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The Lauricella hypergeometric function $F_D^{(N)}$, the Riemann–Hilbert problem, and some applications

S. I. Bezrodnykh

Abstract. The problem of analytic continuation is considered for the Lauricella function $F_D^{(N)}$, a generalized hypergeometric functions of N complex variables. For an arbitrary N a complete set of formulae is given for its analytic continuation outside the boundary of the unit polydisk, where it is defined originally by an N-variate hypergeometric series. Such formulae represent $F_D^{(N)}$ in suitable subdomains of $\mathbb{C}^N$ in terms of other generalized hypergeometric series, which solve the same system of partial differential equations as $F_D^{(N)}$. These hypergeometric series are the N-dimensional analogue of Kummer's solutions in the theory of Gauss's classical hypergeometric equation. The use of this function in the theory of the Riemann–Hilbert problem and its applications to the Schwarz–Christoffel parameter problem and problems in plasma physics are also discussed.

Bibliography: 163 titles.

Keywords: multivariate hypergeometric functions, systems of partial differential equations, analytic continuation, Riemann–Hilbert problem, Schwarz–Christoffel integral, crowding problem, magnetic reconnection phenomenon.

Contents

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1. Introduction

1.1. The Lauricella function $F_D^{(N)}$. Hypergeometric functions of two or more variables arise in many areas of modern mathematics, and they enable one to solve constructively many topical problems important for theory and applications. The basis for the theory of such functions was laid in [1]–[6] at the end of the 19th century, and it was further developed by a number of well-known authors (for instance, see the original papers and monographs [7]–[29]). We should note the significant progress made in the general theory of hypergeometric functions of several variables. Particular functions in this class that are interesting in their own right have also traditionally been objects of great attention.

In this paper we consider the function $F_D^{(N)}(a_1,\dots,a_N;b,c;z_1,\dots,z_N)$ introduced by Lauricella [6] (see also [13], [23], [27]) as one of the most natural generalizations of the Gauss hypergeometric function $F(a,b;c;z)$ to the case of N complex variables $(z_1,\dots,z_N) =: \mathbf{z} \in \mathbb{C}^N$ and complex parameters $(a_1,\dots,a_N) =: \mathbf{a} \in \mathbb{C}^N$, b, and c. Recall that the Gauss function (of a single complex variable z) is defined by the series

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \tag{1.1}$$

which converges in the unit disk $\mathbb{U} := \{z \in \mathbb{C} \colon |z| < 1\}$. Outside $\mathbb{U}$ this function is an analytic continuation of (1.1). Here the expression $(a)_k$, called the Pochhammer symbol, is defined in terms of the gamma function $\Gamma(s)$ by

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{1.2}$$

For an integer $k \geqslant 0$ it is a product of the form

$$(a)_0 = 1, \quad (a)_k = a(a+1)\cdots(a+k-1), \quad k = 1,2,\dots .$$

It is assumed in (1.1) that the parameters a, b, and c can take arbitrary complex values, with the exception that c cannot be a non-positive integer ($c \notin \mathbb{Z}^-$).

The function $F(a,b;c;z)$ is a solution $u(z)$ of Gauss's equation

$$z(1-z)u''(z) + [c-(a+b+1)z]u'(z) - abu(z) = 0, \tag{1.3}$$

which is holomorphic at $z = 0$. This is an equation of Fuchs class with three (regular) singular points $z = 0$, 1, and ∞. There is a detailed presentation of the theory of the Gauss function and equation (1.3) in [30] and [31].

The Lauricella function, which we denote by $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ for brevity, is defined for $c \notin \mathbb{Z}^-$ by the N-variate hypergeometric series

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}|}(a_1)_{k_1}\cdots(a_N)_{k_N}}{(c)_{|\mathbf{k}|}k_1!\cdots k_N!} z_1^{k_1}\cdots z_N^{k_N}, \tag{1.4}$$

which converges in the unit polydisk $\mathbb{U}^N := \{\mathbf{z} \in \mathbb{C}^N \colon |z_j| < 1,\ j = 1,\dots,N\}$. The sum in (1.4) is taken over the multi-indices $\mathbf{k} := (k_1,\dots,k_N)$ with non-negative integer components $k_j \geqslant 0$, $j = 1,\dots,N$, and we define $|\mathbf{k}| := \sum_{j=1}^N k_j$.

The function $F_D^{(N)}$ satisfies the following system of N linear partial differential equations of second order with respect to the variables z_j (see [6], and also [13], [23], [27]):

$$\begin{aligned} &z_j(1-z_j)\frac{\partial^2 u}{\partial {z_j}^2} + (1-z_j)\sideset{}{'}\sum_{k=1}^{N} z_k \frac{\partial^2 u}{\partial z_j\,\partial z_k} \\ &\qquad + [c-(1+a_j+b)z_j]\frac{\partial u}{\partial z_j} - a_j \sideset{}{'}\sum_{k=1}^{N} z_k \frac{\partial u}{\partial z_k} - a_j b u = 0, \quad j = 1,\dots,N, \end{aligned} \tag{1.5}$$

where a prime on a summation sign means that the sum is taken for $k \neq j$, and the parameters $\mathbf{a}$, b, and c appear in the expressions for the coefficients of these equations. It is known [6], [13] that the general solution of (1.5) depends only on $N+1$ arbitrary complex constants, so the system is overdetermined. For short we will occasionally denote the Lauricella system of equations (1.5) by $E_D^{(N)}$. Its singular set $\mathscr{M}$ is the union of the hyperplanes

$$\mathscr{M}_j^{(\tau)} := \{\mathbf{z} \in \overline{\mathbb{C}}^N \colon z_j = \tau\},$$

where $\tau \in \mathscr{S} := \{0, 1, \infty\}$, and the hyperplanes $\mathscr{M}_{j,l} := \{\mathbf{z} \in \overline{\mathbb{C}}^N \colon z_j = z_l\}$; here $j, l = 1, \dots, N$, $l \neq j$, and the extended space $\overline{\mathbb{C}}^N$ is defined by $\overline{\mathbb{C}}^N = \overline{\mathbb{C}} \times \dots \times \overline{\mathbb{C}}$ (N factors).

Points in the singular set $\mathscr{M}$ that lie in the intersection of two or more of the above hyperplanes will be important in what follows. We let

$$\mathbf{z}_{p,q}^{(1,\infty,0)} := (\underbrace{1, \dots, 1}_{p}, \underbrace{\infty, \dots, \infty}_{q}, \underbrace{0, \dots, 0}_{N-p-q})$$

be the point in the singular set with the first p components equal to 1, the next q components equal to ∞, and the remaining $N-p-q$ components equal to 0. Furthermore, we let

$$\mathbf{z}_p^{(1,0)} := \mathbf{z}_{p,0}^{(1,\infty,0)} = (\underbrace{1, \dots, 1}_{p}, \underbrace{0, \dots, 0}_{N-p})$$

and

$$\mathbf{z}_q^{(\infty,0)} := \mathbf{z}_{0,q}^{(1,\infty,0)} = (\underbrace{\infty, \dots, \infty}_{q}, \underbrace{0, \dots, 0}_{N-q}).$$

Finally, let $\mathbf{z}^{(1)} := (1, \dots, 1)$ and $\mathbf{z}^{(\infty)} := (\infty, \dots, \infty)$ denote the points in $\mathscr{M}_j^{(\tau)}$ with all N components z_j equal to 1 or ∞, respectively. Note that, for example, $\mathbf{z}_p^{(1,0)}$ lies in the intersection of the hyperplanes $\mathscr{M}_j^{(1)}$ and $\mathscr{M}_{j,l}$ for $j, l = 1, \dots, p$, and of the hyperplanes $\mathscr{M}_j^{(0)}$ and $\mathscr{M}_{j,l}$ for $j, l = p+1, \dots, N$, where $l \neq j$. In turn, $\mathbf{z}^{(1)}$ lies in the intersection of the hyperplanes $\mathscr{M}_j^{(1)}$ for $j = 1, \dots, N$, and in the intersection of all the hyperplanes $\mathscr{M}_{j,l}$ for $j, l = 1, \dots, N$, where $l \neq j$.

If we look at the system (1.5) assuming that the function $u(z)$ in question is independent of $z_2, \dots, z_N$ and we set the corresponding parameters $a_2, \dots, a_N$ equal to 0, then the system reduces to Gauss's hypergeometric equation. This is in full agreement with the observation that in the case of one variable (that is, for $N = 1$) the Lauricella series (1.4) becomes the hypergeometric series for the Gauss function. We note also that in the case of two variables (for $N = 2$) the generalized hypergeometric series (1.4) has a special name, the Appell function, and is denoted by $F_1(a, a'; b, c; z, \zeta)$ (see [7], [30]).

We also present the integral representation ([13], p. 49),

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{\prod_{j=1}^N (1 - t z_j)^{a_j}} \, dt, \tag{1.6}$$

which holds in the domain

$$\mathbb{L}^N := \{\mathbf{z} \in \mathbb{C}^N \colon |\arg(1 - z_j)| < \pi,\ j = 1, \dots, N\},$$

where the right-hand side of (1.6) is a single-valued function, and it is assumed there that $\operatorname{Re} b > 0$ and $\operatorname{Re}(c - b) > 0$. In the case of one variable, the denominator of the integrand in (1.6) contains a single factor and (1.6) transforms into Euler's well-known formula for the Gauss function (see [30], [31]).

Introduced as a formal generalization of (1.1), the Lauricella function $F_D^{(N)}$ became one of the most commonly used representatives of the class of multivariate hypergeometric functions. The many papers devoted to investigating it (or its special cases for particular N or sets of parameters; see [10], [11], [13], [17], [23], [32], and others) have revealed its deep connections with algebra and partial differential equations. The interest in $F_D^{(N)}$ has also been stimulated by the numerous and diverse applications it has found. These applications include problems in astrophysics [33], quantum field theory [34], [35], relativistic mechanics [36], relativity theory [37], [38], as well as some problems in information transmission theory [39], probability theory and mathematical statistics [13], [40]–[42], modelling Brownian motion [43], string theory [44] and conformal field theory [45]–[47], calculation of Feynman diagrams [48]–[51], and the mechanics of deformable bodies [52]. We remark that many of the above applications are connected with the integral representation (1.6). It is easy to see that hyperelliptic integrals can be expressed in terms of the function $F_D^{(N)}$ with half-integer parameters $\mathbf{a} = (a_1, \dots, a_N)$, b, and c. Thus, the Lauricella function gives us yet another tool (in addition to multidimensional Θ-series [53]) for the study of such integrals. As concerns calculations of hyperelliptic integrals using $F_D^{(N)}$, see [54]–[56].

Let us now discuss the contents of our paper, where analytic continuation of the series (1.4) is one of the central questions. For the extended function we shall use the same notation $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$. We start with an integral representation of Mellin–Barnes type [7], [13]:

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \frac{\Gamma(c)}{(2\pi i)^N \Gamma(b) \prod_{j=1}^{N} \Gamma(a_j)} \times \int_{\mathscr{L}} \frac{\Gamma(b + |\mathbf{t}|)}{\Gamma(c + |\mathbf{t}|)} \biggl(\prod_{j=1}^{N} \Gamma(a_j + t_j)\Gamma(-t_j)(-z_j)^{t_j} \biggr) \, d\mathbf{t}, \qquad (1.7)$$

where $\mathbf{t} = (t_1, \dots, t_N)$, $|\mathbf{t}| = \sum_{j=1}^{N} t_j$, $d\mathbf{t} = dt_1 \cdots dt_N$, and $\mathscr{L} = \mathscr{L}_1 \times \cdots \times \mathscr{L}_N$, with $\mathscr{L}_j$ a standard contour in the t_j-plane which is a deformed imaginary axis, that is, it connects $-i\infty$ and $+i\infty$ but is curved so that among all the poles of the integrand only the poles of $\Gamma(-t_j)$ lie to the right of $\mathscr{L}_j$. Formally speaking, this representation, like the Euler representation (1.6), gives an analytic continuation of the series (1.4). However, (1.6) and (1.7) can in fact only be regarded as intermediate constructions, and the most adequate tools for the qualitative analysis and calculation of $F_D^{(N)}$ outside $\mathbb{U}^N$ are its representations by certain other generalized hypergeometric series converging in suitable subdomains of $\mathbb{C}^N \setminus \mathbb{U}^N$ and solving the system (1.5). We will call such representations of $F_D^{(N)}$ *formulae for analytic continuation*. We derive such formulae in § 2, where we present results from [57]–[61]. It was noted in [62] that an effective calculation of the Lauricella function $F_D^{(N)}$ outside $\mathbb{U}^N$ is a key aspect in solving the well-known problem of finding the parameters of the Schwarz–Christoffel integral. We consider this application of the theory of the function $F_D^{(N)}$ in § 5.

Section 3, which reproduces results in [63] and [64], is devoted to applications of the Lauricella function $F_D^{(N)}$ to the derivation of a new representation for the solution of the Riemann–Hilbert problem. This possibility has been opened by the Jacobi-type formula for $F_D^{(N)}$ found in [57], [65], [66]. In § 4 we show how these advances in the Riemann–Hilbert problem can be used in the solution of a particular problem of this type in a complicated domain arising in plasma physics; the results in § 4 mostly follow [67].

Before describing the central topics of this paper in greater detail (see §§ 1.3, 1.4, and 1.5), we consider the place of the Lauricella function $F_D^{(N)}$ in the general theory of hypergeometric functions of several variables.

1.2. Multivariate hypergeometric functions and systems of equations. According to Horn's approach [5], a power series

$$\chi(z_1,\dots,z_N)=\sum_{\mathbf{k}\in\mathbb{Z}^n}\Lambda(k_1,\dots,k_N)z_1^{k_1}\cdots z_N^{k_N},$$

or briefly

$$\chi(\mathbf{z})=\sum_{\mathbf{k}\in\mathbb{Z}^n}\Lambda(\mathbf{k})\mathbf{z}^{\mathbf{k}}, \tag{1.8}$$

is said to be hypergeometric if the ratio of any two adjacent coefficients is a rational function of the components of the summation index $\mathbf{k}:=(k_1,\dots,k_N)$, that is, for all $j=1,\dots,N$

$$\frac{\Lambda(\mathbf{k}+\mathbf{e}_j)}{\Lambda(\mathbf{k})}=\frac{P_j(\mathbf{k})}{Q_j(\mathbf{k})}\,, \tag{1.9}$$

where $P_j(\mathbf{x})$ and $Q_j(\mathbf{x})$ are some polynomials in the N variables $(x_1,\dots,x_N)=:\mathbf{x}$, and $\mathbf{e}_j:=(0,\dots,1,\dots,0)$ denote the vectors with jth component equal to 1 and the others equal to 0 (for instance, see [15], [20], [28]).

The general form of the coefficients $\Lambda(\mathbf{k})$ satisfying (1.9) is given by the Ore–Sato theorem [8], [22], which shows that $\Lambda(\mathbf{k})$ is a certain product of Γ-functions and a multiplier of the form

$$\lambda_1^{k_1}\cdots\lambda_N^{k_N}R(k_1,\dots,k_N) \tag{1.10}$$

(which has no crucial importance for the properties of the series), where R is a rational function and $\lambda_j\in\mathbb{C}$. The series $\chi(\mathbf{z})$ defined by (1.8) and (1.9) can be shown to solve the following system of partial differential equations [5], [7], [68]:

$$Q_j(\theta)\bigl(z_j^{-1}\chi(\mathbf{z})\bigr)=P_j(\theta)\chi(\mathbf{z}),\qquad j=1,\dots,N, \tag{1.11}$$

where the differential operators $P_j(\theta)$ and $Q_j(\theta)$ are obtained by substituting the components of the vector $\theta:=(\theta_1,\dots,\theta_N)$, $\theta_s:=z_s\,\partial/\partial z_s$, as the arguments of the polynomials P_j and Q_j in (1.9).

It is easy to see that the coefficients of the Lauricella series (1.4), which are given by

$$\Lambda(\mathbf{k}) = \frac{(b)_{|\mathbf{k}|}(a_1)_{k_1}\cdots(a_N)_{k_N}}{(c)_{|\mathbf{k}|}k_1!\cdots k_N!}, \tag{1.12}$$

satisfy the relations (1.9) for

$$P_j(\mathbf{x}) = (b+|\mathbf{x}|)(a_j+x_j) \quad \text{and} \quad Q_j(\mathbf{x}) = (c+|\mathbf{x}|)(1+x_j), \qquad |\mathbf{x}| = \sum_{l=1}^{N} x_l,$$

so that (1.4) belongs to the family of Horn hypergeometric series. The system (1.11) corresponding to such P_j and Q_j has the form

$$\biggl(c+\sum_{m=1}^{N}\theta_m\biggr)(1+\theta_j)\bigl(z_j^{-1}\chi(\mathbf{z})\bigr) = \biggl(b+\sum_{m=1}^{N}\theta_m\biggr)(a_j+\theta_j)\chi(\mathbf{z}), \qquad j=1,\dots,N.$$

Setting $\theta_s = z_s\,\partial/\partial z_s$, removing parentheses, and bearing in mind that

$$\biggl(1+z_j\frac{\partial}{\partial z_j}\biggr)\bigl(z_j^{-1}\chi(\mathbf{z})\bigr) = \frac{\partial}{\partial z_j}\chi(\mathbf{z}),$$

we arrive at (1.5), which therefore is a system in the class of Horn hypergeometric systems.

We note that the property (1.9) holds also for the other series F_A, F_B, and F_C introduced by Lauricella, and of course for their special cases, the Appell series, the Kampé de Feriet functions, and many other well-known hypergeometric series (see [7], [13], [15], [30]).

General hypergeometric functions can alternatively be defined as solutions of the hypergeometric $\mathfrak{A}$-systems of Gelfand, Kapranov, and Zelevinsky ([16], [18], [24]). A system of this type is defined by an $r\times M$ integer matrix $\mathfrak{A}=\{a_{sj}\}$ and a set of complex parameters $(b_1,\dots,b_r) =: \mathbf{b}\in\mathbb{C}^r$, where it is assumed that the columns of $\mathfrak{A}$ generate the lattice $\mathbb{Z}^r$ and that for some vector $(h_1,\dots,h_r)\in\mathbb{Z}^r$ we have $\sum_{s=1}^{r} h_s a_{sj} = 1$, $j=1,\dots,M$. The matrix $\mathfrak{A}$ is associated with the sublattice $\mathbb{L}\subset\mathbb{Z}^M$ defined by

$$\mathbb{L} := \biggl\{(g_1,\dots,g_M) =: \mathbf{g}\in\mathbb{Z}^M\colon \sum_{j=1}^{M} g_j a_{sj} = 0,\ s=1,\dots,r\biggr\}. \tag{1.13}$$

The Gelfand–Kapranov–Zelevinsky system consists of r first-order equations

$$\sum_{j=1}^{M} a_{sj}w_j\frac{\partial\psi(\mathbf{w})}{\partial w_j} = b_s\psi(\mathbf{w}), \qquad s=1,\dots,r, \tag{1.14}$$

and an infinite set of equations of order at most M:

$$\prod_{j:g_j>0}\biggl(\frac{\partial}{\partial w_j}\biggr)^{g_j}\psi(\mathbf{w}) = \prod_{j:g_j<0}\biggl(\frac{\partial}{\partial w_j}\biggr)^{-g_j}\psi(\mathbf{w}), \qquad \mathbf{g}\in\mathbb{L}, \tag{1.15}$$

each of which corresponds to an element $\mathbf{g} = (g_1, \ldots, g_M)$ of $\mathbb{L}$, where the product on the left-hand side of (1.15) involves only the positive components g_j of $\mathbf{g}$ and the one on the right-hand side involves the negative components. The unknown in (1.14) and (1.15) is a complex scalar function $\psi(\mathbf{w})$ of the vector-valued argument $(w_1, \ldots, w_M) =: \mathbf{w} \in \mathbb{C}^M$.

We note that one solution of (1.14), (1.15), that is, one generalized hypergeometric function in the sense of Gelfand, Kapranov, and Zelevinsky, is given by the (formal) power series

$$\psi(\mathbf{w}) = \sum_{\mathbf{g}\in\mathbb{L}} \prod_{j=1}^{M} \frac{w_j^{g_j+\gamma_j}}{\Gamma(1+g_j+\gamma_j)}, \tag{1.16}$$

where $\mathbb{L}$ is the lattice (1.13), and the vector $\gamma = (\gamma_1, \ldots, \gamma_M)$ is connected with the matrix $\mathfrak{A}$ and the parameter vector $\mathbf{b} = (b_1, \ldots, b_r)$ by the equalities

$$b_s = \sum_{j=1}^{M} a_{sj}\gamma_j, \qquad s = 1, \ldots, r. \tag{1.17}$$

The systems (1.14), (1.15) gave new impetus to the development of the multidimensional theory of hypergeometric functions in the 1980s. These systems and the corresponding series (1.16) were the subject of many investigations, some aspects of which were reflected in [69]–[71]. We remark that the systems arise in a natural way in the theory of algebraic equations [70] (see [28] for other applications). The systems (1.14), (1.15) are holonomic (have a finite number of linearly independent solutions) for a fairly general matrix $\mathfrak{A}$ with the above properties and an arbitrary set of parameters $\mathbf{b}$ (see [18]). Concerning the monodromy groups of these systems, see [71] and the literature cited there.

The Horn series (1.8) and the Gelfand–Kapranov–Zelevinsky series (1.16) are closely connected, as pointed out in [20]: each series of the form (1.16) can be represented as a product of a monomial and a Horn series whose coefficients do not contain 'non-essential' multipliers of the form (1.10). We can demonstrate this connection in the case of the Lauricella function $F_D^{(N)}$: to do this we reformulate its definition (1.4) in terms of a series (1.16) and find the corresponding system (1.14) (see also [70]). Using (1.2) and the equality

$$\frac{\Gamma(a+k)}{\Gamma(a)} = \frac{(-1)^k\Gamma(1-a)}{\Gamma(1-a-k)}\,, \tag{1.18}$$

we can easily rewrite (1.4) as

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \Gamma(c)\Gamma(1-b)\prod_{j=1}^{N}\Gamma(1-a_j)\biggl(\prod_{j=1}^{2(N+1)} w_j^{-\gamma_j}\biggr)\psi_D(\mathbf{w}), \tag{1.19}$$

where $\psi_D(\mathbf{w})$ is the series of form (1.16)

$$\psi_D(\mathbf{w}) := \sum_{\mathbf{g}\in\mathbb{L}_D}^{\infty} \prod_{j=1}^{2(N+1)} \frac{w_j^{g_j+\gamma_j}}{\Gamma(1+g_j+\gamma_j)}, \tag{1.20}$$

with $(\gamma_1, \ldots, \gamma_{2N+2}) =: \gamma$ the parameter vector defined in terms of $\mathbf{a}$, b, and c by

$$\gamma = (-b, c-1, -a_1, \ldots, -a_N, 0, \ldots, 0) \tag{1.21}$$

and with the sum taken over the lattice $\mathbb{L}_D$ generated by the rows of the $N \times (2N+2)$ matrix

$$\mathfrak{L}_D = \begin{pmatrix} -1 & 1 & -1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ -1 & 1 & 0 & -1 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & 0 \\ -1 & 1 & 0 & 0 & \ldots & -1 & 0 & 0 & \ldots & 1 \end{pmatrix}, \tag{1.22}$$

that is, any $\mathbf{g} \in \mathbb{L}_D$ has the form $\mathbf{g} = \mathbf{k}\mathfrak{L}_D$, where $\mathbf{k} = (k_1, \ldots, k_N) \in \mathbb{Z}^N$, and the variables $\mathbf{z} = (z_1, \ldots, z_N)$ and $\mathbf{w} = (w_1, \ldots, w_{2N+2})$ in (1.19) and (1.20) are connected by the relations

$$z_s = \frac{w_2 w_{N+s+2}}{w_1 w_{s+2}}, \qquad s = 1, \ldots, N. \tag{1.23}$$

Now we write out a Gelfand–Kapranov–Zelevinsky system satisfied by the series (1.20). By using (1.13) it is easy to see that the rows $\mathbf{a}_s = (a_{s1}, \ldots, a_{sM})$ of the matrix $\mathfrak{A}_D = \{a_{sj}\}$, $s = 1, \ldots, N+2$, $j = 1, \ldots, M$, where $M = 2(N+1)$, satisfy the following linear system of algebraic equations with matrix $\mathfrak{L}_D$ in (1.22):

$$\mathfrak{L}_D \mathbf{a}_s^{\mathrm{T}} = 0 \tag{1.24}$$

(the superscript $^{\mathrm{T}}$ denotes transposition). From (1.24) we see that we can take $\mathfrak{A}_D$ to be the $(N+2) \times (2N+2)$ matrix

$$\mathfrak{A}_D = \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & 1 & 1 & \ldots & 1 \\ 0 & 1 & \ldots & 0 & 0 & -1 & -1 & \ldots & -1 \\ \vdots & \vdots & \ddots & 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \ddots & 0 & 1 & 0 & 0 & \ldots & 1 \end{pmatrix} \tag{1.25}$$

whose first $N+2$ columns form the corresponding identity matrix, and the remaining N columns form a matrix with two rows consisting of 1s and minus 1s, respectively, while the other rows form the $N \times N$ identity submatrix. We note that $\mathfrak{A}_D = \{\alpha_{sj}\}$ satisfies the above general conditions for matrices defining $\mathfrak{A}$-systems, and in particular, we can set $(h_1, \ldots, h_r) = (1, \ldots, 1)$ because it is obvious that $\sum_{s=1}^{N+2} \alpha_{sj} = 1$ for $j = 1, \ldots, 2N+2$.

From (1.21), (1.25), and (1.17) we see that the first equations (1.14) of the Gelfand–Kapranov–Zelevinsky system for $\psi_D(\mathbf{w})$ in the representation (1.19) have the form

$$
\begin{aligned}
w_1\,\frac{\partial\psi_D}{\partial w_1}+\sum_{j=N+3}^{2N+2} w_j\,\frac{\partial\psi_D}{\partial w_j}+b\psi_D=0,\\
w_2\,\frac{\partial\psi_D}{\partial w_2}-\sum_{j=N+3}^{2N+2} w_j\,\frac{\partial\psi_D}{\partial w_j}+(1-c)\psi_D=0,\\
w_s\,\frac{\partial\psi_D}{\partial w_s}+w_{s+N}\,\frac{\partial\psi_D}{\partial w_{s+N}}+a_{s-2}\psi_D=0,\qquad s=3,\dots,N+2.
\end{aligned}
\tag{1.26}
$$

In the second group of equations (1.15) the lattice $\mathbb{L}$ must be taken to be the lattice $\mathbb{L}_D$ generated by the rows of the matrix (1.22).

An important part of the theory of hypergeometric functions of several variables concerns their representations by contour integrals of Euler–Pochhammer type, Mellin–Barnes type, or other types (for instance, see [7], [12], [13], [27], [72]). Such an integral representation can be taken as the definition of a certain class of hypergeometric functions. Integral representations will be important for our purposes in this paper.

The above arguments show that the Lauricella function can be viewed both from the standpoint of Horn systems and from the standpoint of $\mathfrak{A}$-systems. In this paper we take the first point of view.

We note further that for Horn series, and for the Lauricalla function in particular, authors often use notation indicating the basis in the lattice (1.13). For example, the coefficients (1.12) of the series defining $F_D^{(N)}$ can be written in the form

$$
\Lambda(\mathbf{k})=\Gamma(c)\Gamma(1-b)\prod_{j=1}^{N}\Gamma(1-a_j)\prod_{j=1}^{2(N+1)}\frac{1}{\Gamma(1+\mathbf{k}\cdot\mathbf{q}_j+\gamma_j)}\,,
$$

where $\mathbf{k}=(k_1,\dots,k_N)$ is a multi-index, $\mathbf{q}_j$ is the jth column of the matrix $\mathfrak{L}_D$ in (1.22), the dot $\cdot$ denotes the scalar product, and the vector $\gamma=(\gamma_1,\dots,\gamma_{2N+2})$ is expressed using (1.21) in terms of the parameters of the Lauricella function (1.4). Nevertheless, in what follows we hold to the traditional notation involving products of Pochhammer symbols (1.2), because then the results we are going to discuss can be expressed in a more compact form.

1.3. Formulae for analytic continuation of the Lauricella function. In spite of the great progress made in the general theory of hypergeometric functions, quite a few important questions which are well understood for the Gauss function have long remained unresolved in the multidimensional case.

One of the unresolved questions for the Lauricella function $F_D^{(N)}$, already mentioned in § 1.1, is the *problem of its analytic continuation.* This is the problem of finding representations of the form

$$
F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})=\sum_{j=0}^{N}\lambda_j u_j(\mathbf{a};b,c;\mathbf{z}),\qquad \mathbf{z}\notin\mathbb{U}^N,
\tag{1.27}
$$

outside the polydisk $\mathbb{U}^N$, where the functions $u_j(\mathbf{a};b,c;\mathbf{z})$ are generalized hypergeometric series (distinct from the original function) which satisfy the same system (1.5) as $F_D^{(N)}$, and the coefficients λ_j are independent of $a_1,\dots,a_N$, b, and c and do not vanish simultaneously. We call representations of the form (1.27) *formulae for analytic continuation*. This is the sense in which analytic continuation of hypergeometric functions is understood in the fundamental papers [73], [74], as well as in [7], [10], [11], [13], [30], [31], [75], and other papers. Note that the right-hand side of (1.27) contains $N+1$ terms, because this is the number of linearly independent solutions of (1.5). These formulae are a direct generalization of the corresponding representations for the Gauss function (see [30], [31]), which we discuss below in this subsection.

The problem of finding representations (1.27) for $F_D^{(N)}$ is a particular case of the general problem of analytic continuation arising in the theory of multivariate hypergeometric functions and the theory of systems of equations satisfied by these functions. This problem (which is closely related to calculating the monodromy group) consists in finding a complete set of solutions of a hypergeometric system of differential equations in a neighbourhood of each point of $\mathbb{C}^N$ and in finding explicit formulae expressing the connections between two such sets defined in neighbourhoods of different points. Questions of the monodromy of the Lauricella function $F_D^{(N)}$ were investigated in [17], [76]–[78]. The problem of its analytic continuation has been considered by many authors. In the cases $N=2$ and $N=3$ important results were obtained by Erdélyi [10], Olsson [11], and Exton [13]. For arbitrary N a complete set of formulae of the form (1.27) was constructed in [57]–[61]. Such representations hold in domains that, in totality, cover the whole of $\mathbb{C}^N$ away from certain hyperplanes. It should be noted that the methods of analytic continuation in those papers made essential use of the form of the coefficients of the hypergeometric series (1.4). A way to effectively construct analytic continuations of general power series without relying on the specific form of their coefficients is opened by methods based on Padé approximations and their generalizations developed in [79]–[81].

In the single-variable case, that is, for the Gauss function, the problem of analytic continuation was brought to conclusion in the 19th century in well-known works. First of all, for the hypergeometric equation (1.1) we have the set of Kummer's canonical solutions [30], [31], [73], which are the 'simplest' solutions of this equation. We present here two functions in this set which (for $c-a-b\notin\mathbb{Z}$) form a complete system in a neighbourhood of the singular point $z=1$:

$$u_1^{(1)}(a,b;c;z)=F(a,b;a+b-c+1;1-z), \tag{1.28}$$

$$u_2^{(1)}(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c-a-b+1;1-z), \tag{1.29}$$

and also another pair of functions which (for $b-a\notin\mathbb{Z}$) form a complete system in a neighbourhood of $z=\infty$:

$$u_1^{(\infty)}(a,b;c;z)=(-z)^{-a}F(a,1-c+a;1-b+a;z^{-1}), \tag{1.30}$$

$$u_2^{(\infty)}(a,b;c;z)=(-z)^{-b}F(b,1-c+b;1-a+b;z^{-1}). \tag{1.31}$$

Near $z = 0$ the following two functions form a complete system (provided that $c \notin \mathbb{Z}$):

$$u_1^{(0)}(a,b;c;z) = F(a,b;c;z), \quad u_2^{(0)}(a,b;c;z) = z^{1-c}F(1+a-c,1+b-c;2-c;z). \tag{1.32}$$

In (1.28)–(1.32), F denotes the hypergeometric series (1.1), and the superscripts in the notation for the functions $u_j^{(0)}$, $u_j^{(1)}$, and $u_j^{(\infty)}$, $j = 1, 2$, indicate the point near which they are defined. The general solution of (1.3) is expressed near these points as a linear combination of the corresponding two canonical solutions.

For example, in constructing an analytic continuation of the function $u_1^{(0)}(z) = F(a,b;c;z)$, which is a solution of (1.3) holomorphic at $z = 0$, into the domain

$$\mathbb{K} := \{z \in \mathbb{C} \colon |z-1| < 1,\ |\arg(1-z)| < \pi\} \tag{1.33}$$

we obtain a representation of the form (1.27):

$$F(a,b;c;z) = \mathrm{A}_1 u_1^{(1)}(a,b;c;z) + \mathrm{A}_2 u_2^{(1)}(a,b;c;z), \tag{1.34}$$

where the functions $u_1^{(1)}$ and $u_2^{(1)}$ can be found from (1.28) and (1.29), and the coefficients A_1 and A_2 are given by

$$\mathrm{A}_1 = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{and} \quad \mathrm{A}_2 = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{1.35}$$

with $c-a-b$ assumed not to be an integer.

In a similar way, for analytic continuation of the function $u_1^{(0)}(z) = F(a,b;c;z)$ into the domain

$$\mathbb{V} := \{z \in \mathbb{C} \colon |z| > 1,\ |\arg(-z)| < \pi\} \tag{1.36}$$

we have the following formula of the form (1.27):

$$F(a,b;c;z) = \mathrm{B}_1 u_1^{(\infty)}(a,b;c;z) + \mathrm{B}_2 u_2^{(\infty)}(a,b;c;z), \tag{1.37}$$

where $u_1^{(\infty)}$ and $u_2^{(\infty)}$ can be found from (1.30) and (1.31), and the coefficients B_1 and B_2 are given by

$$\mathrm{B}_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} \quad \text{and} \quad \mathrm{B}_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} \tag{1.38}$$

when $a-b$ is not an integer.

