the DG functor F' is well defined up to a canonical DG isomorphism). Below we use this observation implicitly.

We give the following definition.

Definition 4.4. A semi-orthogonal decomposition of a DG category $\mathscr{B}$ is a pair of DG functors $F_1 \colon \mathscr{A}_1 \to \mathscr{B}$ and $F_2 \colon \mathscr{A}_2 \to \mathscr{B}$ such that F_1 and F_2 are quasi-fully-faithful and the triangulated category $D_{\mathrm{perf}}(\mathscr{B})$ has a semi-orthogonal decomposition

$$D_{\mathrm{perf}}(\mathscr{B}) = \langle F_1^* D_{\mathrm{perf}}(\mathscr{A}_1), F_2^* D_{\mathrm{perf}}(\mathscr{A}_2) \rangle.$$

If these conditions are satisfied, then we write $\mathscr{B} = \langle F_1(\mathscr{A}_1), F_2(\mathscr{A}_2)\rangle$. We also write $\mathscr{B} = \langle \mathscr{A}_1, \mathscr{A}_2\rangle$ if the DG functors F_1 and F_2 are either clear from the context or are irrelevant.

We will use the following special case of the gluing construction.

Definition 4.5. Suppose that we are given a pair of DG functors between small DG categories

$$\mathscr{A} \xleftarrow{F_1} \mathscr{C} \xrightarrow{F_2} \mathscr{B}.$$

Take the $\mathscr{B}$-$\mathscr{A}$-bimodule

$$M := \mathscr{B} \overset{\mathbf{L}}{\otimes}_{\mathscr{C}} \mathscr{A},$$

where the derived tensor product is computed via bar resolution. We put

$$\mathscr{A} \sqcup_{(\mathscr{C})} \mathscr{B} := \mathscr{A} \sqcup_M \mathscr{B} \quad \text{and} \quad \mathscr{A} \oplus_{(\mathscr{C})} \mathscr{B} := \mathscr{A} \oplus_M \mathscr{B}.$$

5. Exotic derived categories

In this section we recall the notions of absolute derived and coderived categories. Exotic derived categories were introduced by Positselski ([29], [30]). The case of a locally Noetherian Abelian category was also studied earlier by Krause [20].

Definition 5.1. Let $\mathscr{C}$ be an exact category with exact small coproducts. Denote by $K(\mathscr{C})$ the homotopy category of complexes of objects in $\mathscr{C}$. Define the full subcategory of co-acyclic complexes co-Acycl$(\mathscr{C}) \subset K(\mathscr{C})$ to be the localizing subcategory generated by the totalizations of short exact sequences of complexes. Then the coderived category is defined as the quotient

$$D^{\mathrm{co}}(\mathscr{C}) := K(\mathscr{C})/\operatorname{co-Acycl}(\mathscr{C}).$$

Here by 'totalization' of a short exact sequence of complexes we mean the sum-total complex of the corresponding bicomplex (in this bicomplex the non-zero columns are given by the terms of the short exact sequence, and they are placed in degrees -1, 0, and 1). A localizing subcategory of a triangulated category with small coproducts (that is, a cocomplete category) is by definition a full triangulated subcategory which is closed under small coproducts.

Note that an acyclic complex bounded below is always co-acyclic, hence we have a natural functor

$$D^+(\mathscr{C}) \to D^{\mathrm{co}}(\mathscr{C}).$$

We will be mostly interested in locally Noetherian Abelian categories ([12], [13]). Recall that an Abelian category $\mathscr{C}$ is locally Noetherian if it has exact directed colimits (AB5), and a (small) generating set of Noetherian objects. For such a category $\mathscr{C}$ we denote by $\mathscr{C}_f \subset \mathscr{C}$ the full (essentially small) subcategory of Noetherian objects. Also, we denote by $D^b_f(\mathscr{C}) \subset D^b(\mathscr{C})$ the subcategory of complexes with bounded Noetherian cohomology.

The basic example of a locally Noetherian category is the category $\mathscr{C} = \mathrm{QCoh}(X)$ of quasi-coherent sheaves on a Noetherian scheme X. In this case $\mathscr{C}_f = \mathrm{Coh}(X)$.

Theorem 5.2. *Let $\mathscr{C}$ be a locally Noetherian Abelian category. Then the following statements hold.*

1) *There is a semi-orthogonal decomposition*

$$K(\mathscr{C}) = \langle K(\mathrm{Inj}(\mathscr{C})), \text{co-Acycl}(\mathscr{C})\rangle,$$

where $K(\mathrm{Inj}(\mathscr{C}))$ is the homotopy category of complexes with injective components.

2) *The functor $D^+(\mathscr{C}) \to D^{\mathrm{co}}(\mathscr{C})$ is fully faithful.*

3) *The category $D^{\mathrm{co}}(\mathscr{C})$ is compactly generated, and the subcategory of compact objects coincides with the essential image of the composition*

$$D^b_f(\mathscr{C}) \to D^+(\mathscr{C}) \to D^{\mathrm{co}}(\mathscr{C}).$$

Proof. The statement 1) is proved in [30], § 3.7, in a different but analogous context. It also follows from [31], Corollary A.6.2, after reversion of arrows.

The statement 2) is proved in [31], Lemma A.1.2, (a).

The statement 3) follows from 1) and [20], Proposition 2.3. □

We will use the following terminology borrowed from [7], § 2.

Definition 5.3. 1) A $\mathbb{Z}_+$-category is a pair $(\mathscr{C}, W)$, where $\mathscr{C}$ is a category and $W\colon \mathrm{id}_{\mathscr{C}} \to \mathrm{id}_{\mathscr{C}}$ is a natural transformation.

2) A $\mathbb{Z}_+$-functor between $\mathbb{Z}_+$-categories $(\mathscr{C}_1, W_1)$ and $(\mathscr{C}_2, W_2)$ is a functor $F\colon \mathscr{C}_1 \to \mathscr{C}_2$ such that $(W_2)_{F(X)} = F((W_1)_X)$ for any object $X \in \mathscr{C}_1$.

We will say that a $\mathbb{Z}_+$-category $(\mathscr{C}, W)$ is additive (respectively, Abelian, exact, …) if the underlying category $\mathscr{C}$ is additive (respectively, Abelian, exact, …), and similarly for $\mathbb{Z}_+$-functors.

Definition 5.4. Let $(\mathscr{C}, W)$ be an additive $\mathbb{Z}_+$-category. A matrix factorization is a pair (F, δ), where $F = F^{\mathrm{ev}} \oplus F^{\mathrm{odd}}$ is a $\mathbb{Z}/2$-graded object in $\mathscr{C}$ and $\delta\colon F \to F$ is an odd morphism such that $\delta^2 = W_F$.

The matrix factorizations form a $\mathbb{Z}/2$-graded DG category which we denote by $\mathrm{MF}_{\mathrm{dg}}(\mathscr{C}, W)$. This DG category is strongly pre-triangulated. Let $K(\mathscr{C}, W)$ be its homotopy category.

Suppose that $\mathscr{C}$ is moreover Abelian. If $\mathscr{C}$ is small, then we define the triangulated subcategory $\mathrm{Acycl}(W) \subset K(\mathscr{C}, W)$ of *absolutely acyclic* matrix factorizations to be the subcategory generated by the totalizations of short exact sequences of matrix factorizations. The absolute derived category is defined as the quotient

$$D^{\mathrm{abs}}(\mathscr{C}, W) := K(\mathscr{C}, W)/\mathrm{Acycl}(W).$$

Remark 5.5. Note that by the definition of the category $D^{\mathrm{abs}}(\mathscr{C}, W)$, it has a natural DG enhancement given by the DG quotient $\mathrm{MF}_{\mathrm{dg}}(\mathscr{C}, W)/\mathrm{Acycl}_{\mathrm{dg}}(W)$, where $\mathrm{Acycl}_{\mathrm{dg}}(W)$ is the full DG subcategory of absolutely acyclic matrix factorizations.

