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To our Teacher
Sergei Petrovich Novikov
on his 80th birthday

Conway topograph, $\mathbf{PGL_2(\mathbb{Z})}$-dynamics and two-valued groups

V. M. Buchstaber and A. P. Veselov

Abstract. Conway's topographic approach to binary quadratic forms and Markov triples is reviewed from the point of view of the theory of two-valued groups. This leads naturally to a new class of commutative two-valued groups, which we call involutive. It is shown that the two-valued group of Conway's lax vectors plays a special role in this class. The group $\mathrm{PGL}_2(\mathbb{Z})$ describing the symmetries of the Conway topograph acts by automorphisms of this two-valued group. Binary quadratic forms are interpreted as primitive elements of the Hopf 2-algebra of functions on the Conway group. This fact is used to construct an explicit embedding of the Conway two-valued group into $\mathbb{R}$ and thus to introduce a total group ordering on it. The two-valued algebraic involutive groups with symmetric multiplication law are classified, and it is shown that they are all obtained by the coset construction from the addition law on elliptic curves. In particular, this explains the special role of Mordell's modification of the Markov equation and reveals its connection with two-valued groups in K-theory. The survey concludes with a discussion of the role of two-valued groups and the group $\mathrm{PGL}_2(\mathbb{Z})$ in the context of integrability in multivalued dynamics.

Bibliography: 104 titles.

Keywords: Conway topograph, modular group, two-valued groups, algebraic discrete-time dynamics, integrability.

Contents

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1. Introduction

The study of the values of binary quadratic forms is a very classical subject going back to Legendre and Gauss. The question of numbers N which can be represented as a sum of squares, $N = m^2 + n^2$, is now part of standard undergraduate courses in number theory (see [81], for example).

In his lectures at the beginning of the 1990s [32] Conway described a remarkable way to 'visualise' the values of a binary quadratic forms

$$Q(x, y) = ax^2 + hxy + by^2$$

on the integer lattice $(x, y) \in \mathbb{Z}^2$, by proposing what is now called *Conway's topograph*.

Following Conway, we define a *lax* vector as a pair $(\pm v)$ with $v \in \mathbb{Z}^2$, and a *superbase* of the integer lattice $\mathbb{Z}^2$ as a triple of lax vectors $(\pm e_1, \pm e_2, \pm e_3)$ such that (e_1, e_2) is a basis of the lattice and $e_1 + e_2 + e_3 = 0$.

Similar triples of ordinary vectors were first used in the theory of quadratic forms by Selling in [84] and later by Voronoi and Delone. Every basis can be included in exactly two superbases, which we can represent using the binary tree embedded in the plane (see the left-hand side of Fig. 1, where, following Conway, we drop the plus-minus signs).

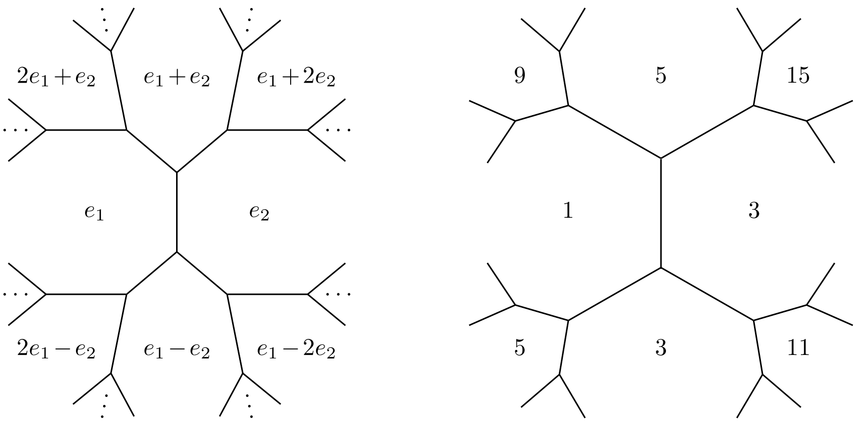

Figure 1. The superbase tree and the Conway topograph of $Q = x^2 + xy + 3y^2$.

The complement of the Conway topographic tree consists of domains marked by *primitive* lax vectors (that is, lax vectors having coprime coordinates), while the superbases correspond to the vertices.

The group $\mathrm{PGL}_2(\mathbb{Z})$ acts naturally on the integer lax vectors and plays the role of the full symmetry group of the Conway tree, while its modular subgroup $\mathrm{PSL}_2(\mathbb{Z})$ describes the rotational symmetries of the tree (see [32]).

Explicitly, we have a well-known isomorphism $\mathbb{Z}_2 * \mathbb{Z}_3 = \mathrm{PSL}_2(\mathbb{Z})$ with the generators

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

acting on the tree as the rotations through the angle π about an edge centre and through the angle $2\pi/3$ about a vertex, respectively. The element $\mathrm{diag}(-1,1) \in \mathrm{PGL}_2(\mathbb{Z})$ acts by a natural mirror reflection on the tree. This explains why $\mathrm{PGL}_2(\mathbb{Z})$ and its subgroups play such an important role in our paper.

By taking values of the quadratic form Q on the vectors of the superbase, we get the Conway topograph of Q containing the values of Q on all primitive lattice vectors. One can construct the topograph of Q recursively, using the *parallelogram rule*:

$$Q(u+v) + Q(u-v) = 2(Q(u) + Q(v)), \qquad u, v \in \mathbb{R}^2.$$

This is shown graphically on Fig. 2, where

$$a = Q(u), \qquad b = Q(v), \qquad c_1 = Q(u-v), \quad \text{and} \quad c_2 = Q(u+v)$$

satisfy the parallelogram relation $c_1 + c_2 = 2(a+b)$ (which Conway called the 'arithmetic progression rule', since numbers c_1, $a+b$, and c_2 form the arithmetic progression with $h = a + b - c_1$).

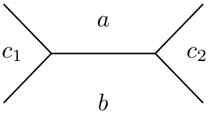

Figure 2. The parallelogram rule $c_1 + c_2 = 2(a+b)$ for the values of quadratic forms.

In particular, for the quadratic form $Q(x,y) = x^2 + xy + 3y^2$ we have the Conway topograph fragment shown in Fig. 1.

A similar topographic representation (see Fig. 11 in § 4) can be used to describe the celebrated *Markov triples*, which are the positive integer solutions of the Markov equation [74]

$$x^2 + y^2 + z^2 = 3xyz, \tag{1}$$

where the parallelogram rule is replaced by the Vieta relation

$$z_1 + z_2 = 3xy$$

(see [42] and [89], for example). There is a natural action of $\mathrm{PGL}_2(\mathbb{Z})$ on the solutions of the Markov equation, generated by the Vieta involution

$$(x, y, z) \to (x, y, 3xy - z)$$

and the permutations of the variables. If we also add the possibility of changing the sign of any two variables, then it follows from results of Èl'-Huti [39] that this generates the full group of automorphisms of the affine cubic surface given by the Markov equation.

Markov showed that set of all positive integer solutions of the equation (1) is just one $\mathrm{PGL}_2(\mathbb{Z})$-orbit of the solution $(1,1,1)$:

$$(1,1,1),\ (1,1,2),\ (1,2,5),\ (1,5,13),\ (2,5,29),\ \dots .$$

Elements of Markov triples are known as *Markov numbers*:

$$1,\ 2,\ 5,\ 13,\ 29,\ 34,\ 89,\ 169,\ 194,\ 233,\ 433,\ 610,\ 985,\ \dots .$$

There is a (still unproven) conjecture due to Frobenius (see Aigner [2]) claiming that the maximal element z of a Markov triple (x,y,z), $x \leqslant y \leqslant z$, determines the triple uniquely (so that there are as many Markov triples as Markov numbers).

Initially, the Markov equation appeared in number theory to describe an indefinite analogue of the dense circle packing problem [74]. After the paper [58] of Hurwitz, Markov triples have usually been regarded as the Diophantine analysis of the most irrational numbers (see Delone [35] and Aigner [2] for surveys of this beautiful area).

The *Markov constant* $\mu(\alpha)$ of an irrational number α is defined as the minimal number c such that the inequality

$$\left|\alpha - \frac{p}{q}\right| \leqslant \frac{c}{q^2} \tag{2}$$

holds for infinitely many rationals p/q. The function $\mu(\alpha)$ is $\mathrm{PGL}_2(\mathbb{Z})$-invariant and can be regarded as a measure of the irrationality of α. Hurwitz [58] proved that $\mu(\alpha) \leqslant 1/\sqrt{5}$ for all real α, with equality only for the golden ratio $\alpha = (1+\sqrt{5})/2$ and its $\mathrm{PGL}_2(\mathbb{Z})$-equivalents.

Markov [74] described all irrational numbers α with $\mu(\alpha) > 1/3$ (*Markov irrationalities*), linking them with Markov triples (1). Namely, for any such triple (x,y,z), $x \leqslant y \leqslant z$, the formula

$$\alpha = \frac{b}{x} + \frac{y}{xz} - \frac{3}{2} + \frac{\sqrt{9z^2-4}}{2z}, \qquad by - ax = z, \tag{3}$$

defines a Markov irrationality with

$$\mu(\alpha) = \frac{z}{\sqrt{9z^2-4}}.$$

For example, the smallest triple $(1,1,1)$ corresponds to the golden ratio, $(1,1,2)$ corresponds to $\alpha = \sqrt{2}$, and $(1,2,5)$ to $(9+\sqrt{221})/10$. The converse is also true: any Markov irrationality is $\mathrm{PGL}_2(\mathbb{Z})$-equivalent to a number α of the form (3). Several proofs of this truly remarkable theorem of Markov can be found in [2], [33], and [93].

From work in the 1950s by Gorshkov [48] (see also [49]) and Cohn [31], this story became part of hyperbolic geometry and the Teichmüller space theory (see, for example, [42], [47], and [72]).

The main idea goes back to work of Fricke and Klein on the uniformisation problem for the punctured tori T^2_*. To solve this problem we need to define a suitable monomorphism of $\pi_1(T^2_*)$ into the group of motions $\mathrm{PSL}_2(\mathbb{R})$ of the hyperbolic plane $\mathbb{H}^2$.

Since the fundamental group $\pi_1(T^2_*)$ is the free group $F_2 = \mathbb{Z} * \mathbb{Z}$ with two generators, we need to find two hyperbolic matrices A and B in $\mathrm{SL}_2(\mathbb{R})$ such that their commutator $ABA^{-1}B^{-1}$ is parabolic (which is the condition for a puncture). One can show [6] that this commutator must have two eigenvalues equal to -1, therefore,

$$\mathrm{tr}(ABA^{-1}B^{-1}) = -2.$$

The classical Fricke identity [43] states that for any $A, B \in \mathrm{SL}_2(\mathbb{R})$ and for $C = AB$ we have

$$(\mathrm{tr}\, A)^2 + (\mathrm{tr}\, B)^2 + (\mathrm{tr}\, C)^2 = \mathrm{tr}\, A\ \mathrm{tr}\, B\ \mathrm{tr}\, C + \mathrm{tr}(ABA^{-1}B^{-1}) + 2. \tag{4}$$

It is interesting that this identity is a particular case of the Frobenius relation for traces of two-dimensional representations $f\colon G \to \mathbb{C}$:

$$\begin{aligned}\Phi_3(f)(a_1, a_2, a_3) := f(a_1)f(a_2)f(a_3) - f(a_1)f(a_1a_2) - f(a_1a_2)f(a_3)\\ - f(a_2)f(a_1a_3) + f(a_1a_2a_3) + f(a_2a_1a_3) = 0\end{aligned}$$

(see Frobenius [44] and the survey of further developments in Buchstaber–Rees [25]). Applying this to

$$G = \mathrm{SL}_2(\mathbb{R}), \qquad f(a) = \mathrm{tr}\, a, \qquad a_1 = AB, \qquad a_2 = A^{-1}, \qquad a_3 = B^{-1},$$

we have the Fricke identity (4).

The puncture condition implies that $X = \mathrm{tr}\, A$, $Y = \mathrm{tr}\, B$, and $Z = \mathrm{tr}\, C$ satisfy the real Markov equation

$$X^2 + Y^2 + Z^2 = XYZ, \qquad X, Y, Z \in \mathbb{R}_{>0}, \tag{5}$$

which defines the Teichmüller space of one-punctured tori (see, for instance, Goldman [47]). The group $\mathrm{SL}_2(\mathbb{Z})$ appears here naturally as the orientation-preserving mapping class group of both the torus and the punctured torus.

Note that the substitution $X = 3x$, $Y = 3y$, $Z = 3z$ leads to the Markov equation (1). Hurwitz showed that the Diophantine equation

$$x^2 + y^2 + z^2 = bxyz \tag{6}$$

has non-trivial integer solutions only for $b = 1$ and $b = 3$.