Note that all canonical solutions of Gauss's equation (1.3) are expressed in terms of hypergeometric series of the form (1.1). But the system (1.5) lacks a similar property even for $N = 2$. In the case of two variables (recall that this system is then denoted by $E_D^{(2)}$) it is satisfied by the Appell function $F_D^{(2)} = F_1(a,a';b,c;z,\zeta)$,

$$F_1(a,a';b,c;z,\zeta) = \sum_{k,n=0}^{\infty} \frac{(b)_{k+n}(a)_k(a')_n}{(c)_{k+n}k!\,n!} z^k \zeta^n, \qquad |z| < 1, \quad |\zeta| < 1 \tag{1.39}$$

(z and ζ are variables and a, a', b, and c are parameters), which is a holomorphic solution in a neighbourhood of $(z,\zeta) = (0,0)$. The problem of constructing an analogue of Kummer's canonical solutions for $E_D^{(2)}$ has been considered beginning with [4] (see [10] for a detailed survey of investigations in this direction). A set of canonical solutions of $E_D^{(2)}$ which can be expressed in terms of the Appell series (1.39) is presented in [7]. Erdélyi [10] showed that the functions in [7] are not sufficient for describing the general solution of the system $E_D^{(2)}$, and he proved that the required additional canonical solutions can be expressed in terms of a bivariate hypergeometric series

$$G(a,a';b,c;z,\zeta)=\sum_{k,n=0}^{\infty}\frac{(b)_{n-k}(a)_k(a')_n}{(c)_{n-k}k!\,n!}z^k\zeta^n,\qquad |z|<1,\quad |\zeta|<1,\tag{1.40}$$

which cannot be reduced to (1.39). This series, which is of central importance in the theory of the Appell function F_1, had previously been indicated in [9] (in a slightly different form) as the function G_2 in the so-called Horn's list, a list of essentially different hypergeometric series of two variables (see also [30]). The difference of indices $n-k$ in (1.40) can take negative values. Note that, in view of the property (1.18) of the gamma function, the Pochhammer symbol $(a)_k$ defined in (1.2) can be expressed for negative integer values of k by

$$(a)_k=(-1)^k[(1-a)(2-a)\cdots((1-a)-k-1)]^{-1},\qquad k=-1,-2,\dots.\tag{1.41}$$

Let us consider the system (1.5) for $N=2$ more closely in a neighbourhood of the point $(z,\zeta)=(\infty,\infty)$. For this system the analogue of Kummer's canonical solutions (1.30), (1.31) in the domain

$$\mathbb{V}^2:=\{(z,\zeta)\in\mathbb{C}^2\colon |z|>|\zeta|>1,\ |\arg(-z)|<\pi;\ |\arg(-\zeta)|<\pi\}\tag{1.42}$$

is given by the functions (see [10] and [11])

$$\begin{aligned}\mathscr{U}_0^{(\infty)}(a,a';b,c;z,\zeta)&=(-z)^{-a}(-\zeta)^{-a'}\\&\quad\times F_1\biggl(a,a';1-c+a+a',1-b+a+a';\frac{1}{z},\frac{1}{\zeta}\biggr),\end{aligned}\tag{1.43}$$

$$\begin{aligned}\mathscr{U}_1^{(\infty)}(a,a';b,c;z,\zeta)&=(-z)^{-a}(-\zeta)^{a-b}\\&\quad\times G\biggl(a,1-c+b;b-a,1+b-a-a';\frac{\zeta}{z},\frac{1}{\zeta}\biggr),\end{aligned}\tag{1.44}$$

$$\mathscr{U}_2^{(\infty)}(a,a';b,c;z,\zeta)=(-z)^{-b}F_1\biggl(1-c+b,a';b,1+b-a;\frac{1}{z},\frac{\zeta}{z}\biggr),\tag{1.45}$$

two of which can be expressed in terms of the Appell series (1.39), while the third can be expressed in terms of the Horn series (1.40). (In the notation of $\mathbb{V}^2$ the superscript indicates that we are in the case of two variables.) The functions (1.43)–(1.45) form a complete system of linearly independent solutions of the system $E_D^{(2)}$ in $\mathbb{V}^2$, provided that

$$b-a\notin\mathbb{Z},\qquad b-a-a'\notin\mathbb{Z}.\tag{1.46}$$

Furthermore, the series F_1 on the right-hand side of (1.43) converges in a whole neighbourhood of the point (∞,∞), namely, for $|z|>1$ and $|\zeta|>1$, while the two series G and F_1 in (1.44) and (1.45) converge only for $|z|>|\zeta|>1$. Thus, bearing in mind that the right-hand sides of (1.43)–(1.45) contain branching factors, we see that $\mathbb{V}^2$ is the domain where all three functions $\mathscr{U}_j^{(\infty)}$, $j=0,1,2$, are well defined (and single valued). Since the system $E_D^{(2)}$ is clearly preserved by interchanging z and ζ (and interchanging the parameters a and a' at the same time), by using (1.43)–(1.45) we also can indicate a complete system of canonical solutions in the domain

$$\widetilde{\mathbb{V}}^2 := \{(z,\zeta)\in\mathbb{C}^2\colon (\zeta,z)\in\mathbb{V}^2\} = \{|\zeta|>|z|>1,\ |\arg(-z)|<\pi;\ |\arg(-\zeta)|<\pi\}. \tag{1.47}$$

Such solutions there are given by

$$\mathscr{U}_0^{(\infty)}(a,a';b,c;z,\zeta) \ \text{ and }\ \widetilde{\mathscr{U}_j}^{(\infty)}(a,a';b,c;z,\zeta) := \mathscr{U}_j^{(\infty)}(a',a;b,c;\zeta,z), \quad j=1,2, \tag{1.48}$$

provided that $b-a'\notin\mathbb{Z}$ and $b-a-a'\notin\mathbb{Z}$. We note that the canonical solutions of $E_D^{(2)}$ in a neighbourhood of $(z,\zeta)=(1,1)$ that were constructed in [10] and [11], like (1.43)–(1.45) and (1.48), have the form of power series, but now in powers of $1-z$ and $1-\zeta$.

On the other hand, it is possible that the general solution of the system $E_D^{(2)}$ contains not only powers of z, ζ, $1-z$, and $1-\zeta$, but also logarithms of these quantities. Such cases are said to be *resonant* or *logarithmic*, and they occur when any of the following numbers is an integer:

$$c-a-a'-b,\quad c-a-b,\quad c-a'-b,\quad b-a-a',\quad b-a,\quad b-a'. \tag{1.49}$$

For example, if $b-a\in\mathbb{Z}$ or $b-a-a'\in\mathbb{Z}$, then we cannot define a complete system of canonical solutions of $E_D^{(2)}$ using the formulae (1.43)–(1.45), and the corresponding modified functions (see [61] and also § 2.5) will contain the logarithms $\log z$ and $\log\zeta$ in addition to powers z^m and ζ^k.

Olsson [11] constructed a complete system of formulae for analytic continuation (1.27) of the Appell function F_1 in *non-resonant*, or *non-logarithmic* cases. For instance, if (1.46) holds, then we have the following representation in the domain $\mathbb{V}^2$ defined in (1.42) (see [11]):

$$F_1(a,a';b,c;z,\zeta) = \sum_{j=0}^{2}\lambda_j\mathscr{U}_j^{(\infty)}(z,\zeta), \tag{1.50}$$

where the functions $\mathscr{U}_j^{(\infty)}(z,\zeta)$ are given by (1.43)–(1.45) and the coefficients λ_j have the expressions

$$\lambda_0 = \frac{\Gamma(c)\Gamma(b-a-a')}{\Gamma(b)\Gamma(c-a-a')}\,, \qquad \lambda_1 = \frac{\Gamma(c)\Gamma(b-a)\Gamma(a+a'-b)}{\Gamma(a')\Gamma(c-b)\Gamma(b)}\,,$$
$$\lambda_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,. \tag{1.51}$$

A formula for analytic continuation into the domain $\widetilde{\mathbb{V}}^2$ defined in (1.47) can be obtained from (1.50), (1.51) by replacing $\mathscr{U}_j^{(\infty)}(z,\zeta)$ by $\widetilde{\mathscr{U}_j}^{(\infty)}(z,\zeta)$, $j=1,2$, and interchanging a and a' in (1.51). We underscore that, generally speaking, the conditions (1.46) ensuring (1.50) do not rule out resonant cases. For example, if (1.46) is satisfied but $b-a'\in\mathbb{Z}$, then the representation for F_1 in the domain $\widetilde{\mathbb{V}}^2$ contains the logarithms of z and ζ.

The resonant cases of the system $E_D^{(2)}$ must be considered separately, because the results in [10] and [11] do not extend directly to these cases. Such an investigation was carried out in [61], where complete systems of canonical solutions and formulae for analytic continuation of F_1 were constructed in resonant cases of $E_D^{(2)}$ corresponding to integer numbers in (1.49).

In [11] the Appell function F_1 was analytically continued using a procedure based on re-expanding series (see also [7], [82]). The approach in [11] was used by Exton [13] to deduce formulae for analytic continuation in the case of three variables ($N=3$). In [13] an (essentially complete) set of representations of the form (1.27) was found for $N=3$ in the non-logarithmic case. However, this is a laborious approach, which meets with considerable difficulties even for $N=3$, and moreover, we actually cannot use this method to treat logarithmic cases of the system $E_D^{(N)}$ for $N\geqslant 2$.

In our § 2 we present a complete set of formulae for analytic continuation of the Lauricella function $F_D^{(N)}$ of an arbitrary number N of variables. The subdomains of $\mathbb{C}^N$ where these formulae hold totally cover $\mathbb{C}^N$ (away from certain hyperplanes). To derive continuation formulae we use representations in the form of Mellin–Barnes type integrals for $F_D^{(N)}$ (see § 2.1). Formulae for analytic continuation into neighbourhoods of the points $\mathbf{z}_q^{(\infty,0)}$, $\mathbf{z}_p^{(1,0)}$, and $\mathbf{z}_{p,q}^{(1,\infty,0)}$ are derived in § 2.2, 2.3, and 2.4, respectively. These subsections are an extended version of [57]–[60]. Some facts concerning the resonant case are given in § 2.5 (the reader can find a detailed analysis for $N=2$ in [61]).

1.4. Schwarz–Christoffel parameter problem and analytic continuation of $F_D^{(N)}$. The effective construction of a conformal mapping of a complicated domain $\mathscr{B}$ onto a canonical domain (a half-plane, a disk, the exterior of a disk) is usually a difficult problem. However, when such a mapping is known, substantive new possibilities sometimes emerge for the investigation of many theoretical and applied problems (for instance, see [83]–[91]), in particular, for the solution of boundary-value problems in the original domain $\mathscr{B}$. In this connection many papers developing and improving methods for conformal mappings have appeared (for example, [92]–[100]).

As is well known, the general approach to constructing conformal mappings of simply connected polygonal domains is based on the Schwarz–Christoffel integral (see [92], [93], [97], [99], [101]–[104]). For a mapping $\mu\colon \mathbb{H}^+ \xrightarrow{\text{conf}} \mathscr{B}$ of a half-plane onto an N-gonal domain $\mathscr{B}$ with internal angles $\pi\beta_j$ at the vertices z_j this integral has the form

$$z=\mu(\zeta)=\mathscr{K}_0\int_{\widetilde{\zeta}}^{\zeta}\prod_{j=1}^{N}(t-\zeta_j)^{\beta_j-1}\,dt+\mathscr{K}_1, \tag{1.52}$$

where $\mathscr{K}_0$ and $\mathscr{K}_1=\mu(\widetilde{\zeta})$ are constants and the $\zeta_j:=\mu^{-1}(z_j)$ are the inverse images of the vertices of $\mathscr{B}$. According to Riemann's theorem [102], [104], three of the quantities ζ_j can be prescribed arbitrarily (with preservation of the direction of a circuit around the domain) on the real line $\mathbb{R} = \partial\mathbb{H}^+$, but finding the other $N-3$ points is difficult. This limits the scope of applications of the analytic representation (1.52) (for example, see [97], [102], [105]–[108] on this topic).

Methods for calculating the unknown parameters of the Schwarz–Christoffel integral (besides the inverse images ζ_j, these parameters include the coefficient $\mathscr{K}_0$) are indicated in [92]–[94], [97], [105], [109]. Finding them becomes especially difficult in the case of *crowding*, when the ζ_j are very unevenly distributed (see [107] and also [97], [105], [110], [111]). In our case of a mapping of a half-plane, crowding in $\mathbb{R}$ is understood with respect to the spherical metric [93], [97], [103]. It should be noted that in applications the situation of crowding is most often encountered in the use of the Schwarz–Christoffel integral. The crowding problem has been treated in [62], [99], [105], and [111]–[113], but it is still far from a comprehensive solution.

Our hopes are that the results in § 4 on analytic continuation of the Lauricella function $F_D^{(N)}$ can be instrumental for making significant progress in the solution of the crowding problem. We remark that connections between the theory of the function $F_D^{(N)}$ and the Schwarz–Christoffel parameter problem were pointed out in [62] (and also in [111], in the particular case of a pentagonal domain). Here we present the corresponding arguments from [62] (see also § 5).

We can form a system of equations for the parameters (see [93]) by integrating in (1.52) over the intervals (ζ_k, ζ_{k+1}) and equating the absolute values of the integrals obtained to the corresponding lengths $L_k := |z_{k+1} - z_k|$ of the sides of the boundary $\partial\mathscr{B}$:

$$\left|\mathscr{K}_0 \int_{\zeta_k}^{\zeta_{k+1}} \prod_{j=1}^{N} (t-\zeta_j)^{\beta_j - 1}\, dt\right| = L_k, \qquad k = 1, \dots, N-2, \tag{1.53}$$

where we assume that all the vertices z_j, $j = 1, \dots, N-2$, are finite. Such non-linear systems are usually solved via Newton-type iterative procedures, and we need a high-precision algorithm for calculating the left-hand sides of the equations (1.53) in order that such procedures converge efficiently. After obvious changes of variables and use of an Euler-type representation (1.6), we express the left-hand sides of (1.53) in terms of the Lauricella function $F_D^{(n)}$ with $n = N-3$, where each equation is characterized by its own set of parameters and the variables of $F_D^{(n)}$, the parameters are expressed in terms of the characteristics β_j of the angles of the polygon, and the variables are expressed in terms of the inverse images ζ_j of its vertices. It should be stressed that when crowding occurs, the arguments of the functions $F_D^{(n)}$ vary outside the unit polydisk, so that we cannot use the representation (1.4) to calculate the values of these functions. A fairly effective algorithm for such calculations is provided by the formulae of type (1.27) for analytic continuation which we present in § 2. In § 5 we show that the left-hand sides of (1.53), regarded as functions of $\zeta_1, \dots, \zeta_N$, are related one to another in the sense that they are solutions of the same Lauricella system (1.5), and we present explicit expressions for them in terms of the functions involved in the representations (1.27). In § 5 we also give an example of solving the Schwarz–Christoffel parameter problem and of constructing a conformal mapping in the case of crowding.

We note that to get a sound initial approximation for the unknown inverse images we can use asymptotic expressions for them corresponding to limiting cases of the structure of the polygon $\mathscr{B}$. Such asymptotic formulae can be deduced using the constructive results in [114] and [115] on variation of a conformal mapping under singular deformations of the domain. For a pentagonal domain such formulae were found in [62] and [111].

1.5. Riemann–Hilbert problem and Jacobi-type formulae. The differential relations holding for hypergeometric functions are very important (see [116]). One such relation in the theory of the Gauss function $F(a,b;c;z)$ is the familiar Jacobi identity [117] (see also [30]). It has a direct generalization to the case of $F_D^{(N)}$, in the form of a system of differential relations found in [57], [65], [66], and we call them *Jacobi-type formulae.* By using such formulae we can find [63], [64] a new type of representation for the solutions of the Riemann–Hilbert problem with piecewise constant data, which we discuss below. This representation has the form of a Schwarz–Christoffel integral, which is significantly different from Cauchy-type integrals arising in commonly used representations for solutions of boundary-value problems for analytic functions.

Starting from the fundamental papers [74] and [118], many authors ([119]–[124]) have considered the *Riemann–Hilbert* boundary problem which consists in finding an analytic function $\mathscr{F}(z) = u(x,y) + iv(x,y)$ in a domain $\mathscr{B} \subset \overline{\mathbb{C}}$ from a given relation

$$pu - qv = r \tag{1.54}$$

between its real and imaginary parts on the boundary $\partial\mathscr{B}$ (where p, q, and r are real functions). Results in the classical theory of this problem and methods for solving it are presented, for instance, in the treatises [125]–[127] (see also the books [97] and [128]–[130]). The problem (1.54) has many applications to mechanics, electrodynamics, stochastic processes, approximation theory, and so on; some of its applications are listed in [126] and [131]–[140]. For contemporary theoretical investigations of the problem and some of its generalizations, see [141]–[146], for example.

To solve the Riemann–Hilbert problem (1.54) constructively, it is useful to use a conformal mapping of the original domain $\mathscr{B}$ onto, say, the half-plane $\mathbb{H}^+$ (or a disk, or the exterior of a disk). Then the solution of the transformed problem can be written out in a closed form in terms of Cauchy-type integrals (see [125]–[127]). In particular, using this approach we can adequately take into account the complicated geometry of the domain $\mathscr{B}$.

Starting from Riemann's paper [74] (see also [93], [131], [133], [147]), many authors have noted that the solution of a Riemann–Hilbert problem has a clear geometric interpretation. For example, in the simplest case when p, q, and r are constant, the condition (1.54) is the equation of a straight line in the plane $w = u + iv$. This observation suggests that a solution of the Riemann–Hilbert problem with piecewise constant data can be interpreted geometrically as a conformal mapping of the original domain onto a (not necessarily schlicht) polygonal domain. We note that the representation of a solution $\mathscr{P}^+(\zeta)$ as a Schwarz–Christoffel integral that was constructed in [63] and [64] (see also [62]) on the basis of a Jacobi-type formula for the Lauricella function [57], [65], [66], is a realization of this interpretation of a Riemann–Hilbert problem with piecewise constant data.

Before we present this representation, we introduce some further notation. We write the boundary condition in the Riemann–Hilbert problem (1.54) as

$$\operatorname{Re}\big[h(z')\mathscr{F}(z')\big] = r(z'), \qquad z' \in \partial\mathscr{B},$$

where $h(z') := p(z') + iq(z')$. After a conformal mapping $z = \Phi(\zeta)$ of the domain $\mathscr{B}$ onto $\mathbb{H}^+$ the unknown function $\mathscr{F}(z)$ is transformed into

$$\mathscr{P}^+(\zeta) = \mathscr{F} \circ \Phi^{-1}(\zeta),$$

and the piecewise constant functions $h(z')$ and $r(z')$ become

$$\chi(\xi) = h \circ \Phi^{-1}(\xi) \quad \text{and} \quad \sigma(\xi) = r \circ \Phi^{-1}(\xi), \quad \xi \in \mathbb{R},$$

respectively, which are also piecewise constant (on the real line $\mathbb{R} = \partial\mathbb{H}^+$). Thus, in view of this notation the boundary condition in the Riemann–Hilbert problem in $\mathbb{H}^+$ has the form

$$\operatorname{Re}[\chi(\xi)\mathscr{P}^+(\xi)] = \sigma(\xi), \qquad \xi \in \mathbb{R}. \tag{1.55}$$

We denote the set of points of discontinuity of $\chi(\xi)$ or $\sigma(\xi)$ by

$$\Xi := \{\xi_0, \xi_1, \dots, \xi_N\}, \tag{1.56}$$

where $\xi_1, \dots, \xi_N$ are finite points in $\mathbb{R}$, $\xi_{k+1} > \xi_k$, and ξ_0 is the (unique) point at infinity. A representation of $\mathscr{P}^+(\zeta)$ which realizes the geometric interpretation of the solution of the Riemann–Hilbert problem has the form (see [63], [64])

$$\mathscr{P}^+(\zeta) = \mathscr{K}_0 \int^{\zeta} \prod_{j=1}^{N} (t - \xi_j)^{\gamma_j - 1} P(t)\, dt + \mathscr{K}_1, \tag{1.57}$$

where $P(\zeta)$ is a polynomial with real coefficients and with degree depending on the number of points of discontinuity of $\chi(\xi)$ and on the index $\varkappa$ of the problem, the fractional parts of the exponents γ_j are expressed in terms of the jumps of the argument of $\chi(\xi)$ at points of discontinuity, and the integer parts of γ_j are determined by some additional conditions.

The Schwarz–Christoffel integral in (1.57) has two important (and beneficial) properties. First, as we have already mentioned, it provides a clear geometric interpretation for $\mathscr{P}^+(\zeta)$ by showing that this function realizes a conformal mapping of the upper half-plane $\mathbb{H}^+$ onto a non-schlicht polygonal domain $\mathscr{M}$ (see [93]). Here the mapping $w = \mathscr{P}^+(\zeta)$ takes points in Ξ and real zeros of $P(\zeta)$ to corner points of the boundary of $\mathscr{M}$, while complex zeros of $P(\zeta)$ in $\mathbb{H}^+$ are taken to interior branch points of this domain. Second, the integral (1.57) is much more effective for calculations than the traditional representations of a solution of the Riemann–Hilbert problem via Cauchy-type integrals (see [111], [140] on this). These features are useful for applications. For example, many important problems in mechanics ([131], [132], [148]–[150]) and plasma physics ([67], [134], [151]) reduce to a Riemann–Hilbert problem with piecewise constant data. We should note that a solution $\mathscr{P}^+(\zeta)$ of a Riemann–Hilbert problem in $\mathbb{H}^+$ that has these properties can also be constructed for boundary data $h(z')$ and $\sigma(z')$ in (1.55) which belong to certain wider classes.

Furthermore, in many papers (for instance, [93], [131], [133], [148]–[153]) solutions of Riemann–Hilbert boundary problems, including ones arising in connection with applied problems, were expressed by means of Schwarz–Christoffel integrals, but the proof of such a representation for arbitrary piecewise constant data of the problem and formulae for the parameters of the integrand were apparently first obtained in [63] and [64]. We devote our § 3 to a presentation of results from these papers.

2. Analytic continuation of the Lauricella function

2.1. Representations by Mellin–Barnes contour integrals. In this subsection we present two representations by (one-dimensional) contour integrals of Mellin–Barnes type for the function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$. Then on this basis we obtain in §§ 2.2–2.4 a system of analytic continuation formulae of the form (1.27). We remark that Mellin–Barnes integrals and various generalizations of them play an important role in the theory of hypergeometric and other special functions (see [28]–[31], [154]).

2.1.1. *The first representation of $F_D^{(N)}$.* We consider the domain

$$\mathbb{S}_1^N := \{\mathbf{z}\in\mathbb{C}^N\colon |\arg(-z_1)|<\pi;\ |z_k|<1,\ k=2,\dots,N\}, \tag{2.1}$$

introduce the notation $\mathbf{z}_1' := (z_2,\dots,z_N)$ and $\mathbf{a}_1' := (a_2,\dots,a_N)$, and define the function

$$f(\mathbf{a};b,c;\mathbf{z},s) := \frac{\Gamma(a_1+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} \times(-z_1)^s F_D^{(N-1)}(\mathbf{a}_1';b+s,c+s;\mathbf{z}_1'),\qquad \mathbf{z}\in\mathbb{S}_1^N,\quad s\in\mathbb{C}. \tag{2.2}$$

The function $F_D^{(N-1)}(\mathbf{a}_1';b+s,c+s;\mathbf{z}_1')$ in (2.2) is defined by the series (1.4) for the corresponding values of the parameters and variables.

The following result establishes the *first* integral representation of Mellin–Barnes type for the Lauricella function $F_D^{(N)}$.

Proposition 1. *The Lauricella function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ defined by the series* (1.4) *can be represented for $\mathbf{z}\in\mathbb{U}^N\cap\mathbb{S}_1^N$ as a contour integral of Mellin–Barnes type*:

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = \frac{\Gamma(c)}{2\pi i\Gamma(a_1)\Gamma(b)}\int_{-i\infty}^{+i\infty} f(\mathbf{a};b,c;\mathbf{z},s)\,ds, \tag{2.3}$$

where f is given by (2.2) *and the contour of integration in* (2.3) *is chosen so that the poles $s_k^{(0)}=k$, $k\in\mathbb{Z}^+$, and the poles $s_k^{(1)}=-a_1-k$, $s_k^{(2)}=-b-k$, $k\in\mathbb{Z}^+$, of the function $f(s)$ lie to the right and to the left of it, respectively* (*see Fig.* 1).

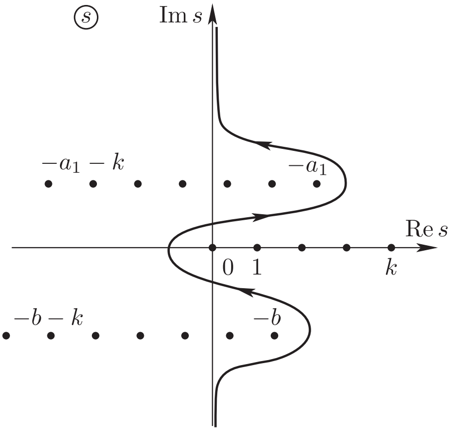

Figure 1. The contour of integration in the Mellin–Barnes representation (2.3).

Proof. Let L denote the contour of integration in (2.3), and let L_n be the part of it between the points $-(n+1/2)i$ and $(n+1/2)i$, where $n \in \mathbb{N}$, that is, L_n is the curve

$$L_n := \left\{ s \in L \colon |\operatorname{Im} s| \leqslant n + \frac{1}{2} \right\},$$

and moreover, let C_n^+ be a semicircle (centred at the origin) with radius $n+1/2$ in the right half-plane which is oriented clockwise. Using the well-known identity

$$\Gamma(-s) = \frac{-\pi}{\Gamma(1+s)\sin \pi s}$$

(see [30]), we can write the function in (2.2) in a form

$$f(\mathbf{a}; b, c; \mathbf{z}, s) = -\frac{\Gamma(a_1+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)} F_D^{(N-1)}(\mathbf{a}_1'; b+s, c+s; \mathbf{z}_1') \frac{\pi(-z_1)^s}{\sin \pi s} \tag{2.4}$$

more convenient for asymptotic analysis, and we can consider the integral

$$I_n(\mathbf{a}; b, c; \mathbf{z}) := \int_{L_n \cup C_n^+} f(\mathbf{a}; b, c; \mathbf{z}, s)\, ds. \tag{2.5}$$

The known asymptotic formula

$$\Gamma(a+s) = \mathscr{O}(s^{s+a-1/2} e^{-s}), \qquad s \to \infty, \quad |\arg s| < \pi$$

(see [30], § 1.18), for the gamma function gives us the following estimate for the first fraction in (2.4):

$$\frac{\Gamma(a_1+s)\Gamma(b+s)}{\Gamma(c+s)\Gamma(1+s)} = \mathscr{O}(s^{a_1+b-c-1}), \qquad s \to \infty, \quad |\arg s| < \pi. \tag{2.6}$$

It is easy to verify that for the Lauricella function in (2.4) we have the asymptotic relation

$$F_D^{(N-1)}(\mathbf{a}_1'; b+s, c+s; \mathbf{z}_1') = \mathscr{O}(1), \qquad s \to \infty. \tag{2.7}$$

We find an estimate on the curve C_n^+ for the third factor on the right-hand side of (2.4). Let $s=(n+1/2)e^{i\theta}$ and $(-z_1)^s=\exp(s\log(-z_1))$, where $\log(-z_1):=\log|z_1|+i\arg(-z_1)$ and $|\arg(-z_1)|\leqslant\pi-\delta$ for some small positive δ by the conditions of the proposition. Then it is easy to verify that

$$\frac{(-z_1)^s}{\sin\pi\theta}=\mathscr{O}\biggl(\exp\biggl[-\biggl(n+\frac{1}{2}\biggr)(-\cos\theta\log|z_1|+\delta|\sin\theta|)\biggr]\biggr),\qquad n\to\infty. \tag{2.8}$$

From (2.6)–(2.8) we obtain the following asymptotic formula for the function (2.4) on the curve C_n^+:

$$|f(s)|=\mathscr{O}\bigl(n^{a_1+b-c-1}\exp\bigl[-n(-\cos\theta\log|z_1|+\delta|\sin\theta|)\bigr]\bigr), \tag{2.9}$$
$$s\in C_n^+,\quad n\to\infty,$$

where we use the notation $f(s):=f(\mathbf{a};b,c;\mathbf{z},s)$. Hence if $|z_1|<1$ and therefore $\log|z_1|<0$, then for all values of $s=(n+1/2)e^{i\theta}$ and $\theta\in[-\pi/2,\pi/2]$ the integrand in (2.5) tends exponentially to 0 as $n\to\infty$.

We write the integral in (2.5) as

$$\int_{L_n\cup C_n^+}f(s)\,ds=\int_{L_n}f(s)\,ds+\int_{C_n^+}f(s)\,ds,$$

where the first integral on the right approaches $\int_L f(s)\,ds$ as $n\to\infty$, while the second tends to 0 because of (2.9). Thus,

$$\int_L f(s)\,ds=\lim_{n\to\infty}\int_{L_n\cup C_n^+}f(s)\,ds. \tag{2.10}$$

To calculate the integral on the right-hand side of (2.10), we discuss the properties of the function (2.2) in its dependence on the complex variable s with the other variables fixed. Recall that the gamma function $\Gamma(s)$ has simple zeros at non-positive integer points $s=-k$, $k\in\mathbb{Z}^+$, and the residues there are given by

$$\operatorname*{res}_{s=-k}\Gamma(s)=\frac{(-1)^k}{k!}\,,\qquad k\in\mathbb{Z}^+ \tag{2.11}$$

(see [30] and [31]). Also note that the function $\widetilde{F}(s):=F_D^{(N-1)}(\mathbf{a}_1';b+s,c+s;\mathbf{z}_1')/\Gamma(c+s)$ is clearly regular with respect to s on the whole finite plane. In view of the above, it follows from (2.2) that $f(s)$ has simple zeros at the points $s_k^{(0)}=k$, $k\in\mathbb{Z}^+$, and its residues there are given by

$$\operatorname*{res}_{s=s_k^{(0)}}f(s)=-\frac{\Gamma(a_1+k)\Gamma(b+k)}{\Gamma(c+k)k!}z_1^kF_D^{(N-1)}(\mathbf{a}_1';b+k,c+k;\mathbf{z}_1'). \tag{2.12}$$

The integral on the right-hand side of (2.10) is $-2\pi i$ times the sum of the residues of the integrand $f(s)$ in the domain bounded by the contour $L_n\cup C_n^+$:

$$\int_{L_n\cup C_n^+}f(s)\,ds=-2\pi i\sum_{k=0}^{n}\operatorname*{res}_{s=s_k^{(0)}}f(s). \tag{2.13}$$

Using (2.10)–(2.13) and the definition of the Pochhammer symbol (1.2), we obtain

$$\int_L f(s)\,ds = 2\pi i \frac{\Gamma(a_1)\Gamma(b)}{\Gamma(c)} \sum_{k=0}^{\infty} \frac{(a_1)_k (b)_k}{(c)_k k!} z_1^k F_D^{(N-1)}(\mathbf{a}_1'; b+k, c+k; \mathbf{z}_1'). \tag{2.14}$$

We verify (2.3) by dividing both sides by the coefficient of the sum on the right and observing that this sum is equal to $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$:

$$F_D^{(N)}(\mathbf{a}; b; c; \mathbf{z}) = \sum_{k=0}^{\infty} \frac{(a_1)_k (b)_k}{(c)_k k!} z_1^k F_D^{(N-1)}(\mathbf{a}_1'; b+k, c+k; \mathbf{z}_1'). \qquad \square$$

A particular case of the representation (2.3) for $N = 2$, that is, for the Appell function F_1, was presented in [7], for instance. For arbitrary N this representation is perhaps not new either. By applying it successively to functions F_D with fewer variables in the integrand in (2.3), we can obtain the N-fold Mellin–Barnes integral (1.7), which was derived (in a somewhat different way), for example, in [13].

It is easy to see that the integral representation (2.3) in Proposition 1 realizes an analytic continuation of the Lauricella function originally defined by the series (1.4), into the domain $\mathbb{S}_1^N$, in which the right-hand side of (2.3) is a holomorphic function of $\mathbf{z}$.

2.1.2. *The second representation of* $F_D^{(N)}$. We now derive another Mellin–Barnes type representation for $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$, which will be used for constructing an analytic continuation of it in cases when one or more variables z_j are close to 1.

We recall a well-known identity established by Barnes's lemma [30], [31]:

$$\frac{\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma+\delta)} = \frac{1}{\Gamma(\alpha+\delta)\Gamma(\beta+\delta)\cdot 2\pi i} \times \int_{-i\infty}^{+i\infty} \Gamma(\alpha+t)\Gamma(\beta+t)\Gamma(\gamma-t)\Gamma(\delta-t)\,dt, \tag{2.15}$$

where the integration path has been deformed (if necessary) so that the poles of the product $\Gamma(\gamma-t)\Gamma(\delta-t)$, that is, the points $t = \gamma + k$ and $t = \delta + k$ with $k \in \mathbb{Z}^+$, and the poles of the product $\Gamma(\alpha+t)\Gamma(\beta+t)$, that is, the points $t = -\alpha - k$ and $t = -\beta - k$ with $k \in \mathbb{Z}^+$, lie to the right and to the left of it, respectively. We write a factor in (2.2) as a series

$$\frac{\Gamma(a_1+s)\Gamma(b+s)}{\Gamma(c+s)} F_D^{(N-1)}(\mathbf{a}_1'; b+s, c+s; \mathbf{z}_1') = \sum_{|\mathbf{k}_{2,N}|=0}^{\infty} \frac{\Gamma(a_1+s)\Gamma(b+s+|\mathbf{k}_{2,N}|)}{\Gamma(c+s+|\mathbf{k}_{2,N}|)} \frac{(a_2)_{k_2}\cdots(a_N)_{k_N}}{k_2!\cdots k_N!} z_2^{k_2}\cdots z_N^{k_N},$$

where $|\mathbf{k}_{2,N}| = \sum_{j=2}^{N} k_j$. Applying (2.15) with $\alpha = a_1$, $\beta = b + |\mathbf{k}_{2,N}|$, $\gamma = s$, and $\delta = c - a_1 - b$ to the combination of gamma functions on the right-hand side, we establish the equality

$$\frac{\Gamma(a_1+s)\Gamma(b+s)}{\Gamma(c+s)} F_D^{(N-1)}(\mathbf{a}_1'; b+s, c+s; \mathbf{z}_1') = \frac{1}{2\pi i \Gamma(c-a_1)\Gamma(c-b)} \times \int_{-i\infty}^{+i\infty} \Gamma(a_1+t)\Gamma(b+t)\Gamma(s-t)\Gamma(c-a_1-b-t) \times F_D^{(N-1)}(\mathbf{a}_1'; b+t; c-a_1; \mathbf{z}_1')\,dt. \tag{2.16}$$

Rewriting the definition (2.2) of f with (2.16) taken into account and substituting the new expression for f into (2.3), we obtain a representation of $F_D^{(N)}$ as a double integral. Interchanging the integrations with respect to s and t, using the known formula

$$\int_{-i\infty}^{+i\infty} \Gamma(-s)\Gamma(s-t)(-z)^s\,ds = \Gamma(-t)(1-z)^t$$

(see [31]), and setting

$$g(\mathbf{a}; b, c; \mathbf{z}, s) := \Gamma(a_1+s)\Gamma(b+s)\Gamma(-s)\Gamma(c-a_1-b-s)(1-z_1)^s \times F_D^{(N-1)}(\mathbf{a}_1'; b+s, c-a_1; \mathbf{z}_1'), \tag{2.17}$$

we arrive at the following result, which establishes the *second* representation of Mellin–Barnes type for $F_D^{(N)}$ in our paper.

Proposition 2. *The Lauricella function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ defined by (1.4) has a representation for $\mathbf{z} \in \{|\arg(1-z_1)| < \pi,\ |z_k| < 1,\ k = 2, \dots, N\}$ as a contour integral of Mellin–Barnes type*

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \frac{\Gamma(c)}{2\pi i \Gamma(a_1)\Gamma(b)\Gamma(c-a_1)\Gamma(c-b)} \int_{-i\infty}^{+i\infty} g(\mathbf{a}; b, c; \mathbf{z}, s)\,ds, \tag{2.18}$$

where the integrand has the form (2.17) and the contour of integration is chosen so that the poles

$$s_k^{(1)} = k \quad \text{and} \quad s_k^{(2)} = c - a_1 - b + k, \qquad k \in \mathbb{Z}^+, \tag{2.19}$$

and the poles $s_k^{(3)} = -a_1 - k$, $s_k^{(4)} = -b - k$, $k \in \mathbb{Z}^+$, of the function $g(s) := g(\mathbf{a}; b, c; \mathbf{z}, s)$ lie to the right and to the left of it, respectively (see Fig. 2).

The integrand $g(s)$ has the poles indicated in Proposition 2 because the gamma function $\Gamma(s)$ has poles at $s \in \mathbb{Z}^-$. The function $\widetilde{F}(s) := F_D^{(N-1)}(\mathbf{a}_1'; b+s, c-a_1; \mathbf{z}_1')$ is clearly regular in the finite part of the s-plane, hence the integrand $g(s)$ has no singular points other than the poles (2.19) to the right of the contour of integration in (2.18).

2.2. Analytic continuation into a neighbourhood of $\mathbf{z}_q^{(\infty,0)}$. In this subsection, we present a complete set of formulae for analytic continuation of the form (1.27) for the function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ for all $N \geqslant 2$ into a neighbourhood of $\mathbf{z}_q^{(\infty,0)} \in \mathbb{C}^N$, where $q = 1, \dots, N$. First of all, on the basis of proposition 1 we construct such formulae for a neighbourhood of $\mathbf{z}_1^{(\infty,0)} = (\infty, 0, \dots, 0)$.

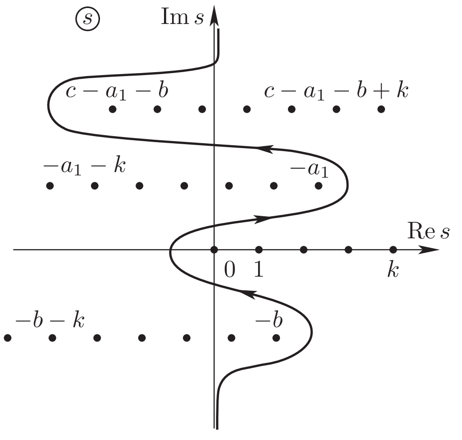

Figure 2. The contour of integration in the Mellin–Barnes representation (2.18).

