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Dedicated to our teachers and friends
Andrei Alexandrovich Gonchar,
Eugene Mikhailovich Nikishin, *and*
Herbert Stahl

On Nikishin systems with discrete components and weak asymptotics of multiple orthogonal polynomials

A. I. Aptekarev, G. López Lagomasino, and A. Martínez-Finkelshtein

Abstract. This survey considers multiple orthogonal polynomials with respect to Nikishin systems generated by a pair (σ_1, σ_2) of measures with unbounded supports ($\operatorname{supp}(\sigma_1) \subseteq \mathbb{R}_+$, $\operatorname{supp}(\sigma_2) \subset \mathbb{R}_-$) and with σ_2 discrete. A Nikishin-type equilibrium problem in the presence of an external field acting on $\mathbb{R}_+$ and a constraint on $\mathbb{R}_-$ is stated and solved. The solution is used for deriving the contracted zero distribution of the associated multiple orthogonal polynomials.

Bibliography: 56 titles.

Keywords: Hermite–Padé approximants, multiple orthogonal polynomials, orthogonality with respect to a discrete measure, weak asymptotics, vector equilibrium problem, Nikishin systems.

Contents

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1. Introduction

In a celebrated paper published in 1980, Nikishin [42] introduced a general class of systems of measures, now called Nikishin systems. Let Δ_α and Δ_β be two disjoint bounded intervals of the real line $\mathbb{R}$, and let $\sigma_\alpha \in \mathscr{M}(\Delta_\alpha)$ and $\sigma_\beta \in \mathscr{M}(\Delta_\beta)$, where $\mathscr{M}(\Delta)$ denotes the set of all finite Borel measures with constant sign on an interval Δ. With σ_α and σ_β we construct a third measure $\langle \sigma_\alpha, \sigma_\beta \rangle$ given by

$$d\langle \sigma_\alpha, \sigma_\beta \rangle(x) := \widehat{\sigma}_\beta(x)\, d\sigma_\alpha(x), \qquad \widehat{\sigma}_\beta(x) = \int \frac{1}{x-t}\, d\sigma_\beta(t). \tag{1.1}$$

Definition 1.1. Take a collection Δ_j, $j = 1, \dots, m$, of intervals such that

$$\Delta_j \cap \Delta_{j+1} = \varnothing, \qquad j = 1, \dots, m-1,$$

and a system of measures $(\sigma_1, \dots, \sigma_m)$ with $\sigma_j \in \mathscr{M}(\Delta_j)$, $j = 1, \dots, m$. We assume in addition that for each j the convex hull of the support $\operatorname{supp}(\sigma_j)$ of σ_j coincides with Δ_j. Let

$$s_1 = \sigma_1, \quad s_2 = \langle \sigma_1, \sigma_2 \rangle, \quad \dots, \quad s_m = \langle \sigma_1, \sigma_2, \dots, \sigma_m \rangle = \langle \sigma_1, \langle \sigma_2, \dots, \sigma_m \rangle \rangle.$$

We call $(s_1, \dots, s_m)$ the *Nikishin system of measures* generated by $(\sigma_1, \dots, \sigma_m)$, and denote it by $\mathscr{N}(\sigma_1, \dots, \sigma_m)$.

This model system was introduced in order to study general properties of *Hermite–Padé approximants* and *multiple orthogonal polynomials*.

We fix

$$\mathbf{n} := (n_1, \dots, n_m) \in \mathbb{Z}_+^m \setminus \{\mathbf{0}\},$$

where $\mathbf{0}$ is the m-dimensional zero vector. Define $P_{\mathbf{n}}$ to be a non-zero polynomial of degree $\deg(P_{\mathbf{n}}) \leqslant |\mathbf{n}| := n_1 + \dots + n_m$ such that

$$\int x^\nu P_{\mathbf{n}}(x)\, ds_j(x) = 0, \qquad \nu = 0, \dots, n_j - 1, \quad j = 1, \dots, m.$$

The existence of $P_{\mathbf{n}}$ reduces to solving a homogeneous linear system of $|\mathbf{n}|$ equations in the $|\mathbf{n}|+1$ coefficients of $P_{\mathbf{n}}$, so a non-trivial solution is guaranteed. However, in contrast with the scalar case $m = 1$ of standard *orthogonal polynomials*, the uniqueness of $P_{\mathbf{n}}$ (up to a constant factor) is not a trivial matter (and in general is not true for systems of arbitrary measures $(s_1, \dots, s_m)$). In connection with this question it was shown in [42] that for a Nikishin system uniqueness holds, with $\deg P_{\mathbf{n}} = |\mathbf{n}|$, for multi-indices of the form $(n+1, \dots, n+1, n, \dots, n)$, and it was stated (without proof) that the same is true when

$$n_1 \geqslant \dots \geqslant n_m.$$

Below we assume that $P_{\mathbf{n}}$ is monic.

Motivated by the structure of Nikishin systems, Stahl studied their analytic and algebraic properties (see [9]). In a series of papers [21]–[23] Driver and Stahl showed, among other results, that uniqueness remains valid when

$$n_j \leqslant n_k + 1, \qquad 1 \leqslant k < j \leqslant m.$$

The problem for arbitrary multi-indices was definitively solved in [25] (and in [26] when the generating measures have unbounded and/or touching supports).

A remarkable property of Nikishin orthogonal polynomials is that they not only satisfy orthogonality relations with respect to several measures but also satisfy the full system of orthogonality relations with respect to a single (varying with respect to $\mathbf{n}$) measure. For $m = 2$ and $n_2 \leqslant n_1 + 1$ this was first observed by Andrei Aleksandrovich Gonchar,[1] who showed that the function of the second kind

$$R_{\mathbf{n},1}(z) = \int \frac{P_{\mathbf{n}}(x)}{z - x}\, d\sigma_1(x)$$

satisfies the orthogonality relations

$$\int x^\nu R_{\mathbf{n},1}(x)\, d\sigma_2(x) = 0, \qquad \nu = 0, \dots, n_2 - 1. \tag{1.2}$$

From this it follows that $R_{\mathbf{n},1}$ has exactly n_2 zeros in $\mathbb{C} \setminus \Delta_1$, they are all simple, and they lie in the interior of Δ_2. If $P_{\mathbf{n},2}$ denotes the monic polynomial of degree n_2 vanishing at these points, then

$$\int x^\nu P_{\mathbf{n}}(x)\, \frac{d\sigma_1(x)}{P_{\mathbf{n},2}(x)} = 0, \qquad \nu = 0, \dots, n_1 + n_2 - 1. \tag{1.3}$$

The study of the asymptotic behavior of multiple orthogonal polynomials is greatly indebted to Gonchar. In joint papers with Rakhmanov [27]–[29], they introduced the notion of vector equilibrium problem for describing the asymptotic zero distribution of such polynomials. For a Nikishin system of two measures and $n_1 = n_2 = n$ the result can be stated as follows. Define the normalized zero-counting measure ν_P of a polynomial P by

$$\nu_P = \frac{1}{\deg P} \sum_{P(x)=0} \delta_x,$$

where δ_x denotes the Dirac measure with mass 1 at the point x, and each zero of P is taken with its multiplicity counted, so that the total variation $|\nu_P|$ of ν_P is 1. Assume that $\sigma_j \in \mathbf{Reg}$, $j = 1, 2$ (for the definition of the class $\mathbf{Reg}$ of measures, see [56], Chap. 3). Then there exist unique (positive) measures $\lambda_j \in \mathscr{M}(\Delta_j)$, $j = 1, 2$, with $|\lambda_1| = 2$ and $|\lambda_2| = 1$ such that

$$\lim_n \nu_{P_{\mathbf{n}}} = \frac{\lambda_1}{2} \quad \text{and} \quad \lim_n \nu_{P_{\mathbf{n},2}} = \lambda_2, \tag{1.4}$$

[1] At one of the regular Monday seminars at the Steklov Mathematical Institute Gonchar was reporting on some results in [42], but after a short while he had to leave to attend an important meeting. After an hour or so he returned and started his presentation anew, proving (1.2) and (1.3), and from them he deduced the convergence of the corresponding Hermite–Padé approximants.

in the vague topology of measures, and λ_1 and λ_2 are uniquely determined as the solution of the vector equilibrium problem

$$\begin{aligned} 2U^{\lambda_1}(x) - U^{\lambda_2}(x) &\begin{cases} = w_1, & x \in \operatorname{supp}(\lambda_1), \\ \geqslant w_1, & x \in \Delta_1 \setminus \operatorname{supp}(\lambda_1), \end{cases} \\ 2U^{\lambda_2}(x) - U^{\lambda_1}(x) &\begin{cases} = w_2, & x \in \operatorname{supp}(\lambda_2), \\ \geqslant w_2, & x \in \Delta_2 \setminus \operatorname{supp}(\lambda_2), \end{cases} \end{aligned} \tag{1.5}$$

where w_1 and w_2 are some constants, and U^λ denotes the logarithmic potential of λ (see the definition below). At the time (in the 1980s–90s), this result and its extensions were well known within a small circle of specialists. With some variations it appeared for general Nikishin systems in a paper by Stahl [55], and with the highest degree of generality in a paper by Gonchar, Rakhmanov, and Sorokin [30]. For other extensions and generalizations see [5], [7], [11], [15], [24], [43], [47], [48].

In recent years, Nikishin systems have attracted new attention because this construction has been identified in different *models of random matrix theory* and multiple orthogonal polynomial ensembles (see [6], [37], [38]). In some of these models new ingredients appear in which some of the generating measures turn out to be discrete and/or have unbounded support. Sorokin has studied the asymptotic distribution of the zeros for several multiple orthogonal polynomials of this type (see [53], [54]).

Orthogonal polynomials with respect to discrete measures have the characteristic that between two consecutive mass points there may be at most one zero of the polynomial. This fact induces a constraint on the equilibrium problem for the logarithmic potential whose solution describes the asymptotic zero distribution of the orthogonal polynomials. This effect was first pointed out by Rakhmanov in [46] (see also [20] and [41]). A similar situation occurs in the case of multiple orthogonal polynomials.

The present paper is devoted to the study of multiple orthogonal polynomials with respect to Nikishin systems generated by two measures (σ_1, σ_2) with unbounded supports

$$\operatorname{supp}(\sigma_1) \subseteq \mathbb{R}_+ := [0, +\infty) \quad \text{and} \quad \operatorname{supp}(\sigma_2) \subset (-\infty, 0).$$

The second measure σ_2 is discrete. To obtain the limiting zero distribution (1.4) of such multiple orthogonal polynomials we state and solve a Nikishin-type equilibrium problem which generalizes (1.5) by having an external field acting on $\mathbb{R}_+$ and a constraint on $\mathbb{R}_- := (-\infty, 0]$. The main results are stated in § 2. In § 3 we review some examples of explicit solutions of the type of equilibrium problems that we consider. Section 4 contains new results concerning potentials with unbounded support and scalar equilibrium problems. The last two sections include proofs of the main results.

There is a long story behind this paper. It began in 2011 while the first author was visiting Spain in the framework of 'The Excellence Chair Program' sponsored by Universidad Carlos III de Madrid and the Bank of Santander. Then essential progress on this project was achieved in 2014 when the editorial boards of Sbornik Mathematics and the Journal of Approximation Theory were preparing the special issues [34] and [35] of their journals in memory of H. Stahl (1945–2012) and A. A. Gonchar (1931–2013). However, we were not able to complete the task in due time. Finally, the 70th anniversary in 2015 of E. M. Nikishin's birth and the 30th anniversary in 2016 of his death motivated the authors to finish the work, which is dedicated to the memory of these three outstanding analysts.

2. Statement of the main results

Let $d\sigma_1(x) = \sigma_1'(x)\,dx$ be a positive, absolutely continuous measure on $\mathbb{R}_+$ and σ_2 a purely discrete measure with support in $(-\infty, 0)$ given by

$$\sigma_2 = \sum_{k\geqslant 1} \beta_k\,\delta_{t_k}, \qquad 0 > t_k \searrow -\infty, \quad \beta_k > 0, \quad \sum_{k\geqslant 1} \frac{\beta_k}{|t_k|} < +\infty. \tag{2.1}$$

All the moments of σ_1 are assumed to be finite. We note that $\widehat{\sigma}_2$ is integrable with respect to σ_1. Let $(s_1, s_2) = \mathscr{N}(\sigma_1, \sigma_2)$ be the Nikishin system generated by these measures. For $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}_+^2 \setminus \{\mathbf{0}\}$ we define $P_{\mathbf{n}}$ as the monic polynomial of degree $|\mathbf{n}|$ which satisfies

$$\int x^\nu P_{\mathbf{n}}(x)\,ds_j(x) = 0, \qquad \nu = 0, \dots, n_j - 1, \quad j = 1, 2. \tag{2.2}$$

The zeros of $P_{\mathbf{n}}$ are simple and lie in the interior of $\mathbb{R}_+$. We will restrict our attention to sequences of multi-indices of the form $\mathbf{n} = (n, n)$. In order to simplify the notation we write P_n instead of $P_{\mathbf{n}}$. Thus, $\deg P_n = 2n$. Our goal is to describe the (rescaled) asymptotic zero distribution of the polynomials (P_n), $n \in \mathbb{N}$, under appropriate assumptions on the generating measures σ_j, $j = 1, 2$.

From the properties of Nikishin systems (see [26] and [30]) it is easy to deduce that there exists a monic polynomial $P_{n,2}$ with $\deg P_{n,2} = n$ whose zeros are simple and contained in the interior of the convex hull of $\operatorname{supp}(\sigma_2)$, such that

$$\int x^\nu \frac{P_n(x)}{P_{n,2}(x)}\,d\sigma_1(x) = 0, \qquad \nu = 0, \dots, 2n-1, \tag{2.3}$$

and

$$\int t^\nu \frac{P_{n,2}(t)}{P_n(t)} \int \frac{P_n^2(x)}{P_{n,2}(x)} \frac{d\sigma_1(x)}{x-t}\,d\sigma_2(t) = 0, \qquad \nu = 0, \dots, n-1. \tag{2.4}$$

In other words, P_n and $P_{n,2}$ satisfy the full system of orthogonality relations with respect to varying measures.

Let $(d_n)_{n\in\mathbb{Z}_+}$, $d_n \geqslant 1$, be an increasing sequence of numbers with $\lim_n d_n^{1/n} = 1$, and let

$$Q_n(x) = \frac{P_n(d_n x)}{d_n^{2n}} \quad \text{and} \quad Q_{n,2}(t) = \frac{P_{n,2}(d_n t)}{d_n^n}\,. \tag{2.5}$$

After the change of variables $x \to d_n x$ and $t \to d_n t$ the monic polynomials Q_n and $Q_{n,2}$ satisfy the orthogonality relations

$$\int x^\nu \frac{Q_n(x)}{Q_{n,2}(x)} \sigma_1'(d_n x)\,dx = 0, \qquad \nu = 0, \dots, 2n-1, \tag{2.6}$$

and

$$\int t^{\nu} \frac{Q_{n,2}(t)}{Q_n(t)} \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \frac{\sigma_1'(d_n x)\, dx}{x-t}\, d\sigma_{2,n}(t) = 0, \qquad \nu = 0, \dots, n-1, \tag{2.7}$$

where

$$\sigma_{2,n} = \sum_{k \geqslant 1} \beta_k \delta_{\xi_{k,n}}, \qquad \xi_{k,n} = \frac{t_k}{d_n}. \tag{2.8}$$

The asymptotic zero distribution of the multiple orthogonal polynomials Q_n and $Q_{n,2}$ is described in terms of an associated vector equilibrium problem that we now present.

For a closed subset $\Delta \subset \mathbb{R}$ we denote by $\mathscr{M}^+(\Delta)$ the class of all finite positive Borel measures μ such that $\operatorname{supp}(\mu) \subset \Delta$. We write $\mu \in \mathscr{M}_c^+(\Delta)$ if in addition $|\mu| = c$. Let $\mu \in \mathscr{M}^+(\mathbb{R})$. Its logarithmic potential and energy are given by

$$U^{\mu}(x) := \int \log \frac{1}{|x-y|}\, d\mu(y) \quad \text{and} \quad I(\mu) := \iint \log \frac{1}{|x-y|}\, d\mu(x)\, d\mu(y), \tag{2.9}$$

respectively, whenever these integrals are well defined.

Assume that $\mu_1, \mu_2 \in \mathscr{M}^+(\mathbb{R})$ satisfy the conditions

$$I(\mu) < +\infty, \qquad \int \log(1 + |x|^2)\, d\mu(x) < +\infty. \tag{2.10}$$

Their mutual energy can be defined as

$$I(\mu_1, \mu_2) := \iint \log \frac{1}{|x-y|}\, d\mu_1(x)\, d\mu_2(y).$$

One can define the potential, energy, and mutual energy of signed measures similarly. In particular, if μ_1 and μ_2 satisfy (2.10), then

$$I(\mu_1 - \mu_2) = I(\mu_1) + I(\mu_2) - 2I(\mu_1, \mu_2).$$

Moreover, if $\mu_1, \mu_2 \in \mathscr{M}_c^+(\mathbb{R})$ (only finiteness of the energy is required), then

$$I(\mu_1 - \mu_2) \geqslant 0, \tag{2.11}$$

with equality if and only if $\mu_1 = \mu_2$ (see Theorem 2.5 in [16], Theorem 4.1 in [52], and also Lemma 1.1.8 in [50] if the measures have compact support).

Let σ be a positive Borel measure with $\operatorname{supp}(\sigma) = \mathbb{R}_-$ and $|\sigma| > 1$ such that for every compact subset $K \subset \mathbb{R}_-$ the function $U^{\sigma|_K}$ is continuous on $\mathbb{C}$, where $\sigma|_K$ denotes the restriction of σ to K. We define the class

$$\mathfrak{M}(\sigma) := \{\, \vec{\mu} = (\mu_1, \mu_2)^t \in \mathscr{M}_2^+(\mathbb{R}_+) \times \mathscr{M}_1^+(\mathbb{R}_-) \colon \mu_2 \leqslant \sigma \}, \tag{2.12}$$

where the superscript ^{t} stands for transposition. By $\mu_2 \leqslant \sigma$ we mean that $\sigma - \mu_2$ is a positive measure. Since we have assumed that $U^{\sigma|_K}$ is continuous on $\mathbb{C}$ for every compact set K, it readily follows that U^{μ_2} is continuous on $\mathbb{C}$. Eventually we will require that a measure μ on $\mathbb{R}$ (in particular σ) satisfies the following condition: for any $\varepsilon > 0$ there exist $0 < \delta < 1/2$ and $R_0 > 0$ such that

$$\sup_{|R| \geqslant R_0} \int_{R-\delta}^{R+\delta} \log \frac{1}{|R-y|}\, d\mu(y) < \varepsilon. \tag{2.13}$$

Let φ be a real-valued continuous function on $\mathbb{R}_+$ satisfying the condition

$$\lim_{x\to\infty} (\varphi(x) - 4\log x) = +\infty, \tag{2.14}$$

and let

$$\begin{aligned} \mathfrak{M}^*(\sigma) &:= \{\vec{\mu} \in \mathfrak{M}(\sigma) \colon \mu_1, \mu_2 \text{ satisfy } (2.10)\}, \\ J_\varphi &:= \inf\{J_\varphi(\vec{\mu}) \colon \vec{\mu} \in \mathfrak{M}^*(\sigma)\}, \\ J_\varphi(\vec{\mu}) &:= 2\biggl(I(\mu_1) - I(\mu_1,\mu_2) + I(\mu_2) + \int \varphi\, d\mu_1\biggr), \end{aligned}$$

and

$$W_1^{\vec{\mu}}(x) := 2U^{\mu_1}(x) - U^{\mu_2}(x) + \varphi(x), \qquad W_2^{\vec{\lambda}}(x) := 2U^{\lambda_2}(x) - U^{\lambda_1}(x).$$

Theorem 2.1. *Let σ be a (positive) Borel measure with $\operatorname{supp}(\sigma) = \mathbb{R}_-$ and $|\sigma| > 1$ such that for every compact subset $K \subset \mathbb{R}_-$ the function $U^{\sigma|_K}$ is continuous on $\mathbb{C}$. Let φ be a continuous function on $\mathbb{R}_+$ which satisfies (2.14). Then the following statements are equivalent and all have the same unique solution.*

(A) *There exists a $\vec{\lambda} \in \mathfrak{M}^*(\sigma)$ such that $J_\varphi(\vec{\lambda}) = J_\varphi > -\infty$.*

(B) *There exists a $\vec{\lambda} \in \mathfrak{M}^*(\sigma)$ such that for all $\vec{\nu} \in \mathfrak{M}^*(\sigma)$,*

$$\int W_1^{\vec{\lambda}}\, d(\nu_1 - \lambda_1) + \int W_2^{\vec{\lambda}}\, d(\nu_2 - \lambda_2) \geqslant 0.$$

(C) *There exist $\vec{\lambda} = (\lambda_1, \lambda_2) \in \mathfrak{M}^*(\sigma)$ and constants $w_1 = w_1(\sigma,\varphi)$ and $w_2 = w_2(\sigma,\varphi)$ such that*

$$2U^{\lambda_1}(x) - U^{\lambda_2}(x) + \varphi(x) \begin{cases} = w_1, & x \in \operatorname{supp}(\lambda_1), \\ \geqslant w_1, & x \in \mathbb{R}_+, \end{cases} \tag{2.15}$$

$$2U^{\lambda_2}(x) - U^{\lambda_1}(x) \begin{cases} \leqslant w_2, & x \in \operatorname{supp}(\lambda_2) = \mathbb{R}_-, \\ = w_2, & x \in \operatorname{supp}(\sigma - \lambda_2). \end{cases} \tag{2.16}$$

The constants w_1 and w_2 are uniquely determined by (2.15) and (2.16). The functions U^{λ_1} and U^{λ_2} are continuous on $\mathbb{C}$ and $\operatorname{supp}(\lambda_1)$ is compact.

If $x\varphi'(x) > 0$ is increasing on $\mathbb{R}_+$, then $\operatorname{supp}(\lambda_1)$ is also connected. If φ is increasing on $\mathbb{R}_+$, then $0 \in \operatorname{supp}(\lambda_1)$. If

$$\int \log(1+y^2)\, d\sigma(y) = +\infty$$

and σ satisfies (2.13), then $w_2(\sigma,\varphi) = 0$.

Results of this nature (in a more general setting regarding the dimension of the vector equilibrium problem and the supports of the corresponding measures) may be seen in [10]. There, the action of constraints on the measures is not considered, and the external fields, which satisfy restrictions of the form (2.18), act on all the components of the vector of measures. This implies in turn that all the components of the equilibrium vector measure have compact support. Nevertheless, taking into consideration certain applications, we are especially interested in allowing the support of the second component of the equilibrium measure to be unbounded. For this reason, in the proof of Theorem 2.1 (see also Theorem 5.1) we follow the approach in [32], where results similar to Theorem 2.1, except for part (C), also appear. It is worth mentioning that when dealing with vector potentials involving measures with overlapping supports, there is in general no reason for the Euler–Lagrange variational conditions to hold everywhere, even if the interaction matrix[2] is positive-definite and the external fields are strongly confining (see the interesting examples in [10]). In our case, the solution is due to the Nikishin-type structure of the problem and the action of the constraint σ satisfying adequate conditions.

In order to study the contracted zero distribution of the polynomials Q_n and $Q_{n,2}$, we must impose some restrictions on the points $\xi_{k,n}$ and the numbers β_k and d_n. These conditions are inspired by similar ones introduced for the study of the contracted zero distribution of discrete orthogonal polynomials in the scalar case, as one can see in Theorem 2 of [46], Definition 3.1 of [20], § 6 of [41], and Theorem 7.1 of [40], whose model we follow closely. Below we assume the following conditions.