If $\mathscr{C}$ has exact small coproducts, then we define the subcategory $\text{co-Acycl}(W) \subset K(\mathscr{C}, W)$ to be the localizing subcategory generated by totalizations of short exact sequences. In this case the coderived category is defined as the quotient

$$K(\mathscr{C}, W)/\text{co-Acycl}(W).$$

Theorem 5.6. *Let $(\mathscr{C}, W)$ be a locally Noetherian Abelian $\mathbb{Z}_+$-category. Then the following statements hold.*

1) *There is a semi-orthogonal decomposition*

$$K(\mathscr{C}, W) = \langle K(\operatorname{Inj}(\mathscr{C}), W), \text{co-Acycl}(W)\rangle,$$

where $K(\operatorname{Inj}(\mathscr{C}), W)$ is the homotopy category of matrix factorizations with injective components.

2) *The category $D^{\mathrm{co}}(\mathscr{C}, W)$ is compactly generated, and the subcategory of compact objects is the Karoubi completion of the essential image of the functor*

$$D^{\mathrm{abs}}(\mathscr{C}_f, W) \to D^{\mathrm{co}}(\mathscr{C}, W),$$

which is fully faithful.

Proof. The statement 1) is proved in [30], § 3.7, in a different but analogous context.

The statement 2) is proved in the same way as Proposition 1.5, (d) in [11]. □

In the special case when $\mathscr{C} = \operatorname{QCoh} X$ for some Noetherian separated scheme X and W is given by multiplication by a regular function on X, we write $D^{\mathrm{co}}(\operatorname{QCoh}(X, W))$ or just $D^{\mathrm{co}}(X, W)$ instead of $D^{\mathrm{co}}(\operatorname{QCoh} X, W)$. We also write $D^{\mathrm{abs}}(\operatorname{Coh}(X, W))$ or $D^{\mathrm{abs}}_{\mathrm{coh}}(X, W)$ instead of $D^{\mathrm{abs}}(\operatorname{Coh} X, W)$.

6. Coherent sheaves and coherent matrix factorizations

6.1. Coherent sheaves. Fix some base field k. Let X be a separated Noetherian scheme over k. Recall that the category $\operatorname{QCoh}(X)$ is locally Noetherian and $\operatorname{Coh}(X) \subset \operatorname{QCoh}(X)$ is exactly its subcategory of Noetherian objects. Hence, by 3) in Theorem 5.2 the triangulated category $D^b_{\mathrm{coh}}(X)$ is exactly the subcategory of compact objects in the coderived category $D^{\mathrm{co}}(X) := D^{\mathrm{co}}(\operatorname{QCoh}(X))$.

We will need the following enhancement of $D^{\mathrm{co}}(X)$. Denote by $\operatorname{Flasque}(X) \subset \operatorname{QCoh}(X)$ the subcategory of all flasque quasi-coherent sheaves. It is closed under small coproducts, extensions, and cokernels of injections. It contains the category of injective acyclic quasi-coherent sheaves: $\operatorname{Inj}(X) \subset \operatorname{Flasque}(X)$. It follows that one has the semi-orthogonal decomposition

$$K(\operatorname{Flasque}(X)) = \langle K(\operatorname{Inj}(X)), \text{co-Acycl}(\operatorname{Flasque}(X))\rangle.$$

Therefore, we have

$$D^{\mathrm{co}}(X) \cong K(\operatorname{Flasque}(X)) / \text{co-Acycl}(\operatorname{Flasque}(X)).$$

Let us denote by $\operatorname{Com}(\mathscr{A})$ the DG category of (unbounded) complexes of objects in $\mathscr{A}$, where $\mathscr{A}$ is any additive k-linear category. Then one has a natural DG enhancement of $D^{\mathrm{co}}(X)$:

$$D^{\mathrm{co}}(X) \cong \operatorname{Ho}\bigl(\operatorname{Com}(\operatorname{Flasque}(X)) / \operatorname{Com}^{\text{co-ac}}(\operatorname{Flasque}(X))\bigr).$$

The set-theoretic issues are resolved in the same way as in Appendix A of [23].

Further, for a morphism $f\colon X \to Y$ of Noetherian separated k-schemes one has a natural DG functor

$$f_*\colon \operatorname{Com}(\operatorname{Flasque}(X)) \to \operatorname{Com}(\operatorname{Flasque}(Y)),$$

which takes $\operatorname{Com}^{\text{co-ac}}(\operatorname{Flasque}(X))$ to $\operatorname{Com}^{\text{co-ac}}(\operatorname{Flasque}(Y))$. Hence we have a natural DG functor

$$\begin{aligned} f_*\colon \operatorname{Com}(\operatorname{Flasque}(X))/\operatorname{Com}^{\text{co-ac}}(\operatorname{Flasque}(X)) \\ \to \operatorname{Com}(\operatorname{Flasque}(Y))/\operatorname{Com}^{\text{co-ac}}(\operatorname{Flasque}(Y)). \end{aligned}$$

Now let $\mathfrak{D}^b(X) := \operatorname{Com}^b_{\text{coh}}(\operatorname{Flasque}(X)) \subset \operatorname{Com}(\operatorname{Flasque}(X))$ be a full DG subcategory consisting of complexes which are isomorphic to an object of $D^b(\operatorname{Coh}(X))$ in $D^{\text{co}}(X)$. As in Appendix A of [23], we may assume that these DG categories are small, and we have well-defined pushforward DG functors $f_*\colon \mathfrak{D}^b(X') \to \mathfrak{D}^b(X)$ for any proper morphism $f\colon X' \to X$. We clearly have

$$D^b_{\text{coh}}(X) \cong \operatorname{Ho}(\mathfrak{D}^b(X)).$$

Moreover, we have natural isomorphisms of DG functors $(fg)_* \cong f_*g_*$ for composable proper morphisms f and g.

We recall some results on the triangulated category $D^b_{\text{coh}}(X)$ and the DG category $\mathfrak{D}^b(X)$.

Definition 6.1 ([5] and [33]). Let $\mathscr{T}$ be a small triangulated category and $E \in \mathscr{T}$ an object. Take the recursively defined sequence $\{\mathscr{T}_n\}_{n\geqslant 0}$ of full subcategories of $\mathscr{T}$, where

- $\mathscr{T}_0$ consists of direct summands of finite direct sums of shifts of E;
- $\mathscr{T}_{n+1}$ consists of direct summands of objects F such that there exists an exact triangle

$$F' \to F \to F'' \to F'[1],$$

  with $F' \in \mathscr{T}_n$ and $F'' \in \mathscr{T}_0$.

Then E is called a strong generator if $\mathscr{T}_n = \mathscr{T}$ for $n \gg 0$.

Theorem 6.2 [33]. *If a scheme X is separated of finite type over a field* k, *then the triangulated category $D^b_{\text{coh}}(X)$ has a strong generator.*

In the case of perfect fields there is a stronger result by Lunts.

Theorem 6.3 [22]. *Let X be a separated scheme of finite type over a perfect field* k. *Then the DG category $\mathfrak{D}^b(X)$ is smooth.*

Our main result on derived categories of coherent sheaves (Theorem 1.3) is stronger than Theorem 6.3, but it requires the assumption that the base field has characteristic zero.

6.2. Coherent matrix factorizations. For matrix factorizations the general picture looks roughly similar. By Theorem 5.6 the ($\mathbb{Z}/2$-graded) category of coherent matrix factorizations $D^{\mathrm{abs}}_{\mathrm{coh}}(X, W)$ is (up to direct summands) the subcategory of compact objects in $D^{\mathrm{co}}(X, W)$.

For the derived functors between the (absolute derived and coderived) categories of matrix factorizations there are two approaches: the technique in Appendix A of [8] and the approach in [1]. They yield the same result (see [1], the proof of Proposition 2.22 and Remark 4.4). Below we freely use these derived functors just as for the usual derived categories of (quasi-)coherent sheaves.