2.2.1. *A formula for analytic continuation with respect to z_1 into a neighbourhood of* $(\infty, 0, \dots, 0)$. Using the integral (2.3), we obtain a representation of $F_D^{(N)}$ as a sum of two hypergeometric series which converge exponentially in the domain

$$\mathbb{D}_1^N := \{\mathbf{z} \in \mathbb{C}^N\colon |z_1| > 1,\ |\arg(-z_1)| < \pi;\ |z_k| < 1,\ k = 2, \dots, N\};$$

this is a part of the domain $\mathbb{S}_1^N$ defined in (2.1). Assuming that $b - a_1$ is not an integer and expressing the integral in (2.3) as an (infinite) sum of the residues at the simple poles $s_k^{(1)}$ and $s_k^{(2)}$ of $f(s)$, $k \in \mathbb{Z}^+$, we arrive at the following result.

Proposition 3. *If the function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ has parameters such that $b - a_1$ is not an integer, then it has the representation*

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = C_0 u_0(\mathbf{a}; b, c; \mathbf{z}) + C_1 u_1(\mathbf{a}; b, c; \mathbf{z}), \tag{2.20}$$

where u_0 and u_1 are defined by

$$u_0(\mathbf{a}; b, c; \mathbf{z}) = (-z_1)^{-a_1} \sum_{k=0}^{\infty} \frac{(a_1)_k (1 + a_1 - c)_k}{k!\,(1 + a_1 - b)_k} \times z_1^{-k} F_D^{(N-1)}(\mathbf{a}_1'; b - a_1 - k, c - a_1 - k; \mathbf{z}_1') \tag{2.21}$$

and

$$u_1(\mathbf{a}; b, c; \mathbf{z}) = (-z_1)^{-b} F_D^{(N)}\left(1 - c + b, a_2, \dots, a_N; b, 1 + b - a_1; \frac{1}{z_1}, \frac{z_2}{z_1}, \dots, \frac{z_N}{z_1}\right), \tag{2.22}$$

and the coefficients C_0 and C_1 are

$$C_0 = \frac{\Gamma(c)\Gamma(b - a_1)}{\Gamma(b)\Gamma(c - a_1)} \quad \text{and} \quad C_1 = \frac{\Gamma(c)\Gamma(a_1 - b)}{\Gamma(a_1)\Gamma(c - b)}. \tag{2.23}$$

The formulae (2.20)–(2.23) give an analytic continuation of the series (1.4) into the domain $\mathbb{D}_1^N$.

We can show that the series (2.21) converges in $\mathbb{D}_1^N$ using methods presented, for instance, in [13]. The representation (2.22) holds in $\mathbb{D}_1^N$ because it is obvious that for a vector $\mathbf{z}$ in this domain the argument $(1/z_1, z_2/z_1, \dots, z_N/z_1)$ of the function $F_D^{(N)}$ in (2.22) lies in the polydisk $\mathbb{U}^N$.

Note that Proposition 2 gives a formula for analytic continuation of (1.4) with respect to the variable z_1, whose absolute value for $\mathbf{z} \in \mathbb{D}_1^N$ is greater than 1. On the other hand, the function u_1 in (2.22) is obviously defined in the wider domain

$$\widetilde{\mathbb{D}}_1^N := \{\mathbf{z} \in \mathbb{C}^N \colon |z_1| > 1,\ |\arg(-z_1)| < \pi;\ |z_1| > \dots > |z_N|\},$$

where all the variables z_j, $j = 1, \dots, N$, can simultaneously take values with absolute value greater than 1. Thus, in (2.20) only the function u_0 must be continued with respect to the variables z_j with $j = 2, \dots, N$. The case when $b - a_1 \in \mathbb{Z}$, which we excluded in the above proposition, is a resonant case of the Lauricella system (1.5) and must be dealt with separately (see § 2.5), because it is easy to see that the relations (2.20)–(2.23) cannot be applied directly to it.

2.2.2. *Some notation.* Before considering formulae for analytic continuation of $F_D^{(N)}$ with respect to the remaining variables $z_2, \dots, z_N$, we introduce some needed notation. Let

$$\mathbf{h}_j := (a_1, \dots, a_{j-1}, 1-c+b, a_{j+1}, \dots, a_N) \quad \text{and} \quad \mathbf{a}_{s,l} := (a_s, a_{s+1}, \dots, a_l), \tag{2.24}$$

where $a_1, \dots, a_N$, b, and c are the parameters of the Lauricella function. By the modulus of a vector we will mean the sum of its components, so that, for example,

$$|\mathbf{a}_{s,j}| := \sum_{l=s}^{j} a_l \quad \text{and} \quad |\mathbf{a}| := |\mathbf{a}_{1,N}| = \sum_{l=1}^{N} a_l. \tag{2.25}$$

We define the quantities

$$\mathbf{z}^{-1} := \left(\frac{1}{z_1}, \dots, \frac{1}{z_N}\right) \quad \text{and} \quad \mathbf{z}_q^{-1} := \left(\frac{1}{z_1}, \dots, \frac{1}{z_q}, z_{q+1}, \dots, z_N\right), \tag{2.26}$$

and the following transformation of vectors $\mathbf{z} = (z_1, \dots, z_N)$:

$$\mathscr{Y}_j(\mathbf{z}) := \left(\frac{z_1}{z_j}, \dots, \frac{z_{j-1}}{z_j}, z_j, \frac{z_j}{z_{j+1}}, \dots, \frac{z_j}{z_N}\right). \tag{2.27}$$

Thus, for instance,

$$\mathscr{Y}_1(\mathbf{z}) := \left(z_1, \frac{z_1}{z_2}, \dots, \frac{z_1}{z_N}\right) \text{ and } \mathscr{Y}_j(\mathbf{z}^{-1}) := \left(\frac{z_j}{z_1}, \dots, \frac{z_j}{z_{j-1}}, \frac{1}{z_j}, \frac{z_{j+1}}{z_j}, \dots, \frac{z_N}{z_j}\right). \tag{2.28}$$

We will also use the notation

$$|\mathbf{k}_{s,j}| := \sum_{l=s}^{j} k_l \tag{2.29}$$

for a partial sum of components of the multi-index $\mathbf{k} = (k_1, \dots, k_N)$, and moreover, we will use shorthand notation for the following products:

$$\mathbf{k}! := k_1! \cdots k_N!, \qquad (\mathbf{a})_{\mathbf{k}} := (a_1)_{k_1} \cdots (a_N)_{k_N}, \qquad \mathbf{z}^{\mathbf{k}} := z_1^{k_1} \cdots z_N^{k_N}. \tag{2.30}$$

Now we write out a generalized hypergeometric series which has appeared previously, for example, in [13]:

$$G^{(N,j)}(\mathbf{a}; b, c; \mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{|\mathbf{k}_j|}(\mathbf{a})_{\mathbf{k}}}{(c)_{|\mathbf{k}_j|}\mathbf{k}!}\mathbf{z}^{\mathbf{k}}, \tag{2.31}$$

where $|\mathbf{k}_j| := |\mathbf{k}_{j,N}| - |\mathbf{k}_{1,j-1}|$ and the parameter j can take the values $1, \dots, N+1$. In (2.31) the quantity $|\mathbf{k}_j|$ can be negative. Recall that for negative integers k the Pochhammer symbol $(a)_k$ defined in (1.2) is expressed as the product (1.41).

For all $j = 1, \dots, N+1$ the domain of convergence of the series (2.31) is the unit polydisk $\mathbb{U}^N$. For $j = 1$ and $j = N+1$ the function $G^{(N,j)}$, has obvious expressions in terms of the Lauricella function:

$$G^{(N,1)}(\mathbf{a}; b; c; \mathbf{z}) = F_D^{(N)}(\mathbf{a}; b; c; \mathbf{z}), \qquad G^{(N,N+1)}(\mathbf{a}; b; c; \mathbf{z}) = F_D^{(N)}(\mathbf{a}; 1-c; 1-b; \mathbf{z}),$$

and for $N = 2$ and $j = 1$ this function coincides with (1.40).

We also set

$$\begin{aligned}\mathbb{V}_q^N := \{\mathbf{z} \in \mathbb{C}^N : |z_1| > \dots > |z_q| > 1,\ |\arg(-z_j)| < \pi,\ j = 1, \dots, q;\\ |z_l| < 1,\ l = q+1, \dots, N\}, \qquad q = 1, \dots, N,\end{aligned} \tag{2.32}$$

and

$$\mathbb{V}^N := \mathbb{V}_N^N = \{\mathbf{z} \in \mathbb{C}^N : |z_1| > \dots > |z_N| > 1,\ |\arg(-z_j)| < \pi,\ j = 1, \dots, N\}. \tag{2.33}$$

2.2.3. *Formulae for analytic continuation of $F_D^{(N)}$ into a neighbourhood of the point* $(\infty, \dots, \infty, 0, \dots, 0)$. Applying Proposition 2 to the functions $F_D^{(N-1)}$ on the right-hand side of (2.21) and to similar functions of fewer variables arising after such an application, we arrive at the following result, which lets us extend the Lauricella function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ into the domains $\mathbb{V}_q^N$, $q = 1, \dots, N$, defined in (2.32).

Theorem 1. *If the Lauricella function has parameters satisfying*

$$b - |\mathbf{a}_{1,j}| \notin \mathbb{Z}, \qquad j = 1, \dots, q \tag{2.34}$$

(recall that $|\mathbf{a}_{1,j}| = \sum_{l=1}^{j} a_l$), then an analytic continuation of the series (1.4) into the domain $\mathbb{V}_q^N$ is described by the formula

$$F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{j=0}^{q} B_{q,j} \mathscr{U}_{q,j}^{(\infty,0)}(\mathbf{a}; b, c; \mathbf{z}), \tag{2.35}$$

where the functions $\mathscr{U}_{q,j}^{(\infty,0)}$ are defined by

$$\mathscr{U}_{q,0}^{(\infty,0)}(\mathbf{a}; b, c; \mathbf{z}) := \left(\prod_{l=1}^{q} (-z_l)^{-a_l}\right) G^{(N,q+1)}(\mathbf{a}; b - |\mathbf{a}_{1,q}|, c - |\mathbf{a}_{1,q}|; \mathbf{z}_q^{-1}) \tag{2.36}$$

and

$$\mathscr{U}_{q,j}^{(\infty,0)}(\mathbf{a};b,c;\mathbf{z}) := (-z_j)^{|\mathbf{a}_{1,j-1}|-b}\left(\prod_{l=1}^{j-1}(-z_l)^{-a_l}\right) \times G^{(N,j)}\big(\mathbf{h}_j;b-|\mathbf{a}_{1,j-1}|,1+b-|\mathbf{a}_{1,j}|;\mathscr{Y}_j(\mathbf{z}^{-1})\big), \quad j=1,\dots,q, \tag{2.37}$$

$G^{(N,j)}$ *is the series* (2.31), *the vectors* $\mathbf{h}_j$, $\mathbf{z}_q^{-1}$, *and* $\mathscr{Y}_j(\mathbf{z}^{-1})$ *are defined in* (2.24), (2.26), *and* (2.28), *respectively, and the coefficients* $B_{q,j}$ *are*

$$B_{q,0} = \frac{\Gamma(c)\Gamma(b-|\mathbf{a}_{1,q}|)}{\Gamma(b)\Gamma(c-|\mathbf{a}_{1,q}|)}, \quad B_{q,j} = \frac{\Gamma(c)\Gamma(b-|\mathbf{a}_{1,j-1}|)\Gamma(|\mathbf{a}_{1,j}|-b)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}, \quad j=1,\dots,q. \tag{2.38}$$

The functions (2.36) *and* (2.37) *are linearly independent solutions of the Lauricella system of differential equations* (1.5) *in the domain* $\mathbb{V}_q^N$.

In resonant cases, when one or more of the numbers $b-|\mathbf{a}_{1,j}|$, $j=1,\dots,q$, are integers, we cannot use (2.35)–(2.38). These cases require separate consideration and can be treated by carrying out suitable limiting procedures in (2.35)–(2.38). However, the method used in [61] for analytic continuation of the Appell function F_1, that is, for $N=2$, is more convenient. We give an illustration of that method in § 2.5. In those cases formulae for analytic continuation of $F_D^{(N)}$ contain not only powers of the variables z_j, but also their logarithms.

Theorem 1 is proved by induction on the number of variables of the Lauricella function. For example, we give a proof of an important special case of this theorem, a formula for continuation of $F_D^{(N)}$ into the domain $\mathbb{V}^N$ in (2.33), that is, for continuation with respect to all the variables z_j into a neighbourhood of infinity.

Theorem 2. *If* (2.34) *is satisfied for* $q=N$, *that is, none of the numbers*

$$b-|\mathbf{a}_{1,j}|, \qquad j=1,\dots,N,$$

are integers (here $|\mathbf{a}_{1,j}| = \sum_{l=1}^{j} a_l$*), then an analytic continuation of the series* (1.4) *into the domain* $\mathbb{V}^N$ *is given by*

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = \sum_{j=0}^{N} B_j \mathscr{U}_j^{(\infty)}(\mathbf{a};b,c;\mathbf{z}), \tag{2.39}$$

where the functions $\mathscr{U}_0^{(\infty)} := \mathscr{U}_{N,0}^{(\infty,0)}$ *and* $\mathscr{U}_j^{(\infty)} := \mathscr{U}_{N,j}^{(\infty,0)}$ *are defined by*

$$\mathscr{U}_0^{(\infty)}(\mathbf{a};b,c;\mathbf{z}) = \left(\prod_{l=1}^{N}(-z_l)^{-a_l}\right) F_D^{(N)}(\mathbf{a};1+|\mathbf{a}|-c,1+|\mathbf{a}|-b;\mathbf{z}^{-1}) \tag{2.40}$$

and

$$\mathscr{U}_j^{(\infty)}(\mathbf{a};b,c;\mathbf{z}) = (-z_j)^{|\mathbf{a}_{1,j-1}|-b}\left(\prod_{l=1}^{j-1}(-z_l)^{-a_l}\right) \times G^{(N,j)}(\mathbf{h}_j;b-|\mathbf{a}_{1,j-1}|,1+b-|\mathbf{a}_{1,j}|;\mathscr{Y}_j(\mathbf{z}^{-1})), \qquad j=1,\dots,N, \tag{2.41}$$

$F_D^{(N)}$ and $G^{(N,j)}$ in (2.40) and (2.41) are the respective series (1.4) and (2.31), the vectors $\mathbf{h}_j$, $\mathbf{z}^{-1}$, and $\mathscr{Y}_j(\mathbf{z}^{-1})$ are defined in (2.24), (2.26), and (2.28), respectively, and the coefficients B_j are

$$B_0 = \frac{\Gamma(c)\Gamma(b-|\mathbf{a}|)}{\Gamma(b)\Gamma(c-|\mathbf{a}|)}, \qquad B_j = \frac{\Gamma(c)\Gamma(b-|\mathbf{a}_{1,j-1}|)\Gamma(|\mathbf{a}_{1,j}|-b)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}, \quad j=1,\dots,N. \tag{2.42}$$

The functions (2.40) and (2.41) are linearly independent solutions of the Lauricella system of differential equations (1.5) in the domain $\mathbb{V}^N$.

Proof. We prove the relations (2.39)–(2.42) using induction on the number N of variables of the function $F_D^{(N)}$.

First of all, note that for $N=1$, that is, when the Lauricella function coincides with the Gauss function, the relations (2.39)–(2.42) in the theorem are the well-known formulae (1.30), (1.31), (1.37), and (1.38) realizing an analytic continuation of $F(a,b;c;z)$ into the exterior of the unit disk. In fact, for $N=1$ the right-hand side of (2.39), like the right-hand side of (1.37), contains only the two terms $B_0\mathscr{U}_0^{(\infty)}$ and $B_1\mathscr{U}_1^{(\infty)}$, and the formulae (2.40) and (2.41) determining $\mathscr{U}_0^{(\infty)}$ and $\mathscr{U}_1^{(\infty)}$ become the canonical Kummer solutions (1.30) and (1.31), respectively. Furthermore, the equalities (2.42), from which we can determine the coefficients B_0 and B_1, coincide with the equalities (1.38) determining B_1 and B_2. The formula (2.33) for the domain $\mathbb{V}^N$ becomes the equality (1.36) for the domain $\mathbb{V}$, where we have (1.30), (1.31), (1.37), and (1.38).

Now assume that the theorem holds for the Lauricella function of $N-1$ variables and let us verify (2.39)–(2.42) for the Lauricalla function of N variables. To do this we use the representation (2.20) for $F_D^{(N)}$ which was established in Proposition 3. First of all, note that the function u_1 and the coefficient C_1 in (2.20) defined in (2.22) and (2.23), coincide with the function $\mathscr{U}_1^{(\infty)}$ and the coefficient B_1 in (2.39) defined in (2.41) and (2.42) for $j=1$, that is, the second term in (2.20) satisfies the equality

$$C_1 u_1(\mathbf{a};b,c;\mathbf{z}) = B_1\mathscr{U}_1^{(\infty)}(\mathbf{a};b,c;\mathbf{z}). \tag{2.43}$$

Next we verify that an analytic continuation of the first term in (2.20) (which is equal to $C_0u_0(\mathbf{a};b,c;\mathbf{z})$) with respect to $z_2,\dots,z_N$ gives the sum (2.39) without the term $B_1\mathscr{U}_1^{(\infty)}$, that is, we verify the equality

$$C_0 u_0(\mathbf{a};b,c;\mathbf{z}) = B_0\mathscr{U}_0^{(\infty)}(\mathbf{a};b,c;\mathbf{z}) + \sum_{j=2}^{N} B_j\mathscr{U}_j^{(\infty)}(\mathbf{a};b,c;\mathbf{z}). \tag{2.44}$$

Applying the relations (2.39)–(2.42) (which, recall, are assumed to hold for $N-1$ variables) to the functions $F_D^{(N-1)}(\mathbf{a}_1';b-a_1-k_1,c-a_1-k_1;\mathbf{z}_1')$ in (2.21), we obtain formulae for analytic continuation of these functions in the form

$$F_D^{(N-1)}(\mathbf{a}_1';b-a_1-k_1,c-a_1-k_1;\mathbf{z}_1') = {\sum_{j=0}^{N}}' \widetilde{B}_j\widetilde{\mathscr{U}_j}^{(\infty)}(\mathbf{a};b,c,k_1;\mathbf{z}_1'), \tag{2.45}$$

where the prime on a summation sign means that the term corresponding to $j = 1$ is omitted, the functions $\widetilde{\mathscr{U}_j}^{(\infty)}$ are defined by

$$\widetilde{\mathscr{U}_0}^{(\infty)}(\mathbf{a}; b, c, k_1; \mathbf{z}_1') := \left(\prod_{l=2}^{N}(-z_l)^{-a_l}\right) \times F_D^{(N)}\left(\mathbf{a}_2; 1 + |\mathbf{a}| - c + k_1, 1 + |\mathbf{a}| - b + k_1; \frac{1}{\mathbf{z}_1'}\right) \tag{2.46}$$

and

$$\widetilde{\mathscr{U}_j}^{(\infty)}(\mathbf{a}; b, c, k_1; \mathbf{z}_1') := (-z_j)^{|\mathbf{a}_{1,j-1}|-b+k_1}\left(\prod_{l=2}^{j-1}(-z_l)^{-a_l}\right) \times G^{(N-1,j)}\left(\widetilde{\mathbf{h}}_j; b - |\mathbf{a}_{1,j-1}| - k_1, 1 - |\mathbf{a}_{1,j}| + b - k_1; \mathscr{Y}_{j-1}\left(\frac{1}{\mathbf{z}_1'}\right)\right), \quad j = 2, \ldots, N, \tag{2.47}$$

and the coefficients $\widetilde{B}_j = \widetilde{B}_j(k_1)$ are

$$\widetilde{B}_0(k_1) = \frac{\Gamma(c - a_1 - k_1)\Gamma(b - |\mathbf{a}| - k_1)}{\Gamma(b - a_1 - k_1)\Gamma(c - |\mathbf{a}| - k_1)} \tag{2.48}$$

and

$$\widetilde{B}_j(k_1) = \frac{\Gamma(c - a_1 - k_1)\Gamma(b - |\mathbf{a}_{1,j-1}| - k_1)\Gamma(|\mathbf{a}_{1,j}| - b + k_1)}{\Gamma(a_j)\Gamma(b - a_1 - k_1)\Gamma(c - b)}, \quad j = 2, \ldots, N. \tag{2.49}$$

The quantities $\widetilde{\mathbf{h}}_j$ and $\mathscr{Y}_{j-1}(1/\mathbf{z}_1')$ in (2.47) are defined by

$$\widetilde{\mathbf{h}}_j = (a_2, \ldots, a_{j-1}, 1 - c + b, a_{j+1}, \ldots, a_N), \qquad j = 2, \ldots, N,$$
$$\mathscr{Y}_{j-1}\left(\frac{1}{\mathbf{z}_1'}\right) = \left(\frac{z_j}{z_2}, \ldots, \frac{z_j}{z_{j-1}}, \frac{1}{z_j}, \frac{z_{j+1}}{z_j}, \ldots, \frac{z_N}{z_j}\right), \qquad j = 2, \ldots, N.$$

In particular, for $j = 2$

$$\widetilde{\mathbf{h}}_2 = (1 - c + b, a_3, \ldots, a_N), \quad \mathscr{Y}_1\left(\frac{1}{\mathbf{z}_1'}\right) = \left(\frac{1}{z_2}, \frac{z_3}{z_2}, \ldots, \frac{z_N}{z_2}\right).$$

Substituting (2.45)–(2.49) in (2.21) and multiplying by C_0, we obtain

$$C_0 u_0(\mathbf{a}; b, c; \mathbf{z}) = \mathscr{Q}_0(\mathbf{a}; b, c; \mathbf{z}) + \sum_{j=2}^{N} \mathscr{Q}_j(\mathbf{a}; b, c; \mathbf{z}), \tag{2.50}$$

where

$$\mathscr{Q}_j(\mathbf{a}; b, c; \mathbf{z}) := C_0(-z_1)^{-a_1} \sum_{k_1=0}^{\infty} \frac{(a_1)_{k_1}(1 + a_1 - c)_{k_1}}{k_1!\,(1 + a_1 - b)_{k_1}} \times z_1^{-k_1}\widetilde{B}_j(k_1)\widetilde{\mathscr{U}_j}^{(\infty)}(\mathbf{a}; b, c, k_1; \mathbf{z}_1'). \tag{2.51}$$

We will show that for $j = 0$ and $j = 2, \dots, N$ we have

$$\mathscr{Q}_j(\mathbf{a}; b, c; \mathbf{z}) = B_j \mathscr{U}_j^{(\infty)}(\mathbf{a}; b, c; \mathbf{z}), \tag{2.52}$$

where the functions $\mathscr{U}_j^{(\infty)}$ are defined by (2.40) and (2.41) and the coefficients B_j are defined by (2.42).

I) Let us verify (2.52) for $j = 0$. To do this we transform the right-hand side of (2.51) for $j = 0$ and show that it is equal to $B_0 \mathscr{U}_0^{(\infty)}$. Setting $j = 0$ in (2.51) and substituting the values of $\widetilde{\mathscr{U}}_0^{(\infty)}$ and $\widetilde{B}_0$ from (2.46) and (2.48), we obtain

$$\begin{aligned} \mathscr{Q}_0(\mathbf{a}; b, c; \mathbf{z}) = {} & \frac{\Gamma(c)\Gamma(b-a_1)}{\Gamma(b)\Gamma(c-a_1)} (-z_1)^{-a_1} \Biggl(\prod_{l=2}^{N} (-z_l)^{-a_l} \Biggr) \\ & \times \sum_{k_1=0}^{\infty} \frac{(a_1)_{k_1} (1+a_1-c)_{k_1}}{k_1!\,(1+a_1-b)_{k_1}} \, \frac{\Gamma(c-a_1-k_1)\Gamma(b-|\mathbf{a}|-k_1)}{\Gamma(b-a_1-k_1)\Gamma(c-|\mathbf{a}|-k_1)} \\ & \times z_1^{-k_1} F_D^{(N-1)}(\mathbf{a}_1'; 1+|\mathbf{a}|-c+k_1, 1+|\mathbf{a}|-b+k_1; \mathbf{z}_1'). \end{aligned} \tag{2.53}$$

The following equalities are obtained using (1.18):

$$\frac{\Gamma(b-a_1)}{(1+a_1-b)_{k_1}\Gamma(b-a_1-k_1)} = (-1)^{k_1}, \qquad \frac{\Gamma(c-a_1)}{(1+a_1-c)_{k_1}\Gamma(c-a_1-k_1)} = (-1)^{k_1}. \tag{2.54}$$

Taking them into account, we transform (2.53) into

$$\begin{aligned} \mathscr{Q}_0(\mathbf{a}; b, c; \mathbf{z}) = {} & \frac{\Gamma(c)}{\Gamma(b)} \Biggl(\prod_{l=1}^{N} (-z_l)^{-a_l} \Biggr) \sum_{k_1=0}^{\infty} \frac{(a_1)_{k_1}}{k_1!} \, \frac{\Gamma(b-|\mathbf{a}|-k_1)}{\Gamma(c-|\mathbf{a}|-k_1)} \\ & \times z_1^{-k_1} F_D^{(N-1)}(\mathbf{a}_1'; 1+|\mathbf{a}|-c+k_1, 1+|\mathbf{a}|-b+k_1; \mathbf{z}_1'). \end{aligned} \tag{2.55}$$

Expressing the functions $F_D^{(N-1)}$ in (2.55) in terms of the hypergeometric series (1.4), we obtain

$$\begin{aligned} \mathscr{Q}_0(\mathbf{a}; b, c; \mathbf{z}) = {} & \frac{\Gamma(c)}{\Gamma(b)} \Biggl(\prod_{l=1}^{N} (-z_l)^{-a_l} \Biggr) \\ & \times \sum_{|\mathbf{k}|=0}^{\infty} \frac{\Gamma(b-|a|-k_1)(1+|a|-c+k_1)_{|\mathbf{k}_{2,N}|}}{\Gamma(c-|a|-k_1)(1+|a|-b+k_1)_{|\mathbf{k}_{2,N}|}} \, \frac{(a_1)_{k_1} \cdots (a_N)_{k_N}}{k_1! \cdots k_N!} z_1^{-k_1} \cdots z_N^{-k_N}. \end{aligned} \tag{2.56}$$

Now we can use the equalities (which follow from (1.18))

$$\begin{aligned} \frac{\Gamma(b-|a|-k_1)}{(1+|a|-b+k_1)_{|\mathbf{k}_{2,N}|}} &= (-1)^{k_1} \frac{\Gamma(b-|a|)}{(1+|a|-b)_{|\mathbf{k}|}}, \\ \frac{\Gamma(c-|a|-k_1)}{(1+|a|-c+k_1)_{|\mathbf{k}_{2,N}|}} &= (-1)^{k_1} \frac{\Gamma(c-|a|)}{(1+|a|-c)_{|\mathbf{k}|}}, \end{aligned}$$

and rewrite $\mathscr{Q}_0$ in the form

$$\mathscr{Q}_0(\mathbf{a};b,c;\mathbf{z}) = \frac{\Gamma(c)\Gamma(b-|\mathbf{a}|)}{\Gamma(b)\Gamma(c-|\mathbf{a}|)}\left(\prod_{l=1}^{N}(-z_l)^{-a_l}\right)F_D^{(N)}(\mathbf{a};1+|\mathbf{a}|-c,1+|\mathbf{a}|-b;\mathbf{z}^{-1}),$$

so that in view of (2.40) and (2.42) we arrive at the equality (2.52) for $j=0$.

II) Next we verify (2.52) for all $j=2,\dots,N$. To do this we transform the right-hand side of (2.51) to the form $B_j\mathscr{U}_j^{(\infty)}$. Substituting into (2.51) the expressions from (2.47) and (2.48) for $\widetilde{\mathscr{U}}_j^{(\infty)}$ and $\widetilde{B}_j$, $j=2,\dots,N$, we obtain

$$\begin{aligned}\mathscr{Q}_j(\mathbf{a};b,c;\mathbf{z}) &= \frac{\Gamma(c)\Gamma(b-a_1)}{\Gamma(b)\Gamma(c-a_1)\Gamma(a_j)\Gamma(c-b)}(-z_1)^{-a_1}(-z_j)^{|\mathbf{a}_{1,j-1}|-b}\left(\prod_{l=2}^{j-1}(-z_l)^{-a_l}\right)\\ &\times\sum_{k_1=0}^{\infty}\frac{(a_1)_{k_1}(1+a_1-c)_{k_1}}{k_1!\,(1+a_1-b)_{k_1}}\\ &\times\frac{\Gamma(c-a_1-k_1)\Gamma(b-|\mathbf{a}_{1,j-1}|-k_1)\Gamma(|\mathbf{a}_{1,j}|-b+k_1)}{\Gamma(b-a_1-k_1)}\\ &\times z_1^{-k_1}G^{(N-1,j-1)}\left(\widetilde{\mathbf{h}}_j;b-|\mathbf{a}_{1,j-1}|-k_1,1-|\mathbf{a}_{1,j}|+b-k_1;\mathscr{Y}_{j-1}\left(\frac{1}{\mathbf{z}_1'}\right)\right).\end{aligned}\tag{2.57}$$

Transforming this and taking (2.54) into account, we obtain

$$\begin{aligned}\mathscr{Q}_j(\mathbf{a};b,c;\mathbf{z}) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(a_j)\Gamma(c-b)}(-z_j)^{|\mathbf{a}_{1,j-1}|-b}\left(\prod_{l=1}^{j-1}(-z_l)^{-a_l}\right)\\ &\times\sum_{k_1=0}^{\infty}\frac{(a_1)_{k_1}}{k_1!}\Gamma(b-|\mathbf{a}_{1,j-1}|-k_1)\Gamma(|\mathbf{a}_{1,j}|-b+k_1)(-1)^{k_1}\left(\frac{z_j}{z_1}\right)^{k_1}\\ &\times G^{(N-1,j-1)}\left(\widetilde{\mathbf{h}}_j;b-|\mathbf{a}_{1,j-1}|-k_1,1-|\mathbf{a}_{1,j}|+b-k_1;\mathscr{Y}_{j-1}\left(\frac{1}{\mathbf{z}_1'}\right)\right).\end{aligned}\tag{2.58}$$

Expressing the functions $G^{(N-1,j-1)}$ in (2.58) as hypergeometric series (2.31),

$$\begin{aligned}&G^{(N-1,j-1)}\left(\widetilde{\mathbf{h}}_j;b-|\mathbf{a}_{1,j-1}|-k_1,1-|\mathbf{a}_{1,j}|+b-k_1;\mathscr{Y}_{j-1}\left(\frac{1}{\mathbf{z}_1'}\right)\right)\\ &\quad=\sum_{|\mathbf{k}_{2,N}|=0}^{\infty}\frac{(b-|\mathbf{a}_{1,j-1}|-k_1)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{2,j-1}|}(a_2)_{k_2}\cdots(1-c+b)_{k_j}\cdots(a_N)_{k_N}}{(1+b-|\mathbf{a}_{1,j}|-k_1)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{2,j-1}|}k_2!\cdots k_N!}\\ &\qquad\times\left(\frac{z_j}{z_2}\right)^{k_2}\cdots\left(\frac{z_j}{z_{j-1}}\right)^{k_{j-1}}\left(\frac{1}{z_j}\right)^{k_j}\left(\frac{z_{j+1}}{z_j}\right)^{k_{j+1}}\cdots\left(\frac{z_N}{z_j}\right)^{k_N},\end{aligned}$$

we get that

$$\mathscr{Q}_j(\mathbf{a};b,c;\mathbf{z})=\frac{\Gamma(c)}{\Gamma(b)\Gamma(a_j)\Gamma(c-b)}(-z_j)^{|\mathbf{a}_{1,j-1}|-b}\biggl(\prod_{l=1}^{j-1}(-z_l)^{-a_l}\biggr)$$
$$\times\sum_{|\mathbf{k}|=0}^{\infty}\frac{\Gamma(b-|\mathbf{a}_{1,j-1}|-k_1)\Gamma(|\mathbf{a}_{1,j}|-b+k_1)(b-|\mathbf{a}_{1,j-1}|-k_1)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{2,j-1}|}}{(1-|\mathbf{a}_{1,j}|+b-k_1)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{2,j-1}|}}$$
$$\times\frac{(a_1)_{k_1}\cdots(1-c+b)_{k_j}\cdots(a_N)_{k_N}}{k_1!\cdots k_N!}(-1)^{k_1}$$
$$\times\biggl(\frac{z_j}{z_1}\biggr)^{k_1}\cdots\biggl(\frac{z_j}{z_{j-1}}\biggr)^{k_{j-1}}\biggl(\frac{1}{z_j}\biggr)^{k_j}\biggl(\frac{z_{j+1}}{z_j}\biggr)^{k_{j+1}}\cdots\biggl(\frac{z_N}{z_j}\biggr)^{k_N}.$$

From the equalities

$$\Gamma(b-|\mathbf{a}_{1,j-1}|-k_1)(b-|\mathbf{a}_{1,j-1}|-k_1)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{2,j-1}|}$$
$$=\Gamma(b-|\mathbf{a}_{1,j-1}|)(b-|\mathbf{a}_{1,j-1}|)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{1,j-1}|},$$
$$\frac{\Gamma(|\mathbf{a}_{1,j}|-b+k_1)}{(1-|\mathbf{a}_{1,j}|+b-k_1)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{2,j-1}|}}=(-1)^{k_1}\frac{\Gamma(|\mathbf{a}_{1,j}|-b)}{(1-|\mathbf{a}_{1,j}|+b)_{|\mathbf{k}_{j,N}|-|\mathbf{k}_{1,j-1}|}},$$

we can rewrite $\mathscr{Q}_j$ as

$$\mathscr{Q}_j(\mathbf{a};b,c;\mathbf{z})=\frac{\Gamma(c)\Gamma(b-|\mathbf{a}_{1,j-1}|)\Gamma(|\mathbf{a}_{1,j}|-b)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}$$
$$\times(-z_j)^{|\mathbf{a}_{1,j-1}|-b}\biggl(\prod_{l=1}^{j-1}(-z_l)^{-a_l}\biggr)G^{(N,j)}\bigl(\mathbf{g}_j;b-|\mathbf{a}_{1,j-1}|,1-|\mathbf{a}_{1,j}|+b;\mathscr{Y}_j(\mathbf{z}^{-1})\bigr),$$

and thereby verify (2.52) for all $j=2,\dots,N$.

Substituting (2.43), (2.50), and (2.52) into (2.20), we obtain the required representation (2.39) for the Lauricella function.

The fact that the functions $\mathscr{U}_j^{(\infty)}$, $j=0,\dots,N$, are solutions of the Lauricella system of differential equations (1.5) can be checked by direct substitution of (2.40) and (2.41) into (1.5), and Theorem 2 is proved. □

Let S_N be the symmetric group on an N-element set, and let $\sigma(\mathbf{z})$ be the result of the action of some $\sigma\in S_N$ on the vector $\mathbf{z}$, that is, the vector obtained by a rearrangement of the components of $\mathbf{z}$. Using Theorems 1 and 2, we can establish by simple arguments formulae for analytic continuation of the Lauricella function into domains of the form

$$\mathbb{V}^N_{q,\sigma}:=\{\mathbf{z}\in\mathbb{C}^N\colon\sigma(\mathbf{z})\in\mathbb{V}^N_q\}\quad\text{and}\quad\mathbb{V}^N_{\sigma}:=\{\mathbf{z}\in\mathbb{C}^N\colon\sigma(\mathbf{z})\in\mathbb{V}^N\},$$

respectively, where σ is an arbitrary element of S_N. In fact, bearing in mind the equality

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})=F_D^{(N)}(\sigma(\mathbf{a});b,c;\sigma(\mathbf{z})),\tag{2.59}$$

which is a direct consequence of (1.4), and the observation that the inclusion $\mathbf{z}\in\mathbb{V}^N_{q,\sigma}$ (or $\mathbf{z}\in\mathbb{V}^N_{\sigma}$) means by definition that $\sigma(\mathbf{z})\in\mathbb{V}^N_q$ (respectively, $\sigma(\mathbf{z})\in\mathbb{V}^N$), we see that an analytic continuation of $F_D^{(N)}$ into the domain $\mathbb{V}_{q,\sigma}^N$ (respectively, into $\mathbb{V}_\sigma^N$) is realized by the formula (2.35) (respectively, (2.39)), where the vector $\mathbf{a}$ on the right-hand side is replaced by $\sigma(\mathbf{a})$ and $\mathbf{z}$ is replaced by $\sigma(\mathbf{z})$. Furthermore, the functions $\mathscr{U}_{q,j}^{(\infty,0)}(\sigma(\mathbf{a});b,c;\sigma(\mathbf{z}))$, $q,j=0,\dots,N$, obtained from (2.36) and (2.37) by the permutation $\sigma\in S_N$ of the components of $\mathbf{z}$ and $\mathbf{a}$, are linearly independent solutions of the Lauricella system of differential equations (1.5).