Gorshkov and Cohn showed that Markov's $\mathrm{PGL}_2(\mathbb{Z})$-orbit of the solution $(1, 1, 1)$ corresponds to the punctured $\mathbb{Z}_3$-symmetric rhombic (equi-anharmonic) torus with complete hyperbolic metric: Markov numbers can be interpreted as $(2/3)\cosh l(\gamma)$, where $l(\gamma)$ is the length of the corresponding simple closed geodesic. Markov triples describe the lengths of three such geodesics with one-point pairwise intersections (see [31], [48], [49], [51], and [85]).

It is interesting that the corresponding Fuchsian group is precisely the commutator subgroup $\mathrm{SL}_2(\mathbb{Z})'$. This commutator subgroup is the free product $\mathbb{Z} * \mathbb{Z}$ generated by the matrices

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

(for instance, see Aigner [2]). We also mention here the recent paper [79] by Panov and Veryovkin, where the isomorphism of the commutator subgroup of the group $\mathbb{Z}_2 * \mathbb{Z}_3$ and the free group $\mathbb{Z} * \mathbb{Z}$ appears as a particular case of a much more general result, proved using methods of the homotopic theory of polyhedral products (see [21]).

To study the growth of Markov numbers, Zagier [104] considered a linearisable modification of the Markov equation

$$x^2 + y^2 + z^2 = 3xyz + \frac{4}{9}, \tag{7}$$

which can be reduced to the Euclid relation $u \pm v = w$ by the simple change

$$x = \frac{2}{3}\cosh u, \quad y = \frac{2}{3}\cosh v, \quad z = \frac{2}{3}\cosh w. \tag{8}$$

An equivalent Diophantine equation for $X = 3x/2$, $Y = 3y/2$, and $Z = 3z/2$,

$$X^2 + Y^2 + Z^2 = 2XYZ + 1, \tag{9}$$

was first studied by Mordell [76], who reduced it to $u \pm v = w$ using

$$X = \cosh u, \qquad Y = \cosh v, \qquad Z = \cosh w. \tag{10}$$

The equation (9) determines a particular affine realisation of the classical *Cayley cubic surface* with the maximal number (which is 4) of conical singularities (see Fig. 3 and § 5 below).

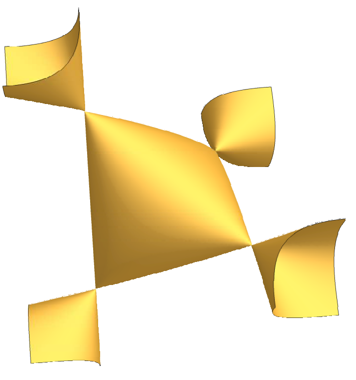

Figure 3. Real Cayley cubic surface.

It is interesting that its middle real part (defined by $|X| \leqslant 1$, $|Y| \leqslant 1$, $|Z| \leqslant 1$), known as a *spectrahedron* (see, for example, [100]) with the trigonometric parametrisation

$$X = \cos u, \quad Y = \cos v, \quad Z = \cos w, \qquad u + v + w = 2\pi,$$

admits a simple geometric interpretation as a set of three coplanar unit vectors in $\mathbb{R}^3$ with pairwise angles u, v, and w. Indeed, the condition that the corresponding

Gram matrix is singular, that is,

$$\det \begin{pmatrix} 1 & X & Y \\ X & 1 & Z \\ Y & Z & 1 \end{pmatrix} = 0,$$

is exactly the Mordell equation. Mordell's parametrisation (10) for $|X| \geqslant 1$, $|Y| \geqslant 1$, $|Z| \geqslant 1$ also has a similar geometric interpretation in Lobachevskian geometry.

The growth of the Markov numbers and of the values of binary quadratic forms on the Conway topograph was studied by Spalding and Veselov in [89] and [90], and this led to a new and interesting function $\Lambda(\xi)$ of the real variable ξ (see § 2 for details).

The aim of this paper is to establish a connection between the Conway topographic approach and the theory of two-valued groups, and to demonstrate a new approach to the rich diversity of results noted above. This leads to a new and interesting class of two-valued groups which we call involutive. The key example of such groups is the Conway group of lax vectors $\mathbb{X}_2$ (see below). The group $\mathrm{PGL}_2(\mathbb{Z})$ appears here as the automorphism group of $\mathbb{X}_2$.

The theory of multivalued groups originated in algebraic topology in the paper [20] by Buchstaber and Novikov and was shaped in the papers [9]–[13] by Buchstaber, with further development due to Buchstaber and Rees [23] and others (see the survey [15]).

Leaving the details for [23] and § 3 below, we can say that a two-valued group is a set X with a two-valued multiplication operation

$$\mu\colon x, y \in X \to z = x * y,$$

where $z = (z_1, z_2)$ is an unordered pair of elements of X. This operation satisfies the natural analogues of group properties: associativity and the existence of an identity and of inverses.

The first important examples of such groups were discovered by Buchstaber and Novikov [20] in connection with problems in algebraic topology (see § 5 below). The simplest non-trivial example is the *Buchstaber–Novikov two-valued group*, which is the set of non-negative integers $\mathbb{X}_1 = \mathbb{Z}_{\geqslant 0}$ with the multiplication

$$m * n = (|m - n|, m + n). \tag{11}$$

The identity here is $m = 0$ and the inverse is $m^{-1} = m$.

Our main idea is very simple: we propose to interpret the fragment of the Conway topograph in Fig. 2 as the multiplication rule in a certain two-valued group X (see Fig. 4).

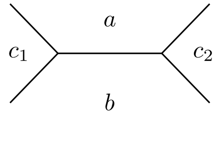

Figure 4. Multiplication rule $a * b = c = (c_1, c_2)$ on the Conway topograph.

The extension of this fragment to the whole Conway topographic tree imposes certain conditions on the two-valued group X, and this leads to a new class of two-valued groups which are commutative and consist of weak involutions. We call such two-valued groups *involutive*.

Using the connection with Conway's topograph, we show that the two-valued group $\mathbb{X}_2$ of lax vectors $\pm u \in \mathbb{Z}^2$ with the multiplication

$$(\pm u) * (\pm v) = (\pm(u - v), \pm(u + v)) \tag{12}$$

(which we will call the *Conway group*) is the universal group in the class of involutive two-valued groups with two generators.

The parallelogram rule for the values of binary quadratic forms Q takes the form

$$Q(a * b) = Q(a) + Q(b), \qquad a, b \in \mathbb{X}_2. \tag{13}$$

Here by the definition of Hopf 2-algebras of functions on two-valued groups (see [23]) we have the formula

$$Q(a * b) = \frac{1}{2}(Q(c_1) + Q(c_2)), \quad \text{where} \;\; a * b = (c_1, c_2). \tag{14}$$

Thus, the binary quadratic forms Q can be interpreted as the primitive elements of such Hopf 2-algebras (see [23]).

Markov triples determine an interesting function m on primitive elements in $\mathbb{X}_2$. At each pair of such elements a and b generating the whole group $\mathbb{X}_2$, m takes values satisfying the relation

$$m(a * b) = m(a)m(b)$$

(see the details in § 5 below). This reminds us of the property of Euler's φ-function in number theory:

$$\varphi(kl) = \varphi(k)\varphi(l),$$

but only for coprime k and l (that is, k and l generate the whole group $\mathbb{Z}$).

Thus, the Markov function m can be regarded as an arithmetic multiplicative function on the *primitive part* $\mathscr{P}(\mathbb{X}_2)$ of the Conway group $\mathbb{X}_2$, which consists of the lax vectors $\pm(p, q)$ with coprime p and q.

Note that $\mathscr{P}(\mathbb{X}_2)$ can be identified with the set of domains for the Conway topograph. There is a natural 'projectivisation' map

$$\pi\colon \mathbb{X}_2^* := \mathbb{X}_2 \setminus (0,0) \to \mathscr{P}(\mathbb{X}_2), \qquad \pi(\pm(m, n)) = (m : n). \tag{15}$$

The fibre over a primitive lax vector $\pm(p, q)$ consists of the lax vectors $\pm l(p, q)$, $l \in \mathbb{N}$ (and thus can be identified with $\mathbb{X}_1^* = \mathbb{X}_1 \setminus 0$).

It is interesting that the projection π implicitly appeared in the theory of modular forms and Eisenstein series [7]. Namely, there are two definitions of Eisenstein series,

$$G_k(z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z},\, (m,n) \neq (0,0)} \frac{1}{(mz + n)^{2k}}, \qquad k > 1, \quad \operatorname{Im} z > 0,$$

and

$$E_k(z)=\frac{1}{2}\sum_{m,n\in\mathbb{Z},\,\mathrm{GCD}(m,n)=1}\frac{1}{(mz+n)^{2k}},\qquad k>1,\quad \operatorname{Im} z>0,$$

where in the second case the summation is only over coprime pairs (see § 2 in Zagier's contribution to [7]). The definitions differ only by a normalisation factor:

$$G_k(z)=\zeta(2k)E_k(z),\tag{16}$$

where

$$\zeta(s)=\sum_{l\in\mathbb{N}}\frac{1}{l^s}$$

is the Riemann zeta-function, but as Zagier argued, both definitions are important in their own way.

We note that (16) can be interpreted as a Fubini formula for the 'integral' over $\mathbb{X}_2^*$ of the function $(mz+n)^{-2k}$ in correspondence with the fibration (15):

$$\int_{\mathbb{X}_2^*}\frac{1}{(mz+n)^{2k}}=\int_{\mathscr{P}(\mathbb{X}_2)}\frac{1}{(mz+n)^{2k}}\int_{X_1^*}\frac{1}{l^{2k}}.\tag{17}$$

It is interesting that Mordell's modification (9) of the Markov equation (1) appears naturally in the theory of two-valued formal groups related to algebraic topology (see the survey [15]). Namely, consider an algebraic version of two-valued groups with the multiplication law given by $F(x,y,z)=0$, where $F(x,y,z)$ is a symmetric polynomial of degree 2 in each variable such that $F(x,y,0)=(x-y)^2$. Note that such groups are automatically commutative and involutive. In § 6 we classify all such group laws and show that they belong to the three-parameter family with

$$F=(x+y+z-a_2xyz)^2-4(1+a_3xyz)(xy+xz+yz+a_1xyz)=0,\tag{18}$$

which is found in [13] and defines the addition law on the two-valued coset group $\mathscr{E}/\sigma$, where $\mathscr{E}$ is the elliptic curve (or its degeneration) given by

$$v^2=u^3+a_1u^2+a_2u+a_3$$

and σ is the involution $v\to -v$.

The Mordell equation (9) corresponds to the degenerate case $a_2=a_3=0$ and is obtained by a simple affine change (see §§ 4 and 5 below). In algebraic topology this case corresponds in complex K-theory to a formal group for which the square modulus construction gives (9). From the algebro-geometric point of view the equation (18) describes the special Kummer surface $\mathrm{Km}(\mathscr{E}\times\mathscr{E})=\mathscr{E}\times\mathscr{E}/\pm I$ of the square of the elliptic curve $\mathscr{E}$.

The plan of this paper is as follows.

In § 2 we discuss in more detail Conway's topographic approach, emphasizing the special role of the modular group.

In § 3 we give the main definitions and examples from the theory of two-valued groups, discuss their connection with Conway's topograph, and introduce the related class of involutive two-valued groups. We demonstrate the special role played by the Buchstaber–Novikov and Conway groups in this class.

In § 4 binary quadratic forms are interpreted as primitive elements of the corresponding Hopf 2-algebra of functions on the Conway group. Using a suitable quadratic form Q with irrational coefficients, we define an embedding of $\mathbb{X}_2$ into $\mathbb{R}$ and thereby introduce on it a total group ordering.

In § 5 we describe the square modulus construction in the theory of formal groups from [20] and explain the relation of the corresponding two-valued group from K-theory with the Mordell equation and the Cayley nodal cubic surface.

In § 6 we classify all two-valued algebraic groups with a symmetric multiplication law and describe their relation to elliptic curves, addition laws for the classical Weierstrass and Jacobi functions, and special Kummer surfaces.

In § 7 we discuss the role of two-valued groups and the group $\mathrm{PGL}_2(\mathbb{Z})$ in the context of integrability in multivalued dynamics.

In § 8 we conclude the paper with a general discussion of some closely related questions and results.

2. Conway topograph and the modular group

Since the paper [84] of Selling in 1874 it has been known that instead of a basis $e_1,\dots,e_n$ in the lattice $\mathbb{Z}^n$ it is useful in the theory of quadratic forms to consider a superbasis $e_1,\dots,e_{n+1}$ containing one additional vector such that

$$e_1+e_2+\dots+e_{n+1}=0$$

(Delone used to call such a collection of vectors *Selling's prickle*).

Conway [32] contributed an essential twist to this idea by considering so-called lax vectors $\pm v$. The key observation here is that in dimension 2 a usual basis of lax vectors $\pm e_1$, $\pm e_2$ gives rise to exactly to two lax superbases: $\pm e_1,\pm e_2,\pm(e_1+e_2)$ and $\pm e_1,\pm e_2,\pm(e_1-e_2)$.