As above, we can construct a system of enhancements $\mathfrak{D}^{\mathrm{abs}}(X, W)$ of the triangulated categories $D^{\mathrm{abs}}_{\mathrm{coh}}(X, W)$. Namely, one has a ($\mathbb{Z}/2$)-graded DG category $\mathrm{Flasque}(X, W) \subset \mathrm{QCoh}(X, W)$ of flasque matrix factorizations, and its full DG subcategory

$$\mathrm{Flasque}(X, W)^{\text{co-ac}} \subset \mathrm{Flasque}(X, W)$$

of co-acyclic flasque matrix factorizations. This gives us a natural enhancement of $D^{\mathrm{co}}(\mathrm{QCoh}(X, W))$:

$$D^{\mathrm{co}}(\mathrm{QCoh}(X, W)) \cong \mathrm{Ho}(\mathrm{Flasque}(X, W)/\, \mathrm{Flasque}(X, W)^{\text{co-ac}}).$$

Again, for any morphism $f\colon X \to Y$ and any function W on Y we have a natural DG functor

$$\begin{aligned} f_*\colon\ & \mathrm{Flasque}(X, f^*W)/\, \mathrm{Flasque}(X, f^*W)^{\text{co-ac}} \\ & \to \mathrm{Flasque}(Y, W)/\, \mathrm{Flasque}(Y, W)^{\text{co-ac}}. \end{aligned}$$

Further, we have a full DG subcategory $\mathrm{Flasque}(X, W)_{\mathrm{coh}} \subset \mathrm{Flasque}(X, W)$ consisting of matrix factorizations which are in the essential image of the inclusion $D^{\mathrm{abs}}_{\mathrm{coh}}(X, W) \subset D^{\mathrm{co}}(\mathrm{QCoh}(X, W))$. Putting

$$\mathfrak{D}^{\mathrm{abs}}(X, W) := \mathrm{Flasque}(X, W)_{\mathrm{coh}}/\, \mathrm{Flasque}(X, W)^{\text{co-ac}},$$

we have

$$D^{\mathrm{abs}}_{\mathrm{coh}}(X, W) \cong \mathrm{Ho}(\mathfrak{D}^{\mathrm{abs}}(X, W)).$$

By Proposition A.4 in [8], for any proper morphism $f\colon X \to Y$ and any function W on Y the functor $f_*\colon D^{\mathrm{co}}(X, f^*W) \to D^{\mathrm{co}}(Y, W)$ takes $D^{\mathrm{abs}}_{\mathrm{coh}}(X, f^*W))$ to $D^{\mathrm{abs}}_{\mathrm{coh}}(Y, W)$.

As in the previous subsection, we may (and will) assume that all the DG categories $\mathfrak{D}^{\mathrm{abs}}(X, W)$ are small, and for any proper morphism $f\colon X \to Y$ and any function W on Y one has the DG functor $f_*\colon \mathfrak{D}^{\mathrm{abs}}(X, f^*W) \to \mathfrak{D}^{\mathrm{abs}}(Y, W)$. Again, we have natural isomorphisms of DG functors $(fg)_* \cong f_* g_*$.

For completeness, we formulate the analogues of Theorems 6.2 and 6.3 which were proved in [8].

Theorem 6.4 ([8], Theorem 7.5). *Let X be a separated scheme of finite type over a field k, and let $W \in \mathscr{O}(X)$ be a regular function. Then the triangulated category $D^{\mathrm{abs}}_{\mathrm{coh}}(X, W)$ has a strong generator.*

Theorem 6.5 ([8], Theorem 7.7). *Let X be a separated scheme of finite type over a perfect field* k, *and let $W\colon X \to \mathbb{A}^1$ be a regular function. Denote by t the coordinate on $\mathbb{A}^1$. Assume that the scheme $(X \times_{\mathrm{k}[t]} \mathrm{k}(t))_{\mathrm{red}}$ has a smooth stratification over* $\mathrm{k}(t)$ (*this holds automatically if* k *has characteristic zero*).

Then the DG category $\mathfrak{D}^{\mathrm{abs}}(X, W)$ is smooth as a $\mathbb{Z}/2$-graded DG category.

Our main result on the absolute derived categories of coherent matrix factorizations (Theorem 1.4) is stronger than Theorem 6.5 but again requires the assumption of zero characteristic.

6.3. Nice ringed spaces. In this subsection we define the category of nice ringed spaces, and we discuss coherent sheaves and matrix factorizations on them.

Definition 6.6. 1) A nice ringed space over k is a pair $(S, \mathscr{A}_S)$ consisting of a Noetherian separated k-scheme S and a coherent sheaf $\mathscr{A}_S$ of (unital) $\mathscr{O}_S$-algebras with the following property:

- there exists a coherent sheaf of nilpotent two-sided ideals $I \subset \mathscr{A}_S$ such that there is an isomorphism of $\mathscr{O}_S$-algebras

$$\mathscr{A}_S/I \cong \bigoplus_{i=1}^{n} \mathscr{O}_S/J_i,$$

where $J_i \subset \mathscr{O}_S$ are some coherent sheaves of ideals.

2) A morphism of nice ringed spaces from $(S, \mathscr{A}_S)$ to $(S', \mathscr{A}_{S'})$ is a pair (f, φ), where $f\colon S \to S'$ is a morphism of schemes and $\varphi\colon f^*\mathscr{A}_Y \to \mathscr{A}_S$ is a (possibly non-unital) morphism of $\mathscr{O}_S$-algebras.

The composition of morphisms is defined in the natural way. Each Noetherian separated k-scheme S can be considered as a nice ringed space $(S, \mathscr{O}_S)$.

For a nice ringed space $(S, \mathscr{A}_S)$ denote by $\mathrm{QCoh}(\mathscr{A}_S)$ (respectively, $\mathrm{Coh}(\mathscr{A}_S)$) the Abelian category of right $\mathscr{A}_S$-modules which are $\mathscr{O}_S$-quasi-coherent (respectively, $\mathscr{O}_S$-coherent). We denote by $\mathrm{Mod}\text{-}\mathscr{A}_S$ the category of all sheaves of right $\mathscr{A}_S$-modules.

Proposition 6.7 ([8], Proposition 7.9). *Let $(S, \mathscr{A}_S)$ be a nice ringed space.*

1) *The Abelian category $\mathrm{QCoh}(\mathscr{A}_S)$ is locally Noetherian, and the Noetherian objects in it form the subcategory $\mathrm{Coh}(\mathscr{A}_S)$. In particular, it has enough injective objects.*

2) *The Abelian category $\mathrm{Mod}\text{-}\mathscr{A}_S$ is locally Noetherian.*

3) *An object $\mathscr{I} \in \mathrm{QCoh}(\mathscr{A}_S)$ is injective in the category $\mathrm{QCoh}(\mathscr{A}_S)$ if and only if it is injective in the category $\mathrm{Mod}\text{-}\mathscr{A}_S$. In particular, in this case the restriction $\mathscr{I}_{|U}$ to any open subscheme $U \subset S$ is injective in $\mathrm{QCoh}((\mathscr{A}_S)_{|U})$.*

For a morphism $(f, \phi)\colon (X, \mathscr{A}_X) \to (Y, \mathscr{A}_Y)$ of nice ringed spaces we have a direct image functor $(f, \phi)_*\colon \mathrm{QCoh}(\mathscr{A}_X) \to \mathrm{QCoh}(\mathscr{A}_Y)$, which is left exact and commutes with small coproducts. More precisely, for $\mathscr{F} \in \mathrm{QCoh}(\mathscr{A}_X)$ we first take the $f_*(\mathscr{A}_X)$-module $f_*(\mathscr{F})$ and then put

$$(f, \phi)_*(\mathscr{F}) := f_*(\mathscr{F}) \cdot \psi(1),$$

where $\psi\colon \mathscr{A}_Y \to f_*(\mathscr{A}_X)$ corresponds to ϕ by adjunction.