2.3. Analytic continuation into a neighbourhood of $\mathbf{z}_p^{(1,0)}$.

2.3.1. *The hypergeometric series $\mathscr{P}^{(N,p)}$ and $\mathscr{Q}_j^{(N,p)}$ and their convergence domains.* In this subsection, for the function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ with arbitrary $N\geqslant 2$ we present a complete set of formulae for analytic continuation of the form (1.27) into a neighbourhood of the points $\mathbf{z}_p^{(1,0)}\in\mathbb{C}^N$, where $p=1,\dots,N$.

We start by writing the hypergeometric series that appear in the formulae for such continuations. It will be shown below in §§ 2.3.2 and 2.3.4 that in the process of analytic continuation there arise not only the series $G^{(N,j)}$ defined by (2.31) but also the two N-variate hypergeometric series

$$\mathscr{P}^{(N,p)}(\mathbf{a};b,c_1,c_2;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty}\frac{(b)_{|\mathbf{k}|}}{(c_1)_{|\mathbf{k}_{1,p}|}(c_2)_{|\mathbf{k}_{p+1,N}|}}\,\frac{(\mathbf{a})_{\mathbf{k}}}{\mathbf{k}!}\mathbf{z}^{\mathbf{k}}, \tag{2.60}$$

$$\mathscr{Q}_j^{(N,p)}(\mathbf{a};b,c;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty}\frac{(b)_{\lambda(\mathbf{k},p,j)}}{(c)_{\lambda(\mathbf{k},p,j)}}\,\frac{(a_j+|\mathbf{k}_{p+1,N}|)_{k_j}}{(a_j)_{k_j}}\,\frac{(\mathbf{a})_{\mathbf{k}}}{\mathbf{k}!}\mathbf{z}^{\mathbf{k}}, \tag{2.61}$$

where $j=1,\dots,p$, the quantities $|\mathbf{k}_{s,l}|$ are defined in (2.29), and

$$\lambda(\mathbf{k},p,j) := |\mathbf{k}_{j,p}|-|\mathbf{k}_{1,j-1}|. \tag{2.62}$$

The convergence domains $\mathbb{P}^{(N,p)}$ and $\mathbb{Q}_j^{(N,p)}$ of the series $\mathscr{P}^{(N,p)}$ and $\mathscr{Q}_j^{(N,p)}$, $j=1,\dots,p$, respectively, are defined by

$$\mathbb{P}^{(N,p)} := \{\mathbf{z}\in\mathbb{C}^N\colon |z_s|+|z_l|<1\ \forall\, s=1,\dots,p,\ \forall\, l=p+1,\dots,N\}, \tag{2.63}$$

and

$$\mathbb{Q}_j^{(N,p)} := \{\mathbf{z}\in\mathbb{C}^N: |z_s|<1,\ s=1,\dots,p;\ |z_j|+|z_l|<1\ \forall\, l=p+1,\dots,N\}. \tag{2.64}$$

It is easy to see that for $p=N$ the series $\mathscr{P}^{(N,p)}$ coincides with the definition (1.4) of the Lauricella function $F_D^{(N)}$, and $\mathscr{Q}_j^{(N,p)}$ becomes the series $G^{(N,j)}$ given by (2.31).

2.3.2. *A formula for analytic continuation with respect to z_1 into a neighbourhood of* $(1,0,\dots,0)$. Assuming that $c-a_1-b$ is not an integer and writing the integral in (2.18) as an (infinite) sum of the residues at the simple poles $s_k^{(1)}$ and $s_k^{(2)}$ of $g(s)$, $k\in\mathbb{Z}^+$, we arrive at the following result.

Proposition 4. *If the function* $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ *given by* (1.4) *has parameters such that* $c-a_1-b$ *is not an integer, then an analytic continuation of it into the domain*

$$\mathbb{G}^N := \{|1-z_1|+|z_j|<1,\ j=2,\dots,N;\ |\arg(1-z_j)|<\pi,\ j=1,\dots,p\}$$

is given by

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = D_0 v_0(\mathbf{a};b,c;\mathbf{z}) + D_1 v_1(\mathbf{a};b,c;\mathbf{z}), \tag{2.65}$$

where v_0 *and* v_1 *are defined by*

$$v_0(\mathbf{a};b,c;\mathbf{z}) = \sum_{k=0}^{\infty} \frac{(a_1)_k(b)_k}{k!\,(1+a_1+b-c)_k}(1-z_1)^k F_D^{(N-1)}(\mathbf{a}_1';b+k,c-a_1;\mathbf{z}_1') \tag{2.66}$$

and

$$\begin{aligned} v_1(\mathbf{a};b,c;\mathbf{z}) = {} & (1-z_1)^{c-a_1-b}\biggl(\prod_{l=2}^{p}(1-z_l)^{-a_l}\biggr) \\ & \times \mathscr{Q}_1^{(N,p)}\biggl(c-|\mathbf{a}_{1,p}|,a_2,\dots,a_N;c-b,1+c-a_1-b; \\ & \qquad 1-z_1,\frac{1-z_1}{1-z_2},\dots,\frac{1-z_1}{1-z_p},z_{p+1},\dots,z_N\biggr) \end{aligned} \tag{2.67}$$

(*here* $|\mathbf{a}_{1,p}| = \sum_{l=1}^{p} a_l$) *and the coefficients* D_0 *and* D_1 *are*

$$D_0 = \frac{\Gamma(c)\Gamma(c-a_1-b)}{\Gamma(c-a_1)\Gamma(c-b)} \quad \text{and} \quad D_1 = \frac{\Gamma(c)\Gamma(a_1+b-c)}{\Gamma(a_1)\Gamma(b)}. \tag{2.68}$$

The convergence of the series (2.66) for v_0 in $\mathbb{G}^N$ can be proved using methods described in [13]. The representation (2.67) for v_1 holds in $\mathbb{G}^N$ because if $\mathbf{z}\in\mathbb{G}^N$, then the argument of the function $\mathscr{Q}_1^{(N,p)}$ in (2.67) is easily seen to vary in the domain $\mathbb{Q}_1^{(N,p)}$ defined in (2.64). The case $c-a_1-b\in\mathbb{Z}$ excluded in this proposition is a resonant case for the Lauricella system (1.5) and must be considered separately (see § 2.5), because (2.65)–(2.68) are easily seen not to be applicable and must be modified.

2.3.3. *Auxiliary notation.* Proposition 4 provides an adequate representation for the Lauricella function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ in the case when z_1 is close to 1 and the other variables z_j, $j=2,\dots,N$, lie in the unit polydisk. Before we derive formulae for analytic continuation of $F_D^{(N)}$ with respect to the variables z_j, $j=2,\dots,N$, into a neighbourhood of 1, we define the vectors

$$\begin{aligned} \mathbf{1}-\mathbf{z} &:= (1-z_1,\dots,1-z_N), \\ \mathbf{1}-\mathbf{z}_p &:= (1-z_1,\dots,1-z_p,z_{p+1},\dots,z_N), \end{aligned} \tag{2.69}$$

$$\begin{aligned} \mathbf{r}_p = \mathbf{r}_p(\mathbf{z}) &:= (z_1,\dots,z_p), \\ \mathbf{g}_{j,p} &:= (a_1,\dots,a_{j-1},c-|\mathbf{a}_{1,p}|,a_{j+1},\dots,a_N), \\ \mathbf{g}_j := \mathbf{g}_{j,N} &= (a_1,\dots,a_{j-1},c-|\mathbf{a}|,a_{j+1},\dots,a_N) \end{aligned} \tag{2.70}$$

(here the quantities $|\mathbf{a}_{s,j}|$ are given in (2.25)), and the transformations of $\mathbf{z}$

$$\mathscr{Z}_j^{N,p}(\mathbf{z}) = \bigl(\mathscr{Y}_j(\mathbf{1}-\mathbf{r}_p(\mathbf{z})), z_{p+1}, \dots, z_N\bigr), \qquad j=1,\dots,p, \tag{2.71}$$

where $\mathscr{Y}_j$ is given by (2.27), that is,

$$\mathscr{Y}_j(\mathbf{1}-\mathbf{r}_p(\mathbf{z})) = \left(\frac{1-z_1}{1-z_j}, \dots, \frac{1-z_{j-1}}{1-z_j}, 1-z_j, \frac{1-z_j}{1-z_{j+1}}, \dots, \frac{1-z_j}{1-z_p}\right) \tag{2.72}$$

for $j=1,\dots,p$, while for $p=N$

$$\mathscr{Y}_j(\mathbf{1}-\mathbf{z}) = \left(\frac{1-z_1}{1-z_j}, \dots, \frac{1-z_{j-1}}{1-z_j}, 1-z_j, \frac{1-z_j}{1-z_{j+1}}, \dots, \frac{1-z_j}{1-z_N}\right) \tag{2.73}$$

for $j=1,\dots,N$. Let $\mathbb{K}_p^N$ be the domain

$$\begin{aligned}\mathbb{K}_p^N = \{\mathbf{z}\in\mathbb{C}^N\colon 0<|1-z_1|<\dots<|1-z_p|<1;\\ |1-z_s|+|z_l|<1,\ |\arg(1-z_s)|<\pi\ \forall\, s=1,\dots,p,\ \forall\, l=p+1,\dots,N\}.\end{aligned} \tag{2.74}$$

For $p=N$ we set $\mathbb{K}^N := \mathbb{K}_N^N$.

2.3.4. *Formulae for analytic continuation of $F_D^{(N)}$ into a neighbourhood of* $(1,\dots,1,0,\dots,0)$. Applying Proposition 4 to the functions $F_D^{(N-1)}$ in (2.66) and to similar functions of fewer variables arising as a result, we arrive at the following theorem, which leads to an analytic continuation of the Lauricella function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ into the domains $\mathbb{K}_p^N$ defined by (2.74).

Theorem 3. *If the Lauricella function $F_D^{(N)}$ has parameters such that*

$$c-|\mathbf{a}_{1,j}|-b\notin\mathbb{Z}, \qquad j=1,\dots,p,$$

that is, none of the numbers $c-|\mathbf{a}_{1,j}|-b$ are integers (here $|\mathbf{a}_{1,j}|=\sum_{l=1}^{j} a_l$), then an analytic continuation of the series (1.4) into the domain $\mathbb{K}_p^N$ is given by

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = \sum_{j=0}^{p} A_{p,j}\,\mathscr{U}_{p,0}^{(1,0)}(\mathbf{a};b,c;\mathbf{z}), \tag{2.75}$$

where the functions $\mathscr{U}_{p,j}^{(1,0)}$ are defined by

$$\mathscr{U}_{p,0}^{(1,0)}(\mathbf{a};b,c;\mathbf{z}) = \mathscr{P}^{(N,p)}(\mathbf{a};b,1+|\mathbf{a}_{1,p}|+b-c,c-|\mathbf{a}_{1,p}|;\mathbf{1}-\mathbf{z}_p) \tag{2.76}$$

and for $j=1,\dots,p$,

$$\begin{aligned}\mathscr{U}_{p,j}^{(1,0)}(\mathbf{a};b,c;\mathbf{z}) = (1-z_j)^{c-|\mathbf{a}_{1,j}|-b}\biggl(\prod_{l=j+1}^{p}(1-z_l)^{-a_l}\biggr)\\ \times\,\mathscr{Q}_j^{(N,p)}\bigl(\mathbf{g}_{j,p};c-|\mathbf{a}_{1,j-1}|-b,1+c-|\mathbf{a}_{1,j}|-b,\mathscr{Z}_j^{N,p}(\mathbf{z})\bigr),\end{aligned} \tag{2.77}$$

the series $\mathscr{P}^{(N,p)}$ *and* $\mathscr{Q}_j^{(N,p)}$ *are defined in* (2.60) *and* (2.61), *the vectors* $\mathbf{g}_j$, $\mathbf{1}-\mathbf{z}_p$, *and* $\mathscr{Z}_j^{N,p}(\mathbf{z})$ *are given in* (2.70), (2.69), *and* (2.71), (2.72), *and the coefficients* $A_{p,j}$ *are*

$$\begin{aligned} A_{p,0} &= \frac{\Gamma(c)\Gamma(c-|\mathbf{a}_{1,p}|-b)}{\Gamma(c-|\mathbf{a}_{1,p}|)\Gamma(c-b)}, \\ A_{p,j} &= \frac{\Gamma(c)\Gamma(c-|\mathbf{a}_{1,j-1}|-b)\Gamma(|\mathbf{a}_{1,j}|+b-c)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}, \qquad j=1,\dots,p. \end{aligned} \tag{2.78}$$

The functions (2.76) *and* (2.77) *are linearly independent particular solutions of the Lauricella system of differential equations* (1.5).

Theorem 3 yields formulae for continuation of $F_D^{(N)}$ into the domain $\mathbb{K}^N$, that is, into a neighbourhood of 1 with respect to all the variables z_j. More precisely, the following theorem holds.

Theorem 4. *If none of the numbers* $c-|\mathbf{a}_{1,j}|-b$, $j=1,\dots,N$, *are integers (here* $|\mathbf{a}_{1,j}|=\sum_{l=1}^{j}a_l$*), then an analytic continuation of the series* (1.4) *into the domain* $\mathbb{K}^N$ *is given by*

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = \sum_{j=0}^{N} A_j \mathscr{U}_0^{(1)}(\mathbf{a};b,c;\mathbf{z}), \tag{2.79}$$

where the functions $\mathscr{U}_0^{(1)} := \mathscr{U}_{N,0}^{(1,0)}$ *and* $\mathscr{U}_j^{(1)} := \mathscr{U}_{N,j}^{(1,0)}$ *are defined by*

$$\mathscr{U}_0^{(1)}(\mathbf{a};b,c;\mathbf{z}) = F_D^{(N)}(\mathbf{a};b,1+|\mathbf{a}|+b-c;\mathbf{1}-\mathbf{z}), \tag{2.80}$$

and for $j=1,\dots,N$ *by*

$$\begin{aligned} \mathscr{U}_j^{(1)}(\mathbf{a};b,c;\mathbf{z}) &= (1-z_j)^{c-|\mathbf{a}_{1,j}|-b}\biggl(\prod_{l=j+1}^{N}(1-z_l)^{-a_l}\biggr) \\ &\quad\times G^{(N,j)}\bigl(\mathbf{g}_j;c-|\mathbf{a}_{1,j-1}|-b,1+c-|\mathbf{a}_{1,j}|-b,\mathscr{Y}_j(\mathbf{1}-\mathbf{z})\bigr), \end{aligned} \tag{2.81}$$

$F_D^{(N)}$ *and* $G^{(N,j)}$ *in* (2.80) *and* (2.81) *are the respective series* (1.4) *and* (2.31), *the vectors* $\mathbf{g}_j$, $\mathbf{1}-\mathbf{z}$, *and* $\mathscr{Y}_j(\mathbf{1}-\mathbf{z})$ *are defined in* (2.70), (2.69), *and* (2.73), *and the coefficients* A_j *are*

$$\begin{aligned} A_0 &= \frac{\Gamma(c)\Gamma(c-|\mathbf{a}|-b)}{\Gamma(c-|\mathbf{a}|)\Gamma(c-b)}, \\ A_j &= \frac{\Gamma(c)\Gamma(c-|\mathbf{a}_{1,j-1}|-b)\Gamma(|\mathbf{a}_{1,j}|+b-c)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}, \qquad j=1,\dots,N. \end{aligned} \tag{2.82}$$

The functions (2.80) *and* (2.81) *are linearly independent solutions of the Lauricella system of differential equations* (1.5).

Theorems 3 and 4 are proved by induction on the number of variables of the Lauricella function. These proofs are quite similar to the proof of Theorem 2, and we do not give them here. The restrictions on the parameters of $F_D^{(N)}$ in Theorems 3 and 4, which exclude resonant cases, can be circumvented, for instance, by carrying out suitable limiting procedures or by using the approach in [61] (see also § 2.5).

It follows from Theorem 3 and the equality (2.59) that formulae for analytic continuation of $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ into domains of the form $\mathbb{K}^N_{p,\sigma} := \{\mathbf{z}\in\mathbb{C}^N\colon \sigma(\mathbf{z})\in\mathbb{K}^N_p\}$, where the $\mathbb{K}^N_p$ are defined by (2.74) and $\sigma\in S_N$, have the form (2.75) with $\mathbf{a}$ replaced by $\sigma(\mathbf{a})$ and $\mathbf{z}$ replaced by $\sigma(\mathbf{z})$ on the right-hand side. Furthermore, the functions $\mathscr{U}^{(1,0)}_{p,j}(\sigma(\mathbf{a});b,c;\sigma(\mathbf{z}))$, $p,j=0,\dots,N$, obtained from (2.76) and (2.77) by applying the permutation $\sigma\in S_N$ to $\mathbf{z}$ and the parameter $\mathbf{a}$, are linearly independent particular solutions of the Lauricella system of differential equations (1.5).

2.4. Analytic continuation into a neighbourhood of $\mathbf{z}^{(1,\infty,0)}_{p,q}$. In this subsection we give a complete set of formulae of the form (1.27) for analytic continuation of the function $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ with arbitrary $N\geqslant 2$ into neighbourhoods of the points $\mathbf{z}^{(1,\infty,0)}_{p,q}\in\overline{\mathbb{C}}^N$, $p,q=0,\dots,N$.

2.4.1. *The hypergeometric series $\mathscr{F}^{(N,p,m)}$, $\mathscr{G}^{(N,p,m)}_j$, and $\mathscr{H}^{(N,p)}_j$ and their convergence domains.* We start with generalized hypergeometric series involved in the formulae for such an analytic continuation. First we introduce notation for the following quantities, which are expressed in terms of the partial sum (2.29) of components of the multi-index $\mathbf{k}$:

$$\begin{gathered}\varkappa(\mathbf{k},p,j) := |\mathbf{k}_{1,p}| - |\mathbf{k}_{p+1,j}| + |\mathbf{k}_{j+1,N}|, \qquad \mu(\mathbf{k},p,j) := -\varkappa(\mathbf{k},p,j),\\ \tau(\mathbf{k},p,j) := |\mathbf{k}_{p+1,j}| - |\mathbf{k}_{j+1,N}|.\end{gathered} \tag{2.83}$$

The required hypergeometric series $\mathscr{F}^{(N,p,m)}(\mathbf{a};b,c_1,c_2;\mathbf{z})$, $\mathscr{G}^{(N,p,m)}_j(\mathbf{a};b,c;\mathbf{z})$, and $\mathscr{H}^{(N,p)}_j(\mathbf{a};b,c;\mathbf{z})$ are defined by the following formulae, where we use the notation (2.29), (2.30), (2.62), and (2.83):

$$\mathscr{F}^{(N,p,m)}(\mathbf{a};b,c_1,c_2;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{\tau(\mathbf{k},p,m)}}{(c_1)_{\mu(\mathbf{k},p,m)}(c_2)_{|\mathbf{k}_{1,p}|}} \frac{(\mathbf{a})_{\mathbf{k}}}{\mathbf{k}!}\mathbf{z}^{\mathbf{k}}, \tag{2.84}$$

$$\mathscr{G}^{(N,p,m)}_j(\mathbf{a};b,c;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{\lambda(\mathbf{k},p,j)}}{(c)_{\lambda(\mathbf{k},p,j)}} \frac{(a_j+|\mathbf{k}_{m+1,N}|-|\mathbf{k}_{p+1,m}|)_{k_j}}{(a_j)_{k_j}} \frac{(\mathbf{a})_{\mathbf{k}}}{\mathbf{k}!}\mathbf{z}^{\mathbf{k}}, \tag{2.85}$$

$$\mathscr{H}^{(N,p)}_j(\mathbf{a};b,c;\mathbf{z}) := \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{\varkappa(\mathbf{k},p,j)}}{(c)_{\varkappa(\mathbf{k},p,j)}} \frac{(a_j+|\mathbf{k}_{1,p}|)_{k_j}}{(a_j)_{k_j}} \frac{(\mathbf{a})_{\mathbf{k}}}{\mathbf{k}!}\mathbf{z}^{\mathbf{k}}. \tag{2.86}$$

In (2.85) we assume that j can take the values $1,2,\dots,p$, while in (2.86) it can take the values $p+1,p+2,\dots,N$, and the index m in (2.84) and (2.85) also lies in the segment from $p+1$ to N.

Proposition 5. *The series* (2.84), (2.85), *and* (2.86) *converge in the domains* $\mathbb{F}^{N,p,m}$, $\mathbb{G}^{N,p,m}_j$, *and* $\mathbb{H}^{N,p}_j$, *respectively, which have the representations*

$$\mathbb{F}^{N,p,m} = \bigcup_{\delta\in(0,1)} \mathbb{F}^{N,p,m}(\delta), \qquad \mathbb{G}^{N,p,m}_j = \bigcup_{\delta\in(0,1)} \mathbb{G}^{N,p,m}_j(\delta), \qquad \mathbb{H}^{N,p}_j = \bigcup_{\delta\in(0,1)} \mathbb{H}^{N,p}_j(\delta),$$

where for each $\delta \in (0,1)$ *the auxiliary multicircular domains* $\mathbb{F}^{N,p,m}(\delta)$, $\mathbb{G}_j^{N,p,m}(\delta)$, *and* $\mathbb{H}_j^{N,p}(\delta)$ *are defined by*

$$\begin{aligned}\mathbb{F}^{N,p,m}(\delta) := \{\mathbf{z} \in \mathbb{C}^N\colon |z_s| < \delta,\ s = 1,\dots,p;\ |z_l| < (1+\delta)^{-1},\\ l = p+1,\dots,m;\ |z_q| < 1-\delta,\ q = m+1,\dots,N\},\\ \mathbb{G}_j^{N,p,m}(\delta) := \{\mathbf{z} \in \mathbb{C}^N\colon |z_l| < 1,\ l = 1,\dots,p,\ l \neq j,\ |z_j| < \delta;\\ |z_s| < (1+\delta)^{-1},\ s = p+1,\dots,m,\\ |z_s| < 1-\delta,\ s = m+1,\dots,N\},\end{aligned}$$

and

$$\begin{aligned}\mathbb{H}_j^{N,p}(\delta) := \{\mathbf{z} \in \mathbb{C}^N\colon |z_l| < \delta,\ l = 1,\dots,p,\ |z_j| < 1-\delta;\\ |z_s| < 1,\ s = p+1,\dots,N,\ s \neq j\}.\end{aligned}$$

Proof. To prove the above convergence properties we use the approach described, for instance, in [13]. For each of the N-variate hypergeometric series (2.84)–(2.86) we consider the set of *conjugate* radii of convergence, which are positive quantities r_j, $j = 1,\dots,N$, such that the corresponding series converges for $|z_j| < r_j$, $j = 1,\dots,N$, and diverges when the reverse inequalities hold (see [155] and [156] for details on the convergence of N-fold power series). Following [13], we calculate the quantities r_j for the series (2.84)–(2.86) by the formulae

$$r_j = |\Phi_j(\mathbf{k})|^{-1},$$

where

$$\Phi_j(\mathbf{k}) = \lim_{\varepsilon\to\infty} f_j(\varepsilon\mathbf{k}), \quad f_j(\mathbf{k}) = \frac{A(k_1,\dots,k_j+1,\dots,k_N)}{A(\mathbf{k})}, \qquad j = 1,\dots,N,$$

$A(\mathbf{k}) = A(k_1,\dots,k_N)$ being the general form for the coefficients. For example, consider the series (2.84), with coefficients of the form

$$A(\mathbf{k}) = \frac{(b)_{\tau(\mathbf{k},p,m)}}{(c_1)_{\mu(\mathbf{k},p,m)}(c_2)_{|\mathbf{k}_{1,p}|}}\,\frac{(\mathbf{a})_{\mathbf{k}}}{\mathbf{k}!}\,.$$

We see that

$$\begin{aligned} r_s &= \frac{|\mathbf{k}_{1,p}|}{r} && \text{for } s = 1,\dots,p,\\ r_l &= \frac{r}{\big|\,|\mathbf{k}_{p+1,m}| - |\mathbf{k}_{m+1,N}|\,\big|} && \text{for } l = p+1,\dots,m,\\ r_q &= \frac{\big|\,|\mathbf{k}_{p+1,m}| - |\mathbf{k}_{m+1,N}|\,\big|}{r} && \text{for } q = m+1,\dots,N,\end{aligned}$$

where $r := \big|\,|\mathbf{k}_{1,p}| - |\mathbf{k}_{p+1,m}| + |\mathbf{k}_{m+1,N}|\,\big|$. Hence $(1+r_s)r_l = 1$ and $r_q - r_s = 1$ for s, l, and q in the indicated segments. In this way we show that the series (2.84) is convergent on each set $\mathbb{F}^{N,p,m}(\delta)$ for $\delta \in (0,1)$, and therefore on $\mathbb{F}^{N,p,m}$. The proof of the other two assertions of the proposition is similar. □

2.4.2. *Domains and elementary transformations.* Let

$$\mathbb{W}^{N,p,m} := \bigcup_{\delta\in(0,1)} \mathbb{W}^{N,p,m}(\delta), \tag{2.87}$$

where for each fixed $\delta \in (0,1)$ the auxiliary domain $\mathbb{W}^{N,p,m}(\delta)$ is given by

$$\begin{aligned} \mathbb{W}^{N,p,m}(\delta) := \{\mathbf{z} \in \mathbb{C}^N\colon\ & 0 < |1-z_1| < \cdots < |1-z_p| < \delta; \\ & |\arg(1-z_j)| < \pi,\ j = 1,\dots,p; \\ & |z_{p+1}| > \cdots > |z_m| > 1+\delta; \\ & |\arg(-z_j)| < \pi,\ j = p+1,\dots,m; \\ & |z_j| < 1-\delta,\ j = m+1,\dots,N\}. \end{aligned} \tag{2.88}$$

Here the integer parameter p takes the values $0,\dots,m$, where $m = 0,\dots,N$, and if $p = 0$, then in (2.88) there are no restrictions on the variables z_j for $j = 1,\dots,p$, while if $p = m$, then there are no restrictions on the z_j for $j = p+1,\dots,m$.

We define cone domains coinciding with $\mathbb{W}^{N,p,m}$ up to certain symmetries by

$$\mathbb{W}^{N,p,m}_\sigma := \{\mathbf{z} \in \mathbb{C}^N\colon \sigma(\mathbf{z}) \in \mathbb{W}^{N,p,m}\}, \tag{2.89}$$

where we recall that $\sigma(\mathbf{z})$ is the result of the action on $\mathbf{z}$ of some element σ of the symmetric group S_N over an N-element set.

For the vectors $\mathbf{r}_p(\mathbf{z})$ in (2.70) and for

$$\mathbf{s}_{p,m} = \mathbf{s}_{p,m}(\mathbf{z}) := (z_{p+1},\dots,z_m) \tag{2.90}$$

the transformations analogous to (2.26) and (2.27) and compositions of them are defined in an obvious way. For example, we have the equality (2.72) and

$$\mathscr{Y}_j(\mathbf{s}_{p,m}^{-1}) = \biggl(\frac{z_j}{z_{p+1}},\dots,\frac{z_j}{z_{j-1}},\frac{1}{z_j},\frac{z_{j+1}}{z_j},\dots,\frac{z_m}{z_j}\biggr), \qquad j = p+1,\dots,m. \tag{2.91}$$

We will also use the auxiliary functions $\mathscr{Z}_j^{(N,p,m)}(\mathbf{z})$ defined for $m = 1,\dots,N$ and $p, j = 0,\dots,m$ by the formulae

$$\mathscr{Z}_0^{(N,p,m)}(\mathbf{z}) := (z_1-1,\dots,z_p-1,z_{p+1}^{-1},\dots,z_m^{-1},z_{m+1},\dots,z_N), \tag{2.92}$$

$$\mathscr{Z}_j^{(N,p,m)}(\mathbf{z}) := \bigl(\mathscr{Y}_j(\mathbf{1}-\mathbf{r}_p(\mathbf{z})),z_{p+1}^{-1},\dots,z_m^{-1},z_{m+1},\dots,z_N\bigr) \tag{2.93}$$

for $j = 1,\dots,p$, and

$$\mathscr{Z}_j^{(N,p,m)}(\mathbf{z}) := \biggl(\frac{z_1-1}{z_j},\dots,\frac{z_p-1}{z_j},\mathscr{Y}_j(\mathbf{s}_{p,m}^{-1}),z_{m+1},\dots,z_N\biggr) \tag{2.94}$$

for $j = p+1,\dots,m$; here we have used the definitions (2.72), (2.90), and (2.91), and expressions of the form $\mathbf{f} = (p_1,\dots,p_n,\mathbf{q})$ or $\mathbf{f} = (\mathbf{p},q_1,\dots,q_m)$, where $\mathbf{p} = (p_1,\dots,p_n)$ and $\mathbf{q} = (q_1,\dots,q_m)$, mean that $\mathbf{f} = (p_1,\dots,p_n,q_1,\dots,q_m)$. If $p = 0$, then we use the equalities (2.92) and (2.94) to define the functions $\mathscr{Z}_j^{(N,p,m)}(\mathbf{z})$, but if $p = N$, then these functions are found from (2.92) and (2.93), while (2.94) is not used in the definition.

Now consider the vectors $\mathbf{g}_{j,m}$ and $\mathbf{h}_{j,p}$, which are expressed in terms of the parameters $a_1, \dots, a_N$, b, and c of the Lauricella function by means of the equalities

$$\mathbf{g}_{j,m} := (a_1, \dots, a_{j-1}, c - |\mathbf{a}_{1,m}|, a_{j+1}, \dots, a_N) \tag{2.95}$$

for $j = 1, \dots, p$ and

$$\mathbf{h}_{j,p} := (a_1, \dots, a_{j-1}, 1 - c + |\mathbf{a}_{1,p}| + b, a_{j+1}, \dots, a_N) \tag{2.96}$$

for $j = p+1, \dots, m$, where the quantities $|\mathbf{a}_{s,l}|$ are defined by (2.25). In § 2.4.3 the vectors (2.92)–(2.94) and (2.95), (2.96) play the role of the variables and the parameters, respectively, of the generalized hypergeometric functions used in the formulae for analytic continuation of the series (1.4).

2.4.3. *Formulae for analytic continuation.* Using Theorem 1, we extend the functions $\mathscr{U}_{p,j}^{(1,0)}$ mentioned in Theorem 3, $j = 0, \dots, p$, into a neighbourhood of infinity and thereby obtain formulae for analytic continuation of the Lauricella function $F_D^{(N)}$ into a neighbourhood of the point $\mathbf{z}_{p,q}^{(1,\infty,0)}$.

We express the definitions (2.60) and (2.61) of the series $\mathscr{P}^{(N,p)}$ and $\mathscr{Q}_j^{(N,p)}$ in the form

$$\mathscr{P}^{(N,p)}(\mathbf{a}; b, c_1, c_2; \mathbf{z}) = \sum_{|\mathbf{k}_{1,p}|=0}^{\infty} \frac{(b)_{|\mathbf{k}_{1,p}|}}{(c_1)_{|\mathbf{k}_{1,p}|}} \frac{(a_1)_{k_1} \cdots (a_p)_{k_p}}{k_1! \cdots k_p!} \times z_1^{k_1} \cdots z_p^{k_p} F_D^{(N-p)}(a_{p+1}, \dots, a_N; b + |\mathbf{k}_{1,p}|, c_2; z_{p+1}, \dots, z_N),$$

$$\mathscr{Q}_j^{(N,p)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{(b)_{\lambda(\mathbf{k},p,j)}}{(c)_{\lambda(\mathbf{k},p,j)}} \frac{(a_1)_{k_1} \cdots (a_p)_{k_p}}{k_1! \cdots k_p!} \times z_1^{k_1} \cdots z_p^{k_p} F_D^{(N-p)}(a_{p+1}, \dots, a_N; a_j + k_j, a_j; z_{p+1}, \dots, z_N),$$

and then we represent the functions $\mathscr{U}_{p,j}^{(1,0)}$, $j = 0, \dots, p$, given by (2.76) and (2.77) in the form

$$\mathscr{U}_{p,0}^{(1,0)}(\mathbf{a}; b, c; \mathbf{z}) = \sum_{|\mathbf{k}_{1,p}|=0}^{\infty} \frac{(b)_{|\mathbf{k}_{1,p}|}}{(1 + |\mathbf{a}_{1,p}| + b - c)_{|\mathbf{k}_{1,p}|}} \frac{(a_1)_{k_1} \cdots (a_p)_{k_p}}{k_1! \cdots k_p!} \times (1 - z_1)^{k_1} \cdots (1 - z_p)^{k_p} \times F_D^{(N-p)}(a_{p+1}, \dots, a_N; b + |\mathbf{k}_{1,p}|, c - |\mathbf{a}_{1,p}|; z_{p+1}, \dots, z_N) \tag{2.97}$$

and

$$\mathscr{U}_{p,j}^{(1,0)}(\mathbf{a};b,c;\mathbf{z}) = (1-z_j)^{c-|\mathbf{a}_{1,j}|-b}\left(\prod_{l=j+1}^{p}(1-z_l)^{-a_l}\right)$$
$$\times \sum_{|\mathbf{k}_{1,p}|=0}^{\infty} \frac{(c-|\mathbf{a}_{1,j-1}|-b)_{\lambda(\mathbf{k},p,j)}}{(1+c-|\mathbf{a}_{1,j}|-b)_{\lambda(\mathbf{k},p,j)}} \frac{(a_1)_{k_1}\cdots(c-|\mathbf{a}_{1,p}|)_{k_j}\cdots(a_p)_{k_p}}{k_1!\cdots k_p!}$$
$$\times \left(\frac{1-z_1}{1-z_j}\right)^{k_1}\cdots\left(\frac{1-z_{j-1}}{1-z_j}\right)^{k_{j-1}}(1-z_j)^{k_j}\left(\frac{1-z_j}{1-z_{j+1}}\right)^{k_{j+1}}\cdots\left(\frac{1-z_j}{1-z_p}\right)^{k_p}$$
$$\times F_D^{(N-p)}(a_{p+1},\ldots,a_N;c-|\mathbf{a}_{1,p}|+k_j,c-|\mathbf{a}_{1,p}|;z_{p+1},\ldots,z_N), \quad j=1,\ldots,p. \tag{2.98}$$

Continuing the functions $F_D^{(N-p)}$ in (2.97) and (2.98) analytically by the formulae (2.35)–(2.38), we arrive at the following result establishing formulae for analytic continuation of $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ into the domains $\mathbb{W}^{N,p,m}$ of the form (2.87), where below we use the notation from §§ 2.4.1 and 2.4.2 and the quantities $|\mathbf{a}_{s,j}|$ are defined by (2.25).

