A. B. Aleksandrov, V. V. Peller, Operator Lipschitz functions, *Russian Mathematical Surveys*, 2016, Volume 71, Issue 4, 605–702

DOI: 10.1070/RM9729

Russian Math. Surveys **71**:4 605–702 *Uspekhi Mat. Nauk* **71**:4 3–106

DOI 10.1070/RM9729

Operator Lipschitz functions

A. B. Aleksandrov and V. V. Peller

Abstract. The goal of this survey is a comprehensive study of operator Lipschitz functions. A continuous function f on the real line $\mathbb{R}$ is said to be operator Lipschitz if $\|f(A)-f(B)\| \leqslant \text{const}\|A-B\|$ for arbitrary self-adjoint operators A and B. Sufficient conditions and necessary conditions are given for operator Lipschitzness. The class of operator differentiable functions on $\mathbb{R}$ is also studied. Further, operator Lipschitz functions on closed subsets of the plane are considered, and the class of commutator Lipschitz functions on such subsets is introduced. An important role for the study of such classes of functions is played by double operator integrals and Schur multipliers.

Bibliography: 77 titles.

Keywords: functions of operators, operator Lipschitz functions, operator differentiable functions, self-adjoint operators, normal operators, divided differences, double operator integrals, Schur multipliers, linear-fractional transformations, Besov classes, Carleson measures.

Contents

The first author was supported by the Russian Foundation for Basic Research (grant no. 14-01-00198), and the second author by the National Science Foundation (grant no. DMS-130092).

AMS 2010 *Mathematics Subject Classification*. Primary 26A16, 47A56; Secondary 47B15.

1. Introduction

One of the most important problems in perturbation theory is the study of how much a function $f(A)$ of an operator A can change under small perturbations of the operator. In particular, in a natural way the problem arises of describing the class of continuous functions f on the real line $\mathbb{R}$ such that

$$\|f(A) - f(B)\| \leqslant \mathrm{const}\,\|A - B\| \tag{1.1}$$

for arbitrary (bounded) self-adjoint operators A and B on a Hilbert space. Such functions are said to be *operator Lipschitz*. We recall that functions of self-adjoint (normal) operators are defined as the integrals of these functions with respect to the spectral measures of the operators (see [68]).

We will denote the class of operator Lipschitz functions on $\mathbb{R}$ by $\mathrm{OL}(\mathbb{R})$. If f is an operator Lipschitz function, then the inequality (1.1) also holds for unbounded self-adjoint operators A and B with bounded difference (see Theorem 3.2.1 below), and moreover, the constant on the right-hand side remains the same. The minimal value of this constant is, by definition, the norm $\|f\|_{\mathrm{OL}} = \|f\|_{\mathrm{OL}(\mathbb{R})}$ of the function f in the space $\mathrm{OL}(\mathbb{R})$ (strictly speaking, it is a seminorm that becomes a norm after the identification of functions that differ from each other by a constant function).

Clearly, if f is an operator Lipschitz function, then it is *Lipschitz*, that is,

$$|f(x) - f(y)| \leqslant \mathrm{const}\,|x - y|$$

for any real x and y (we use the notation $\mathrm{Lip}(\mathbb{R})$ for the class of Lipschitz functions on $\mathbb{R}$). The converse is false. In [25] Farforovskaya constructed an example of a Lipschitz function that is not operator Lipschitz. Later it was shown in [45] and [34] that the Lipschitz function $x \mapsto |x|$ is not operator Lipschitz.

Operator Lipschitz functions play an important role in operator theory and mathematical physics. In particular, they appear when studying the applicability of the Lifshits–Krein trace formula:

$$\operatorname{trace}\bigl(f(A) - f(B)\bigr) = \int_{\mathbb{R}} f'(t)\xi(t)\,dt \tag{1.2}$$

(see [41]). Here A and B are self-adjoint operators acting in a Hilbert space such that $A - B$ is a trace class operator (that is, $A - B \in \boldsymbol{S}_1$) and ξ is a function of class $L^1(\mathbb{R})$ (the *spectral shift function*) which is determined only by A and B. Obviously, the right-hand side of (1.2) makes sense for an arbitrary Lipschitz function f. As for the left-hand side, the conditions $A - B \in \boldsymbol{S}_1$ and $f \in \operatorname{Lip}(\mathbb{R})$ do not guarantee that $f(A) - f(B) \in \boldsymbol{S}_1$, as seen from the example of Farforovskaya in [26]. Thus, for applicability of the trace formula (1.2) for all pairs of self-adjoint operators with trace class difference, one has to impose a stronger condition on f. At the least, f must have the following property:

$$A - B \in \boldsymbol{S}_1 \quad \Rightarrow \quad f(A) - f(B) \in \boldsymbol{S}_1 \tag{1.3}$$

for self-adjoint operators A and B. For a function f on $\mathbb{R}$ the property (1.3) holds for any (not necessarily bounded) self-adjoint operators if and only if f is operator Lipschitz (see Theorem 3.6.5 below). It turns out (see the recent paper [64]) that the operator Lipschitzness of f is not only necessary for the validity of the trace formula (1.2) for any (not necessarily bounded) self-adjoint operators A and B with trace class difference, but also sufficient.

The class of operator Lipschitz functions possesses certain specific properties. For example, operator Lipschitz functions must be differentiable everywhere, but not necessarily continuously differentiable (see Theorem 3.3.3 and Example 7 in § 1.1).

It turns out that operator Lipschitzness can be characterized in terms of Schur multipliers (see § 3.3). We will see that a continuous function f on $\mathbb{R}$ is operator Lipschitz if and only if it is differentiable everywhere and the divided difference $\mathfrak{D}f$,

$$(\mathfrak{D}f)(x,y) \stackrel{\text{def}}{=} \frac{f(x) - f(y)}{x - y}, \qquad x, y \in \mathbb{R},$$

is a Schur multiplier.

Similarly, one can consider the same problem for functions on the circle and for unitary operators. A continuous function f on the unit circle $\mathbb{T}$ is said to be *operator Lipschitz* if $\|f(U) - f(V)\| \leqslant \operatorname{const} \|U - V\|$ for any unitary operators U and V.

In Chapter I below we discuss necessary conditions and sufficient conditions for functions on the line $\mathbb{R}$ and on the circle $\mathbb{T}$ to be operator Lipschitz. In the case of self-adjoint operators a key role is played by the inequality

$$\|f(A) - f(B)\| \leqslant \operatorname{const} \sigma \|f\|_{L^\infty} \|A - B\| \tag{1.4}$$

for any self-adjoint operators A and B with bounded difference and any bounded function f on $\mathbb{R}$ whose Fourier transform is supported in $[-\sigma, \sigma]$, $\sigma > 0$. This inequality was obtained in [56] and [58]. Later, it was shown in [10] that (1.4) holds with the constant 1.

By analogy with operator Lipschitz functions, it would be natural to consider operator Hölder functions. Let $0 < \alpha < 1$. We say that a function f on $\mathbb{R}$ is *operator Hölder of order* α if the inequality

$$\|f(A) - f(B)\| \leqslant \text{const}\, \|A - B\|^{\alpha}$$

holds for arbitrary self-adjoint operators A and B acting in a Hilbert space. However (see § 1.7), the situation here is quite different from the case of operator Lipschitz estimates: a function f is operator Hölder of order α if and only if it belongs to the class $\Lambda_\alpha(\mathbb{R})$ of *Hölder functions of order* α, that is, $|f(x) - f(y)| \leqslant \text{const}\, |x - y|^{\alpha}$, $x, y \in \mathbb{R}$.

In Chap. II we discuss double operator integrals, that is, expressions of the form

$$\iint \Phi(x, y)\, dE_1(x) T\, dE_2(y).$$

Here Φ is a bounded measurable function, T is a bounded linear operator on a Hilbert space, and E_1 and E_2 are spectral measures. Double operator integrals appeared in the paper [23] by Daletskii and S. G. Krein and were studied systematically by Birman and Solomyak in [19]–[21]. Already in these papers the important role that double operator integrals play in perturbation theory was noted. Double operator integrals are defined for arbitrary bounded linear operators T in the case when the function Φ is a *Schur multiplier* with respect to E_1 and E_2. In Chap. II we study the space of such Schur multipliers. We begin by studying the so-called *discrete Schur multipliers*, and then we use them to study Schur multipliers with respect to spectral measures.

Next, in Chap. III we consider the class $\mathrm{OL}(\mathfrak{F})$ of *operator Lipschitz functions on an arbitrary closed subset* $\mathfrak{F}$ of the complex plane $\mathbb{C}$, which consists of continuous functions f on $\mathfrak{F}$ such that

$$\|f(N_1) - f(N_2)\| \leqslant \text{const}\, \|N_1 - N_2\| \tag{1.5}$$

for any normal operators N_1 and N_2 whose spectra are contained in $\mathfrak{F}$. We also study in detail the class of *commutator Lipschitz functions* on $\mathfrak{F}$, that is, the class of continuous functions f on $\mathfrak{F}$ such that

$$\|f(N_1)R - Rf(N_2)\| \leqslant \text{const}\, \|N_1 R - R N_2\|$$

for any bounded linear operator R and any normal operators N_1 and N_2 with spectra in $\mathfrak{F}$. To study these classes of functions, we use results from Chap. II.

As in the case of self-adjoint operators, in the study of the class of operator Lipschitz functions on the whole plane a key role is played by the following generalization of the inequality (1.4):

$$\|f(N_1) - f(N_2)\| \leqslant \text{const}\, \sigma \|f\|_{L^\infty} \|N_1 - N_2\| \tag{1.6}$$

for any normal operators N_1 and N_2 with bounded difference and any bounded function f on $\mathbb{R}^2$ whose Fourier transform is supported in $[-\sigma,\sigma]\times[-\sigma,\sigma]$. We remark that the proof of the inequality (1.4) obtained in [56] and [58] cannot be generalized to the case of normal operators. A new method for obtaining such estimates was found in [14].

We also obtain a sufficient condition (found in [3]) for the commutator Lipschitzness of functions on a proper closed subset of the plane in terms of Cauchy integrals of measures on the complement of the set. We use this condition to deduce the Arazy–Bartman–Friedman sufficient condition [15] for commutator Lipschitzness of functions analytic in the disk, as well as its analogue for the upper half-plane.

Finally, in Chap. III we study properties of commutator Lipschitz functions on the unit circle $\mathbb{T}$ that admit analytic extensions to the unit disk $\mathbb{D}$; these results are grouped around the results of Kissin and Shulman in [39].

In the final section "Concluding remarks" we mention briefly certain results not covered in the survey.

The authors would like to express their sincere gratitude to V. S. Shulman for helpful remarks.

2. Preliminaries and notation

1. Besov classes. Let w be an infinitely differentiable function on $\mathbb{R}$ such that

$$w\geqslant 0,\quad \operatorname{supp} w\subset\left[\frac12,2\right],\quad\text{and}\quad w(s)=1-w\left(\frac{s}{2}\right)\quad\text{for } s\in[1,2]. \tag{2.1}$$

We define functions W_n, $n\in\mathbb{Z}$, on $\mathbb{R}^d$ by

$$(\mathscr{F}W_n)(x)=w\left(\frac{\|x\|}{2^n}\right),\qquad n\in\mathbb{Z},\quad x=(x_1,\dots,x_d),\quad \|x\|\stackrel{\text{def}}{=}\left(\sum_{j=1}^d x_j^2\right)^{1/2},$$

where $\mathscr{F}$ is the *Fourier transform* defined on $L^1(\mathbb{R}^d)$ by

$$(\mathscr{F}f)(t)=\int_{\mathbb{R}^d} f(x)e^{-\mathrm{i}(x,t)}\,dx,\qquad x=(x_1,\dots,x_d),$$
$$t=(t_1,\dots,t_d),\quad (x,t)\stackrel{\text{def}}{=}\sum_{j=1}^d x_jt_j.$$

Clearly, $\sum_{n\in\mathbb{Z}}(\mathscr{F}W_n)(t)=1$, $t\in\mathbb{R}^d\setminus\{0\}$.

With each tempered distribution f in $\mathscr{S}'(\mathbb{R}^d)$ we associate the sequence $\{f_n\}_{n\in\mathbb{Z}}$ with

$$f_n\stackrel{\text{def}}{=}f*W_n. \tag{2.2}$$

The formal series $\sum_{n\in\mathbb{Z}}f_n$, being a Paley–Wiener type expansion of f, does not necessarily converge to f. We first define the (homogeneous) Besov class $\dot B^s_{p,q}(\mathbb{R}^d)$ with $s\in\mathbb{R}$ and $0<p,q\leqslant\infty$ to be the space of distributions f such that

$$\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}}\in\ell^q(\mathbb{Z}),\quad \|f\|_{B^s_{p,q}}\stackrel{\text{def}}{=}\big\|\{2^{ns}\|f_n\|_{L^p}\}_{n\in\mathbb{Z}}\big\|_{\ell^q(\mathbb{Z})}. \tag{2.3}$$

In accordance with this definition, $\dot{B}^s_{p,q}(\mathbb{R}^d)$ contains all polynomials and $\|f\|_{B^s_{p,q}} = 0$ for every polynomial f. Moreover, a distribution f is uniquely determined by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ up to a polynomial. It is easy to see that the series $\sum_{n\geqslant 0} f_n$ converges in $\mathscr{S}'(\mathbb{R}^d)$.[1] However, the series $\sum_{n<0} f_n$ can diverge in general. Nevertheless, it can be proved that the series

$$\sum_{n<0}\frac{\partial^r f_n}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} \qquad \text{for} \quad r_j\geqslant 0, \quad 1\leqslant j\leqslant d, \quad \sum_{j=1}^d r_j = r, \tag{2.4}$$

converge uniformly on $\mathbb{R}^d$ for $r\in\mathbb{Z}_+$ and $r > s - d/p$. We note that for $q\leqslant 1$ the series in (2.4) converge uniformly under the weaker assumption that $r\geqslant s-d/p$.

We can now define the modified (homogeneous) Besov space $B^s_{p,q}(\mathbb{R}^d)$: $f\in B^s_{p,q}(\mathbb{R}^d)$ if (2.3) holds and

$$\frac{\partial^r f}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} = \sum_{n\in\mathbb{Z}}\frac{\partial^r f_n}{\partial x_1^{r_1}\cdots\partial x_d^{r_d}} \quad \text{for} \quad r_j\geqslant 0, \quad 1\leqslant j\leqslant d, \quad \sum_{j=1}^d r_j = r,$$

in the space $\mathscr{S}'(\mathbb{R}^d)$, where r is the minimal non-negative integer such that $r > s - d/p$ ($r\geqslant s-d/p$ if $q\leqslant 1$). Now f is uniquely determined by the sequence $\{f_n\}_{n\in\mathbb{Z}}$ up to a polynomial of degree less than r. Also, a polynomial g belongs to $B^s_{p,q}(\mathbb{R}^d)$ if and only if $\deg g < r$.