Proposition 6.8. *Let $(f,\phi)\colon (X,\mathscr{A}_X)\to (Y,\mathscr{A}_Y)$ be a morphism of nice ringed spaces. Suppose that the morphism $f\colon X\to Y$ is proper. Then the functor $(f,\phi)_*\colon D^{\mathrm{co}}(\mathrm{QCoh}(\mathscr{A}_X))\to D^{\mathrm{co}}(\mathrm{QCoh}(\mathscr{A}_Y))$ takes $D^b_{\mathrm{coh}}(\mathscr{A}_X)$ to $D^b_{\mathrm{coh}}(\mathscr{A}_Y)$.*

Proof. By 3) in Proposition 6.7 each injective quasi-coherent $\mathscr{A}_X$-module is a flasque sheaf. It follows that for any $\mathscr{F}\in D^b_{\mathrm{coh}}(\mathscr{A}_X)$ we have $\mathbf{R}(f,\phi)_*(\mathscr{F})=\mathbf{R}f_*(\mathscr{F})\cdot\psi(1)$, where $\psi\colon \mathscr{A}_Y\to f_*\mathscr{A}_X$ is as above. But the restriction of scalars of $\mathbf{R}f_*(\mathscr{F})$ (from $f_*\mathscr{A}_X$ to $\mathscr{O}_Y$) is in $D^b_{\mathrm{coh}}(Y)$ (since f is proper). Hence we have $\mathbf{R}(f,\phi)_*(\mathscr{F})\in D^b_{\mathrm{coh}}(\mathscr{A}_Y)$. $\square$

As in § 6.1, we define a small DG category $\mathfrak{D}^b(\mathscr{A}_X)$, which is an enhancement of $D^b_{\mathrm{coh}}(\mathscr{A}_X)$ for any nice ringed space $(X,\mathscr{A}_X)$. Moreover, for any proper morphism $f\colon (X,\mathscr{A}_X)\to (Y,\mathscr{A}_Y)$ we have a DG functor $f_*\colon \mathfrak{D}^b(\mathscr{A}_X)\to\mathfrak{D}^b(\mathscr{A}_Y)$, and $(fg)_*\cong f_*g_*$.

Now let $(S,\mathscr{A}_S)$ be a ringed space and let $W\in\mathscr{O}(S)$ be a regular function. We regard W also as a central section of $\mathscr{A}_S$ (that is, for any open subset $U\subset S$ the element $W_{|U}\in\mathscr{A}_S(U)$ lies in the centre of the algebra $\mathscr{A}_S(U)$). Then by Theorem 5.6 the category $D^{\mathrm{abs}}_{\mathrm{coh}}(\mathscr{A}_X,W)=D^{\mathrm{abs}}(\mathrm{Coh}(\mathscr{A}_X,W))$ is (up to direct summands) the subcategory of compact objects in $D^{\mathrm{co}}(\mathrm{QCoh}(\mathscr{A}_X,W))$.

Proposition 6.9. *Let $f\colon (X,\mathscr{A}_X)\to (Y,\mathscr{A}_Y)$ be a morphism of nice ringed spaces and let $W\in\mathscr{O}(Y)$ be a regular function. Suppose that the morphism $f\colon X\to Y$ is proper. Then the functor*

$$f_*\colon D^{\mathrm{co}}(\mathrm{QCoh}(\mathscr{A}_X,f^*W))\to D^{\mathrm{co}}(\mathrm{QCoh}(\mathscr{A}_Y,W))$$

takes $D^{\mathrm{abs}}_{\mathrm{coh}}(\mathscr{A}_X,W)$ to $D^{\mathrm{abs}}_{\mathrm{coh}}(\mathscr{A}_Y,W)$.

Proof. This follows from Proposition A.4 in [8] and Propositions 6.7 and 6.8. $\square$

As in § 6.2, we define a $\mathbb{Z}/2$-graded small DG category $\mathfrak{D}^{\mathrm{abs}}_{\mathrm{coh}}(\mathscr{A}_X,W)$ for each nice ringed space $(X,\mathscr{A}_X)$ and each $W\in\mathscr{O}(X)$. For any proper morphism $f\colon (X,\mathscr{A}_X)\to (Y,\mathscr{A}_Y)$ and any $W\in\mathscr{O}(Y)$ we have a DG functor $\mathfrak{D}^{\mathrm{abs}}_{\mathrm{coh}}(\mathscr{A}_X,f^*W)\to\mathfrak{D}^{\mathrm{abs}}_{\mathrm{coh}}(\mathscr{A}_Y,W)$, and $(fg)_*\cong f_*g_*$.

7. Smooth categorical compactifications of geometric categories

In this section we construct smooth categorical compactifications (see Definition 1.5) for derived categories of coherent sheaves and absolute derived categories of coherent matrix factorizations.

7.1. Auslander-type construction: coherent sheaves. The construction we present in this subsection is due to Kuznetsov and Lunts [21].

Let S be a Noetherian scheme, let $\tau\subset\mathscr{O}_S$ be a sheaf of ideals, and let $n>0$ be an integer such that $\tau^n=0$. Starting with such a triple (S,τ,n), we define a coherent sheaf of $\mathscr{O}_S$-algebras:

$$\mathscr{A}=\mathscr{A}_S=\mathscr{A}_{S,\tau,n}:=\bigoplus_{1\leqslant i,j\leqslant n}\mathscr{A}_{ij},\qquad \mathscr{A}_{ij}:=\tau^{\max(i-j,0)}/\tau^{n+1-j}.$$

The multiplication

$$\mathscr{A}_{jk} \otimes_{\mathscr{O}_S} \mathscr{A}_{ij} \to \mathscr{A}_{ik}$$

is induced by the multiplication in $\mathscr{O}_S$. We denote by

$$e_i = 1 \in \mathscr{A}_{ii} = \mathscr{O}_S/\tau^{n+1-i}$$

the orthogonal idempotents such that $e_1 + \cdots + e_n = 1_{\mathscr{A}}$.

Let $S_0 \subset S$ be the subscheme defined by the ideal τ, so that $S_{\mathrm{red}} \subseteq S_0 \subseteq S$.

Proposition 7.1. *The pair $(S, \mathscr{A}_{S,\tau,n})$ is a nice ringed space.*

Below we will say that $(S, \mathscr{A}_S)$ is obtained from the triple (S, τ_S, n) by the Auslander-type construction.

We have a morphism of nice ringed spaces

$$\rho_S \colon (S, \mathscr{A}_S) \to S,$$

which is the identity on S, and the map $\mathscr{O}_S \to \mathscr{A}_S$ is $f \mapsto f \cdot e_1$.

Also, for each $1 \leqslant k \leqslant n$ we denote by $i_k \colon S_0 \to (S, \mathscr{A}_S)$ the natural morphism which is given by the inclusion $S_0 \hookrightarrow S$ and the projection

$$\mathscr{A}_S \to \mathscr{A}_S/\langle e_1, \dots, e_{k-1}, e_{k+1}, \dots, e_n\rangle \cong \mathscr{O}_{S_0}.$$

Proposition 7.2 ([8], Proposition 8.2). *The exact functor $\rho_{S*} \colon \mathrm{Coh}(\mathscr{A}_S) \to \mathrm{Coh}(S)$ is a localization of Abelian categories, and its kernel is generated (as a Serre subcategory) by the subcategories $i_{k*}(\mathrm{Coh}\, S_0)$, $2 \leqslant k \leqslant n$.*

In particular, the DG functor $\rho_{S*} \colon \mathfrak{D}^b(\mathscr{A}_S) \to \mathfrak{D}^b(S)$ is a localization, and its kernel is generated by $i_{k*}(\mathfrak{D}^b(S_0))$, $2 \leqslant k \leqslant n$.

Suppose that $n > 1$. Denote by $S' \subset S$ the subscheme defined by the ideal τ^{n-1}. Applying the Auslander-type construction to the triple $(S', \tau, n-1)$, we get a nice ringed space $(S', \mathscr{A}_{S'})$. Note that $\mathscr{A}_{S'}$ is identified with the (non-unital) subalgebra $(1-e_1)\mathscr{A}_S(1-e_1) \subset \mathscr{A}_S$. Hence we have a natural morphism

$$e \colon (S, \mathscr{A}_S) \to (S, \mathscr{A}_{S'}).$$

Proposition 7.3. *The functors*

$$i_{1*} \colon \mathfrak{D}^b(S_0) \to \mathfrak{D}^b(\mathscr{A}_S) \quad \text{and} \quad e^* \colon \mathfrak{D}^b(\mathscr{A}_{S'}) \to \mathfrak{D}^b(\mathscr{A}_S)$$

are quasi-fully-faithful, and they give a semi-orthogonal decomposition

$$\mathfrak{D}^b(\mathscr{A}_S) := \langle i_{1*}(\mathfrak{D}^b(S_0)), e^*(\mathfrak{D}^b(\mathscr{A}_{S'}))\rangle.$$

Induction yields a semi-orthogonal decomposition

$$\mathfrak{D}^b(\mathscr{A}_S) = \langle \mathfrak{D}^b(S_0), \dots, \mathfrak{D}^b(S_0)\rangle,$$

where the number of copies is n.