Theorem 5. *If none of the numbers*

$$c-|\mathbf{a}_{1,j}|-b, \quad j=1,\ldots,p, \qquad b-|\mathbf{a}_{p+1,j}|, \quad j=p+1,\ldots,N,$$

are integers, then an analytic continuation of (1.4) *into the domains* $\mathbb{W}^{N,p,m}$ *with arbitrary* $m=0,\ldots,N$ *and* $p=0,\ldots,m$ *is given by the formula*

$$F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) = \sum_{j=0}^{m} A_j^{p,m}\mathscr{U}_j^{(N,p,m)}(\mathbf{a};b,c;\mathbf{z}), \tag{2.99}$$

where for $j=0,\ldots,p$ *the functions* $\mathscr{U}_j^{(N,p,m)}$ *are defined by*

$$\mathscr{U}_0^{(N,p,m)}(\mathbf{a};b,c;\mathbf{z}) := \left(\prod_{l=p+1}^{m}(-z_l)^{-a_l}\right)$$
$$\times \mathscr{F}^{(N,p,m)}\big(\mathbf{a};1+|\mathbf{a}_{1,m}|-c,1+|\mathbf{a}_{p+1,m}|-b,1+|\mathbf{a}_{1,p}|+b-c;\mathscr{Z}_0^{(N,p,m)}(\mathbf{z})\big), \tag{2.100}$$

$$\mathscr{U}_j^{(N,p,m)}(\mathbf{a};b,c;\mathbf{z}) := (1-z_j)^{c-|\mathbf{a}_{1,j}|-b}\left(\prod_{l=j+1}^{p}(1-z_l)^{-a_l}\right)\left(\prod_{l=p+1}^{m}(-z_l)^{-a_l}\right)$$
$$\times \mathscr{G}_j^{(N,p,m)}\big(\mathbf{g}_{j,m};c-|\mathbf{a}_{1,j-1}|-b,1+c-|\mathbf{a}_{1,j}|-b;\mathscr{Z}_j^{(N,p,m)}(\mathbf{z})\big), \tag{2.101}$$

and for $j=p+1,\ldots,m$ *they are defined by*

$$\mathscr{U}_j^{(N,p,m)}(\mathbf{a};b,c;\mathbf{z}) := (-z_j)^{|\mathbf{a}_{p+1,j-1}|-b}\left(\prod_{l=p+1}^{j-1}(-z_l)^{-a_l}\right)$$
$$\times \mathscr{H}_j^{(N,p,m)}\big(\mathbf{h}_{j,p};b-|\mathbf{a}_{p+1,j-1}|,1+b-|\mathbf{a}_{p+1,j}|;\mathscr{Z}_j^{(N,p,m)}(\mathbf{z})\big), \tag{2.102}$$

the series $\mathscr{F}^{(N,p,m)}$, $\mathscr{G}_j^{(N,p,m)}$, *and* $\mathscr{H}_j^{(N,p)}$ *are defined in* (2.84)–(2.86), *and the vectors* $\mathscr{Z}_j^{(N,p,m)}(\mathbf{z})$, $\mathbf{g}_{j,m}$, *and* $\mathbf{h}_{j,p}$ *are given in* (2.92)–(2.96).

For $j = 0,\dots,p$ *the coefficients* $A_j^{p,m}$ *in* (2.99) *are*

$$
\begin{aligned}
A_0^{p,m} &= \frac{\Gamma(c)\Gamma(b-|\mathbf{a}_{p+1,m}|)\Gamma(c-|\mathbf{a}_{1,p}|-b)}{\Gamma(b)\Gamma(c-|\mathbf{a}_{1,m}|)\Gamma(c-b)}\,,\\
A_j^{p,m} &= \frac{\Gamma(c)\Gamma(c-|\mathbf{a}_{1,j-1}|-b)\Gamma(|\mathbf{a}_{1,j}|+b-c)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}\,, \qquad j=1,\dots,p,
\end{aligned}
\tag{2.103}
$$

while for $j = p+1,\dots,m$ *they are*

$$
A_j^{p,m} = \frac{\Gamma(c)\Gamma(b-|\mathbf{a}_{p+1,j-1}|)\Gamma(|\mathbf{a}_{p+1,j}|-b)}{\Gamma(a_j)\Gamma(b)\Gamma(c-b)}\,. \tag{2.104}
$$

The functions $\mathscr{U}_j^{(N,p,m)}$ *given by* (2.100)–(2.102) *are linearly independent solutions of the Lauricella system of differential equations* (1.5).

The proof of the formulae (2.99)–(2.104) for analytic continuation uses induction on the number of variables of the Lauricella function and is quite similar to the proof of Theorem 2. We can see that the functions $\mathscr{U}_j^{(N,p,m)}$, $j = 0,\dots,m$, are particular solutions of (1.5) by substituting (2.100)–(2.102) directly into (1.5). The restrictions on the parameters of the function $F_D^{(N)}$ in Theorem 5, which exclude resonant (logarithmic) cases, can be circumvented by means of suitable limiting procedures or by the approach in [61] (see also § 2.5).

Using simple arguments, we can deduce from Theorem 5 formulae for analytic continuation of the Lauricella function into the domains $\mathbb{W}_\sigma^{N,p,m}$ defined by (2.89) for $m = 0,\dots,N$, $p = 0,\dots,m$, and $\sigma \in S_N$, where we recall that S_N is the symmetric group over an N-element set. In fact, in view of the symmetry property (2.59) of the Lauricella function, which is a direct consequence of the definition (1.4), and the fact that $\mathbf{z} \in \mathbb{W}_\sigma^{N,p,m}$ means by definition that $\sigma(\mathbf{z}) \in \mathbb{W}^{N,p,m}$, we see that an analytic continuation of $F_D^{(N)}$ into the domain $\mathbb{W}_\sigma^{N,p,m}$ is realized by the formula (2.99) with the parameter $\mathbf{a}$ replaced by $\sigma(\mathbf{a})$ and the argument $\mathbf{z}$ replaced by $\sigma(\mathbf{z})$ on the right-hand side (that is, in the coefficients $A_j^{p,m} = A_j^{p,m}(\mathbf{a};b,c;\mathbf{z})$ and the functions $\mathscr{U}_j^{(N,p,m)}(\mathbf{a};b,c;\mathbf{z})$ given by (2.100)–(2.104)). Also, the functions $\mathscr{U}_{j,\sigma}^{(N,p,m)} := \mathscr{U}_j^{(N,p,m)}\bigl(\sigma(\mathbf{a});b,c;\sigma(\mathbf{z})\bigr)$ obtained from (2.100)–(2.102) by applying $\sigma \in S_N$ are linearly independent particular solutions of (1.5).

We can show that

$$
\mathfrak{A}^{(N)} := \{\mathscr{U}_{j,\sigma}^{(N,p,m)},\ m=1,\dots,N,\ p=0,\dots,m,\ \sigma\in S^N\} \tag{2.105}
$$

is a complete set of solutions of the Lauricella system of differential equations (1.5) in the domain $\mathbb{W} := \bigcup_{m,p,\sigma} \mathbb{W}_\sigma^{(N,p,m)}$. For $N = 1$ the functions in $\mathfrak{A}^{(N)}$ become the well-known solutions found by Kummer for the classical hypergeometric equation [30], [31]. For $N = 2$ such a system of solutions was constructed in [10] and [11], and for $N = 3$ it was indicated in [13], apart from certain exceptions. For $N \geqslant 3$ the complete set of functions in $\mathfrak{A}^{(N)}$ was found in [57]–[60].

2.5. Logarithmic case. As mentioned above, the formulae in § 1.3 for analytic continuation of the Gauss function, as well as the formulae in Theorems 1–5 for continuation of the Lauricella function, must be modified for the special *resonant* sets of values of the parameters. Such resonant cases requiring special consideration are also said to be *logarithmic* because the solutions of Gauss's equation (1.3) and of the Lauricella system of differential equations (1.5) contain logarithmic terms in addition to power terms. We start with several known results for the Gauss function.

2.5.1. *Analytic continuation of the Gauss function in the logarithmic case.* If the parameters a, b, and c of the hypergeometric equation (1.3) are such that $c-a-b$ is an integer, then we cannot define two linearly independent solutions $u_1^{(1)}$ and $u_2^{(1)}$ of (1.3) using (1.28) and (1.29). In fact, if $c-a-b=0$, then it is easy to see that the right-hand sides of (1.28) and (1.29) are equal. Further, if $c-a-b\in\mathbb{Z}\setminus\{0\}$, then the third parameter of some function F in these formulae is a non-positive integer $-m$, and all the terms of the hypergeometric series (1.1) for this function, beginning with the mth, become infinite. Clearly, an analogous remark also applies to (1.30) and (1.31), which do not define two linearly independent solutions $u_1^{(\infty)}$ and $u_2^{(\infty)}$ for integer $b-a$.

In the above special cases, when at least one of the relations

$$c-a-b\in\mathbb{Z},\quad b-a\in\mathbb{Z}$$

holds for a,b, and c, a solution of the hypergeometric equation (1.3) contains logarithms of z and $1-z$ in addition to powers of them. To define analogues of the canonical solutions (1.28)–(1.31) in the logarithmic case it is convenient to consider the series

$$F^{\pm}_{\log}(a,b;1-m;z):=\sum_{k=0}^{m-1}\frac{(a)_k(b)_k}{k!\,(1-m)_k}z^k \\ +\frac{(-1)^m}{(m-1)!}\sum_{k=m}^{\infty}\frac{(a)_k(b)_k}{k!\,(k-m)!}[h_k^{\pm}(a,b,m)-\log(\pm z)]z^k, \tag{2.106}$$

where the numbers $h_k^{\pm}(a,b,m)$ are defined by

$$h_k^{+}(a,b,m):=\widetilde{h}_k-\psi(b+k),\qquad h_k^{-}(a,b,m):=\widetilde{h}_k-\psi(1-b-k), \\ \widetilde{h}_k:=\psi(1-m+k)+\psi(1+k)-\psi(a+k), \tag{2.107}$$

$\psi(s)=\Gamma'(s)/\Gamma(s)$ being the logarithmic derivative of the gamma function, with the first sum in (2.106) taken to be 0 for $m=0$ and 1 for $m=1$. Using the series $F^{\pm}_{\log}$ defined by (2.106) and (2.107), we can simplify the standard notation for canonical solutions and the formulae for analytic continuation that can be found in [30], for instance.

Let $c=a+b+m$, where $m\in\mathbb{Z}^+$ is arbitrary. Then the following functions play the role of the canonical solutions (1.28) and (1.29) of the equation (1.3) in a neighbourhood of $z=1$:

$$u_1^{(1)}(a,b;a+b+m;z) = F_{\log}^{+}(a,b;1-m;1-z), \tag{2.108}$$

$$u_2^{(1)}(a,b;a+b+m;z) = (1-z)^m F(a+m,b+m;1+m;1-z). \tag{2.109}$$

On the other hand, if $c=a+b-m$, where $m\in\mathbb{Z}^+$, then

$$u_1^{(1)}(a,b;a+b-m;z) = F(a,b;1+m;1-z), \tag{2.110}$$

$$u_2^{(1)}(a,b;a+b-m;z) = (1-z)^{-m} F_{\log}^{+}(a-m,b-m;1-m;1-z). \tag{2.111}$$

Let $b=a+m$ for some non-negative integer m. Then the role of the canonical solutions (1.30) and (1.31) of (1.3) in a neighbourhood of $z=\infty$ is played by

$$u_1^{(\infty)}(a,a+m;c;z) = (-z)^{-a} F_{\log}^{-}(a,1-c+a;1-m;z^{-1}), \tag{2.112}$$

$$u_2^{(\infty)}(a,a+m;c;z) = (-z)^{-a-m} F(a+m,1-c+a+m;1+m;z^{-1}). \tag{2.113}$$

But if $a=b+m$, where $m\in\mathbb{Z}^+$, then the system of canonical solutions of (1.3) is

$$u_1^{(\infty)}(b+m,b;c;z) = (-z)^{-b-m} F(b+m,1-c+b+m;1+m;z^{-1}), \tag{2.114}$$

$$u_2^{(\infty)}(b+m,b;c;z) = (-z)^{-b} F_{\log}^{-}(b,1-c+b;1-m;z^{-1}). \tag{2.115}$$

The functions $u_j^{(1)}$ and $u_j^{(\infty)}$ defined by (2.108)–(2.115), $j=1,2$, form a basis for analytic continuation of the series (1.1) into the exterior of the unit disk in the case when the parameters a, b, and c are connected by the above special relations. Namely, the formula for analytic continuation of $F(a,b;c;z)$ into the domain (1.33) in the case when $c=a+b+m$ for $m\in\mathbb{Z}^+$ has the form

$$F(a,b;a+b+m;z) = \mathrm{A}_1 u_1^{(1)}(a,b;a+b+m;z), \qquad \mathrm{A}_1 = \frac{\Gamma(a+b+m)(m-1)!}{\Gamma(a+m)\Gamma(b+m)}\,, \tag{2.116}$$

where $u_1^{(1)}$ is given by (2.108), while if $c=a+b-m$ for $m\in\mathbb{N}$, then

$$F(a,b;a+b-m;z) = A_2 u_2^{(1)}(a,b;a+b-m;z), \qquad A_2 = \frac{\Gamma(a+b-m)(m-1)!}{\Gamma(a)\Gamma(b)}\,. \tag{2.117}$$

The formula for analytic continuation of the series (1.1) into the domain (1.36) in the case when $b=a+m$ with $m\in\mathbb{Z}^+$ has the form

$$F(a,a+m;c;z) = B_1 u_1^{(\infty)}(a,a+m;c;z), \qquad B_1 = \frac{\Gamma(c)(m-1)!}{\Gamma(a+m)\Gamma(c-a)}\,, \tag{2.118}$$

where the function $u_1^{(\infty)}$ is given by (2.112). But if $a=b+m$ with $m\in\mathbb{N}$, then a continuation into the domain (1.36) is given by

$$F(b+m,b;c;z) = B_2 u_2^{(\infty)}(b+m,a;c;z), \qquad B_2 = \frac{\Gamma(c)(m-1)!}{\Gamma(b+m)\Gamma(c-b)}\,, \tag{2.119}$$

where the function $u_2^{(\infty)}$ is defined in (2.115).

Note that the formulae (2.116)–(2.119) for analytic continuation in the logarithmic case contain only one canonical solution of (1.3), while in the general case the right-hand sides of (1.34) and (1.37) involve two canonical solutions.

2.5.2. *Analytic continuation of the Appell function F_1 in the logarithmic case.* In Theorems 1–5 giving formulae for analytic continuation of $F_D^{(N)}$ the restrictions on the parameters of the function exclude the *resonant*, or *logarithmic* case. We discuss this case in this subsection.

We show the necessity of a separate discussion of this case by taking the example of Theorems 1 and 2. We will show that the formulae (2.35) and (2.39) for analytic continuation cannot be used directly in the case when for some index k and some integer m we have

$$b-|\mathbf{a}_{1,k}|=m, \qquad m\in\mathbb{Z}. \tag{2.120}$$

Note that the formulae (2.35) and (2.39) in Theorems 1 and 2 for analytic continuation into domains with large absolute values of the variables z_k contain two types of hypergeometric series $F_D^{(N)}$ and $G^{(N,j)}$, $j=1,\dots,N$, defined by (1.4) and (2.31), respectively. It is easy to see that non-positive integer values of c are 'singular' for the Lauricella series (1.4). In fact, if $c=-m$ with $m\in\mathbb{Z}^+$, then from the definition (1.2) of the Pochhammer symbol it follows that all the terms in (1.4) with $|\mathbf{k}|$ (the sum of the components of the multi-index) greater than m become infinite. As regards the series $G^{(N,j)}$, $j=1,\dots,N$, non-positive integer values $c\in\mathbb{Z}^-$ and positive integer values $b\in\mathbb{N}$ are singular for them. Recalling also that the gamma function $\Gamma(s)$ has poles at $s\in\mathbb{Z}^-$, we easily see that if for some index k the restrictions on the parameters of the Lauricella function indicated in Theorems 1 and 2 do not hold, so that $b-|\mathbf{a}_{1,k}|=m$ for some $m\in\mathbb{Z}$, then the quantity $B_{q,k}\mathscr{U}_{q,k}^{(\infty,0)}$ and one of the neighbouring terms in the formula (2.35) are not defined, and similarly, the quantity $B_k\mathscr{U}_k^{(\infty)}$ and one of the neighbouring terms are not defined in (2.39). Hence, if in Theorems 1 and 2 the conditions imposed on the parameters are not satisfied, so that a resonant case occurs, then the corresponding representations (2.35) and (2.39) cannot be used. In particular, the formula (1.50), (1.51) for analytic continuation of the Appell function F_1 (which is the special case of Theorem 2 for $N=2$) cannot be applied directly. Thus, the resonant case of values of the parameters must be considered separately.

To construct formulae for analytic continuation of the Lauricella function $F_D^{(N)}$ in resonant cases, we use first of all the Mellin–Barnes representations in Propositions 1 and 2, and then with their help we obtain analogues of Propositions 3 and 4. Here one must take into account that the integrands in the formulae (2.3) and (2.18) have not only simple but also double poles. The rest of the argument is mostly the same as in deducing the analytic continuation results in Theorems 1–5.

In this paper we do not write out the complete set of formulae for analytic continuation of $F_D^{(N)}$ in the resonant case. In § 5 we use such formulae (for resonant cases of the parameters of $F_D^{(N)}$) for high-precision calculation of the parameters of the Schwarz–Christoffel integral when there is crowding. One example here is a modification of the formula (1.50) for analytic continuation of the Appell function F_1 in the logarithmic case (for a complete set of formulae for analytic continuation of F_1 see [61]).

We start with the following analogue of the function G in (1.40) for non-positive integer values of c:

$$G^-_{\log}(a,a';b,1-m;z,\zeta) := \sum_{k=0}^{\infty}\biggl\{\sum_{n=0}^{m+k-1}\frac{(b)_{n-k}(a)_k(a')_n}{(1-m)_{n-k}k!\,n!}z^k\zeta^n + \frac{(-1)^m}{(m-1)!}\sum_{n=m+k}^{\infty}\frac{(b)_{n-k}(a)_k(a')_n}{(n-k-m)!\,k!\,n!}[\varkappa^-_{k,n}-\log(-\zeta)]z^k\zeta^n\biggr\}, \tag{2.121}$$

where the quantities $\varkappa^-_{k,n}$ are given by

$$\varkappa^-_{k,n} := \psi(1+n)+\psi(1-m+n-k)-\psi(1-a'-n)+\psi(b+n-k). \tag{2.122}$$

Now let $b = a+m$, where $m \in \mathbb{Z}^+$. In this case the following three functions are the analogues of the solutions (1.43)–(1.45):

$$\mathscr{U}^{(\infty)}_0(a,a';a+m,c;z,\zeta) = (-z)^{-a}(-\zeta)^{-a'} \times F_1\biggl(a,a';1-c+a+a',1+a'-m;\frac{1}{z},\frac{1}{\zeta}\biggr), \tag{2.123}$$

$$\mathscr{U}^{(\infty)}_1(a,a';a+m,c;z,\zeta) = (-z)^{-a}(-\zeta)^{-m} \times G^-_{\log}\biggl(a,1-c+a+m;m,1-a'+m;\frac{\zeta}{z},\frac{1}{\zeta}\biggr), \tag{2.124}$$

$$\mathscr{U}^{(\infty)}_2(a,a';a+m,c;z,\zeta) = (-z)^{-b}F_1\biggl(1-c+a+m,a';a+m,1+m;\frac{1}{z},\frac{\zeta}{z}\biggr), \tag{2.125}$$

where the generalized hypergeometric series $G^-_{\log}$ is defined in (2.121), (2.122). A formula extending the Appell function F_1 into the domain $\mathbb{V}^2$ defined in (2.33) has the form

$$F_1(a,a';a+m,c;z,\zeta) = B_0\mathscr{U}^{(\infty)}_0(a,a';a+m,c;z,\zeta)+B_1\mathscr{U}^{(\infty)}_1(a,a';a+m,c;z,\zeta), \tag{2.126}$$

where the coefficients B_0 and B_1 are

$$B_0 = \frac{\Gamma(c)\Gamma(m-a')}{\Gamma(a+m)\Gamma(c-a-a')} \quad\text{and}\quad B_1 = \frac{\Gamma(c)\Gamma(a'-m)(m-1)!}{\Gamma(a')\Gamma(a+m)\Gamma(c-a-m)}.$$

Note that only two (of the three) linearly independent solutions of the system $E^{(2)}_D$ are involved in (2.126).

3. Jacobi-type formulae for the Lauricella function $F^{(N)}_D$ and their application to the Riemann–Hilbert problem

3.1. Jacobi identity for the Gauss function $F(a,b;c;z)$ and its generalization for $F^{(N)}_D$. In Gauss's equation (1.3) we replace the parameters a, b, and c by

$a-1$, $b-1$, and $c-1$, respectively, so that the function $u(z) = F(a-1, b-1; c-1; z)$ is now a solution, and we rewrite the new equation as

$$\frac{d}{dz}\left[z^{c-1}(1-z)^{a+b-c}\,\frac{du(z)}{dz}\right] = (a-1)(b-1)z^{c-2}(1-z)^{a+b-c-1}u(z). \tag{3.1}$$

Using the differentiation formula

$$\frac{d}{dz}F(a-1, b-1; c-1; z) = \frac{(a-1)(b-1)}{c-1}F(a, b; c; z),$$

we arrive at the familiar Jacobi identity for the Gauss function [117] (see also [30]):

$$\frac{d}{dz}[z^{c-1}(1-z)^{a+b-c}F(a, b; c; z)] = (c-1)z^{c-2}(1-z)^{a+b-c-1}F(a-1, b-1; c-1; z). \tag{3.2}$$

The above arguments (also presented in [30], for instance) show that (3.2) is a consequence of Gauss's differential equation (1.3).

To state a generalization of the Jacobi identity (3.2) for the Lauricella function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ (see [57], [65], [66]), we introduce some further notation. First of all, let

$$\mathbf{e}_j := (0, \dots, 1, \dots, 0)$$

denote the vector with jth component 1 and the others equal to 0. Subtracting $\mathbf{e}_j$ from a vector $\mathbf{a} = (a_1, \dots, a_N)$ decreases the jth component of $\mathbf{a}$ by 1, that is,

$$\mathbf{a} - \mathbf{e}_j = (a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_N). \tag{3.3}$$

Similarly, the vector obtained from the one in (3.3) by increasing its sth component ($s \neq j$) by 1 can be expressed as

$$\mathbf{a} - \mathbf{e}_j + \mathbf{e}_s = (a_1, \dots, a_j - 1, \dots, a_s + 1, \dots, a_N). \tag{3.4}$$

The vectors $\mathbf{a}'_j$ and $\mathbf{z}'_j$ are obtained from $\mathbf{a}$ and $\mathbf{z}$ by eliminating the jth component:

$$\mathbf{a}'_j := (a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_N), \qquad \mathbf{z}'_j := (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_N). \tag{3.5}$$

As in the previous sections, the modulus of a vector is understood to be the sum of its components; for example, for $\mathbf{a}'_j$ in (3.5) we have

$$|\mathbf{a}'_j| := \sum_{1\leqslant s\leqslant N,\ s\neq j} a_s.$$

The following statement establishes an analogue of the identity (3.2) for the Lauricella function.

Theorem 6. *The Lauricella function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ satisfies the following differential equations of Jacobi type*:

$$\frac{\partial}{\partial z_j}\left\{\left[\prod_{p=1}^{N}{}'(z_j - z_p)^{a_p}\right] z_j^{c-|\mathbf{a}'_j|-1}(1-z_j)^{a_j+b-c}F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})\right\}$$
$$= \left[\prod_{p=1}^{N}{}'(z_j - z_p)^{a_p-1}\right] z_j^{c-|\mathbf{a}'_j|-2}(1-z_j)^{a_j+b-c-1}\mathscr{R}_j(\mathbf{a}; b, c; \mathbf{z}), \quad j = 1, \dots, N, \tag{3.6}$$

where $\mathscr{R}_j$ *is given by*

$$\mathscr{R}_j(\mathbf{a};b,c;\mathbf{z}) = \Big[\prod_{p=1}^{N}{}'(z_j - z_p)\Big]\Big[(c-1)F_D^{(N)}(\mathbf{a}-\mathbf{e}_j;b-1,c-1;\mathbf{z}) \\ + \sum_{s=1}^{N}{}' a_s \frac{z_s(1-z_s)}{z_j - z_s} F_D^{(N)}(\mathbf{a}-\mathbf{e}_j+\mathbf{e}_s;b,c;\mathbf{z})\Big], \tag{3.7}$$

and a prime on a summation (*or a product*) *sign means that* $s \neq j$ $(p \neq j)$.

A detailed proof of Theorem 6 was given in [59] and [66] using induction on the number N of variables of the Lauricella function $F_D^{(N)}$, and we do not present it here. We show only that if $N = 1$, then the relations (3.6), (3.7) in Theorem 6 become the Jacobi identity (3.2) for the Gauss function F. In fact, for $N = 1$ the system of formulae (3.6), (3.7) reduces to a single equality. The vector-valued parameter $\mathbf{a}$ and argument $\mathbf{z}$ of the Lauricella function now consist of one component each and become the scalar parameter a and argument z of the Gauss function. Thus, we must set $|\mathbf{a}'_j| = 0$ in (3.6). Furthermore, the products with respect to p involved in (3.6) and (3.7) do not contain factors, and the sum with respect to s in (3.7) does not contain terms. By the standard convention such products should be set equal to 1 and such sums should be set equal to 0. In view of the above, we arrive at the identity (3.2) by substituting $F(a,b;c;z)$ in place of $F_D^{(1)}$ in the left-hand side of (3.6) and substituting $\mathscr{R}_1(a;b,c;z) = (c-1)F(a-1,b-1;c-1;z)$ from (3.7) in the right-hand side.

We note that some differential relations for the Appell function F_1 considered in [157] are close to the ones established in Theorem 6 when we take $N = 2$ there.

Just as in the derivation of (1.3) from (3.2), we can start from the Jacobi-type formulae (3.6), (3.7) and obtain a system of partial differential equations for $F_D^{(N)}$. This system has a form different from the standard one (1.5) and is given by the following theorem.

Theorem 7. *The Lauricella function* $F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$ *satisfies the following system of partial differential equations with respect to the variables* z_j, $j = 1,\dots,N$:

$$\frac{\partial^2 u}{\partial z_j^2} + \Big(\frac{c - |\mathbf{a}'_j|}{z_j} + \frac{a_j + b - c + 1}{z_j - 1} + \sum_{s=1}^{N}{}' \frac{a_s}{z_j - z_s}\Big)\frac{\partial u}{\partial z_j} \\ + \frac{a_j}{z_j(z_j-1)} \sum_{s=1}^{N}{}' \frac{z_s(1-z_s)}{z_j - z_s}\frac{\partial u}{\partial z_s} + \frac{a_j b u}{z_j(z_j-1)} = 0, \qquad j = 1,\dots,N, \tag{3.8}$$

where a prime on a summation sign indicates that the sum is taken for $s \neq j$.

Proof. We consider the function

$$\widetilde{u}(\mathbf{a};b,c;\mathbf{z}) := F_D^{(N)}(\mathbf{a}-\mathbf{e}_j;b-1,c-1;\mathbf{z}) \tag{3.9}$$

and observe that the functions $F_D^{(N)}$ on the right-hand side of (3.7) are expressed as follows in terms of the derivatives of $\widetilde{u}$:

$$\begin{aligned} F_D^{(N)}(\mathbf{a}-\mathbf{e}_j+\mathbf{e}_s;b,c;\mathbf{z}) &= \frac{c-1}{a_s(b-1)}\,\frac{\partial\widetilde{u}}{\partial z_s}, \qquad j\neq s, \\ F_D^{(N)}(\mathbf{a};b,c;\mathbf{z}) &= \frac{c-1}{(a_j-1)(b-1)}\,\frac{\partial\widetilde{u}}{\partial z_j}\,, \end{aligned} \tag{3.10}$$

as follows from the definition (3.9) and the relation

$$\frac{\partial}{\partial z_j}F_D^{(N)}(\mathbf{a};b;c;\mathbf{z}) = \frac{a_j b}{c}F_D^{(N)}(\mathbf{a}+\mathbf{e}_j;b+1,c+1;\mathbf{z}), \tag{3.11}$$

where we recall that the vectors $\mathbf{a}-\mathbf{e}_j$ and $\mathbf{a}-\mathbf{e}_j+\mathbf{e}_s$ in (3.9) and (3.10) were defined in (3.3) and (3.4), respectively. Substituting (3.9) and (3.10) into the formulae (3.6) and (3.7) with index j, we arrive at an equation satisfied by $\widetilde{u}$. In it we replace a_j, b, and c by a_j+1, $b+1$, and $c+1$, respectively, bearing in mind that $\widetilde{u}$ is then transformed into $u = F_D^{(N)}(\mathbf{a};b,c;\mathbf{z})$. Then Theorem 7 is obtained by using the property (2.59) of $F_D^{(N)}$. $\square$

Note also that if $a_j = 1$, that is, the vector parameter of the Lauricella function has the form $\widetilde{\mathbf{a}} = (a_1,\dots,a_{j-1},1,a_{j-1},\dots,a_N)$, then $\mathscr{R}_j$ on the right-hand side of (3.6) is a polynomial of degree $N-1$ in z_j. In fact, then the functions

$$F_D^{(N)}(\widetilde{\mathbf{a}}-\mathbf{e}_j;b-1,c-1;\mathbf{z}), \qquad F_D^{(N)}(\widetilde{\mathbf{a}}-\mathbf{e}_j+\mathbf{e}_s;b,c;\mathbf{z})$$

in the definition of $\mathscr{R}_j$ are independent of z_j, because (3.3) and (3.4) show that

$$\widetilde{\mathbf{a}}-\mathbf{e}_j = (a_1,\dots,a_{j-1},0,a_{j+1},\dots,a_N)$$

and

$$\widetilde{\mathbf{a}}-\mathbf{e}_j+\mathbf{e}_s = (a_1,\dots,a_{j-1},0,a_{j+1},\dots,a_s+1,\dots,a_N).$$

Thus, $\mathscr{R}_j$ is a polynomial of the form

$$\begin{aligned} \mathscr{R}_j(\mathbf{a};b,c;\mathbf{z}) = \prod_{1\leqslant p\leqslant N,\ p\neq j}(z_j-z_p)\biggl[(c-1)F_D^{(N-1)}(\mathbf{a}'_j;b-1,c-1;\mathbf{z}'_j) \\ + \sum_{1\leqslant s\leqslant N,\ s\neq j} a_s\frac{z_s(1-z_s)}{z_j-z_s}F_D^{(N-1)}(\mathbf{a}'_{j,s};b,c;\mathbf{z}'_j)\biggr], \end{aligned} \tag{3.12}$$

where $\mathbf{a}'_{j,s}$ is the vector obtained from $\mathbf{a}$ by adding one to the sth component and omitting the jth component, that is, $\mathbf{a}'_{j,s} := (a_1,\dots,a_{j-1},a_{j+1},\dots,a_s+1,\dots,a_N)$, $s\neq j$, and the vectors $\mathbf{a}'_j$ and $\mathbf{z}'_j$ are defined in (3.5).

It follows from the above that the right-hand side of (3.6) with index j becomes much simpler when $a_j = 1$: it is then a product of binomials and the explicit polynomial (3.12). This important special case of Theorem 6 is a basis for the representation (1.57) for the solution of the Riemann–Hilbert problem (which we mentioned in the Introduction and will also discuss below in this section). Thus, it is a link between the theory of the Lauricella function and the theory of the Riemann–Hilbert problem.

For a more convenient further presentation we assume that $a_N = 1$, and we write the $(j = N)$th Jacobi-type identity for the function $F_D^{(N)}$ of the form

$$F_D^{(N)}(\underbrace{a_1, \dots, a_{N-1}}_{=\mathfrak{a}}, 1; b, c; \underbrace{z_1, \dots, z_{N-1}}_{=\mathfrak{z}}, w), \tag{3.13}$$

where we have changed to w the notation for the variable z_N corresponding to $a_N = 1$.

In what follows we use the following vector notation for the parameters and arguments of the function (3.13):

$$\mathfrak{a} := (a_1, \dots, a_{N-1}), \qquad \mathfrak{z} := (z_1, \dots, z_{N-1}), \tag{3.14}$$

and we also need the vector gotten from $\mathfrak{a}$ by increasing the sth component by 1:

$$\mathfrak{a} + \mathbf{e}_s = (a_1, \dots, a_{s-1}, a_s + 1, a_{s+1}, \dots, a_{N-1}); \tag{3.15}$$

here $\mathbf{e}_s = (0, \dots, 1, \dots, 0)$ is the $(N-1)$-dimensional vector with sth component 1 and the others equal to 0. By the modulus of a vector we mean (as above) the sum of its components; for example, for the vector $\mathfrak{a}$ in (3.14) we have $|\mathfrak{a}| := \sum_{s=1}^{N-1} a_s$.

Now we state a needed consequence of Theorem 6.

Theorem 8. *The following formula of Jacobi type holds for the Lauricella function* (3.13):

$$\begin{aligned} &\frac{\partial}{\partial w}\biggl\{\biggl[\prod_{j=1}^{N-1}(w - z_j)^{a_j}\biggr] w^{c-|\mathfrak{a}|-1}(1-w)^{1+b-c} F_D^{(N)}(\mathfrak{a}, 1; b, c; \mathfrak{z}, w)\biggr\} \\ &\qquad = \biggl[\prod_{j=1}^{N-1}(w - z_j)^{a_j - 1}\biggr] w^{c-|\mathfrak{a}|-2}(1-w)^{b-c}\mathscr{R}(\mathfrak{a}; b, c; \mathfrak{z}, w), \end{aligned} \tag{3.16}$$

where $\mathscr{R}(\mathfrak{a}; b, c; \mathfrak{z}, w)$ is a polynomial in w of degree $N-1$ defined by

$$\mathscr{R}(\mathfrak{a}; b, c; \mathfrak{z}, w) = \biggl[\prod_{j=1}^{N-1}(w - z_j)\biggr]\biggl(\lambda_0 + \sum_{s=1}^{N-1}\frac{\lambda_s}{w - z_s}\biggr), \tag{3.17}$$

with coefficients λ_s, $s = 0, \dots, N-1$, independent of w and expressible in terms of the Lauricella function of $N-1$ variables by the formulae

$$\begin{aligned} \lambda_0 &:= (c-1)F_D^{(N-1)}(\mathfrak{a}; b-1, c-1; \mathfrak{z}), \\ \lambda_s &:= a_s z_s(1 - z_s)F_D^{(N-1)}(\mathfrak{a} + \mathbf{e}_s; b, c; \mathfrak{z}), \qquad s = 1, \dots, N-1, \end{aligned} \tag{3.18}$$

with the vectors $\mathfrak{a}$, $\mathfrak{z}$, and $\mathfrak{a} + \mathbf{e}_s$ defined in (3.14) *and* (3.15).

It is easy to see that for $N = 1$ the relations (3.16)–(3.18) become the following equality, which is a consequence of the Jacobi identity (3.2) for the Gauss function with $a = 1$:

$$\frac{d}{dz}[z^{c-1}(1-z)^{1+b-c}F(1,b;c;z)] = (c-1)z^{c-2}(1-z)^{b-c}. \tag{3.19}$$

In fact, let $N = 1$, so that the Lauricella function (3.13) coincides with $F(1,b;c;z)$. Then since $\mathfrak{a}$ in (3.14) has $N-1$ components, we set $|\mathfrak{a}| = 0$ in (3.16), and we take the products with respect to j and the sums with respect to s in (3.16) and (3.17) to be equal to 1 and 0, respectively (because the upper limit is smaller than the lower limit). Taking this into account, we get from (3.17) and (3.18) that $\mathscr{R} = \lambda_0 = c-1$, and thus we see that the identity (3.16) for $N = 1$ coincides with (3.19).

We can obtain N similar representations for the polynomial $\mathscr{R}$ in the following way. Differentiating on the left-hand side in (3.16) and taking (3.11) into account, we compare the result with the right-hand side and find the following expression for $\mathscr{R}$:

$$\begin{aligned}\mathscr{R}(\mathfrak{a};b,c;\mathfrak{z},w) &= w(1-w)\biggl[\prod_{j=1}^{N-1}(w-z_j)\biggr]\biggl[\frac{b}{c}F_D^{(N)}(\mathfrak{a},2;b+1,c+1;\mathfrak{z},w)\\ &+\biggl(\sum_{s=1}^{N-1}\frac{a_s}{w-z_s}+\frac{c-|\mathfrak{a}|-1}{w}-\frac{1+b-c}{1-w}\biggr)F_D^{(N)}(\mathfrak{a},1;b,c;\mathfrak{z},w)\biggr].\end{aligned} \tag{3.20}$$

Since this expression is a polynomial of degree $N-1$ by Theorem 8, we can represent it in the Lagrange form in terms of the values at N arbitrary points. It is convenient to take points in the set $\{0, 1, a_1, \dots, a_{N-1}\}$ because the values of $\mathscr{R}$ there are easy to calculate using (3.20). In this way we find N representations for $\mathscr{R}$, one of which is

$$\mathscr{R}(\mathfrak{a};b,c;\mathfrak{z},w) = w\biggl[\prod_{j=1}^{N-1}(w-z_j)\biggr]\biggl(\frac{c-|\mathfrak{a}|-1}{w}\Lambda_0+\sum_{s=1}^{N-1}\frac{a_s(1-z_s)}{w-z_s}\Lambda_s\biggr),$$

where $\Lambda_0 = F_D^{(N)}(\mathfrak{a};b,c;\mathfrak{z})$, $\Lambda_s = F_D^{(N)}(\mathfrak{a}+\mathbf{e}_s;b,c;\mathfrak{z})$, $s = 1,\dots,N-1$, and the vector of parameters $\mathfrak{a}+\mathbf{e}_s$ is defined by (3.15).