This observation allows us to construct in an obvious way a plane embedded graph (*Conway's topograph*) with edges corresponding to lax bases and vertices corresponding to lax superbases, in fact an infinite tree.

Theorem 2.1 (Conway [32]). *Conway's topograph is an infinite plane-embedded trivalent tree, which describes all the lax superbases of the lattice* $\mathbb{Z}^2$.

By construction Conway's tree is embedded in the plane, so we have not only vertices and edges but also the *domains*, and it is the domains that we populate with data. Following Conway, we drop the signs $\pm$ and choose representatives of the lax vectors to mark the domains in the complement of the tree (see Fig. 1).

Note that all the (lax) vectors $u=me_1+ne_2=(m,n)$ on the Conway topograph are *primitive* in the sense that their coordinates m and n are coprime. From elementary number theory it is known that any such vector u can be included in a basis (u,v) with the corresponding vector $v=(x,y)$ found as a solution of the linear Diophantine equation

$$mx+ny=\pm1.$$

The solution can be found using the Euclidean algorithm, which can actually be interpreted as a walk along the edges of the Conway tree that bound the domain marked by the vector (m,n).

Recall that the primary goal of Conway [32] was to 'visualise' the values of such a form

$$Q(x, y) = ax^2 + hxy + by^2$$

on the integer lattice $(x, y) \in \mathbb{Z}^2$.

By taking values of the form Q on vectors in the superbase we get what Conway called the topograph of Q, which contains the values of Q at all primitive lattice vectors. In particular, for the standard superbasis e_1, e_2, $e_3 = -e_1 - e_2$ we have the values

$$Q(e_1) = a, \qquad Q(e_2) = b, \qquad Q(e_3) = c := a + b + h.$$

On the Conway topograph of the quadratic form Q one can define the direction of the edges, depending on the sign of the number $h = c - a - b$ (see [32]). Assume that the coefficients a, b, and h of the form are all positive. In this case Conway's Climbing Lemma [32] guarantees a constant growth of values of Q, whichever upward path we choose (see Fig. 5).

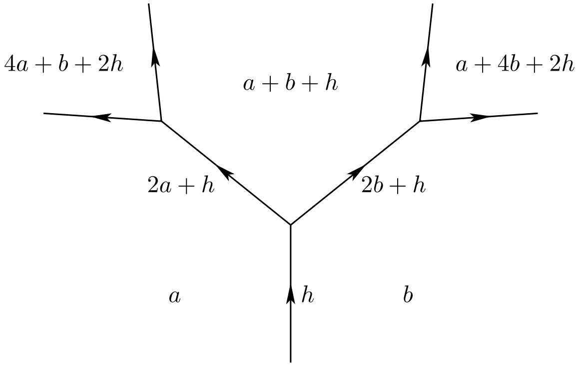

Figure 5. Conway's Climbing Lemma.

These arguments imply that the superbase graph cannot have loops and thus must be an infinite trivalent tree.

The Conway topographic tree has a natural realisation in hyperbolic geometry as the *Farey tree*, which is dual to the tesselation of $\mathbb{H}^2$ by ideal triangles. The domains in the complement of the tree can be naturally labelled by rational numbers using Farey 'addition' (the mediant rule):

$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d}$$

(see Fig. 6 and Hatcher [54]).

We note that an ideal triangle is the fundamental domain for the action of the subgroup $\Lambda \subset \mathrm{PGL}_2(\mathbb{Z})$ of index 6 that is isomorphic to $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ and is freely generated by the reflections with respect to the sides of the triangle (see the formula (57) below). This explains the natural action of $\mathrm{PGL}_2(\mathbb{Z})$ on the Conway topograph, as described in the Introduction.

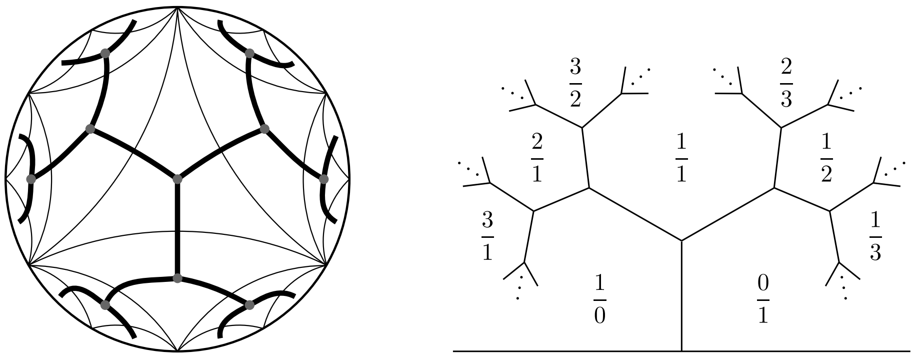

Figure 6. The dual tree for the Farey tessellation and the positive part of the Farey tree.

Thus, from the point of view of Klein's Erlangen programme, the Conway tree plays the same role in the theory of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ as the Lobachevsky plane plays for the group $\mathrm{PSL}_2(\mathbb{R})$.[1]

We remark that the positive part of the Farey tree (see Fig. 6) gives a nice parametrisation of the monoid $\mathrm{SL}_2(\mathbb{N}) \subset \mathrm{SL}_2(\mathbb{Z})$ consisting of the matrices with non-negative entries. Indeed, for every edge we have adjacent to it two Farey fractions, a/c and b/d, so we can consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

which belongs to $\mathrm{SL}_2(\mathbb{N})$. One can easily show that every matrix $A \in \mathrm{SL}_2(\mathbb{N})$ appears exactly once in this way.

In $\mathrm{PSL}_2(\mathbb{Z})$-dynamics the infinite paths on the positive part of the Farey tree correspond to all possible 'futures' (values on the absolute) and are connected naturally with the continued fraction expansion of the corresponding limit point $\xi \in \mathbb{RP}^1$ on the absolute of the hyperbolic plane (see the details in [89]).

In particular, one can consider the following natural function describing the growth of the monoid $\mathrm{SL}_2(\mathbb{N})$ along a path γ_ξ on this tree [89]:

$$\Lambda(\xi) = \limsup_{n\to\infty} \frac{\log \rho(A_n(\xi))}{n}, \tag{19}$$

where $A_n(\xi) \in \mathrm{SL}_2(\mathbb{N})$ is attached to the nth edge of the path $\gamma(\xi)$ and $\rho(A)$ is the spectral radius of a matrix A and is defined as the maximum of the moduli of its eigenvalues.

Theorem 2.2 (Spalding and Veselov [89]). *The function $\Lambda(\xi)$ is defined for all $\xi \in \mathbb{RP}^1$ and has the following properties*:

[1] S. P. Novikov told us that the fact that a plane-embedded universal trivalent tree can be considered as a discrete version of the hyperbolic plane was first explained to him by M. Gromov many years ago.

1) *it is* $\mathrm{PGL}_2(\mathbb{Z})$*-invariant, that is,*

$$\Lambda\left(\frac{a\xi+b}{c\xi+d}\right)=\Lambda(\xi), \qquad \xi\in\mathbb{RP}^1, \quad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathbb{Z});$$

2) *it vanishes almost everywhere on* $\mathbb{RP}^1$;

3) *it takes all the values in the interval* $[0,\log\varphi]$, *where* $\varphi=(1+\sqrt{5})/2$ *is the golden ratio*;

4) *it describes the growth of the Markov numbers on the Conway topograph, namely,*

$$\Lambda(\xi)=\limsup_{n\to\infty}\frac{\log\log z_n(\xi)}{n}, \tag{20}$$

where $z_n(\xi)$ *is the largest number in the nth Markov triple along the path* γ_ξ.

The action of the group $\mathrm{SL}_2(\mathbb{Z})$ on the binary quadratic forms is a very classical subject, studied, in particular, by Gauss. Conway's topograph contributes naturally to this theory (see the details in the nicely written recent book by Weissman [102]).

This action has an invariant, the *discriminant* of the quadratic form $Q=ax^2+hxy+by^2$, which in terms of the Conway parameters a, b, and $c=a+h+b$ has the symmetric form

$$D=h^2-4ab=(c-a-b)^2-4ab=a^2+b^2+c^2-2ab-2ac-2bc.$$

A well-known problem is to find the number $h(D)$ of equivalence classes of quadratic forms with given discriminant

$$a^2+b^2+c^2-2ab-2ac-2bc=D. \tag{21}$$

As we will see below, the equation $D(a,b,c)=0$ is the multiplication law in the simplest two-valued algebraic group!

The generating involution in $\mathrm{PSL}_2(\mathbb{Z})=\mathbb{Z}_2*\mathbb{Z}_3$ acts on this set as

$$(a,b,c)\to(b,a,c'=2a+2b-c).$$

We note that the $\mathrm{PGL}_2(\mathbb{Z})$-action contains the more natural Vieta involution

$$(a,b,c)\to(a,b,c'=2a+2b-c),$$

which is nothing else but the parallelogram rule (see Fig. 2).

In [90] the growth

$$\Lambda_Q(\xi)=\limsup_{n\to\infty}\frac{\log|Q_n(\xi)|}{n} \tag{22}$$

of the values of quadratic forms along the path γ_ξ was studied, where $|Q_n(\xi)|=\max(|a_n(\xi)|,|b_n(\xi)|,|c_n(\xi)|)$ on the path γ_ξ (see Fig. 7 for the path leading to the golden ratio $\xi=(1+\sqrt{5})/2$).

It was shown in [90] that for positive-definite quadratic forms Q with $D<0$ we always have

$$\Lambda_Q(\xi)=2\Lambda(\xi),$$

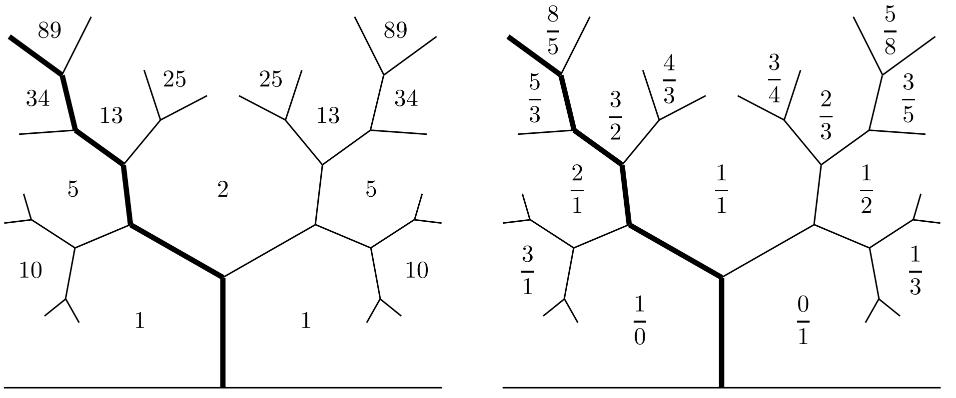

Figure 7. The topograph of $Q = x^2 + y^2$ and the corresponding positive part of the Farey tree with marked 'golden' path.

where $\Lambda(\xi)$ was defined above by (19). For indefinite forms with $D > 0$ we have the so-called *Conway river* separating positive and negative values of the form [32]. In this case the same formula works with the exception of the 'ends' of the Conway river on the absolute, which are defined as the roots $\xi_\pm$ of the corresponding quadratic equation $a\xi^2 + h\xi + b = 0$, and where Λ_Q vanishes (see the details in [90]).

In this connection we would like to mention the following remarkable correspondence, due to Klein [66], between indefinite binary quadratic forms and geodesics in the hyperbolic plane.

Note first of all that the projectivised vector space of real binary quadratic forms is a real projective plane. The degenerate forms make up a conic section determined by the equation $D(a, b, c) = 0$. The definite forms correspond to the points inside this conic, forming Klein's model of the hyperbolic plane. The indefinite forms are points outside the conic, which by polarity correspond to geodesic lines in Klein's model. Conway rivers on the topographic tree can be regarded as analogues of these geodesics.

3. Two-valued groups and the Conway topograph

Let X be a set. Consider a generalisation of the notion of group by allowing the product of two elements to be a subset of X.

A classical example is the set X_G of all irreducible representations of a compact group G. The product of two irreducible representations V and W is the set of n irreducible representations $V_{k_1}, \dots, V_{k_n}$ such that

$$V \otimes W = \bigoplus_{i=1}^{n} V_{k_i}.$$

This operation is associative and has an identity, the one-dimensional trivial representation. In the case when G is commutative, one has the well-known group of one-dimensional representations (the Pontryagin dual of the group G). But in the non-commutative case one has a multivalued operation, and with a number n that depends on the choice of the representations V and W.

Such examples have led to the theory of hypergroups, which combines many deep results, including the Delsarte–Levitan theory of generalised shift operators (see [36], [69], and [71]).

In 1971 Novikov and Buchstaber [20] introduced a construction suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is an *n-multiset*, that is, an unordered set of n elements, possibly with multiplicities. This construction led to the notion of an *n-valued group*.