In the case $p=q$ we use the notation $B^s_p(\mathbb{R}^d)$ for $B^s_{p,p}(\mathbb{R}^d)$.

We now consider the scale $\Lambda_\alpha(\mathbb{R}^d)$, $\alpha>0$, of *Hölder–Zygmund classes*. They can be defined by $\Lambda_\alpha(\mathbb{R}^d)\stackrel{\text{def}}{=}B^\alpha_\infty(\mathbb{R}^d)$.

The Besov classes admit many other descriptions. We give the one in terms of finite differences. For h in $\mathbb{R}^d$ we define the difference operator Δ_h by $(\Delta_h f)(x) = f(x+h) - f(x)$, $x\in\mathbb{R}^d$.

Let $s>0$, $m\in\mathbb{Z}$, and $m-1\leqslant s<m$. For $p,q\in[1,+\infty]$ the Besov class $B^s_{p,q}(\mathbb{R}^d)$ can be defined as the set of functions f in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}|h|^{-d-sq}\|\Delta_h^m f\|^q_{L^p}\,dh<\infty, \quad q<\infty; \qquad \sup_{h\neq 0}\frac{\|\Delta_h^m f\|_{L^p}}{|h|^s}<\infty, \quad q=\infty.$$

However, with this definition the Besov classes can contain polynomials of degree higher than in the case of the definition in terms of convolutions with the functions W_n.

The space B^s_{pq} can be defined in terms of the Poisson integral. Let $P_d(x,t)$ be the Poisson kernel on $\mathbb{R}^{d+1}_+\stackrel{\text{def}}{=}\{(x,t)\colon x\in\mathbb{R}^d,\ t>0\}$, that is, $P_d(x,t) = c_d t(|x|^2+t^2)^{-(d+1)/2}$, where $c_d=\pi^{-(d+1)/2}\Gamma((d+1)/2)$. With each function f in $L^1\big(\mathbb{R}^d,(\|x\|+1)^{-(d+1)}\,dx\big)$ we can associate the Poisson integral $\mathcal{P}f$,

$$(\mathcal{P}f)(x,t)=\int_{\mathbb{R}^d}P_d(x-y,t)f(y)\,dy.$$

[1]Here and in what follows we assume that the space $\mathscr{S}'(\mathbb{R}^d)$ is equipped with the weak topology $\sigma\big(\mathscr{S}'(\mathbb{R}^d),\mathscr{S}(\mathbb{R}^d)\big)$.

Then for every positive integer m,

$$\frac{\partial^m(\mathcal{P}f)}{\partial t^m}(x,t)=\int_{\mathbb{R}^d}\frac{\partial^m P_d(x-y,t)}{\partial t^m}f(y)\,dy. \tag{2.5}$$

Note that the integral in (2.5) makes sense for all $f\in L^1\big(\mathbb{R}^d,(\|x\|+1)^{-(d+m+1)}\,dx\big)$, which lets us define $\dfrac{\partial^m}{\partial t^m}\mathcal{P}f$.

Let $m\in\mathbb{Z}$, $m-1\leqslant s<m$, and $1\leqslant p,q\leqslant+\infty$. We can define B^s_{pq} as the set of functions $f\in L^1\big(\mathbb{R}^d,(\|x\|+1)^{-(d+m+1)}\,dx\big)$ such that

$$\left(\int_0^\infty t^{(m-s)q-1}\left\|\left(\frac{\partial^m}{\partial t^m}\mathcal{P}f\right)(\,\cdot\,,t)\right\|^q_{L^p(\mathbb{R}^d)}dt\right)^{1/q}<+\infty,\qquad q<+\infty,$$

$$\sup_{t>0}t^{m-s}\left\|\left(\frac{\partial^m}{\partial t^m}\mathcal{P}f\right)(\,\cdot\,,t)\right\|_{L^p(\mathbb{R}^d)}<+\infty,\qquad q=+\infty.$$

It is true that also with this definition Besov classes can contain polynomials of degree higher than in the case of the definition in terms of the convolutions with W_n. We note further that this definition in terms of the Poisson integral can also be used under certain provisions in the case when $p<1$ or $q<1$.

We now proceed to *Besov classes of functions on the unit circle* $\mathbb{T}$. Let w be a function satisfying (2.1). We define the trigonometric polynomials W_n, $n\geqslant 0$, by

$$W_n(\zeta)\stackrel{\text{def}}{=}\sum_{j\in\mathbb{Z}}w\left(\frac{|j|}{2^n}\right)\zeta^j,\quad n\geqslant 1,\quad\text{and}\quad W_0(\zeta)\stackrel{\text{def}}{=}\sum_{\{j:\,|j|\leqslant 1\}}\zeta^j,\qquad \zeta\in\mathbb{T}.$$

If f is a distribution on $\mathbb{T}$, then we put $f_n=f*W_n$ for $n\geqslant 0$ and say that f belongs to the Besov class $B^s_{p,q}(\mathbb{T})$ with $s\in\mathbb{R}$ and $0<p,q\leqslant\infty$ if

$$\big\{2^{ns}\|f_n\|_{L^p}\big\}_{n\geqslant 0}\in\ell^q. \tag{2.6}$$

Let $s\in\mathbb{R}$ with $s>\max\{0,1/p-1\}$, and let m be a positive integer such that $m>\max\{s,s+1/p-1\}$. Then a distribution f on $\mathbb{T}$ belongs to $B^s_{p,q}(\mathbb{T})$ if and only if

$$\int_0^1 r(1-r^2)^{(m-s)q-1}\left\|\frac{\partial^m}{\partial r^m}\big((\mathcal{P}f)(r\zeta)\big)\right\|^q_{L^p(\mathbb{T})}dr<+\infty,\qquad q<+\infty,$$

$$\sup_{r\in[0,1)}(1-r^2)^{m-s}\left\|\frac{\partial^m}{\partial r^m}\big((\mathcal{P}f)(r\zeta)\big)\right\|_{L^p(\mathbb{T})}<+\infty,\qquad q=+\infty,$$

where $\mathcal{P}f$ denotes the Poisson integral of the distribution f.

In the definitions of Besov classes in terms of the Poisson integral we considered the mth derivative with respect to the variable t in the first case and with respect to the variable r in the second case. It is well known that in both cases we would get an equivalent definition if we required that the analogous expressions for all the partial derivatives (including mixed) of order m be finite.

We refer the reader to [52] and [74] for more detailed information about the Besov classes.

2. Schatten–von Neumann classes. The *singular values* $s_j(T)$ $(j \geqslant 0)$ of a bounded linear operator T on a Hilbert space are defined by

$$s_j(T) \stackrel{\text{def}}{=} \inf\bigl\{\|T-R\|\colon\ \operatorname{rank} R \leqslant j\bigr\}.$$

The *Schatten–von Neumann class* $\boldsymbol{S}_p$ with $0<p<\infty$ consists by definition of the operators T for which

$$\|T\|_{\boldsymbol{S}_p} \stackrel{\text{def}}{=} \biggl(\sum_{j\geqslant 0}\bigl(s_j(T)\bigr)^p\biggr)^{1/p} < \infty.$$

For $p \geqslant 1$ this is a normed ideal of operators on a Hilbert space. The class $\boldsymbol{S}_1$ is called the *trace class*. If T is a trace class operator on a Hilbert space $\mathscr{H}$, then its *trace* $\operatorname{trace} T$ is defined by $\operatorname{trace} T \stackrel{\text{def}}{=} \sum_{j\geqslant 0}(Te_j, e_j)$, where $\{e_j\}_{j\geqslant 0}$ is an orthonormal basis in $\mathscr{H}$. The right-hand side does not depend on the choice of a basis.

The class $\boldsymbol{S}_2$ is called the *Hilbert–Schmidt class*. It forms a Hilbert space with inner product $(T,R)_{\boldsymbol{S}_2} \stackrel{\text{def}}{=} \operatorname{trace}(TR^*)$.

For $p \in (1,\infty)$ the dual space $(\boldsymbol{S}_p)^*$ can be identified isometrically with the space $\boldsymbol{S}_{p'}$ with $1/p+1/p'=1$ via the bilinear form $\langle T,R\rangle \stackrel{\text{def}}{=} \operatorname{trace}(TR)$. The space dual to $\boldsymbol{S}_1$ can be identified with the space of bounded linear operators via the same bilinear form, while the space dual to the space of compact operators can be identified with $\boldsymbol{S}_1$.

We refer the reader to [28] for more detailed information.

3. Hankel operators. For a function φ of class L^2 on the unit circle $\mathbb{T}$, the *Hankel operator* H_φ is defined on the dense subset of polynomials in the Hardy class H^2 by $H_\varphi f \stackrel{\text{def}}{=} \mathbb{P}_-\varphi f$, where $\mathbb{P}_-$ is the orthogonal projection from L^2 onto $H^2_- \stackrel{\text{def}}{=} L^2 \ominus H^2$. By Nehari's theorem, H_φ extends to a bounded operator from H^2 to H^2_- if and only if there exists a function ψ of class L^∞ on $\mathbb{T}$ whose Fourier coefficients $\widehat{\psi}(n)$ satisfy the equality $\widehat{\psi}(n) = \widehat{\varphi}(n)$ for $n<0$. The last property, in turn, is equivalent by Ch. Fefferman's theorem to the condition that $\mathbb{P}_-\varphi$ belongs to the class BMO.

A Hankel operator H_φ belongs to the Schatten–von Neumann class $\boldsymbol{S}_p$ if and only if the function $\mathbb{P}_-\varphi$ belongs to the Besov class $B_p^{1/p}(\mathbb{T})$. For $p \geqslant 1$ this was proved in [54], and for $p \in (0,1)$ in [55] (see also [53] and [69], where other proofs are given for $p<1$).

It is easy to see that the operator H_φ has the matrix $\{\widehat{\varphi}(-j-k-1)\}_{j\geqslant 0,\ k>1}$ in the bases $\{z^j\}_{j\geqslant 0}$ and $\{\overline{z}^k\}_{k\geqslant 1}$. Such matrices, that is, matrices of the form $\{\alpha_{j+k}\}_{j,k\geqslant 0}$, are called *Hankel matrices*. The criterion for Hankel operators to belong to $\boldsymbol{S}_p$ can be reformulated as follows: *an operator on ℓ^2 with Hankel matrix $\{\alpha_{j+k}\}_{j,k\geqslant 0}$ belongs to $\boldsymbol{S}_p$ with $p>0$ if and only if the function $\sum_{j\geqslant 0}\alpha_j z^j$ belongs to $B_p^{1/p}(\mathbb{T})$.*

We refer the reader to the monograph [60] for proofs of the above results and for more detailed information on Hankel operators.

4. Notation. Here is a list of some of the notation used in this survey.

- $\mathrm{OL}(\mathfrak{F})$ is the space of operator Lipschitz functions on a closed subset $\mathfrak{F}$ of the complex plane $\mathbb{C}$;
- $\mathrm{CL}(\mathfrak{F})$ is the space of commutator Lipschitz functions on a closed subset $\mathfrak{F}$ of $\mathbb{C}$;
- $\mathrm{OD}(\mathbb{R})$ is the space of operator differentiable functions on $\mathbb{R}$;
- $\mathcal{B}(\mathscr{H}_1,\mathscr{H}_2)$ is the space of bounded linear operators from a Hilbert space $\mathscr{H}_1$ to a Hilbert space $\mathscr{H}_2$, and $\mathcal{B}(\mathscr{H})\stackrel{\mathrm{def}}{=}\mathcal{B}(\mathscr{H},\mathscr{H})$;
- $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ is the space of bounded self-adjoint operators on a Hilbert space $\mathscr{H}$;
- $\boldsymbol{m}$ is normalized Lebesgue measure on the circle $\mathbb{T}$;
- $\boldsymbol{m}_2$ is Lebesgue measure on the plane.

Chapter I. Operator Lipschitz functions on the line and on the circle. The first round

In this introductory chapter we consider operator Lipschitz functions on the real line $\mathbb{R}$ and on the unit circle $\mathbb{T}$. Later, in Chap. III, we will subject the class of operator Lipschitz functions to a more detailed study, and we will also study operator Lipschitz functions on closed subsets of the complex plane $\mathbb{C}$.

We use the notation $\mathrm{OL}(\mathbb{R})$ for the class of operator Lipschitz functions on $\mathbb{R}$, and for $f\in\mathrm{OL}(\mathbb{R})$ we put

$$\|f\|_{\mathrm{OL}(\mathbb{R})}\stackrel{\mathrm{def}}{=}\sup\left\{\frac{\|f(A)-f(B)\|}{\|A-B\|}: A \text{ and } B \text{ are self-adjoint operators, } A\ne B\right\}.$$

Similarly, we introduce the space $\mathrm{OL}(\mathbb{T})$ of operator Lipschitz functions on $\mathbb{T}$, replacing self-adjoint operators by unitary operators.

It turns out that the class $\mathrm{OL}(\mathbb{R})$ has somewhat unusual properties. In particular, functions in this class must be differentiable everywhere on $\mathbb{R}$ and also must have a derivative at infinity, that is, the limit $\lim_{|t|\to\infty}f(t)/t$ must exist (see Theorem 3.3.3 below). This implies the McIntosh–Kato result mentioned in the Introduction: the function $x\mapsto|x|$ is not operator Lipschitz. On the other hand, functions of class $\mathrm{OL}(\mathbb{R})$ do not have to be continuously differentiable. In particular, the function $x\mapsto x^2\sin(1/x)$, while not continuously differentiable, is operator Lipschitz (see Theorem 1.1.4 below).

We begin this chapter with elementary examples of operator Lipschitz functions (see § 1.1).

In § 1.2 we introduce the class of operator differentiable functions and the class of locally operator differentiable functions. It turns out that for the definition of these classes, it does not matter whether we consider differentiability in the sense of Gâteaux or in the sense of Fréchet. We will see that (locally) operator differentiable functions must be continuously differentiable and that operator differentiable functions must be operator Lipschitz. However, not every operator Lipschitz function is operator differentiable.

Besides operator Lipschitz functions, we consider in § 1.3 *commutator Lipschitz functions*, that is, functions f on $\mathbb{R}$ such that

$$\|f(A)R-Rf(B)\|\leqslant\mathrm{const}\,\|AR-RB\|$$

for any self-adjoint operators A and B (again, no matter whether bounded or not) and any bounded linear operator R. The *commutator Lipschitz norm* $\|f\|_{\mathrm{CL}(\mathbb{R})}$ of f is defined as the minimal constant for which the inequality holds. Similarly, we can define commutator Lipschitz functions on the unit circle if instead of self-adjoint operators we consider unitary operators.

It turns out that for functions *on the line* (*as well as for functions on the circle*) *the class of commutator Lipschitz functions coincides with the class of operator Lipschitz functions.* In Chap. III we will see that for functions on an arbitrary closed subset of the plane $\mathbb{R}^2$ this is no longer true.

In this chapter we obtain a sufficient condition for operator Lipschitzness on the line and on the circle (see § 1.6) as well as a necessary condition (see § 1.5), and we compare them.