3.2. Statement of the Riemann–Hilbert problem in $\mathbb{H}^+$ with piecewise constant data. Let L_k denote the intervals of the real line $\mathbb{R}$ lying between consecutive points in the set Ξ in (1.56): $L_k = (\xi_k, \xi_{k+1})$, $k = 0,\dots,N$, where we recall that $\xi_0 = \xi_{N+1}$ is the point at infinity. By the formulae

$$\chi(\xi) = \chi_k \quad \text{and} \quad \sigma(\xi) = \sigma_k, \quad \xi \in L_k, \quad k = 0,\dots,N, \tag{3.21}$$

we introduce complex and real piecewise constant functions $\chi(\xi)$ and $\sigma(\xi)$ on $\mathbb{R}$, which will be the data in the problem below, where the $\chi_k \neq 0$ and σ_k are some constants. We also fix a set of non-negative integers

$$\mathfrak{G} := \{n_0, n_1, \dots, n_N\}, \qquad n_k \in \mathbb{Z}^+, \quad k = 0,\dots,N, \tag{3.22}$$

and with each point $\xi_k \in \Xi$ we associate the number $n_k \in \mathfrak{G}$; in what follows this number will characterize the integer part of the growth exponent at ξ_k of the solution $\mathscr{P}^+(\zeta)$ of the Riemann–Hilbert problem.

We remark that on each interval L_k, $k = 0, \dots, N$, the argument of the function $\chi(\xi)$ in (3.21) takes the constant value $\arg \chi_k$, which is determined up to a quantity $2\pi m_k$, where m_k is an arbitrary integer. Since $\chi(\xi)$ is discontinuous, the integers m_k corresponding to different intervals L_k are in no way connected. We fix arbitrary values of the m_k, $k = 0, \dots, N$, and thereby branches of the function $\arg \chi(\xi)$ on each interval L_k, and then we calculate the jumps of $\arg \chi(\xi)$ (more precisely, of the branches chosen) at points in Ξ by

$$\delta_k := \frac{\arg \chi_k - \arg \chi_{k-1}}{\pi}, \quad k = 1, \dots, N, \qquad \delta_0 := -\frac{\arg \chi_0 - \arg \chi_N}{\pi}. \tag{3.23}$$

We also calculate the fractional and integer parts of the jumps δ_k defined in (3.23):

$$[0,1) \ni \alpha_k := \{\delta_k\}, \quad \beta_k := [\delta_k], \qquad k = 0, \dots, N; \tag{3.24}$$

and we consider the jumps of the function $\rho(\xi) = \sigma(\xi)/\chi(\xi)$ at the points in Ξ:

$$\rho_k = \frac{\sigma_{k+1}}{\chi_{k+1}} - \frac{\sigma_k}{\chi_k}, \qquad k = 0, \dots, N.$$

We also consider the quantity

$$\Theta_N := \frac{\pi}{2} - \arg \chi_N. \tag{3.25}$$

It is assumed that the numbers n_0 and α_0 corresponding to the point ξ_0 at infinity do not vanish simultaneously. Since $\alpha_0 \in [0,1)$ and $n_0 \in \mathbb{Z}^+$, this means that $\alpha_0 + n_0 \neq 0$, and moreover, there are no finite points $\xi_k \in \Xi$ such that $n_k = 0$ and $\alpha_k = 0$ but at the same time $\rho_k \neq 0$. Thus, we assume that we always have the conditions

$$1) \quad \alpha_0 + n_0 \neq 0, \qquad 2) \not\exists\, k = 1, \dots, N \colon n_k = 0, \ \alpha_k = 0, \ \rho_k \neq 0. \tag{3.26}$$

Let $\mathscr{H}^+$ denote the class of functions analytic in $\mathbb{H}^+$ that are continuous in $\overline{\mathbb{H}^+} \setminus \Xi$, where Ξ is the set of points (1.56) on the real line at which $\chi(\xi)$ or $\sigma(\xi)$ is discontinuous.

The Riemann–Hilbert problem under consideration is the problem of finding a function $\mathscr{P}^+ \in \mathscr{H}^+$ which is analytic in the upper half-plane from the boundary condition

$$\operatorname{Re}[\chi(\xi)\mathscr{P}^+(\xi)] = \sigma(\xi), \qquad \xi \in \mathbb{R} \setminus \Xi, \tag{3.27}$$

on the real line, where χ and σ are defined in (3.21), and it is assumed that at points in Ξ the function $\mathscr{P}^+$ satisfies the growth conditions

$$\mathscr{P}^+(\zeta) = \begin{cases} \mathscr{O}\big((\zeta - \xi_k)^{\alpha_k - n_k}\big) & \text{if } \ n_k \neq 0, \\ \mathscr{O}(1) & \text{if } \ n_k = 0, \end{cases} \quad \zeta \to \xi_k \quad (k = 1, \dots, N), \tag{3.28}$$

$$\mathscr{P}^+(\zeta) = \mathscr{O}(\zeta^{\alpha_0 + n_0}), \qquad \zeta \to \infty. \tag{3.29}$$

Because of the presence of the numbers n_k, the conditions (3.28) and (3.29) allow the solution $\mathscr{P}^+(\zeta)$ to have non-integrable powerlike growth in general. This version of the Riemann–Hilbert problem can appropriately be called *singular*.

The following theorem (see [140] and [64]) establishes the solvability of the Riemann–Hilbert problem and a representation for its solution in terms of Cauchy-type integrals. It was proved using methods going back to Gakhov [126] and Muskhelishvili [125].

Theorem 9. *The following results hold for the Riemann–Hilbert problem (3.27)–(3.29) under consideration with piecewise constant data (3.21) satisfying the conditions (3.26).*

i) *If the index $\varkappa$ defined by*

$$\varkappa := n_0 - \beta_0 + \sum_{k=1}^{N}(\beta_k + n_k) \tag{3.30}$$

is non-negative, then the solution $\mathscr{P}^+ \in \mathscr{H}^+$ has the form

$$\mathscr{P}^+(\zeta) = X^+(\zeta)P_\varkappa(\zeta) + \mathscr{N}^+(\zeta), \tag{3.31}$$

where $X^+(\zeta)P_\varkappa(\zeta) =: \Psi^+(\zeta)$ is the general solution of the homogeneous problem, $X^+(\zeta)$ is the canonical function defined by

$$X^+(\zeta) = e^{i\Theta_N}\prod_{k=1}^{N}(\zeta-\xi_k)^{\alpha_k - n_k}, \tag{3.32}$$

the constant Θ_N is given by (3.25), $P_\varkappa(\zeta)$ is an arbitrary polynomial of degree $\varkappa$ with real coefficients, and the function $\mathscr{N}^+(\zeta)$ is a particular solution of the inhomogeneous problem and is found as follows:

$$\mathscr{N}^+(\zeta) = \sum_{k=0}^{N}\mathscr{N}_k^+(\zeta), \tag{3.33}$$

$$\mathscr{N}_k^+(\zeta) = \frac{\sigma_k X^+(\zeta)}{\chi_k \pi i}\int_{L_k}\frac{dt}{X^+(t)(t-\zeta)}, \qquad k = 1,\dots,N-1, \tag{3.34}$$

$$\begin{aligned} \mathscr{N}_0^+(\zeta) &= \frac{\sigma_0 X^+(\zeta)(\zeta-\tau_*)^\varkappa}{\chi_0\pi i}\int_{L_0}\frac{(t-\tau_*)^{-\varkappa}}{X^+(t)(t-\zeta)}\,dt,\\ \mathscr{N}_N^+(\zeta) &= \frac{\sigma_N X^+(\zeta)(\zeta-\tau^*)^\varkappa}{\chi_N\pi i}\int_{L_N}\frac{(t-\tau^*)^{-\varkappa}}{X^+(t)(t-\zeta)}\,dt, \end{aligned} \tag{3.35}$$

$\tau_*, \tau^* \in \mathbb{R}$ being arbitrary points on the respective intervals $(\xi_1, +\infty)$ and $(-\infty, \xi_N)$.

ii) *If $\varkappa = -1$, then the unique solution of the problem is*

$$\mathscr{P}^+(\zeta) = \mathscr{N}^+(\zeta), \tag{3.36}$$

where the function $\mathscr{N}^+(\zeta)$ is defined by (3.33)–(3.35) with $\varkappa = 0$ set formally in (3.35).

iii) *If* $\varkappa < -1$ *and the solvability conditions*

$$\sum_{m=0}^{N} B_{km}\frac{\sigma_m}{\chi_m} = 0, \quad k = 0, 1, \dots, |\varkappa| - 2, \quad \textit{where} \quad B_{km} := \int_{L_m} \frac{t^k}{X^+(t)}\, dt, \tag{3.37}$$

are satisfied, then the unique solution of the problem is (3.36). *But if* $\varkappa < -1$ *and the conditions* (3.37) *fail, then this Riemann–Hilbert problem is unsolvable.*

Assume that the first condition in (3.26) does not hold and $\alpha_0 = n_0 = 0$. Then it is easy to see from the equality

$$\alpha_0 + n_0 = \varkappa + \sum_{k=1}^{N}(\alpha_k - n_k), \tag{3.38}$$

which follows from (3.23), (3.24), and (3.30), that the integrals $\mathscr{N}_0^+(\zeta)$ and $\mathscr{N}_N^+(\zeta)$ over infinite intervals in (3.35) are divergent (we consider the case when $\varkappa \geqslant 0$). If in addition $\rho_0 \neq 0$, then the condition (3.29) in Theorem 9 must be replaced by $\mathscr{P}^+(\zeta) = \mathscr{O}(\log\zeta)$ as $\zeta \to \infty$, and $\varkappa + 1$ must replace $\varkappa$ in the formulae (3.35) for $\mathscr{N}_0^+(\zeta)$ and $\mathscr{N}_N^+(\zeta)$. Then all the assertions of the theorem hold. But if $\alpha_0 = n_0 = 0$ and $\rho_0 = 0$, then in (3.35) we set $\tau^* = \tau_*$ and regard the sum $\mathscr{N}_0^+(\zeta) + \mathscr{N}_N^+(\zeta) =: \mathscr{S}(\zeta)$ in the sense of the principal value of the integral, which is easily shown to exist. Now the function $\mathscr{N}^+(\zeta)$ involved in the theorem will be calculated by the formula

$$\mathscr{N}^+(\zeta) = \mathscr{S}(\zeta) + \sum_{k=1}^{N-1} \mathscr{N}_k^+(\zeta),$$

where, as before, $\mathscr{N}_k^+(\zeta)$ is defined by (3.34). After this modification of the function $\mathscr{N}^+(\zeta)$ all the assertions of Theorem 9 hold. The relation (3.29) is transformed into $\mathscr{P}^+(\zeta) = \mathscr{O}(1)$ as $\zeta \to \infty$. We note also that if the second condition in (3.26) fails at one of the finite points $\xi_k \in \Xi$, then the form of the solution given in Theorem 9 is preserved, but at ξ_k the asymptotic behaviour of the function $\mathscr{P}^+(\zeta)$ will be logarithmic rather than powerlike as in (3.28).

We can show that if in the formula (3.35) for $\mathscr{N}_0^+$ we replace τ_* by $\widetilde{\tau} \neq \tau_*$ and denote this function by $\widetilde{\mathscr{N}_0^+}$, then the difference $\widetilde{\mathscr{N}_0^+} - \mathscr{N}_0^+$ is the product of a polynomial of degree $\varkappa - 1$ with real coefficients and the canonical function $X^+(\zeta)$ (so that it satisfies the conditions of the homogeneous Riemann–Hilbert boundary problem). A similar observation is valid for $\mathscr{N}_N^+$. Thus, the presence of τ_* and τ^* in (3.35) does not affect the total number ($= \varkappa + 1$) of arbitrary real constants determining the solution of the Riemann–Hilbert problem in Theorem 9.

3.3. Schwarz–Christoffel integral representation of the solution of the Riemann–Hilbert problem. The aim of this subsection is a theorem on representing the solution of the Riemann–Hilbert problem by an integral (1.52), a theorem which gives explicitly all the quantities in the integrand. Before stating this theorem, we introduce some notation.

Consider the vector $\mathbf{a} := (a_0, \dots, a_N)$ with components a_j connected with the data of the Riemann–Hilbert problem (3.27)–(3.29) by the relations

$$a_0 := \varkappa, \qquad a_j := \alpha_j - n_j, \quad j = 1, \dots, N, \tag{3.39}$$

where we recall that the quantities α_j are to be found from (3.24) and (3.23), and the n_j are non-negative integers in the set (3.22). We let $\mathbf{a}_k$, $k = 1, \dots, N-1$, be the vectors obtained from $\mathbf{a}$ by omitting the components a_0, a_k, and a_{k+1}, that is,

$$\mathbf{a}_k := (a_1, \dots, a_{k-1}, a_{k+2}, \dots, a_N), \tag{3.40}$$

and let $\mathbf{a}_0$ and $\mathbf{a}_N$ be defined by

$$\mathbf{a}_0 = \mathbf{a}_N := (a_0, a_2, \dots, a_{N-1}). \tag{3.41}$$

Consider the vectors $\mathbf{a}_k^s$ obtained by increasing the component a_s of $\mathbf{a}_k$ by 1 (provided that $s \neq k, k+1$ if $k = 1, \dots, N-1$ and $s \neq 1, N$ if $k = 0$ or N), that is,

$$\begin{aligned} &\mathbf{a}_k^s := (a_1, \dots, a_{k-1}, a_{k+2}, \dots, a_{s-1}, a_s + 1, a_{s+1}, \dots, a_N), \qquad k = 1, \dots, N-1, \\ &\mathbf{a}_0^s = \mathbf{a}_N^s := (a_0, a_2, \dots, a_{s-1}, a_s + 1, a_{s+1}, \dots, a_{N-1}), \\ &\mathbf{a}_0^0 = \mathbf{a}_N^0 := (a_0 + 1, a_2, \dots, a_{N-1}). \end{aligned} \tag{3.42}$$

Let b_k and c_k, $k = 0, \dots, N$, be numbers defined as follows:

$$b_0 := |\alpha| - |\mathbf{n}| + \varkappa, \quad c_0 := |\alpha_{2,N}| - |\mathbf{n}_{2,N}| + \varkappa + 1; \tag{3.43}$$

$$b_k := 1 + n_k - \alpha_k, \quad c_k := 2 + n_k + n_{k+1} - \alpha_k - \alpha_{k+1}, \quad k = 1, \dots, N-1; \tag{3.44}$$

$$b_N := |\alpha| - |\mathbf{n}| + \varkappa, \quad c_N := |\alpha_{1,N-1}| - |\mathbf{n}_{1,N-1}| + \varkappa + 1; \tag{3.45}$$

as usual, here

$$|\alpha_{k,l}| = \sum_{j=k}^{l} \alpha_j, \quad |\alpha| = |\alpha_{1,N}|, \qquad |\mathbf{n}_{k,l}| = \sum_{j=k}^{l} n_j, \quad |\mathbf{n}| = |\mathbf{n}_{1,N}|. \tag{3.46}$$

We define the quantities $|\beta_{k,l}|$ and $|\beta|$, where $\beta := (\beta_1, \dots, \beta_N)$, in a similar way. The vectors $\mathbf{u}_k$, $k = 0, \dots, N$, have the form

$$\mathbf{u}_0 := (u_0^0, u_2^0, \dots, u_{N-1}^0), \quad \mathbf{u}_N := (u_0^N, u_2^N, \dots, u_{N-1}^N), \tag{3.47}$$

$$\mathbf{u}_k := (u_1^k, \dots, u_{k-1}^k, u_{k+2}^k, \dots, u_N^k), \qquad k = 1, \dots, N-1, \tag{3.48}$$

where the u_j^k are the quantities defined by

$$u_0^0 := \frac{\xi_N - \tau_*}{\xi_N - \xi_1}, \qquad u_j^0 := \frac{\xi_N - \xi_j}{\xi_N - \xi_1}, \quad j = 2, \dots, N-1, \tag{3.49}$$

$$u_j^k := \frac{\xi_{k+1} - \xi_k}{\xi_j - \xi_k}, \qquad k = 1, \dots, N-1, \quad j = 1, \dots, N, \quad j \neq k, k+1, \tag{3.50}$$

$$u_0^N := \frac{\tau^* - \xi_1}{\xi_N - \xi_1}, \qquad u_j^N := \frac{\xi_j - \xi_1}{\xi_N - \xi_1}, \quad j = 2, \dots, N-1, \tag{3.51}$$

τ_* and τ^* are the same as in Theorem 9, and the points ξ_j, $j = 1, \ldots, N$, are in the set Ξ in (1.56) of points of discontinuity of the boundary data $\chi(\xi)$ and $\sigma(\xi)$ in the Riemann–Hilbert problem (3.27)–(3.29).

Let Λ_k be the quantities defined by

$$\begin{aligned} \Lambda_0 &:= -e^{i\pi(\beta_0 - n_0)} \frac{\sigma_0}{\pi|\chi_0|} \mathrm{B}(b_0, c_0 - b_0)(\xi_N - \xi_1)^{-b_0}, \\ \Lambda_N &:= \frac{\sigma_N}{\pi|\chi_N|} \mathrm{B}(b_N, c_N - b_N)(\xi_N - \xi_1)^{-b_N}, \\ \Lambda_k &:= -e^{i\pi(|\beta_{k+1,N}| + |\mathbf{n}_{k+1,N}|)} \frac{\sigma_k}{\pi|\chi_k|} \mathrm{B}(b_k, c_k - b_k) \\ &\qquad \times (\xi_{k+1} - \xi_k)^{c_k - 1} \prod_{\substack{1 \leqslant j \leqslant N \\ j \neq k, k+1}} |\xi_k - \xi_j|^{-a_j}, \qquad k = 1, \ldots, N-1, \end{aligned} \tag{3.52}$$

where $\mathrm{B}(\alpha, \beta)$ is the beta function [30]

$$\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}\,.$$

We also define the quantities μ_s^k by

$$\begin{aligned} \mu_{-1}^0 &= (c_0 - 1)(\xi_1 - \xi_N) F_D^{(N-1)}(\mathbf{a}_0; b_0 - 1, c_0 - 1; \mathbf{u}_0), \\ \mu_0^0 &= a_0(\xi_N - \tau_*)(\tau_* - \xi_1) F_D^{(N-1)}(\mathbf{a}_0^0; b_0, c_0; \mathbf{u}_0), \\ \mu_s^0 &= a_s(\xi_N - \xi_s)(\xi_s - \xi_1) F_D^{(N-1)}(\mathbf{a}_0^s; b_0, c_0; \mathbf{u}_0), \qquad s = 2, \ldots, N-1, \\ \mu_{-1}^N &= (c_N - 1)(\xi_N - \xi_1) F_D^{(N-1)}(\mathbf{a}_N; b_N - 1, c_N - 1; \mathbf{u}_N), \\ \mu_0^N &= a_0(\tau^* - \xi_1)(\xi_N - \tau^*) F_D^{(N-1)}(\mathbf{a}_N^0; b_N, c_N; \mathbf{u}_N), \\ \mu_s^N &= a_s(\xi_s - \xi_1)(\xi_N - \xi_s) F_D^{(N-1)}(\mathbf{a}_N^s; b_N, c_N; \mathbf{u}_N), \qquad s = 2, \ldots, N-1, \\ \mu_{-1}^k &= (c_k - 1) F_D^{(N-2)}(\mathbf{a}_k; b_k - 1, c_k - 1; \mathbf{u}_k), \\ \mu_s^k &= a_s \frac{\xi_s - \xi_{k+1}}{\xi_k - \xi_s} F_D^{(N-2)}(\mathbf{a}_k^s; b_k, c_k; \mathbf{u}_k), \qquad s = 1, \ldots, N, \quad s \neq k, k+1. \end{aligned} \tag{3.53}$$

Recall that the quantities $\mathbf{a}_k$, $\mathbf{a}_k^s$, b_k, c_k, and $\mathbf{u}_k$ in these formulae are defined by (3.39)–(3.51) in terms of the data of the Riemann–Hilbert problem (3.27)–(3.29).

The next theorem gives a representation of the solution of the Riemann–Hilbert problem as a Schwarz–Christoffel integral.

Theorem 10. *The following assertions hold for the solution $\mathscr{P}^+(\zeta)$ of the Riemann–Hilbert problem* (3.27)*–*(3.29) *in $\mathbb{H}^+$ with piecewise constant data* (3.21) *satisfying* (3.26)*.*

i) *If the index $\varkappa$ defined by* (3.30) *is non-negative, then the solution $\mathscr{P}^+ \in \mathscr{H}^+$ has a representation as a Schwarz–Christoffel integral:*

$$\mathscr{P}^+(\zeta) = e^{i\Theta_N} \int_{\zeta^*}^{\zeta} \prod_{j=1}^{N} (t - \xi_j)^{\alpha_j - n_j - 1} \mathscr{R}(t)\, dt + w^*, \tag{3.54}$$

where $\mathscr{R}(\zeta)$ *is a polynomial of degree* $N+\varkappa-1$ *with real coefficients which has the form*

$$\mathscr{R}(\zeta)=\mathscr{Q}(\zeta)+\mathscr{T}(\zeta), \tag{3.55}$$

$\mathscr{Q}(\zeta)$ *is a polynomial of degree* $N+\varkappa-1$ *defined in terms of an arbitrary polynomial* $P_\varkappa(\zeta)$ *of degree* $\varkappa$ *with real coefficients by*

$$\mathscr{Q}(\zeta)=\prod_{j=1}^{N}(\zeta-\xi_j)\biggl(P_\varkappa(\zeta)\sum_{s=1}^{N}\frac{\alpha_s-n_s}{\zeta-\xi_s}+P'_\varkappa(\zeta)\biggr), \tag{3.56}$$

and

$$\begin{aligned}\mathscr{T}(\zeta)=\biggl[\prod_{j=2}^{N-1}(\zeta-\xi_j)\biggr]\biggl[&\Lambda_0(\zeta-\tau_*)^\varkappa\biggl(\mu_{-1}^0+\frac{\mu_0^0}{\zeta-\tau_*}+\sum_{s=2}^{N-1}\frac{\mu_s^0}{\zeta-\xi_s}\biggr)\\&+\Lambda_N(\zeta-\tau^*)^\varkappa\biggl(\mu_{-1}^N+\frac{\mu_0^N}{\zeta-\tau^*}+\sum_{s=2}^{N-1}\frac{\mu_s^N}{\zeta-\xi_s}\biggr)\biggr]\\&+\sum_{k=1}^{N-1}\Lambda_k\biggl[\prod_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}}(\zeta-\xi_j)\biggr]\biggl[\mu_{-1}^k+(\zeta-\xi_k)\sum_{\substack{1\leqslant s\leqslant N\\ s\neq k,k+1}}\frac{\mu_s^k}{\zeta-\xi_s}\biggr]\end{aligned} \tag{3.57}$$

is a real polynomial of degree $N+\varkappa-2$, *with the coefficients* Λ_k *and* μ_s^k *defined by* (3.52) *and* (3.53). *The formula* (3.54) *involves the constants* Θ_N *in* (3.25) *and constants* ζ^* *and* w^* *such that* $\mathscr{P}^+(\zeta^*)=w^*$.

ii) *If* $\varkappa=-1$, *then the unique solution of the problem is expressed by the Schwarz–Christoffel integral* (3.54), (3.55), *where* $\mathscr{Q}(\zeta)\equiv 0$ *in the formula* (3.55) *for* $\mathscr{R}(\zeta)$.

iii) *If* $\varkappa<-1$, *then a solution exists if and only if the conditions* (3.37) *are satisfied. If they are, then the solution can be found using the same formula as for* $\varkappa=-1$.

Before turning to a discussion of the proof of this theorem, we note that if the solution $\mathscr{P}^+$ of the Riemann–Hilbert problem in question is finite at some point $\xi_k\in\Xi$ (and therefore is continuous in the intersection of a neighbourhood of ξ_k with the closed half-plane $\overline{\mathbb{H}^+}$), then we can set $\zeta^*=\xi_k$ in (3.55), and we can find $w^*=\mathscr{P}^+(\zeta^*)$ directly from the boundary condition of the Riemann–Hilbert problem. In fact, extending the boundary condition (3.27) to ξ_k by continuity from the left and right and setting $w^*=\mathscr{P}^+(\xi_k)$, we obtain the following system of two (linear) equations with respect to w^*:

$$\operatorname{Re}(\chi_{k-1}w^*)=\sigma_{k-1},\qquad \operatorname{Re}(\chi_k w^*)=\sigma_k.$$

It is easy to verify that the system is satisfied by

$$w^*=\mathscr{P}^+(\xi_k)=i\frac{\overline{\chi}_k\sigma_{k-1}-\overline{\chi}_{k-1}\sigma_k}{\operatorname{Im}(\chi_k\overline{\chi}_{k-1})}\,, \tag{3.58}$$

which we take to be the constant of integration in (3.54) for $\zeta^*=\xi_k$; in the formula (3.58) we assume that $\operatorname{Im}(\chi_k\overline{\chi}_{k-1})\neq 0$.

We note also that the function

$$\mathscr{N}^+(\zeta) = e^{i\Theta_N} \int_{\zeta_*}^{\zeta} \prod_{j=1}^{N} (t-\xi_j)^{\alpha_j - n_j - 1} \mathscr{T}(t)\, dt + w_*, \tag{3.59}$$

where the polynomial $\mathscr{T}(\zeta)$ can be found from (3.57), is a particular solution of the inhomogeneous Riemann–Hilbert problem (3.27)–(3.29), and the functions

$$\Psi_m^+(\zeta) = e^{i\Theta_N} \int_{\zeta_*}^{\zeta} \prod_{j=1}^{N} (t-\xi_j)^{\alpha_j - n_j - 1} \mathscr{Q}_m(t)\, dt, \qquad m = 0, \dots, \varkappa,$$

with the polynomials $\mathscr{Q}_m(\zeta)$ given by

$$\mathscr{Q}_m(\zeta) = \zeta^m \prod_{j=1}^{N} (\zeta - \xi_j) \Biggl(\sum_{s=1}^{N} \frac{\alpha_s - n_s}{\zeta - \xi_s} + \frac{m}{\zeta} \Biggr),$$

are (for $\varkappa \geqslant 0$) linearly independent solutions of the corresponding homogeneous problem.

If we formally set $\varkappa = 0$ in the expression (3.57) for the polynomial $\mathscr{T}(\zeta)$, then (3.54) and (3.59) will still produce the general solution of the (inhomogeneous) Riemann–Hilbert problem (3.27)–(3.29) and a particular solution of it, respectively. Then the expression for $\mathscr{T}(\zeta)$ will be slightly simpler; in particular, there will be no additional quantities τ_* and τ^* in it.

The representation (3.54) as a Schwarz–Christoffel integral shows that the function $\mathscr{P}^+(\zeta)$ realizes a conformal mapping of the upper half-plane $\mathbb{H}^+$ onto some simply connected non-schlicht polygonal domain $\mathscr{M}$ (for instance, see [93]). The interior branch points of $\mathscr{M}$ are the images of complex zeros of $\mathscr{R}(\zeta)$ (in $\mathbb{H}^+$) under the mapping $w = \mathscr{P}^+(\zeta)$, and the boundary corner points of $\mathscr{M}$ are the images of the points $\xi_k \in \Xi$, and also of the real zeros of $\mathscr{R}(\zeta)$ under this mapping. The internal angle of $\mathscr{M}$ at the boundary point $w_k = \mathscr{P}^+(\xi_k)$, $k \neq 0$, is equal to $\pi\gamma_k := \pi(\alpha_k - n_k)$ if $\mathscr{R}(\xi_k) \neq 0$, and to $\pi(\gamma_k + \rho)$ if $\mathscr{R}(\xi_k) = 0$, where ρ is the order of the zero of $\mathscr{R}$ at ξ_k. The angle at a point $\widetilde{w} := \mathscr{P}^+(\widetilde{\xi})$, where $\widetilde{\xi} \in \mathbb{R}$ and $\mathscr{R}(\widetilde{\xi}) = 0$ but $\widetilde{\xi} \notin \Xi$, is equal to $\pi(\widetilde{\rho} + 1)$, where $\widetilde{\rho}$ is the order of the zero of $\mathscr{R}$ at $\widetilde{\xi}$. In this way Theorem 10 gives a clear geometric interpretation of the solution $\mathscr{P}^+(\zeta)$ of this Riemann–Hilbert problem.

3.4. Application of Jacobi-type formulae to the derivation of a new representation for the solution of the Riemann–Hilbert problem. First of all, avoiding technical details, we describe our approach to the derivation of the representation (3.54) under the assumption that the index $\varkappa$ defined by (3.30) is non-negative, that is, the condition i) in Theorem 10 is satisfied. We transform the derivatives of $\Psi^+(\zeta)$ and $\mathscr{N}^+(\zeta)$ into products of binomials and a polynomial.

Differentiating $\Psi^+(\zeta) = X^+(\zeta) P_\varkappa(\zeta)$, where $X^+(\zeta)$ is the canonical function (3.32) and $P_\varkappa(\zeta)$ is a real polynomial of degree $\varkappa$, we get after simple transformations that

$$\frac{d}{d\zeta} \Psi^+(\zeta) = \prod_{j=1}^{N} (t-\xi_j)^{\alpha_j - n_j - 1} \mathscr{Q}(\zeta), \tag{3.60}$$

where $\mathscr{Q}(\zeta)$ is the polynomial defined by (3.56).

Next we transform the derivative of each function $\mathscr{N}_k^+(\zeta)$ in (3.33). Using a change of variables that takes the interval L_k into $(0,1)$, and using the integral representation (1.6) for the Lauricella function, we express all the functions $\mathscr{N}_k^+(\zeta)$ in terms of the functions $F_D^{(N)}$ (with sets of parameters and variables depending on k). Applying the version of a Jacobi-type formula indicated in Theorem 8 to the expressions obtained for $\mathscr{N}_k^+(\zeta)$, we transform the derivatives $d\mathscr{N}_k^+(\zeta)/d\zeta$ into the required form of a product of binomials and some explicit polynomial $T_k(\zeta)$:

$$\frac{d}{d\zeta}\mathscr{N}_k^+(\zeta) = \prod_{j=1}^{N}(\zeta-\xi_j)^{\alpha_j-n_j-1}T_k(\zeta). \tag{3.61}$$

Noting that for different values of k the derivatives (3.61) differ only in the form of the polynomial $T_k(\zeta)$, we add the equalities (3.61) and, in view of (3.33), obtain the desired representation for $d\mathscr{N}^+(\zeta)/d\zeta$:

$$\frac{d}{d\zeta}\mathscr{N}^+(\zeta) = \prod_{j=1}^{N}(t-\xi_j)^{\alpha_j-n_j-1}T(\zeta), \qquad T(\zeta)=\sum_{k=0}^{N}T_k(\zeta). \tag{3.62}$$

Adding (3.60) and (3.62), we find the desired representation for $d\mathscr{P}^+(\zeta)/d\zeta$. Integrating it, we obtain (3.54).

All these transformations were thoroughly described in [59] and [64]. For example, we present the argument for $\mathscr{N}_0^+$. By (3.32) and (3.35) the function $\mathscr{N}_0^+(\zeta)$ has the following representation in terms of a Cauchy-type integral:

$$\mathscr{N}_0^+(\zeta) = \frac{\sigma_0}{\chi_0\pi i}\left[(\zeta-\tau_*)^{\varkappa}\prod_{j=1}^{N}(\zeta-\xi_j)^{\alpha_j-n_j}\right]\int_{-\infty}^{\xi_1}\frac{(t-\tau_*)^{-\varkappa}\,dt}{\prod_{j=1}^{N}(t-\xi_j)^{\alpha_j-n_j}(t-\zeta)}. \tag{3.63}$$

Making a change of the variables t, ζ to the new variables τ, w by the formulae

$$t(\tau)=\xi_N+(\xi_1-\xi_N)\tau^{-1} \quad\text{and}\quad \zeta(w)=\xi_N+(\xi_1-\xi_N)w, \tag{3.64}$$

we obtain

$$\begin{aligned}\mathscr{N}_0^+(\zeta(w)) &= -\frac{\sigma_0}{\chi_0\pi i}w^{\alpha_N-n_N}(w-1)^{\alpha_1-n_1}\\ &\quad\times\left[\left(w-\frac{\xi_N-\tau_*}{\xi_N-\xi_1}\right)^{\varkappa}\prod_{j=2}^{N-1}\left(w-\frac{\xi_N-\xi_j}{\xi_N-\xi_1}\right)^{\alpha_j-n_j}\right]\\ &\quad\times\int_0^1\left[\left(1-\frac{\xi_N-\tau_*}{\xi_N-\xi_1}\tau\right)^{\varkappa}\prod_{j=2}^{N-1}\left(1-\frac{\xi_N-\xi_j}{\xi_N-\xi_1}\tau\right)^{\alpha_j-n_j}(1-w\tau)\right]^{-1}\\ &\quad\times\tau^{|\alpha|-|\mathbf{n}|+\varkappa-1}(1-\tau)^{n_1-\alpha_1}\,d\tau,\end{aligned}$$

where we recall that $|\alpha|=\sum_{s=1}^{N}\alpha_s$ and $|\mathbf{n}|=\sum_{s=1}^{N}n_s$. Rewriting the above formula for $\mathscr{N}_0^+$ while taking into account the notation (3.42), (3.43), (3.49) and using the Euler-type representation (1.6) for the Lauricella function, we obtain the expression

$$\mathscr{N}_0^+\big(\zeta(w)\big) = -\frac{\sigma_0}{\chi_0 \pi i}\mathrm{B}(b_0, c_0 - b_0) w^{c_0 - |\mathbf{a}_0| - 1}(w-1)^{1+b_0-c_0} \times \Bigg[\prod_{\substack{0\leqslant j\leqslant N-1\\ j\neq 1}} (w - u_j^0)^{a_j}\Bigg] F_D^{(N)}(\mathbf{a}_0, 1; b_0, c_0; \mathbf{u}_0, w). \tag{3.65}$$

Differentiating (3.65) and using the Jacobi-type formula (3.16)–(3.18), we obtain an expression for the derivative:

$$\frac{d}{dw}\mathscr{N}_0^+\big(\zeta(w)\big) = \frac{\sigma_0}{\chi_0 \pi i}\mathrm{B}(b_0, c_0 - b_0) w^{c_0 - |\mathbf{a}_0| - 2}(w-1)^{b_0-c_0} \times \Bigg[\prod_{\substack{0\leqslant j\leqslant N-1\\ j\neq 1}} (w - u_j^0)^{a_j - 1}\Bigg] R_0(w), \tag{3.66}$$

where $R_0(w)$ is a polynomial in w defined by

$$R_0(w) = \Bigg[\prod_{\substack{0\leqslant j\leqslant N-1\\ j\neq 1}} (w - u_j^0)\Bigg]\Bigg(\lambda_{-1}^0 + \sum_{\substack{0\leqslant j\leqslant N-1\\ j\neq 1}} \frac{\lambda_s^0}{w - u_s^0}\Bigg), \tag{3.67}$$

with

$$\begin{aligned} \lambda_{-1}^0 &= (c_0 - 1) F_D^{(N-1)}(\mathbf{a}_0; b_0 - 1, c_0 - 1; \mathbf{u}_0), \\ \lambda_s^0 &= a_s u_s^0 (1 - u_s^0) F_D^{(N-1)}(\mathbf{a}_0^s; b_0, c_0; \mathbf{u}_0), \qquad s = 0, 2, \dots, N-1. \end{aligned} \tag{3.68}$$

In (3.66) and (3.67) we make the substitution inverse to $\zeta(w)$ in (3.64):

$$w(\zeta) = \frac{\zeta - \xi_N}{\xi_1 - \xi_N}, \qquad \frac{d}{d\zeta} = \frac{1}{\xi_1 - \xi_N}\frac{d}{dw}, \tag{3.69}$$

and then in view of the equality $-\arg\chi_0 - \pi(|\alpha| - |\mathbf{n}|) = -\arg\chi_N + \pi(|\beta| + |\mathbf{n}|)$ following from (3.23), (3.24), and (3.30), we obtain

$$\frac{d}{d\zeta}\mathscr{N}_0^+(\zeta) = e^{i\Theta_N}\Bigg[\prod_{j=1}^{N} (\zeta - \xi_j)^{\alpha_j - n_j - 1}\Bigg]\mathscr{T}_0(\zeta), \tag{3.70}$$

where Θ_N is the constant in (3.25) and $\mathscr{T}_0(\zeta)$ is a polynomial of degree $N + \varkappa - 2$ defined by

$$\mathscr{T}_0(\zeta) = \Lambda_0\Bigg[(\zeta - \tau_*)^{\varkappa}\prod_{j=2}^{N-1}(\zeta - \xi_j)\Bigg]\Bigg(\mu_{-1}^0 + \frac{\mu_0^0}{\zeta - \tau_*} + \sum_{s=2}^{N-1}\frac{\mu_s^0}{\zeta - \xi_s}\Bigg), \tag{3.71}$$

with coefficients Λ_0 and μ_s^0 determined from the data of the Riemann–Hilbert problem (3.27)–(3.29) by the equalities (3.52) and (3.53) in the notation (3.39)–(3.50).