(i) There exists a positive continuous function ρ on $\mathbb{R}_-$ such that

$$|\xi_{k+1,n} - \xi_{k,n}| > \frac{\rho(\xi_{k,n})}{n}, \qquad k \geqslant 0 \quad (\xi_{0,n} = 0).$$

(ii)

$$\lim_{n\to\infty} \big(\min\{\beta_k : \xi_{k,n} \in [-n, 0]\}\big)^{1/n} = 1.$$

(iii) There exists a positive Borel measure σ with $\operatorname{supp}(\sigma) = \mathbb{R}_-$ and $|\sigma| > 1$ such that:

- for every compact subset $K \subset \mathbb{R}_-$, the logarithmic potential $U^{\sigma|_K}$ of the restriction of σ to K is continuous on $\mathbb{C}$;
- $$\int \log(1 + y^2)\, d\sigma(y) = +\infty;$$
- for any $\varepsilon > 0$ there exist $0 < \delta < 1/2$ and $R_0 < 0$ such that

$$\sup_{R\leqslant R_0} \int_{R-\delta}^{R+\delta} \log \frac{1}{|R-y|}\, d\sigma(y) < \varepsilon,$$

[2] See § 5 for the definition of the interaction matrix relevant in our case.

and

$$\lim_{n\to\infty}\frac{1}{n}\int f(x)\,d\biggl(\sum_{k\geqslant 1}\delta_{\xi_{k,n}}\biggr)(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{k\geqslant 1}f(\xi_{k,n})=\int f(x)\,d\sigma(x) \tag{2.17}$$

for every continuous function f with compact support in $\mathbb{R}_-$.

(iv) There exists a continuous function φ on $\mathbb{R}_+$ satisfying

$$\liminf_{x\to+\infty}\frac{\varphi(x)}{4\log x}>1 \tag{2.18}$$

such that for a certain $\alpha<1$

$$\lim_{n\to\infty}\frac{1}{n}\log(x^\alpha\sigma_1'(d_nx))=-\varphi(x) \tag{2.19}$$

uniformly on each compact subset of $\mathbb{R}_+$, and

$$\liminf_{n\to\infty,\ x\to+\infty}\frac{-\log(x^\alpha\sigma_1'(d_nx))}{4n\log x}>1. \tag{2.20}$$

We are now ready to formulate the main result about the zero asymptotics of the multiple orthogonal polynomials for the Nikishin system in question.

Theorem 2.2. *Let the above assumptions (i)–(iv) hold, and let $\vec\lambda=(\lambda_1,\lambda_2)\in\mathfrak{M}^*(\sigma)$ be the solution of the extremal problem in Theorem 2.1. Assume that*

$$0\notin\operatorname{supp}(\sigma-\lambda_2),\qquad \int|y|^\alpha\,d\lambda_2(y)<\infty,\quad \alpha>\frac{1}{2}\,. \tag{2.21}$$

Then

$$\lim_n\nu_{Q_n}=\frac{\lambda_1}{2}\quad\text{and}\quad\lim_n\nu_{Q_{n,2}}=\lambda_2 \tag{2.22}$$

in the vague topology of measures. That is, for every bounded continuous functions f and g on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively,

$$\lim_n\int f\,d\nu_{Q_n}=\frac{1}{2}\int f\,d\lambda_1\quad\text{and}\quad\lim_n\int g\,d\nu_{Q_{n,2}}=\int g\,d\lambda_2.$$

Although the assumptions of this theorem may seem too restrictive, it encompasses many interesting examples. Some of them will be discussed in the next section. In particular, we will analyze briefly the case of modified Bessel weights (appearing in the analysis of non-intersecting squared Bessel paths), multiple Hermite polynomials (useful in the study of ensembles of random matrices with an external source), and finally, the multiple Pollaczek polynomials studied previously in [53], which will be discussed in more detail and for which an alternative method for explicitly solving the equilibrium problem in Theorem 2.1 will be presented. These examples satisfy all the assumptions of Theorem 2.2 except the integral condition in (2.21). It remains a difficult unsolved problem to eliminate this condition from a general theorem like Theorem 2.2.

Let us finish this section by noting that we can easily translate the results of Theorems 2.1 and 2.2 to an equivalent setting on the whole real axis $\mathbb{R}$ (with symmetric measures with respect to the origin). Indeed, let $\{P_m\}$ be a sequence of multiple orthogonal polynomials satisfying (2.2) with respect to a Nikishin system (2.3)–(2.4) on the semiaxis $\mathbb{R}_+$. Define the polynomial sequence $\{\widetilde{P}_n\}$ with polynomials of even degrees by

$$\widetilde{P}_n(x) := P_m(x^2), \qquad m = \frac{n}{2}, \quad n \in 2\mathbb{N}. \tag{2.23}$$

Then the $\widetilde{P}_n$ are multiple orthogonal polynomials satisfying conditions of the form (2.2) with respect to what can be seen as a natural generalization of a Nikishin system: now the first generating measure σ_1 is supported on the whole real axis $\mathbb{R}$, while the second generating measure σ_2 is a discrete measure on the imaginary axis. Then for the rescaled polynomials

$$\widetilde{Q}_n(x) := \frac{P_n(d_n x^2)}{d_n^{2n}}$$

we have straightforward analogues of Theorems 2.1 and 2.2, but now in terms of the solution of the following equilibrium problem: there exists a unique pair of measures (λ_1, λ_2) with $|\lambda_1| = 2$, $|\lambda_2| = 1$, and $\lambda_2(x) \leqslant \tilde{\sigma}$ and unique constants w_1 and w_2 such that

$$2U^{\lambda_1}(x) - U^{\lambda_2}(x) + \widetilde{\varphi}(x) \begin{cases} = w_1, & x \in \operatorname{supp}(\lambda_1) \subset \mathbb{R}, \\ \geqslant w_1, & x \in \mathbb{R}, \end{cases} \tag{2.24}$$

$$2U^{\lambda_2}(x) - U^{\lambda_1}(x) \begin{cases} \leqslant w_2, & x \in \operatorname{supp}(\lambda_2) = i\mathbb{R}, \\ \geqslant w_2, & x \in \operatorname{supp}(\tilde{\sigma} - \lambda_2). \end{cases} \tag{2.25}$$

The external field and the constraint are related to their analogues in (2.15)–(2.16) by the formulae

$$\widetilde{\varphi}(x) = \varphi(x^2) \quad \text{and} \quad \tilde{\sigma}'(x) = 2x\sigma'(x^2).$$

We note that the polynomials $\widetilde{Q}_n(x)$ are multiple orthogonal with respect to the varying weights $s'_{j,n}(x) := s'_j(d_n x)$:

$$\int_{\mathbb{R}} x^k \widetilde{Q}_n(x)\, s'_{j,n}(x)\, dx = 0, \qquad k = 0, \dots, m-1, \quad j = 1, 2. \tag{2.26}$$

3. Examples of explicit solutions of the equilibrium problem

As already mentioned in the Introduction, in recent years various models from random matrix theory have been reformulated in terms of multiple orthogonal polynomials corresponding to Nikishin systems of type (2.2)–(2.4). In all these examples, the generated weights are given by entire functions whose ratio is a meromorphic function which can be regarded as the Cauchy transform of a discrete measure σ_2 as in (2.1).

In this section we discuss three examples of this type of Nikishin systems for which explicit solutions of the associated equilibrium problems stated in Theorem 2.1 are available. One of them (see § 3.3 below) is analyzed in more detail, along with a new approach for expressing the density of the equilibrium measure as a jump of the logarithm of an algebraic function. In this representation, the component of the equilibrium measure constrained by Lebesgue measure is modelled as the jump of the logarithm of a negative function. In contrast to the standard approach using either the underlying differential equations or the recurrence relations of the corresponding multiple orthogonal polynomials, we derive this representation directly from the equilibrium conditions.

3.1. Modified Bessel weights (and non-intersecting squared Bessel paths). In [17] and [18] multiple orthogonal polynomials $\{P_n\}$ satisfying (2.2) for the system of weights

$$\begin{aligned} s_1'(x) &= x^{\nu/2}e^{-x/2}I_\nu\bigl(\sqrt{x}\,\bigr), \\ s_2'(x) &= x^{(\nu+1)/2}e^{-x/2}I_{\nu+1}\bigl(\sqrt{x}\,\bigr), \end{aligned} \qquad x\in\mathbb{R}_+,$$

were introduced and studied, where I_ν is the modified Bessel function with $\nu>-1$. This system has found applications (see [38], [39], [33]) in the description of ensembles of particles following non-intersecting squared Bessel paths (that is, the radial component of a multidimensional Brownian motion [51]). For these applications one must take the multiple orthogonal polynomials with respect to varying measures, depending on n, of the form

$$\begin{aligned} s_{1,n}'(x) &= x^{\nu/2}e^{-C_1nx}I_\nu\bigl(\sqrt{C_2\,nx}\,\bigr), \\ s_{2,n}'(x) &= x^{(\nu+1)/2}e^{-C_1nx}I_{(\nu+1)}\bigl(\sqrt{C_2\,nx}\,\bigr). \end{aligned} \tag{3.1}$$

Since this system of multiple orthogonal polynomials has been studied in depth, we just note briefly that the polynomials $\{P_n\}$, rescaled as in (2.5), have the asymptotic zero distribution given in (2.22).

The ratio of the two weights in (3.1) is a meromorphic function which has its poles at the squares of the zeros of the modified Bessel functions, that is, t_k in (2.1) equals

$$t_k:=-j_{k,\nu+1}^2,\qquad k\in\mathbb{Z}_+,$$

where $j_{k,\nu}$ is the kth zero of the Bessel function J_ν. To apply Theorem 2.2 we do not need to have explicit expressions for the mass points t_k and the values of the masses β_k for the measure σ_2, but we will need the asymptotics of the zeros of the Bessel function (see [1], p. 192):

$$j_{k,\nu}=\pi\biggl(k+\frac{\nu}{2}-\frac14\biggr)+O\biggl(\frac1k\biggr),\qquad k\to\infty, \tag{3.2}$$

and for estimating the values of the masses β_k we can use the asymptotics of the modulus M_ν of the amplitude of the Bessel function $J_\nu=:M_\nu\cos\theta_\nu$ (see [1], p. 186):

$$M_\nu(x)=\sqrt{\frac{2}{\pi x}}\biggl(1+O\biggl(\frac{1}{x^2}\biggr)\biggr),\qquad x\to+\infty. \tag{3.3}$$

Choosing the scaling coefficient in (2.5) to be $d_n = n^2$ for the measure $\sigma_{2,n}$ (see (2.8)), we have $\xi_{k,n} = -(j_{k,\nu}/n)^2$. By using (3.2) and (3.3) and the asymptotic expression

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}}\left(1 + O\left(\frac{1}{|z|}\right)\right), \qquad |\arg z| < \frac{\pi}{2},$$

for the modified Bessel function on the right-hand half-plane (see [1], p. 199) it is possible to verify that the conditions (i)–(iv) in § 2 are satisfied with

$$\rho(x) \sim \sqrt{|x|}, \qquad \alpha = \frac{1}{2}$$

(here $f \sim g$ means that $0 < C_1 < |f/g| < C_2 < \infty$, where C_1 and C_2 do not depend on x), and

$$\varphi(x) = \frac{x}{2} - \sqrt{x}, \quad x > 0, \quad \text{and} \quad \frac{d\sigma}{dx} = \frac{1}{\pi\sqrt{|x|}}, \quad x < 0. \tag{3.4}$$

(The condition (i) follows from the fact that (3.2) implies that $\lim_{n\to\infty}(\xi_{k+1,n} - \xi_{k,n}) = \pi$; see the proof of Lemma 4.4 in [33].) In § 3.3 below we give more details verifying some of the limits in the conditions (iii) and (iv) in a similar situation.

The rescaled weak asymptotics of the polynomial sequence $\{P_n\}$ is described by means of the extremal problem solved in Theorem 2.1, with the particular choice of the external field φ and the upper constraint σ indicated in (3.4). We note that the example in this subsection and some other relevant examples were also discussed in [33], providing insight into why such a vector equilibrium problem should appear.

An explicit solution of the equilibrium problem (2.15), (2.16), and (3.4) is known (see [38] or [6], p. 1188). The measures λ_j, $j = 1, 2$, are absolutely continuous with respect to Lebesgue measure, with densities that can be expressed in terms of solutions of the cubic equation (also known as the spectral curve)

$$H^3 - 2H^2 + H - \frac{2}{z} = 0. \tag{3.5}$$

Equation (3.5) has three solutions, enumerated so that

$$\begin{aligned} H_0(z) &= \frac{2}{z} + O(z^{-2}), \\ H_1(z) &= 1 - \frac{\sqrt{2}}{z^{1/2}} - \frac{1}{z} + O(z^{-3/2}), \\ H_2(z) &= 1 + \frac{\sqrt{2}}{z^{1/2}} - \frac{1}{z} + O(z^{-3/2}) \end{aligned}$$

as $z \to \infty$. Then as shown in [38], λ_1 and λ_2 can be written as

$$\begin{aligned} \lambda_1'(x) &= \frac{1}{\pi}\operatorname{Im} H_{0,+}(x), && x > 0, \\ \lambda_2'(x) &= \frac{d\sigma}{dx} - \frac{1}{\pi}\operatorname{Im} H_{1,+}(x), && x < 0, \end{aligned} \tag{3.6}$$

where the + subindices indicate the boundary values from the upper half-plane.

3.2. Multiple Hermite polynomials (and random matrices with an external source). Another set of multiple orthogonal polynomials was described in [4]. It turns out to be more convenient to deal with the polynomials $\{\widetilde{Q}_n\}$ defined by (2.26) with respect to the system of varying weights

$$s'_{j,n}(x) = \exp\biggl\{-n\biggl(\frac{1}{2}x^2 - a_j x\biggr)\biggr\}, \qquad x \in \mathbb{R}, \quad j = 1, \dots, p.$$

This system has found applications in the description of ensembles of non-intersecting Brownian bridges or random matrices with an external source [3], [14]. For the case

$$p = 2 \quad \text{and} \quad a_1 = -a_2 = a$$

it was proved in these papers that the zero-counting measures of the rescaled polynomials $\{\widetilde{Q}_n\}$ (corresponding to $\{\widetilde{P}_n\}$) have a weak limit λ which can be described by means of the spectral curve

$$H^3 - zH^2 + (2 - a^2)H + za^2 = 0. \tag{3.7}$$

This equation is due to Pastur [45]. If we enumerate the branches in (3.7) so that as $z \to \infty$

$$\begin{aligned} H_0(z) &= z - \frac{2}{z} + O(z^{-2}), \\ H_1(z) &= a + \frac{1}{z} + O(z^{-2}), \\ H_2(z) &= -a + \frac{1}{z} + O(z^{-2}), \end{aligned}$$

then λ is an absolutely continuous measure with density

$$\lambda'(x) = \frac{1}{\pi} \operatorname{Im} H_{0,+}(x), \qquad x \in \mathbb{R}. \tag{3.8}$$

A generalization of Pastur's curve for arbitrary p can be found in [31].

It was remarked in [13] (see also [8]) that the measure λ with density in (3.8) coincides with the component λ_1 in the solution of the equilibrium problem (2.24), (2.25) corresponding to the following external field $\widetilde{\varphi}$ and constraint $\tilde{\sigma}$:

$$\widetilde{\varphi}(x) = \frac{x^2}{2} - a|x|, \quad x \in \mathbb{R}, \qquad d\tilde{\sigma}(z) = \frac{a}{\pi}\,|dz|, \quad z \in i\mathbb{R}.$$

Indeed, multiple Hermite polynomials are also orthogonal with respect to the weights

$$\begin{aligned} \widetilde{s}'_{1,n}(x) &:= s'_{1,n}(x) + s'_{2,n}(x) = e^{-n\,x^2/2}\cosh(nax), \qquad & x \in \mathbb{R}, \\ \widetilde{s}'_{2,n}(x) &:= s'_{1,n}(x) - s'_{2,n}(x) = \tanh(nax)\widetilde{s}'_1(x), \qquad & x \in \mathbb{R}. \end{aligned}$$

Since

$$\tanh(nax) = \lim_{N\to\infty} \sum_{k=-N}^{N} \biggl(\frac{1}{na}\,\frac{1}{x + i\pi(k-1/2)/(na)} - \frac{1}{i\pi\,(k-1/2)}\biggr),$$

$(\widetilde{s}_{1,n}, \widetilde{s}_{2,n})$ is a Nikishin system generated by $\tilde{\sigma}_{1,n} := \widetilde{s}_{1,n}$ and the discrete measure

$$d\tilde{\sigma}_{2,n} := \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{1}{na}\delta_{\xi_{k,n}}, \qquad \xi_{k,n} := \frac{i\pi}{na}\left(k - \frac{1}{2}\right).$$

It is clear that

$$\#\{k\colon \xi_{k,n} \in [-ix,\, ix]\} \sim \left[\frac{2nax}{\pi}\right];$$

thus

$$\frac{1}{n} \lim_{N\to\infty} \sum_{k=-N}^{N} \delta_{\xi_{k,n}} \xrightarrow[n\to\infty]{*} d\tilde{\sigma}(z) = \frac{a}{\pi}\,|dz|, \qquad z \in i\mathbb{R},$$

and the conditions (ii), (iii) of (the symmetric analogue on the real axis and the imaginary axis of) Theorem 2.2 are satisfied. Regarding (iv), one can use the fact that

$$-\frac{1}{n}\log \widetilde{s}_{1,n}'(x) = \frac{x^2}{2} \begin{cases} -ax - \dfrac{1}{n}\log(1+e^{-2nax}), & x \geqslant 0, \\ +ax - \dfrac{1}{n}\log(1+e^{+2nax}), & x \leqslant 0, \end{cases}$$

which leads, in particular, to the uniform convergence

$$-\frac{1}{n}\log \widetilde{s}_{1,n}'(x) \rightrightarrows \widetilde{\varphi}(x) := \frac{x^2}{2} - a|x|, \qquad n\to\infty,$$

on compact subsets of $\mathbb{R}$. As for (i), it can be derived as in the previous example.

Actually, [13] contains a more general result for the multiple orthogonal polynomials $\{\widetilde{Q}_n\}$ given by (2.26) and corresponding to the system of varying weights

$$s_{j,n}'(x) = \exp\{-n(V(x) - a_j x)\}, \qquad x\in\mathbb{R}, \quad j = 1, 2,$$

where $V(x) = \sum_{j=1}^d v_j x^{2j}$ is an even polynomial potential with $v_d > 0$. It was shown there that the zero-counting measure of the rescaled polynomials $\{\widetilde{Q}_n\}$ converges (in the vague sense) to the first component $\lambda = \lambda_1$ of the solution to the equilibrium problem (2.24), (2.25) with the constraint $\tilde{\sigma}$ and the external field $\widetilde{\varphi}$ given by

$$\widetilde{\varphi}(x) = V(x) - a|x|, \quad x\in\mathbb{R}, \qquad d\tilde{\sigma}(z) = \frac{a}{\pi}\,|dz|, \quad z\in i\mathbb{R}. \tag{3.9}$$

For a detailed proof of the existence and uniqueness of the solution of this equilibrium problem, see [32].

Moreover, it was also proved in [13] that the equilibrium problem (2.24), (2.25) with input data (3.9) always has a unique solution (λ_1, λ_2) with $|\lambda_1| = 2$ and $|\lambda_2| = 1$, and that the functions

$$\begin{aligned} H_0(z) &= V'(z) - \int \frac{d\lambda_1(s)}{z-s}, && z\in\mathbb{C}\setminus S(\lambda_1), \\ H_1(z) &= \pm a + \int \frac{d\lambda_1(s)}{z-s} - \int \frac{d\lambda_2(s)}{z-s}, && z\in\mathbb{C}\setminus \big(S(\lambda_1)\cup S(\sigma-\lambda_2)\big), \quad \pm\operatorname{Re} z > 0, \\ H_2(z) &= \mp a + \int \frac{d\lambda_2(s)}{z-s}, && z\in\mathbb{C}\setminus S(\sigma-\lambda_2), \quad \pm\operatorname{Re} z>0, \end{aligned}$$

are the three solutions of the equation

$$H^3 + p_2(z)H^2 + p_1(z)H + p_0(z) = 0 \tag{3.10}$$

with polynomial coefficients whose degrees can easily be determined from the degree of the potential V. However, finding the coefficients of these polynomials explicitly in the most general situation is a very difficult problem. In [8] (see also [13]) this was done for a general even quartic potential,

$$V(x) = \frac{1}{4}x^4 - \frac{b}{2}x^2$$

in the case when the Riemann surface of (3.10) has genus either 0 or 1. For instance, when the genus is 1, we have from [8] that

$$H^3 - (z^3 + bz)H^2 + z^2 H + a^2 z^3 = 0,$$

where a and b belong to the triangular domain on the (a, b)-plane bounded by the curves

$$a_m(b) := \frac{\sqrt{6b^3 - 27b - 6(b^2-3)^{3/2}}}{9} > 0, \qquad b \in (-2, -\sqrt{3}\,),$$
$$a_M(b) := \frac{\sqrt{6b^3 - 27b + 6(b^2-3)^{3/2}}}{9} > 0, \qquad b \in (-\infty, -\sqrt{3}\,),$$

and the b-axis ($a = 0$).

3.3. Multiple Pollaczek polynomials. We have come to the main example as discussed at the end of § 2.

The sequence of polynomials studied in [53] is defined by the multiple orthogonality conditions (2.2) on $\mathbb{R}_+$ with the measures

$$ds_1(x) = \frac{dx}{\sinh(\pi\sqrt{x}/2)}\,, \quad ds_2(x) = \frac{1}{\cosh(\pi\sqrt{x}/2)}\,\frac{dx}{\sqrt{x}} = \frac{\tanh(\pi\sqrt{x}/2)}{\sqrt{x}}\,ds_1(x). \tag{3.11}$$

Decomposing $\dfrac{1}{z}\tanh\dfrac{\pi z}{2}$ into partial fractions, it is easy to check that

$$\frac{\tanh(\pi\sqrt{z}/2)}{\sqrt{z}} = \frac{4}{\pi}\sum_{k\geqslant 0}\frac{1}{z+(2k+1)^2} = \int\frac{d\sigma_2(x)}{z-x}\,,$$

where

$$\sigma_2 = \frac{4}{\pi}\sum_{k\in\mathbb{Z}_+}\delta_{-(2k+1)^2}$$

(cf. (2.1)). Hence, $(s_1, s_2) = \mathscr{N}(\sigma_1, \sigma_2)$ is a Nikishin system generated by $\sigma_1 = s_1$ supported on $\mathbb{R}_+$ and the discrete measure σ_2 consisting of equal masses of size $4/\pi$ distributed on $(-\infty, 0)$. In this case the rescaling (2.5) is done by taking $d_n = 4n^2$. This yields the measure $\sigma_{2,n}$ (see (2.8)) with

$$\xi_{k,n} = -\left(\frac{2k+1}{2n}\right)^2 \quad \text{and} \quad \beta_k = \frac{4}{\pi}\,.$$

It is easy to verify that the conditions (i)–(iv) in § 2 are satisfied with

$$\rho(x) = \sqrt{|x|}\,, \qquad d\sigma(x) = \frac{dx}{2\sqrt{|x|}}\,, \qquad \varphi(x) = \pi\sqrt{x}\,, \qquad \alpha = \frac{1}{2}\,. \tag{3.12}$$

For example, to derive the expression for σ (the condition (iii)), let $T \in (-\infty, 0)$. Then

$$\lim_n \frac{1}{n}\int_{[T,0]} d\biggl(\sum_{k\geqslant 1}\delta_{k,n}(t)\biggr) = \lim_n \frac{\sharp\{k : (2k+1)^2 \leqslant 4n^2|T|\}}{n} = \sqrt{|T|} = \int_{[T,0]} \frac{|dt|}{2\sqrt{|t|}}\,.$$

Since $d\sigma$ has no mass points, this is sufficient to prove convergence in the vague topology (for continuous functions with compact support). We wish to underscore that the constraint comes purely from the fact that between two mass points of $\sigma_{2,n}$ there is at most one zero of $Q_{n,2}$. Only the positions of the mass points of $\sigma_{2,n}$ are relevant in this property, not their weights, and therefore the constant $4/\pi$ can be disregarded.