Fixing the number n of elements in the product is in fact very restrictive and not very natural from the point of view of hypergroup theory, and initially it was not clear if there are enough interesting examples of this kind.

However, the subsequent development, mainly in papers of Buchstaber and his school ([9]–[12], [17], [64], [65]) and later papers by Buchstaber and Rees ([22]–[24]), revealed not only the richness of this theory but also its very interesting applications. For example, it was shown that the set of all irreducible unitary representations of any compact Lie group G determines an n-valued group with n given by the order of the Weyl group of G (see the survey [15] and the important example of $G = \mathrm{SU}_2$ given below).

In our paper we will concentrate on the case of two-valued groups, leaving possible generalisations aside.

Following [23], we define a two-valued group as a set X with a two-valued multiplication operation

$$\mu\colon x, y \in X \to z = x * y \in (X)^2,$$

where $(X)^2 = X \times X/S_2$ is the symmetric square of X, consisting of unordered pairs $z = (z_1, z_2)$, $z_1, z_2 \in X$. This operation has the following natural properties:

1) associativity

$$x * (y * z) = (x * y) * z \tag{23}$$

(understood as equality of the corresponding unordered multisets);

2) existence of a (strong) identity $e \in X$ such that for every $x \in X$

$$e * x = x * e = (x, x); \tag{24}$$

3) existence of a (weak) inverse x^{-1}, such that

$$x^{-1} * x = x * x^{-1} = (e, \phi(x)) \tag{25}$$

for some map $\phi\colon X \to X$.

We assume that the inverse is unique, so that if $x * y = (e, z)$, then $y = x^{-1}$ and $z = \phi(x)$.

We call an element $x \in X$ a *weak involution* if $x = x^{-1}$, or equivalently, $x * x = (e, y)$ for some $y \in X$, and a *strong involution* if $x * x = (e, e)$.

Proposition 3.1. *If x is a strong involution, then $x * y = (z, z)$ for any $y \in X$.*

Proof. Let x be a strong involution and let $x * y = (z_1, z_2)$. We multiply both sides by x from the left: $x * (x * y) = (x * z_1, x * z_2)$. On the other hand, associativity implies that

$$x * (x * y) = (x * x) * y = (e, e) * y = (y, y, y, y).$$

Thus, $x * z_1 = (y, y) = x * z_2$. Multiplying this again by x from the left and using associativity, we see that $z_1 = z_2$. $\square$

We say that X is *strongly two-valued* if the relation $x * y = (z, z)$ implies that either x or y is a strong involution. We will see that our main examples (see below) are strongly two-valued.

An important class of the two-valued groups is given by the following so-called *coset construction*. Let G be a (usual) group and let $\sigma\colon G \to G$ be an involutive automorphism of it: $\sigma^2 = \mathrm{Id}$. Then the corresponding coset two-valued group (in short, *coset group*) on the set $X = G/\sigma$ consists of the orbits $(g, h = \sigma(g))$ of σ with the multiplication

$$(g_1, h_1) * (g_2, h_2) = \bigl((g_1g_2, h_1h_2 = \sigma(g_1g_2)), (g_1h_2, h_1g_2 = \sigma(g_1h_2))\bigr).$$

For the Abelian groups G there is a canonical involution $\tau\colon g \to g^{-1}$. In this case every element in G/τ is a weak involution, since for $x = (g, g^{-1})$

$$x * x = (g, g^{-1}) * (g, g^{-1}) = ((e, e), (g^2, g^{-2})).$$

Then strong involutions now have the form $x = (s, s)$, where the s are involutions in the original group G, that is, $s^2 = e$, and they are also the fixed points of τ.

It is easy to see that all such coset groups G/τ satisfy the strong two-valuedness condition above. Indeed, if $x = (g, g^{-1})$ and $y = (h, h^{-1})$, then

$$x * y = \bigl((gh, (gh)^{-1}), (gh^{-1}, g^{-1}h)\bigr) = (z_1, z_2).$$

Therefore, if $z_1 = z_2$, then either $g^2 = e$ or $h^2 = e$.

The first non-trivial example of a coset group is the Buchstaber–Novikov group $\mathbb{X}_1 = \mathbb{Z}/\tau$, where $\tau(m) = -m$. The corresponding set of orbits can be naturally identified with the set of non-negative integers with the multiplication $m * n = (|m - n|, m + n)$, for which the identity is $m = 0$ and the inverse is $m^{-1} = m$. In this case the identity 0 is the only strong involution.

This group originated in topology [20], but it was later found to play an important role in multivalued dynamics: the classical Euler–Chasles correspondence related to the geometric Poncelet porism can be regarded as an algebraic action of it (see [27]).

The group $\mathbb{X}_1$ is isomorphic also to the two-valued group of irreducible unitary representations of SU_2 (see [15]). Let V_s, $\dim V_s = s$, $s \in \mathbb{Z}_+$, be the list of all irreducible representations of SU_2, where V_1 is the trivial one-dimensional representation and V_2 is the canonical representation on $\mathbb{C}^2$. In the ring of virtual unitary representations of $\mathrm{SU}(2)$ we consider the set of elements

$$\Psi_0 = V_1, \qquad \Psi_1 = V_2, \qquad \Psi_k = V_{k+1} - V_{k-1}, \quad k > 1.$$

Note that $\dim \Psi_0 = 1$ and $\dim \Psi_k = 2$, $k > 0$. We introduce the set

$$X = (x_0, x_1, \dots, x_k, \dots), \qquad x_0 = \Psi_0, \quad x_k = \frac{1}{2}\Psi_k, \quad k > 0.$$

Proposition 3.2 (Buchstaber [15]). *The tensor product of unitary representations of* SU_2 *defines the multiplication*

$$x_k * x_m = (x_{k+m}, x_{|k-m|})$$

on X *and thus defines the structure of a two-valued group which is isomorphic to* $\mathbb{X}_1$.

An example closely related to the Buchstaber–Novikov group $\mathbb{X}_1$ is the two-valued *Conway group* $\mathbb{X}_2 = \mathbb{Z}^2/\sigma$ with $\sigma(u) = -u$ and with multiplication given by (12). Although Conway did not consider two-valued groups, he introduced the key notion of *lax vectors* $\pm u$.

We point out that the symmetry groups $\mathrm{PSL}_2(\mathbb{Z})$ and $\mathrm{PGL}_2(\mathbb{Z})$ of the Conway tree appear here naturally as automorphism groups of the two-valued group $\mathbb{X}_2$ (as counterparts of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z})$ being automorphism groups of the integer lattice group $\mathbb{Z}^2$).

We now show that the group $\mathbb{X}_2$ plays a very special role in the theory of commutative two-valued groups.

Our approach is quite natural: let us replace Fig. 2 in Conway's construction by Fig. 4, which defines the product in a commutative two-valued group X, and see whether this fragment can be consistently extended to the whole topographic tree.

For simplicity of the arguments we assume that our two-valued group X does not contain strong involutions apart from the identity. In this case the strong two-valuedness condition means that the relation $a * b = (c, c)$ implies that either a or b is the identity. Under this assumption we prove the following result.

Proposition 3.3. *The multiplication rule from Fig. 4 can be extended to the fragment of the topograph in Fig. 8 for any two elements* a *and* b *if and only if the two-valued group* X *is commutative and consists of weak involutions.*

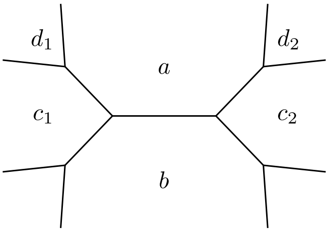

Figure 8. A fragment of the topograph of the two-valued group X.

Proof. First we prove the necessity of the conditions. Commutativity follows directly from the multiplication rule (see Fig. 4). Let $a * b = (c_1, c_2)$. From Fig. 8 we see that

$$a * (a * b) = a * (c_1, c_2) = (a * c_1, a * c_2) = (b, d_1, b, d_2).$$

On the other hand, the associativity condition in X implies that

$$a * (a * b) = (a * a) * b.$$

Let $a*a = (x, y)$. Then the set of elements $(a*a)*b = (x*b, y*b)$ must be equal to the set (b, d_1, b, d_2). If $x*b$ is equal to (b, b), then by our strong two-valuedness assumption x must be the identity. Thus, $a*a = (e, y)$, and hence a is a weak involution. The same is true if $y*b$ is equal to (b, b).

Assume now that

$$x*b = (b, d_1) \quad \text{and} \quad y*b = (b, d_2).$$

Multiplying both sides of the first equality by b^{-1}, we have the set of elements

$$(x*b)*b^{-1} = x*(b*b^{-1}) = x*(e, b') = (x, x, x*b'),$$

which must be equal to

$$(b*b^{-1}, d_1*b^{-1}) = (e, b', u, v)$$

and thus must contain the identity. If x is the identity, then $a*a = (e, y)$ and hence a is a weak involution. On the other hand, if $e \in x*b'$ then $x = b'^{-1}$. The same arguments for the relation $y*b = (b, d_2)$ give us that $y = b'^{-1}$ and hence $x = y$, so $a*a = (x, x)$. By the strong two-valuedness a must be the identity, which proves that a is a weak involution in all cases.

Let us now prove that the conditions are sufficient. We assume that each element a is a weak involution and show that the fragment in Fig. 4 can be extended as shown in Fig. 8. For this we have to show that if $a*b = (c_1, c_2)$ then both sets $a*c_1$ and $a*c_2$ contain b. Since $a*a = (e, a')$, the associativity $a*(a*b) = (a*a)*b$ implies that $(a*c_1, a*c_2) = (b, b, a'*b)$. If both $a*c_1$ and $a*c_2$ contain b, then we are done. If not, then without loss of generality we can assume that $a*c_1 = (b, b)$. But then by our assumption either a or c_1 is the identity. If $a = e$, then $c_1 = c_2 = d_1 = d_2 = b$, which corresponds to the topograph. If $c_1 = e$, then $b = a^{-1} = a = d_1$ and $a*b = a*a = (e, a')$, so $c_2 = a'$. We have to show that $a*a'$ contains a.

Let $a*a' = (u, v)$ and multiply this by a:

$$a*(a*a') = (a*a)*a' = (e, a')*a' = (a', a', e, a''),$$

since a' is a weak involution. This means that either $a*u$ or $a*v$ must contain the identity, so either $u = a^{-1} = a$ or $v = a^{-1} = a$. □

This leads us naturally to the following notion.[2]

Definition 3.4. A commutative two-valued group X is said to be *involutive* if it consists of weak involutions.

All cosets G/τ, where G is an Abelian group and $\tau\colon x \to x^{-1}$, are examples of such groups, and we conjecture that (up to an isomorphism of two-valued groups) there are no other examples.

We prove this now in two of the most important cases of involutive groups with one or two generators. We assume that the commutativity and involutivity are the only relations in the group, leaving aside the case of additional relations.

[2] A similar terminology was used in [26] in a slightly more general situation; we hope that this will not cause special problems in our case.

This can be made more precise by considering the growth of the two-valued involutive groups X generated by some elements $a_1, \dots, a_n$.

Let $B(N)$ be the set of elements of X that appear in all possible products of generators of length not exceeding N, and define $S(N) = B(N) \setminus B(N-1)$. The sets $B(N)$ and $S(N)$ can be considered as the ball and sphere of radius N in X, respectively. We define the corresponding *volumes* $V_X^B(N)$ and $V_X^S(N)$ as the number of elements in $B(N)$ and $S(N)$, respectively.

Define the *degree* $\deg x$ of an element $x \in X$ to be the minimal N such that $x \in B(N)$. The sphere $S(N)$ consists of the elements of degree N.

Since $B(N) * B(M) \subset B(N+M)$, it follows that the degree satisfies the relation

$$\deg(x * y) \leqslant \deg x + \deg y, \tag{26}$$

where the degree of the product $x * y = (z_1, z_2)$ is defined as the maximum of the degrees of z_1 and z_2.

It is easy to see that the groups $\mathbb{X}_1$ and $\mathbb{X}_2$ have volumes

$$V_1^B(N) = N + 1, \qquad V_1^S(N) = 1,$$

and

$$V_2^B(N) = N^2 + N + 1, \qquad V_2^S(N) = 2N,$$

respectively.

We say that an involutive group X is *freely generated* by one or two generators if it has the same growth characteristics.

For such groups X the notion of the topograph can be defined more precisely as follows. Denote by $Y \subset X \times X$ the set of pairs (a, b) generating X, that is, every element $x \in X$ is contained in the product $a^k * b^l$ for some $k \geqslant 0$ and $l \geqslant 0$.

We call a triple (a, b, c) of elements a *Conway triple* if

$$(a, b) \in Y \quad \text{and} \quad c \in a * b.$$

From the proof of Proposition 3.3 it follows that this condition is symmetric with respect to a, b, and c.

Consider the graph $\Gamma(X)$ with vertices corresponding to Conway triples and with edges connecting the vertices labelled by triples with two common elements.