Similar arguments show that

$$\frac{d}{d\zeta}\mathscr{N}_k^+(\zeta) = e^{i\Theta_N}\left[\prod_{j=1}^{N}(\zeta-\xi_j)^{\alpha_j-n_j-1}\right]\mathscr{T}_k(\zeta), \tag{3.72}$$

where $\mathscr{T}_k(\zeta)$ is the polynomial defined by

$$\mathscr{T}_k(\zeta) = \Lambda_k\Bigg[\prod_{\substack{1\leqslant j\leqslant N\\ j\neq k,k+1}}(\zeta-\xi_j)\Bigg] \times\Bigg[\mu_{-1}^k + (\zeta-\xi_k)\sum_{\substack{1\leqslant s\leqslant N\\ s\neq k,k+1}}\frac{\mu_s^k}{\zeta-\xi_s}\Bigg], \qquad k=1,\dots,N-1, \tag{3.73}$$

$$\mathscr{T}_N(\zeta) = \Lambda_N\left[(\zeta-\tau^*)^{\varkappa}\prod_{j=2}^{N-1}(\zeta-\xi_j)\right]\left(\mu_{-1}^N + \frac{\mu_0^N}{\zeta-\tau^*} + \sum_{s=2}^{N-1}\frac{\mu_s^N}{\zeta-\xi_s}\right). \tag{3.74}$$

Adding the equalities (3.60) and (3.70), (3.72) for $k=0,\dots,N$, taking (3.71) and (3.74) into account, and integrating the result, we arrive at the representation (3.54).

Parts ii) and iii) of Theorem 10 are proved quite similarly. There we must bear in mind the assertion of Theorem 9 in the case when the index $\varkappa$ is negative. Theorem 10 is proved.

4. Applications to plasma physics

4.1. A model of magnetic reconnection and the statement of the corresponding Riemann–Hilbert problem. Many explosion-like processes studied in stellar physics take place in a rarefied plasma, when magnetic forces dominate other forces (gas-dynamic, gravitational, and so on), and large amounts of energy are released as a result of the *magnetic reconnection phenomenon*, which means a fundamental change in the configuration of the magnetic field [158]–[160]. The central mathematical problem in the investigation of such processes is often an effective calculation of the magnetic field for this or that plasma configuration [160].

In this section we present a solution of the Riemann–Hilbert problem in a complicated polygonal domain (see [67], [111]) which arises in modelling magnetic reconnection near a disintegrating current layer in the corona of the Sun. According to contemporary views, it is the destruction of this layer that leads to solar flares [160]. The unknown analytic function $\mathscr{F}(z)$ in this problem describes the magnetic field in the exterior of the configuration of currents shown in Fig. 3. The horizontal cuts Γ_0^- and Γ_0^+ correspond to two parts of the disintegrating current layer, and the four slanted cuts Γ_j, $j=1,\dots,4$, depict the shock waves associated with the layer. Thus, the configuration of currents $\Gamma = Y^+\cup Y^-$ is a union of two Y-formed components:

$$Y^+ := \Gamma_0^+\cup\Gamma_1\cup\Gamma_4, \qquad Y^- := \{z\colon -\overline{z}\in Y^+\}, \tag{4.1}$$

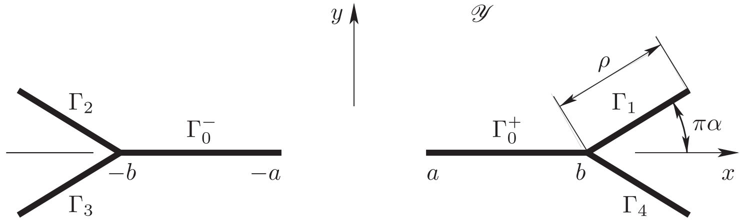

Figure 3. The system of cuts $\Gamma = \partial\mathscr{Y}$.

where Y^+ consists of the horizontal cut Γ_0^+ and the slanted cuts Γ_1 and Γ_4,

$$\Gamma_0^+ = \{z\colon \operatorname{Re} z \in [a,b],\ \operatorname{Im} z = 0\},$$
$$\Gamma_1 = \{z\colon z = b + t\rho e^{i\pi\alpha},\ t \in [0,1]\}, \quad \Gamma_4 = \{z\colon \overline{z} \in \Gamma_1\},$$

while Y^- is obtained as a mirror reflection of Y^+ in the y-axis.

In the domain $\mathscr{Y} := \overline{\mathbb{C}}\setminus\Gamma$ (see Fig. 3) we consider the stationary planar magnetic field

$$\mathbf{B}(x,y) = (B_x(x,y), B_y(x,y), 0), \tag{4.2}$$

which is assumed to be a divergence-free potential field. A physical justification of such assumptions in modelling the phenomenon under consideration was given in [160]. We also assume that the normal component of the field vanishes on the current layer (that is, on $\Gamma_0^\pm$) and is equal to a fixed constant β on the shock waves (that is, on the Γ_j, $j = 1,\dots,4$). At infinity the modulus of the field grows linearly, with coefficient of proportionality γ, and at the endpoints of the segments $\Gamma_0^\pm$, which are free from shock waves, it is also unbounded but has minimal possible growth. Furthermore, it is known [85] that if (4.2) is a divergence-free potential field, then the associated function $\mathscr{F}(z) = B_x(x,y) - iB_y(x,y)$ is analytic in the domain $\mathscr{Y}$.

The above mathematical model leads to the formulation of the Riemann–Hilbert problem of finding a function $\mathscr{F}(z)$ that is analytic in $\mathscr{Y}$ and continuous in $\overline{\mathscr{Y}} \setminus \{\infty, -a, a\}$ and that satisfies the boundary conditions

$$\operatorname{Re}[\nu_j \mathscr{F}(z)] = c_j, \qquad z \in \Gamma_j, \quad j = 0,1,\dots,4, \tag{4.3}$$

where $\Gamma_0 := \Gamma_0^+ \cup \Gamma_0^-$, the complex normals ν_j to the cuts Γ_j, $j = 0,\dots,4$, have the form

$$\nu_0 = i, \quad \nu_1 = ie^{i\pi\alpha}, \quad \nu_2 = -ie^{-i\pi\alpha}, \quad \nu_3 = -ie^{i\pi\alpha}, \quad \nu_4 = ie^{-i\pi\alpha}, \tag{4.4}$$

and the numbers c_j on the right-hand side of (4.3) are given by

$$c_0 = 0 \quad \text{and} \quad c_j = \beta, \quad j = 1,\dots,4. \tag{4.5}$$

At the points $z \in \{\infty, -a, +a\}$ where $\mathscr{F}$ is discontinuous we assume that it satisfies the growth conditions

$$\mathscr{F}(z) = -i\gamma z + o(1), \quad z \to \infty, \quad \text{and} \quad \mathscr{F}(z) = \mathscr{O}\bigl((z \pm a)^{-1/2}\bigr), \quad z \to \pm a. \tag{4.6}$$

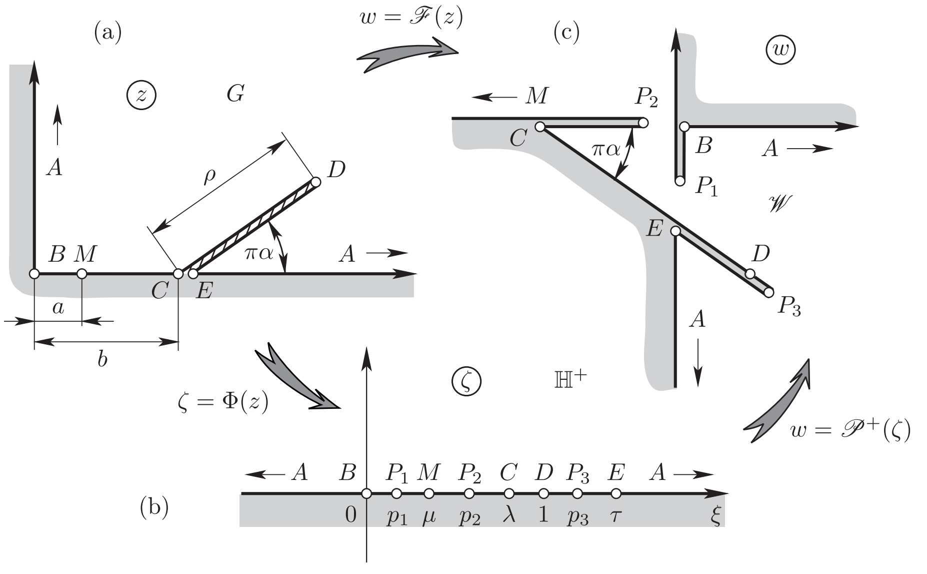

Figure 4. Scheme of the solution of the Riemann–Hilbert problem.

The real quantities β and γ in the statement of the problem along with the numbers a, b, ρ, and α specifying the contour Γ are parameters of the model. This model [67], [160] is a natural development of the model of disintegration of an (infinite) current layer without shock waves which was investigated in [161].

4.2. The problem in one fourth of the original domain and the construction of a conformal mapping. Adding the symmetry conditions

$$\mathscr{F}(\overline{z}) = -\overline{\mathscr{F}}(z) \quad \text{and} \quad \mathscr{F}(-\overline{z}) = \overline{\mathscr{F}}(z) \tag{4.7}$$

to the problem posed, we reduce it to a Riemann–Hilbert problem in one-fourth of the domain $\mathscr{Y}$, which we denote by G, that is, in the first coordinate quadrant cut along Γ_1 (see Fig. 4, (a)):

$$G := \mathscr{Q}_{\mathrm{I}} \setminus \Gamma_1, \qquad \mathscr{Q}_{\mathrm{I}} := \{z\colon \operatorname{Re} z > 0,\ \operatorname{Im} z > 0\}. \tag{4.8}$$

This Riemann–Hilbert problem is stated as follows: find an analytic function $\mathscr{F}$ in G that is continuous on $\overline{G} \setminus \{\infty, a\}$, satisfies the boundary condition

$$\operatorname{Re}[h(z)\mathscr{F}(z)] = c(z), \qquad z \in \partial G, \tag{4.9}$$

in which the coefficients $h(z)$ and $c(z)$ are defined by

$$h(z) = \begin{cases} i & \text{on } (AB) \cup (MC), \\ 1 & \text{on } (BM) \cup (EA), \\ ie^{i\pi\alpha} & \text{on } (CDE), \end{cases} \qquad c(z) = \begin{cases} 0 & \text{on } (ABMC) \cup (EA), \\ \beta & \text{on } (ABC), \end{cases} \tag{4.10}$$

and satisfies the following growth conditions at the points A and M:

$$\mathscr{F}(z) = -i\gamma z + o(1), \quad z \to \infty, \quad \text{and} \quad \mathscr{F}(z) = \mathscr{O}\bigl((z-a)^{-1/2}\bigr), \quad z \to a; \tag{4.11}$$

the quantities β and γ here are real constants which are parameters of the model. The geometric quantities a, b, ρ, and α are also parameters of the model.

After finding the solution $\mathscr{F}$ of the problem (4.9)–(4.11), we extend it to the whole of $\mathscr{Y}$ using the symmetry relations (4.7) and thereby obtain a solution of the original problem (4.3)–(4.6). We present the numerical result for the full domain $\mathscr{Y}$ in the most representative form from a physics standpoint, as a family of level curves $A(x,y) = \text{const}$ of the magnetic potential $A(x,y)$, which we find by the formula

$$A(x,y) := \operatorname{Im}\Psi(z), \qquad \Psi(z) := \int_0^z \mathscr{F}(t)\,dt. \tag{4.12}$$

The function $\Psi(z)$ is called a complex potential of the field. It is easy to see that the magnetic field $\mathbf{B}$ is tangent to level curves of $A(x,y)$.

A conformal mapping of one-fourth of the reconnection domain G onto the upper half-plane was constructed in [111]. Here we present only the general scheme of reasoning in order to indicate connections with § 5. Note that G is an (infinite) simply connected pentagon with vertices $A=\infty$, $B=0$, $C=b$ (to the left of the cut Γ_1), $D=b+\rho e^{i\pi\alpha}$, and $E=b$ (to the right of Γ_1). The internal angles at these points with respect to G have measure

$$\pi\alpha_j, \qquad j\in\{A,B,C,D,E\},$$

where

$$\alpha_A=-\frac{1}{2}\,, \quad \alpha_B=\frac{1}{2}\,, \quad \alpha_C=1-\alpha, \quad \alpha_D=2, \quad \alpha_E=\alpha.$$

We consider the mapping $\Phi^{-1}\colon \mathbb{H}^+ \xrightarrow{\text{conf}} G$ subject to the following conditions (which were used in [111], but are slightly different from the ones we use in § 5):

$$\Phi^{-1}(\infty)=\infty, \qquad \Phi^{-1}(0)=0, \qquad \Phi^{-1}(1)=b+\rho e^{i\pi\alpha}, \tag{4.13}$$

that is, we assume that the vertices A, B, and D of G correspond to the boundary points $\zeta=\infty$, $\zeta=0$, and $\zeta=1$ of the half-plane (see Fig. 4, (a) and (b)). Letting λ and τ denote the unknown inverse images of C and E, we express $z=\Phi^{-1}(\zeta)$ as a Schwarz–Christoffel integral [85], [93], [104]:

$$\Phi^{-1}(\zeta)=\mathscr{K}\int_0^\zeta t^{-1/2}(t-\lambda)^{-\alpha}(t-1)(t-\tau)^{\alpha-1}\,dt, \tag{4.14}$$

where the coefficient $\mathscr{K}$ of the integral is easily seen to be real and positive ($\mathscr{K}>0$).

To find the unknown parameters λ, τ, and $\mathscr{K}$ of this integral in the usual way (for instance, see [93] and also § 5), we form a system of non-linear transcendental equations by equating the three given distances between vertices of the polygonal boundary ∂G to their expressions calculated by (4.14). Integrating over the segments $[0,\lambda]$, $[\lambda,1]$, and $[\lambda,\tau]$ in (4.14), we obtain the system of equations

$$\mathscr{K} I_1(\lambda,\tau)=b, \quad \mathscr{K} I_2(\lambda,\tau)=r, \quad \mathscr{K} I_3(\lambda,\tau)=0, \tag{4.15}$$

where the numbers $I_j(\lambda,\tau)$, $j=1,2,3$, denote the integrals

$$I_1(\lambda,\tau)=\int_0^{\lambda} t^{-1/2}(\lambda-t)^{-\alpha}(1-t)(\tau-t)^{\alpha-1}\,dt, \tag{4.16}$$

$$I_2(\lambda,\tau)=\int_{\lambda}^{1} t^{-1/2}(t-\lambda)^{-\alpha}(1-t)(\tau-t)^{\alpha-1}\,dt, \tag{4.17}$$

$$I_3(\lambda,\tau)=\int_{\lambda}^{\tau} t^{-1/2}(t-\lambda)^{-\alpha}(1-t)(\tau-t)^{\alpha-1}\,dt. \tag{4.18}$$

Geometrically, the third equality in (4.15), which implies that $I_3(\lambda,\tau)=0$, means that the distance between the vertices C and E, the images of the limits of integration λ and τ, is 0 (see Fig. 4, (a) and (b)). Making changes of variables in (4.16)–(4.18) so that the integration is over the interval $[0,1]$, and using the representation (1.6) for the Lauricella function $F_D^{(N)}$, we express the integrals in terms of this function with $N=2$ variables (that is, in terms of the Appell function F_1) by the formulae

$$I_1(\lambda,\tau)=\frac{\sqrt{\pi}\,\Gamma(1-\alpha)}{\Gamma(3/2-\alpha)}\lambda^{1/2-\alpha}\tau^{\alpha-1}F_D^{(2)}\biggl(-1,1-\alpha;\frac{1}{2},\frac{3}{2}-\alpha;\lambda,\frac{\lambda}{\tau}\biggr), \tag{4.19}$$

$$\begin{aligned}I_2(\lambda,\tau)&=[(1-\alpha)(2-\alpha)]^{-1}\lambda^{-1/2}(1-\lambda)^{2-\alpha}(\tau-\lambda)^{\alpha-1}\\&\quad\times F_D^{(2)}\biggl(\frac{1}{2},1-\alpha;1-\alpha,3-\alpha;\frac{\lambda-1}{\lambda},\frac{1-\lambda}{\tau-\lambda}\biggr),\end{aligned} \tag{4.20}$$

$$I_3(\lambda,\tau)=-\frac{\pi}{\sin\pi\alpha}\lambda^{-1/2}(1-\lambda)F_D^{(2)}\biggl(\frac{1}{2},-1;1-\alpha,1;-\frac{\tau-\lambda}{\lambda},\frac{\tau-\lambda}{1-\lambda}\biggr). \tag{4.21}$$

Dividing the second equation by the first in (4.15), we eliminate $\mathscr{K}$ and thus reduce the problem of finding the unknown parameters in the Schwarz–Christoffel formula (4.14) to a system of two equations involving only the inverse images λ and τ:

$$\frac{I_2(\lambda,\tau)}{I_1(\lambda,\tau)}=\frac{\rho}{b}\,,\qquad I_3(\lambda,\tau)=0, \tag{4.22}$$

where the integrals I_j, $j=1,2,3$, are found from (4.19)–(4.21). After solving (4.22), we find the coefficient $\mathscr{K}$ from the first equation in (4.15) by the formula

$$\mathscr{K}=\frac{b}{I_1(\lambda,\tau)}\,. \tag{4.23}$$

Once we have found λ, τ, and $\mathscr{K}$, the mapping $z=\Phi^{-1}(\zeta)$, expressed as the Schwarz–Christoffel integral (4.14), is completely determined and we must invert it, because we need the inverse mapping to solve the Riemann–Hilbert problem $\mathscr{F}(z)=\mathscr{P}\circ\Phi(z)$ in G. In [111] we presented an analytic method for inverting a Schwarz–Christoffel integral in the form of a set of expansions (into power series) with explicitly given coefficients. The method was based on a theory presented in [115]. The convergence sets of these expansions cover in totality the closure of the domain G of the mapping (away from infinity). Moreover, for each point $z\in\overline{G}\setminus\{\infty\}$ there is at least one expansion in this set that converges at z at an exponential rate. Thus, this set of expansions is a convenient and effective tool for calculating and investigating the mapping $\zeta=\Phi(z)$.

4.3. Solving the Riemann–Hilbert problem in the half-plane. We reduce the original boundary-value problem (4.9)–(4.11) for an analytic function $\mathscr{F}(z) = u(z) + iv(z)$ in the domain G to a similar problem in the upper half-plane $\mathbb{H}^+$ (see Fig. 4) with respect to the function

$$w = \mathscr{P}^+(\zeta) = \mathscr{F} \circ \Phi^{-1}(\zeta)$$

by means of the mapping $z = \Phi^{-1}(\zeta)$. The Riemann–Hilbert problem for $\mathscr{P}^+(\zeta)$ has the following statement: find a function $\mathscr{P}^+(\zeta)$ analytic in $\mathbb{H}^+$ that is continuous on $\overline{\mathbb{H}^+} \setminus \{\infty, a\}$ and satisfies on the real line the boundary condition

$$\operatorname{Re}[\chi(\xi)\mathscr{P}^+(\xi)] = \sigma(\xi), \qquad \xi \in \mathbb{R} \setminus \{a\}, \tag{4.24}$$

where $\chi(\xi) = h \circ \Phi^{-1}(\xi)$ and $\sigma(\xi) = c \circ \Phi^{-1}(\xi)$ are the complex and real piecewise constant functions defined by

$$\chi(\xi) = \begin{cases} i, & \xi \in (AB) \cup (MC), \\ 1, & \xi \in (BM) \cup (EA), \\ ie^{i\pi\alpha}, & \xi \in (CDE), \end{cases} \quad \sigma(\xi) = \begin{cases} 0, & \xi \in (ABMC) \cup (EA), \\ \beta, & \xi \in (ABC), \end{cases} \tag{4.25}$$

with the following growth conditions prescribed at $\zeta = \infty$ and $\zeta = \mu := \Phi(a)$:

$$\mathscr{P}^+(\zeta) = -2i\gamma\mathscr{K}\sqrt{\zeta} + o(1), \ \ \zeta \to \infty; \quad \mathscr{P}^+(\zeta) = \mathscr{O}\big((\zeta-\mu)^{-1/2}\big), \ \ \zeta \to \mu. \tag{4.26}$$

Here γ is a given coefficient (a parameter of the model), $\mathscr{K}$ is the coefficient of the integral in (4.14), which we calculated in finding Φ^{-1}, and μ denotes the inverse image of the point M.

The boundary problem (4.24)–(4.26) is a special case of the Riemann–Hilbert problem with piecewise constant coefficients that we considered in § 3. The five points of discontinuity of the coefficients $\{\xi_k\}$ are $\xi_0 = \infty$, 0, μ, λ, τ. Calculating the quantities α_k and the index $\varkappa$ of the problem using the formulae (3.24), (3.23), and (3.30), we find that

$$\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{2}, \qquad \alpha_3 = \alpha, \qquad \alpha_4 = \frac{1}{2} - \alpha, \quad \text{and} \quad \varkappa = 0.$$

By Theorem 9, the equality $\varkappa = 0$, and fact that the coefficient γ in (4.26) is given, the problem (4.24)–(4.26) is uniquely solvable. Since the mapping $\Phi(z)$ is unique, all this also implies that the problem (4.9)–(4.11) is uniquely solvable in G.

Using Theorem 9, i) and taking into account that $\varkappa = 0$, we find the solution of (4.24)–(4.26) as a Cauchy-type integral

$$\mathscr{P}^+(\zeta) = X^+(\zeta)\biggl[2\gamma\mathscr{K} + \frac{\beta}{\pi}\int_\lambda^\tau \frac{t^{-1/2}(t-\mu)^{1/2}(t-\lambda)^{-\alpha}(\tau-t)^{\alpha-1/2}}{t-\zeta}\,dt\biggr], \tag{4.27}$$

where X^+ is the canonical solution of the homogeneous problem and is given by

$$X^+(\zeta) = e^{-i\pi/2}\zeta^{1/2}(\zeta-\mu)^{-1/2}(\zeta-\lambda)^{\alpha}(\zeta-\tau)^{1/2-\alpha}, \tag{4.28}$$

and the first term $2\gamma\mathscr{K}$ in the square brackets in (4.27) is found from the first asymptotic formula in (4.26).

The representation (4.27) can be transformed into a Schwarz–Christoffel integral by means of a Jacobi-type formula for the Lauricella function (see § 3). We do not reproduce this transformation for (4.27), but use at once the final result of § 3 stated in Theorem 10. Applying this theorem to the problem (4.24)–(4.26) and calculating $\mathscr{P}^+(0) = 0$ in accordance with (3.58), we obtain the desired representation for $\mathscr{P}^+(\zeta)$ as a Schwarz–Christoffel integral:

$$\mathscr{P}^+(\zeta) = -i\gamma\mathscr{K}\int_0^\zeta t^{-1/2}(t-\mu)^{-3/2}(t-\lambda)^{\alpha-1}(t-\tau)^{-1/2-\alpha}R_3(t)\,dt, \tag{4.29}$$

where $R_3(\zeta)$ is a third-degree polynomial of the form

$$\begin{aligned} R_3(\zeta) = {} & (\zeta-\mu)(\zeta-\lambda)(\zeta-\tau) - \zeta(\zeta-\lambda)(\zeta-\tau) \\ & + 2\alpha\zeta(\zeta-\mu)(\zeta-\tau) + (1-2\alpha)\zeta(\zeta-\mu)(\zeta-\lambda) \\ & + \frac{\beta}{\gamma}\,\frac{\Gamma(1-\alpha)\Gamma(\alpha+1/2)}{\pi\sqrt{\pi}\,\mathscr{K}}\lambda^{-3/2}(\lambda-\mu)^{-1/2}(\tau-\lambda)^{1/2} \\ & \times [A_0\lambda(\lambda-\mu)\zeta(\zeta-\mu) - A_1\tau(\lambda-\mu)(\zeta-\mu)(\zeta-\lambda) \\ & + A_2\lambda(\tau-\mu)\zeta(\zeta-\lambda)], \end{aligned} \tag{4.30}$$

with A_0, A_1, and A_2 being numbers expressed in terms of the Lauricella function of two variables $F_D^{(2)}$ (that is, the Appell function F_1) by the formulae

$$A_0 = F_D^{(2)}\biggl(\frac{1}{2}, -\frac{1}{2}; -\alpha, \frac{1}{2}; x_1, x_2\biggr), \qquad A_1 = F_D^{(2)}\biggl(\frac{3}{2}, -\frac{1}{2}; 1-\alpha, \frac{3}{2}; x_1, x_2\biggr),$$
$$\text{and}\quad A_2 = F_D^{(2)}\biggl(\frac{1}{2}, \frac{1}{2}; 1-\alpha, \frac{3}{2}; x_1, x_2\biggr), \qquad x_1 = -\frac{\tau-\lambda}{\lambda}, \quad x_2 = -\frac{\tau-\lambda}{\lambda-\mu}. \tag{4.31}$$

Using (3.58), we also see easily that

$$\mathscr{P}^+(\lambda) = \frac{-\beta}{\sin\pi\alpha} \quad\text{and}\quad \mathscr{P}^+(\tau) = \frac{-i\beta}{\cos\pi\alpha}\,. \tag{4.32}$$

It follows from (4.29) that the dependence of the solution $\mathscr{P}^+(\beta,\gamma;\zeta)$ on the parameters β and γ in the conditions (4.24)–(4.26) of the problem can be 'factorized' in the form

$$\mathscr{P}^+(\beta,\gamma;\zeta) = \gamma\widehat{\mathscr{P}}\biggl(\frac{\beta}{\gamma};\zeta\biggr),$$

where

$$\widehat{\mathscr{P}}\biggl(\frac{\beta}{\gamma};\zeta\biggr) := -i\mathscr{K}\int_0^\zeta t^{-1/2}(t-\mu)^{-3/2}(t-\lambda)^{\alpha-1}(t-\tau)^{-1/2-\alpha}R_3(t)\,dt.$$

Hence, to understand this dependence for $\gamma \neq 0$ we can confine ourselves to investigating the dependence of $\mathscr{P}^+$ on β alone with $\gamma = 1$.

We can show that as $\mu \to 0$ the formulae (4.29), (4.30) for $\mathscr{P}^+(\zeta)$ become the solution of the Riemann–Hilbert problem constructed in [111] and corresponding to the model in [134] of magnetic reconnection without rupture of the current layer.

4.4. Hodograph domain for the magnetic field and numerical results for solving the original problem in the reconnection domain. The analytic function $w = \mathscr{F}(z)$ that solves the boundary-value problem (4.9)–(4.11) realizes a conformal mapping of the original domain G onto a domain $\mathscr{W}$ which, following [85], we call the hodograph domain for the magnetic field. It follows from the representation of $\mathscr{P}^+(\zeta)$ as a Schwarz–Christoffel integral (4.29) and the formula

$$\mathscr{F}(z) = \mathscr{P}^+ \circ \Phi(z) \tag{4.33}$$

that the hodograph domain $\mathscr{W}$ is polygonal. This makes the solution of this problem more geometrically clear and simplifies its analysis.

If the Riemann–Hilbert problem has data such that the three zeros p_1, p_2, and p_3 of the polynomial $R_3(\zeta)$ defined by (4.30), (4.31) are real, then it follows from the representation (4.29) for $\mathscr{P}^+(\zeta)$ that $\mathscr{W}$ is an octagonal domain with vertices A, B, P_1, M, P_2, C, P_3, E (see Fig. 4, (c)), where the points C and E have the complex coordinates $\mathscr{P}^+(\lambda)$ and $\mathscr{P}^+(\tau)$ given by (4.32), respectively. The angles at these vertices are

$$\begin{gathered}\pi\delta_A = -\frac{\pi}{2}\,, \quad \pi\delta_B = \frac{\pi}{2}\,, \quad \pi\delta_{P_1} = 2\pi, \quad \pi\delta_M = -\frac{\pi}{2}\,, \\ \pi\delta_{P_2} = 2\pi, \quad \pi\delta_C = \pi\alpha, \quad \pi\delta_{P_3} = 2\pi, \quad \pi\delta_E = \pi\biggl(\frac{1}{2} - \alpha\biggr),\end{gathered} \tag{4.34}$$

and their inverse images are the points

$$\begin{gathered}\xi_A = \infty, \quad \xi_B = 0, \quad \xi_{P_1} = p_1, \quad \xi_M = \mu, \\ \xi_{P_2} = p_2, \quad \xi_C = \lambda, \quad \xi_{P_3} = p_3, \quad \xi_E = \tau\end{gathered}$$

(see Fig. 4, (b)).

But if $R_3(\zeta)$ has two complex zeros p_1 and p_2 ($p_1 \in \mathbb{H}^+$ and $p_2 = \overline{p_1}$) and one real zero p_3, then the hodograph domain $\mathscr{W}$ still has a polygonal boundary, but it does not lie in the plane but rather on a two-sheeted Riemann surface formed by two copies of the w-plane cut along a curve $\mathscr{L}$ from $P_1 = \mathscr{P}^+(p_1)$ to the point at infinity and glued together (in the standard way). We show such a domain in Fig. 5, in the case when $p_3 \in (\lambda, \tau)$; here $\mathscr{W}$ is a (two-sheeted) hexagonal domain with vertices A, B, M, C, P_3, E, where C and E have complex coordinates $\mathscr{P}^+(\lambda)$ and $\mathscr{P}^+(\tau)$ given by (4.32), respectively. The angles at these six vertices are the same as in the case when $\mathscr{W}$ is schlicht (see (4.34)).

To calculate the solution $\mathscr{P}^+(\zeta)$ of the Riemann–Hilbert problem in $\mathbb{H}^+$ as a Schwarz–Christoffel integral (4.29), we represent it by a set of power series converging in neighbourhoods of the inverse images of vertices of $\mathscr{W}$ indicated above and in neighbourhoods of some regular points in this domain and on its boundary. For $\mathscr{P}^+(\zeta)$ such representations can be found by expanding the integrand in (4.29) into series in fractional powers of the variable and integrating these series termwise. This method for calculating $\mathscr{P}^+(\zeta)$ is very convenient and effective in practical usage.

We calculate the solution $\mathscr{F}(z)$ of the original Riemann–Hilbert problem (4.3)–(4.7) in the domain G by the formula (4.33) and extend it into $\mathscr{Y}$ using the symmetry relations (4.7). Here we use the results in [111] to calculate the conformal

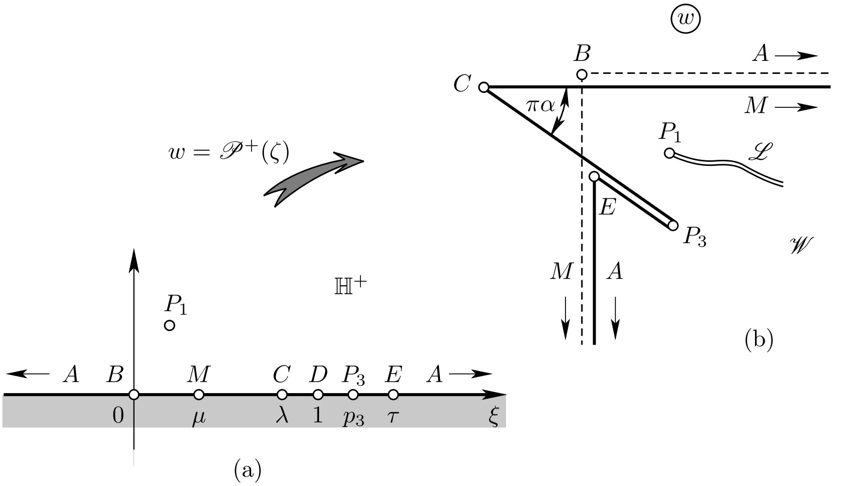

Figure 5. The case when the polynomial $R_3(\zeta)$ in the representation (4.29) has complex zeros: (a) the presence of a branch point in $\mathbb{H}^+$ and (b) an example of a non-schlicht hodograph domain $\mathscr{W}$.

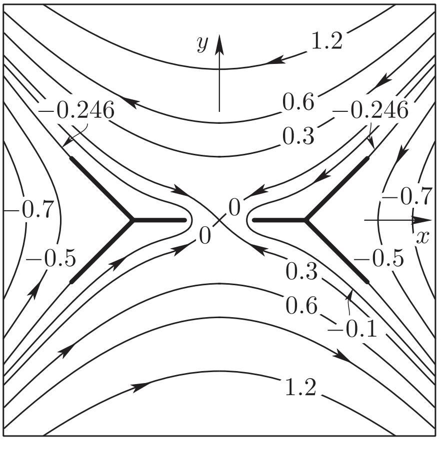

Figure 6. A solution of the Riemann–Hilbert problem; the field picture for $\beta = 0$.

mapping $\Phi\colon G \xrightarrow{\text{conf}} \mathbb{H}^+$ and then calculate the function $\mathscr{P}^+(\zeta)$ as we described above.

In Figs. 6 and 7 we give examples of pictures of magnetic fields, that is, families of level curves of the function $A(x, y)$ in (4.12), which we calculated in $\mathscr{Y}$ by means of the algorithm indicated above. The picture in Fig. 6 is given for $\gamma = 1$ and $\beta = 0$, while in Fig. 7 it is given for $\gamma = 1$ and $\beta = 0.5$. The configuration of currents (see Fig. 3) in Figs. 6 and 7 has the following geometric parameters: $a = 0.4$, $b = 1$, $\rho = 1$, and $\pi\alpha = \pi/4$.

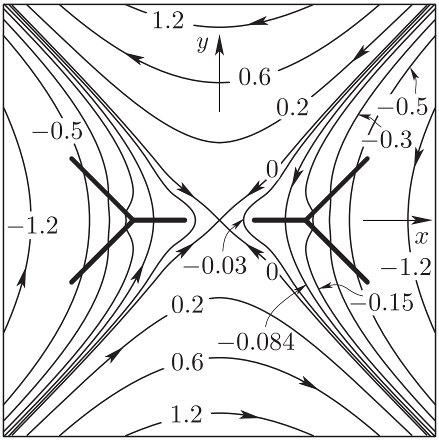

Figure 7. A solution of the Riemann–Hilbert problem; the field picture for $\beta = 0.5$.