Regarding (2.19) (the condition (iv)), we have

$$\frac{1}{n}\log\frac{1}{x^{1/2}s_1'(4n^2x)} = \frac{1}{n}\log\frac{\sinh(\pi n\sqrt{x}\,)}{\sqrt{x}}\,.$$

At $x = 0$ we give this function its limiting value $\log(\pi n)/n$ in order to make it continuous. For the proof of the uniform convergence we make the change of variables $\sqrt{x} = y$. Note that

$$\frac{1}{n}\log\frac{\sinh(\pi n y)}{y} = \pi y + \frac{1}{n}\log\frac{1 - e^{-2n\pi y}}{2y}\,.$$

Obviously, for $y > 0$ the pointwise limit is πy. On the other hand,

$$\left(\frac{1 - e^{-2n\pi y}}{2y}\right)' = \frac{(4n\pi y + 2)e^{-2n\pi y} - 2}{4y^2} < 0, \qquad y > 0,$$

since the numerator equals 0 at $y = 0$, and

$$\bigl((4n\pi y + 2)e^{-2n\pi y} - 2\bigr)' = -8n^2\pi^2 y e^{-2n\pi y} < 0, \qquad y > 0.$$

Consequently, on any interval $[0, T]$, $T > 0$, the function

$$h_n(x) := \frac{1}{n}\log\frac{\sinh(\pi n y)}{y} - \pi y$$

attains its maximum and minimum at the extreme points. We have

$$\lim_{n\to\infty} h_n(0) = \lim_{n\to\infty}\frac{\log(\pi n)}{n} = 0, \qquad \lim_{n\to\infty} h_n(T) = 0.$$

Therefore, the uniform convergence follows.

Obviously, a pair of measures $(f\,ds_1, f\,ds_2)$, where f is any continuous function such that

$$0 < c_1 \leqslant f(x) \leqslant c_2 < +\infty, \qquad x \in \mathbb{R}_+,$$

is associated with the same vector equilibrium problem. Thus, the corresponding multiple orthogonal polynomials exhibit the same rescaled normalized zero distribution as those corresponding to (3.11). Other examples can be constructed by replacing the discrete component of the Nikishin system by a Meixner- or a Charlier-type measure (see, for example, [41], [54], or [2]). A large two-parameter class of Meixner–Pollaczek-type multiple orthogonal polynomials was studied in [12] and [13], and the rescaled logarithmic and ratio asymptotics were given. Our example is a confluent case of those analyzed in [12] and [13].

We will also consider the corresponding polynomials transplanted to the whole real axis for multi-indices of the form (n, n). Using the transformation (2.23), we obtain a sequence of monic polynomials $\widetilde{P}_n$ of degree $2n$ satisfying the orthogonality relations

$$\int_{\mathbb{R}} x^\nu \widetilde{P}_n(x)\, \frac{x\,dx}{\sinh(\pi x)} = 0, \qquad \nu = 0, \dots, n-1, \tag{3.13}$$

$$\int_{\mathbb{R}} x^\nu \widetilde{P}_n(x)\, \frac{dx}{\cosh(\pi x)} = 0, \qquad \nu = 0, \dots, n-1, \tag{3.14}$$

known as multiple (or generalized) Pollaczek polynomials (see [53]). In order to guarantee normality, we will assume in addition that n is even. In this case the zeros of $\widetilde{P}_n$ are real and simple.

As is done for Nikishin systems (on the real line), it can be deduced that there exists a monic polynomial $\widetilde{P}_{n,2}$ with $\deg \widetilde{P}_{n,2} = n$ whose zeros are also simple and contained in $i\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} x^\nu \frac{\widetilde{P}_n(x)}{\widetilde{P}_{n,2}(x)}\, \frac{x\,dx}{\sinh(\pi x)} = 0, \qquad \nu = 0, \dots, 2n-1, \tag{3.15}$$

and

$$\int_{\mathbb{R}} t^\nu \frac{\widetilde{P}_{n,2}(t)}{\widetilde{P}_n(t)} \int_{i\mathbb{R}} \frac{\widetilde{P}_n^2(x)}{\widetilde{P}_{n,2}(x)}\, \frac{x\,dx}{(x-t)\sinh(\pi x)}\, d\beta(t) = 0, \qquad \nu = 0, \dots, n-1, \tag{3.16}$$

where β is a discrete measure supported on the imaginary axis. Let

$$\widetilde{Q}_n(z) = \frac{\widetilde{P}_n(nz)}{n^{2n}} \quad \text{and} \quad \widetilde{Q}_{n,2}(z) = \frac{\widetilde{P}_{n,2}(nz)}{n^n}\,.$$

The logarithmic (weak) asymptotic behavior of these polynomials was studied by Sorokin in [53]. Sorokin's approach is based on the existence of an explicit expression for the generating function of the polynomials $\widetilde{Q}_n(x)$, to which a weak form of the Darboux method can be applied. In this way the weak asymptotics of the polynomials can be deduced from the singularities of the generating function.

By (3.12), the zero-counting measures of the rescaled polynomials $\{\widetilde{Q}_n\}$ (corresponding to $\{\widetilde{P}_n\}$) have a weak limit λ, which is the first component ($\lambda = \lambda_1$) of the solution to the equilibrium problem (2.24), (2.25) with

$$\widetilde{\varphi}(x) = \pi|x|, \quad x \in \mathbb{R}, \quad \text{and} \quad d\tilde{\sigma}(z) = |dz| \quad \text{on } i\mathbb{R} \tag{3.17}$$

in view of (3.12). One of the goals of this section is to obtain λ by solving this equilibrium problem directly.

From electrostatic considerations we expect that $\operatorname{supp}(\lambda_2) = i\mathbb{R}$, because the external field created by U^{λ_1} on $i\mathbb{R}$ is too weak to make $\operatorname{supp}(\lambda_2)$ compact. An alternative argument is that if there were no restrictions on λ_2, then the measure $2\lambda_2$ in (2.25) would coincide with the balayage of λ_1 onto $i\mathbb{R}$. Hence, the upper constraint forces the balayage measure to redistribute its mass precisely where it exceeds σ in order to attain equilibrium on the rest of $i\mathbb{R}$. This consideration makes us look for a solution λ_2 for which there is an equality on $\operatorname{supp}(\sigma - \lambda_2)$ in the equilibrium conditions (2.25).

We shall try to find the Cauchy transform of the equilibrium measure λ_1,

$$H(z) := -\widehat{\lambda}_1(z) = \int_{\mathbb{R}} \frac{d\lambda_1(x)}{x - z}\,. \tag{3.18}$$

If we 'complexify' the equilibrium relations (2.24), (2.25) with (3.17), differentiate them, and take the real parts, then we get that

$$\operatorname{Re}\big(2\widehat{\lambda}_1(x) - \widehat{\lambda}_2(x)\big) = \begin{cases} -\pi & \text{on } \mathbb{R}_- \cap \operatorname{supp}(\lambda_1), \\ \pi & \text{on } \mathbb{R}_+ \cap \operatorname{supp}(\lambda_1) \end{cases}$$

and

$$\operatorname{Re}\big(2\widehat{\lambda}_2(x) - \widehat{\lambda}_1(x)\big) = 0 \quad \text{on } \operatorname{supp}(\sigma - \lambda_2).$$

Using the Riemann–Schwarz symmetry principle, we deduce from the first relation that the function H can be continued analytically from both sides of the cut along $\mathbb{R}_- \cap \operatorname{supp}(\lambda_1)$. Thus, H can be lifted to a Riemann surface, where

$$H(z) = \pi + \widehat{\lambda}_1(z) - \widehat{\lambda}_2(z) =: H_1(z) \tag{3.19}$$

is considered on the next sheet. Similarly, H can be continued analytically from both sides of the cut along $\mathbb{R}_+ \cap \operatorname{supp}(\lambda_1)$, so that

$$H(z) = -\pi + \widehat{\lambda}_1(z) - \widehat{\lambda}_2(z) =: H_2(z) \tag{3.20}$$

is defined on another sheet of the same surface. Let us assume that the complete Riemann surface

$$\mathscr{R} = \{\overline{\mathscr{R}^{(j)}}\}_{j=0}^2, \qquad \overline{\mathscr{R}^{(j)}} = \overline{\mathbb{C}},$$

has three sheets. With appropriate cuts we will have three branches of $H = \{H_j\}_{j=0}^2$, where $H_0(z) = -\widehat{\lambda}_1(z)$ is holomorphic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\lambda_1)$, and (3.18)–(3.20) give us that, as $z \to \infty$,

$$\begin{aligned} H_0(z) &= -\frac{2}{z} + \cdots\,, \\ H_1(z) &= \pi + \frac{1}{z} + \cdots\,, \\ H_2(z) &= -\pi + \frac{1}{z} + \cdots\,. \end{aligned} \tag{3.21}$$

We make an ansatz that the function H can be found in the form

$$H(\zeta) = \frac{2}{i}\log\psi(\zeta) \quad \text{on} \quad \mathscr{R}\setminus\{\zeta\in\mathscr{R}\colon \psi(\zeta)\in\mathbb{R}_-\}, \tag{3.22}$$

where ψ is a meromorphic function on the compact three-sheeted Riemann surface $\mathscr{R}$. Although the explicit form (equation) of the surface $\mathscr{R}$ is not yet known (but it exists), nevertheless the representation (3.22) and the relations (3.21) enable us to show that

$$\psi(\zeta) = \begin{cases} 1 - \dfrac{i}{\zeta} + \cdots\,, & \zeta\to\infty^{(0)}, \\ i - \dfrac{1}{2\zeta} + \cdots\,, & \zeta\to\infty^{(1)}, \\ -i + \dfrac{1}{2\zeta} + \cdots\,, & \zeta\to\infty^{(2)}, \end{cases} \tag{3.23}$$

where $q^{(j)}$ denotes the point on $\mathscr{R}^{(j)}$ whose canonical projection on the plane is $q\in\overline{\mathbb{C}}$. We try to take ψ as the simplest meromorphic function that maps $\mathscr{R}$ conformally onto $\overline{\mathbb{C}}$. The inverse of this function is a rational function $\zeta = r(\psi)$. From the main term in the asymptotic expansion (3.23) we have

$$\zeta = \frac{A}{\psi-1} + \frac{B}{\psi-i} + \frac{C}{\psi+i}\,,$$

and the second term gives us that

$$A = -i, \qquad B = \frac{-1}{2}\,, \quad \text{and} \quad C = \frac{1}{2}\,.$$

Thus,

$$\zeta = -i\frac{\psi(\psi+1)}{(\psi^2+1)(\psi-1)}\,, \tag{3.24}$$

or what is the same,

$$\psi^3 + \frac{i-\zeta}{\zeta}\psi^2 + \frac{i+\zeta}{\zeta}\psi - 1 = 0. \tag{3.25}$$

The discriminant of (3.25) is equal to

$$16\zeta^4 - 44\zeta^2 - 1.$$

Therefore, the algebraic function $\psi(\zeta)$ has four branch points $\pm e_1$ and $\pm e_2$, where

$$e_1 = \frac{1}{4}\sqrt{22-10\sqrt{5}} \quad \text{and} \quad e_2 = \frac{i}{4}\sqrt{-22+10\sqrt{5}}\,.$$

Taking (3.23) into account, we fix the following sheet structure of $\mathscr{R}$ (see Fig. 1):

$$\mathscr{R}^{(0)} := \overline{\mathbb{C}} \setminus [-e_1, e_1], \qquad \begin{aligned} \mathscr{R}^{(1)} &:= \overline{\mathbb{C}} \setminus ([-e_1, 0] \cup [-e_2, e_2]), \\ \mathscr{R}^{(2)} &:= \overline{\mathbb{C}} \setminus ([0, e_1] \cup [-e_2, e_2]). \end{aligned} \tag{3.26}$$

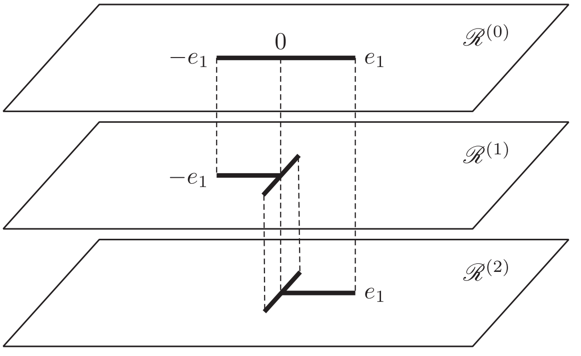

Figure 1. Sheet structure of the Riemann surface $\mathscr{R}$.

Therefore, the algebraic function ψ has the following single-valued meromorphic branches (in fact, holomorphic since $\psi(0) = \{0, -1, \infty\}$):

$$\psi_0(\zeta) \in \mathscr{H}(\overline{\mathbb{C}} \setminus [-e_1, e_1]), \qquad \begin{aligned} \psi_1(\zeta) &\in \mathscr{H}\big(\overline{\mathbb{C}} \setminus ([-e_1, 0] \cup [-e_2, e_2])\big), \\ \psi_2(\zeta) &\in \mathscr{H}\big(\overline{\mathbb{C}} \setminus ([0, e_1] \cup [-e_2, e_2])\big), \end{aligned}$$

where $\mathscr{H}(\Omega)$ stands for the class of functions holomorphic (and single-valued) in a domain Ω. From the analysis of the roots of (3.25) it follows that

$$\{i\mathbb{R}\}^{(0)} = \{\zeta \in \mathscr{R} \colon \psi(\zeta) \in \mathbb{R}_+\}, \tag{3.27a}$$

$$\{[-e_2, e_2]\}^{(1)} \cup \{[-e_2, e_2]\}^{(2)} = \{\zeta \in \mathscr{R} \colon \psi(\zeta) \in \mathbb{R}_-\}. \tag{3.27b}$$

Thus, if we cut our compact Riemann surface $\mathscr{R}$ along the set (3.27b) and define

$$\widetilde{\mathscr{R}} := \mathscr{R} \setminus \big(\{[-e_2, e_2]\}^{(1)} \cup \{[-e_2, e_2]\}^{(2)}\big), \tag{3.28}$$

then we get that the function H in (3.22) is single-valued and holomorphic in the open Riemann surface $\widetilde{\mathscr{R}}$. We can now formulate our result about the solution of the equilibrium problem (2.24), (2.25).

Proposition 3.1. *Let*

$$H_j(\zeta) = \frac{2}{i} \log \psi_j(\zeta), \qquad \zeta \in \mathscr{R}^{(j)}, \quad j = 0, 1, 2,$$

where the ψ_j are the solutions of (3.25) satisfying (3.23). Define the absolutely continuous measures

$$d\lambda_1(x) = \lambda_1'(x)\, dx, \qquad d\lambda_2(x) = \lambda_2'(x)\, |dx|,$$

by

$$\lambda_1'(x) = \frac{1}{\pi} \lim_{\varepsilon\to 0+} |\operatorname{Im} H_0(x+i\varepsilon)|, \qquad x \in \mathbb{R},$$
$$\lambda_2'(x) = -1 + \frac{1}{\pi} \lim_{\varepsilon\to 0+} \operatorname{Re} H_1(x-\varepsilon), \quad x \in i\mathbb{R} = \operatorname{supp}(\lambda_2). \tag{3.29}$$

Then the pair (λ_1, λ_2) *is the solution of the equilibrium problem* (2.24), (2.25) *with the external field and constraint* (3.17). *More precisely,* $|\lambda_1| = 2$ *and* $|\lambda_2| = 1$, *and these measures satisfy*

$$d\sigma(x) = |dx|, \qquad \lambda_2 \leqslant \sigma, \quad \text{and} \quad \lambda_2'(x) = 1 \quad \text{for } x \in [-e_2, e_2], \tag{3.30}$$

$$2U^{\lambda_1}(x) - U^{\lambda_2}(x) + \pi|x| \begin{cases} = w_1, & x \in [-e_1, e_1] = \operatorname{supp}(\lambda_1) \subset \mathbb{R}, \\ > w_1, & x \in \mathbb{R} \setminus [-e_1, e_1], \end{cases} \tag{3.31}$$

and

$$2U^{\lambda_2}(x) - U^{\lambda_1}(x) \begin{cases} = w_2, & x \in \operatorname{supp}(\sigma - \lambda_2) = i\mathbb{R} \setminus (-e_2, e_2), \\ < w_2, & x \in (-e_2, e_2). \end{cases} \tag{3.32}$$

Before proving Proposition 3.1 we discuss some properties of the primitive function G defined by

$$G' = H, \tag{3.33}$$

which we now consider on the open Riemann surface $\widetilde{\mathscr{R}}$, that is,

$$G(\zeta) = \int_{\zeta_0}^{\zeta} H(t)\, dt, \qquad \zeta_0, \zeta, t \in \widetilde{\mathscr{R}}. \tag{3.34}$$

The uniformization of $\mathscr{R}$ defined in (3.24) allows us to integrate (3.34) by parts, obtaining

$$\begin{aligned} G(\zeta) &= -2 \int_{\psi(\zeta_0)}^{\psi(\zeta)} \log(\psi)\, d\frac{\psi(\psi+1)}{(\psi^2+1)(\psi-1)} \\ &= C + \zeta H(\zeta) + 2\log(\psi(\zeta)-1) - \log(\psi^2(\zeta)+1), \end{aligned} \tag{3.35}$$

where C is a constant which depends on ζ_0. According to (3.35), G is multivalued on $\widetilde{\mathscr{R}}$ and has a local analytic extension to the whole of $\mathscr{R}$ (and beyond), with possible singular points at $\zeta = 0$ and $\zeta = \infty$ (note that $\psi(\infty) = \{1, i, -i\}$ by (3.24)). However, its periods are purely imaginary. Therefore, its real part is a single-valued harmonic function on $\mathscr{R} \setminus \{0, \infty\}$,

$$g := \{g_j = \operatorname{Re} G_j\}_{j=0}^2,$$

which is defined up to an additive constant. We fix the constant so that

$$g_0(\infty) + g_1(\infty) + g_2(\infty) = 0.$$

This normalization in turn implies that

$$g_0(\zeta) + g_1(\zeta) + g_2(\zeta) \equiv 0, \qquad \zeta \in \mathbb{C}. \tag{3.36}$$

Indeed, $g_0+g_1+g_2$ is a symmetric function of g which is harmonic on $\overline{\mathbb{C}}\setminus\{0,\infty\}$. From (3.24) and (3.35), one sees that the singularity it has at $\zeta=0$ is removable. On the other hand, from (3.21) and (3.34) we see that the branches of g at infinity have the following behaviour:

$$g(\zeta)\simeq\begin{cases}-2\log|\zeta|, & \zeta\to\infty^{(0)},\\ \pi\operatorname{Re}\zeta+\log|\zeta|, & \zeta\to\infty^{(1)},\\ -\pi\operatorname{Re}\zeta+\log|\zeta|, & \zeta\to\infty^{(2)}.\end{cases} \tag{3.37}$$

Therefore, $\zeta=\infty$ is also a removable singularity of $g_0+g_1+g_2$. Since $g_0+g_1+g_2$ is harmonic in $\overline{\mathbb{C}}$ and equal to zero at ∞, it is identically equal to zero.

Proof of Proposition 3.1. We must verify that the measures defined by their densities in (3.29) satisfy (3.30)–(3.32). In order to identify the potentials of the measures λ_1 and λ_2, let us change the sheet structure of $\mathscr{R}$. Define

$$g_0^*:=g_0,\quad g_1^*:=\begin{cases}g_1(z), & \operatorname{Re}z<0,\\ g_2(z), & \operatorname{Re}z>0,\end{cases}\quad g_2^*:=\begin{cases}g_2(z), & \operatorname{Re}z<0,\\ g_1(z), & \operatorname{Re}z>0.\end{cases} \tag{3.38}$$

On $i\mathbb{R}$, g^* is defined by continuity. Note that g_1^* and g_2^* have a harmonic continuation across the interval $[-e_2,e_2]$.

We now see that the function g_0^* is superharmonic and that g_2^* is subharmonic (as the maximum of two harmonic functions). Therefore, taking into account their behavior at ∞ (see (3.37)), we get from the Riesz decomposition theorem for superharmonic functions a global representation of the branches of g^* in $\mathbb{C}$ in the form

$$\begin{aligned}g_0^*(z)&=U^{\lambda_1}(z)+\kappa_1,\\ g_2^*(z)&=-U^{\lambda_2}(z)-v(z)+\kappa_2,\end{aligned} \tag{3.39}$$

where λ_1 and λ_2 are measures supported on $[-e_1,e_1]$ and $i\mathbb{R}$, respectively, and $v(z)$ is the superharmonic function

$$v(z)=\begin{cases}\pi\operatorname{Re}z, & \operatorname{Re}z\leqslant 0,\\ -\pi\operatorname{Re}z, & \operatorname{Re}z>0.\end{cases} \tag{3.40}$$

As a consequence of (3.36), we also have

$$g_1^*(z)=-U^{\lambda_1}(z)+U^{\lambda_2}(z)+v(z)-\kappa_1-\kappa_2. \tag{3.41}$$

Using (3.21) and (3.39), we easily verify that

$$|\lambda_1|=2\quad\text{and}\quad|\lambda_2|=1,$$

and by the definition of g, (3.29) follows from the Stieltjes–Perron formula applied to the calculation of the measures.

Since $g_0^*(x)=g_1^*(x)$ for $x\in[-e_1,e_1]$, (3.39) and (3.41) give us the equality in (3.31) with $w_1:=-2\kappa_1-\kappa_2$. The fact that $g_0^*(x)>g_1^*(x)$ on $\mathbb{R}\setminus[-e_1,e_1]$

allows us to verify the inequality in (3.31). Similarly, comparing g_1^* and g_2^* on $i\mathbb{R}$ and using (3.39), (3.41), and the fact that $v(z) \equiv 0$ for $z \in i\mathbb{R}$ (see (3.40)), we obtain (3.32) with $w_2 := 2\kappa_1 + \kappa_2$.

Finally, note that the functions ψ_1 and ψ_2 have negative limit values on $[-e_2, e_2]$ (see (3.27b)). Therefore, it follows from (3.22) that

$$\lim_{\varepsilon\to 0+} \operatorname{Re} H_1(x-\varepsilon) = 2\pi, \qquad x \in [-e_2, e_2],$$

and $\lambda_2'(x) \equiv 1$, $x \in [-e_2, e_2]$. On the rest of the imaginary axis,

$$\pi < \lim_{\varepsilon\to 0+} \operatorname{Re} H_1(x-\varepsilon) < 2\pi$$

(see also (3.21)). Thus, we obtain (3.30). Note that in applying the Stieltjes–Perron formula to the second half of (3.29) we took the imaginary part because $|dx| = -i\,dx$, $x \in i\mathbb{R}$. $\square$

4. Scalar case

4.1. Potentials of measures with unbounded support. In all that follows, finite positive Borel measures μ supported on $\mathbb{R}$ and satisfying

$$\int \log(1+y^2)\,d\mu(y) < +\infty \tag{4.1}$$

will play a central role. It is easy to see that (4.1) is equivalent to

$$\int (1+|y|)\,d\mu(y) < +\infty \quad \text{or} \quad \int_{|y|\geqslant 1} \log|y|\,d\mu(y) < +\infty.$$

Another important assumption about μ which we will use is that for any $\varepsilon > 0$ there exist $0 < \delta < 1/2$ and $R_0 > 0$ such that

$$\sup_{|R|\geqslant R_0} \int_{R-\delta}^{R+\delta} \log\frac{1}{|R-y|}\,d\mu(y) < \varepsilon. \tag{4.2}$$

Obviously, if $\mu \leqslant \mu^*$ and μ^* satisfies (4.2), then μ satisfies (4.2). In particular, a sufficient condition for (4.2) is that there exists an $R_0 > 0$ such that

$$d\mu|_{\mathbb{R}\setminus(-R_0,R_0)} \leqslant |f|\,dm,$$

where $f \in L_\infty(m)$ and m is Lebesgue measure. We have the following lemma.

Lemma 4.1. *Let μ be a finite positive Borel measure on $\mathbb{R}_+$ such that U^μ is continuous at some point $x_0 \in \operatorname{supp}(\mu)$. Then for any compact set $K \subset \mathbb{C}$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sup_{x\in K} \int_{x_0-\delta}^{x_0+\delta} \bigl|\log|x-y|\bigr|\,d\mu(y) < \varepsilon. \tag{4.3}$$

Suppose that (4.1) *and* (4.2) *hold. Then for any* $\varepsilon > 0$ *there exists an* R_0 *such that*

$$\sup_{R\geqslant R_0}\ \sup_{x\in[0,R]} \int_R^{+\infty} \bigl|\log|x-y|\,\bigr|\,d\mu(y) < \varepsilon \tag{4.4}$$

and

$$\lim_{x\to\infty} \int \left|\log\left|1-\frac{y}{x}\right|\right|\,d\mu(y) = 0, \tag{4.5}$$

where $x\to\infty$ *in any direction in* $\mathbb{C}$.