Definition 3.5. *The topograph $T(X)$ of the two-valued involutive group X* is an embedding of the graph $\Gamma(X)$ into the plane dividing it into domains labelled by elements of X in such a way that the labels in the three domains joining at a vertex form Conway triples (a, b, c), and the neighbours across edges correspond to pairs $(a, b) \in Y$.

We now show that the Conway topograph is universal in our class of two-valued groups.

Proposition 3.6. *The commutative involutive two-valued groups freely generated by one or two elements are isomorphic to the Buchstaber–Novikov group $\mathbb{X}_1$ or the Conway group $\mathbb{X}_2$, respectively.*

Such a group X admits a topograph $T(X)$ which in the case of two generators is isomorphic to the Conway topograph.

Proof. First consider a group X generated by one element a.

By definition the degree of an element $x \in X$ is the minimal power k of a such that $x \in a^k$. By our assumption there is precisely one element of degree $k \geqslant 0$, which we denote by a_k (we assume that $a_0 = e$ and $a_1 = a$). The degree of the product $x * y = (z_1, z_2)$ is defined as the maximum of the degrees of z_1 and z_2 and in our case satisfies the relation $\deg x * y = \deg x + \deg y$.

We first prove by induction on k that

$$a * a_k = (a_{k-1}, a_{k+1}). \tag{27}$$

For this we have to show that $a_{k-1} \in a * a_k$. This is certainly true for $k = 1$ since a is a weak involution: $a * a = (e, a_2)$.

Using associativity and the induction hypothesis, for $k \geqslant 2$ we have the two equal sets

$$a * (a * a_k) = (a * a) * a_k = (e, a_2) * a_k = (a_k, a_k, a_2 * a_k)$$

and

$$a * (a * a_k) = (a * (a_{k-1}, a_{k+1})) = (a * a_{k-1}, a * a_{k+1}) = (a_{k-2}, a_k, a * a_{k+1}).$$

This implies that indeed $a_k \in a * a_{k+1}$.

Similar arguments show that the elements a_k satisfy the multiplication rule

$$a_k * a_l = (a_{|k-l|}, a_{k+l}), \qquad k, l \in \mathbb{Z}_{\geqslant 0},$$

which finishes the proof in the one-generator case.

The topograph of the two-valued group $\mathbb{X}_1$ is shown in Fig. 9. In this case it is isomorphic to the so-called Euclid tree (see the right-hand side of Fig. 9), describing the Euclidean algorithm and the pairwise coprime triples x, y, z such that $z \in x * y$ in $\mathbb{X}_1$.

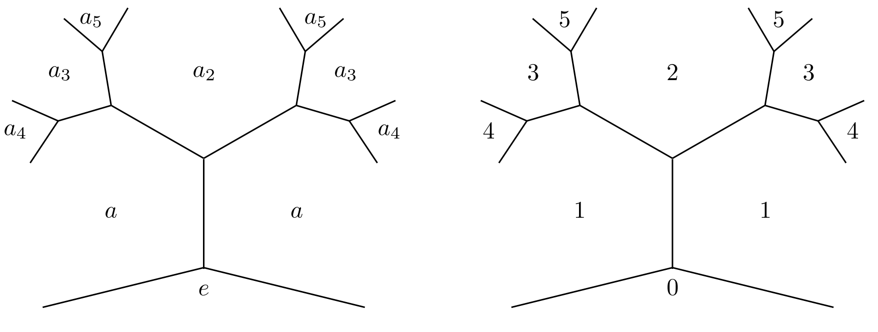

Figure 9. Topograph for $\mathbb{X}_1$ and Euclid tree.

We now sketch the proof in the case when X is freely generated by two elements a and b.[3]

[3] A. A. Gaifullin showed us a complete proof of this result without using the Conway topograph.

First note that, as in the previous case, we have uniquely defined elements a_k and b_l, and that any product $a^k * b^l := a * a * \cdots * a * b * \cdots * b$ contains only two elements, namely $a_k * b_l$, which can have maximal degree $N = k + l$. By our growth assumption they do indeed have degree $N = k + l$, and the sphere $S(N)$ consists of the $2N$ distinct elements a_N, b_N, and $a_k * b_l$, with $k + l = N$ and $k, l \geqslant 1$.

This can clearly be seen on the topograph of X shown in Fig. 10 next to the Conway superbase topograph (which is the same as the Conway topograph of the two-valued group $\mathbb{X}_2$). Namely, when we are inductively constructing the topograph of X, there will at each step be exactly two new elements which have not appeared before. Moreover, once we have made the choice of $c_+ = c_1$ and $c_- = c_2$ in the product $a * b = (c_1, c_2)$ at the first step, we uniquely determine the corresponding new elements $(a^k * b^l)_+$ and $(a^k * b^l)_-$ on the topograph (in Fig. 10 the sign subscripts are dropped).

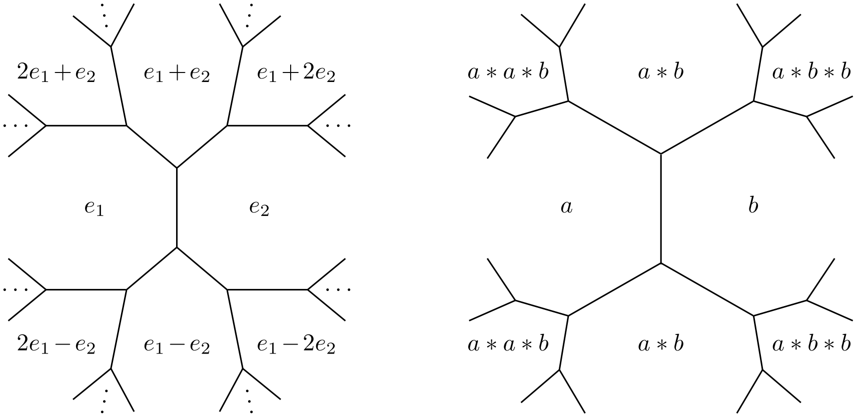

Figure 10. The Conway topograph and the topograph of the free involutive group X.

For example, in the product

$$a * a * b = (a * a) * b = (e, a_2) * b = (b, b, a_2 * b) = (a * c_+, a * c_-)$$

we have exactly two new elements $a_2 * b$. Depending on whether a new element belongs to $a * c_+$ or $a * c_-$, we denote it by $(a * a * b)_\pm$.

Since both the topographs in Fig. 10 are constructed by the same rule, we have an isomorphism of the topographs X and $\mathbb{X}_2$, with

$$x = (a^k * b^l)_\pm \in X \to u = ke_1 \pm le_2 \in \mathbb{X}_2 \tag{28}$$

for coprime k and l. This correspondence can be uniquely extended to an isomorphism of the two-valued groups X and $\mathbb{X}_2$, with

$$a_k * b_l \in X \to ke_1 \pm le_2 \in \mathbb{X}_2 \tag{29}$$

for all $k, l \in \mathbb{Z}$. $\square$

We finish this section with an interesting example from [15], showing that involutive commutative groups can be obtained from non-commutative groups by the coset-construction.

Consider the infinite dihedral group $G = \mathbb{Z}_2 * \mathbb{Z}_2$ with

$$G = \{a, b \mid a^2 = b^2 = e\}$$

and the involution σ of it interchanging a and b. Then

$$X = G/\sigma = \{u_{2n}, u_{2n+1}\}, \qquad n \geqslant 0,$$

where

$$u_{2n} = \{(ab)^n, (ba)^n\} \quad \text{and} \quad u_{2n+1} = \{b(ab)^n, a(ba)^n\}.$$

It is easy to check that

$$u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell|}],$$

and hence this group is isomorphic to the Buchstaber–Novikov group $\mathbb{X}_1$.

This does not contradict our conjecture that every involutive two-valued group is the quotient of an Abelian group by the involution $\tau\colon x \to x^{-1}$, since $G/\sigma \cong \mathbb{Z}/\tau$, but it shows that the classification of two-valued coset groups cannot be simply reduced to the classification of pairs (G, σ).

4. Quadratic forms and the orderings of the Conway group

We return to Conway's original motivation [32] of visualizing the values of binary quadratic forms (see Fig. 1). Using the isomorphism in the previous section, we can view quadratic forms Q as functions of the Conway two-valued group.

It is interesting that the parallelogram rule

$$Q(u+v) + Q(u-v) = 2(Q(u) + Q(v)), \qquad u, v \in \mathbb{Z}^2,$$

has a very natural interpretation from this point of view.

Recall that for ordinary groups X we have a Hopf algebra structure on the ring of functions $\mathscr{F}(X)$, with the diagonal homomorphism

$$\Delta\colon \mathscr{F}(X) \to \mathscr{F}(X \times X), \quad (\Delta f)(x, y) = f(x * y),$$

which has the properties of the generalised shift operator [69] and is a ring homomorphism, that is, $\Delta f_1 f_2 = (\Delta f_1)(\Delta f_2)$ for any two functions f_1 and f_2.

In the case of two-valued groups X it was shown in [11] that on the ring of functions $\mathscr{F}(X)$ the linear homomorphism $\Delta\colon \mathscr{F}(X) \to \mathscr{F}(X \times X)$ acting by the rule

$$(\Delta f)(x, y) = f(x * y) := \frac{1}{2}(f(z_1) + f(z_2)), \tag{30}$$

where $(z_1, z_2) = x * y$, also has the properties of the generalised shift operator.

In [23] the notion of a Hopf n-algebra was introduced, with the diagonal homomorphism Δ being an n-homomorphism, that is, the value $\Delta f_1 \cdots f_n$ is a polynomial in the values $\Delta f_{i_1} \cdots f_{i_k}$, $1 \leqslant k \leqslant n-1$, for any set of functions $f_1, \dots, f_n$. In particular, it was proved that for any two-valued group X there is a Hopf 2-algebra structure on the ring of functions $\mathscr{F}(X)$ with diagonal homomorphism (30) satisfying the identity

$$\Delta f_1 f_2 f_3 = \Delta f_1 \Delta f_2 f_3 + \Delta f_2 \Delta f_3 f_1 + \Delta f_3 \Delta f_1 f_2 - 2\Delta f_1 \Delta f_2 \Delta f_3.$$

A function $f(x)$ on the two-valued group X is called a *primitive element* of the Hopf 2-algebra $\mathscr{F}(X)$ (or simply said to be *primitive*) if

$$f(x * y) = f(z_1) + f(z_2),$$

where $(z_1, z_2) = x * y$, and it is said to be *multiplicative* if

$$f(x * y) = f(x) f(y).$$

For example, in the case of the two-valued group $\mathbb{X}_1$ the function αk^2 is primitive for any α, while the function $\cosh \alpha k$ is multiplicative.

The parallelogram rule for a quadratic form simply means that

$$Q(a * b) = Q(a) + Q(b), \qquad a, b \in \mathbb{X}_2, \tag{31}$$

so binary quadratic forms Q can be interpreted as primitive elements of the corresponding Hopf 2-algebra of the Conway two-valued group.

We can use this fact to embed the Conway group in $\mathbb{R}_{\geqslant 0}$ and to introduce on it a total group ordering in the following way.

Recall that a total ordering on an ordinary group G satisfies the group condition: if $f < g$ with $f, g \in G$, then for any $h \in G$

$$fh < gh \quad \text{and} \quad hf < hg. \tag{32}$$

Consider the map $f_Q \colon \mathbb{X}_2 \to \mathbb{R}$ given by the positive-definite quadratic form

$$Q(u) = x^2 + \sqrt{2}xy + \sqrt{3}y^2, \qquad u = \pm(x, y), \quad (x, y) \in \mathbb{Z}^2,$$

and consider its image $X_Q = f_Q(\mathbb{X}_2) \subset \mathbb{R}$. We say that an $a \in X_Q$ with $a \neq 0$ is primitive if there are no elements $b \in X_Q$ and $k \in \mathbb{Z}$ with $k > 1$ such that $a = k^2 b$.

Proposition 4.1. *The map $f_Q \colon \mathbb{X}_2 \to \mathbb{R}$ is injective and induces a total ordering on the two-valued group $\mathbb{X}_2$, where by definition*

$$u < v \iff Q(u) < Q(v).$$

This ordering satisfies the following analogue of the group condition (32): if $u < v$, then for any $w \in \mathbb{X}_2$

$$Q(u * w) < Q(v * w). \tag{33}$$

The equivalence class of the form Q with respect to transformations in $\mathrm{GL}_2(\mathbb{Z})$ (and hence the structure of a two-valued group on the set X_Q) is uniquely determined by the three smallest primitive elements of the subset $X_Q \subset \mathbb{R}$.