5. Applications to the Schwarz–Christoffel parameter problem

5.1. Representing the system of non-linear equations for the parameters in terms of the Lauricella function. In this subsection we discuss in greater detail the connection mentioned in § 1.4 between the theory of the function $F_D^{(N)}$ and the Schwarz–Christoffel parameter problem. In what follows we consider a conformal mapping $\mu\colon \mathbb{H}^+ \xrightarrow{\text{conf}} \mathscr{B}$ of the half-plane $\mathbb{H}^+$ onto a polygonal domain $\mathscr{B}$ with $N+3$ vertices z_j (see Fig. 8). We number the points z_j, the corresponding angles $\pi\beta_j$, and the inverse images $\zeta_j = \mu^{-1}(z_j)$ by indices from 0 to $N+2$. The number of vertices is such that the equations in the system for ζ_j can conveniently be expressed in terms of the Lauricella function of N variables.

The mapping μ subject to the conditions

$$\zeta_0 = \mu^{-1}(z_0) = 0, \qquad \zeta_{N+1} = \mu^{-1}(z_{N+1}) = 1, \quad \text{and} \quad \zeta_{N+2} = \mu^{-1}(z_{N+2}) = \infty,$$

is expressed as a Schwarz–Christoffel integral:

$$z = \mu(\zeta) = \mathscr{K}_0 \int_{\widetilde{\zeta}}^{\zeta} t^{\beta_0 - 1} \Biggl[\prod_{j=1}^{N} (t-\zeta_j)^{\beta_j - 1}\Biggr] (t-1)^{\beta_{N+1}-1}\, dt + \mathscr{K}_1, \tag{5.1}$$

where the inverse images ζ_j, $j = 1, \dots, N$, and the coefficient $\mathscr{K}_0$ are the unknowns. Assuming that the z_j are finite, $j = 0, \dots, N+1$, so that $\beta_j \in (0,2)$, we obtain the following system of $N+1$ equations for determining the vector $\mathbf{x} := (\zeta_1, \dots, \zeta_N)$ of inverse images and the coefficient $\mathscr{K}_0$ (see [92], [93], [97]):

$$\mathscr{K}_0 I_k(\mathbf{x}) = L_k, \qquad k = 0, \dots, N, \tag{5.2}$$

where the $L_k = |z_{k+1} - z_k|$ are the lengths of the sides of the polygon and

$$I_k(\mathbf{x}) := \biggl|\int_{\zeta_k}^{\zeta_{k+1}} t^{\beta_0 - 1} \Biggl[\prod_{j=1}^{N} (t-\zeta_j)^{\beta_j - 1}\Biggr] (t-1)^{\beta_{N+1}-1}\, dt\biggr|. \tag{5.3}$$

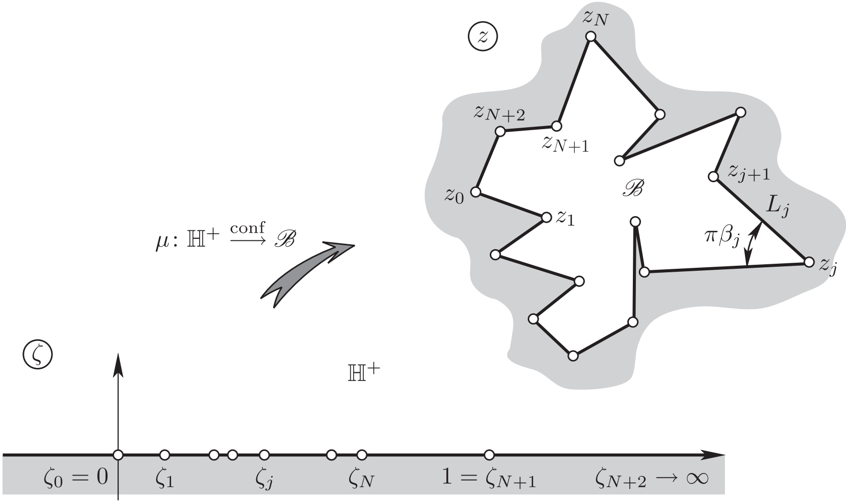

Figure 8. A conformal mapping of the half-plane onto a polygonal domain.

For further arguments it is convenient to introduce the vector $\mathbf{a}$ and numbers b and c connected with the characteristics β_j of the angles of the polygon by the formulae

$$\begin{gathered}\mathbf{a} = (a_1, \dots, a_N), \qquad a_j := 1 - \beta_j, \quad j = 1, \dots, N, \\ b := -1 + \sum_{j=0}^{N+1} (1 - \beta_j), \qquad c := \sum_{j=0}^{N} (1 - \beta_j).\end{gathered} \tag{5.4}$$

Making the change of variable $t = \zeta_k + (\zeta_{k+1} - \zeta_k)\tau$, $\tau \in (0, 1)$, in the integrals (5.3) and using the Euler-type representation (1.6), we express these integrals in the form

$$I_k(\mathbf{x}) = C_k \mathscr{I}_k(\mathbf{a}; b, c; \mathbf{x}), \qquad k = 0, \dots, N, \tag{5.5}$$

where the coefficients are given by

$$\begin{gathered}C_0 = \frac{\Gamma(1 + |\mathbf{a}| - c)\Gamma(1 - a_1)}{\Gamma(2 + |\mathbf{a}_{2,N}| - c)}, \qquad C_N = \frac{\Gamma(1 - a_N)\Gamma(c - b)}{\Gamma(1 + c - b - a_N)}, \\ C_k = \frac{\Gamma(1 - a_k)\Gamma(1 - a_{k+1})}{\Gamma(2 - a_k - a_{k+1})}, \qquad k = 1, \dots, N-1,\end{gathered} \tag{5.6}$$

while the functions $\mathscr{I}_k(\mathbf{x}) = \mathscr{I}_k(\mathbf{a}; b, c; \mathbf{x})$ are expressed in terms of $F_D^{(N)}$ by

$$\mathscr{I}_0(\mathbf{a}; b, c; \mathbf{x}) := \zeta_1^{1+|\mathbf{a}_{2,N}|-c} \left(\prod_{j=2}^{N} \zeta_j^{-a_j} \right) F_D^{(N)}(\mathbf{a}_0; b_0, c_0; \mathbf{x}_0), \tag{5.7}$$

$$\begin{aligned}\mathscr{I}_k(\mathbf{a}; b, c; \mathbf{x}) := {} & \zeta_k^{|\mathbf{a}|-c} (\zeta_{k+1} - \zeta_k)^{1-a_k-a_{k+1}} (1 - \zeta_k)^{c-b-1} \\ & \times \prod_{j=1}^{k-1} (\zeta_k - \zeta_j)^{-a_j} \prod_{j=k+2}^{N} (\zeta_j - \zeta_k)^{-a_j} F_D^{(N)}(\mathbf{a}_k; b_k, c_k; \mathscr{Y}_1(\mathbf{1} - \mathbf{x}_k)),\end{aligned} \tag{5.8}$$

and

$$\mathscr{I}_N(\mathbf{a};b,c;\mathbf{x}) := \zeta_N^{|\mathbf{a}|-c}(1-\zeta_N)^{c-b-a_N} \times \prod_{j=1}^{N-1}(\zeta_N-\zeta_j)^{-a_j} F_D^{(N)}(\mathbf{a}_N;b_N,c_N;\mathscr{Y}_1(\mathbf{1}-\mathbf{x}_N)), \tag{5.9}$$

with $\mathbf{a}_k$, b_k, and c_k expressed in terms of the quantities in (5.4) by

$$\mathbf{a}_0 := (a_2,\dots,a_N,1+b-c), \quad b_0 := 1+|\mathbf{a}|-c, \quad c_0 := 2+|\mathbf{a}_{2,N}|-c, \tag{5.10}$$

$$\mathbf{a}_k := (c-|\mathbf{a}|,a_1,\dots,a_{k-1},a_{k+2},\dots,a_N,1+b-c), \quad b_k := 1-a_k, \quad c_k := 2-a_k-a_{k+1}, \tag{5.11}$$

$$\mathbf{a}_N := (c-|\mathbf{a}|,a_1,\dots,a_{N-1}), \quad b_N := 1-a_N, \quad c_N := 1+c-b-a_N, \tag{5.12}$$

the vectors $\mathbf{x}_k$ expressed in terms of the ζ_j, $j=1,\dots,N$, by

$$\mathbf{x}_0 := \left(\frac{\zeta_1}{\zeta_2},\dots,\frac{\zeta_1}{\zeta_N},\zeta_1\right), \quad \mathbf{x}_N := \left(\frac{1}{\zeta_N},\frac{\zeta_1}{\zeta_N},\dots,\frac{\zeta_{N-1}}{\zeta_N}\right),$$
$$\mathbf{x}_k := \left(\frac{\zeta_{k+1}}{\zeta_k},\frac{\zeta_1}{\zeta_k},\dots,\frac{\zeta_{k-1}}{\zeta_k},\frac{\zeta_{k+2}}{\zeta_k},\dots,\frac{\zeta_N}{\zeta_k},\frac{1}{\zeta_k}\right), \qquad k=1,\dots,N-1, \tag{5.13}$$

and $\mathscr{Y}_1(\mathbf{x})$ defined by (2.28).

It is easy to see that the quantities (5.7)–(5.9) are expressed in terms of the solution $\mathscr{U}_1^{(1)}$ of the Lauricella system of equations (1.5) with parameters (5.4) by the formulae

$$\mathscr{I}_0(\mathbf{a};b,c;\mathbf{x}) = \mathscr{U}_1^{(1)}(\mathbf{a};b,1+|\mathbf{a}|+b-c;\mathbf{1}-\mathbf{x}),$$
$$\mathscr{I}_k(\mathbf{a};b,c;\mathbf{x}) = \zeta_k^{-b}\mathscr{U}_1^{(1)}(\widetilde{\mathbf{a}}_k;b,1+b-a_k;\mathbf{x}_k), \qquad k=1,\dots,N,$$

where $\widetilde{\mathbf{a}}_k := (a_{k+1},a_1,\dots,a_{k-1},a_{k+2},\dots,a_N,1-c+b)$, and the function $\mathscr{U}_1^{(1)}$ defined by (2.81) has the form

$$\mathscr{U}_1^{(1)}(\mathbf{a};b,c;\mathbf{z}) = (1-z_1)^{c-a_1-b}\left(\prod_{l=2}^{N}(1-z_l)^{-a_l}\right) \times F_D^{(N)}\big(c-|\mathbf{a}|,a_2,\dots,a_N;c-b,1+c-a_1-b;\mathscr{Y}_1(\mathbf{1}-\mathbf{z})\big). \tag{5.14}$$

Using Theorems 2 and 4, we can show that the quantities $\mathscr{I}_k$, regarded as functions of $(\zeta_1,\dots,\zeta_N)$, are solutions of the system (1.5) with one and the same set of parameters connected with the angles of the polygon by (5.4).

In a similar way it is easy to see that the integral in (5.1) over the interval between any two points ζ_j and ζ_k $(k\neq j)$ can, like I_k, be expressed in terms of $F_D^{(N)}$. For example,

$$I_{N+1}(\mathbf{x}) := \int_1^\infty t^{\beta_0-1}\prod_{j=1}^{N}(t-\zeta_j)^{\beta_j-1}(t-1)^{\beta_{N+1}-1}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F_D^{(N)}(\mathbf{a};b,c;\mathbf{x}), \tag{5.15}$$

where we use the notation (5.4), and in deducing (5.15) we made the substitution $t = 1/\tau$ in the integral and then used the Euler-type formula (1.6).

In the case when some vertex z_k is infinite, so that the characteristic β_k of some angle is negative, the integrals I_k and I_{k+1} diverge and therefore the system (5.2), (5.3) must be modified. For example, if $\beta_k \in (-2, 0)$, then the kth and $(k+1)$st equations in this system can be replaced by the two equations

$$\mathscr{K}_0 I_k^{\pm}(\mathbf{x}) = H_k^{\pm}, \tag{5.16}$$

where

$$I_k^{\pm}(\mathbf{x}) := \biggl|\int_{\Gamma_k^{\pm}} t^{\beta_0-1} \prod_{j=1}^{N} (t-\zeta_j)^{\beta_j-1}(t-1)^{\beta_{N+1}-1}\,dt\biggr|, \tag{5.17}$$

the contour Γ_k^- begins at ζ_{k-1}, continues into the upper half-plane, goes clockwise around ζ_k, and (continuing into the lower half-plane) returns to ζ_{k-1}, while the contour Γ_k^+ begins at ζ_{k+1}, continues into the upper half-plane, goes anticlockwise around ζ_k, and (continuing into the lower half-plane) returns to ζ_{k+1}. The quantity $H_k^- := |z_{k-1}^* - z_{k-1}|$ on the right-hand side of (5.16) is the distance between the vertex z_{k-1} and its reflection z_{k-1}^* in the side (z_k, z_{k+1}), while $H_k^+ := |z_{k+1}^* - z_{k+1}|$ is the distance between z_{k+1} and its reflection z_{k+1}^* in the side (z_{k-1}, z_k).

On the other hand, if $\beta_k = 0$ and $\beta_{k-1}, \beta_{k+1} \in (0,1)$, then the kth and $(k+1)$st equations above can be replaced by the following equation (we use the complex notation):

$$\mathscr{K}_0 \widetilde{I}_k(\mathbf{x}) = z_{k+1} - z_{k-1}, \tag{5.18}$$

$$\widetilde{I}_k(\mathbf{x}) := \int_{\Gamma_k} t^{\beta_0-1} \prod_{k=1}^{N} (t-\zeta_k)^{\beta_k-1}(t-1)^{\beta_{N+1}-1}\,dt, \tag{5.19}$$

where the integration contour Γ_k joining ζ_{k-1} and ζ_{k+1} lies in $\overline{\mathbb{H}^+} \setminus \{\zeta_0, \dots, \zeta_{N+2}\}$ (except for the endpoints). The left-hand sides of (5.16), (5.17) and (5.18), (5.19) can be expressed in terms of the Lauricella function if instead of the representation (1.6) we use the representation for this function via integrals over Pochhammer loop contours indicated, for instance, in [13].

The cases considered do not exhaust all possible configurations of polygonal domains $\mathscr{B}$. The reader can find details on the formation of systems of equations for the inverse images in [93]. Below we give an example of the construction of a conformal mapping in the case when the domain is finite and the system of equations for the parameters of the Schwarz–Christoffel integral (5.1) has the form (5.2), (5.3).

5.2. An example of the construction of a conformal mapping in the situation of crowding. We illustrate the results in the previous subsection, § 5.1, by giving an example of the construction of a conformal mapping of a 10-gonal domain $\mathscr{M}$ which is a rectangle with two cuts[1] (see Fig. 9, (a)). The vertices of $\mathscr{M}$ are denoted by M_j, $j = 0, \dots, 9$, and the angles at these vertices are

$$\pi\beta_j = \frac{\pi}{2}, \quad j = 0, \dots, 9, \quad j \neq 3, 6, \qquad \pi\beta_3 = \pi\beta_6 = 2\pi. \tag{5.20}$$

[1] We have borrowed the shape of the domain $\mathscr{M}$ from Grigor'ev's Ph.D. thesis [162].

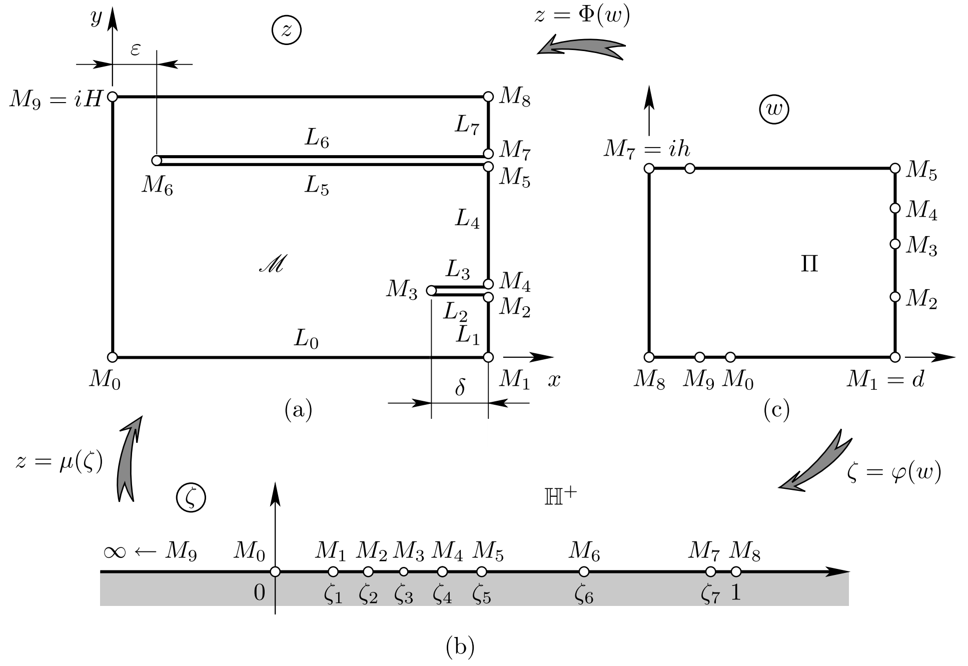

Figure 9. The domains $\mathscr{M}$, $\mathbb{H}^+$, and Π and the correspondence of points under the mappings $\mu\colon \mathbb{H}^+ \xrightarrow{\text{conf}} \mathscr{M}$, $\varphi\colon \Pi \xrightarrow{\text{conf}} \mathbb{H}^+$, and $\Phi\colon \Pi \xrightarrow{\text{conf}} \mathscr{M}$.

The domain $\mathscr{M}$ is determined by the lengths of the sides L_j, $j = 0, \dots, 7$, where we set

$$L_2 = L_3 = \delta \quad \text{and} \quad L_5 = L_6 = L_0 - \varepsilon.$$

We consider the case when ε and δ are sufficiently small, so that the long cut 'almost' partitions the domain into two disconnected parts, leaving only a narrow isthmus between the vertex M_6 and the side (M_0, M_9), and the short cut only 'slightly' affects the behaviour of the conformal mapping near this isthmus.

The mapping $\mu\colon \mathbb{H}^+ \xrightarrow{\text{conf}} \mathscr{M}$ is normalized by the conditions

$$\mu(0) = 0, \qquad \mu(1) = L_0 + iH, \qquad \mu(\infty) = iH \tag{5.21}$$

(see Fig. 9, (a), and (b)), where $H = L_1 + L_4 + L_7$ is the height of the rectangle. The function $\mu(\zeta)$ satisfying (5.21) is expressed as a Schwarz–Christoffel integral

$$z = \mu(\zeta) = -\mathscr{K} \int_0^\zeta \frac{(t - \zeta_3)(t - \zeta_6)\, dt}{\sqrt{t(t - \zeta_1)(t - \zeta_2)(t - \zeta_4)(t - \zeta_5)(t - \zeta_7)(t - 1)}}\,, \tag{5.22}$$

where the vector of inverse images $(\zeta_1, \dots, \zeta_7) =: \mathbf{x}$ and the coefficient $\mathscr{K} > 0$ are the unknowns. To find the ζ_j, $k = 1, \dots, 7$, we have the system of equations

$$\frac{I_k(\mathbf{x})}{I_0(\mathbf{x})} = \frac{L_k}{L_0}\,, \qquad k = 1, \dots, 7, \tag{5.23}$$

where

$$I_k = \left| \int_{\zeta_k}^{\zeta_{k+1}} \frac{(t-\zeta_3)(t-\zeta_6)\,dt}{\sqrt{t(t-\zeta_1)(t-\zeta_2)(t-\zeta_4)(t-\zeta_5)(t-\zeta_7)(t-1)}} \right|, \qquad k=0,\dots,7.$$

After it is solved, we calculate $\mathscr{K}$ by the formula $\mathscr{K} = L_0/I_0(\mathbf{x})$, where $\mathbf{x}$ is the vector of inverse images which is found from the system (5.23).

To express the elements of this system in terms of $F_D^{(7)}$, we start by calculating the quantities in (5.4), taking (5.20) into account:

$$\mathbf{a} = (a_1,\dots,a_7) = \left(\frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}\right), \quad b = \frac{1}{2}, \quad c = 1. \tag{5.24}$$

Then we use (5.24) to find the vectors in (5.11):

$$\begin{aligned}
\mathbf{a}_0 &= \left(\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}\right), &\quad \mathbf{a}_1 &= \left(\frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}\right),\\
\mathbf{a}_2 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}\right), &\quad \mathbf{a}_3 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}\right),\\
\mathbf{a}_4 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, -1, \frac{1}{2}, \frac{1}{2}\right), &\quad \mathbf{a}_5 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right),\\
\mathbf{a}_6 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), &\quad \mathbf{a}_7 &= \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, \frac{1}{2}, \frac{1}{2}, -1\right),
\end{aligned} \tag{5.25}$$

together with the scalar quantities in (5.11):

$$\begin{gathered}
b_0 = \frac{1}{2}, \quad c_0 = 1; \quad b_1 = \frac{1}{2}, \quad c_1 = 1; \quad b_2 = \frac{1}{2}, \quad c_2 = \frac{5}{2}; \quad b_3 = 2, \quad c_3 = \frac{5}{2};\\
b_4 = \frac{1}{2}, \quad c_4 = 1; \quad b_5 = \frac{1}{2}, \quad c_5 = \frac{5}{2}; \quad b_6 = 2, \quad c_6 = \frac{5}{2}; \quad b_7 = \frac{1}{2}, \quad c_7 = 1.
\end{gathered} \tag{5.26}$$

Next we calculate the coefficients C_k in (5.5) by the formulae (5.6):

$$C_0 = C_1 = C_4 = C_7 = \Gamma^2\left(\frac{1}{2}\right) = \pi, \qquad C_2 = C_3 = C_5 = C_6 = \frac{\Gamma(1/2)}{\Gamma(5/2)} = \frac{4}{3}. \tag{5.27}$$

We write out the vectors $\mathbf{x}_k$, found from (5.13):

$$\begin{gathered}
\mathbf{x}_0 := \left(\frac{\zeta_1}{\zeta_2}, \dots, \frac{\zeta_1}{\zeta_7}, \zeta_1\right), \qquad \mathbf{x}_7 := \left(\frac{1}{\zeta_7}, \frac{\zeta_1}{\zeta_7}, \dots, \frac{\zeta_6}{\zeta_7}\right),\\
\mathbf{x}_k := \left(\frac{\zeta_{k+1}}{\zeta_k}, \frac{\zeta_1}{\zeta_k}, \dots, \frac{\zeta_{k-1}}{\zeta_k}, \frac{\zeta_{k+2}}{\zeta_k}, \dots, \frac{\zeta_7}{\zeta_k}, \frac{1}{\zeta_k}\right), \qquad k = 1,\dots,6.
\end{gathered} \tag{5.28}$$

Finally, we obtain expressions for the integrals I_k in (5.23) in the form

$$I_k(\mathbf{x}) = C_k \mathscr{I}_k(\mathbf{a}; b, c; \mathbf{x}), \qquad k = 0,\dots,7,$$

where $\mathscr{I}_k$ is found by the formulae (5.7)–(5.9) with $N=7$, in which we substitute the parameters and variables $\mathbf{a}_k$, b_k, c_k, and $\mathbf{x}_k$ corresponding to the domain $\mathscr{M}$

and calculated by the formulae (5.25)–(5.28):

$$
\begin{aligned}
I_0(\mathbf{x}) &= C_0\biggl(\prod_{j=2}^{7}\zeta^{-a_j}\biggr)F_D^{(7)}(\mathbf{a}_0;b_0,c_0;\mathbf{x}_0),\\
I_7(\mathbf{x}) &= C_7\zeta^{-1/2}\biggl(\prod_{j=2}^{7}\zeta^{-a_j}\biggr)F_D^{(7)}(\mathbf{a}_7;b_7,c_7;\mathscr{Y}_1(\mathbf{1}-\mathbf{x}_7)),\\
I_k(\mathbf{x}) &:= C_k\frac{(\zeta_{k+1}-\zeta_k)^{1-a_k-a_{k+1}}}{\zeta_k^{1/2}(1-\zeta_k)^{1/2}}\\
&\qquad\times\biggl(\prod_{\substack{1\leqslant j\leqslant 7\\ j\ne k,k+1}}(|\zeta_k-\zeta_j|)^{-a_j}\biggr)F_D^{(7)}(\mathbf{a}_k;b_k,c_k;\mathscr{Y}_1(\mathbf{1}-\mathbf{x}_k)),\qquad k=1,\dots,6.
\end{aligned}
\tag{5.29}
$$

We solve the system (5.23), (5.29) numerically by Newton's method, using the (known) method of continuation with respect to a parameter to find an initial approximation. The key thing here is the formulae for analytic continuation of the Lauricella function $F_D^{(7)}$ in (5.29): they enable us to make a high-precision (essentially, machine-precision) calculation of this function at each step of Newton's iterative algorithm. We note that, because of the half-integer values of the characteristics of the angles of $\mathscr{M}$, we are in the resonant (logarithmic) case for the Lauricella function.

To illustrate the mapping $\mu\colon \mathbb{H}^+\xrightarrow{\text{conf}}\mathscr{M}$ it is convenient to use the auxiliary conformal mapping $\varphi\colon \Pi\xrightarrow{\text{conf}}\mathbb{H}^+$ of the rectangle Π onto the half-plane $\mathbb{H}^+$ (see Fig. 9) and first map the natural Cartesian grid for Π into $\mathbb{H}^+$, then use the function $z=\mu(\zeta)$ to map it into $\mathscr{M}$. Thus we obtain in $\mathscr{M}$ the image of the Cartesian grid (originally constructed in Π) under the mapping $\Phi\colon \Pi\xrightarrow{\text{conf}}\mathscr{M}$. Imposing on φ the conditions

$$
\varphi(ih)=\zeta_7,\qquad \varphi(0)=1,\qquad \varphi(d)=\zeta_1,\qquad \varphi(d+ih)=\zeta_5 \tag{5.30}
$$

(so that the points M_7, M_8, M_1, M_5 on $\partial\Pi$ are taken to points with the same names on $\partial\mathbb{H}^+$), we find φ in the form

$$
\varphi(w)=\frac{\zeta_7(1-\zeta_1)\,\mathrm{sn}^2(k,w)-(\zeta_7-\zeta_1)}{(1-\zeta_1)\,\mathrm{sn}^2(k,w)-(\zeta_7-\zeta_1)},\quad k=\biggl[\frac{(\zeta_7-\zeta_5)(1-\zeta_1)}{(\zeta_7-\zeta_1)(1-\zeta_5)}\biggr]^{1/2}, \tag{5.31}
$$

where $\mathrm{sn}(k,w)$ is the Jacobi elliptic function with modulus k (see [163]), calculated in terms of the parameters of the conformal mapping (5.22) by the formula in (5.31). The length d and height h of the rectangle are equal to the elliptic integrals $K(k)$ and $K'(k)$, respectively (see [163] about these integrals).

We proceed to the calculation of the conformal mapping of $\mathscr{M}$ when the lengths of sides L_j (see Fig. 9, (a)) are as follows:

$$
\begin{gathered}
L_0=2,\qquad L_1=0.25,\qquad L_2=L_3=0.1,\qquad L_4=0.5,\\
L_5=L_6=2-\varepsilon,\quad \varepsilon=0.001,\qquad L_7=0.25.
\end{gathered}
$$

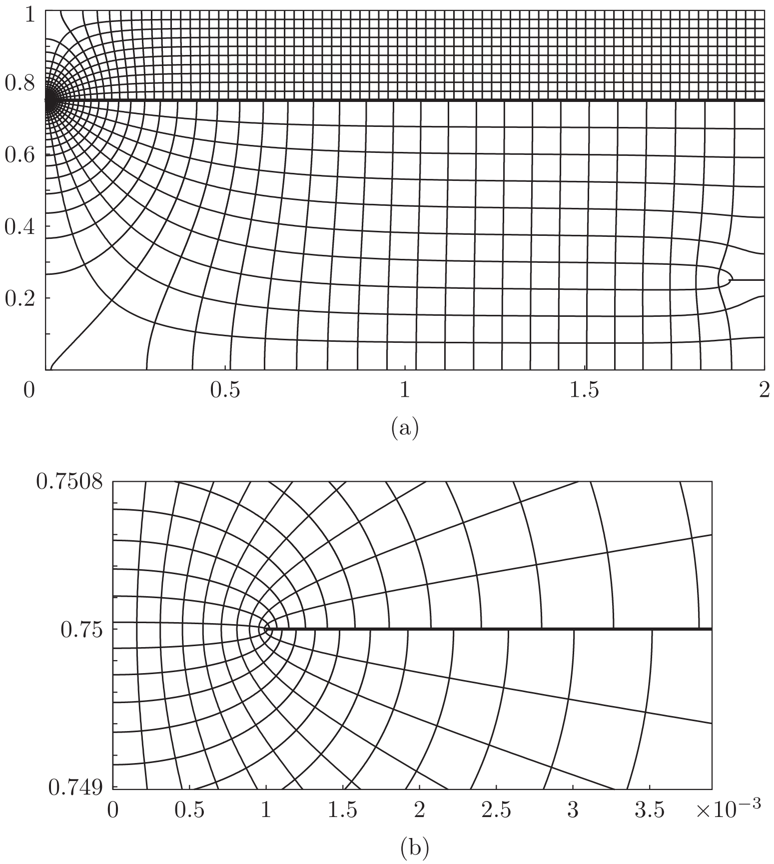

Figure 10. Results of the calculation of the mapping $\Phi\colon \Pi \xrightarrow{\text{conf}} \mathscr{M}$: (a) the grid in the whole of $\mathscr{M}$ and (b) the scaled grid near the narrow isthmus.

We found the parameters of the Schwarz–Christoffel integral (5.22) by solving the system (5.23), (5.29) numerically with an accuracy of 14 significant digits, using the above method (we used the standard 16-digit significand in our computations). It should be pointed out that, as our calculations showed, the points ζ_j, $j=1,\dots,7$, are extremely unevenly distributed in the interval $(0,1)$, that is, we have the crowding phenomenon discussed in § 1.4. Most of the successive inverse images lie very close together, for example,

$$\zeta_1 = 1.08006360840606\times 10^{-11}, \quad \zeta_3-\zeta_2 = 6.61626298018142\times 10^{-15},$$
$$\zeta_6-\zeta_5 = 9.86947089484393\times 10^{-6},$$

and the order of the 'smallness' can be quite different. On the other hand, the distance

$$\zeta_7-\zeta_6 = 0.999990130323672$$

is close to the length of the full unit interval containing all seven inverse images. These results illustrate nicely the term 'crowding': the points ζ_j, $j=1,\dots,5$, 'cluster' on the interval $(0,\zeta_6)$, whose length is less than that of $(0,1)$ by five orders of magnitude, and furthermore the distance between the closest of the ζ_k is less than the length of $(0,\zeta_6)$ by nine orders of magnitude.

In Fig. 10, (a) we give the image of the Cartesian grid under the mapping $\Phi(w) = \mu\circ\varphi(w)$, where $z=\mu(\zeta)$ is the Schwarz–Christoffel integral (5.22) and $\zeta=\varphi(w)$ is the auxiliary mapping (5.31) of the rectangle onto the half-plane. In Fig. 10, (b) we give a scaled piece of Fig. 10, (a) showing in detail the conformal grid in the narrow isthmus between the endpoint of the 'long' cut M_6 and the side (M_9, M_0) (see the notation in Fig. 9, (a)).

6. Conclusion

Integration of general hypergeometric systems of partial differential equations is a very topical problem, interesting both theoretically and in applications. For systems in the Horn class (1.11) we can write a particular solution in the form of a hypergeometric series (1.8) with coefficients expressible directly in terms of the polynomials P_j and Q_j. For instance, in the case of the system (1.5) the Lauricella function $F_D^{(N)}$ defined in (1.4) and considered above is such a solution. On the other hand, describing a basis of the space of solutions of the system (1.11) and an analytic continuation of this basis is a well-known but difficult problem.

In § 2 we presented a solution of this problem for the Lauricella system of differential equations (1.5). The set of solutions given by Theorems 1–5, and symmetries (2.105) of these solutions provide a basis in the space of solutions of (1.5) in corresponding subdomains of $\mathbb{C}^N$ and are the N-dimensional analogue of Kummer's solutions (1.28)–(1.32) of Gauss's hypergeometric equation. These theorems present a complete set of formulae of the type (1.27) for analytic continuation of $F_D^{(N)}$ into the exterior of the polydisk $\mathbb{U}^N$. The construction of such formulae has long attracted the attention of many authors. These formulae are an effective tool for a qualitative analysis and calculation of $F_D^{(N)}$, and thus for computing integrals of the form (1.6) for all values of $\mathbf{z}\in\mathbb{C}^N$ away from certain hyperplanes. In this paper we did not consider the question of analytic continuation of solutions of (1.5) other than $F_D^{(N)}$. Formulae for such a continuation (which is realized using the same methods) are of interest because they provide an effective machinery for calculating the monodromy group of the Lauricella system (1.5). We remark also that our approach can be carried over to some other hypergeometric systems and can also be used for analytic continuation of multivariate hypergeometric functions in an (apparently, quite broad) class including the three other functions $F_A^{(N)}$, $F_B^{(N)}$, and $F_C^{(N)}$ introduced by Lauricella [6], [13].

The particular solutions of the system (1.11) presented in Theorems 1–5 are rather complicated power series. Constructing such solutions using the method of indeterminate coefficients is very laborious even for two variables (see [10] on this question). It is known that multiple Mellin–Barnes integrals can be efficiently used to find particular solutions of hypergeometric systems (see [13], [28]). However, in our paper we have used one-dimensional integrals of this type indicated in Propositions 1 and 2, and they seem to provide a more convenient approach to the construction of a complete set of solutions of the system (1.11) and to the analytic continuation of it in the form (1.27). Representations of Mellin–Barnes type (2.3) and (2.18) reduce the problem of analytic continuation of the Lauricella function $F_D^{(N)}$ into the exterior of $\mathbb{U}^N$ to the continuation of it with respect to each of the N variables in succession. As a result, the required formulae for analytic continuation is obtained in N steps.

Section 3 is devoted to applications of the Lauricella function to the theory of the Riemann–Hilbert problem. As noted in § 1.5, many authors have observed that this problem (with piecewise constant boundary data) is connected with the Schwarz–Christoffel integral (3.54). However, the question of finding such a representation explicitly, including finding the polynomial $\mathscr{R}(\zeta)$, has remained unresolved for quite a while. For all the quantities involved in this representation we gave expressions for them in Theorem 10 in terms of the data of the Riemann–Hilbert problem. It is important to note we gave a closed representation for the polynomial $\mathscr{R}(\zeta)$ in terms of the Lauricella function $F_D^{(N)}$ without using any numerical procedures. The derivation of the representation (3.54) was based on Jacobi-type formulae for $F_D^{(N)}$ that amount to new advances in the theory of this function. Jacobi-type formulae can also be interpreted as relations between so-called 'contiguous' Lauricella functions. Moreover, they make it possible to give the hypergeometric system of partial differential equations in the alternative form of (1.11)

The seemingly unexpected connections between the Lauricella function $F_D^{(N)}(\mathbf{a}; b, c; \mathbf{z})$ and other areas in mathematics and its applications are to a significant extent explained by the fact that any integral of the form

$$I(\mathbf{w}) = \int_{\mathscr{L}} \prod_{k=1}^{N+3} (t - w_k)^{\alpha_k - 1}\, dt \tag{6.1}$$

can be expressed in terms of this function with N variables (see [13]), where $\alpha := (\alpha_1, \dots, \alpha_{N+3})$ and $\mathbf{w} := (w_1, \dots, w_{N+3})$ are vectors in $\mathbb{C}^{N+3}$, and the curve of integration $\mathscr{L}$ either joins a pair of points w_n and w_m with $n \neq m$ or is a certain closed loop contour on the Riemann surface of the integrand in (6.1). The parameters $\mathbf{a}$, b, and c and the variable $\mathbf{z}$ are expressed in terms of the vectors α and $\mathbf{w}$, respectively, by simple expressions. The Euler-type integral representation (1.6) for $F_D^{(N)}$ is an example of such a result.

The prospective applications of the function $F_D^{(N)}$ include the problem of 'crowding' for the parameters of the Schwarz–Christoffel integral, which we again point out as a problem that has attracted the attention of many researchers (for instance, see [97], [99]). It has many very interesting theoretical and computational aspects, and formulae for analytic continuation of $F_D^{(N)}$ are a key to the analysis of these aspects.

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Sergei I. Bezrodnykh
Dorodnicyn Computing Centre of Russian Academy of Sciences;
Peoples' Friendship University of Russia
E-mail: sbezrodnykh@mail.ru

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