Proof. Let us prove (4.3). Consider the closed disk $B = \{x\colon |x-x_0| \leqslant 1/2\}$. For all $x \in B$

$$0 < \int_B \bigl|\log|x-y|\,\bigr|\,d\mu(y) = \int_B \log\frac{1}{|x-y|}\,d\mu(y).$$

Obviously, $U^{\mu|_B}$ is continuous at x_0. Therefore,

$$\log\frac{1}{|x-x_0|} \in L_1\bigl(\mu\big|_B\bigr)$$

and x_0 is not a mass point of $\mu\big|_B$. Consequently, for every $\varepsilon > 0$ there exists a δ with $0 < \delta_1 < 1/2$ such that

$$0 < \int_{x_0-\delta_1}^{x_0+\delta_1} \log\frac{1}{|x_0-y|}\,d\mu(y) < \frac{\varepsilon}{2}\,.$$

The potential of the measure $\mu\big|_{[x_0-\delta_1,x_0+\delta_1]}$ is also continuous at x_0, so there exists δ_2 with $0 < \delta_2 < 1/2$ such that

$$\left|\int_{x_0-\delta_1}^{x_0+\delta_1} \log\frac{1}{|x-y|}\,d\mu(y) - \int_{x_0-\delta_1}^{x_0+\delta_1} \log\frac{1}{|x_0-y|}\,d\mu(y)\right| < \frac{\varepsilon}{2}\,, \qquad |x-x_0| < \delta_2.$$

Using these two inequalities, we get that

$$0 < \int_{x_0-\delta_1}^{x_0+\delta_1} \log\frac{1}{|x-y|}\,d\mu(y) < \varepsilon, \qquad |x-x_0| < \delta_2.$$

Fix a compact set $K \subset \mathbb{C}$ and take

$$K_1 = K \setminus \{x\colon |x-x_0| < \delta_2\}.$$

The distance from K_1 to x_0 is positive and x_0 is not a mass point of $\mu\big|_{[x_0-\delta_1,x_0+\delta_1]}$, so there exists $0 < \delta_3 < \delta_1$ such that

$$\int_{x_0-\delta_3}^{x_0+\delta_3} \bigl|\log|x-y|\,\bigr|\,d\mu(y) < \varepsilon, \qquad x \in K_1.$$

On the other hand,

$$0 < \int_{x_0-\delta_3}^{x_0+\delta_3} \log\frac{1}{|x-y|}\,d\mu(y) \leqslant \int_{x_0-\delta_1}^{x_0+\delta_1} \log\frac{1}{|x-y|}\,d\mu(y) < \varepsilon, \qquad |x-x_0| < \delta_2.$$

The last two relations imply (4.3).

If μ has compact support, then the assertions (4.4) and (4.5) are trivial, so in their proof we restrict our attention to measures with unbounded support in $\mathbb{R}_+$. We will analyze (4.4) by subdividing the real line. Take $R > 1$.

Assume that $x \in [0, R-1]$; then $y - x \geqslant 1$ for all $y \in [R, +\infty)$. Using the monotonicity of the logarithm and (4.1), we find that

$$
\begin{aligned}
0 &\leqslant \lim_{R\to+\infty} \sup_{x\in[0,R-1]} \int_R^{+\infty} \left|\log \frac{1}{|x-y|}\right| d\mu(y) \\
&= \lim_{R\to+\infty} \sup_{x\in[0,R-1]} \int_R^{+\infty} \log(y-x)\, d\mu(y) \\
&\leqslant \lim_{R\to+\infty} \int_R^{+\infty} \log(y)\, d\mu(y) = 0.
\end{aligned}
$$

By the same token,

$$
\lim_{R\to+\infty} \sup_{x\in[0,R]} \int_{R+1}^{+\infty} \left|\log \frac{1}{|x-y|}\right| d\mu(y) = 0.
$$

Choose a constant δ with $0 < \delta < 1/2$. For $x \in [R-1, R-\delta]$ and $y \in [R, R+1]$,

$$
\log \frac{1}{2} \leqslant \log \frac{1}{|x-y|} \leqslant \log \frac{1}{\delta},
$$

which implies that

$$
\left|\log \frac{1}{|x-y|}\right| \leqslant \log \frac{1}{\delta}\,.
$$

Consequently,

$$
0 \leqslant \lim_{R\to+\infty} \sup_{x\in[R-1,R-\delta]} \int_R^{R+1} \left|\log \frac{1}{|x-y|}\right| d\mu(y) \leqslant \log \frac{1}{\delta} \lim_{R\to+\infty} \mu([R, R+1]) = 0,
$$

since μ is finite. Similarly,

$$
\lim_{R\to+\infty} \sup_{x\in[R-1,R]} \int_{R+\delta}^{R+1} \left|\log \frac{1}{|x-y|}\right| d\mu(y) = 0.
$$

Fix $\varepsilon > 0$. By (4.2) there exist δ with $0 < \delta < 1/2$ and $R_0 > 0$ such that

$$
\begin{aligned}
&\sup_{R\geqslant R_0} \sup_{x\in[R-\delta,R]} \int_R^{R+\delta} \left|\log \frac{1}{|x-y|}\right| d\mu(y) \\
&\qquad \leqslant \sup_{R\geqslant R_0} \sup_{x\in[R-\delta,R]} \int_R^{R+\delta} \log \frac{1}{y-R}\, d\mu(y) < \varepsilon.
\end{aligned}
$$

Putting everything together, we immediately get (4.4).

To prove (4.5) we first restrict ourselves to the limiting case when $x \in \mathbb{R}_+$, and without loss of generality we assume that $x > 2$. For the moment, fix x. As a function of y on $\mathbb{R}_+$, the non-negative function $\big|\log|1-y/x|\big|$ has a vertical asymptote at $y=x$ and zeros at $y \in \{0, 2x\}$. It is convex on $[0,x)$ and $(x,2x]$ and concave on $[2x,+\infty)$. The functions $\log(1+y)$ and $\log(y-1)$ are concave in their domain of definition. On the interval $[0,x]$ it is easy to verify that

$$\left|\log\left|1-\frac{y}{x}\right|\right| = \log(1+y)$$

if and only if $y=0$ or $y=x-1$. On the interval $[x,2x]$,

$$\left|\log\left|1-\frac{y}{x}\right|\right| = \log(y-1)$$

if and only if $y=x+1$. By the concavity properties of the functions in the specified intervals and the monotonicity of the logarithm it follows that

$$\left|\log\left|1-\frac{y}{x}\right|\right| \begin{cases} \leqslant \log(1+y), & 0 \leqslant y \leqslant x-1, \\ \leqslant \log(y-1) \leqslant \log(1+y), & x+1 \leqslant y \leqslant 2x, \\ = \log\left(\dfrac{y}{x}-1\right) \leqslant \log(1+y), & 2x \leqslant y < +\infty. \end{cases} \tag{4.6}$$

Let

$$E_x = \left[\frac{x^\alpha}{2}, +\infty\right) \setminus (x-1, x+1), \qquad 0<\alpha<1.$$

Fix $\varepsilon > 0$ and take δ with $0<\delta<1/2$ such that (4.2) holds. We have

$$0 \leqslant \int \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) \leqslant \int_0^{x^\alpha/2} + \int_{E_x} + \int_{x-1}^{x+1} \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y).$$

Let us consider these integrals separately.

First,

$$\begin{aligned} 0 \leqslant \int_0^{x^\alpha/2} \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) &= \int_0^{x^\alpha/2} \left|\log\left(1-\frac{y}{x}\right)\right| d\mu(y) \\ &\leqslant |\mu| \left|\log\left(1-\frac{1}{2x^{1-\alpha}}\right)\right| \leqslant C\left|\frac{1}{x^{1-\alpha}}\right| \to 0, \qquad x \to +\infty. \end{aligned}$$

Taking (4.6) and (4.1) into account, we get that on E_x

$$0 \leqslant \int_{E_x} \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) \leqslant \int_{y \geqslant x^\alpha/2} \log(1+y)\, d\mu(y) \to 0, \qquad x \to +\infty.$$

Finally, on $[x-1, x+1]$

$$\begin{aligned} 0 \leqslant \int_{x-1}^{x+1} \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) &\leqslant \mu([x-1,x+1]) \log x + \int_{x-1}^{x+1} \log\frac{1}{|y-x|}\, d\mu(y) \\ &\leqslant \mu([x-1,x+1]) \left(\log x + \log\frac{1}{\delta}\right) + \int_{x-\delta}^{x+\delta} \log\frac{1}{|y-x|}\, d\mu(y), \end{aligned}$$

where the first term tends to zero as $x \to +\infty$ in view of (4.1), and the second term is bounded by ε for all sufficiently large x in view of (4.2).

Summarizing, we have

$$0 \leqslant \liminf_{x\to+\infty} \int \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) \leqslant \limsup_{x\to+\infty} \int \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) \leqslant \varepsilon$$

for each $\varepsilon > 0$. Letting $\varepsilon \to 0$, we obtain (4.5) in the case when $x \in \mathbb{R}_+$.

Now take θ with $0 < \theta < \pi/2$, and define the region

$$F_\theta = \mathbb{C} \setminus \{x \colon |\arg(x)| \leqslant \theta\}.$$

Assume that $x \to \infty$ with $x \in F_\theta$. In this case $y/x \in F_\theta$ for all $y \geqslant 0$ and $x \in F_\theta$. Consequently, $|1 - y/x| \geqslant |\sin\theta| > 0$. Therefore, if $|x| \geqslant 1$, then

$$0 < |\sin\theta| \leqslant \left|1-\frac{y}{x}\right| \leqslant 1 + \left|\frac{y}{x}\right| \leqslant 1 + y.$$

Thus,

$$\left|\log\left|1-\frac{y}{x}\right|\right| \leqslant \max\{-\log|\sin\theta|, \log(1+y)\}, \qquad x \geqslant 1, \quad y \in \mathbb{R}_+.$$

A function defined as the maximum of integrable functions is in $L_1(\mu)$. From Lebesgue's dominated convergence theorem it follows that

$$\lim_{x\to\infty,\ x\in F_\theta} \int \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) = 0.$$

Let $a_x = \arg x$, and assume that $x \to \infty$, $a_x \not\to 0$, and

$$\limsup_{x\to\infty} \int \left|\log\left|1-\frac{y}{x}\right|\right| d\mu(y) > 0.$$

Then we can find a sufficiently small θ with $0 < \theta < \pi/2$ and a sequence $x_n \in F_\theta$ with $x_n \to \infty$ such that

$$\limsup_{n\to\infty} \int \left|\log\left|1-\frac{y}{x_n}\right|\right| d\mu(y) > 0,$$

in contrast to what we proved before. Consequently, to prove (4.5) it remains to show that the assertion is true as $x \to \infty$ and $a_x \to 0$. This case is similar to the one when $x \to \infty$ with $x \in \mathbb{R}_+$, so we focus on the main ingredients of the proof.

Without loss of generality we can assume that $|x| \geqslant 1$ and $\operatorname{Re} x > 2$. Let $|1 - y/x| \geqslant 1$. This implies that $y \geqslant 2\operatorname{Re} x$. Then

$$\left|\log\left|1-\frac{y}{x}\right|\right| = \log\left|1-\frac{y}{x}\right| \leqslant \log(1+y), \qquad y \geqslant 2\operatorname{Re} x. \tag{4.7}$$

Note that

$$\left|1-\frac{y}{x}\right|^2 = \left|e^{ia_x}-\frac{y}{|x|}\right|^2 = \left(\cos a_x - \frac{y}{|x|}\right)^2 + \sin^2 a_x$$
$$\geqslant \left(\cos a_x - \frac{y}{|x|}\right)^2 = (\cos^2 a_x)\left(1-\frac{y}{\operatorname{Re} x}\right)^2.$$

Consequently, when $0 < |1-y/x| \leqslant 1$, that is, $0 \leqslant y \leqslant 2\operatorname{Re} x$, we have

$$0 \geqslant \log\left|1-\frac{y}{x}\right| \geqslant \log\left|(\cos a_x)\left(1-\frac{y}{\operatorname{Re} x}\right)\right|$$

and

$$\left|\log\left|1-\frac{y}{x}\right|\right| \leqslant \left|\log\left|(\cos a_x)\left(1-\frac{y}{\operatorname{Re} x}\right)\right|\right|$$
$$\leqslant |\log|\cos a_x|| + \left|\log\left|1-\frac{y}{\operatorname{Re} x}\right|\right|, \qquad 0 \leqslant y \leqslant 2\operatorname{Re} x. \tag{4.8}$$

Analyzing the cases

$$y \in [0, \operatorname{Re} x - 1], \quad y \in [\operatorname{Re} x + 1, 2\operatorname{Re} x], \quad \text{and} \quad y \in [2\operatorname{Re} x, +\infty]$$

separately and reasoning as in the proof of (4.6) (with x replaced by $\operatorname{Re} x$), we get from (4.7) and (4.8) that

$$\left|\log\left|1-\frac{y}{x}\right|\right| \leqslant \log(1+y) + \log\sec a_x, \qquad y \in \mathbb{R}_+ \setminus (\operatorname{Re} x - 1, \operatorname{Re} x + 1). \tag{4.9}$$

In the final part of the proof we take

$$E_x = \left[\frac{(\operatorname{Re} x)^\alpha}{2}, +\infty\right) \setminus (\operatorname{Re} x - 1, \operatorname{Re} x + 1), \qquad 0 < \alpha < 1,$$

and proceed as in the case when $x \in \mathbb{R}_+$, observing that

$$\lim_{x\to\infty,\ a_x\to 0} \int \log\sec a_x \, d\mu(y) = \lim_{x\to\infty,\ a_x\to 0} \log\sec a_x = 0. \qquad \square$$

With the aid of (4.4) we prove a version of the principle of domination for measures with unbounded support.

Lemma 4.2. *Suppose that μ and ν are finite positive Borel measures supported on $\mathbb{R}_+$ such that $|\mu| = |\nu|$ and $I(\mu) < \infty$, and let (4.1) and (4.2) hold. If* $\operatorname{supp}(\mu)$ *is unbounded and* $\operatorname{supp}(\nu)$ *is compact, then we also suppose that U^ν is continuous at some point $x_0 \in \operatorname{supp}(\nu)$. Assume that for some constant $c \in \mathbb{R}$*

$$U^\mu(x) \leqslant U^\nu(x) + c \qquad \mu\text{-almost everywhere.} \tag{4.10}$$

Then

$$U^\mu(x) \leqslant U^\nu(x) + c, \qquad x \in \mathbb{C}. \tag{4.11}$$

Proof. If the supports of μ and ν are compact, then the lemma gives the standard statement of the principle of domination (see, for example, [50], Theorem II.3.2), so this result is new when at least one of the two measures has unbounded support. We will reduce the proof to the case of measures with compact support. We consider in detail the case when the supports of μ and ν are both unbounded and then mention how to proceed when one of them is bounded and the other is unbounded.

Assume that $\operatorname{supp}(\mu)$ and $\operatorname{supp}(\nu)$ are unbounded. Fix $\varepsilon > 0$. According to (4.4) there exist $R_1(\varepsilon)$ and $R_2(\varepsilon)$ such that $\mu([0,R_1]) = \nu([0,R_2])$ and

$$\max\biggl\{\sup_{x\in[0,R_1]}\biggl|\int_{R_1}^{+\infty}\log\frac{1}{|x-y|}\,d\mu(y)\biggr|,\ \sup_{x\in[0,R_2]}\biggl|\int_{R_2}^{+\infty}\log\frac{1}{|x-y|}\,d\nu(y)\biggr|\biggr\} < \varepsilon. \tag{4.12}$$

We can take $R_1(\varepsilon)$ and $R_2(\varepsilon)$ such that

$$\lim_{\varepsilon\to 0} R_1(\varepsilon) = +\infty \quad \text{and} \quad \lim_{\varepsilon\to 0} R_2(\varepsilon) = +\infty.$$

Let

$$\mu_1 = \mu_1(\varepsilon) = \mu\big|_{[0,R_1(\varepsilon)]} \quad \text{and} \quad \nu_1 = \nu_1(\varepsilon) = \nu\big|_{[0,R_2(\varepsilon)]}.$$

Then $|\mu_1| = |\nu_1|$. Since $\mu_1 \leqslant \mu$, it follows from (4.10) and (4.12) that

$$U^{\mu_1}(x) \leqslant U^{\nu_1}(x) + c + 2\varepsilon \qquad \mu_1\text{-almost everywhere.}$$

Note that $I(\mu_1) < +\infty$. Using Theorem II.3.2 from [50], we have

$$U^{\mu_1}(x) \leqslant U^{\nu_1}(x) + c + 2\varepsilon, \qquad x \in \mathbb{C}. \tag{4.13}$$

Fix an arbitrary compact set $K \subset \mathbb{C}$ and let $M = \sup_{x\in K}|x|$. For all sufficiently large R

$$|\log|x-y|\,| = \log|x-y| \leqslant \log(M+y), \qquad y \geqslant R, \quad x \in K,$$

and from (4.1) it follows that

$$\lim_{\varepsilon\to 0} U^{\mu_1(\varepsilon)} = U^{\mu} \quad \text{and} \quad \lim_{\varepsilon\to 0} U^{\nu_1(\varepsilon)} = U^{\nu}$$

uniformly on K. Letting ε tend to zero, we get (4.11) from (4.13), and we are done.

When only $\operatorname{supp}(\nu)$ is unbounded, we proceed as before to reduce ν to a measure ν_1 with compact support, but we can leave μ as it is because the principle of domination for logarithmic potentials of measures with compact support only needs $|\nu_1| \leqslant |\mu|$ to deduce (4.13). If only $\operatorname{supp}(\mu)$ is unbounded, then we take μ_1 as before, but we must reduce ν so that $|\nu_1| \leqslant |\mu_1|$ $(< |\mu|)$. In order to achieve this, since $\operatorname{supp}(\mu)$ is a compact set, we must take away some mass from a neighbourhood of a point $x_0 \in \operatorname{supp}(\nu)$ where U^{ν} is continuous and use (4.3) instead of (4.4). $\square$

Remark 4.3. Lemmas 4.1 and 4.2 are valid for measures supported on the whole of $\mathbb{R}$. In fact, Lemma 4.2 will be used in the next section for measures supported on $\mathbb{R}_-$.

4.2. Equilibrium measures with a constraint and an external field. This question has been considered by several authors (see, for example, [10], [20], [29], [32], [40], and [46]). Our contribution to the theory consists in studying the corresponding variational problem in cases when the equilibrium measure does not have compact support. We will state the corresponding results for measures supported on $\mathbb{R}_-$ because this is the setting in which they will be needed in the proof of Theorem 2.1, but they can be restated for measures supported on $\mathbb{R}$.

In order to deal with measures with unbounded support it is convenient to follow the approach used in [32]. For arbitrary $\mu_1, \mu_2 \in \mathscr{M}^+(\mathbb{R})$ we define the *modified* logarithmic potential and mutual energy as follows:

$$\mathscr{U}^{\mu_1}(x) := \int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\mu_1(y), \tag{4.14}$$

$$\mathscr{I}(\mu_1, \mu_2) := \iint \log \frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|}\, d\mu_1(y)\, d\mu_2(x). \tag{4.15}$$

The *modified* energy of μ is then given by

$$\mathscr{I}(\mu) := \mathscr{I}(\mu, \mu).$$

The new kernel is connected with the inverse stereographic projection from the ball in $\mathbb{R}^3$ with centre at $(0,0,1/2)$ and radius $1/2$ on the extended complex plane. Therefore,

$$\frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|} \geqslant 1 \tag{4.16}$$

(for more details see (2.9)–(2.11) in [32]). Consequently, the modified potential and mutual energy are uniformly bounded below for all $\mu_1, \mu_2 \in \mathscr{M}^+(\mathbb{R})$. When μ_1 and μ_2 have finite mutual energy and satisfy (4.1), then the modified and the ordinary energy are related by

$$\mathscr{I}(\mu_1, \mu_2) = I(\mu_1, \mu_2) + \frac{|\mu_2|}{2} \int \log(1+x^2)\, d\mu_1(x) + \frac{|\mu_1|}{2} \int \log(1+x^2)\, d\mu_2(x).$$

In what follows, σ denotes a (positive) Borel measure with $\operatorname{supp}(\sigma) = \mathbb{R}_-$ and $|\sigma| > 1$ such that $U^{\sigma|_K}$ is continuous on $\mathbb{C}$ for every compact subset $K \subset \mathbb{R}_-$. Let

$$\mathscr{M}(\sigma) := \{\mu \in \mathscr{M}_1^+(\mathbb{R}_-) \colon \mu \leqslant \sigma\}, \qquad \widetilde{\mathscr{M}}(\sigma) := \{\mu \in \mathscr{M}(\sigma) \colon \mathscr{I}(\mu) < \infty\}.$$

Lemma 4.4. *For any $\mu \in \mathscr{M}(\sigma)$ the potential $\mathscr{U}^\mu$ is continuous on $\mathbb{C}$.*

Proof. Take any $\mu \in \mathscr{M}(\sigma)$. Obviously, $\mathscr{U}^\mu$ is continuous on $\mathbb{C} \setminus \operatorname{supp}(\mu)$, so we only have to check for continuity on $\mathbb{R}_-$. Let $x_0 \in \mathbb{R}_-$, and take a compact set $K \subset \mathbb{R}_-$ that contains x_0 in its interior. Since

$$\mathscr{U}^\mu = \mathscr{U}^{\mu|_K} + \mathscr{U}^{\mu - \mu|_K}$$

and $x_0 \notin \operatorname{supp}(\mu - \mu|_K)$, it follows that $\mathscr{U}^{\mu-\mu|_K}$ is continuous at x_0. However, $\mu\big|_K \leqslant \sigma\big|_K$ and $U^{\sigma|_K}$ is continuous on $\mathbb{C}$, so (see [20], Lemma 5.2) $U^{\mu|_K}$ and $\mathscr{U}^{\mu|_K}$ are continuous on $\mathbb{C}$ and, in particular, at x_0. Thus, $\mathscr{U}^\mu$ is continuous at any $x_0 \in \mathbb{R}_-$. $\square$

Let ϕ be a real-valued continuous function on $\mathbb{R}_-$ such that

$$\liminf_{x\to-\infty} \phi^*(x) > -\infty, \qquad \phi^*(x) := \phi(x) - \log(1+x^2). \tag{4.17}$$

For $\mu \in \mathscr{M}_1^+(\mathbb{R}_-)$ define

$$\mathscr{W}^\mu(x) := 2\int \log\frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|}\,d\mu(y) + \phi^*(x) = 2\mathscr{U}^\mu(x) + \phi(x)$$

and

$$\begin{aligned}\mathscr{J}_{\phi^*}(\mu) :&= 2\int\biggl(\int \log\frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|}\,d\mu(y) + \phi^*(x)\biggr)\,d\mu(x)\\ &= 2\mathscr{I}(\mu) + 2\int \phi^*(x)\,d\mu(x).\end{aligned}$$

If $\mathscr{I}(\mu) = +\infty$, then we take $\mathscr{J}_{\phi^*}(\mu) = +\infty$.

The condition (4.17) guarantees that the energy minimization problem for the functional $\mathscr{J}_{\phi^*}(\mu)$ is weakly admissible as defined in § 2.1 of [32], and according to Corollary 2.7 in [32] there exists a unique solution $\lambda \in \widetilde{\mathscr{M}}(\sigma)$ such that

$$\mathscr{J}_{\phi^*}(\lambda) = \inf\{\mathscr{J}_{\phi^*}(\mu)\colon \mu \in \mathscr{M}(\sigma)\}. \tag{4.18}$$

The measure λ is said to be *extremal*.

For $\mu \in \widetilde{\mathscr{M}}(\sigma)$ we also introduce the characteristic quantity

$$\mathscr{F}_\mu := \max\{C \in \mathbb{R}\colon \mathscr{W}^\mu(x) \geqslant C \text{ holds } (\sigma-\mu)\text{-almost everywhere}\}.$$

Theorem 4.5. *Let ϕ satisfy (4.17) and let σ be a positive Borel measure with* $\operatorname{supp}(\sigma) = \mathbb{R}_-$ *and $|\sigma| > 1$ such that $U^{\sigma|_K}$ is continuous on $\mathbb{C}$ for every compact subset $K \subset \mathbb{R}_-$. The following statements are equivalent and concern the same unique solution.*

(A′) *There exists an extremal measure $\lambda \in \widetilde{\mathscr{M}}(\sigma)$.*

(B′) *There exists a $\lambda \in \widetilde{\mathscr{M}}(\sigma)$ such that for all $\nu \in \mathscr{M}(\sigma)$*

$$\int \mathscr{W}^\lambda\,d(\nu-\lambda) \geqslant 0.$$

(C′) *There exist a $\lambda \in \widetilde{\mathscr{M}}(\sigma)$ and a constant $\mathfrak{w} = \mathfrak{w}(\sigma,\phi)$ such that*

$$\mathscr{W}^\lambda(x) = 2\mathscr{U}^\lambda(x) + \phi(x)\begin{cases}\leqslant \mathfrak{w}, & x \in \operatorname{supp}(\lambda),\\ \geqslant \mathfrak{w}, & x \in \operatorname{supp}(\sigma-\lambda).\end{cases}$$

The constant $\mathfrak{w}$ is uniquely determined and equals $\mathscr{F}_\lambda$. The extremal measure satisfies (4.1).