Proof. This is proved in [21], Proposition 5.14 and Corollary 5.15. □

Proposition 7.4. *Suppose that* S_0 *is smooth and proper over* k*. Then the DG category* $\mathfrak{D}^b(\mathscr{A}_S)$ *is smooth and proper.*

Proof. This is proved in [21], Theorem 5.20 and Proposition 5.17. □

By definition, a morphism of triples $f\colon (T,\tau_T,n)\to(S,\tau_S,n)$ is a morphism $f\colon T\to S$ such that $f^{-1}(\tau_S)\subset\tau_T$. It induces a natural morphism of nice ringed spaces $\widetilde{f}\colon (T,\mathscr{A}_T)\to(S,\mathscr{A}_S)$.

Proposition 7.5 ([8], Proposition 8.5). *Let* $f\colon (T,\tau_T,n)\to(S,\tau_S,n)$ *be a morphism of triples. Then*:

1) $\rho_S\widetilde{f}=f\rho_T$;

2) $\widetilde{f}i_k=i_kf_0$ *for* $1\leqslant k\leqslant n$;

3) *the functor* $\widetilde{f}_*\colon\mathfrak{D}^b(\mathscr{A}_T)\to\mathfrak{D}^b(\mathscr{A}_S)$ *is compatible with the semi-orthogonal decompositions*

$$\mathfrak{D}^b(\mathscr{A}_T)=\langle\mathfrak{D}^b(T_0),\dots,\mathfrak{D}^b(T_0)\rangle\quad\textit{and}\quad\mathfrak{D}^b(\mathscr{A}_S)=\langle\mathfrak{D}^b(S_0),\dots,\mathfrak{D}^b(S_0)\rangle.$$

Moreover, all the induced functors on the semi-orthogonal components are isomorphic to f_{0*}.

7.2. Auslander-type construction: coherent matrix factorizations. Now let (S,τ,n) be a triple as above, let $(S,\mathscr{A}_S)$ be the corresponding nice ringed space, and let $W\in\mathscr{O}(S)$ be a regular function on S. Denote by W_0 (respectively, W') the pullback of W on S_0 (respectively, S').

Proposition 7.6 ([8], Proposition 8.6). *The DG functor*

$$\rho_{S*}\colon\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)\to\mathfrak{D}^{\mathrm{abs}}(S,W)$$

is a localization, and its kernel is generated by $i_{k*}(\mathfrak{D}^{\mathrm{abs}}(S_0,W_{S_0}))$, $2\leqslant k\leqslant n$.

Proof. This follows from Proposition 7.2 and [8], Proposition A.6. □

Proposition 7.7 ([8], Proposition 8.7). *The functors*

$$i_{1*}\colon\mathfrak{D}^{\mathrm{abs}}(S_0,W_0)\to\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)\quad\textit{and}\quad e^*\colon\mathfrak{D}^{\mathrm{abs}}(S_0,W_0)\to\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)$$

are quasi-fully-faithful, and give a semi-orthogonal decomposition

$$\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W):=\langle i_{1*}(\mathfrak{D}^{\mathrm{abs}}(S_0,W_0)),e^*(\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_{S'},W'))\rangle.$$

Induction yields a semi-orthogonal decomposition

$$\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)=\langle\mathfrak{D}^{\mathrm{abs}}(S_0,W_0),\dots,\mathfrak{D}^{\mathrm{abs}}(S_0,W_0)\rangle,$$

where the number of copies is n.

Proposition 7.8 ([8], Proposition 8.8). *Suppose that* S_0 *is smooth and the morphism* $W_0\colon S_0\to\mathbb{A}^1$ *is proper. Then the* $\mathbb{Z}/2$*-graded DG category* $\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)$ *is smooth and proper.*

Proposition 7.9. *Let* $f\colon (T,\tau_T,n)\to(S,\tau_S,n)$ *be a morphism of triples and let* $\widetilde{f}\colon (T,\mathscr{A}_T)\to(S,\mathscr{A}_S)$ *be the corresponding morphism of nice ringed spaces. Let* W *be a regular function on* S*.*

Then the functor $\widetilde{f}_*\colon \mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_T,f^*W)\to\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)$ *is compatible with the semi-orthogonal decompositions*

$$\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_T,f^*W)=\langle\mathfrak{D}^{\mathrm{abs}}(T_0,f_0^*W_0),\dots,\mathfrak{D}^{\mathrm{abs}}(T_0,f_0^*W_0)\rangle$$

and

$$\mathfrak{D}^{\mathrm{abs}}(\mathscr{A}_S,W)=\langle\mathfrak{D}^{\mathrm{abs}}(S_0,W_0),\dots,\mathfrak{D}^{\mathrm{abs}}(S_0,W_0)\rangle.$$

Moreover, all the induced functors on the semi-orthogonal components are isomorphic to f_{0*}*.*

Proof. This is completely analogous to the proof of 3) in Proposition 7.5. □

7.3. Categorical blow-ups: coherent sheaves. Let $f\colon X\to Y$ be a proper morphism of Noetherian separated schemes. The following definition is taken from [21].

Definition 7.10 ([21], Definition 6.1). Let $f\colon X\to Y$ be a proper morphism. A closed subscheme $S\subset Y$ is called a non-rational locus of Y with respect to f if the natural morphism

$$I_S\to\mathbf{R}f_*I_{f^{-1}(S)}$$

is an isomorphism in $D^b_{\mathrm{coh}}(Y)$.

Proposition 7.11 ([21], Lemma 6.3). *Let* $f\colon X\to Y$ *be the blow-up of a sheaf of ideals* I *on* Y*. Then for* m *sufficiently large the closed subscheme of* Y *defined by the ideal* I^m *is a non-rational locus of* Y *with respect to* f*.*

Let S be a non-rational locus of Y with respect to a *proper* morphism $f\colon X\to Y$. Let $T:=f^{-1}(S)\subset X$, and denote by $i\colon S\to Y$ and $j\colon T\to X$ the closed embeddings and by $p\colon T\to S$ the morphism induced by f. We have a commutative diagram

$$\begin{array}{ccc}\mathfrak{D}^b(X) & \xleftarrow{\ j_*\ } & \mathfrak{D}^b(T)\\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle p_*}\\ \mathfrak{D}^b(Y) & \xleftarrow{\ i_*\ } & \mathfrak{D}^b(S)\end{array}\tag{7.1}$$

of DG functors.

We put

$$\mathscr{D}_{\mathrm{coh}}(X,S):=\mathfrak{D}^b(X)\oplus_{(\mathfrak{D}^b(T))}\mathfrak{D}^b(S)$$

(as in Definition 4.5). By 1) in Lemma 5.6 of [8] this commutative diagram induces a natural DG functor

$$\pi_*\colon\mathscr{D}_{\mathrm{coh}}(X,S)\to\mathfrak{D}^b(Y).$$

Similarly, we have the DG category

$$\mathscr{D}_{\mathrm{coh},T}(X,S):=\mathfrak{D}^b_T(X)\oplus_{(\mathfrak{D}^b(T))}\mathfrak{D}^b(S)$$

and a DG functor

$$\pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S)\to \mathfrak{D}^b_S(Y),$$

which we denote by the same symbol. We have a commutative diagram

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{coh},T}(X,S) & \longrightarrow & \mathscr{D}_{\mathrm{coh}}(X,S) \\ {\scriptstyle\pi_*}\downarrow & & {\scriptstyle\pi_*}\downarrow \\ \mathfrak{D}^b_S(Y) & \longrightarrow & \mathfrak{D}^b(Y) \end{array} \tag{7.2}$$

of DG functors.

Clearly, the horizontal arrows in (7.2) are quasi-fully-faithful DG functors.

Proposition 7.12 ([8], Proposition 8.12). *The DG functors*

$$\pi_*\colon \mathscr{D}_{\mathrm{coh}}(X,S)\to \mathfrak{D}^b(Y) \quad \textit{and} \quad \pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S)\to \mathfrak{D}^b_S(Y)$$

are homological epimorphisms.