It would also be natural to consider the class of *operator Hölder functions of order* α with $0 < \alpha < 1$, that is, the class of functions f such that

$$\|f(A) - f(B)\| \leqslant \mathrm{const}\, \|A - B\|^\alpha$$

for self-adjoint operators A and B acting in a Hilbert space. However, this term turned out to be short-lived, because *every function f on $\mathbb{R}$ of Hölder class of order α is necessarily operator Hölder of order α* (see § 1.7).

1.1. Elementary examples of operator Lipschitz functions

In this section we give examples of operator Lipschitz functions on the line and circle and obtain simple sufficient conditions for operator Lipschitzness.

Example 1. For every λ in $\mathbb{C}\setminus\mathbb{R}$ the function $(\lambda-x)^{-1}$ is operator Lipschitz on $\mathbb{R}$, and $\|(\lambda - x)^{-1}\|_{\mathrm{OL}(\mathbb{R})} = |\operatorname{Im}\lambda|^{-2}$.

Proof. The Hilbert resolvent identity

$$(\lambda I - A)^{-1} - (\lambda I - B)^{-1} = (\lambda I - A)^{-1}(A - B)(\lambda I - B)^{-1}$$

immediately implies that $\|(\lambda - x)^{-1}\|_{\mathrm{OL}(\mathbb{R})} \leqslant |\operatorname{Im}\lambda|^{-2}$. It remains to observe that $\|(\lambda - x)^{-1}\|_{\mathrm{OL}(\mathbb{R})} \geqslant \|(\lambda - x)^{-1}\|_{\mathrm{Lip}(\mathbb{R})} = |\operatorname{Im}\lambda|^{-2}$. $\square$

Example 1′. For every λ in $\mathbb{C}\setminus\mathbb{T}$ the function $(\lambda - z)^{-1}$ is operator Lipschitz on $\mathbb{T}$, and $\|(\lambda - z)^{-1}\|_{\mathrm{OL}(\mathbb{T})} = (|\lambda| - 1)^{-2}$.

Example 2. The function $x \mapsto \log(1 + \mathrm{i}x)$ is operator Lipschitz on $\mathbb{R}$, and $\|\log(1 + \mathrm{i}x)\|_{\mathrm{OL}(\mathbb{R})} = 1$. Here log means the principal branch of the logarithm.

Proof. Clearly, $\log(1 + \mathrm{i}x) = \int_0^{+\infty}\left(\dfrac{1}{1+t} - \dfrac{1}{1+t+\mathrm{i}x}\right) dt$. It follows that

$$\begin{aligned}\|\log(1 + \mathrm{i}x)\|_{\mathrm{OL}(\mathbb{R})} &\leqslant \int_0^{+\infty}\left\|\frac{1}{1+t} - \frac{1}{1+t+\mathrm{i}x}\right\|_{\mathrm{OL}(\mathbb{R})} dt \\ &= \int_0^{+\infty}\left\|\frac{1}{1+t+\mathrm{i}x}\right\|_{\mathrm{OL}(\mathbb{R})} dt = \int_0^{+\infty}\frac{dt}{(1+t)^2} = 1.\end{aligned}$$

On the other hand, the inequality $\|\log(1 + \mathrm{i}x)\|_{\mathrm{OL}(\mathbb{R})} \geqslant 1$ is obvious because $\|\log(1 + \mathrm{i}x)\|_{\mathrm{OL}(\mathbb{R})} \geqslant \|\log(1 + \mathrm{i}x)\|_{\mathrm{Lip}(\mathbb{R})} = 1$. $\square$

In a similar way one can prove that for every λ in $\mathbb{C}\setminus\mathbb{R}$ we have the equality $\|\log(\lambda-x)\|_{\mathrm{OL}(\mathbb{R})}=|\operatorname{Im}\lambda|^{-1}$, where $\log(\lambda-x)$ denotes any regular branch of the function $\log(\lambda-z)$ on $\mathbb{R}$.

Example 3. The function arctan is operator Lipschitz and $\|\arctan\|_{\mathrm{OL}(\mathbb{R})}=1$.

Proof. It suffices to verify that $\|\arctan\|_{\mathrm{OL}(\mathbb{R})}\leqslant 1$. To this end, we observe that $\arctan x=\operatorname{Im}\log(1+\mathrm{i}x)$, $x\in\mathbb{R}$. □

Example 4. For every positive integer n, $\|(\lambda-x)^{-n}\|_{\mathrm{OL}(\mathbb{R})}=n|\operatorname{Im}\lambda|^{-n-1}$ for all $\lambda\in\mathbb{C}\setminus\mathbb{R}$.

Proof. Substituting $X=(\lambda I-A)^{-1}$ and $Y=(\lambda I-B)^{-1}$ in the elementary identity

$$X^n-Y^n=\sum_{k=1}^{n}X^{n-k}(X-Y)Y^{k-1}, \tag{1.1.1}$$

we obtain

$$(\lambda I-A)^{-n}-(\lambda I-B)^{-n}=\sum_{k=1}^{n}(\lambda I-A)^{k-n}\big((\lambda I-A)^{-1}-(\lambda I-B)^{-1}\big)(\lambda I-B)^{1-k}.$$

Therefore, for any self-adjoint operators A and B

$$\begin{aligned}&\|(\lambda I-A)^{-n}-(\lambda I-B)^{-n}\|\\ &\qquad\leqslant\sum_{k=1}^{n}\|(\lambda I-A)^{k-n}\|\cdot\big\|(\lambda I-A)^{-1}-(\lambda I-B)^{-1}\big\|\cdot\|(\lambda I-B)^{1-k}\|\\ &\qquad\leqslant\sum_{k=1}^{n}|\operatorname{Im}\lambda|^{k-n}|\operatorname{Im}\lambda|^{-2}\|A-B\|\cdot|\operatorname{Im}\lambda|^{1-k}=n|\operatorname{Im}\lambda|^{-n-1}\|A-B\|.\end{aligned}$$

Thus, we have proved that $\|(\lambda I-x)^{-n}\|_{\mathrm{OL}(\mathbb{R})}\leqslant n|\operatorname{Im}\lambda|^{-n-1}$. It remains to observe that $\|(\lambda-x)^{-n}\|_{\mathrm{OL}(\mathbb{R})}\geqslant\|(\lambda-x)^{-n}\|_{\mathrm{Lip}(\mathbb{R})}=n|\operatorname{Im}\lambda|^{-n-1}$. □

Example 5. The function $x\mapsto e^{\mathrm{i}ax}$, $a\in\mathbb{R}$, is operator Lipschitz, and $\|e^{\mathrm{i}ax}\|_{\mathrm{OL}(\mathbb{R})}=|a|$.

Proof. Again, it suffices to establish only the upper estimate. We can assume that $a=1$. Let A and B be self-adjoint operators. Then

$$\big(e^{\mathrm{i}tA}e^{-\mathrm{i}tB}\big)'=\mathrm{i}Ae^{\mathrm{i}tA}e^{-\mathrm{i}tB}-\mathrm{i}e^{\mathrm{i}tA}e^{-\mathrm{i}tB}B=\mathrm{i}e^{\mathrm{i}tA}(A-B)e^{-\mathrm{i}tB},$$

whence

$$\begin{aligned}\|e^{\mathrm{i}A}-e^{\mathrm{i}B}\|&=\|e^{\mathrm{i}A}e^{-\mathrm{i}B}-I\|=\left\|\mathrm{i}\int_0^1e^{\mathrm{i}tA}(A-B)e^{-\mathrm{i}tB}\,dt\right\|\\ &\leqslant\int_0^1\big\|e^{\mathrm{i}tA}(A-B)e^{-\mathrm{i}tB}\big\|\,dt=\int_0^1\|A-B\|\,dt=\|A-B\|.\end{aligned}$$ □

In all the above examples we have the equality $\|f\|_{\mathrm{OL}(\mathbb{R})}=\|f'\|_{L^\infty(\mathbb{R})}$, which is rather an exception than a rule.

Example 5 immediately implies the following assertion.

Theorem 1.1.1. *Let f be a primitive of the Fourier transform $\mathscr{F}\mu$ of a complex Borel measure μ on $\mathbb{R}$. Then $f \in \mathrm{OL}(\mathbb{R})$ and $\|f\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|\mu\|$.*

Proof. We can assume that $f(0) = 0$. Then

$$f(x) = \int_0^x (\mathscr{F}\mu)(t)\,dt = \int_0^x \left(\int_{\mathbb{R}} e^{-\mathrm{i}st}\,d\mu(s)\right)dt$$
$$= \int_0^1 \left(\int_{\mathbb{R}} xe^{-\mathrm{i}stx}\,d\mu(s)\right)dt = \mathrm{i}\int_{\mathbb{R}} \frac{e^{-\mathrm{i}sx}-1}{s}\,d\mu(s).$$

Hence, $\|f\|_{\mathrm{OL}(\mathbb{R})} \leqslant \displaystyle\int_{\mathbb{R}} \left\|\frac{e^{-\mathrm{i}sx}-1}{s}\right\|_{\mathrm{OL}(\mathbb{R})} d|\mu|(s) \leqslant \int_{\mathbb{R}} d|\mu|(s) = \|\mu\|$. □

Corollary 1.1.2. *Let $f \in C^1(\mathbb{R})$. Suppose that the function f' is positive definite. Then $\|f\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{Lip}(\mathbb{R})} = f'(0)$.*

Proof. By the classical theorem of Bochner (see, for example, [77]), f' can be represented in the form $f' = \mathscr{F}\mu$, where μ is a finite positive Borel measure on $\mathbb{R}$. It remains to observe that $\|\mu\| = f'(0) = |f'(0)| \leqslant \|f\|_{\mathrm{Lip}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|\mu\|$, where the last inequality follows from Theorem 1.1.1. □

In this section almost all the above examples of explicit calculation of the seminorm in $\mathrm{OL}(\mathbb{R})$ are based more or less on Corollary 1.1.2. Nevertheless, one can construct an example of a function f in $\mathrm{OL}(\mathbb{R})$ such that $\|f\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{Lip}(\mathbb{R})} = f'(0)$ and f does not satisfy the conditions of Corollary 1.1.2.

On the other hand, if $\|f\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{Lip}(\mathbb{R})} = (f(a) - f(0))/a = 1$ for an $a \in \mathbb{R}$ with $a \neq 0$, then $f(x) = x + f(0)$ for all $x \in \mathbb{R}$.

Example 5 admits one more generalization, the so-called operator Bernstein inequality. This will be discussed in § 1.4. In particular, it will be shown there that $L^\infty(\mathbb{R}) \cap \mathscr{E}_\sigma \subset \mathrm{OL}(\mathbb{R})$, where the symbol $\mathscr{E}_\sigma$ denotes the space of entire functions of exponential type at most σ.

We now consider examples of operator Lipschitz functions on the unit circle $\mathbb{T}$.

Example 6. Let $n \in \mathbb{Z}$. Then $\|z^n\|_{\mathrm{OL}(\mathbb{T})} = |n|$ for all $n \in \mathbb{Z}$.

Proof. It suffices to consider the case when $n > 0$, and then everything reduces to verifying the inequality $\|U^n - V^n\| \leqslant n\|U - V\|$ for any unitary operators U and V. For a proof it suffices to substitute $X = U$ and $Y = V$ in (1.1.1). □

This example immediately leads to an analogue of Theorem 1.1.1 for the circle.

Theorem 1.1.3. *Let f be a continuous function on the unit circle $\mathbb{T}$ such that $\sum_{n\in\mathbb{Z}} |n| \cdot |\widehat{f}(n)| < \infty$. Then*

$$f \in \mathrm{OL}(\mathbb{T}) \quad \textit{and} \quad \|f\|_{\mathrm{OL}(\mathbb{T})} \leqslant \sum_{n\in\mathbb{Z}} |n| \cdot |\widehat{f}(n)|.$$

We remark that stronger results will soon be given in § 1.6.

Example 7. The function $x \mapsto x^2 \sin(1/x)$ is operator Lipschitz. To see this, we prove the following theorem.

Theorem 1.1.4. *Let $f \in \mathrm{OL}(\mathbb{R})$ and $f(0) = 0$. Put $g(x) = x^2 f(x^{-1})$ for $x \neq 0$ and $g(0) = 0$. Then $g \in \mathrm{OL}(\mathbb{R})$ and*

$$\frac{1}{3}\|f\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|g\|_{\mathrm{OL}(\mathbb{R})} \leqslant 3\|f\|_{\mathrm{OL}(\mathbb{R})}. \tag{1.1.2}$$

Proof. It suffices to prove the second inequality, because the first inequality reduces to the second. We can assume that $\|f\|_{\mathrm{OL}(\mathbb{R})} = 1$. As we noted in the introduction to this chapter, for functions on the line, operator Lipschitzness is equivalent to commutator Lipschitzness, and the corresponding norms coincide: $\|f\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{CL}(\mathbb{R})}$ (see § 1.3 and § 3.1). Therefore, it suffices to prove that the inequality

$$\|f(A)R - Rf(A)\| \leqslant \|AR - RA\| \tag{1.1.3}$$

for any bounded operator R and any bounded self-adjoint operator A implies that $\|g(A)R - Rg(A)\| \leqslant 3\|AR - RA\|$ for any bounded operator R and any self-adjoint operator A. Suppose first that A is invertible. This case reduces to the assertion that

$$\|A^2 f(A^{-1})R - RA^2 f(A^{-1})\| \leqslant 3\|AR - RA\| \tag{1.1.4}$$

for any bounded operator R and any invertible self-adjoint operator A. We have

$$\begin{aligned} f(A^{-1})A^2R - RA^2 f(A^{-1}) = f(A^{-1})A(AR - RA) + f(A^{-1})ARA \\ - ARAf(A^{-1}) + (AR - RA)Af(A^{-1}). \end{aligned}$$

Clearly, $\|Af(A^{-1})\| \leqslant \sup_{t\neq 0} |t^{-1}f(t)| \leqslant \|f\|_{\mathrm{Lip}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})} = 1$. Consequently,

$$\|f(A^{-1})A(AR - RA)\| \leqslant \|AR - RA\|, \quad \|(AR - RA)Af(A^{-1})\| \leqslant \|AR - RA\|.$$

Substituting the operators ARA and A^{-1} in (1.1.3), we get that

$$\|f(A^{-1})ARA - ARAf(A^{-1})\| \leqslant \|A^{-1}ARA - ARAA^{-1}\| = \|AR - RA\|,$$

which immediately implies (1.1.4). To consider the general case, it is sufficient to observe that for any positive number δ there exists an invertible self-adjoint operator A_δ such that $AA_\delta = A_\delta A$ and $\|A - A_\delta\| < \delta$. Then for all $\delta > 0$,

$$\begin{aligned} \|g(A)R - Rg(A)\| &\leqslant \|g(A) - g(A_\delta)\| \cdot \|R\| + \|g(A_\delta)R - Rg(A_\delta)\| \\ &\quad + \|g(A_\delta) - g(A)\| \cdot \|R\| \\ &\leqslant 2\delta\|R\| \cdot \|g\|_{\mathrm{Lip}(\mathbb{R})} + 3\|A_\delta R - RA_\delta\| \\ &\leqslant 6\delta\|R\| \cdot \|f\|_{\mathrm{Lip}(\mathbb{R})} + 3\|AR - RA\| + 6\delta\|R\| \\ &\leqslant 3\|AR - RA\| + 12\delta\|R\|. \qquad \square \end{aligned}$$

Remark. It is now clear that in view of Example 5 the function g defined by $g(x) = x^2 \sin(1/x)$ is operator Lipschitz. *The function g gives an example of an operator Lipschitz function that is not continuously differentiable.* The problem of the existence of such functions was posed in [76] and solved in [36]. The fact that g is operator Lipschitz on every finite interval was established in [38]. Recall (see Theorem 3.3.3 below) that every operator Lipschitz function on $\mathbb{R}$ must be differentiable everywhere.