Proof. As mentioned above, the existence of a unique extremal measure follows from Corollary 2.7 in [32]. The equivalence of (A′) and (B′) follows from the identity

$$\mathscr{J}_{\phi^*}(\nu_\varepsilon) - \mathscr{J}_{\phi^*}(\lambda) = \varepsilon^2\mathscr{J}_0(\nu-\lambda) + 2\varepsilon\int \mathscr{W}^\lambda\,d(\nu-\lambda),$$

valid for all $\lambda, \nu \in \widetilde{\mathscr{M}}(\sigma)$, where $\nu_\varepsilon = \varepsilon\nu + (1-\varepsilon)\lambda$, $0 \leqslant \varepsilon \leqslant 1$, and $\mathscr{J}_0(\nu - \lambda)$ is the energy functional applied to $\nu - \lambda$ with $\phi^* \equiv 0$.

Assume that λ is extremal. From the above identity it follows that

$$\varepsilon^2 \mathscr{J}_0(\nu - \lambda) + 2\varepsilon \int \mathscr{W}^\lambda \, d(\nu - \lambda) \geqslant 0.$$

Dividing by ε and letting $\varepsilon \to 0$, we find that

$$\int \mathscr{W}^\lambda \, d(\nu - \lambda) \geqslant 0, \qquad \nu \in \widetilde{\mathscr{M}}(\sigma), \tag{4.19}$$

so (A′) implies (B′). Taking $\varepsilon = 1$, we obtain

$$\mathscr{J}_{\phi^*}(\nu) - \mathscr{J}_{\phi^*}(\lambda) = \mathscr{J}_0(\nu - \lambda) + 2 \int \mathscr{W}^\lambda \, d(\nu - \lambda).$$

From Theorem 2.5 in [16], we have $\mathscr{J}_0(\nu - \lambda) \geqslant 0$, with equality if and only if $\nu = \lambda$. Therefore, (B′) implies (A′), and the solution of (B′) is unique.

We now prove that any solution to (C′) solves (B′). Let λ satisfy (C′) and take $\nu \in \widetilde{\mathscr{M}}(\sigma)$. Since $|\lambda| = |\nu| = 1$, it follows that

$$\int \mathscr{W}^\lambda \, d(\nu - \lambda) = \int (\mathscr{W}^\lambda - \mathfrak{w}) \, d(\nu - \lambda).$$

Define

$$E_+ = \{t \in \mathbb{R}_- : \mathscr{W}^\lambda(t) - \mathfrak{w} > 0\} \quad \text{and} \quad E_- = \{t \in \mathbb{R}_- : \mathscr{W}^\lambda(t) - \mathfrak{w} < 0\}.$$

According to (C′), $\lambda(E_+) = 0$, hence

$$\int_{E_+} (\mathscr{W}^\lambda - \mathfrak{w}) \, d(\nu - \lambda) = \int_{E_+} (\mathscr{W}^\lambda - \mathfrak{w}) \, d\nu \geqslant 0.$$

Moreover, $(\sigma - \lambda)(E_-) = 0$. Take an increasing sequence of compact sets $K_n \subset E_-$ such that

$$\lim_{n\to\infty} (\sigma - \lambda)(K_n) = (\sigma - \lambda)(E_-).$$

By Lemma 4.4, $\mathscr{W}^\lambda$ is continuous on the whole of $\mathbb{C}$, and in particular on K_n, and therefore $\mathscr{W}^\lambda - \mathfrak{w}$ is bounded on K_n. From Lebesgue's monotone convergence theorem it follows that

$$\int_{E_-} |\mathscr{W}^\lambda - \mathfrak{w}| \, d(\sigma - \lambda) = \lim_{n\to\infty} \int_{E_-} 1_{K_n} |\mathscr{W}^\lambda - \mathfrak{w}| \, d(\sigma - \lambda) = 0,$$

where 1_{K_n} is the function which equals 1 on K_n and 0 elsewhere. Consequently, taking into account that $\nu \leqslant \sigma$, we obtain

$$\int_{E_-} (\mathscr{W}^\lambda - \mathfrak{w}) \, d(\nu - \lambda) = \int_{E_-} (\mathscr{W}^\lambda - \mathfrak{w}) \, d(\nu - \sigma) + \int_{E_-} (\mathscr{W}^\lambda - \mathfrak{w}) \, d(\sigma - \lambda) \geqslant 0.$$

Putting all these relations together, we obtain

$$\int \mathscr{W}^{\lambda}\, d(\nu-\lambda) \geqslant 0, \qquad \nu \in \widetilde{\mathscr{M}}(\sigma),$$

as claimed. Therefore, (C′) has a unique solution. Let us see that (B′) implies (C′).

Suppose that λ solves (B′) and consider the quantity

$$\mathscr{F}_{\lambda} = \max\{C \in \mathbb{R} \colon \mathscr{W}^{\lambda} \geqslant C \text{ holds } (\sigma-\lambda)\text{-almost everywhere}\}.$$

Suppose that there exists an $x_0 \in \operatorname{supp}(\lambda)$ such that $\mathscr{W}^{\lambda}(x_0) > \gamma > \mathscr{F}_{\lambda}$ for some γ. By the definition of $\mathscr{F}_{\lambda}$, there exists a compact set $K_1 \subset \operatorname{supp}(\sigma-\lambda)$ such that

$$\mathscr{W}^{\lambda}(x) < \gamma, \quad x \in K_1, \quad \text{and} \quad (\sigma-\lambda)(K_1) > 0.$$

On the other hand, $\mathscr{W}^{\lambda}(x)$ is continuous on $\mathbb{R}_-$, so there exists a sufficiently small $\delta > 0$ such that $\mathscr{W}^{\lambda}(x) > \gamma$ for $|x-x_0| < \delta$, and by the same token there exists a compact set K_2 with

$$\lambda(K_2) > 0 \quad \text{and} \quad \mathscr{W}^{\lambda}(x) > \gamma \quad \text{for } x \in K_2.$$

Obviously, $K_1 \cap K_2 = \varnothing$. Choose $\alpha, \beta \in (0,1)$ so that

$$\beta(\sigma-\lambda)(K_1) = \alpha\lambda(K_2).$$

Define a signed measure η equal to $-\alpha\lambda$ on K_2, equal to $\beta(\sigma-\lambda)$ on K_1, and equal to zero otherwise.

We prove that $\nu := \lambda + \eta \in \widetilde{\mathscr{M}}(\sigma)$. Indeed,

$$0 \leqslant \nu\big|_{K_2} = (1-\alpha)\lambda\big|_{K_2} \leqslant \sigma\big|_{K_2},$$
$$0 \leqslant \nu\big|_{K_1} = \beta\sigma\big|_{K_1} + (1-\beta)\lambda\big|_{K_1} \leqslant \sigma\big|_{K_1},$$

and since $\operatorname{supp}(\nu) = \operatorname{supp}(\lambda)$, we have

$$\nu(\operatorname{supp}(\nu)) = \nu(\operatorname{supp}(\lambda)) = \lambda(\operatorname{supp}(\lambda)) - \alpha\mu(K_2) + \beta(\sigma-\lambda)(K_1) = 1.$$

The energy of ν is bounded since λ and $(\sigma-\lambda)\big|_{K_1}$ have finite energy. Then

$$\int \mathscr{W}^{\lambda}\, d(\nu-\lambda) = \int \mathscr{W}^{\lambda}\, d\eta < \gamma\beta(\sigma-\lambda)(K_1) - \gamma\alpha\lambda(K_2) = 0,$$

contradicting (B′). Consequently, $\mathscr{W}^{\lambda}(x) \leqslant \mathscr{F}_{\lambda}$ for $x \in \operatorname{supp}(\lambda)$. By definition, $\mathscr{W}^{\lambda}(x) \geqslant \mathscr{F}_{\lambda}$ holds $(\sigma-\lambda)$-almost everywhere. Since $\mathscr{W}^{\lambda}$ is continuous on $\mathbb{C}$, we have

$$\mathscr{W}^{\lambda}(x) \geqslant \mathscr{F}_{\lambda}, \qquad x \in \operatorname{supp}(\sigma-\lambda).$$

Thus, λ solves (C′) with $\mathfrak{w} = \mathscr{F}_{\lambda}$.

The uniqueness of λ and the fact that $\operatorname{supp}(\sigma-\lambda) \cap \operatorname{supp}(\lambda) \neq \varnothing$ imply that $\mathfrak{w}$ is uniquely determined.

If the extremal measure λ has compact support, then it obviously satisfies (4.1). Now suppose that $\operatorname{supp}(\lambda)$ is unbounded. Using (4.16), we get that

$$\frac{\sqrt{1+y^2}}{|1-y/x|} \geqslant \frac{|x|}{\sqrt{1+x^2}}\,.$$

Therefore, for all $x \leqslant -1$

$$\log \frac{\sqrt{1+y^2}}{|1-y/x|} \geqslant -\frac{1}{2}\log 2.$$

Using Fatou's lemma ([49], p. 22), (C′), and (4.17), we obtain

$$\begin{aligned}
\int \log(1+y^2)\,d\lambda(y) &= 2\int \liminf_{x\to-\infty} \log \frac{\sqrt{1+y^2}}{|1-y/x|}\,d\lambda(y) \\
&\leqslant \liminf_{x\to-\infty} 2\int \log \frac{\sqrt{1+y^2}}{|1-y/x|}\,d\lambda(y) \\
&\leqslant \liminf_{x\to-\infty,\ x\in\operatorname{supp}(\lambda)} 2\int \log \frac{\sqrt{1+y^2}}{|1-y/x|}\,d\lambda(y) \\
&\leqslant \mathfrak{w} + \limsup_{x\to-\infty}(2\log|x| - \phi(x)) < +\infty.
\end{aligned}$$

Thus, in this case (4.1) is also satisfied by λ. □

We are ready to return to standard potentials. Define

$$\mathscr{M}^*(\sigma) := \left\{\mu \in \mathscr{M}(\sigma)\colon I(\mu) < +\infty,\ \int \log(1+y^2)\,d\mu(y) < +\infty\right\}.$$

Note that

$$\mathscr{M}^*(\sigma) \subset \widetilde{\mathscr{M}}(\sigma) \subset \mathscr{M}(\sigma).$$

According to the last assertion of Theorem 4.5, $\lambda \in \mathscr{M}^*(\sigma)$. Therefore, under the assumptions of Theorem 4.5, (4.18) admits the same solution when we minimize the functional over $\mathscr{M}^*(\sigma)$.

Let

$$J_\phi = \inf\{J_\phi(\mu)\colon \mu \in \mathscr{M}^*(\sigma)\}, \qquad J_\phi(\mu) := 2\left(I(\mu) + \int \phi(x)\,d\mu(x)\right).$$

We take $J_\phi(\mu) = +\infty$ when $I(\mu) = +\infty$. It is easy to see that

$$\mathscr{J}_{\phi^*}(\mu) = J_\phi(\mu), \qquad \mu \in \mathscr{M}^*(\sigma).$$

Moreover,

$$W^\mu(x) := 2U^\lambda(x) + \phi(x) = 2\mathscr{U}^\lambda(x) + \phi(x) - \int \log(1+y^2)\,d\mu(y), \qquad \mu \in \mathscr{M}^*(\sigma).$$

For $\mu \in \mathscr{M}^*(\sigma)$ let

$$F_\mu := \max\{C \in \mathbb{R}\colon 2U^\mu(x) + \phi(x) \geqslant C \text{ holds } (\sigma-\mu)\text{-almost everywhere}\}. \quad (4.20)$$

Note that

$$F_\mu = \mathscr{F}_\mu - \int \log(1+y^2)\, d\mu(y), \qquad \mu \in \mathscr{M}^*(\sigma).$$

The next result follows from Theorem 4.5.

Corollary 4.6. *Under the assumptions of Theorem 4.5, the following statements are equivalent and have the same unique solution.*

(A″) *There exists a $\lambda \in \mathscr{M}^*(\sigma)$ that is extremal.*

(B″) *There exists a $\lambda \in \mathscr{M}^*(\sigma)$ such that for all $\nu \in \mathscr{M}^*(\sigma)$*

$$\int W^\lambda \, d(\nu - \lambda) \geqslant 0.$$

(C″) *There exist a $\lambda \in \mathscr{M}^*(\sigma)$ and a constant $w = w(\sigma, \phi)$ such that*

$$2U^\lambda(x) + \phi(x) \begin{cases} \leqslant w, & x \in \operatorname{supp}(\lambda), \\ \geqslant w, & x \in \operatorname{supp}(\sigma - \lambda). \end{cases}$$

(D″) *If σ also satisfies (4.2), then the solution λ of (A″)–(C″) satisfies*

$$F_\lambda = \max\{F_\mu \colon \mu \in \mathscr{M}^*(\sigma)\},$$

and if also

$$\lim_{x\to+\infty} \sqrt{x} \int \log\left(1 - \frac{y}{x}\right) d\lambda(y) = 0, \quad (4.21)$$

then λ is the unique measure that satisfies (D″). A sufficient condition for (4.21) is that

$$\int (-y)^\alpha \, d\lambda(y) < \infty, \qquad \alpha > \frac{1}{2}\,. \quad (4.22)$$

The constant $w(\sigma, \phi) = F_\lambda$ is uniquely determined.

Proof. The equivalence of the statements (A″), (B″), and (C″) and the uniqueness of the extremal measure for the functional $J_\phi(\,\cdot\,)$ is immediate from Theorem 4.5 and the connections with ordinary potentials established above. For (D″) we have assumed that σ also satisfies (4.2). Then all the measures in $\mathscr{M}^*(\sigma)$ satisfy (4.1) and (4.2) (see the sentence right after (4.2)).

Note that (C″) implies that $F_\lambda = w(\sigma, \phi)$. We must show that

$$F_\mu \leqslant F_\lambda \quad \text{for all } \mu \in \mathscr{M}^*(\sigma).$$

Assume that $F_\mu > F_\lambda$ for some $\mu \in \mathscr{M}^*(\sigma)$. Following the proof of Theorem 2.1.e in [20], but using Lemma 4.2 instead of the standard principle of domination, we get that there exists a $c > 0$ such that

$$U^\lambda(x) \leqslant U^\mu(x) - c, \qquad x \in \mathbb{C}.$$

Subtracting $\log(1/|x|)$ from both sides and letting $x \to +\infty$, one obtains the contradiction $0 \leqslant -c$. Therefore,

$$\max\{F_\mu \colon \mu \in \mathscr{M}^*(\sigma)\} = F_\lambda.$$

If $F_\lambda = F_\mu$, then by repeating the scheme in the proof of Theorem 2.1.e in [20], we arrive at the inequality

$$U^\lambda(x) \leqslant U^\mu(x), \qquad x \in \mathbb{C}.$$

In other words,

$$U^{\mu-\lambda}(x) \geqslant 0, \qquad x \in \mathbb{C}.$$

If $\operatorname{supp}(\lambda)$ and $\operatorname{supp}(\mu)$ are compact, then since $\lim_{x\to\infty} U^{\mu-\lambda}(x) = 0$, this inequality and the minimum principle for harmonic functions give us that

$$U^{\mu-\lambda}(x) \equiv 0, \qquad x \in \mathbb{C} \setminus (\operatorname{supp}(\lambda) \cup \operatorname{supp}(\mu)), \tag{4.23}$$

which in turn implies that $\mu = \lambda$. Note that (4.22) obviously holds when λ has compact support.

Suppose that there exists an $x_0 \in \mathbb{C} \setminus \mathbb{R}_-$ with $U^{\mu-\lambda}(x_0) = 0$. Then by the minimum principle $U^{\lambda-\mu}(x) \equiv 0$ for $x \in \mathbb{C} \setminus (\operatorname{supp}(\lambda) \cup \operatorname{supp}(\mu))$, since on the whole boundary (including ∞) this harmonic function has limit values $\geqslant 0$. In this case, as in the compact case, we conclude that $\mu = \lambda$.

Assume that $U^{\mu-\lambda}(x) > 0$ for $x \in \mathbb{C} \setminus \mathbb{R}_-$, and define

$$G^{\mu-\lambda}(x) = \int \log \frac{1}{x-y}\, d(\mu-\lambda)(y)$$

to be the corresponding complex potential. This analytic function is non-zero in $\mathbb{C} \setminus \mathbb{R}_-$. Let

$$\widetilde{G}^{\mu-\lambda}(z) := iG^{\mu-\lambda}(-z^2).$$

We note that $\widetilde{G}^{\mu-\lambda}$ is analytic and non-zero in the upper half-plane $\operatorname{Im} z > 0$. Moreover,

$$\operatorname{Im} \widetilde{G}^{\mu-\lambda}(z) = \operatorname{Re} G^{\mu-\lambda}(-z^2) = U^{\mu-\lambda}(-z^2) > 0, \qquad \operatorname{Im} z > 0.$$

Therefore, $\widetilde{G}^{\mu-\lambda}$ takes the upper half-plane into itself. From this we get an integral representation for $\widetilde{G}^{\mu-\lambda}(z)$.

Indeed, from Theorem A.2 in [36] we know that

$$\widetilde{G}^{\mu-\lambda}(z) = \kappa + \beta z + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\rho(t), \tag{4.24}$$

where $\kappa \in \mathbb{R}$, $\beta \geqslant 0$, and ρ is a positive Borel measure on $\mathbb{R}$ such that

$$\int \frac{1}{1+t^2}\, d\rho(t) < \infty.$$

Similarly, from Theorem A.3 in [36] it follows that

$$\log \widetilde{G}^{\mu-\lambda}(z) = \gamma + \int_{\mathbb{R}} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) f(t)\, dt, \tag{4.25}$$

where $\gamma \in \mathbb{R}$ and f is an integrable function on $\mathbb{R}$ such that $0 \leqslant f(t) \leqslant 1$ almost everywhere. Let us simplify these representations a bit.

If $z = iu$ with $u > 0$, then it follows from the definition of $\widetilde{G}^{\mu-\lambda}$ that

$$\begin{aligned}\widetilde{G}^{\mu-\lambda}(iu) &= i \int \log \frac{1}{|u^2 - y|}\, d(\mu-\lambda)(y) - \int \arg \frac{1}{u^2 - y}\, d(\mu-\lambda)(y) \\ &= i \int \log \frac{1}{|u^2 - y|}\, d(\mu-\lambda)(y) = iU^{\mu-\lambda}(u^2)\end{aligned} \tag{4.26}$$

takes purely imaginary values. By the symmetry principle, $\widetilde{G}^{\mu-\lambda}$ is symmetric with respect to the imaginary axis, that is, for $\operatorname{Im} z > 0$,

$$\operatorname{Im} \widetilde{G}^{\mu-\lambda}(z) = \operatorname{Im} \widetilde{G}^{\mu-\lambda}(-\overline{z}) \quad \text{and} \quad \operatorname{Re} \widetilde{G}^{\mu-\lambda}(z) = -\operatorname{Re} \widetilde{G}^{\mu-\lambda}(-\overline{z}). \tag{4.27}$$

In particular,

$$\arg \widetilde{G}^{\mu-\lambda}(z) = \pi - \arg \widetilde{G}^{\mu-\lambda}(-\overline{z}), \qquad \operatorname{Im} z > 0. \tag{4.28}$$

Actually, $\widetilde{G}^{\mu-\lambda}$ can be extended continuously to $\mathbb{R}$ from the upper half plane, and therefore the last relation implies that

$$\arg\big(\widetilde{G}^{\mu-\lambda}(t)\big)_+ = \pi - \arg\big(\widetilde{G}^{\mu-\lambda}(-t)\big)_+, \qquad t \in \mathbb{R}. \tag{4.29}$$

By the Stieltjes inversion formula, the first relation in (4.27) implies that the measure ρ is symmetric with respect to the origin: $d\rho(t) = d\rho(-t)$. Therefore, (4.24) can be transformed as follows:

$$\begin{aligned}\widetilde{G}^{\mu-\lambda}(z) &= \kappa + \beta z + \int_{-\infty}^{0} + \int_{0}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\rho(t) \\ &= \kappa + \beta z + \int_{-\infty}^{0} \frac{2z}{t^2 - z^2}\, d\rho(t).\end{aligned}$$

Taking $z = iu$ in this representation, we obtain a purely imaginary number (see (4.26)). Comparing both sides, we see that $\kappa = 0$. Dividing by u and letting u tend to ∞, we now get that $\beta = 0$. Consequently,

$$\widetilde{G}^{\mu-\lambda}(z) = iG^{\mu-\lambda}(-z^2) = \int_{-\infty}^{0} \frac{2z}{t^2 - z^2}\, d\rho(t).$$

Changing variables $-z^2 = x$ and $-t^2 = y$, we obtain

$$G^{\mu-\lambda}(x) = \int_{-\infty}^{0} \frac{2\sqrt{x}}{x - y}\, d\widetilde{\rho}(y), \qquad x \in \mathbb{C} \setminus \mathbb{R}_-, \tag{4.30}$$

where we fix the branch of the square root by setting $\sqrt{1}=1$ and $d\widetilde{\rho}(y)=d\rho(\sqrt{-y}\,)$. Note that $\int(1+|y|)^{-1}\,d\widetilde{\rho}(y)<\infty$.

Take $x>0$ and $N>0$. From (4.30) we have

$$\sqrt{x}\,G^{\mu-\lambda}(x)\geqslant\int_{-N}^{0}\frac{2x}{x-y}\,d\widetilde{\rho}(y),\qquad x\in\mathbb{C}\setminus\mathbb{R}_-.$$

Assume that $\sqrt{x}\,G^{\mu-\lambda}(x)\leqslant M$ for all sufficiently large x. Taking lim sup as $x\to\infty$, we see that $\widetilde{\rho}([-N,0])\leqslant M/2$. If this held for all $N>0$, then we could conclude that $\widetilde{\rho}$ is bounded, with total mass $\leqslant M/2$. Then if $\lim_{x\to+\infty}\sqrt{x}\,G^{\mu-\lambda}(x)=0$, we would get that $\widetilde{\rho}$ is the zero measure and (4.30) would become the identity $G^{\mu-\lambda}(x)\equiv 0$ for $x\in\mathbb{C}\setminus\mathbb{R}_-$, implying that $\mu=\lambda$, as we want.

For $x>0$ we have

$$\begin{aligned}0&\leqslant\sqrt{x}\,G^{\mu-\lambda}(x)=\sqrt{x}\,U^{\mu-\lambda}(x)\\&=\sqrt{x}\left(\int\log\left(1-\frac{y}{x}\right)d(\lambda-\mu)(y)\right)\leqslant\sqrt{x}\int\log\left(1-\frac{y}{x}\right)d\lambda(y).\end{aligned}$$

Therefore, $\lim_{x\to+\infty}\sqrt{x}\,G^{\mu-\lambda}(x)=0$ under the condition (4.21).