By 1) in Lemma 5.7 of [8] we have a DG quasi-functor

$$\Phi\colon \mathfrak{D}^b(T)\to \mathscr{D}_{\mathrm{coh},T}(X,S).$$

Moreover, the composition $\pi_*\Phi\colon \mathfrak{D}^b(T)\to \mathfrak{D}^b_S(Y)$ is zero in $\mathrm{Ho}(\mathrm{dgcat}_{\mathrm{k}})$.

Our main result in this subsection reduces to the following statement.

Theorem 7.13 ([8], Theorem 8.13). *In the above notation, suppose that all the infinitesimal neighbourhoods $S_n\subset Y$, $n\geqslant 1$, are non-rational loci of Y with respect to f. Then the DG functor*

$$\pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S)\to \mathfrak{D}^b(S)$$

is a localization, and its kernel is generated by the image of the DG quasi-functor $\Phi\colon \mathfrak{D}^b(T)\to \mathscr{D}_{\mathrm{coh},T}(X,S)$.

2) *Suppose that, moreover, the morphism f is an isomorphism outside S. Then the DG functor*

$$\pi_*\colon \mathscr{D}_{\mathrm{coh}}(X,S)\to \mathfrak{D}^b(Y)$$

is a localization, and again the kernel of π_* is generated by the image of the composition $\Phi\colon \mathfrak{D}^b(T)\to \mathscr{D}_{\mathrm{coh},T}(X,S)\hookrightarrow \mathscr{D}_{\mathrm{coh}}(X,S)$.

The proof of Theorem 7.13 consists of several steps.

Lemma 7.14. *Part 2) of Theorem 7.13 follows from part 1).*

Proof. Note that we have quasi-equivalences

$$\mathscr{D}_{\mathrm{coh}}(X,S)/\mathscr{D}_{\mathrm{coh},T}(X,S)\xrightarrow{\sim}\mathfrak{D}^b(X\setminus T) \quad \text{and} \quad \mathfrak{D}^b(Y)/\mathfrak{D}^b(S)\xrightarrow{\sim}\mathfrak{D}^b(Y\setminus S).$$

By the assumption of part 2) of Theorem 7.13, the pushforward $\mathfrak{D}^b(X\setminus T)\to \mathfrak{D}^b(Y\setminus S)$ is a quasi-equivalence. The assertion is obtained from a direct application of Lemma 3.8 to the commutative square (7.2). $\square$

Lemma 7.15 ([8], Lemma 8.15). *Let Q be a Noetherian separated scheme and let $Z \subset Q$ be a closed subscheme. Then the natural DG functor*

$$\operatorname{colim}_n \mathfrak{D}^b(Z_n) \to \mathfrak{D}^b_Z(Q)$$

is a quasi-equivalence.

Denote by $i_{m,n}\colon S_m \to S_n$, $i_n\colon S_n \to Y$, $j_{m,n}\colon T_m \to T_n$, and $j_n\colon T_n \to X$ the natural inclusions. Also, denote by $p_n\colon T_n \to S_n$ the natural projections. For any $0 < m < n$ we have a commutative diagram

$$\begin{array}{ccccc} T_m & \xrightarrow{j_{m,n}} & T_n & \xrightarrow{j_n} & X \\ {\scriptstyle p_m}\downarrow & & {\scriptstyle p_n}\downarrow & & {\scriptstyle f}\downarrow \\ S_m & \xrightarrow{i_{m,n}} & S_n & \xrightarrow{i_n} & Y \end{array}$$

Let

$$\mathscr{D}_n := \mathscr{D}_{\mathrm{coh}}(T_n, S).$$

We have natural DG functors $J_{m,n}\colon \mathscr{D}_m \to \mathscr{D}_n$ and $J_n\colon \mathscr{D}_n \to \mathscr{D}$. Also, we have the functors $P_n\colon \mathscr{D}_n \to \mathfrak{D}^b(S_n)$ defined in the same way as the functors π_* above. Moreover, all these DG functors fit into the commutative diagrams

$$\begin{array}{ccccc} \mathscr{D}_m & \xrightarrow{J_{m,n}} & \mathscr{D}_n & \xrightarrow{J_n} & \mathscr{D}_{\mathrm{coh},T}(X,S) \\ {\scriptstyle P_m}\downarrow & & {\scriptstyle P_n}\downarrow & & {\scriptstyle \pi_*}\downarrow \\ \mathfrak{D}^b(S_m) & \xrightarrow{i_{m,n*}} & \mathfrak{D}^b(S_n) & \xrightarrow{i_{n*}} & \mathfrak{D}^b_S(Y) \end{array}$$

Corollary 7.16 ([8], Corollary 8.16). *The natural DG functor*

$$\operatorname{colim}_n \mathscr{D}_n \to \mathscr{D}_{\mathrm{coh},T}(X,S)$$

is a quasi-equivalence.

Since by our assumption, S is a non-rational locus of S_n with respect to $p_n\colon T_n \to S_n$, we have by Proposition 7.12 that the DG functor $P_n\colon \mathscr{D}_n \to \mathfrak{D}^b(S_n)$ is a homological epimorphism.

As above, we have the DG quasi-functors $\Phi_n\colon \mathfrak{D}^b(T) \to \mathscr{D}_n$, $n \geqslant 1$. Furthermore,

$$J_{m,n}\Phi_m = \Phi_n \quad \text{and} \quad J_n\Phi_n = \Phi \quad \text{in } \operatorname{Ho}(\mathrm{dgcat}_{\mathrm{k}}).$$

Lemma 7.17. *Suppose that all the DG functors*

$$P_n\colon \mathscr{D}_n \to \mathfrak{D}^b(S_n)$$

are localizations, and the kernel of P_n is generated by $\Phi_n(\mathfrak{D}^b(T))$. Then the functor

$$\pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S) \to \mathfrak{D}^b_{\mathrm{coh},S}(Y)$$

is also a localization and its kernel is generated by $\Phi(\mathfrak{D}^b(T))$.

Proof. Indeed, by assumption the DG functor

$$P_n\colon \mathscr{D}_n/\Phi_n(\mathfrak{D}^b(T)) \to \mathfrak{D}^b(S_n)$$

is a quasi-equivalence. Hence the DG functor

$$\begin{aligned}\mathscr{D}_{\mathrm{coh},T}(X,S)/\Phi(\mathfrak{D}^b(T)) &= (\mathrm{colim}_n\,\mathscr{D}_n)/\Phi(\mathfrak{D}^b(T))\\ &= \mathrm{colim}_n(\mathscr{D}_n/\Phi_n(\mathfrak{D}^b(T))) \to \mathrm{colim}_n\,\mathfrak{D}^b(S_n) \cong \mathfrak{D}^b_S(Y)\end{aligned}$$

is a quasi-equivalence (because the DG quotient commutes with colimits, which is seen from the explicit construction in [6], § 3.1). □

Hence, to finish the proof of the theorem we need to show that the DG functors P_n are localizations with prescribed kernels. Let us start with the functor P_1.

Lemma 7.18. *The DG quasi-functor $\Phi_1\colon \mathfrak{D}^b(T) \to \mathscr{D}_1$ is quasi-fully-faithful, there is a semi-orthogonal decomposition*

$$[\mathscr{D}_1] = \langle D^b_{\mathrm{coh}}(S), \Phi_1(D^b_{\mathrm{coh}}(T))\rangle,$$

and the functor $[P_1]$ is the left semi-orthogonal projection onto $D^b_{\mathrm{coh}}(S)$. In particular, the DG functor P_1 is a localization, and its kernel is generated by $\Phi_1(\mathfrak{D}^b(T))$.

Proof. This is a direct application of Lemma 5.10 in [8], with $\mathscr{A} = \mathfrak{D}^b(T)$, $\mathscr{B} = \mathfrak{D}^b(S)$, and $F = \pi_*$. □

The following lemma is the key technical point in the proof of Theorem 7.13. Its proof is quite involved technically and uses the Auslander-type construction, although this construction is not mentioned in the formulation.