We note also that it was proved in [3] that a set on the line is the set of discontinuities of the derivative of an operator Lipschitz function if and only if it is an F_σ set of the first category. In other words, the sets of discontinuity points of the derivatives of operator Lipschitz functions are the same as the sets of discontinuity points of functions in the first Baire class.

In § 3.10 Theorem 1.1.4 will be generalized to the case of arbitrary linear-fractional transformations.

1.2. Operator Lipschitzness in comparison with operator differentiability

Let H be a function on a subset Λ of the real line $\mathbb{R}$ and with values in a Banach space X. It is said to be *Lipschitz* if there is a non-negative number c such that

$$\|H(s)-H(t)\|_X \leqslant c|s-t|, \qquad s,t\in\Lambda. \tag{1.2.1}$$

We denote the set of all such functions by $\operatorname{Lip}(\Lambda,X)$. Let $\|H\|_{\operatorname{Lip}(\Lambda,X)}$ denote the least constant c satisfying (1.2.1). As usual, we put $\|H\|_{\operatorname{Lip}(\Lambda,X)} \stackrel{\text{def}}{=} \infty$ if $H \notin \operatorname{Lip}(\Lambda,X)$.

Let f be a continuous function on $\mathbb{R}$. With each self-adjoint operator A and each bounded self-adjoint operator K we associate the function $H_{f,A,K}$ with $H_{f,A,K}(t) = f(A+tK)-f(A)$, which is defined for those t in $\mathbb{R}$ for which $f(A+tK)-f(A)\in \mathcal{B}(\mathscr{H})$.

Note that if $f\in \operatorname{OL}(\mathbb{R})$, then

$$H_{f,A,K}\in \operatorname{Lip}(\mathbb{R},\mathcal{B}(\mathscr{H})) \quad \text{and} \quad \|H_{f,A,K}\|_{\operatorname{Lip}(\mathbb{R},\mathcal{B}(\mathscr{H}))} \leqslant \|K\|\cdot\|f\|_{\operatorname{OL}(\mathbb{R})}.$$

It is easy to see that the following result holds.

Lemma 1.2.1. *Let f be a continuous function on $\mathbb{R}$. Then*

$$\begin{aligned}\|f\|_{\operatorname{OL}(\mathbb{R})} &= \sup\bigl\{\|H_{f,A,K}\|_{\operatorname{Lip}(\mathbb{R},\mathcal{B}(\mathscr{H}))} \colon A,K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H}),\ \|K\|=1\bigr\}\\ &= \sup\bigl\{\|H_{f,A,K}\|_{\operatorname{Lip}(\mathbb{R},\mathcal{B}(\mathscr{H}))} \colon K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H}),\ \|K\|=1,\ A^*=A\bigr\}.\end{aligned}$$

We need the following well-known elementary fact. For the reader's convenience we give one of its existing proofs.

Lemma 1.2.2. *Let H be a function with values in a Banach space X, defined on a non-degenerate interval $\Lambda\subset\mathbb{R}$. Then*

$$\|H\|_{\operatorname{Lip}(\Lambda,X)} = \sup_{t\in\Lambda}\ \varlimsup_{h\to 0}\frac{\|H(t+h)-H(t)\|_X}{|h|}.$$

Proof. The $\geqslant$ inequality is evident. To prove the reverse inequality, it suffices to show that the inequality (1.2.1) holds if c satisfies the condition

$$c > \sup_{t\in\Lambda}\ \varlimsup_{h\to 0}\frac{\|H(t+h)-H(t)\|_X}{|h|}. \tag{1.2.2}$$

We fix such a number c and an arbitrary point t in Λ. Let Λ_t be the set of points s in Λ satisfying the inequality (1.2.1). It follows immediately from (1.2.2) that the set Λ_t is at the same time open and closed in Λ. Moreover, $\Lambda_t\ne\varnothing$ since $t\in\Lambda$. Consequently, $\Lambda_t=\Lambda$ because Λ is connected. $\square$

Theorem 1.2.3. *Let f be a continuous function on $\mathbb{R}$. Suppose that*

$$\overline{\lim_{t\to 0}}\frac{\|f(A+tK)-f(A)\|}{|t|}<+\infty$$

for any (not necessarily bounded) self-adjoint operator A and any bounded self-adjoint operator K. Then $f\in \mathrm{OL}(\mathbb{R})$.

Proof. It follows easily from Lemmas 1.2.1 and 1.2.2 that

$$\|f\|_{\mathrm{OL}(\mathbb{R})}=\sup\left\{\overline{\lim_{t\to 0}}\frac{\|f(A+tK)-f(A)\|}{|t|}:\ A,K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H}),\ \|K\|=1\right\}.$$

Thus, if we assume that $\|f\|_{\mathrm{OL}(\mathbb{R})}=\infty$, then for each n in $\mathbb{Z}_+$ there exist operators $A_n,K_n\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ such that $\|K_n\|=1$ and

$$\overline{\lim_{t\to 0}}\frac{\|f(A_n+tK_n)-f(A_n)\|}{|t|}>n.$$

Consider the self-adjoint operators $\mathscr{A}$ and $\mathcal{K}$ acting in the Hilbert space $\ell^2(\mathscr{H})$ as follows:

$$\mathscr{A}(x_0,x_1,x_2,\dots)=(A_0x_0,A_1x_1,A_2x_2,\dots),\qquad (x_0,x_1,x_2,\dots)\in\ell^2(\mathscr{H}),\tag{1.2.3}$$

$$\mathcal{K}(x_0,x_1,x_2,\dots)=(K_0x_0,K_1x_1,K_2x_2,\dots),\qquad (x_0,x_1,x_2,\dots)\in\ell^2(\mathscr{H}).\tag{1.2.4}$$

Then

$$\overline{\lim_{t\to 0}}\frac{\|f(\mathscr{A}+t\mathcal{K})-f(\mathscr{A})\|}{|t|}\geqslant\overline{\lim_{t\to 0}}\frac{\|f(A_n+tK_n)-f(A_n)\|}{|t|}>n$$

for any non-negative integer n, and we arrive at a contradiction. □

Remark. It can be seen from the proof of Theorem 1.2.3 that

$$\begin{aligned}\|f\|_{\mathrm{OL}(\mathbb{R})}&=\sup\left\{\overline{\lim_{t\to 0}}\frac{\|f(A+tK)-f(A)\|}{|t|}:\ A,K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H}),\ \|K\|_{\mathcal{B}_{\mathrm{sa}}(\mathscr{H})}=1\right\}\\&=\sup\bigl\{\|H_{f,A,K}\|_{\mathrm{Lip}(\mathbb{R})}:\ A,K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H}),\ \|K\|_{\mathcal{B}_{\mathrm{sa}}(\mathscr{H})}=1\bigr\}.\end{aligned}$$

To state the next theorem, we observe that the function $H_{f,A,K}$ is differentiable for any self-adjoint operators A and K if and only if it is differentiable at $\mathbf{0}$ for any self-adjoint operators A and K (as usual, the operator K is assumed to be bounded).

The proof of the following theorem uses Theorem 3.5.6, proved in Chap. III.

Theorem 1.2.4. *Let f be a continuous function on $\mathbb{R}$. Then the condition*

(a) $f\in\mathrm{OL}(\mathbb{R})$

is equivalent to each of the following conditions for any self-adjoint operator A and any bounded self-adjoint operator K:

(b) $H_{f,A,K}\in\mathrm{Lip}(\mathbb{R},\mathcal{B}(\mathscr{H}))$;

(c) *the function $H_{f,A,K}$ is differentiable as a function from $\mathbb{R}$ to the space $\mathcal{B}(\mathscr{H})$, equipped with the weak operator topology;*

(d) *the function* $H_{f,A,K}$ *is differentiable as a function from* $\mathbb{R}$ *to the space* $\mathcal{B}(\mathscr{H})$, *equipped with the strong operator topology.*

Proof. The implications (a) ⇒ (b) and (d) ⇒ (c) are obvious. The implication (a) ⇒ (d) follows from Theorem 3.5.6 below. Finally, the implications (c) ⇒ (a) and (b) ⇒ (a) follow immediately from Theorem 1.2.3. □

We denote by $\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$ the space of continuous functions f on $\mathbb{R}$ such that $f\big|[-a,a] \in \mathrm{OL}([-a,a])$ for all $a>0$, and by $\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R},\mathcal{B}(\mathscr{H}))$ the space of continuous functions f on $\mathbb{R}$ such that $f\big|[-a,a] \in \mathrm{Lip}([-a,a],\mathcal{B}(\mathscr{H}))$ for all $a>0$. All the results of this section also have natural analogues for these spaces.

Theorem 1.2.5. *Let* f *be a continuous function on* $\mathbb{R}$. *Suppose that*

$$\varlimsup_{t\to 0}\frac{\|f(A+tK)-f(A)\|}{|t|}<\infty$$

for all $A,K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$. *Then* $f\in\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$.

Proof. Suppose that $f\notin\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$. Then $f\notin\mathrm{OL}([-a,a])$ for some $a>0$. Thus, for each $c\geqslant 0$ there exist operators A and K in $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ such that $\|A\|\leqslant a$, $\|A+K\|\leqslant a$, and $\|f(A+K)-f(A)\|>c\|K\|$. Repeating the reasoning in the proof of Theorem 1.2.3, we arrive at a contradiction by constructing self-adjoint operators $\mathscr{A}$ and $\mathscr{A}+\mathcal{K}$ such that

$$\|\mathscr{A}\|\leqslant a,\quad \|\mathscr{A}+\mathcal{K}\|\leqslant a\quad\text{and}\quad \varlimsup_{t\to 0}\frac{\|f(\mathscr{A}+t\mathcal{K})-f(\mathscr{A})\|}{|t|}=\infty. \qquad \square$$

Theorem 1.2.6. *Let* f *be a continuous function on* $\mathbb{R}$. *The following assertions are equivalent*:

(a) $f\in\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$;

(b) $H_{f,A,K}\in\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R},\mathcal{B}(\mathscr{H}))$ *for all* $A,K\in\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$;

(c) *for all* A, K *in* $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ *the function* $H_{f,A,K}$ *is differentiable as a function from* $\mathbb{R}$ *to the space* $\mathcal{B}(\mathscr{H})$ *equipped with the weak operator topology*;

(d) *for all* A, K *in* $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ *the function* $H_{f,A,K}$ *is differentiable as a function from* $\mathbb{R}$ *to the space* $\mathcal{B}(\mathscr{H})$ *equipped with the strong operator topology.*

This theorem can be proved by analogy with Theorem 1.2.4, except that instead of Theorem 1.2.3 one has to use Theorem 1.2.5.

We note that in [37] it was shown that (a) in Theorem 1.2.6 is equivalent to differentiability in the norm in all compact directions for all bounded self-adjoint operators.

It follows from Theorem 1.2.4 that if f is a continuous function on $\mathbb{R}$, then $f\in\mathrm{OL}(\mathbb{R})$ if and only if for every self-adjoint operator A and every bounded self-adjoint operator K the limit

$$\lim_{t\to 0}\frac{1}{t}\big(f(A+tK)-f(A)\big)\stackrel{\mathrm{def}}{=}\boldsymbol{d}_{f,A}K \tag{1.2.5}$$

exists in the strong operator topology. It will also follow from Theorem 3.5.6 in Chap. III that $\boldsymbol{d}_{f,A}$ is a bounded linear operator from $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ to $\mathcal{B}(\mathscr{H})$.

Similar results hold for functions $f \in \mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$, with the only difference being that the operator A must be bounded.

It follows from Theorem 1.2.6 that if f is a continuous function on $\mathbb{R}$, then $f \in \mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$ if and only if for any operators A and K in $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ the limit in (1.2.5) exists in the strong operator topology, and thus $\boldsymbol{d}_{f,A}$ is a bounded linear operator from $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ to $\mathcal{B}(\mathscr{H})$.

Theorem 1.2.7. *Let* $f \in \mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$. *Then*

$$\|f\|_{\mathrm{OL}(\mathbb{R})} = \sup_{A \in \mathcal{B}_{\mathrm{sa}}(\mathscr{H})} \|\boldsymbol{d}_{f,A}\| = \sup\{\|\boldsymbol{d}_{f,A}\| \colon A \text{ a self-adjoint operator}\}.$$

As usual, the equality $\|f\|_{\mathrm{OL}(\mathbb{R})} = \infty$ means that $f \notin \mathrm{OL}(\mathbb{R})$.

Proof. It suffices to use Lemma 1.2.1. □

Theorem 1.2.8. *Let f be a continuous function on $\mathbb{R}$. Suppose that the limit in* (1.2.5) *exists in the operator norm for every self-adjoint operator A and every bounded self-adjoint operator K. Then $f \in \mathrm{OL}(\mathbb{R}) \cap C^1(\mathbb{R})$, the map $K \mapsto f(A+K) - f(A)$ $(K \in \mathcal{B}_{\mathrm{sa}}(\mathscr{H}))$ is Fréchet differentiable at $\mathbf{0}$ for every self-adjoint operator A, and its differential at $\mathbf{0}$ is equal to $\boldsymbol{d}_{f,A}$.*

Proof. Theorem 1.2.4 implies that $f \in \mathrm{OL}(\mathbb{R})$. It follows from Theorem 3.5.6 that $\boldsymbol{d}_{f,A}$ is a bounded linear operator from $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ to $\mathcal{B}(\mathscr{H})$. Let us verify *Fréchet differentiability*, that is, that $\boldsymbol{d}_{f,A}$ is a bounded linear operator (already proved) and $\lim_{t\to 0} t^{-1}\|f(A+tK) - f(A) - t\boldsymbol{d}_{f,A}K\| = 0$ uniformly with respect to K in the unit sphere of $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$.