On the other hand, if (4.22) holds, then we can assume that for $1/2<\alpha\leqslant 1$

$$\begin{aligned}0&\leqslant\sqrt{x}\int\log\left(1-\frac{y}{x}\right)d\lambda(y)=\frac{\sqrt{x}}{\alpha}\int\log\left(1-\frac{y}{x}\right)^{\alpha}d\lambda(y)\\&\leqslant\frac{\sqrt{x}}{\alpha}\int\log\left(1+\left(-\frac{y}{x}\right)^{\alpha}\right)d\lambda(y)\leqslant\frac{\sqrt{x}}{\alpha}\int\left(-\frac{y}{x}\right)^{\alpha}d\lambda(y).\end{aligned}$$

Consequently, (4.22) is a sufficient condition for (4.21). □

Let τ denote the distribution function of the measure $f(t)\,dt$ (see (4.25)). By the Stieltjes inversion formula,

$$\tau(t_2)-\tau(t_1)=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{t_1}^{t_2}\arg\widetilde{G}^{\mu-\lambda}(t+i\varepsilon)\,dt,\qquad t_1<t_2.$$

From (4.28) and (4.29) it follows that for $\infty<t_1<t_2\leqslant 0$

$$\begin{aligned}\tau(t_2)-\tau(t_1)&=\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{t_1}^{t_2}\arg\widetilde{G}^{\mu-\lambda}(t+i\varepsilon)\,dt\\&=\frac{1}{\pi}\int_{t_1}^{t_2}\arg\big(\widetilde{G}^{\mu-\lambda}(t)\big)_+\,dt=t_2-t_1-\frac{1}{\pi}\int_{t_1}^{t_2}\arg\big(\widetilde{G}^{\mu-\lambda}(-t)\big)_+\,dt\\&=t_2-t_1-\frac{1}{\pi}\int_{-t_2}^{-t_1}\arg\big(\widetilde{G}^{\mu-\lambda}(t)\big)_+\,dt=t_2-t_1-[\tau(-t_1)-\tau(-t_2)].\end{aligned}$$

Consequently,

$$f(t)=\frac{d\tau(t)}{dt}=1-f(-t)\tag{4.31}$$

almost everywhere on $\mathbb{R}$.

From (4.25) and (4.31) we get that

$$\begin{aligned}\log \widetilde{G}^{\mu-\lambda}(z) &= \gamma + \int_{-\infty}^{0}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) f(t)\,dt \\ &\quad + \int_{0}^{\infty}\left(\frac{1}{t-z} - \frac{t}{1+t^2}\right)(1 - f(-t))\,dt \\ &= \gamma + \int_{-\infty}^{0}\left(\frac{1}{t-z} + \frac{1}{t+z} - \frac{2t}{1+t^2}\right) f(t)\,dt + \int_{0}^{\infty}\frac{1+tz}{t-z}\,\frac{dt}{1+t^2} \\ &= \gamma + 2(1+z^2)\int_{-\infty}^{0}\frac{tf(t)}{t^2-z^2}\,\frac{dt}{1+t^2} + \int_{0}^{\infty}\frac{1+tz}{t-z}\,\frac{dt}{1+t^2}\,.\end{aligned}$$

Integrating the function $(1+tz)\log(t)/(t-z)$ with respect to t over the closed contour consisting of the circles $\{t\colon |t| = R\}$ and $\{t\colon |t| = \varepsilon\}$ and the positively oriented interval $[\varepsilon, R]$, where the branch of the logarithm in $\mathbb{C}\setminus\mathbb{R}_+$ such that $\log(-1) = i\pi$ is taken, and using the residue theorem, we find that

$$\int_{0}^{\infty}\frac{1+tz}{t-z}\,\frac{dt}{1+t^2} = i\pi - \log z, \qquad z \in \mathbb{C}\setminus\mathbb{R}_+.$$

Therefore,

$$\log \widetilde{G}^{\mu-\lambda}(z) = \gamma + i\pi - \log z + 2(1+z^2)\int_{-\infty}^{0}\frac{tf(t)}{t^2-z^2}\,\frac{dt}{1+t^2}\,, \qquad \operatorname{Im} z > 0, \quad (4.32)$$

or what is the same,

$$\log G^{\mu-\lambda}(-z^2) = \gamma + \frac{i\pi}{2} - \log z + 2(1+z^2)\int_{-\infty}^{0}\frac{tf(t)}{t^2-z^2}\,\frac{dt}{1+t^2}\,, \qquad \operatorname{Im} z > 0.$$

After the change of variables $-z^2 = x$ and $-t^2 = y$ this becomes

$$\log G^{\mu-\lambda}(x) = \gamma - \log\sqrt{x} + \left(1 - \frac{1}{x}\right)\int_{-\infty}^{0}\frac{x}{x-y}\,\frac{f(-\sqrt{|y|}\,)\,dy}{1+|y|}\,, \qquad x \in \mathbb{C}\setminus\mathbb{R}_-,$$

where $\sqrt{1} = 1$ and $\log 1 = 0$. Evaluating at $x = 1$, it follows that $\gamma = \log G^{\mu-\lambda}(1)$. Therefore,

$$\log\frac{\sqrt{x}\,G^{\mu-\lambda}(x)}{G^{\mu-\lambda}(1)} = \left(1 - \frac{1}{x}\right)\int_{-\infty}^{0}\frac{x}{x-y}\,\frac{f(-\sqrt{|y|}\,)\,dy}{1+|y|}\,, \qquad x \in \mathbb{C}\setminus\mathbb{R}_-.$$

For $x > 1$ the right-hand side is positive, hence,

$$\sqrt{x}\,G^{\mu-\lambda}(x) > G^{\mu-\lambda}(1) \quad \text{for all } x > 1,$$

which contradicts (4.21) unless $G^{\mu-\lambda}(1) = 0$, which we know means that $\mu = \lambda$.

Let us look at some other properties of extremal measures.

Corollary 4.7. *Suppose that the assumptions of Theorem 4.5 hold. Then the following assertions hold.*

(a) *If* $\liminf_{x\to-\infty}\phi^*(x)=+\infty$, *then* $\operatorname{supp}(\lambda)$ *is compact.*

(b) *If* $\operatorname{supp}(\lambda)$ *is unbounded and* λ *satisfies* (4.2), *then*

$$\liminf_{x\to-\infty}\phi^*(x)\leqslant w(\sigma,\phi).$$

(c) *If* $\int\log(1+y^2)\,d\sigma(y)=+\infty$, *then* $\operatorname{supp}(\sigma-\lambda)$ *is unbounded.*

(d) *If* $\operatorname{supp}(\sigma-\lambda)$ *is unbounded and* λ *satisfies* (4.2), *then*

$$\liminf_{x\to-\infty,\ x\in\operatorname{supp}(\sigma-\lambda)}\phi^*(x)\geqslant w(\sigma,\phi).$$

(e) *If* $\operatorname{supp}(\sigma-\lambda)$ *and* $\operatorname{supp}(\lambda)$ *are unbounded, then* λ *satisfies* (4.2) *and the limit* $\lim_{x\to-\infty}\phi^*(x)$ *exists and equals* $w(\sigma,\phi)$.

(f) *If* $\phi(x)$ *is decreasing on* $\mathbb{R}_-$, *then* $0\in\operatorname{supp}(\lambda)$.

(g) *If* $x\phi'(x)$ *is decreasing on* $\mathbb{R}_-$, *then* $\operatorname{supp}(\lambda)$ *is connected.*

(h) *If the function* $\phi(x)=-U^\tau(x)$, *where* $\tau\in\mathscr{M}_2^+(\mathbb{R}_+)$, *has compact support and the potential* $U^\tau(x)$ *is continuous at* $x=0$, *then* $\operatorname{supp}(\lambda)=\mathbb{R}_-$. *If* $\operatorname{supp}(\sigma-\lambda)$ *is unbounded and* λ *satisfies* (4.2), *then* $w(\sigma,\phi)=0$.

Proof. (a) According to (C′)

$$\phi^*(x)\leqslant 2\int\log\frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|}\,d\lambda(y)+\phi^*(x)\leqslant\mathfrak{w}(\sigma,\phi),\qquad x\in\operatorname{supp}(\lambda).$$

If $\operatorname{supp}(\lambda)$ is unbounded, then

$$\limsup_{x\to-\infty,\ x\in\operatorname{supp}(\lambda)}\phi^*(x)\leqslant\mathfrak{w}.$$

Therefore, if $\liminf_{x\to-\infty}\phi^*(x)=+\infty$, then we get a contradiction. Thus, (a) holds.

(b) According to (C″)

$$W^\lambda(x)=2\int\log\frac{1}{|1-y/x|}\,d\lambda(y)+\phi^*(x)+\log\frac{1+x^2}{x^2}\leqslant w,\qquad x\in\operatorname{supp}(\lambda).$$

If $\operatorname{supp}(\lambda)$ is unbounded and λ satisfies (4.2), then it follows from (4.5) that

$$\liminf_{x\to-\infty}\phi^*(x)\leqslant\liminf_{x\to-\infty,\ x\in\operatorname{supp}(\lambda)}\phi^*(x)\leqslant w.$$

Therefore, (b) is valid.

(c) Suppose that $\operatorname{supp}(\sigma-\lambda)$ is a compact set K. Then

$$\lambda\big|_{\mathbb{R}_-\setminus K}=\sigma\big|_{\mathbb{R}_-\setminus K}.$$

However, $\int\log(1+y^2)\,d\lambda(y)<+\infty$, and thus $\int\log(1+y^2)\,d\sigma(y)<+\infty$. This contradiction implies (c).

(d) From (C$''$) we see that

$$W^\lambda(x) = 2\int \log\frac{1}{|1-y/x|}\,d\lambda(y) + \phi^*(x) + \log\frac{1+x^2}{x^2} \geqslant w, \qquad x \in \operatorname{supp}(\sigma-\lambda).$$

Thus, if $\operatorname{supp}(\sigma-\lambda)$ is unbounded and λ satisfies (4.2), then letting $x \to -\infty$ with $x \in \operatorname{supp}(\sigma-\lambda)$, we obtain (d) from (4.5).

(e) is a direct consequence of (b) and (d).

(f) For $x \in \mathbb{R}\setminus\operatorname{supp}(\lambda)$

$$\big(U^\lambda(x)\big)' = -\int\frac{d\lambda(y)}{x-y} \quad\text{and}\quad \big(x\big(U^\lambda(x)\big)'\big)' = \int\frac{y\,d\lambda(y)}{(x-y)^2}\,. \tag{4.33}$$

If ϕ decreases on $\mathbb{R}_-$ and $0 \notin \operatorname{supp}(\lambda)$, then the first of these formulae implies that $W^\lambda(x)$ decreases immediately to the right of $\operatorname{supp}(\lambda)$, but this contradicts (C$''$), so (f) follows.

(g) If $x\phi'(x)$ is decreasing, then $x(W^\lambda(x))'$ is decreasing on any connected component of $\mathbb{R}_-\setminus\operatorname{supp}(\lambda)$ by the second formula in (4.33). From this it follows that $(W^\lambda(x))'$ cannot change sign from plus to minus on any such connected component. Suppose that $\operatorname{supp}(\lambda)$ is disconnected. Then there exist $x_1, x_2 \in \operatorname{supp}(\lambda_2)$ with $x_2 < 0$ such that $(x_1,x_2)\cap\operatorname{supp}(\lambda_2) = \varnothing$. According to (C$''$), $(W^\lambda(x))'$ changes sign from plus to minus on (x_1,x_2); thus, $\operatorname{supp}(\lambda)$ must be connected and we obtain (g).

(h) Finally, it is easy to check that $\phi = -\mathscr{U}^\tau$ is decreasing on $\mathbb{R}_-$ and $x\phi'(x)$ is decreasing on $\mathbb{R}_-$; therefore, according to (f) and (g), $\operatorname{supp}(\lambda)$ is a closed interval in $\mathbb{R}_-$ which contains $x=0$. Suppose that $\operatorname{supp}(\lambda)$ is bounded. Then $W^\lambda(x)$ is subharmonic in $\overline{\mathbb{C}}\setminus\operatorname{supp}(\lambda)$ and continuous on $\operatorname{supp}(\lambda)$, and by the second part of (C$''$)

$$W^\lambda(\infty) = \lim_{x\to-\infty} W^\lambda(x) = 0 \geqslant w.$$

However, $W^\lambda(x) \leqslant w$ for $x \in \operatorname{supp}(\lambda)$, as asserted in the first part of (C$''$). From the maximum principle for subharmonic functions it follows that $2U^\lambda(x) \equiv U^\tau(x)$ for $x \in \mathbb{C}\setminus\operatorname{supp}(\lambda)$, which is impossible. Therefore, $\operatorname{supp}(\lambda) = \mathbb{R}_-$. If λ satisfies (4.2) and $\operatorname{supp}(\sigma-\lambda)$ is unbounded, then we get from (e) that $w(\sigma,\phi) = 0$. □

Remark 4.8. In Corollary 4.7 we assumed more than once that λ satisfies (4.2). One way to ensure this condition is to impose it on the constraint σ. However, it is possible that λ satisfies (4.2) but σ does not (for example, when $\operatorname{supp}(\lambda)$ is compact, see part (a) of the corollary). In connection with (h), note that $\operatorname{supp}(\sigma-\lambda)$ is unbounded when

$$\int\log(1+y^2)d\sigma(y) = +\infty$$

(see part (c)).

Remark 4.9. We wish to call attention to the case when $\sigma \equiv +\infty$, which corresponds to an equilibrium problem with no constraint. This case was considered in [27]. In this situation, one cannot rely on σ to guarantee that λ satisfies (4.2) or to imply the continuity of $\mathscr{U}^\lambda$. Nevertheless, if $\liminf\phi^* = +\infty$, then we can assert that λ

has compact support, which in turn trivially implies (4.2) for λ, and the continuity of $\mathscr{U}^\lambda$ follows from (C′) since $2\mathscr{U}^\lambda$ is equal to the continuous function $\mathfrak{w} - \phi$ on $\operatorname{supp}(\lambda)$.

5. Proof of Theorem 2.1

In this section we use again the notions of modified potential (4.14) and modified energy (4.15) introduced in § 4.2.

Let φ be a continuous function on $\mathbb{R}_+$ satisfying

$$\liminf_{x\to+\infty}\bigl(2\varphi(x) - 3\log(1+x^2)\bigr) > -\infty. \tag{5.1}$$

This assumption is much weaker than (2.18). Let

$$\varphi^*(x) := \varphi(x) - \frac{3}{2}\log(1+x^2), \qquad \mathscr{A} := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \text{and} \quad f := \begin{pmatrix} \varphi^* \\ 0 \end{pmatrix}.$$

For $\vec{\mu} = (\mu_1, \mu_2)^t \in \mathfrak{M}(\sigma)$ (see the definition in (2.12)), we introduce the vector function

$$\mathscr{W}^{\vec{\mu}}(x) = (\mathscr{W}_1^{\vec{\mu}}(x), \mathscr{W}_2^{\vec{\mu}}(x))^t := \int \log\frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|}\, d\mathscr{A}\vec{\mu}(y) + f(x)$$

and the functional

$$\mathscr{J}_{\varphi^*}(\vec{\mu}) := \int (\mathscr{W}^{\vec{\mu}} + f)\cdot d\vec{\mu} = \int (\mathscr{W}_1^{\vec{\mu}} + \varphi^*)\, d\mu_1 + \int \mathscr{W}_2^{\vec{\mu}}\, d\mu_2 \tag{5.2}$$

(when either $\mathscr{I}(\mu_1) = +\infty$ or $\mathscr{I}(\mu_2) = +\infty$, we take $\mathscr{J}_{\varphi^*}(\vec{\mu}) = +\infty$). Thus,

$$\mathscr{J}_{\varphi^*}(\vec{\mu}) = 2(\mathscr{I}(\mu_1) - \mathscr{I}(\mu_1,\mu_2) + \mathscr{I}(\mu_2)) + \int (2\varphi - 3\log(1+x^2))\, d\mu_1.$$

The condition (5.1) and the fact that $\mathscr{A}$ is positive-definite guarantee that the corresponding vector equilibrium problem is weakly admissible as defined in Assumption 2.1 of [32]. In particular (see Corollary 2.7 in [32] and the sentence that follows it), this gives us that

$$\mathscr{J}_{\varphi^*} = \inf\{\mathscr{J}_{\varphi^*}(\vec{\mu})\colon \vec{\mu}\in\mathfrak{M}(\sigma)\} > -\infty.$$

Let

$$\widetilde{\mathfrak{M}}(\sigma) = \{\,\vec{\mu}\in\mathfrak{M}(\sigma)\colon \mathscr{I}(\mu_1) < \infty,\ \mathscr{I}(\mu_2) < \infty\},$$
$$\mathfrak{M}^*(\sigma) = \{\,\vec{\mu}\in\mathfrak{M}(\sigma)\colon \mu_1,\mu_2 \text{ satisfy } (2.10)\}.$$

A vector measure $\vec{\lambda}\in\widetilde{\mathfrak{M}}(\sigma)$ is said to be extremal if

$$-\infty < \mathscr{J}_{\varphi^*}(\vec{\lambda}) = \mathscr{J}_{\varphi^*} < +\infty. \tag{5.3}$$

In the case $\vec{\mu}\in\mathfrak{M}^*(\sigma)$ it is easy to check that

$$\mathscr{J}_{\varphi^*}(\vec{\mu}) = 2\left(I(\mu_1) - I(\mu_1,\mu_2) + I(\mu_2) + \int \varphi\, d\mu_1\right) =: J_\varphi(\vec{\mu}).$$

The next theorem complements results in [32] in our context.

Theorem 5.1. *Let φ satisfy (5.1) and let σ be a positive Borel measure with* $\operatorname{supp}(\sigma) = \mathbb{R}_-$ *and $|\sigma| > 1$ such that $U^{\sigma|_K}$ is continuous on $\mathbb{C}$ for any compact subset $K \subset \mathbb{R}_-$. The following statements are equivalent and concern the same unique solution.*

(A‴) *There exists an extremal measure $\vec{\lambda} \in \widetilde{\mathfrak{M}}(\sigma)$.*

(B‴) *There exists a $\vec{\lambda} \in \widetilde{\mathfrak{M}}(\sigma)$ such that for all $\vec{\nu} \in \widetilde{\mathfrak{M}}(\sigma)$ the following inner product is non-negative*:

$$\int \mathscr{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}) := \int \mathscr{W}_1^{\vec{\lambda}}\, d(\nu_1 - \lambda_1) + \int \mathscr{W}_2^{\vec{\lambda}}\, d(\nu_2 - \lambda_2) \geqslant 0.$$

(C‴) *There exist $\vec{\lambda} = (\lambda_1, \lambda_2) \in \widetilde{\mathfrak{M}}(\sigma)$ and constants $\mathfrak{w}_1 = \mathfrak{w}_1(\sigma, \varphi)$ and $\mathfrak{w}_2 = \mathfrak{w}_2(\sigma, \varphi)$ such that*

$$(\mathrm{C}'''.\mathrm{i})\ \ \mathscr{W}_1^{\vec{\lambda}}(x) = 2\mathscr{U}^{\lambda_1}(x) - \mathscr{U}^{\lambda_2}(x) + \varphi(x) \begin{cases} = \mathfrak{w}_1, & x \in \operatorname{supp}(\lambda_1), \\ \geqslant \mathfrak{w}_1, & x \in \mathbb{R}_+, \end{cases}$$

$$(\mathrm{C}'''.\mathrm{ii})\ \ \mathscr{W}_2^{\vec{\lambda}}(x) = 2\mathscr{U}^{\lambda_2}(x) - \mathscr{U}^{\lambda_1}(x) \begin{cases} \leqslant \mathfrak{w}_2, & x \in \operatorname{supp}(\lambda_2), \\ \geqslant \mathfrak{w}_2, & x \in \operatorname{supp}(\sigma - \lambda_2). \end{cases}$$

The constants $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are uniquely determined, and $\mathscr{U}^{\lambda_1}$ and $\mathscr{U}^{\lambda_2}$ are continuous on $\mathbb{C}$.

Proof. This is similar to the proof of Theorem 4.5, so we will be brief. As shown in Theorem 2.6 of [32], the functional $\mathscr{J}_{\varphi^*}$ is lower semicontinuous and strictly convex on $\mathfrak{M}(\sigma)$, and this ensures the existence of a unique solution of the extremal problem (5.3) (see Corollary 2.7 in [32]). By the definition of the functional, the extremal measure must belong to $\widetilde{\mathfrak{M}}(\sigma)$.

The equivalence of (A‴) and (B‴) comes from the identity

$$\mathscr{J}_{\varphi^*}(\vec{\nu}_\varepsilon) - \mathscr{J}_{\varphi^*}(\vec{\lambda}) = \varepsilon^2 \mathscr{J}_0(\vec{\nu} - \vec{\lambda}) + 2\varepsilon \int \mathscr{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}),$$

valid for any $\vec{\lambda}, \vec{\nu} \in \widetilde{\mathfrak{M}}(\sigma)$ and $0 \leqslant \varepsilon \leqslant 1$, where $\vec{\nu}_\varepsilon = \varepsilon\vec{\nu} + (1-\varepsilon)\vec{\lambda}$ and $\mathscr{J}_0(\vec{\nu} - \vec{\lambda})$ is the energy functional applied to $\vec{\nu} - \vec{\lambda}$ with $\varphi^* \equiv 0$. To prove that (B‴) $\Rightarrow$ (A‴) we also use the fact that $\mathscr{J}_0(\vec{\nu} - \vec{\lambda}) \geqslant 0$, with equality if and only if $\vec{\nu} = \vec{\lambda}$ (see Proposition 3.5 in [32] and Theorem 2.5 in [16]).

If $\vec{\lambda} = (\lambda_1, \lambda_2)^t$ satisfies (C‴) and $\vec{\nu} = (\nu_1, \nu_2)^t \in \widetilde{\mathfrak{M}}(\sigma)$, then from (C‴.i) we have

$$\int \mathscr{W}_1^{\vec{\lambda}}\, d(\nu_1 - \lambda_1) = \int \mathscr{W}_1^{\vec{\lambda}}\, d\nu_1 - \int \mathscr{W}_1^{\vec{\lambda}}\, d\lambda_1 \geqslant \mathfrak{w}_1 - \mathfrak{w}_1 = 0.$$

On the other hand, $|\lambda_2| = |\nu_2| = 1$, and therefore

$$\int \mathscr{W}_2^{\vec{\lambda}}\, d(\nu_2 - \lambda_2) = \int (\mathscr{W}_2^{\vec{\lambda}} - \mathfrak{w}_2)\, d(\nu_2 - \lambda_2).$$

To show that this integral is also $\geqslant 0$, one uses the same arguments as in proving that (C′) $\Rightarrow$ (B′), now with

$$E_+ = \{t \in \mathbb{R}_- : \mathscr{W}_2^{\vec{\lambda}}(t) - \mathfrak{w}_2 > 0\} \quad \text{and} \quad E_- = \{t \in \mathbb{R}_- : \mathscr{W}_2^{\vec{\lambda}}(t) - \mathfrak{w}_2 < 0\}.$$

Putting these relations together, we obtain

$$\int \mathscr{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}) \geqslant 0, \qquad \nu \in \widetilde{\mathfrak{M}}(\sigma).$$

Consequently, (C‴) implies (B‴).

Assume that $\vec{\lambda} = (\lambda_1, \lambda_2)^t$ solves (B‴), and let

$$\mathfrak{w}_1 := \frac{1}{2} \int \mathscr{W}_1^{\vec{\lambda}}\, d\lambda_1.$$

We prove that

$$\mathscr{W}_1^{\vec{\lambda}}(x) \geqslant \mathfrak{w}_1 \quad \text{quasi-everywhere on } \mathbb{R}_+, \tag{5.4}$$

where 'quasi-everywhere' means everywhere except on a set of capacity zero. If this were not so, then there would exist a compact subset $K_1 \subset \mathbb{R}_+$ with $\operatorname{cap}(K_1) > 0$ such that $\mathscr{W}_1^{\vec{\lambda}}(x) < \mathfrak{w}_1$ for $x \in K_1$. Then taking

$$\nu_1 \in \mathscr{M}_2^+(\mathbb{R}_+), \quad \operatorname{supp}(\nu_1) \subset K_1, \quad \text{and} \quad \nu_2 = \lambda_2,$$

we would get that

$$\int \mathscr{W}^{\vec{\lambda}} \cdot d(\vec{\nu} - \vec{\lambda}) = \int \mathscr{W}_1^{\vec{\lambda}}\, d(\nu_1 - \lambda_1) < 2\mathfrak{w}_1 - 2\mathfrak{w}_1 = 0,$$

which contradicts (B‴). Let us now prove that

$$\mathscr{W}_1^{\vec{\lambda}}(x) \leqslant \mathfrak{w}_1, \qquad x \in \operatorname{supp}(\lambda_1).$$

To the contrary, assume that there exists an $x_0 \in \operatorname{supp}(\lambda_1)$ such that $\mathscr{W}_1^{\vec{\lambda}}(x_0) > \mathfrak{w}_1$. By the lower semicontinuity of $\mathscr{W}_1^{\vec{\lambda}}$ on $\mathbb{R}_+$ ($\mathscr{U}^{\lambda_2}$ is continuous by Lemma 4.4 and φ is continuous by assumption), it follows that there exists a $\delta > 0$ such that $\mathscr{W}_1^{\vec{\lambda}}(x) > \mathfrak{w}_1$ for $|x - x_0| \leqslant \delta$. For

$$K_2 = \operatorname{supp}(\lambda_1) \cap \{x \colon |x - x_0| \leqslant \delta\}$$

we have $\lambda_1(K_2) > 0$ and

$$2\mathfrak{w}_1 = \int_{\operatorname{supp}(\lambda_1)\setminus K_2} \mathscr{W}_1^{\vec{\lambda}}\, d\lambda_1 + \int_{K_2} \mathscr{W}_1^{\vec{\lambda}}\, d\lambda_1 > \mathfrak{w}_1\big(\lambda_1(\operatorname{supp}(\lambda_1)\setminus K_2) + \lambda_1(K_2)\big) = 2\mathfrak{w}_1,$$

a contradiction. By reasoning as in Theorem 5.4.1 of [44] it follows from (5.4) that $\mathscr{W}_1^{\vec{\lambda}} \geqslant \mathfrak{w}_1$ on the whole of $\mathbb{R}_+$. Hence, (C‴.i) holds. We have also shown that $\mathscr{U}^{\lambda_1}$ is continuous on the whole of $\mathbb{C}$, because it is equal to the continuous function $(\mathfrak{w}_2 - \varphi + \mathscr{U}^{\lambda_2})/2$ on $\operatorname{supp}(\lambda_1)$.