Proof. Let $u = \pm(x, y)$, $v = \pm(x', y') \in \mathbb{X}_2$. If $Q(u) = Q(v)$, then

$$x^2 + \sqrt{2}xy + \sqrt{3}y^2 = (x')^2 + \sqrt{2}x'y' + \sqrt{3}(y')^2.$$

Since 1, $\sqrt{2}$, and $\sqrt{3}$ are linearly independent over $\mathbb{Q}$, we must have

$$x^2 = (x')^2, \qquad xy = x'y', \quad \text{and} \quad y^2 = (y')^2,$$

which implies that $u = u'$.

The condition $u < v$ by definition means that $Q(u) < Q(v)$. Thus, for every $w \in \mathbb{X}_2$ we have

$$Q(u * w) = Q(u) + Q(w) < Q(v) + Q(w) = Q(v * w).$$

To prove the last assertion we use Conway's observation that the three minimal primitive values a, b, and c of a positive-definite form Q are its values on some superbasis corresponding to a well on the Conway topograph (see [32]). This enables one to uniquely reconstruct the equivalence class of Q from its representative

$$Q'(x, y) = ax^2 + (c - a - b)xy + by^2,$$

and hence to obtain the structure of a two-valued group on the subset $X_Q \subset \mathbb{R}$. $\square$

In the theory of ordered groups there is a famous theorem of Hölder asserting that groups with an Archimedean ordering can be embedded in $\mathbb{R}$ (see [30], for example). It would be interesting to develop a similar theory for two-valued groups, starting from our example.

Note also that the explicit form of the positive-definite quadratic form Q is not important, provided that its coefficients are independent over $\mathbb{Q}$. For embedding and ordering the group $\mathbb{X}_2$ one can also use indefinite forms Q, for example

$$Q(u) = x^2 + \sqrt{2}xy - \sqrt{3}y^2.$$

In this case we have the so-called *Conway river* separating the positive and negative values of Q (see [32], [90], and [91]).

We now consider Markov numbers from the point of view of the Conway topograph and the Conway two-valued group. These numbers appear in Markov triples, giving integer solutions of the Markov Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz.$$

Since for given x and y there are exactly two values z_1 and z_2 satisfying the Vieta relation

$$z_1 + z_2 = 3xy,$$

one can uniquely construct the corresponding Markov topographic tree by using this rule, starting from the initial triple $(1, 1, 1)$ and permuting the numbers in the triple (x, y, z).

The corresponding branch of the Markov tree starting from $(1, 2, 5)$ is shown in Fig. 11 next to a branch of the Farey tree for which at each vertex we have fractions $0 \leqslant k/l$, $m/n \leqslant 1/2$ and their Farey mediant

$$\frac{k}{l} * \frac{m}{n} = \frac{k+m}{l+n}.$$

We note that the irreducible fractions k/l are in a one-to-one correspondence with the primitive elements $\pm(k,l)$ in the Conway group, and the Farey mediant corresponds to addition in $\mathbb{Z}^2$. This gives a parametrisation of the Markov numbers, which actually goes back to Frobenius ([33], [45]).

Thus, the Markov numbers define on the primitive part of the Conway group $\mathbb{X}_2$ a function M which satisfies the relation

$$M(a*b)=\frac{3}{2}M(a)M(b) \tag{34}$$

for any a and b generating the whole group $\mathbb{X}_2$.

The Markov function $M(k/l)$ was considered by Fock in [42] as a component of a new function he introduced, which was studied further in [88] in connection with the Federer–Gromov stable norm.

The simple rescaling $m=3M/2$ satisfies the multiplicative relation

$$m(a*b)=m(a)m(b) \tag{35}$$

for such a and b.

Recall that in number theory a function f of a natural number argument is said to be *multiplicative* if it satisfies the relation

$$f(kl)=f(k)f(l)$$

for all coprime natural numbers k and l. A classical example is the Euler function $\varphi(k)$ giving the number of natural numbers not exceeding k and coprime to k. Thus, we can view the function m as an '*arithmetic multiplicative*' function on the Conway group $\mathbb{X}_2$.

We note that the same arguments can be used to get analogous functions for any $\mathrm{PSL}_2(\mathbb{Z})$-orbit for solutions of the Markov equation and of the modified Markov equation

$$x^2+y^2+z^2=Axyz+B$$

with any A and B, in particular, for the Mordell equation (9) with $A=2$ and $B=1$.

We will now show that Mordell's modification of the Markov equation is connected in a natural way with the two-valued algebraic group structure on $\mathbb{C}^*/\mathbb{Z}_2$.

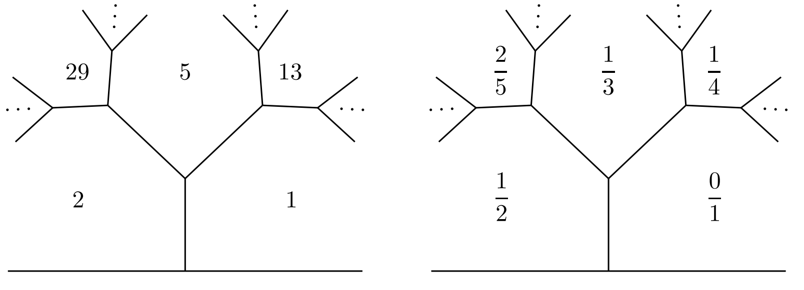

Figure 11. Correspondence between the Markov numbers and the Farey fractions.

5. Square modulus of a formal group, *K*-theory, and the Cayley cubic

The following construction was initiated by Novikov [78]. It was introduced in [20], found applications in algebraic topology, and played an important role in the creation of the theory of two-valued groups.

Let R be a commutative ring with unit. Depending on the context, it is enough to assume that $R=\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, or the polynomial rings over $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{C}$.

Let the equation

$$w-\Phi(u,v)=0, \tag{36}$$

where $\Phi(u,v)$ is a formal series, define a classical formal group law over the ring R.

This means that the series $\Phi(u,v)$ satisfies the following conditions:

1) $\Phi(u,0)=u$;
2) $\Phi(u,\Phi(v,w))=\Phi(\Phi(u,v),w)$.

It was shown by Abel [1] that the conditions 1) and 2) imply that there exists an invertible series $g(x)=x+\cdots$ such that

$$g(\Phi(u,v))=g(u)+g(v), \tag{37}$$

and in particular, $\Phi(u,v)=\Phi(v,u)$, so that multiplication is commutative. The series $g(x)$ is uniquely defined by the Abel relation

$$g'(u)\frac{\partial\Phi(u,v)}{\partial v}\bigg|_{v=0}=1 \tag{38}$$

and is called the *logarithm* of the corresponding formal group.

Consider the two-valued formal group with multiplication defined by the relation

$$F(x,y,z):=z^2-\Theta_1(x,y)z+\Theta_2(x,y)=0, \tag{39}$$

where $\Theta_1(x,y)=Z_++Z_-$ and $\Theta_2(x,y)=Z_+Z_-$, with

$$Z_+=\Phi(u,v)\Phi(\bar u,\bar v)=|\Phi(u,v)|^2 \quad\text{and}\quad Z_-=\Phi(\bar u,v)\Phi(u,\bar v)=|\Phi(\bar u,v)|^2.$$

Here we have $x=u\bar u$ and $y=v\bar v$, where $\bar u$ is the inverse of u, which is the series with $\Phi(u,\bar u)=0$. The two-valued group defined by (39) is called the 'square modulus' of the original formal group (36).

The equation (39) defines a formal two-valued commutative group law satisfying the associativity condition. The identity is $x=0$, and the inverse of x is x itself, since $\Theta_2(x,x)=0$.

This construction is non-trivial since Z_+ and Z_- are series in u and v, but not in x and y.

The equation (39) has the solutions in the set $\{\varphi_k(x),\ k=0,1,\dots\}$, where $\varphi_0(x)=0$, $\varphi_1(x)=x$, $\varphi_2(x)=\Theta_1(x,x)$, and then the series are recursively defined by the Vieta rule:

$$\varphi_{k+1}(x)=\Theta_1(x,\varphi_k(x))-\varphi_{k-1}(x), \qquad k\geqslant 3. \tag{40}$$

This determines a so-called power system of type 2, that is,

$$\varphi_k(x)=k^2x+\cdots, \qquad \varphi_k(\varphi_l(x))=\varphi_{kl}(x).$$

By definition $\varphi_k(\varphi_l(x)) = \varphi_l(\varphi_k(x))$, so that this power system consists of series commuting under substitution of a series into a series. In [20] it was shown that the set $\{\varphi_k(x),\, k = 0, 1, \dots\}$ of series has the structure of a two-valued group with the product

$$(\varphi_k * \varphi_l)(x) = \bigl(\varphi_{k+l}(x), \varphi_{|k-l|}(x)\bigr),$$

which is isomorphic to the Buchstaber–Novikov group $\mathbb{X}_1$ (see the details in [20]). On the topograph of the group $\mathbb{X}_1$ (see Fig. 9) the element a corresponds to the variable x, and the elements a_k correspond to the series $\varphi_k(x)$.

The square modulus construction has the following universal property with respect to the operation of substitution of a series into a series.

Theorem 5.1 (Buchstaber and Novikov [20]). *Let*

$$\{\varphi_k(x) \subset \mathbb{R}[[x]],\, k = 0, 1, \dots\},$$

be a set of series such that

$$\varphi_0(x) = 0, \qquad \varphi_1(x) = x, \quad \textit{and} \quad \varphi_k(\varphi_l(x)) = \varphi_{kl}(x).$$

If $\varphi_k(x) = k^2x + \cdots$, then there exists a formal group $w = \Phi(u, v)$ such that

$$\varphi_{k+1}(x) = \Theta_1(x, \varphi_k(x)) - \varphi_{k-1}(x), \qquad k \geqslant 2,$$

where $\Theta_1(x, y)$ is the coefficient of z in the equation

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$$

defining the two-valued group that is the square modulus of this formal group.

In particular, for the elementary formal group structure with $\Phi(u, v) = u + v$ connected with cohomology theory we have the two-valued group determined by

$$F(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 0; \tag{41}$$

this was the very first example of a two-valued group, appearing in the Buchstaber–Novikov paper [20]. In [10] and [11] it was shown that every two-valued formal group over R which is a $\mathbb{Q}$-module and such that $\Theta_2(x, x) = 0$ can be reduced to the equation (41) by a change of variables. This can be viewed as an analogue of our Proposition 3.6 in the one-generator case.

Since (41) coincides with (21), we see that in this case the group law $F = 0$ describes the quadratic forms Q with zero discriminant $D = 0$.

In K-theory the formal group is defined over the ring $\mathbb{Z}[a, a^{-1}]$, where a is an element such that multiplication by a in $K(X)$ is a Bott periodicity operator. In this case according to [19] we have

$$\Phi(u, v) = u + v - auv,$$

and the square modulus construction gives

$$F(x, y, z) := x^2 + y^2 + z^2 - 2xy - 2xz - 2yz - a^2xyz = 0. \tag{42}$$

Geometrically, this equation determines the classical *Cayley nodal cubic surface* [29], which is a cubic surface with maximal number (four) of conical singularities (see Fig. 3). Indeed, one can check that the singularities of the surface are conical and located at the four points

$$(0,0,0), \quad (-4a^{-2},-4a^{-2},0), \quad (-4a^{-2},0,-4a^{-2}), \quad (0,-4a^{-2},-4a^{-2}).$$

It is remarkable that the equation (42) is easily reduced to the Mordell equation (9), which is also known to determine the Cayley nodal cubic surface (see [92], for example).

Proposition 5.2. *The two-valued algebraic group law* (42) *and the Mordell equation* (9) *are connected by the change of variables*

$$x = \frac{1}{a^2}(2X+1), \qquad y = \frac{1}{a^2}(2Y+1), \qquad z = \frac{1}{a^2}(2Z+1). \tag{43}$$

It is interesting to note that $x = y = z = 0$ corresponds to $X = Y = Z = -1/2 = \cos(2\pi/3)$, which geometrically represents three coplanar unit vectors with pairwise angles $2\pi/3$.

See [92] for a study of the tropical version of the Cayley cubic and the corresponding $\mathrm{PSL}_2(\mathbb{Z})$-dynamics.

6. Two-valued algebraic groups and elliptic curves

Consider now the following family of two-valued algebraic groups, when the multiplication law (39) is given by $F(x,y,z) = 0$, where $F(x,y,z)$ is a polynomial which is symmetric in all three variables x, y, z and has degree 2 in each of them. We assume here that $x,y,z \in \mathbb{C}$, though much of this is valid over any field.

The equation $F(x,y,z) = 0$ determines a two-valued map (correspondence)

$$(x,y) \to z := x * y$$

which must satisfy the associativity relation $(x*y)*z = x*(y*z)$ in the sense of correspondences: after elimination of u and v the systems of equations

$$F(x,y,u) = 0, \qquad F(u,z,w) = 0,$$

and

$$F(y,z,v) = 0, \qquad F(x,v,w) = 0$$

determine the same set of $(x,y,z,w) \in \mathbb{C}^4$.

We assume also that $F(0,y,z) = (z-y)^2$, which means that 0 is the strong identity of the group.