It suffices to see that

$$\lim_{n\to\infty} \frac{1}{t_n}\|f(A+t_nK_n) - f(A) - t_n\boldsymbol{d}_{f,A}K_n\| = 0 \tag{1.2.6}$$

for an arbitrary sequence $\{t_n\}_{n\geqslant 0}$ of non-zero real numbers that tends to zero and an arbitrary sequence of self-adjoint operators $\{K_n\}_{n\geqslant 0}$ with $\|K_n\| = 1$ for all n.

Consider the self-adjoint operator $\mathscr{A}$ and the bounded self-adjoint operator $\mathcal{K}$ on $\ell^2(\mathscr{H})$ defined by (1.2.3) with $A_n = A$ and (1.2.4). Applying the assumptions of the theorem to the operators $\mathscr{A}$ and $\mathcal{K}$, we obtain

$$\lim_{n\to\infty} \frac{1}{t_n}\|f(\mathscr{A}+t_n\mathcal{K}) - f(\mathscr{A}) - t_n\boldsymbol{d}_{f,\mathscr{A}}\mathcal{K}\| = 0. \tag{1.2.7}$$

Obviously, $\boldsymbol{d}_{f,\mathscr{A}}\mathcal{K}$ is the orthogonal sum of the operators $\boldsymbol{d}_{f,A}K_n$, $n \geqslant 0$, and so (1.2.6) is a consequence of (1.2.7).

Finally, let us prove that $f \in C^1(\mathbb{R})$. We have to verify the continuity of the derivative f' at an arbitrary point t_0. Let A be the operator of multiplication by x on $L^2([x_0-1, x_0+1])$. Put $K \overset{\mathrm{def}}{=} I$. Then by the assumptions of the theorem the limit $\lim_{t\to 0} t^{-1}(f(A+tI) - f(A))$ exists in the operator norm. Thus, the limit $\lim_{t\to 0} t^{-1}(f(x+t) - f(x)) = f'(x)$ exists in $L^\infty([x_0-1, x_0+1])$, and hence $f \in C^1(t_0-1, t_0+1)$. □

Definition. A function f satisfying the conditions of Theorem 1.2.8 is said to be *operator differentiable*. We denote by $\mathrm{OD}(\mathbb{R})$ the set of all operator differentiable functions on $\mathbb{R}$.

Recall that there are various notions of differentiability for functions defined on Banach spaces: the existence of a weak derivative in the sense of Gâteaux; the existence of a Gâteaux differential; Fréchet differentiability. However, as one can see from Theorem 1.2.8, all these definitions are equivalent in the case of operator differentiability of functions on the line. We remark that the equivalence of operator Fréchet differentiability and the existence of a Gâteaux differential that is a bounded linear operator is proved in [37].

The following result can be proved in about the same way as Theorem 1.2.8.

Theorem 1.2.9. *Let f be a continuous function on $\mathbb{R}$. Suppose that for any A and K in $\mathcal{B}_{\mathrm{sa}}$, the limit in (1.2.5) exists in the operator norm. Then $f\in\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})\cap C^1(\mathbb{R})$, the map $K\mapsto f(A+K)-f(A)$, $K\in\mathcal{B}_{\mathrm{sa}}$, is Fréchet differentiable at $\mathbf{0}$ for every A in $\mathcal{B}_{\mathrm{sa}}$, and its differential at $\mathbf{0}$ is equal to $\boldsymbol{d}_{f,A}$.*

If a function f satisfies the conditions of Theorem 1.2.9, then we say that it is *locally operator differentiable* and we write $f\in\mathrm{OD}_{\mathrm{loc}}(\mathbb{R})$.

Observe that Theorems 1.2.8 and 1.2.9 affirm, in particular, that *if $f\in\mathrm{OD}_{\mathrm{loc}}(\mathbb{R})$, then f is continuously differentiable and belongs to the class $\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$, and if $f\in\mathrm{OD}(\mathbb{R})$, then $f\in\mathrm{OL}(\mathbb{R})$.*

Remark. The function $g(x)=x^2\sin(1/x)$, not being continuously differentiable, cannot be operator differentiable. Thus, it is impossible to replace the class of operator Lipschitz functions by the class of operator differentiable functions in Theorem 1.1.4. Indeed, it is easy to verify that the function $x\mapsto\sin x=\operatorname{Im}e^{\mathrm{i}x}$ is operator differentiable.

Our immediate goal is to prove the continuous dependence (in the operator norm) of the differential $\boldsymbol{d}_{f,A}$ on the operator A for (locally) operator differentiable functions f. The following result was obtained in [37].

Theorem 1.2.10. *Let f be a locally operator differentiable function and let $c>0$. Then for any $\varepsilon>0$ there exists a $\delta>0$ such that $\|\boldsymbol{d}_{f,A}-\boldsymbol{d}_{f,B}\|\leqslant\varepsilon$ whenever A and B are self-adjoint operators such that $\|A\|\leqslant c$, $\|B\|\leqslant c$, and $\|A-B\|\leqslant\delta$.*

First, we prove the following lemma obtained in [37].

Lemma 1.2.11. *Let f be a locally operator differentiable function. Then for any positive numbers c and ε there exists a $\delta>0$ such that*

$$\|f(A+K)-f(A)-\boldsymbol{d}_{f,A}K\|\leqslant\varepsilon\|K\|$$

whenever A and K are self-adjoint operators such that $\|K\|\leqslant\delta$ and $\|A\|\leqslant c$.

Proof. Assume the contrary. Then for some positive numbers c and ε there are sequences of self-adjoint operators $\{A_n\}_{n\geqslant0}$ and $\{K_n\}_{n\geqslant0}$ such that $\|K_n\|\to0$, $\|A_n\|\leqslant c$, and

$$\|f(A_n+K_n)-f(A_n)-\boldsymbol{d}_{f,A_n}K_n\|>\varepsilon\|K_n\|,\qquad n\geqslant0. \tag{1.2.8}$$

Let $\mathscr{A}$ be the bounded self-adjoint operator on $\ell^2(\mathscr{H})$ defined by (1.2.3). Then $\|\mathscr{A}\|\leqslant c$. Since f is Fréchet differentiable at the point A, there exists a $\delta>0$ such that

$$\|f(\mathscr{A}+K)-f(\mathscr{A})-\boldsymbol{d}_{f,\mathscr{A}}K\|\leqslant\varepsilon\|K\| \tag{1.2.9}$$

for every self-adjoint operator K with $\|K\| \leqslant \delta$. We now define the operator $\mathcal{K}_n$ on $\ell^2(\mathscr{H})$ by

$$\mathcal{K}_n(x_0, x_1, x_2, \dots) = (\mathbf{0}, \dots, \mathbf{0}, K_n x_n, \mathbf{0}, \mathbf{0}, \dots), \quad (x_0, x_1, x_2, \dots) \in \ell^2(\mathscr{H}). \tag{1.2.10}$$

Using the inequality (1.2.9) for sufficiently large n, we obtain

$$\begin{aligned}\|f(A_n + K_n) - f(A_n) - \boldsymbol{d}_{f,A_n} K_n\| &= \|f(\mathscr{A} + \mathcal{K}_n) - f(\mathscr{A}) - \boldsymbol{d}_{f,\mathscr{A}} \mathcal{K}_n\| \\ &\leqslant \varepsilon \|\mathcal{K}_n\| = \varepsilon \|K_n\|,\end{aligned}$$

which contradicts the inequality (1.2.8). □

Proof of Theorem 1.2.10. Let c, ε, and δ mean the same as in Lemma 1.2.11. Consider self-adjoint operators A and B such that $\|A\| \leqslant c$, $\|B\| \leqslant c/2$, and $\|B - A\| \leqslant \min\{\delta/2, c/2\}$. Let K be a self-adjoint operator such that $\|K\| = \delta/2$. Then $\|B + K\| \leqslant c$, $\|B - A\| \leqslant \|K\|$, and $\|B - A + K\| \leqslant 2\|K\|$. Therefore,

$$\begin{aligned}\|f(B + K) - f(B) - \boldsymbol{d}_{f,B} K\| &\leqslant \varepsilon \|K\|, \\ \|f(B) - f(A) - \boldsymbol{d}_{f,A}(B - A)\| &\leqslant \varepsilon \|B - A\| \leqslant \varepsilon \|K\|, \\ \|f(B + K) - f(A) - \boldsymbol{d}_{f,A}(B - A + K)\| &\leqslant \varepsilon \|B - A + K\| \leqslant 2\varepsilon \|K\|.\end{aligned}$$

Using the equality $\boldsymbol{d}_{f,A}(B - A + K) = \boldsymbol{d}_{f,A}(B - A) + \boldsymbol{d}_{f,A} K$, we obtain

$$\begin{aligned}\|\boldsymbol{d}_{f,B} K - \boldsymbol{d}_{f,A} K\| \leqslant \|&\boldsymbol{d}_{f,B} K - f(B + K) + f(B)\| \\ &+ \|f(B + K) - f(A) - \boldsymbol{d}_{f,A}(B - A + K)\| \\ &+ \|\boldsymbol{d}_{f,A}(B - A) - f(B) + f(A)\| \leqslant 4\varepsilon \|K\|,\end{aligned}$$

whence it follows that $\|\boldsymbol{d}_{f,B} - \boldsymbol{d}_{f,A}\| \leqslant 4\varepsilon$. □

We now proceed to the case of operator differentiable functions. We say that *not necessarily bounded self-adjoint operators A and B are equivalent* if there exists an operator K in $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ such that $B = A + K$. For operators in the same equivalence class we can introduce the metric $\operatorname{dist}(A, B) \stackrel{\text{def}}{=} \|B - A\|$.

Theorem 1.2.12. *Let f be an operator differentiable function on $\mathbb{R}$. Then on each equivalence class the map $A \mapsto \boldsymbol{d}_{f,A}$ is continuous in the operator norm.*

Lemma 1.2.13. *Let f be an operator differentiable function on $\mathbb{R}$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\|f(A + K) - f(A) - \boldsymbol{d}_{f,A} K\| \leqslant \varepsilon \|K\|$$

for every (not necessarily bounded) self-adjoint operator A and every self-adjoint operator K with norm at most δ.

Proof. Assume the contrary. Then for some $\varepsilon > 0$, there exist two sequences of self-adjoint operators $\{A_n\}_{n=1}^{\infty}$ and $\{K_n\}_{n=1}^{\infty}$ such that $\|K_n\| \to 0$ and

$$\|f(A_n + K_n) - f(A_n) - \boldsymbol{d}_{f,A_n} K_n\| > \varepsilon \|K_n\| \tag{1.2.11}$$

for all $n \geqslant 1$. Let $\mathscr{A}$ and $\mathcal{K}_n$ be the operators on $\ell^2(\mathscr{H})$ defined by (1.2.3) and (1.2.10). Since f is Fréchet differentiable at $\mathscr{A}$, there exists a $\delta > 0$ such that

$$\|f(\mathscr{A}+K) - f(\mathscr{A}) - \boldsymbol{d}_{f,\mathscr{A}}K\| \leqslant \varepsilon\|K\|$$

for all self-adjoint operators K of norm at most δ. Applying this inequality to the operator $\mathcal{K}_n$ for sufficiently large n, we obtain

$$\begin{aligned}\|f(A_n+K_n) - f(A_n) - \boldsymbol{d}_{f,A_n}K_n\| \\ = \|f(\mathscr{A}+\mathcal{K}_n) - f(\mathscr{A}) - \boldsymbol{d}_{f,\mathscr{A}}\mathcal{K}_n\| \leqslant \varepsilon\|\mathcal{K}_n\| = \varepsilon\|K_n\|,\end{aligned}$$

which contradicts the inequality (1.2.11). □

Proof of Theorem 1.2.12. Let ε and δ mean the same as in Lemma 1.2.13. Consider self-adjoint operators A and B such that $\|B-A\| \leqslant \delta/2$. Let K be a self-adjoint operator such that $\|K\| = \delta/2$. Then $\|B-A\| \leqslant \|K\|$ and $\|B-A+K\| \leqslant 2\|K\|$. Therefore,

$$\begin{gathered}\|f(B+K) - f(B) - \boldsymbol{d}_{f,B}K\| \leqslant \varepsilon\|K\|, \\ \|f(B) - f(A) - \boldsymbol{d}_{f,A}(B-A)\| \leqslant \varepsilon\|B-A\| \leqslant \varepsilon\|K\|, \\ \|f(B+K) - f(A) - \boldsymbol{d}_{f,A}(B-A+K)\| \leqslant \varepsilon\|B-A+K\| \leqslant 2\varepsilon\|K\|.\end{gathered}$$

Using the equality $\boldsymbol{d}_{f,A}(B-A+K) = \boldsymbol{d}_{f,A}(B-A) + \boldsymbol{d}_{f,A}K$, we get that

$$\begin{aligned}\|\boldsymbol{d}_{f,B}K - \boldsymbol{d}_{f,A}K\| \leqslant \|\boldsymbol{d}_{f,B}K - f(B+K) + f(B)\| \\ + \|f(B+K) - f(A) - \boldsymbol{d}_{f,A}(B-A+K)\| \\ + \|\boldsymbol{d}_{f,A}(B-A) - f(B) + f(A)\| \leqslant 4\varepsilon\|K\|\end{aligned}$$

for all self-adjoint operators K such that $\|K\| = \delta/2$, whence it follows that $\|\boldsymbol{d}_{f,B} - \boldsymbol{d}_{f,A}\| \leqslant 4\varepsilon$ if $\|B-A\| \leqslant \delta/2$. □

Theorem 1.2.14. *Let $f \in \mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$. Then f is locally operator differentiable if and only if the map $A \mapsto \boldsymbol{d}_{f,A}$ is continuous as a map from the Banach space $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ to the Banach space of bounded operators from $\mathcal{B}_{\mathrm{sa}}(\mathscr{H})$ to $\mathcal{B}(\mathscr{H})$.*

Proof. By Theorem 1.2.10 it suffices to verify that the continuity of the map $A \mapsto \boldsymbol{d}_{f,A}$ implies operator differentiability. Note that $H'_{f,A,K}(s) = \boldsymbol{d}_{f,A+sK}K$ (the derivative is taken in the strong operator topology). Therefore,

$$f(A+K) - f(A) = \int_0^1 (\boldsymbol{d}_{f,A+sK}K)\,ds, \qquad (1.2.12)$$

where the integral is understood in the sense that

$$(f(A+K) - f(A))u = \int_0^1 \big((\boldsymbol{d}_{f,A+sK}K)u\big)\,ds$$

for any $u \in \mathscr{H}$. Applying (1.2.12) to the operator tK instead of K, we obtain

$$\frac{1}{t}(f(A+K) - f(A)) - \boldsymbol{d}_{f,A}K = \int_0^1 \big((\boldsymbol{d}_{f,A+stK} - \boldsymbol{d}_{f,A})K\big)\,ds.$$

Assume that $\|K\| = 1$. Then it follows from the last identity that

$$\left\|\frac{1}{t}(f(A+K) - f(A)) - \boldsymbol{d}_{f,A}K\right\| \leqslant \int_0^1 \|\boldsymbol{d}_{f,A+stK} - \boldsymbol{d}_{f,A}\|\,ds.$$

It remains to observe that $\lim_{t\to 0}\int_0^1 \|\boldsymbol{d}_{f,A+stK} - \boldsymbol{d}_{f,A}\|\,ds = 0$ uniformly with respect to self-adjoint operators K of norm 1 in view of the continuity of the map $A \mapsto \boldsymbol{d}_{f,A}$. □

The following result can be proved in a similar way.