Lemma 7.19 ([8], Lemma 8.19). *Let $g\colon U \to V$ be a proper morphism of Noetherian separated schemes, and let $Z \subset V$ be a non-rational locus of V with respect to g. Suppose also that U' (respectively, V') is a square-zero thickening of U (respectively, V), and that the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\iota_U} & U' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ V & \xrightarrow{\iota_V} & V' \end{array}$$

commutes. Assume that Z is also a non-rational locus of V' with respect to g'. Then there is a commutative square

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{coh}}(U,Z) & \xrightarrow{J_U} & \mathscr{D}_{\mathrm{coh}}(U',Z) \\ {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle G'} \\ \mathfrak{D}^b(V) & \xrightarrow{\iota_{V*}} & \mathfrak{D}^b(V') \end{array}$$

of DG functors. If the DG functor G is a localization, then the DG functor G' is also a localization, and its kernel is generated by $J_U(\ker G)$.

Proof of Theorem 7.13. By Lemmas 7.18 and 7.19 we get by induction that each DG functor $P_n\colon \mathscr{D}_{\mathrm{coh}}(T_n,S) \to \mathfrak{D}^b(S_n)$ is a localization, and $\ker P_n$ is generated by $J_{1,n}\Phi_1(\mathfrak{D}^b(T)) = \Phi_n(\mathfrak{D}^b(T))$. Therefore, by Lemma 7.17 the DG functor $\pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S) \to \mathfrak{D}^b_S(Y)$ is also a localization, and $\Phi(\mathfrak{D}^b(T))$ generates its kernel. This proves part 1) of the theorem.

By Lemma 7.14 part 1) implies part 2). $\square$

We also formulate here a result analogous to Theorem 7.13 but technically simpler to prove.

Theorem 7.20. 1) *Let $f\colon X \to Y$ be a proper morphism such that $\mathbf{R}f_*\mathscr{O}_X \cong \mathscr{O}_Y$. Assume that there is a subscheme $S \subset Y$ such that all its infinitesimal neighbourhoods S_n, $n \geqslant 1$, are non-rational loci of Y with respect to f. Again, there is a Cartesian square*

$$\begin{array}{ccc} X & \xleftarrow{\ j\ } & T \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xleftarrow{\ i\ } & S \end{array}$$

Assume that the functor $\mathbf{R}p_*\colon D^b_{\mathrm{coh}}(T) \to D^b_{\mathrm{coh}}(S)$ is a localization. Then the functor $\mathbf{R}f_*\colon D^b_{\mathrm{coh},T}(X) \to D^b_{\mathrm{coh},S}(Y)$ is also a localization, and $\ker(\mathbf{R}f_*)$ *is generated by $j_*(\ker(\mathbf{R}p_*))$.*

2) *Suppose that, moreover, the morphism f is an isomorphism outside S. Then the functor $\mathbf{R}f_*\colon D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$ is a localization, and* $\ker(\mathbf{R}f_*)$ *is generated by $j_*(\ker(\mathbf{R}p_*))$.*

Proof. By analogy with Lemma 7.14, we reduce part 2) to part 1).

The proof of part 1) follows essentially the same steps as for part 1) in Theorem 7.13, but it simplifies considerably because we do not need to consider gluings. Thus, instead of $\mathscr{D}_{\mathrm{coh}}(X,S)$ we have $\mathfrak{D}^b(X)$, and instead of $\mathscr{D}_{\mathrm{coh}}(T_n,S)$ we have $\mathfrak{D}^b(T_n)$. Lemma 7.18 is not needed in this context because we have already assumed that $\mathbf{R}p_*$ is a localization. In the key technical step, Lemma 7.19, we assume that the morphisms g and g' satisfy $\mathbf{R}g_*\mathscr{O}_U = \mathscr{O}_V$ and $\mathbf{R}g'_*\mathscr{O}_{U'} = \mathscr{O}_{V'}$, and we take $\mathfrak{D}^b(U)$ (respectively, $\mathfrak{D}^b(U')$) instead of $\mathscr{D}_{\mathrm{coh}}(U,Z)$ (respectively, $\mathscr{D}_{\mathrm{coh}}(U',Z)$). All the other arguments are the same. $\square$

7.4. Categorical blow-ups: matrix factorizations. As above, suppose that S is a non-rational locus of Y with respect to a proper morphism $f\colon X \to Y$. Again we have $T := f^{-1}(S) \subset X$, and we denote by $i\colon S \to Y$ and $j\colon T \to X$ the closed embeddings and by $p\colon T \to S$ the morphism induced by f.

We fix a regular function $W \in \mathscr{O}(Y)$, and we denote by $W_X \in \mathscr{O}(X)$, $W_S \in \mathscr{O}(S)$, $W_T \in \mathscr{O}(T)$, and so on, the pullbacks of W under the natural morphisms.

We have a commutative diagram

$$\begin{array}{ccc} \mathfrak{D}^{\mathrm{abs}}(X,W_X) & \xleftarrow{\ j_*\ } & \mathfrak{D}^{\mathrm{abs}}(T,W_T) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle p_*} \\ \mathfrak{D}^{\mathrm{abs}}(Y,W) & \xleftarrow{\ i_*\ } & \mathfrak{D}^{\mathrm{abs}}(S,W_S) \end{array}$$

of DG functors. Let

$$\mathscr{D}_{\mathrm{coh}}(X,S,W) := \mathfrak{D}^{\mathrm{abs}}(X,W_X) \oplus_{(\mathfrak{D}^{\mathrm{abs}}(T,W_T))} \mathfrak{D}^{\mathrm{abs}}(S,W_S).$$

Again, we have a natural DG functor

$$\pi_*\colon \mathscr{D}_{\mathrm{coh}}(X,S,W) \to \mathfrak{D}^{\mathrm{abs}}(Y,W).$$

Similarly, we have the DG category

$$\mathscr{D}_{\mathrm{coh},T}(X,S,W) := \mathfrak{D}_T^{\mathrm{abs}}(X,W_X) \oplus_{(\mathfrak{D}^{\mathrm{abs}}(T,W_T))} \mathfrak{D}^{\mathrm{abs}}(S,W_S)$$

and the DG functor

$$\pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S,W) \to \mathfrak{D}_S^{\mathrm{abs}}(Y,W),$$

which we denote by the same letter. There is a commutative diagram

$$\begin{array}{ccc} \mathscr{D}_{\mathrm{coh},T}(X,S,W) & \longrightarrow & \mathscr{D}_{\mathrm{coh}}(X,S,W) \\ {\scriptstyle \pi_*}\downarrow & & {\scriptstyle \pi_*}\downarrow \\ \mathfrak{D}_S^{\mathrm{abs}}(Y,W) & \longrightarrow & \mathfrak{D}^{\mathrm{abs}}(Y,W) \end{array} \tag{7.3}$$

of DG functors, and the horizontal arrows in (7.3) are quasi-fully-faithful.

Proposition 7.21 ([8], Proposition 8.23). *The DG functors*

$$\pi_*\colon \mathscr{D}_{\mathrm{coh}}(X,S,W) \to \mathfrak{D}^{\mathrm{abs}}(Y,W) \quad \textit{and} \quad \pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S,W) \to \mathfrak{D}_S^{\mathrm{abs}}(Y)$$

are homological epimorphisms.

Again we have a quasi-functor

$$\Phi\colon \mathfrak{D}^{\mathrm{abs}}(T,W_T) \to \mathscr{D}_{\mathrm{coh},T}(X,S,W),$$

and the composition $\pi_*\Phi\colon \mathfrak{D}^b(T) \to \mathfrak{D}_S^b(Y)$ is zero in $\mathrm{Ho}(\mathrm{dgcat}_{\mathrm{k}})$.

Our main result in this subsection reduces to the following statement.

Theorem 7.22 ([8], Theorem 8.24). 1) *In the above notation, suppose that all infinitesimal neighbourhoods $S_n \subset Y$, $n \geqslant 1$, are non-rational loci of Y with respect to f. Then the DG functor*

$$\pi_*\colon \mathscr{D}_{\mathrm{coh},T}(X,S,W) \to \mathfrak{D}^{\mathrm{abs}}(Y,W)$$

is a localization, and its kernel is generated by the image of the quasi-functor

$$\Phi\colon \mathfrak{D}^{\mathrm{abs}}(T,W_T) \to \mathscr{D}_{\mathrm{coh},T}(X,S,W).$$

2) *Suppose that, moreover, the morphism f is an isomorphism outside S. Then the DG functor*

$$\pi_*\colon \mathscr{D}_{\mathrm{coh}}(X,S,W) \to \mathfrak{D}^{\mathrm{abs}}(Y,W)$$

is a localization, and again the kernel of π_* is generated by the image of the composition

$$\Phi\colon \mathfrak{D}^{\mathrm{abs}}(T,W_T) \to \mathscr{D}_{\mathrm{coh},T}(X,S,W) \to \mathscr{D}_{\mathrm{coh}}(X,S,W).$$

The proof essentially follows the same plan as the proof of Theorem 7.13 (see § 8.4 in [8]).