We say that such two-valued groups are *symmetric algebraic*. Note that the symmetry condition implies that $F(x,y,0) = (x-y)^2$, which means that all such groups are automatically involutive.

In [13] one of the authors classified the two-valued algebraic groups coming from the square modulus construction for formal groups with the multiplication law suggested by the addition theorem for Baker–Akhiezer elliptic functions (see [68]).

The main result of that classification is the following family depending on three arbitrary parameters[4] a_1, a_2, and a_3:

$$F := (x + y + z - a_2xyz)^2 - 4(1 + a_3xyz)(xy + xz + yz + a_1xyz) = 0, \qquad (44)$$

which was proved to be universal in this class. The proof reduces to the solution of a functional equation whose general solution was shown to be expressible in terms of elliptic functions [13].

The parameters in the general form (44) are connected with the standard Weierstrass elliptic parameters g_2, g_3 and a point α on the corresponding elliptic curve $v^2 = 4u^3 - g_2u - g_3$ by

$$a_1 = 3\wp(\alpha), \qquad a_2 = 3\wp(\alpha)^2 - \frac{g_2}{4}, \qquad a_3 = \frac{1}{4}(4\wp(\alpha)^3 - g_2\wp(\alpha) - g_3), \qquad (45)$$

where $\wp$ is the Weierstrass elliptic $\wp$-function satisfying the differential equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

As a corollary of the results of [13] we have the following.

Theorem 6.1. *The multiplication law $F(x, y, z) = 0$ with F of the form (44) with parameters (45) can be reduced to $X \pm Y \pm Z = 0$ by the change of variables*

$$x = (\wp(X) + \wp(\alpha))^{-1}, \qquad y = (\wp(Y) + \wp(\alpha))^{-1}, \qquad z = (\wp(Z) + \wp(\alpha))^{-1}. \quad (46)$$

The corresponding two-valued group is the coset group $\mathscr{E}/\sigma = \mathbb{CP}^1$, where $\mathscr{E}$ is the elliptic curve determined by the equation

$$v^2 = P(u) := u^3 + a_1u^2 + a_2u + a_3,$$

and σ is the involution $v \to -v$.

In the degenerate cases when the roots of the polynomial $P(u)$ coincide, the coset groups $\mathbb{C}^*/\sigma$ and $\mathbb{C}/\sigma$ are obtained, where $\sigma(z) = z^{-1}$ and $\sigma(z) = -z$, respectively.

The proof is based on the addition formula for the Weierstrass elliptic functions

$$\left(xy + xz + yz + \frac{1}{4}g_2\right)^2 = 4(x + y + z)\left(xyz - \frac{1}{4}g_3\right), \qquad (47)$$

with $x = \wp(X)$, $y = \wp(Y)$, and $z = \wp(X \pm Y)$ (see Example 10 in Chap. XX of Whittaker and Watson [103]), and on the following lemma.

Lemma 6.2. *The family (44) is invariant under the Möbius transformation*

$$\tilde{x} = \frac{x}{1 + bx}, \qquad \tilde{y} = \frac{y}{1 + by}, \qquad \tilde{z} = \frac{z}{1 + bz},$$

which reduces to the change of parameters

$$\tilde{a}_1 = a_1 + 3b, \qquad \tilde{a}_2 = a_2 + 2a_1b + 3b^2, \qquad \tilde{a}_3 = a_3 + a_1b^2 + a_2b + b^3. \qquad (48)$$

[4] Our parameter a_3 is a quarter of the parameter a_3 in [13]. We changed it to make the formulae a bit more natural.

The elliptic family (44) includes all the cases of two-valued groups which have appeared from topology: the usual cohomology case (41) corresponds to $a_1 = a_2 = a_3 = 0$, the K-theory case (42) corresponds to $a_2 = a_3 = 0$, and the elliptic cohomology case (47) corresponds to the condition $a_1 = 0$.

We will now show that the same family (44) gives a classification of all symmetric two-valued algebraic groups.

Examples of such groups are given by the elliptic family (44). We prove now that they actually give the complete list.

Theorem 6.3. *For a symmetric polynomial $F(x,y,z)$ with $F(x,y,0) = (x-y)^2$ the equation $F(x,y,z) = 0$ defines a two-valued multiplication law if and only if F is of the form* (44).

Proof. We start with the following lemma, which can be easily checked.

Lemma 6.4. *Any symmetric polynomial $F(x,y,z)$ with $F(x,y,0) = (x-y)^2$ has the form*

$$F(x,y,z) = x^2 + y^2 + z^2 - 2(xy + xz + yz) - A_1(x,y)z - B_1(x,y)z^2, \tag{49}$$

where

$$A_1(x,y) = c_1 xy + c_2(x+y)xy + c_3 x^2y^2 \tag{50}$$

and

$$B_1(x,y) = c_2 xy + c_3(x+y)xy + c_4 x^2y^2. \tag{51}$$

Using this lemma, we can rewrite the equation $F(x,y,z) = 0$ as

$$z^2 - \Theta_{1,1}(x,y)z + \Theta_{2,1}(x,y) = 0,$$

where

$$\Theta_{1,1}(x,y) = \frac{2(x+y) + A_1(x,y)}{1 - B_1(x,y)} \quad \text{and} \quad \Theta_{2,1}(x,y) = \frac{(x-y)^2}{1 - B_1(x,y)}.$$

We compare this with the corresponding formulas for the family (44):

$$z^2 - \Theta_{1,2}(x,y)z + \Theta_{2,2}(x,y) = 0,$$

where

$$\Theta_{1,2}(x,y) = \frac{2(x+y) + A_2(x,y)}{1 - B_2(x,y)}, \qquad \Theta_{2,2}(x,y) = \frac{(x-y)^2}{1 - B_2(x,y)},$$
$$A_2(x,y) = 4a_1 xy + 2a_2(x+y)xy + a_3 x^2y^2,$$

and

$$B_2(x,y) = 2a_2 xy + a_3(x+y)xy + (a_1a_3 - a_2^2)x^2y^2.$$

If we put $c_1 = 4a_1$, $c_2 = 2a_2$, and $c_3 = a_3$, then we see that $A_1(x,y) = A_2(x,y)$. To complete the proof we have to show that $c_4 = a_1a_3 - a_2^2$, which is indeed the case and follows from the associativity property. We can show this without direct calculations using the following result from [11], Corollary 6.5.

Theorem 6.5 (Buchstaber [11]). *Let the equations*

$$z^2 - \Theta_{1,q}(x,y)z + \Theta_{2,q}(x,y) = 0, \qquad q = 1,2, \tag{52}$$

define two-valued algebraic groups. If $\Theta_{2,q}(x,x) \equiv 0$, $q = 1,2$, *then these two groups coincide if and only if*

$$\frac{\partial}{\partial y}(\Theta_{1,1}(x,y) - \Theta_{1,2}(x,y))\Big|_{y=0} \equiv 0.$$

In our situation this implies that $c_4 = a_1a_3 - a_2^2$ and $B_1(x,y) = B_2(x,y)$. This completes the proof of Theorem 6.3. □

It is instructive to compare this to the classical addition formula for the Jacobi elliptic function $\mathrm{cn}(u,k)$ (see [103]) in the following form:

$$X^2 + Y^2 + Z^2 = 1 + 2XYZ - k^2(1-X^2)(1-Y^2)(1-Z^2), \tag{53}$$

where $X = \mathrm{cn}(u,k)$, $Y = \mathrm{cn}(v,k)$, and $Z = \mathrm{cn}(u \pm v, k)$. One can check that the change $x = X+1$, $y = Y+1$, $z = Z+1$ reduces equation (53) to the equation (44) with

$$a_1 = \frac{1}{2} + 2k^2, \qquad a_2 = -2k^2, \qquad a_3 = \frac{1}{2}k^2. \tag{54}$$

Note that in the trigonometric limit $k \to 0$ of (53) we have precisely the Mordell equation.

From the algebro-geometric point of view the equation (44) defines the affine part of the special Kummer surface $\mathrm{Km}(\mathscr{E} \times \mathscr{E}) = \mathscr{E} \times \mathscr{E}/\pm I$ of the square of the elliptic curve $\mathscr{E}$. Indeed, from the parametrisation (46) we see that it is the quotient of the surface defined in $\mathscr{E} \times \mathscr{E} \times \mathscr{E}$ by $z_1 + z_2 + z_3 = 0$, $z_i \in \mathscr{E}$, by the involution $z_i \to -z_i$, $i = 1,2,3$.

It is known that, like the Cayley cubic surface, the Kummer surfaces have the maximal number (which is 16) of nodal singularities among quartic surfaces in $\mathbb{P}^3$. Figure 12 shows the `Mathematica` image of the real affine version of the surface (44) corresponding to the lemniscatic elliptic curve $v^2 = u(u^2-1)$:

$$(x + y + z + xyz)^2 - 4(xy + xz + yz) = 0.$$

In the figure we can see seven singular points, located at the coordinates

$$(0,0,0),\ (-1,1,0),\ (1,1,0),\ (0,1,1),\ (0,-1,1),\ (1,0,1),\ (-1,0,-1).$$

General Kummer surfaces corresponding to the Jacobi varieties of genus-2 hyperelliptic curves were recently discussed by Buchstaber and Dragovich [16] in connection with two-valued groups and integrable billiards. The corresponding two-valued group structure on a Kummer surface embedded in $\mathbb{P}^3$ was described explicitly in terms of the addition laws for Abelian functions of genus 2 (Klein's $\wp$-functions) [18].

In our case the natural compactification space for the surface (44) is not $\mathbb{P}^3$ but $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where the surface is defined by a $(2,2,2)$-form.

In the generic case such a form defines a $K3$ surface with the action of the group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ generated by three Vieta involutions. This action was studied by Baragar [5], who showed in particular that the existence of an infinite integral orbit is an obstruction to the smoothness of the surface 'at infinity' (see Theorem 5.1 in [5]).

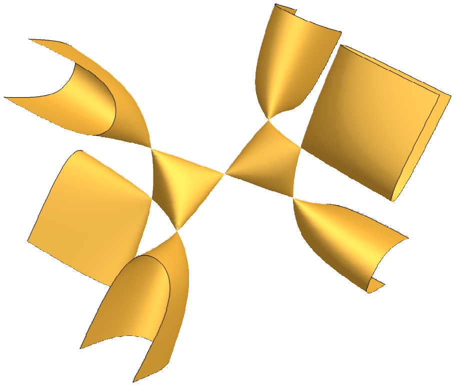

Figure 12. Real surface $F = 0$ with $a_1 = a_3 = 0$, and $a_2 = -1$.

7. Integrability in $\mathbf{PGL_2}(\mathbb{Z})$-dynamics

The notion of the integrability in the dynamics of maps is still quite far from a clear definition.

Several approaches to this problem were proposed in [95] and [96]; in one of them integrability was defined as the existence of suitable commuting maps. This worked well for polynomial maps and led to interesting examples of such maps connected with simple Lie algebras [94].

The motivation was from classical results of Julia and Fatou in the 1920s. They showed that for polynomial maps $z \to f(z), z \in \mathbb{C}$, the existence of a non-trivial commuting polynomial map $z \to g(z)$ is very restrictive, essentially reducing the list to $f = z^m$, $g = z^n$, and $f = T_m(z)$, $g = T_n(z)$, where the $T_m(z)$ are the Chebyshev polynomials. The classification of commuting rational maps of $\mathbb{CP}^1$ was done by Ritt and related to the multiplication laws for elliptic functions and complex multiplications (see the details and references in [96]).

It is interesting that sequences of power series commuting with respect to substitution play a key role in the Buchstaber–Novikov theory of power systems [20]. This theory is closely related to the theory of the two-valued formal groups obtained by the square modulus construction (see § 5 above). In the case of two-valued algebraic groups this leads to algebraic commuting maps defined by the recursion (40). It would be interesting to explore further the connection between commuting maps and the theory of two-valued groups, including the possible connection between elliptic curves with complex multiplication and generalised Adams–Novikov operations in cobordism theory (see [17]).

For polynomial correspondences $\Phi(x, y) = 0$ between the variables x and y, an alternative approach was proposed in [97], based on the slow (polynomial) growth of the number of images. A classical example is the *Euler–Chasles correspondence*

$$\Phi(x, y) = Ax^2y^2 + Bxy(x + y) + C(x^2 + y^2) + Dxy + E(x + y) + F = 0,$$

where instead of exponential growth we have linear growth of the number of images. Darboux used this fact to prove the classical Poncelet porism in geometry [34].

In [27] we showed that this correspondence has a natural interpretation in the theory of two-valued groups, namely, it defines an algebraic representation of the Buchstaber–Novikov group $\mathbb{X}_1$. Moreover, we showed that all algebraic actions of $\mathbb{X}_1$ on $\mathbb{P}^1$ are defined either by the Euler–Chasles correspondence, or by the reducible correspondence

$$\Phi(x,y)=(x-y)(Axy+B(x+y)+C)=0.$$

This led to the problem of classifying all commutative two-valued groups with one generator, which still remains largely open (see some steps in this direction in [27] and [23]).