Theorem 1.2.15. *Let $f \in \mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$. Then f is operator differentiable if and only if the map $A \mapsto \boldsymbol{d}_{f,A}$ is continuous in the operator norm on every equivalence class.*

Theorem 1.2.16. *The set* $\mathrm{OD}(\mathbb{R})$ *is a closed subspace of* $\mathrm{OL}(\mathbb{R})$.

Proof. We have to prove that if $\lim_{n\to\infty} f_n = f$ in $\mathrm{OL}(\mathbb{R})$ and $f_n \in \mathrm{OD}(\mathbb{R})$ for all n, then $f \in \mathrm{OD}(\mathbb{R})$. It follows from Theorems 3.5.6 and 3.3.6 that

$$\|\boldsymbol{d}_{f_n,A} - \boldsymbol{d}_{f,A}\| = \|\boldsymbol{d}_{f_n-f,A}\| \leqslant \|\mathfrak{D}(f_n - f)\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})} = \|f_n - f\|_{\mathrm{OL}(\mathbb{R})} \to 0$$

as $n \to \infty$. Thus, $\lim_{n\to\infty} \boldsymbol{d}_{f_n,A} = \boldsymbol{d}_{f,A}$ in the norm uniformly with respect to self-adjoint operators A. It remains to apply Theorem 1.2.15, because continuity is preserved under uniform convergence. □

Here we remark that in the case of functions on finite intervals the fact that the set of operator differentiable functions is closed in the space of operator Lipschitz functions was established in [37]. Moreover, it was also shown there that in this case the space of operator differentiable functions coincides with the closure of the set of polynomials in the space of operator Lipschitz functions. We remark also that the question of operator differentiability of differentiable functions was posed by Widom in [75].

1.3. Commutator Lipschitzness

Recall that a continuous function f on $\mathbb{R}$ is said to be *commutator Lipschitz* if

$$\|f(A)R - Rf(A)\| \leqslant \mathrm{const}\,\|AR - RA\| \tag{1.3.1}$$

for any bounded self-adjoint operator A and any bounded linear operator R. As in the definition of operator Lipschitz functions, if f is commutator Lipschitz, then the inequality (1.3.1) holds for any (not necessarily bounded) self-adjoint operator A and any bounded linear operator R (see Theorem 3.2.1).

Later we will see that the following result holds.

Theorem 1.3.1. *Let f be a continuous function on $\mathbb{R}$. Then the following statements are equivalent*:

(a) $\|f(A) - f(B)\| \leqslant \|A - B\|$ *for any self-adjoint operators A and B*;

(b) $\|f(A)R - Rf(A)\| \leqslant \|AR - RA\|$ *for any self-adjoint operator A and any bounded linear operator R*;

(c) $\|f(A)R - Rf(B)\| \leqslant \|AR - RB\|$ *for any self-adjoint operators A and B and any bounded linear operator R.*

Operators of the form $f(A)R - Rf(B)$ are called *quasi-commutators*.

We will deduce Theorem 1.3.1 from a more general result for normal operators in § 3.1. We note, however, that commutator Lipschitzness is by no means equivalent to operator Lipschitzness in the case of functions of normal operators.

1.4. Operator Bernstein inequalities

In this section we give an elementary proof of the result in [58] that functions in $L^\infty(\mathbb{R})$ whose Fourier transforms have compact support must be operator Lipschitz. Moreover, we obtain the so-called operator Bernstein inequality with constant 1. We follow the approach in [10], and we also obtain analogous results for functions on the circle.

In § 1.6 we deduce from these results that membership in the Besov class $B^1_{\infty,1}(\mathbb{R})$ is a sufficient condition for operator Lipschitzness.

Let $\sigma > 0$. Recall that an entire function f has *exponential type at most* σ if for any $\varepsilon > 0$ there is a $c > 0$ such that $|f(z)| \leqslant ce^{(\sigma+\varepsilon)|z|}$ for all $z \in \mathbb{C}$.

We denote by $\mathscr{E}_\sigma$ the set of entire functions of exponential type at most σ. It is well known that

$$\mathscr{E}_\sigma \cap L^\infty(\mathbb{R}) = \{f \in L^\infty(\mathbb{R}) \colon \operatorname{supp} \mathscr{F} f \subset [-\sigma, \sigma]\}.$$

We note also that the space $\mathscr{E}_\sigma \cap L^\infty(\mathbb{R})$ coincides with the set of entire functions f such that $f \in L^\infty(\mathbb{R})$ and

$$|f(z)| \leqslant e^{\sigma|\operatorname{Im} z|}\|f\|_{L^\infty(\mathbb{R})}, \qquad z \in \mathbb{C} \tag{1.4.1}$$

(see, for example, [42], p. 97).

The Bernstein inequality (see [18]) says that

$$\sup_{x\in\mathbb{R}} |f'(x)| \leqslant \sigma \sup_{x\in\mathbb{R}} |f(x)|$$

for all f in $\mathscr{E}_\sigma \cap L^\infty(\mathbb{R})$. It implies that

$$|f(x) - f(y)| \leqslant \sigma\|f\|_{L^\infty(\mathbb{R})}|x - y|, \qquad f \in \mathscr{E}_\sigma \cap L^\infty(\mathbb{R}), \quad x, y \in \mathbb{R}, \tag{1.4.2}$$

where $\|f\|_{L^\infty(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x\in\mathbb{R}} |f(x)|$.

Bernstein also proved in [18] the following improvement of (1.4.2):

$$|f(x) - f(y)| \leqslant \beta(\sigma(|x - y|))\|f\|_{L^\infty(\mathbb{R})}, \qquad f \in \mathscr{E}_\sigma \cap L^\infty(\mathbb{R}), \quad x, y \in \mathbb{R}, \tag{1.4.3}$$

where

$$\beta(t) \stackrel{\text{def}}{=} \begin{cases} 2\sin(t/2) & \text{if } 0 \leqslant t \leqslant \pi, \\ 2 & \text{if } t > \pi. \end{cases}$$

Note that $\beta(t) \leqslant \min(t, 2)$ for all $t \geqslant 0$.

Let X be a complex Banach space. We denote by $\mathscr{E}_\sigma(X)$ the space of entire X-valued functions f of exponential type at most σ, that is, satisfying the following condition: for any $\varepsilon > 0$ there exists a $c > 0$ such that $\|f(z)\|_X \leqslant ce^{(\sigma+\varepsilon)|z|}$ for all $z \in \mathbb{C}$.

The Bernstein inequality for vector-valued functions. *Let f be a function in $\mathscr{E}_\sigma(X)\cap L^\infty(\mathbb{R},X)$, where $\sigma>0$. Then*

$$\|f(x)-f(y)\|_X\leqslant\beta(\sigma(|x-y|))\|f\|_{L^\infty(\mathbb{R},X)}\leqslant\sigma\|f\|_{L^\infty(\mathbb{R},X)}|x-y| \tag{1.4.4}$$

for all $x,y\in\mathbb{R}$.

The vector version of the Bernstein inequality can be reduced to the scalar version with the help of the Hahn–Banach theorem.

Theorem 1.4.1. *Let $f\in\mathscr{E}_\sigma\cap L^\infty(\mathbb{R})$. Then*

$$\|f(A)-f(B)\|\leqslant\beta(\sigma(\|A-B\|))\|f\|_{L^\infty}\leqslant\sigma\|f\|_{L^\infty}\|A-B\| \tag{1.4.5}$$

for any (bounded) self-adjoint operators A and B. In particular, $\|f\|_{\mathrm{OL}(\mathbb{R})}\leqslant\sigma\|f\|_{L^\infty(\mathbb{R})}$.

Proof. Let A and B be self-adjoint operators acting in a Hilbert space $\mathscr{H}$. We have to show that

$$\|f(A)-f(B)\|\leqslant\beta(\sigma\|A-B\|)\|f\|_{L^\infty}.$$

Let $F(z)=f(A+z(B-A))$. Clearly, F is an entire function with values in the space of operators $\mathcal{B}(\mathscr{H})$, and $\|F(t)\|\leqslant\|f\|_{L^\infty(\mathbb{R})}$ for any $t\in\mathbb{R}$. It follows from von Neumann's inequality (see [72]) that $F\in\mathscr{E}_{\sigma\|B-A\|}(\mathcal{B}(\mathscr{H}))$. To complete the proof, it remains to apply the Bernstein inequality (1.4.4) to the vector function F for $x=0$ and $y=1$. □

It was shown previously in [58] that

$$\|f\|_{\mathrm{OL}(\mathbb{R})}\leqslant\mathrm{const}\,\sigma\|f\|_{L^\infty(\mathbb{R})},\qquad f\in\mathscr{E}_\sigma\cap L^\infty(\mathbb{R}). \tag{1.4.6}$$

In particular, $\mathscr{E}_\sigma\cap L^\infty(\mathbb{R})\subset\mathrm{OL}(\mathbb{R})$. It follows that for any $f\in\mathscr{E}_\sigma\cap\mathrm{Lip}(\mathbb{R})$ the function f' is operator Lipschitz.

The next example shows that $\mathscr{E}_\sigma\cap\mathrm{Lip}(\mathbb{R})\not\subset\mathrm{OL}(\mathbb{R})$.

Example. *Consider the function $f(x)\stackrel{\mathrm{def}}{=}\int_0^x\mathrm{Si}(t)\,dt$, where* Si *is the integral sine,*

$$\mathrm{Si}(x)\stackrel{\mathrm{def}}{=}\int_0^x\frac{\sin t}{t}\,dt.$$

Obviously, $f\in\mathscr{E}_1\cap\mathrm{Lip}(\mathbb{R})$, but f cannot be operator Lipschitz (see Theorems 3.3.2 and 3.3.3 below) since the limit $\lim_{|x|\to\infty}x^{-1}f(x)$ does not exist (actually, $\lim_{x\to\infty}x^{-1}f(x)=\lim_{x\to\infty}\mathrm{Si}(x)=\pi/2=-\lim_{x\to-\infty}x^{-1}f(x)$).

It is interesting to observe that if we slightly 'corrupt' the function f in this example by replacing it by the function $g(x)\stackrel{\mathrm{def}}{=}\int_0^x\mathrm{Si}(|t|)\,dt$, then it becomes operator Lipschitz. It suffices to see that the function $g(x)-\pi x/2$ is operator Lipschitz. This follows (see Proposition 7.8 of [21]) from the fact that the derivative of this function belongs to the space $L^2(\mathbb{R})\cap\mathrm{Lip}(\mathbb{R})$ (this can also be deduced from Theorem 1.6.4 below).

We now obtain analogues of the Bernstein inequality for unitary operators.

Lemma 1.4.2. *Let U and V be unitary operators. Then there exists a self-adjoint operator A such that $V=e^{\mathrm{i}A}U$, $\|A\|\leqslant\pi$, and $\beta(\|A\|)=\|U-V\|$.*

Proof. We define the operator A by $A = \arg(VU^{-1})$, where the function arg is defined on $\mathbb{T}$ by $\arg(e^{is}) = s$, $s \in [-\pi, \pi)$. Obviously, $\beta(\|A\|) = \|I - e^{iA}\| = \|U - V\|$. $\square$

Theorem 1.4.3. *Let f be a trigonometric polynomial of degree at most n. Then*

$$\|f(U) - f(V)\| \leqslant n\|f\|_{L^\infty(\mathbb{T})}\|U - V\|$$

for any unitary operators U and V.

Proof. Let A be a self-adjoint operator whose existence is ensured by Lemma 1.4.2. Put $\Phi(z) \overset{\text{def}}{=} f(e^{izA}U)$, $z \in \mathbb{C}$, where the same symbol f stands for the analytic extension of f to $\mathbb{C} \setminus \{0\}$. Clearly, Φ is an entire function with values in $\mathcal{B}(\mathscr{H})$ and $\|\Phi(t)\| \leqslant \|f\|_{L^\infty(\mathbb{T})}$ for all $t \in \mathbb{R}$. It follows from von Neumann's inequality (see [72]) that $\Phi \in \mathscr{E}_{n\|A\|}(\mathcal{B}(\mathscr{H}))$. Applying the Bernstein inequality for vector functions, we obtain

$$\|f(U) - f(V)\| = \|\Phi(1) - \Phi(0)\| \leqslant \beta(n\|A\|)\|f\|_{L^\infty(\mathbb{T})}.$$

It remains to observe that $\beta(n\|A\|) \leqslant n\beta(\|A\|) = n\|U - V\|$. $\square$

We remark that it was shown in [56] that $\|f(U) - f(V)\| \leqslant \text{const}\, n\|f\|_{L^\infty(\mathbb{T})} \times \|U - V\|$ for any trigonometric polynomial f of degree n and any unitary operators U and V.

Remark. It can be seen from the proof of Theorem 1.4.3 that

$$\|f(U) - f(V)\| \leqslant \beta(n\|A\|)\|f\|_{L^\infty(\mathbb{T})} = \beta\left(2n \arcsin \frac{\|U - V\|}{2}\right)\|f\|_{L^\infty(\mathbb{T})}.$$

This estimate is best possible for the function $f(z) = z^n$, because $\sup\bigl\{|z_1^n - z_2^n| \colon z_1 \in \mathbb{T},\ z_2 \in \mathbb{T},\ |z_1 - z_2| < n\bigr\} = \beta\bigl(2n \arcsin(\delta/2)\bigr)$, $\delta \in (0, 2]$.

1.5. Necessary conditions for operator Lipschitzness

In this section we obtain necessary conditions for operator Lipschitzness for functions on the line and on the circle. These necessary conditions were essentially contained in the papers [56] and [58], in which other methods were used. In addition to the trace class criterion for Hankel operators (see Subsection 3 in § 2) also employed in [56] and [58], we use here for our purpose results in § 3.12 below on the behavior of derivatives of operator Lipschitz functions under linear-fractional transformations.

To prove the next result, we are going to use results from § 3.6 on the behaviour of functions of operators under trace class perturbations.

Theorem 1.5.1. *Let f be an operator Lipschitz function on $\mathbb{T}$. Then $f \in B_1^1(\mathbb{T})$.*

Proof. By the remark after Theorem 3.6.5, the function f has the property that $f(U) - f(V) \in \boldsymbol{S}_1$ if U and V are unitary operators such that $U - V \in \boldsymbol{S}_1$.