For the proof of (C‴.ii) take

$$\mathfrak{w}_2 := \sup\{\mathfrak{w} \in \mathbb{R} \colon \mathscr{W}_2^{\vec{\lambda}} \geqslant \mathfrak{w} \text{ holds } (\sigma - \lambda_2)\text{-almost everywhere}\}.$$

If there exists an $x_0 \in \operatorname{supp}(\lambda_2)$ such that $\mathscr{W}_2^{\vec\lambda}(x_0) > \mathfrak{w}_2$, then by proceeding as in the scalar case we can construct a signed measure η of total mass 1 supported on a compact subset of $\mathbb{R}_-$ such that $\vec\nu := (\lambda_1, \lambda_2 + \eta)^t \in \widetilde{\mathfrak{M}}(\sigma)$ and

$$\int \mathscr{W}^{\vec\lambda} \cdot d(\vec\nu - \vec\lambda) = \int \mathscr{W}_2^{\vec\lambda}\, d\eta < 0,$$

which contradicts (B‴). By the continuity of $\mathscr{W}_2^{\vec\lambda}$ on $\mathbb{C}$, the inequality in the second part of (C‴.ii) holds for all $x \in \operatorname{supp}(\sigma - \lambda_2)$. Thus, (C‴) is proved.

From the uniqueness of $\vec\lambda$ and the fact that

$$\operatorname{supp}(\sigma - \lambda_2) \cap \operatorname{supp}(\lambda_2) \neq \varnothing,$$

it readily follows that $\mathfrak{w}_1$ and $\mathfrak{w}_2$ are uniquely determined. □

Corollary 5.2. *Under the assumptions of Theorem 5.1, let $\vec\lambda$ be extremal. Then* $\operatorname{supp}(\lambda_2)$ *is connected and* $0 \in \operatorname{supp}(\lambda_2)$*. If $x\varphi'(x)$ is an increasing function on $\mathbb{R}_+$, then* $\operatorname{supp}(\lambda_1)$ *is connected. If φ is increasing on $\mathbb{R}_+$, then* $0 \in \operatorname{supp}(\lambda_1)$*. If*

$$\lim_{x\to+\infty} \big(\varphi(x) - 4\log x\big) = +\infty, \tag{5.5}$$

then $\operatorname{supp}(\lambda_1)$ *is a compact set*, $\operatorname{supp}(\lambda_2) = \mathbb{R}_-$, *and λ_1 and λ_2 satisfy* (4.1).

Proof. Note that for any bounded measure μ on the real line $\big(\mathscr{U}^\lambda(x)\big)' = \big(U^\lambda(x)\big)'$, and thus

$$\big(x\big(\mathscr{U}^\lambda(x)\big)'\big)' = \big(x\big(U^\lambda(x)\big)'\big)' \quad \text{for all } x \in \mathbb{R} \setminus \operatorname{supp}(\lambda).$$

Arguing as in Corollary 4.7 (see (f) and (g)), one proves that $\operatorname{supp}(\lambda_2)$ is connected and $0 \in \operatorname{supp}(\lambda_2)$. Similarly, one proves that $\operatorname{supp}(\lambda_1)$ is connected and $0 \in \operatorname{supp}(\lambda_1)$ when $x\varphi'$ and φ are increasing, respectively.

The first relation in (C‴.i) of Theorem 5.1 can be rewritten as follows:

$$\begin{aligned} 2\int \log \frac{\sqrt{1+x^2}\,\sqrt{1+y^2}}{|x-y|}\, d\lambda_1(y) - \int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2(y) \\ + \varphi(x) - 2\log(1+x^2) = \mathfrak{w}_1, \qquad x \in \operatorname{supp}(\lambda_1). \end{aligned}$$

If $x \geqslant 1$, then $\sqrt{1+y^2}/|x-y| \leqslant 1$ for $y \in \mathbb{R}_-$, and by using (4.16) we get from the previous equality that

$$\varphi(x) - 2\log(1+x^2) \leqslant \mathfrak{w}_1, \qquad x \in \operatorname{supp}(\lambda_1), \quad x \geqslant 1.$$

Consequently, $\operatorname{supp}(\lambda_1)$ must be a compact set when(5.5) holds. The condition (4.1) follows immediately for λ_1.

Now assume that $\operatorname{supp}(\lambda_2)$ is also compact. Then λ_2 satisfies (4.1) and

$$\lim_{x\to\infty} \mathscr{W}_2^{\vec\lambda}(x) = \int \log(1+y^2)\, d\lambda_2(y) - \frac{1}{2}\int \log(1+y^2)\, d\lambda_1(y).$$

In particular, taking the limit as $x \to -\infty$ along $\mathbb{R}_-$ in the second part of (C‴.ii), we have

$$\int \log(1+y^2)\,d\lambda_2(y) - \frac{1}{2}\int \log(1+y^2)\,d\lambda_1(y) \geqslant \mathfrak{w}_2.$$

According to the first part of (C‴.ii), $\mathscr{W}_2^{\vec{\lambda}}(x) \leqslant \mathfrak{w}_2$ on $\operatorname{supp}(\lambda_2)$. However, $\mathscr{W}_2^{\vec{\lambda}}$ is subharmonic in $\overline{\mathbb{C}} \setminus \operatorname{supp}(\lambda_2)$ and continuous on $\mathbb{C}$. By the maximum principle for subharmonic function this means that $\mathscr{W}_2^{\vec{\lambda}} \equiv \mathfrak{w}_2$ on $\mathbb{C}$, which is false. Therefore, $\operatorname{supp}(\lambda_2) = \mathbb{R}_-$, as claimed.

In order to prove that λ_2 satisfies (4.1) we use (C‴.ii) and argue as in Theorem 4.5 for proving that λ satisfies (4.1). □

Proof of Theorem 2.1. Under the present assumptions, we know from the last assertion of Corollary 5.2 that

$$\vec{\lambda} \in \mathfrak{M}^*(\sigma) \subset \widetilde{\mathfrak{M}}(\sigma).$$

The combined statements of Theorem 5.1 and Corollary 5.2 give all but the last assertion of Theorem 2.1. Taking into account that

$$2\mathscr{U}^{\lambda_1} - \mathscr{U}^{\lambda_2} + \varphi = 2U^{\lambda_1} - U^{\lambda_2} + \varphi + C_1 \quad \text{and} \quad 2\mathscr{U}^{\lambda_2} - \mathscr{U}^{\lambda_1} = 2U^{\lambda_2} - U^{\lambda_1} + C_2,$$

where

$$C_1 = \int \log(1+y^2)\,d\lambda_1(y) - \frac{1}{2}\int \log(1+y^2)\,d\lambda_2(y),$$

$$C_2 = \int \log(1+y^2)\,d\lambda_2(y) - \frac{1}{2}\int \log(1+y^2)\,d\lambda_1(y),$$

we get that

$$w_1(\sigma,\varphi) = \mathfrak{w}_1(\sigma,\varphi) - C_1 \quad \text{and} \quad w_2(\sigma,\varphi) = \mathfrak{w}_2(\sigma,\varphi) - C_2.$$

If $\int \log(1+y^2)\,d\sigma(x) = +\infty$, then by combining the arguments employed in the proof of (c) and (h) in Corollary 4.7 we find that $w_2(\sigma,\varphi) = 0$. □

6. Proof of Theorem 2.2

The sequences of zero-counting measures (ν_{Q_n}) and $(\nu_{Q_{n,2}})$, $n \in \mathbb{Z}_+$, of the polynomials Q_n and $Q_{n,2}$ belong to $\mathscr{M}_1^+(\mathbb{R}_+)$ and $\mathscr{M}_1^+(\mathbb{R}_-)$, respectively. By the Banach–Alaoglu theorem there exists a sequence of indices $\Lambda \subset \mathbb{Z}_+$ and positive measures λ_1^* and λ_2^* with $|\lambda_1^*| \leqslant 1$ and $|\lambda_2^*| \leqslant 1$ such that

$$\lim_{n\in\Lambda} \nu_{Q_n} = \lambda_1^* \quad \text{and} \quad \lim_{n\in\Lambda} \nu_{Q_{n,2}} = \lambda_2^* \tag{6.1}$$

in the vague topology of the space of measures. That is, for any continuous functions f and g with compact support on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively,

$$\lim_{n\in\Lambda} \int f\,d\nu_{Q_n} = \int f\,d\lambda_1^* \quad \text{and} \quad \lim_{n\in\Lambda} \int g\,d\nu_{Q_{n,2}} = \int g\,d\lambda_2^*. \tag{6.2}$$

It easily follows that (6.2) also holds for any $f \in \mathscr{C}_0(\mathbb{R}_+)$ and $g \in \mathscr{C}_0(\mathbb{R}_-)$ (the class of continuous functions on the indicated sets with zero limit at infinity).

In principle, it may occur that $|\lambda_1^*| < 1$ or $|\lambda_2^*| < 1$, but we will show that this is not the case under our assumptions. Moreover, we will show that $(2\lambda_1^*, \lambda_2^*)$ is in $\mathfrak{M}^*(\sigma)$ and solves the problem (C) in Theorem 2.1. After this is done, it follows from uniqueness that all the convergent subsequences have the same limit satisfying (6.1), and the corresponding measures are precisely $\lambda_1/2$ and λ_2, where (λ_1, λ_2) is the solution in Theorem 2.1. Then since the limit measures in (6.1) have mass 1, it follows from Theorems 6.21 and 6.22 in [19] that (6.2) holds for all bounded continuous functions f and g on $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively, which amounts to (2.22).

We begin by showing that $\lambda_2^* \leqslant \sigma$. Indeed, between two consecutive mass points of the discrete measure $\sigma_{2,n}$ there can be at most one zero of $Q_{n,2}$. For $-\infty < T_1 < T_2 \leqslant 0$ it follows from (2.17) that

$$\limsup_n \int_{[T_1,T_2]} d\nu_{Q_{n,2}} \leqslant \lim_n \frac{1}{n} \int_{[T_1,T_2]} d\biggl(\sum_{k\geqslant 1} \delta_{\xi_{k,n}}\biggr) = \int_{[T_1,T_2]} d\sigma.$$

On the other hand, since $U^{\sigma|_K}$ is continuous on $\mathbb{C}$ for any compact subset K of $\mathbb{R}_-$, it follows that σ has no mass points, and therefore

$$\limsup_n \nu_{Q_{n,2}}(\{T\}) = 0 = \sigma(\{T\}) \quad \text{for any } T \in \mathbb{R}_-.$$

These facts and the second part of (6.2) imply that $\lambda_2^* \leqslant \sigma$, whence $\mathscr{U}^{\lambda_2^*}$ is continuous on $\mathbb{C}$ by Lemma 4.4. Moreover, λ_2^* satisfies (4.2) since σ satisfies it (see the assumption (iii) in § 2).

Our next goal is to deduce the variational relations. To this end, we use the theorem on p. 124 of [29].

We start with $\mathbb{R}_+$. From (2.6) it follows that for any monic polynomials Q with $\deg Q = 2n$

$$\int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} C_n x^\alpha s_1'(d_n x)\, \frac{dx}{x^\alpha} \leqslant \int \frac{|Q(x)|^2}{|Q_{n,2}(x)|} C_n x^\alpha s_1'(d_n x)\, \frac{dx}{x^\alpha}\,,$$

where

$$C_n = \prod_{Q_{n,2}(x_{n,k})=0} \sqrt{1 + x_{n,k}^2}\,.$$

Therefore, in this class Q_n is the monic polynomial of degree $2n$ that minimizes the L_2-norm with respect to the varying weight

$$\frac{C_n x^\alpha s_1'(d_n x)}{|Q_{n,2}(x)|}\, \frac{dx}{x^\alpha}\,.$$

Since $\alpha < 1$, the measure dx/x^α is locally integrable on $\mathbb{R}_+$.

We have

$$g_n(x) := \frac{1}{n} \log \frac{|Q_{n,2}(x)|}{C_n} = -\int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y),$$

and $\log \dfrac{\sqrt{1+y^2}}{|x-y|} \in \mathscr{C}_0(\mathbb{R}_-)$ for any $x>0$. From (6.2),

$$\lim_{n\in\Lambda} \frac{1}{2n} \log\left(\frac{|Q_{n,2}(x)|}{C_n}\right)^{1/2} = -\frac{1}{4}\int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y) = -\frac{1}{4}\mathscr{U}^{\lambda_2^*}(x) \tag{6.3}$$

pointwise on $(0,+\infty)$. On the other hand, if $0<x<x'<+\infty$, then

$$\left|\int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y) - \int \log \frac{\sqrt{1+y^2}}{|x'-y|}\, d\nu_{Q_{n,2}}(y)\right| = \int \log \frac{x'-y}{x-y}\, d\nu_{Q_{n,2}}(y)$$

$$= \int \log\left(1+\frac{x'-x}{x-y}\right) d\nu_{Q_{n,2}}(y) < (x'-x)\int \frac{d\nu_{Q_{n,2}}(y)}{x-y} \leqslant \frac{x'-x}{x},$$

which means that the family of functions (g_n), $n\in\mathbb{N}$, is equicontinuous on compact subsets of $(0,+\infty)$. Therefore, (6.3) holds uniformly on each compact subset of $(0,+\infty)$. Let us show that indeed (6.3) holds uniformly on each compact subset of $\mathbb{R}_+ := [0,+\infty)$. It remains to show that this is true, for example, on the interval $[0,1/2]$.

Take $\delta\in(0,1/2)$ and $x\in[0,1/2]$. Then

$$\left|\int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y) - \int \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y)\right|$$

$$\leqslant \left|\int_{|y|\geqslant\delta} \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y) - \int_{|y|\geqslant\delta} \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y)\right|$$

$$+ \left|\int_{|y|\leqslant\delta} \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y)\right| + \left|\int_{|y|\leqslant\delta} \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y)\right|$$

$$\leqslant \left|\int_{|y|\geqslant\delta} \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y) - \int_{|y|\geqslant\delta} \log \frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y)\right|$$

$$+ \int_{|y|\leqslant\delta} \log \frac{\sqrt{1+y^2}}{|y|}\, d\lambda_2^*(y) + \left|\int_{|y|\leqslant\delta} \log \frac{1}{|y|}\, d\nu_{Q_{n,2}}(y)\right| + \log\sqrt{1+\delta^2}\,.$$

We consider the terms on the last line. Fix $\varepsilon>0$. Since $\mathscr{U}^{\lambda_2^*}$ is continuous on $\mathbb{C}$ (in particular at $x=0$), $\log \dfrac{\sqrt{1+y^2}}{|y|}$ is integrable with respect to λ_2^* and 0 is not a mass point of λ_2^*. Consequently, for all sufficiently small δ it follows that

$$\int_{|y|\leqslant\delta} \log \frac{\sqrt{1+y^2}}{|y|}\, d\lambda_2^*(y) < \varepsilon.$$

Obviously, $\log\sqrt{1+\delta^2} < \varepsilon$ for all sufficiently small δ.

We show that the same is true for the middle term. Between two mass points of $\sigma_{2,n}$ there is at most one zero of $Q_{n,2}$, and therefore

$$\left|\int_{|y|\leqslant\delta} \log \frac{1}{|y|}\, d\nu_{Q_{n,2}}(y)\right| = \frac{1}{n}\left|\log \prod_{|x_{n,k}|\leqslant\delta} |x_{n,k}|\right| \leqslant \left|\log\Big(\prod_{|\xi_{n,k}|\leqslant\delta} |\xi_{n,k}|\Big)^{1/n}\right|.$$

Let

$$\rho = \min\{\rho(x) \colon x \in [-\delta, 0]\} \quad (> 0),$$

where $\rho(x)$ is the function that appears in the condition (i) in § 2. According to (i),

$$|\xi_{k,n}| = |\xi_{k,n} - \xi_{k-1,n}| + \dots + |\xi_{1,n}| \geqslant \frac{k\rho}{n} \,. \tag{6.4}$$

Let ℓ_n be the number of points $\xi_{k,n}$ in $[-\delta, 0]$:

$$\ell_n := \xi_{k,n} \in [-\delta, 0].$$

By (2.17), $\lim_{n\to\infty} \ell_n/n = \sigma([-\delta, 0])$. The condition (i) also implies that $\ell_n \leqslant n\delta/\rho$, hence $\lim_n \ell_n^{1/n} = 1$. By (6.4),

$$1 > \biggl(\prod_{|\xi_{n,k}|\leqslant\delta} |\xi_{n,k}|\biggr)^{1/n} \geqslant \biggl(\frac{\rho}{n}\,\frac{2\rho}{n} \cdots \frac{\ell_n\rho}{n}\biggr)^{1/n} = \biggl(\frac{\rho}{n}\biggr)^{\ell_n/n} (\ell_n!)^{1/n}.$$

Consequently, by Stirling's formula,

$$\begin{aligned} \limsup_{n\to\infty} \biggl|\int_{|y|\leqslant\delta} \log\frac{1}{|y|}\, d\nu_{Q_{n,2}}(y)\biggr| &\leqslant \biggl|\log\biggl(\lim_{n\to\infty} \biggl(\frac{\rho}{e}\biggr)^{\ell_n/n} \biggl(\frac{\ell_n}{n}\biggr)^{\ell_n/n} \ell_n^{1/(2n)} (\mathscr{O}(1))^{1/n}\biggr)\biggr| \\ &= \biggl|\sigma([-\delta, 0]) \log\frac{\rho\sigma[-\delta, 0]}{e}\biggr| \to 0 \quad \text{as } \delta \to 0. \end{aligned} \tag{6.5}$$

Therefore, we can fix a $\delta \in (0, 1/2)$ such that

$$\int_{|y|\leqslant\delta} \log\frac{\sqrt{1+y^2}}{|y|}\, d\lambda_2^*(y) + \biggl|\int_{|y|\leqslant\delta} \log\frac{1}{|y|}\, d\nu_{Q_{n,2}}(y)\biggr| + \log\sqrt{1+\delta^2} < 3\varepsilon.$$

For such a δ it is easy to show that

$$\lim_{n\in\Lambda} \int_{|y|\geqslant\delta} \log\frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y) = \int_{|y|\geqslant\delta} \log\frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y)$$

uniformly with respect to $x \in [0, 1/2]$. Putting all this together, we find that for any $\varepsilon > 0$ there exists an n_0 such that if $n \geqslant n_0$, $n \in \Lambda$, then

$$\biggl|\int \log\frac{\sqrt{1+y^2}}{|x-y|}\, d\lambda_2^*(y) - \int \log\frac{\sqrt{1+y^2}}{|x-y|}\, d\nu_{Q_{n,2}}(y)\biggr| \leqslant 4\varepsilon,$$

and the left-hand side is independent of $x \in [0, 1/2]$. Thus, (6.3) holds uniformly on any compact subset of $\mathbb{R}_+$, as we wanted to prove.

Let

$$f_n(x) := \frac{1}{2n} \log\biggl(\frac{|Q_{n,2}(x)|}{C_n x^\alpha s_1'(d_n x)}\biggr)^{1/2}.$$

From (6.3) and (2.19),

$$\lim_{n\in\Lambda} \frac{1}{2n} \log\biggl(\frac{|Q_{n,2}(x)|}{C_n x^\alpha s_1'(d_n x)}\biggr)^{1/2} = \frac{1}{4}(\varphi(x) - \mathscr{U}^{\lambda_2^*}(x))$$

uniformly on each compact subset of $\mathbb{R}_+$. In particular, for any closed interval $\Delta \subset \mathbb{R}_+$

$$\lim_{n\in\Lambda} \min_{x\in\Delta} f_n(x) = \min_{x\in\Delta} \frac{1}{4}(\varphi(x) - \mathscr{U}^{\lambda_2^*}(x)). \tag{6.6}$$

For $x \geqslant 1$ and $y \leqslant 0$ we have $\log \sqrt{1+y^2}/(x-y) \leqslant 0$, and therefore it follows from (2.18) and (2.20) that

$$\liminf_{x\to+\infty} \frac{\varphi(x) - \mathscr{U}^{\lambda_2^*}(x)}{4\log x} > 1 \quad \text{and} \quad \liminf_{n\in\Lambda,\ x\to+\infty} \frac{f_n(x)}{\log x} > 1. \tag{6.7}$$

The relations (6.6) and (6.7) say that a) and b) in § 3 of [29] are satisfied. Therefore, it follows from the lemma and the theorem in [29] that λ_1^* is the unique probability measure on $\mathbb{R}_+$ which solves the extremal problem

$$U^{\lambda_1^*}(x) + \frac{1}{4}(\varphi(x) - \mathscr{U}^{\lambda_2^*}(x)) \begin{cases} = w_1^*, & x \in \operatorname{supp}(\lambda_1^*), \\ \geqslant w_1^*, & x \in \mathbb{R}_+, \end{cases} \tag{6.8}$$

for some constant w_1^*, and (recall that $\deg Q_n = 2n$)

$$\lim_{n\in\Lambda} \biggl(\int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} C_n s_1'(d_n x)\, dx\biggr)^{1/(4n)} = e^{-w_1^*}. \tag{6.9}$$

The arguments employed in [29] to prove the main theorem let us conclude that for any $\varepsilon > 0$ there exists an $R > 0$ such that

$$\liminf_{n\in\Lambda} \biggl(\int_0^R \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} C_n s_1'(d_n x)\, dx\biggr)^{1/(4n)} \geqslant e^{-w_1^*-\varepsilon}. \tag{6.10}$$

The first part of (6.7) guarantees that $\operatorname{supp}(\lambda_1^*)$ is a compact subset of $[0, +\infty)$. This is shown in [27] (see also Theorem 1.3.1 in [50], or even Corollary 4.7, (a), applied to measures supported on $\mathbb{R}_+$). We note that (6.8) and the continuity of φ and $\mathscr{U}^{\lambda_2^*}$ on $\mathbb{R}_+$ imply that $\mathscr{U}^{\lambda_1^*}$ is continuous on $\operatorname{supp}(\lambda_1^*)$, and thus on the whole of $\mathbb{C}$. Using the compactness of $\operatorname{supp}(\lambda_1^*)$, we get that

$$I(\lambda_1^*) < +\infty \quad \text{and} \quad \int \log(1+y^2)\, d\lambda_1^*(y) < \infty.$$

We now obtain variational relations on $\mathbb{R}_-$. The varying discrete measure with respect to which the $Q_{n,2}$ are orthogonal (see (2.7) and (2.8)) can be written in the form

$$\sum_{k=1}^{\infty} \frac{\beta_k \eta_{n,k}}{|\xi_{k,n}|} \frac{D_n}{|Q_n(\xi_{k,n})|} \delta_{\xi_{k,n}}(t),$$

where

$$\eta_{n,k} = \int_{\mathbb{R}_+} \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} \frac{C_n s_1'(d_n x)\, dx}{1 - x/\xi_{k,n}} \quad \text{and} \quad D_n = \prod_{Q_n(y_{n,k})=0} \sqrt{1+y_{n,k}^2}\,.$$

Since $\sum_{k=1}^{\infty} \frac{\beta_k}{t_k} < +\infty$ and $\lim_n d_n^{1/n} = 1$, we have

$$\lim_{n\to\infty} \left(\sum_{k=1}^{\infty} \frac{\beta_k}{|\xi_{k,n}|} \right)^{1/n} = 1.$$

By (6.9),

$$\limsup_{n\in\Lambda} \eta_{n,k}^{1/n} \leqslant e^{-4w_1^*}. \tag{6.11}$$

On the other hand, in view of (6.10) for any $\varepsilon > 0$ we can choose an $R > 0$ such that

$$\liminf_{n\in\Lambda} \eta_{n,k}^{1/n} \geqslant \liminf_{n\in\Lambda} \left(\int_0^R \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} \frac{C_n s_1'(d_n x)\, dx}{1 - Rd_n/t_1} \right)^{1/n} \geqslant e^{-4w_1^* - 4\varepsilon}. \tag{6.12}$$

From (6.11) and (6.12) it follows that

$$\lim_{n\in\Lambda} \eta_{n,k}^{1/n} = e^{-4w_1^*} \quad \text{uniformly with respect to } k. \tag{6.13}$$

Since $\log \frac{\sqrt{1+y^2}}{|x-y|} \in \mathscr{C}_0(\mathbb{R}_+)$ for any $x < 0$, arguing as we did for the sequence of polynomials $(Q_{n,2})$, we conclude that

$$\lim_{n\in\Lambda} \left(\frac{|Q_n(x)|}{D_n} \right)^{1/n} = e^{-2\mathscr{U}^{\lambda_1^*}(x)} \tag{6.14}$$

uniformly on any compact subset of $(-\infty, 0)$. Let

$$\phi(x) := 4w_1^* - \mathscr{U}^{2\lambda_1^*}(x).$$

Using (6.13) and (6.14), we find that

$$\lim_{n\in\Lambda} \left(\frac{\eta_{n,k} D_n}{|Q_n(\xi_{k,n})|} \right)^{1/n} - e^{-\phi(\xi_{k,n})} = 0 \tag{6.15}$$

uniformly on any compact subset $K \Subset (-\infty, 0)$ and for k such that $\xi_{k,n} \in K$.