7.5. The construction of a smooth categorical compactification. In this subsection we sketch the proof of the following theorem.

Theorem 7.23. *Let Y be a smooth separated scheme of finite type over a field* k *of characteristic zero. Then*:

1) *the DG category $\mathfrak{D}^b(Y)$ has a smooth categorical compactification of the form $\mathfrak{D}^b(\widetilde{Y}) \to \mathfrak{D}^b(Y)$, where $\widetilde{Y}$ is a smooth and proper variety*;

2) *for any regular function $W \in \mathscr{O}(Y)$ the $D(\mathbb{Z}/2\text{-})G$ category $\mathfrak{D}^{\mathrm{abs}}(Y, W)$ has a $\mathbb{Z}/2$-graded smooth categorical compactification $C_W \to \mathfrak{D}^{\mathrm{abs}}(X, W)$ with a semi-orthogonal decomposition*

$$C_W = \langle \mathfrak{D}^{\mathrm{abs}}(V_1, W_1), \dots, \mathfrak{D}^{\mathrm{abs}}(V_m, W_m) \rangle,$$

where each V_i is a k*-smooth variety, and the morphisms $W_i \colon V_i \to \mathbb{A}^1_{\mathrm{k}}$ are smooth.*

Sketch of the proof. 1) First, we may assume that the scheme Y is proper. Indeed, if Y is not proper, then by the Nagata theorem [24] we can take some compactification $Y \subset \overline{Y}$, so that the restriction DG functor $\mathfrak{D}^b(\overline{Y}) \to \mathfrak{D}^b(Y)$ is a localization, and the kernel $\mathfrak{D}^b_{\overline{Y}\setminus Y}(\overline{Y})$ is generated by a single object. Thus, Y can be replaced by $\overline{Y}$.

From now on, we assume that Y is proper. By Theorem 4.15 in [27], it is sufficient to construct a smooth categorical compactification $C \to \mathfrak{D}^b(Y)$ such that C has a semi-orthogonal decomposition $C = \langle \mathfrak{D}^b(X_1), \dots, \mathfrak{D}^b(X_m) \rangle$, where each X_i is a smooth and proper variety. We will obtain the DG category C by the same construction as the Kuznetsov–Lunts categorical resolution [21], with a slight restriction on the choice of the integer parameters (see below). Also, our description is a bit different because we are dealing with derived categories of coherent sheaves instead of perfect complexes.

By Theorem 1.6 in [2] there is a sequence of blow-ups with smooth centres

$$Y_n \to Y_{n-1} \to \dots \to Y_1 \to Y$$

such that $(Y_n)_{\mathrm{red}}$ is smooth. We proceed by induction on n.

The base of induction is $n = 0$. In this case Y_{red} is smooth and proper. Take the nilpotent radical $\mathscr{I} \subset \mathscr{O}_Y$ and assume that $\mathscr{I}^l = 0$. Applying the Auslander-type construction to the triple $(Y, \mathscr{I}, l)$, we get a nice ringed space $(Y, \mathscr{A}_Y)$ with a morphism $\rho_Y \colon (Y, \mathscr{A}_Y) \to Y$. By Proposition 7.2 the DG functor $\rho_{Y*} \colon \mathfrak{D}^b(\mathscr{A}_Y) \to \mathfrak{D}^b(Y)$ is a localization, and the kernel is generated by a single object. By Proposition 7.4 the DG category $\mathfrak{D}^b(\mathscr{A}_Y)$ is smooth and proper, and by Proposition 7.3 it has a semi-orthogonal decomposition

$$\mathfrak{D}^b(\mathscr{A}_Y) = \langle \mathfrak{D}^b(Y_{\mathrm{red}}), \dots, \mathfrak{D}^b(Y_{\mathrm{red}}) \rangle,$$

where the number of components is l. This proves the induction base.

Now assume that the assertion has been proved for some n. We prove it for $n+1$. Assume that the first blow-up $f \colon X = Y_1 \to Y$ has a smooth centre $Z \subset Y$. By Proposition 7.11 there is an $l > 0$ such that for all $k \geqslant l$ the infinitesimal neighbourhood Z_k of Z is a non-rational locus of Y with respect to f. As in § 7.3 we have a DG category $\mathscr{D}_{\mathrm{coh}}(X, Z_l)$, and by Theorem 7.13 the DG functor

$$\pi_* \colon \mathscr{D}_{\mathrm{coh}}(X, Z_l) \to \mathfrak{D}^b(Y)$$

is a localization, and $\ker(\pi_*)$ is generated by a single object.

We would like to modify the DG category $\mathscr{D}_{\mathrm{coh}}(X, Z_l)$. Let $D := f^{-1}(Z)$. Then $D_l = f^{-1}(Z_l)$. Take the nice ringed spaces $(D_l, \mathscr{A}_{D_l})$ and $(Z_l, \mathscr{A}_{Z_l})$ associated to the triples (D_l, I_D, l) and (Z_l, I_Z, l), respectively. We have a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\ j\ } & D_l & \xleftarrow{\ \rho_{D_l}\ } & (D_l, \mathscr{A}_{D_l}) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle p}\big\downarrow & & {\scriptstyle \widetilde{p}}\big\downarrow \\ Y & \xleftarrow{\ i\ } & Z_l & \xleftarrow{\ \rho_{Z_l}\ } & (Z_l, \mathscr{A}_{Z_l}) \end{array}$$

Let

$$\mathscr{D}_{\mathrm{coh}}(X, \mathscr{A}_{Z_l}) := \mathfrak{D}^b(X) \oplus_{(\mathfrak{D}^b(\mathscr{A}_{D_l}))} \mathfrak{D}^b(\mathscr{A}_{Z_l}).$$

By 1) in Lemma 5.8 of [8] we have the DG functor

$$\rho(X, Z_l) : \mathscr{D}_{\mathrm{coh}}(X, \mathscr{A}_{Z_l}) \to \mathscr{D}_{\mathrm{coh}}(X, Z_l),$$

and by 2) in the same lemma together with Proposition 7.2 the functor $\rho(X, Z_l)$ is a localization, with its kernel generated by a single object. Hence the composition

$$\widetilde{\pi}_* := \pi_* \circ \rho(X, Z_l) \colon \mathscr{D}_{\mathrm{coh}}(X, \mathscr{A}_{Z_l}) \to \mathfrak{D}^b(Y)$$

is also a localization, with its kernel generated by a single object.

By the induction hypothesis, there is a smooth categorical compactification $\pi'_* \colon C' \to \mathfrak{D}^b(X)$, with a semi-orthogonal decomposition

$$C' = \langle \mathfrak{D}^b(X_1), \dots, \mathfrak{D}^b(X_{m'}) \rangle.$$

One can define a smooth and proper DG category C as a suitable gluing of $\mathfrak{D}^b(\mathscr{A}_{Z_l})$ and C', and then construct the desired localization $C \to \mathfrak{D}^b(Y)$ (the details are contained in the proof of Theorem 8.31 in [8]). This completes the induction step, and with it the proof of part 1) of the theorem.

Part 2) is proved in a completely analogous way. We first reduce to the case when the morphism $W \colon Y \to \mathbb{A}^1$ is proper, and then proceed as in the proof of part 1).

Theorem 7.23 is proved. □

Remark 7.24. In the proof of Theorem 7.23 we chose a parameter $k > 0$ such that the subscheme $Z_k \subset Y$ is a non-rational locus of Y with respect to $f \colon X \to Y$, and so are all the infinitesimal neighbourhoods Z_l with $l \geqslant k$. In the construction of a categorical resolution in [21] it is only needed that Z_k is a non-rational locus.

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