We define the operators U and V on $L^2(\mathbb{T})$ by

$$Uf = \overline{z}f \quad \text{and} \quad Vf = \overline{z}f - 2(f, \mathbf{1})\overline{z}, \qquad f \in L^2.$$

It is easy to see that U and V are unitary operators and $\operatorname{rank}(V-U)=1$. It is also easy to verify that for $n\geqslant 0$

$$V^n z^j=\begin{cases} z^{j-n}, & j\geqslant n \text{ or } j<0,\\ -z^{j-n}, & 0\leqslant j<n.\end{cases}$$

Hence, for every continuous function f on $\mathbb{T}$,

$$\begin{aligned}\big((f(V)-f(U))z^j,z^k\big)&=\sum_{n>0}\widehat{f}(n)\big((V^nz^j,z^k)-(z^{j-n},z^k)\big)\\&\qquad+\sum_{n<0}\widehat{f}(n)\big((V^nz^j,z^k)-(z^{j-n},z^k)\big)\\&=-2\begin{cases}\widehat{f}(j-k), & j\geqslant 0,\ k<0,\\ \widehat{f}(j-k), & j<0,\ k\geqslant 0,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Thus, if $f(U)-f(V)\in \boldsymbol{S}_1$, then the operators on ℓ^2 with Hankel matrices $\{\widehat{f}(j+k)\}_{j\geqslant 0,\,k\geqslant 1}$ and $\{\widehat{f}(-j-k)\}_{j\geqslant 0,\,k\geqslant 1}$ belong to $\boldsymbol{S}_1$. We can now use the trace class criterion for Hankel operators (see § 2, Subsection 3) and conclude that $f\in B_1^1(\mathbb{T})$. □

We remark that the construction in the proof of Theorem 1.5.1 is taken from the paper [9].

It is convenient to introduce in this section the notation $\|\!|M|\!\|$ for the norm of a matrix M.

To state a corollary to Theorem 1.5.1, we need the Banach space $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$, which will be studied in detail in § 3.12. Here we only mention that $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})=\{f'+c\bar{z}\colon f\in \mathrm{OL}(\mathbb{T}),\ c\in\mathbb{C}\}$ (see Corollary 3.12.6). Moreover, as always in this paper, the derivative is understood in the complex sense, that is, $f'(\zeta)\stackrel{\text{def}}{=}\lim_{\tau\to\zeta}(\tau-\zeta)^{-1}(f(\tau)-f(\zeta))$.

Corollary 1.5.2. *Let u be the Poisson integral of a function f in $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$. Then $\|\!|\nabla u|\!\|\in L^1(\mathbb{D})$ and $\big\|\,\|\!|\nabla u|\!\|\,\big\|_{L^1(\mathbb{D})}\leqslant \mathrm{const}\cdot\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}$.*

Proof. Let $f=g'$, where $g\in\mathrm{OL}(\mathbb{T})$. Then $f\in B_1^0(\mathbb{T})$ since $g\in B_1^1(\mathbb{T})$. It suffices to use the characterization of the Besov class $B_1^0(\mathbb{T})$ in terms of harmonic extension (see § 2). It remains to observe that the conclusion of the corollary is obvious for the function $f(z)=z^{-1}=\bar{z}$. □

To state a stronger necessary condition for operator Lipschitzness, we need the notion of Carleson measures. Let μ be a positive Borel measure on the open unit disk $\mathbb{D}$. The well-known Carleson theorem says that the Hardy class H^p is contained in $L^p(\mu)$ $(0<p<+\infty)$ if and only if for any point ζ of the unit circle $\mathbb{T}$ and any $r>0$

$$\mu\{z\in\mathbb{D}\colon |z-\zeta|<r\}\leqslant \mathrm{const}\cdot r.$$

Such measures μ are called *Carleson measures* on the disk $\mathbb{D}$. Note that the Carleson condition does not depend on $p\in(0,+\infty)$. More detailed information about

Carleson measures can be found, for example, in [50] and [51]. We need the following equivalent reformulation of the Carleson condition:

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\frac{1-|a|^2}{|1-\overline{a}z|^2}\,d\mu(z)<+\infty \tag{1.5.1}$$

(see, for example, [50], Lecture VII). The condition (1.5.1) means that $\|k_a\|_{L^2(\mu)}\leqslant \mathrm{const}\,\|k_a\|_{H^2}$ for any $a\in\mathbb{D}$, where $k_a(z)\stackrel{\mathrm{def}}{=}(1-z\overline{a})^{-1}$ is the reproducing kernel of the Hilbert space H^2.

We denote by $\mathrm{CM}(\mathbb{D})$ *the space of complex Radon measures* μ *in* $\mathbb{D}$ *such that* $|\mu|$ *is a Carleson measure, and by* $\|\mu\|_{\mathrm{CM}(\mathbb{D})}$ *the norm of the identity embedding from* H^1 *to* $L^1(|\mu|)$. It is well known that the (quasi-)norm of the identity embedding operator from H^p to $L^p(|\mu|)$ is equal to $\|\mu\|^{1/p}_{\mathrm{CM}(\mathbb{D})}$ for all $p\in(0,+\infty)$.

Everything said above about Carleson measures on $\mathbb{D}$ has natural analogues in the upper half-plane $\mathbb{C}_+$. In this case the Carleson condition for a positive Borel measure μ in $\mathbb{C}_+$ can be rewritten as follows:

$$\mu\{z\in\mathbb{C}_+\colon |z-t|<r\}\leqslant \mathrm{const}\cdot r$$

for all $t\in\mathbb{R}$ and all $r>0$. The analogue of (1.5.1) is the inequality

$$\sup_{a\in\mathbb{C}_+}\int_{\mathbb{C}_+}\frac{\operatorname{Im} a}{|z-\overline{a}|^2}\,d\mu(z)<\infty.$$

In particular, in the same way we can introduce the space $\mathrm{CM}(\mathbb{C}_+)$ and the norm in it.

Let f be a function (a distribution) on the unit circle $\mathbb{T}$. *We denote by* $\mathcal{P}f$ *the Poisson integral of* f.

Theorem 1.5.3. *Let* $f\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$. *Then* $\|\nabla(\mathcal{P}f)\|\,d\boldsymbol{m}_2\in\mathrm{CM}(\mathbb{D})$.

Proof. Let $f\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$. Then it follows from Theorem 3.12.10 and Corollary 1.5.2 that

$$\int_{\mathbb{D}}\|((\nabla u)\circ\varphi)(z)\|\cdot|\varphi'(z)|\,d\boldsymbol{m}_2\leqslant \mathrm{const}\,\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \tag{1.5.2}$$

for every linear-fractional automorphism φ of the unit disk $\mathbb{D}$, where $u=\mathcal{P}f$. Now put $\varphi(z)\stackrel{\mathrm{def}}{=}(1-\overline{a}z)^{-1}(a-z)$, where $a\in\mathbb{D}$. Making the change of variable $z=\varphi(w)$ in the integral in (1.5.2), we find that

$$\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}\|(\nabla u)(w)\|\frac{1-|a|^2}{|1-\overline{a}w|^2}\,d\boldsymbol{m}_2(w)\leqslant \mathrm{const}\,\|f\|_{(\mathrm{OL})'(\mathbb{T})}.$$

Thus, the measure $\|\nabla u\|\,d\boldsymbol{m}_2=\|\nabla(\mathcal{P}f)\|\,d\boldsymbol{m}_2$ satisfies the condition (1.5.1). □

The following result is a reformulation of Theorem 1.5.3.

Theorem 1.5.4. *Let* $f\in\mathrm{OL}(\mathbb{T})$. *Then*[2] $\|\operatorname{Hess}\mathcal{P}f\|\,d\boldsymbol{m}_2\in\mathrm{CM}(\mathbb{D})$.

[2] Here and in what follows Hess denotes the Hessian, that is, the matrix of second order partial derivatives.

We now proceed to Poisson integrals of functions on $\mathbb{R}$. If $f \in L^1(\mathbb{R}, (1+x^2)^{-1}\,dx)$, then the Poisson integral can be defined in the standard way. We need the Poisson integral of functions f such that $f' \in L^1(\mathbb{R}, (1+x^2)^{-1}\,dx)$. Clearly, it suffices to consider a real function f. Let u be the Poisson integral of f', and let v be a harmonic conjugate of u. The function $u + \mathrm{i}v$ has a primitive F such that the boundary-value function of $\operatorname{Re} F$ coincides with f everywhere on $\mathbb{R}$. The function F is not uniquely determined, because the harmonic conjugate v is not uniquely determined. The family $\{v + c\}_{c\in\mathbb{R}}$ consists of all functions harmonically conjugate to u. We need a primitive of $u + \mathrm{i}(v + c)$ in the form $F + c\mathrm{i}z + \mathrm{i}\alpha$, where $\alpha \in \mathbb{R}$. Note that $\operatorname{Re}(F + c\mathrm{i}z + \mathrm{i}\alpha) = \operatorname{Re} F - cy$. Thus, it is natural to define the Poisson integral of f as the class of functions $\{\operatorname{Re} F - cy\}_{c\in\mathbb{R}}$. Since $\operatorname{Hess} y = 0$, the Hessian of the Poisson integral $\operatorname{Hess}\mathcal{P}f$ of f is uniquely determined.

In the next theorem $(\mathrm{OL})'(\mathbb{R}) \stackrel{\text{def}}{=} \{f' \colon f \in \mathrm{OL}(\mathbb{R})\}$.

Theorem 1.5.5. *Let $f \in (\mathrm{OL})'(\mathbb{R})$. Then $\|\nabla\mathcal{P}f\|\, d\boldsymbol{m}_2 \in \mathrm{CM}(\mathbb{C}_+)$.*

Proof. Let $f \in (\mathrm{OL})'(\mathbb{R})$. Then it follows from Theorem 3.12.9 and Corollary 1.5.2 that

$$\int_{\mathbb{D}} \|((\nabla u) \circ \varphi)(z)\| \cdot |\varphi'(z)|\, d\boldsymbol{m}_2 \leqslant \mathrm{const}\, \|f\|_{(\mathrm{OL})'(\mathbb{R})} \tag{1.5.3}$$

for any automorphism φ in $\operatorname{Aut}(\widehat{\mathbb{C}})$ such that $\varphi(\mathbb{D}) = \mathbb{C}_+$, where $u = \mathcal{P}f$. Now take $\varphi(z) \stackrel{\text{def}}{=} (1 - z)^{-1}(a - \overline{a}z)$, where $a \in \mathbb{C}_+$. Making the substitution $z = (w - \overline{a})^{-1}(w - a)$ in the integral in (1.5.2), we get that

$$\sup_{a\in\mathbb{C}_+} \int_{\mathbb{C}_+} \|(\nabla u)(w)\| \frac{2\operatorname{Im} a}{|w - \overline{a}|^2}\, d\boldsymbol{m}_2(w) \leqslant \mathrm{const}\, \|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.$$

The last condition is equivalent to the measure $\|\nabla u\|\, d\boldsymbol{m}_2$ being Carleson. □

Theorem 1.5.6. *Let $f \in \mathrm{OL}(\mathbb{R})$. Then $\|\operatorname{Hess}\mathcal{P}f\|\, d\boldsymbol{m}_2 \in \mathrm{CM}(\mathbb{C}_+)$.*

The necessary conditions for operator Lipschitzness given above were originally obtained in [56] and [58]. Namely, it was shown in [56] that if $f \in \mathrm{OL}(\mathbb{T})$, then both Hankel operators H_f and $H_{\overline{f}}$ map the Hardy class H^1 to the Besov class $B_1^1(\mathbb{T})$ (the class of such functions f is denoted by L in [56]). Semmes observed that $f \in L$ if and only if the measure $\|\operatorname{Hess}\mathcal{P}f\|\, d\boldsymbol{m}_2$ is Carleson (see [59], where the proof of this equivalence is given). A similar result also holds for functions on $\mathbb{R}$ (see [58]). It was also shown in [56] that the necessary condition for operator Lipschitzness discussed above is not sufficient. What is more, it is not even sufficient for Lipschitzness.

We now consider the spaces $\mathbb{P}_+(b_\infty^{-1}(\mathbb{T}))$ and $\mathbb{P}_+(b_{1,\infty}^{-1}(\mathbb{T}))$ which are the closures of the set of analytic polynomials in the Besov spaces $B_\infty^{-1}(\mathbb{T})$ and $B_{1,\infty}^{-1}(\mathbb{T})$. It is well known that these spaces admit the following descriptions in terms of analytic extension to the unit disk:

$$\mathbb{P}_+(b_\infty^{-1}(\mathbb{T})) = \Big\{h\colon \lim_{r\to 1^-} (1 - r)\|h(rz)\|_{L^\infty(\mathbb{T})} = 0\Big\};$$

$$\mathbb{P}_+(b_{1,\infty}^{-1}(\mathbb{T})) = \Big\{h\colon \lim_{r\to 1^-} (1 - r)\|h(rz)\|_{L^1(\mathbb{T})} = 0\Big\}.$$

It was also observed in [56] that the space $\mathbb{P}_+L$ is dual to the space of analytic functions g on $\mathbb{D}$ that admit a representation

$$g=\sum_n \varphi_n\psi_n, \quad \text{where} \quad \varphi_n\in H^1, \quad \psi_n\in\mathbb{P}_+(b_\infty^{-1}(\mathbb{T})), \qquad \sum_n \|\varphi_n\|_{H^1}\|\psi_n\|_{\mathbb{P}_+(b_\infty^{-1}(\mathbb{T}))}<\infty. \tag{1.5.4}$$

Obviously, such functions g belong to the space $\mathbb{P}_+(b_{1,\infty}^{-1}(\mathbb{T}))$ whose dual space can be identified naturally with the Besov space $\mathbb{P}_+B_{\infty,1}^1(\mathbb{T})$ of functions analytic in $\mathbb{D}$. Nevertheless, not every function in $\mathbb{P}_+(b_{1,\infty}^{-1}(\mathbb{T}))$ can be represented in the form (1.5.4). Otherwise, we would have the equality $L=B_{\infty,1}^1(\mathbb{T})$, which is impossible because the condition that $f\in L$, though necessary for operator Lipschitzness, is not sufficient.

Remark. The space

$$L=\{f\in\mathrm{BMO}(\mathbb{R})\colon \|\operatorname{Hess}\mathcal{P}f\|\,d\boldsymbol{m}_2\in\mathrm{CM}(\mathbb{C}_+)\}$$

is the limit space of the scale of Triebel–Lizorkin spaces $F_{p,q}^s(\mathbb{R})$, and is denoted by $F_{\infty,1}^1(\mathbb{R})$ (see [27], § 5). The Triebel–Lizorkin space $F_{\infty,1}^1(\mathbb{T})$ of functions on $\mathbb{T}$ is defined similarly. The necessary conditions for operator Lipschitzness obtained above can be reformulated as follows: $\mathrm{OL}(\mathbb{R})\subset F_{\infty,1}^1(\mathbb{R})$ and $\mathrm{OL}(\mathbb{T})\subset F_{\infty,1}^1(\mathbb{T})$.