Let $\lambda \in \mathscr{M}^*(\sigma)$ be the extremal solution in Corollary 4.6 with σ as in Theorem 2.2 and with the external field $\phi(x) := 4w_1^* - \mathscr{U}^{2\lambda_1^*}(x)$. In Theorem 2.2 we assumed that $0 \notin \operatorname{supp}(\sigma - \lambda_2)$, and here we will also assume that $0 \notin \operatorname{supp}(\sigma - \lambda)$. We will show that $\lambda_2^* = \lambda$ by using suitably modified versions of some results in [20] (Lemmas 5.3, 5.5, and 3.2). There the corresponding λ had compact support, while in our case the support is $\mathbb{R}_-$. More precisely, applying Corollaries 4.6 and 4.7, (c) and (h), it follows that there exist a $\lambda \in \mathscr{M}^*(\sigma)$ and a constant

$$w = w(\sigma, \phi) = 4w_1^* - \int \log(1+y^2)\, d\lambda_1^*(y)$$

such that

$$2U^{\lambda}(x) + \phi(x) \begin{cases} \leqslant w, & x \in \operatorname{supp}(\lambda) = \mathbb{R}_-, \\ = w, & x \in \operatorname{supp}(\sigma - \lambda), \end{cases} \tag{6.16}$$

and $\operatorname{supp}(\sigma - \lambda)$ is unbounded.

Let

$$\|Q_{n,2}\|_{2,n} = \left(\sum_{k=1}^{\infty} |Q_{n,2}(\xi_{k,n})|^2 \frac{\beta_k \eta_{n,k}}{|\xi_{k,n}|} \frac{D_n}{|Q_n(\xi_{k,n})|} \right)^{1/2}.$$

We show that

$$\limsup_{n \in \Lambda} \|Q_{n,2}\|_{2,n}^{1/n} \leqslant e^{-w(\sigma,\phi)/2}. \tag{6.17}$$

For this, we follow the approach in Lemma 5.3 of [20].

Fix $\varepsilon > 0$, and let $w = w(\sigma, \phi)$. Choose $A \supset \operatorname{supp}(\sigma - \lambda)$ to be the union of finitely many closed intervals such that for $x \in A$

$$2U^{\lambda}(x) - \mathscr{U}^{2\lambda_1^*}(x) > w - \varepsilon$$

and $0 < \lambda(A) < 1$. The existence of such a set is guaranteed because, according to (4.5) and Lemma 4.4,

$$\lim_{x \to \infty} \bigl(2U^{\lambda}(x) + \phi(x) \bigr) = w(\sigma, \phi) = 4w_1^* - \int \log(1 + y^2)\, d\lambda_1^*(y) \tag{6.18}$$

as $x \to \infty$ in any direction, and in particular as $x \to -\infty$. Moreover, we have $\mathbb{R}_- \setminus \operatorname{supp}(\sigma - \lambda) \neq \varnothing$ because $|\lambda| = 1 < |\sigma|$, and since $0 \notin \operatorname{supp}(\sigma - \lambda)$, we can take A such that $0 \in \mathbb{R}_- \setminus A$.

Let

$$\widetilde{\lambda} = \lambda\big|_{\mathbb{R}_- \setminus A} \quad \text{and} \quad \tilde{\sigma}_n = \frac{1}{n} \sum_{k \geqslant 1} \delta_{\xi_{k,n}}\big|_{\mathbb{R}_- \setminus A}.$$

The set $\mathbb{R}_- \setminus A$ is compact, and we get from (2.17) that $\lim_{n \in \Lambda} \tilde{\sigma}_n = \widetilde{\lambda}$ in the vague topology. In particular,

$$\lim_{n \in \Lambda} \frac{m_n}{n} = \lambda(\mathbb{R}_- \setminus A) < 1,$$

where m_n is the number of points $\xi_{k,n}$ that lie in $\mathbb{R}_- \setminus A$. Therefore, there exists an n_0 such that $m_n < n$ for $n \geqslant n_0$, $n \in \Lambda$.

Let P_n be a monic polynomial of degree n whose zeros consist of the m_n points $\xi_{k,n} \in \mathbb{R}_- \setminus A$ and $n - m_n$ points in A chosen so that $\lim_{n \in \Lambda} \nu_{P_n} = \lambda$ in the vague topology. It is sufficient to discretize λ on A. Since $\lambda \in \mathscr{M}^*(\sigma)$ and $\log(1 + y^2)$ is positive and decreasing in $\mathbb{R}_-$, we can also ensure that

$$\lim_{n \in \Lambda} \int \log(1 + y^2)\, d\nu_{P_n}(y) = \int \log(1 + y^2)\, d\lambda(y). \tag{6.19}$$

For $n \geqslant n_0$, $n \in \Lambda$, we have

$$\begin{aligned} \|Q_{n,2}\|_{2,n}^{2/n} \leqslant \|P_n\|_{2,n}^{2/n} &\leqslant \left(\sum_{\xi_{k,n} \in A} |P_n(\xi_{k,n})|^2 \frac{\beta_k \eta_{n,k}}{|\xi_{k,n}|} \frac{D_n}{|Q_n(\xi_{k,n})|} \right)^{1/n} \\ &\leqslant \left(\sum_{k=1}^{\infty} \frac{\beta_k}{|\xi_{k,n}|} \right)^{1/n} \exp\left\{ - \left(2U^{\nu_{P_n}}(\xi_n) - 2\mathscr{U}^{\nu_{Q_n}}(\xi_n) + \frac{1}{n} \log \eta_n \right) \right\}, \end{aligned}$$

where ξ_n is a point $\xi_{k,n} \in A$ for which

$$2U^{\nu_{P_n}}(\xi_n) - 2\mathscr{U}^{\nu_{Q_n}}(\xi_n) + \frac{1}{n}\log\eta_n = \min_{\xi_{k,n}\in A}\biggl(2U^{\nu_{P_n}}(\xi_{k,n}) - 2\mathscr{U}^{\nu_{Q_n}}(\xi_{k,n}) + \frac{1}{n}\log\eta_{n,k}\biggr),$$

and η_n is the corresponding point $\eta_{n,k}$.

Let $\xi \in A$ be a limit point of the sequence (ξ_n), $n \in \Lambda$; that is,

$$\lim_{n\in\Lambda'}\xi_n = \xi \quad (\neq 0) \quad \text{for some } \Lambda' \subset \Lambda.$$

Then by (6.15) and (6.19) together with the principle of descent,

$$\liminf_{n\in\Lambda'}\biggl(2U^{\nu_{P_n}}(\xi_n) - 2\mathscr{U}^{\nu_{Q_n}}(\xi_n) + \frac{1}{n}\log\eta_n\biggr) \geqslant 2U^{\lambda}(\xi) + \phi(\xi) \geqslant w(\sigma,\phi) - \varepsilon.$$

Consequently,

$$\limsup_{n\in\Lambda}\|Q_{n,2}\|_{2,n}^{2/n} \leqslant e^{-w(\sigma,\phi)+\varepsilon}.$$

Letting $\varepsilon \to 0$, we obtain (6.17).

Using the scheme employed in [20] to prove Lemmas 3.2 and 5.5, we now prove that

$$\liminf_{n\in\Lambda}\|Q_{n,2}\|_{2,n}^{2/n} \geqslant e^{-F_{\lambda_2^*}}, \tag{6.20}$$

where

$$F_{\lambda_2^*} = \max\{C \in \mathbb{R}\colon 2U^{\lambda_2^*}(x) + \phi(x) \geqslant C \text{ holds } (\sigma - \lambda_2^*)\text{-almost everywhere}\}.$$

Let $x_0 \in \operatorname{supp}(\sigma - \lambda_2^*) \setminus \{0\}$. Fix an ε with $0 < \varepsilon < 1/2$ small enough that $[x_0 - \varepsilon, x_0 + \varepsilon] \subset (-\infty, 0)$. Let $\Delta_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon)$. Now choose δ with $0 < \delta < \varepsilon$ and let $\Delta_\delta := (x_0 - \delta, x_0 + \delta)$. Choose $M > 0$ such that $-M < x_0 - \varepsilon - 1$. Define the polynomials

$$Q_{n,2}^{(1)}(x) := \prod_{y_{n,k}\in\Delta_\varepsilon}(x - y_{n,k}), \qquad Q_{n,2}^{(2)}(x) := \prod_{y_{n,k}\in[-M,0]\setminus\Delta_\varepsilon}(x - y_{n,k}),$$

$$Q_{n,2}^{(3)} := \frac{Q_{n,2}}{Q_{n,2}^{(1)}Q_{n,2}^{(2)}}.$$

Since $x_0 \in \operatorname{supp}(\sigma - \lambda_2^*)$, we have $q := (\sigma - \lambda_2^*)(\Delta_\delta) > 0$. Let ℓ_n be the number of zeros of $Q_{n,2}$ in Δ_δ and let m_n be the number of mass points $\xi_{k,n}$ in Δ_δ. Then $\lim_{n\to\infty}(m_n - \ell_n)/n = q$. Since the intervals

$$\biggl(\frac{\xi_{k,n} + \xi_{k-1,n}}{2}, \frac{\xi_{k,n} + \xi_{k+1,n}}{2}\biggr)$$

around the mass point $\xi_{k,n}$ are disjoint for all sufficiently large n, there exists at least one interval containing no zeros of $Q_{n,2}$ whose corresponding mass point is in Δ_δ. Denote this mass point by ξ_n^* and its adjacent mass points by $\xi_n^{(1)}$ and $\xi_n^{(2)}$. Using again the fact that there is at most one zero of $Q_{n,2}$ between two mass points of $\sigma_{2,n}$, we now find that

$$|Q_{n,2}^{(1)}(\xi_n^*)|^{1/n} \geqslant \left(\frac{|\xi_n^* - \xi_n^{(1)}|\,|\xi_n^* - \xi_n^{(1)}|}{4}\right)^{1/n} \Biggl(\prod_{\xi_n^* \neq \xi_{k,n} \in \Delta_\varepsilon} |\xi_n^* - \xi_{k,n}|\Biggr)^{1/n}$$
$$\geqslant \left(\frac{1}{4}\right)^{1/n} \Biggl(\prod_{\xi_n^* \neq \xi_{k,n} \in \Delta_\varepsilon} |\xi_n^* - \xi_{k,n}|\Biggr)^{2/n}.$$

Let p_n be the number of $\xi_{k,n} > \xi_n^*$ in Δ_ε, and let q_n be the number of $\xi_{k,n} < \xi_n^*$ in Δ_ε. Using (2.17), we have $\lim_{n\to\infty}(p_n + q_n)/n = \sigma(\Delta_\varepsilon)$. Let

$$\rho := \inf\{\rho(x) \colon x \in \Delta_\varepsilon\}.$$

The previous inequalities and the assumption (i) in § 2 imply that

$$|Q_{n,2}^{(1)}(\xi_n^*)|^{1/n} \geqslant \left(\frac{1}{4}\right)^{1/n} \left(\frac{\rho}{n}\right)^{2p_n/n} (p_n!)^{2/n} \left(\frac{\rho}{n}\right)^{2q_n/n} (q_n!)^{2/n}$$
$$\geqslant \left(\frac{1}{4}\right)^{1/n} \left(\frac{\rho}{n}\right)^{2(p_n+q_n)/n} \bigl((r_n - 1)!\bigr)^{2/n},$$

where r_n denotes the integer part of $(p_n + q_n)/2$. From this, using Stirling's formula, we easily deduce that

$$\liminf_{n\to\infty} |Q_{n,2}^{(1)}(\xi_n^*)|^{1/n} \geqslant \left(\frac{\rho\sigma(\Delta_\varepsilon)}{2e}\right)^{2\sigma(\Delta_\varepsilon)}. \tag{6.21}$$

Note that the right-hand side tends to 1 as $\varepsilon \to 0$.

We have

$$\|Q_{n,2}\|_{2,n}^{2/n} = \left(\sum_{k=1}^{\infty} |Q_{n,2}(\xi_{k,n})|^2 \,\frac{\beta_k \eta_{n,k}}{|\xi_{k,n}|}\, \frac{D_n}{|Q_n(\xi_{k,n})|}\right)^{1/n}$$
$$\geqslant \left(|Q_{n,2}(\xi_n^*)|^2 \,\frac{\beta_n^* \eta_n^*}{|\xi_n^*|}\, \frac{D_n}{|Q_n(\xi_n^*)|}\right)^{1/n}$$
$$\geqslant \left(\bigl|Q_{n,2}^{(1)}(\xi_n^*) Q_{n,2}^{(2)}(\xi_n^*)\bigr|^2 \,\frac{\beta_n^* \eta_n^*}{|\xi_n^*|}\, \frac{D_n}{|Q_n(\xi_n^*)|}\right)^{1/n}, \tag{6.22}$$

where β_n^* and η_n^* are the values of β_k and $\eta_{n,k}$ corresponding to $\xi_{k,n} = \xi_n^*$, respectively. In the last inequality we omitted $Q_{n,2}^{(3)}$, because all its zeros are at a distance greater than 1 from ξ_n^*. Let us find a lower bound for

$$\left(|Q_{n,2}^{(2)}(\xi_n^*)|^2 \,\frac{\beta_n^* \eta_n^*}{|\xi_n^*|}\, \frac{D_n}{|Q_n(\xi_n^*)|}\right)^{1/n}.$$

Since $\nu_{Q_{n,2}^{(2)}}$ converges vaguely to $\lambda_2^*\big|_{[-M,0]\setminus\Delta_\varepsilon}$, $n \in \Lambda$, the potential $U^{\nu_{Q_{n,2}^{(2)}}}$ converges uniformly on Δ_δ to $U^{\lambda_2^*|_{[-M,0]\setminus\Delta_\varepsilon}}$, $n \in \Lambda$, and the potential $U^{\lambda_2^*|_{[-M,0]}}$ is continuous on $\mathbb{R}_-$ (and in particular, at x_0; recall that $\xi_n^* \in \Delta_\delta$), for a given ε we can find δ with $0 < \delta < \varepsilon$ such that

$$\liminf_{n\in\Lambda} |Q_{n,2}^{(2)}(\xi_n^*)|^{2/n} \geqslant \exp\{-2U^{\lambda_2^*|_{[-M,0]}}(x_0) - 2\varepsilon\}. \tag{6.23}$$

Also, because of the continuity of ϕ and (6.15), δ can be chosen so that

$$|\phi(x) - \phi(x_0)| < \varepsilon, \qquad x \in \delta_\delta,$$

and for all sufficiently large $n \in \Lambda$ and k with $\xi_{k,n} \in \Delta_\delta$

$$\left|\left(\frac{\eta_{n,k} D_n}{|Q_n(\xi_{k,n})|}\right)^{1/n} - e^{-\phi(\xi_{k,n})}\right| < \varepsilon \tag{6.24}$$

(so (6.24) holds, in particular, for ξ_n^* and η_n^*). Since $\xi_n^* \in \Delta_\delta$, it follows that $\lim_{n\to\infty} |\xi_n^*|^{1/n} = 1$.

On the other hand, by the assumption (i) in § 2, if $\xi_{k,n} \in (x_0 - \delta, x_0 + \delta)$, then

$$k < \frac{n|x_0 - \delta|}{\rho^*}, \quad \text{where } \rho^* = \inf\{\rho(x)\colon x \in [x_0 - \delta, 0]\} > 0,$$

which together with the assumption (ii) in § 2 implies that $\liminf_{n\to\infty} |\beta_n^*|^{1/n} \geqslant 1$.

Using (6.21)–(6.24), we get that for all sufficiently small $\varepsilon > 0$ and sufficiently large $M > 0$

$$\liminf_{n\in\Lambda} \|Q_{n,2}\|_{2,n}^{2/n} \geqslant \left(\frac{\rho\sigma(\Delta_\varepsilon)}{2e}\right)^{2\sigma(\Delta_\varepsilon)} \exp\{-2U^{\lambda_2^*|_{[-M,0]}}(x_0) - \phi(x_0) - 4\varepsilon\}. \tag{6.25}$$

Now (6.17) and (6.25) imply that

$$2U^{\lambda_2^*|_{[-M,0]}}(x_0) + \phi(x_0) + 4\varepsilon - 2\sigma(\Delta_\varepsilon) \log \frac{\rho\sigma(\Delta_\varepsilon)}{2e} \geqslant w(\sigma, \phi) > -\infty. \tag{6.26}$$

Suppose that $\int \log(1 + y^2)\, d\lambda_2^*(y) = \infty$. In this case it is easy to prove that $U^{\lambda_2^*|_{[-M,0]}}(x_0)$ tends to $-\infty$ as $M \to +\infty$, which contradicts (6.26). Consequently, $\int \log(1 + y^2)\, d\lambda_2^*(y) < \infty$. In this case $U^{\lambda_2^*}$ is well defined on the whole of $\mathbb{C}$ and is continuous on $\mathbb{R}_-$, and moreover,

$$\lim_{M\to\infty} U^{\lambda_2^*|_{[-M,0]}}(x) = U^{\lambda_2^*}(x)$$

uniformly on any compact subset of $\mathbb{C}$. Letting $M \to \infty$ and $\varepsilon \to 0$, we get from (6.26) that

$$2U^{\lambda_2^*}(x_0) + \phi(x_0) \geqslant w(\sigma, \phi) = F_\lambda.$$

This holds for every $x_0 \in \operatorname{supp}(\sigma - \lambda_2^*) \setminus \{0\}$, and by continuity also at 0 if this is an accumulation point of $\operatorname{supp}(\sigma - \lambda_2^*)$. Consequently, $F_{\lambda_2^*} \geqslant F_\lambda$. From (D″) in Corollary 4.6 we conclude that $F_\lambda = w(\sigma, \phi) = F_{\lambda_2^*}$, therefore,

$$\lim_{n\in\Lambda} \|Q_{n,2}\|_{2,n}^{2/n} = e^{-F_{\lambda_2^*}}, \tag{6.27}$$

and $\lambda = \lambda_2^*$ according to the uniqueness statement in that part of Corollary 4.6.

Since

$$\int (1+y^2)\, d\lambda_1^*(y) < +\infty \quad \text{and} \quad \int (1+y^2)\, d\lambda_2^*(y) < +\infty,$$

we can rewrite (6.8) and (6.16) in terms of $U^{\lambda_1^*}$, $U^{\lambda_2^*}$, and ϕ as follows:

$$2U^{2\lambda_1^*}(x) - U^{\lambda_2^*}(x) + \varphi(x) \begin{cases} = 4w_1^* + \dfrac{1}{2}\displaystyle\int \log(1+y^2)\, d\lambda_2^*(y), & x \in \operatorname{supp}(\lambda_1^*), \\ \geqslant 4w_1^* + \dfrac{1}{2}\displaystyle\int \log(1+y^2)\, d\lambda_2^*(y), & x \in \mathbb{R}_+, \end{cases}$$

$$2U^{\lambda_2^*}(x) - U^{2\lambda_1^*}(x) \begin{cases} \leqslant 0, & x \in \operatorname{supp}(\lambda_2^*) = \mathbb{R}_-, \\ = 0, & x \in \operatorname{supp}(\sigma - \lambda_2^*). \end{cases}$$

Therefore, the pair $(2\lambda_1^*, \lambda_2^*)$ satisfies (2.15) and (2.16) in part (C) of Theorem 2.1. This means that $(2\lambda_1^*, \lambda_2^*) = (\lambda_1, \lambda_2)$ is the extremal solution in Theorem 2.1, and the extremal constants are

$$w_1 = 4w_1^* + \frac{1}{2}\int \log(1+y^2)\, d\lambda_2(y) \quad \text{and} \quad w_2 = 0.$$

In particular, $|\lambda_1^*| = |\lambda_2^*| = 1$ and, as explained in the beginning of the proof, (2.22) follows from (6.1). With this we conclude the proof of Theorem 2.2.

Remark 6.1. From (6.9) and (6.27) we also have

$$\lim_n \left(\int \frac{|Q_n(x)|^2}{|Q_{n,2}(x)|} C_n s_1'(d_n x)\, dx \right)^{1/n} = e^{-4w_1^*} \quad \text{and} \quad \lim_n \|Q_{n,2}\|_{2,n}^{2/n} = e^{-F_{\lambda_2^*}}, \tag{6.28}$$

where $F_{\lambda_2^*} = 4w_1^* - \dfrac{1}{2}\displaystyle\int \log(1+y^2)\, d\lambda_1(y)$ (see (6.18)). A direct computation gives

$$\|Q_{n,2}\|^{2/n} = (D_n C_n)^{1/n} \left| \int \frac{Q_{n,2}^2(t)}{Q_n(t)} \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \frac{\sigma_1'(d_n x)\, dx}{x-t}\, d\sigma_{2,n}(t) \right|^{1/n}.$$

Therefore, using (6.28), we could establish that

$$\lim_n \left| \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \sigma_1'(d_n x)\, dx \right|^{1/n} = e^{-w_1} \tag{6.29}$$

and

$$\lim_n \left| \int \frac{Q_{n,2}^2(t)}{Q_n(t)} \int \frac{Q_n^2(x)}{Q_{n,2}(x)} \frac{\sigma_1'(d_n x)\, dx}{x-t}\, d\sigma_{2,n}(t) \right|^{1/n} = e^{-w_1}, \tag{6.30}$$

where w_1 is the corresponding equilibrium constant in (2.15) (here $w_2=0$), if we could prove that

$$\lim_n C_n^{1/n}=\exp\biggl\{\frac{1}{2}\int\log(1+y^2)\,d\lambda_2(y)\biggr\}$$

and

$$\lim_n D_n^{1/n}=\exp\biggl\{\frac{1}{2}\int\log(1+y^2)\,d\lambda_1(y)\biggr\}.$$

In order to do this, it is necessary to obtain some bound on the rate of growth of the largest zeros of the polynomials Q_n and $Q_{n,2}$.

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Alexander I. Aptekarev
Federal Research Centre
Keldysh Institute of Applied Mathematics
of Russian Academy of Sciences, Moscow
E-mail: aptekaa@keldysh.ru

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Translated by THE AUTHORS

Guillermo López Lagomasino
Carlos III University of Madrid, Madrid, Spain
E-mail: lago@math.uc3m.es

Andrei Martínez-Finkelshtein
Universidad de Almería, Almería, Spain
E-mail: andrei@ual.es