However, already in [96] it was pointed out that there are interesting explicit examples of commuting correspondences with exponential growth of the number of images: *modular correspondences* $\Phi^J_n(x,y)=0$ determining the multiplication law of the modular function $x=J(z)$, $y=J(nz)$. In the simplest non-trivial case $n=2$ it has the form

$$\begin{aligned}\Phi^J_2(x,y)=x^3+y^3-x^2y^2+2^4\cdot 93xy(x+y)-2^4\cdot 3^4\cdot 5^3(x^2+y^2)\\ +3^4\cdot 5^3\cdot 4027xy+2^8\cdot 3^7\cdot 5^6(x+y)-2^{12}\cdot 3^9\cdot 5^9=0.\end{aligned}$$

Modular correspondences Φ^J_m and Φ^J_n with coprime m and n commute, so they should be regarded as integrable. However, we have shown that they cannot be included in an algebraic action of any multivalued group and thus still need to be analysed from the group point of view.

In this paper we have considered a different type of multivalued dynamics determined by an action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$. These actions are all particular cases of the following general situation.

Let $P(x,y,z)$ be a polynomial which is symmetric in x, y, and z and has degree 2 in each variable. Then $\mathrm{PSL}_2(\mathbb{Z})$ acts in a natural way on the surface $P(x,y,z)=0$ by Vieta involutions and cyclic permutations of the variables. As Conway explained in [32], the topograph gives a nice description of the $\mathrm{PSL}_2(\mathbb{Z})$-dynamics, which thus depends on the choice of the path on the Conway tree (and therefore is multivalued).

However, the action of each element of the group is well-defined, and the corresponding dynamics in the case of the generalised Markov equation

$$P(x,y,z)=x^2+y^2+z^2-3xyz-D \tag{55}$$

was studied by Cantat and Loray [28] and by Iwasaki and Uehara [59], [60] in connection with the classical Painlevé-VI equations.

Theorem 7.1 (Cantat and Loray [28]). *The topological entropy of the action of an element $A\in\mathrm{SL}_2(\mathbb{Z})$ on the complex surface $P(x,y,z)=0$ is equal to the logarithm of the spectral radius of the matrix A.*

The proof is based on the calculation in Mordell's 'integrable' case and on the fact that topological entropy is invariant under deformations.

Corollary 7.2 (Spalding and Veselov [89]). *The growth of the average topological entropy of the modular group action on the surface $P(x,y,z)=0$ as a function on the path γ_ξ on the Conway topograph is given by the function $\Lambda(\xi)$ described in* § 2 *above.*

We note that the corresponding action of the modular group preserves the symplectic form on the surface $P(x, y, z) = 0$ given as the Poincaré residue of the form

$$\omega = \frac{dx \wedge dy \wedge dz}{dP(x, y, z)}.$$

This means that one can ask about the usual (Liouville) integrability of the corresponding dynamics [96]. In this case an additional integral is needed but can be shown not to exist, so the dynamics is not integrable in this sense, in agreement with the positiveness of the topological entropy.

However, the question remains as to whether this dynamics may be 'integrable' in some other sense. The precise definition of integrability here is crucial, of course.

In view of our results the following definition seems to be natural.

Let us say that an equation $P(x, y, z) = 0$ is *linearisable* if there exists a change of variables $x = \psi(X)$, $y = \psi(Y)$, $z = \psi(Z)$ with an even meromorphic function $\psi(u)$ that reduces $P(x, y, z) = 0$ to the equation $X \pm Y \pm Z = 0$.

Proposition 7.3. *The equation* (55) *is linearisable if and only if* $D = 4/9$, *which corresponds to Zagier's modification* (7) (*which is equivalent to the Mordell equation* (9)).

Proof. We can use the classification from the previous section, but it is easier to prove this directly. Indeed, assume first that $Z = 0$ is not a pole of $\psi(Z)$ and consider $a = \psi(0)$. The substitution of $z = a$ into (55) gives the equation

$$x^2 + y^2 + a^2 - 3axy - D = 0,$$

which must be equivalent to $X \pm Y = 0$, and hence to $(x - y)^2 = 0$. Thus, we have $3a = 2$ and $a^2 = D$, so $a = 2/3$ and $D = 4/9$. In this case the linearisation is given by (8).

If $Z = 0$ is a pole of ψ, then we should first make the change $x \to x^{-1}$, $y \to y^{-1}$, $z \to z^{-1}$ in (55) and then repeat the same arguments. □

In particular, this means that the Markov equation with $D = 0$ is not linearisable, and thus leaves open the question of integrability of the corresponding dynamics.

In this context the following parametrisation found by Fock ((61) and (62) in [42]) of the generalised Markov cubic

$$x^2 + y^2 + z^2 - 3xyz - D = 0$$

could be useful:

$$\begin{aligned} x &= \frac{1}{3}(e^{u+v} + e^{u-v} + e^{-u-v}), \\ y &= \frac{1}{3}(e^{v+w} + e^{v-w} + e^{-v-w}), \\ z &= \frac{1}{3}(e^{w+u} + e^{w-u} + e^{-w-u}), \end{aligned}$$

with $D = (e^{u+v+w} - e^{-u-v-w})^2/9$. In particular, when $u + v + w = 0$ we have $D = 0$ and a nice parametrisation of the Markov equation.

We mention an intriguing connection between Markov dynamics and enumerative geometry. Let N_d be the number of rational curves in $\mathbb{P}^2$ passing through $3d-1$ points in general position. Kontsevich and Manin [67] showed that a certain generating function of these numbers satisfies the associativity equation and thus the Painlevé-VI equation with very special parameters. Dubrovin [38] showed that the initial data for the corresponding solution is none other than the orbit consisting of Markov triples!

However, the problem is that the theory of the Painlevé-VI equation (see Watanabe [101]) suggests that for these special parameters there are no special solutions in general. We cannot disagree with Manin [73] that from many points of view this is probably the most 'special' solution of the Painlevé-VI equations, but how to make this precise within the theory of the Painlevé equations is still an open question.

We note that the 'integrable' Mordell equation corresponds to the classical Picard solutions of another special Painlevé-VI equation (see Proposition 51 in [70]). Eremenko, Gabrielov, and Hinkkanen [40] recently proved that this special case is the only one where all solutions are exceptional (in some precise sense), which agrees with our results.

We mention here also the interesting paper [50] by Grinevich and Novikov, who argued in favour of the 'hidden integrability' of the special solution of the Painlevé-I equation appearing in quantum gravity and matrix models.

The Conway topograph can also be used to describe the orbits on the generalisations of Markov cubics

$$x^2 + y^2 + z^2 + Kxyz + Ax + By + Cz + D = 0 \tag{56}$$

in the Painlevé-VI theory ([28], [70]). In this case there is no permutation symmetry, but there is an action of the group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ generated by the three Vieta involutions in each variable. This group can be realised as the subgroup $\Lambda \subset \mathrm{PGL}_2(\mathbb{Z})$ of index 6 generated by the matrices

$$\alpha_1 = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \alpha_3 = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}. \tag{57}$$

The action of this group generates the Farey tesselation of the hyperbolic plane by ideal triangles (see Fig. 6). Its action on the absolute consists of three orbits corresponding to the three vertices of a triangle, which explains why the Conway topograph can be used to describe the orbits of Λ on the non-symmetric surface (56).

Due to a general result of Panov and Veryovkin [79] the commutator subgroup Λ' of this subgroup is freely generated by the five matrices

$$K_{ij} = K(\alpha_i, \alpha_j), \qquad 1 \leqslant i < j \leqslant 3,$$
$$K_{123} = K(K_{12}, \alpha_3) \quad \text{and} \quad K_{231} = K(K_{23}, \alpha_1),$$

where $K(\alpha, \beta) = \alpha\beta\alpha^{-1}\beta^{-1}$ is the commutator of the elements α and β :

$$K_{12} = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}, \qquad K_{23} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}, \qquad K_{13} = \begin{pmatrix} 5 & 4 \\ -4 & -3 \end{pmatrix},$$
$$K_{123} = \begin{pmatrix} -71 & -40 \\ 16 & 9 \end{pmatrix}, \qquad K_{231} = \begin{pmatrix} -7 & -16 \\ -24 & -55 \end{pmatrix}.$$

These matrices generate the principal congruence subgroup $\Gamma(4)$ of matrices in $\mathrm{SL}_2(\mathbb{Z})$ that are congruent to the identity modulo 4, which features prominently in the theory of classical theta-functions (see Mumford [77]).

It is interesting that the quotient $\mathbb{H}^2/\Lambda'$ is the sphere with six punctures at the vertices of the regular octahedron (recall that for the commutator subgroup of $\mathrm{SL}_2(\mathbb{Z})$ the analogous quotient is a special punctured torus connected with Markov triples).

8. Discussion

Aside from its well-known arithmetic and group-theoretic characteristics and its intimate connection with the Conway topograph, the modular group $\mathrm{PSL}_2(\mathbb{Z})$ is also very special from the topological point of view. Namely, it is isomorphic to the quotient group B_3/Z, where B_3 is the braid group and Z is its centre, which is isomorphic to $\mathbb{Z}$.

Recall that the braid group B_3 has standard generators σ_1 and σ_2 with the defining relation

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2.$$

The centre Z is generated by the element $(\sigma_1\sigma_2)^3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 = (\sigma_1\sigma_2\sigma_1)^2$. This connects $\mathrm{PSL}_2(\mathbb{Z})$-dynamics with the theory of Yang–Baxter maps (see [37], [14], [98], and [99]), which arose from the theory of integrable systems and quantum groups.

Many interesting arithmetic aspects of the corresponding multivalued $\mathrm{PSL}_2(\mathbb{Z})$-dynamics can be found in Silverman's monograph [86] and in the paper [5] by Baragar mentioned above on the dynamics on $K3$-surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ given by a $(2, 2, 2)$-form.

The arithmetic of the Kummer surfaces $\mathrm{Km}(\mathscr{E}_1 \times \mathscr{E}_2)$ of the product of two elliptic curves was studied by Skorobogatov and Swinnerton-Dyer [87] in connection with the Brauer–Manin obstruction. Very interesting arithmetic aspects of the $\mathrm{PGL}_2(\mathbb{Z})$-action on the Markov surface were considered in the recent paper [80] by Rehmann and Vinberg.

It is interesting that the special case $\mathrm{Km}(\mathscr{E} \times \mathscr{E})$ recently appeared in Gaifullin's paper [46] on the classification problem of flexible cross-polytopes in spaces of constant curvature (see also the important work by Izmestiev ([61] and [62]) in this direction).

The integrability aspects of arithmetic dynamics was discussed by Halburd, who introduced the notion of Diophantine integrability [53] (see also Hone's paper [57]).

Among the more recent results we would like to mention also the very interesting paper [8] by Bourgain, Gamburd, and Sarnak, who argued in favour of the *Strong Approximation Conjecture* asserting that all the non-zero solutions of the Markov equation modulo a prime p also form a single $\mathrm{PGL}_2(\mathbb{Z})$-orbit.

We would like to add that Markov triples have appeared unexpectedly in various problems in algebraic geometry, in particular, as the ranks of exceptional vector bundles over $\mathbb{P}^2$ in Rudakov [82] and in the Hacking–Prokhorov classification of del Pezzo surfaces with quotient singularities [52].

We mention also the paper [56] by Hirzebruch, who discovered a remarkable connection between Markov triples and the equivariant signature of manifolds.

As we have just explained, the Conway topograph can be used to 'vizualise' the ranks of triples of exceptional vector bundles on del Pezzo surfaces, which are described by a non-symmetric generalisation of the Markov equation of the following type:

$$ax^2 + by^2 + cz^2 = dxyz$$

(see Karpov–Nogin [63]). In particular, for $\mathbb{P}^1 \times \mathbb{P}^1$ the ranks x, y, z of the exceptional triples satisfy the Diophantine equation

$$x^2 + y^2 + 2z^2 = 4xyz$$

and can be found from the solution $(1, 1, 1)$ by the action of the group $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$ generated by the Vieta involutions

$$(x, y, z) \to (x, y, 2xy - z), \quad (x, y, z) \to (x, 4xz - y, z), \quad (x, y, z) \to (4yz - x, y, z)$$

(see Rudakov [83]). The corresponding block structure in superconformal quiver gauge theory was studied by Hanany and others in [55].

The Conway topograph can also be used to describe the mutations of rank-3 quivers (see Felikson–Tumarkin [41] and references therein). Similar mutations appeared in the theory of fake projective planes in [3] and [4].

Finally, we mention the recent very interesting paper [75] by Milea, Shelley, and Weissman, who used the fact that $\mathrm{PGL}_2(\mathbb{Z})$ is isomorphic to the Coxeter group of type $(3, \infty)$ to introduce generalisations of the Conway topograph for other arithmetic Coxeter groups.

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