1.6. A sufficient condition for operator Lipschitzness in terms of Besov classes

In this section we show that the functions in the Besov class $B_{\infty,1}^1(\mathbb{R})$ (see § 2) are operator Lipschitz. We also obtain a similar result for functions on the unit circle. The proofs given here differ from the original proofs in [56] and [58] and are based on the operator Bernstein inequalities (see § 1.4).

Theorem 1.6.1. *Let $f\in B_{\infty,1}^1(\mathbb{R})$. Then f is operator Lipschitz and*

$$\|f(A)-f(B)\|\leqslant \operatorname{const}\|f\|_{B_{\infty,1}^1}\|A-B\| \tag{1.6.1}$$

for any self-adjoint operators A and B with bounded difference $A-B$.

Proof. As we observed in the Introduction (see Theorem 3.2.1 below), it suffices to prove (1.6.1) for bounded self-adjoint operators A and B.

Without loss of generality we can assume that $f(0)=0$. Consider the functions $f_n=f*W_n$ defined by (2.2). Put $g_n\stackrel{\text{def}}{=}f_n-f_n(0)$. It follows from the definition of $B_{\infty,1}^1(\mathbb{R})$ (see § 2) that $\sum_{n=-\infty}^\infty g_n'=f'$ and the series converges uniformly on $\mathbb{R}$. Hence the series $\sum_{n=-\infty}^\infty g_n$ converges uniformly on each compact subset of $\mathbb{R}$. Thus,

$$\sum_{n=-\infty}^{\infty} g_n(A)=f(A) \qquad\text{and}\qquad \sum_{n=-\infty}^{\infty} g_n(B)=f(B),$$

with both series absolutely convergent in the operator norm.

Since obviously $f_n \in \mathscr{E}_{2^n+1} \cap L^\infty(\mathbb{R})$, the operator Bernstein inequality (1.4.5) lets us conclude that

$$\|f(A) - f(B)\| \leqslant \left\| \sum_{n=-\infty}^{\infty} \bigl(g_n(A) - g_n(B)\bigr) \right\| = \left\| \sum_{n=-\infty}^{\infty} \bigl(f_n(A) - f_n(B)\bigr) \right\|$$
$$\leqslant \sum_{n=-\infty}^{\infty} 2^{n+1} \|f_n\|_{L^\infty} \|A - B\| \leqslant \text{const}\, \|f\|_{B^1_{\infty,1}} \|A - B\|. \qquad \square$$

In a similar way one can prove the following analogue of Theorem 1.6.1 for functions on the unit circle.

Theorem 1.6.2. *Let $f \in B^1_{\infty,1}(\mathbb{T})$. Then f is operator Lipschitz and*

$$\|f(U) - f(V)\| \leqslant \text{const}\, \|f\|_{B^1_{\infty,1}} \|U - V\|, \qquad f \in B^1_{\infty,1}(\mathbb{T}),$$

for any unitary operators U and V.

The following theorem lets us combine the necessary conditions obtained in § 1.5 with the sufficient conditions of the present section.

Theorem 1.6.3. $B^1_{\infty,1}(\mathbb{T}) \subset \mathrm{OL}(\mathbb{T}) \subset F^1_{\infty,1}(\mathbb{T})$ *and* $B^1_{\infty,1}(\mathbb{R}) \subset \mathrm{OL}(\mathbb{T}) \subset F^1_{\infty,1}(\mathbb{R})$.

It turned out that the functions in $B^1_{\infty,1}(\mathbb{R})$ are not only operator Lipschitz, but also operator differentiable.

Theorem 1.6.4. *Let $f \in B^1_{\infty,1}(\mathbb{R})$. Then f is operator differentiable.*

We refer the reader to [58] and [61] for the proof of this theorem.

1.7. Operator Hölder functions

We discuss here other applications of the operator Bernstein inequalities obtained in § 1.4. We show that the class of operator Hölder functions of order α with $0 < \alpha < 1$ coincides with the class of Hölder functions of order α. We also dwell briefly on the case of arbitrary moduli of continuity. The results of this section were obtained in [7] and [8]. Another approach to these problems was found in [49], where the authors obtained similar results for functions in Hölder classes with somewhat worse constants as well as a somewhat weaker result for arbitrary moduli of continuity.

It is well known that the (scalar) Bernstein inequality plays a key role in approximation theory (see, for example, [1], [24], [47], [73]). We refer to the description of smoothness-type properties in terms of approximation by nice functions. The direct theorems of approximation theory give us estimates of the rate of approximation of functions in a given function space X (usually, of smooth functions in some sense or another) by nice functions. The inverse theorems let us conclude that a given function f belongs to a particular function space if f admits certain estimates of the rate of approximation by nice functions. In the case when the direct theorems 'match' with the inverse theorems for a function space X, we obtain a complete description of X in terms of approximation by such nice functions.

In this section we consider function spaces on the unit circle $\mathbb{T}$ and on the real line $\mathbb{R}$. In the first case the role of nice functions is played by the spaces $\mathcal{P}_n$

of trigonometric polynomials of degree at most n, and in the second case by the spaces $\mathscr{E}_\sigma$ of functions of exponential type at most σ. We shall consider only uniform approximation.

The classical Bernstein inequalities play a decisive role in the proof of inverse theorems of approximation theory. It is easy to see that when we use the operator version of the Bernstein inequality in such a proof, we obtain the corresponding smoothness of a function f on the set of unitary operators if we are dealing with functions on the circle, or on the set of self-adjoint operators if we are dealing with functions on the line.

Let us illustrate this by an example. The classical Jackson theorem says that if f is in the Hölder class $\Lambda_\alpha(\mathbb{T})$ with $0<\alpha<1$, then

$$\operatorname{dist}(f,\mathcal{P}_n)\leqslant \operatorname{const}(n+1)^{-\alpha}\|f\|_{\Lambda_\alpha}. \tag{1.7.1}$$

Bernstein proved that the converse is also true, that is, if for a function f in $C(\mathbb{T})$ the inequalities (1.7.1) hold with $\alpha\in(0,1)$, then $f\in\Lambda_\alpha(\mathbb{T})$.

We give the standard proof of this result of Bernstein. Without loss of generality we can assume that $c=1$. For $n\geqslant 0$ there exists a trigonometric polynomial f_n such that $\deg f_n<2^n$ and $\|f-f_n\|_{C(\mathbb{T})}\leqslant 2^{-\alpha n}$. Clearly,

$$\|f_n-f_{n-1}\|_{C(\mathbb{T})}\leqslant\|f-f_n\|_{C(\mathbb{T})}+\|f-f_{n-1}\|_{C(\mathbb{T})}\leqslant 2^{-\alpha n}(1+2^\alpha)\leqslant 3\cdot 2^{-\alpha n}.$$

Consequently,

$$\|f_n-f_{n-1}\|_{\mathrm{Lip}(\mathbb{T})}\leqslant 2^n\|f_n-f_{n-1}\|_{C(\mathbb{T})}\leqslant 3\cdot 2^{(1-\alpha)n}$$

by the Bernstein inequality. In view of the obvious equality $\|f_0\|_{\mathrm{Lip}(\mathbb{T})}=0$, we get that

$$\|f_N\|_{\mathrm{Lip}(\mathbb{T})}\leqslant\sum_{n=1}^N\|f_n-f_{n-1}\|_{\mathrm{Lip}(\mathbb{T})}\leqslant 3\sum_{n=1}^N 2^{(1-\alpha)n}\leqslant\frac{3}{1-2^{\alpha-1}}2^{(1-\alpha)N},\quad N\in\mathbb{Z}_+.$$

Let $\zeta,\tau\in\mathbb{T}$, and choose $N\in\mathbb{Z}_+$ such that $2^{-N}<|\zeta-\tau|\leqslant 2^{1-N}$. Then

$$\begin{aligned}|f(\zeta)-f(\tau)|&\leqslant|f(\zeta)-f_N(\zeta)|+|f_N(\zeta)-f_N(\tau)|+|f_N(\xi)-f(\tau)|\\&\leqslant 2\|f-f_N\|_{L^\infty}+\|f_N\|_{\mathrm{Lip}}|\zeta-\tau|\\&\leqslant 2\cdot 2^{-\alpha N}+\frac{3}{1-2^{\alpha-1}}2^{(1-\alpha)N}|\zeta-\tau|\\&\leqslant 2|\zeta-\tau|^\alpha+\frac{3\cdot 2^{1-\alpha}}{1-2^{\alpha-1}}|\zeta-\tau|^\alpha\leqslant\frac{8}{1-2^{\alpha-1}}|\zeta-\tau|^\alpha.\end{aligned}$$

The next theorem says that every function in $\Lambda_\alpha(\mathbb{T})$ with $0<\alpha<1$ is operator Hölder of order α, which is in sharp contrast to the case of Lipschitz functions.

Theorem 1.7.1. *Let $f\in\Lambda_\alpha(\mathbb{T})$, where $\alpha\in(0,1)$. Then there exists a constant c such that*

$$\|f(U)-f(V)\|\leqslant c(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha}\|U-V\|^\alpha$$

for any unitary operators U and V.

Proof. Let $f \in \Lambda_\alpha(\mathbb{T})$. First we use a direct theorem of approximation theory, the Jackson theorem in our case. By this theorem,

$$\operatorname{dist}(f, \mathcal{P}_n) \leqslant \operatorname{const}(n+1)^{-\alpha}\|f\|_{\Lambda_\alpha}, \qquad n \in \mathbb{Z}_+.$$

Repeating almost word-for-word the proof of the corresponding inverse theorem, replacing ζ and τ by unitary operators U and V, and using the operator Bernstein inequality instead of the scalar one, we arrive at the desired result. □

A similar result holds also in the case of the line.

Theorem 1.7.2. *Let $f \in \Lambda_\alpha(\mathbb{R})$ with $\alpha \in (0,1)$. Then there exists a constant c such that*

$$\|f(A) - f(B)\| \leqslant c(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha}\|A - B\|^\alpha$$

for any self-adjoint operators A and B.

The proof is based on a similar description of the function class $\Lambda_\alpha(\mathbb{R})$ in terms of approximation by entire functions of exponential type.

The direct theorem for the space $\Lambda_\alpha(\mathbb{R})$. *Let $f \in \Lambda_\alpha(\mathbb{R})$ with $0 < \alpha < 1$. Then there exists a $c > 0$ such that*

$$\inf\{h \in \mathscr{E}_\sigma \colon \|f - h\|_{L^\infty(\mathbb{R})}\} \leqslant c\sigma^{-\alpha}\|f\|_{\Lambda_\alpha(\mathbb{T})} \tag{1.7.2}$$

for all $\sigma > 0$.

The inverse theorem for the space $\Lambda_\alpha(\mathbb{R})$. *Let $0 < \alpha < 1$ and let f be a continuous function on $\mathbb{R}$ such that $\lim_{|x|\to\infty} x^{-1}f(x) = 0$. Suppose that (1.7.2) holds for some $c > 0$ and all $\sigma > 0$. Then $f \in \Lambda_\alpha(\mathbb{R})$ and $\|f\|_{\Lambda_\alpha(\mathbb{R})} \leqslant 5c/(1 - 2^{\alpha-1})$.*

The proofs of Theorems 1.7.1 and 1.7.2 were given in more detail in [8]. In the same paper a series of other results were obtained, based ultimately on certain results in approximation theory.

In particular, analogues of Theorems 1.7.1 and 1.7.2 were obtained there for all $\alpha > 0$. We state here some other results obtained in [8] that also can be proved by methods in approximation theory.

A function $\omega\colon [0, +\infty) \to \mathbb{R}$ is called a *modulus of continuity* if it is a non-negative non-decreasing continuous function such that $\omega(0) = 0$, $\omega(x) > 0$ for $x > 0$, and $\omega(x+y) \leqslant \omega(x) + \omega(y)$ for all $x, y \in [0, +\infty)$.

Denote by $\Lambda_\omega(\mathbb{R})$ the space of continuous functions f on $\mathbb{R}$ such that

$$\|f\|_{\Lambda_\omega(\mathbb{R})} \stackrel{\text{def}}{=} \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega(|x-y|)} < +\infty.$$

We can define the space $\Lambda_\omega(\mathbb{T})$ similarly.

Let

$$\omega_*(x) \stackrel{\text{def}}{=} x \int_x^\infty \frac{\omega(t)}{t^2}\, dt. \tag{1.7.3}$$

Theorem 1.7.3. *Let $f \in \Lambda_\omega(\mathbb{R})$, where ω is a modulus of continuity. Then*

$$\|f(A) - f(B)\| \leqslant c\|f\|_{\Lambda_\omega(\mathbb{R})}\omega_*(\|A - B\|)$$

for any self-adjoint operators A and B, where c is an absolute constant.

A similar result also holds for functions f in $\Lambda_\omega(\mathbb{T})$.

1.8. Hölder functions under perturbations by operators of Schatten–von Neumann classes

In this section we consider another application of the operator Bernstein inequalities in § 1.4. Let f be a function of Hölder class $\Lambda_\alpha(\mathbb{R})$ with $0<\alpha<1$, and let $p\geqslant 1$. Suppose that A and B are self-adjoint operators and $B-A\in\boldsymbol{S}_p$. What can we say about the operator $f(A)-f(B)$? This question was studied in detail in [9]. We state here the result in [9] in the case $p>1$.

Theorem 1.8.1. *Let $p>1$ and $0<\alpha<1$. Then*

$$\|f(A)-f(B)\|_{\boldsymbol{S}_{p/\alpha}}\leqslant \operatorname{const}\|f\|_{\Lambda_\alpha}\|A-B\|^\alpha_{\boldsymbol{S}_p}$$

for any self-adjoint operators A and B with difference in $\boldsymbol{S}_p$.

We omit the proof of Theorem 1.8.1 and refer the reader to [9]. The case $p=1$ is also considered in detail in [9]. The conclusion of Theorem 1.8.1 is false for $p=1$. Also, in [9] there is an analogue of Theorem 1.8.1 for all positive α, and more general problems of perturbations by operators in symmetrically normed ideals are considered.

Chapter II. Schur multipliers and double operator integrals

In this chapter we study Schur multipliers, both discrete ones and Schur multipliers with respect to spectral measures. We use a description of discrete Schur multipliers that is based on Grothendieck's theorem (see Pisier's book [65] and the paper [66]). We refine this result in the case when the initial function is defined on a product of topological spaces and is continuous in each variable. We also obtain a refinement of the general result for Borel functions on a product of topological spaces.

We then define double operator integrals and Schur multipliers with respect to spectral measures. The study of such Schur multipliers in the case of Borel functions on a product of topological spaces can be reduced to discrete Schur multipliers.

2.1. Discrete Schur multipliers

We denote by $\ell^p(\mathcal{T})$ the space of complex functions $\alpha\colon t\mapsto\alpha_t$ defined on a not necessarily countable or finite set $\mathcal{T}$ and such that $\sum_{t\in\mathcal{T}}|\alpha_t|^p<\infty$, with the norm $\|\alpha\|_p=\bigl(\sum_{t\in\mathcal{T}}|\alpha_t|^p\bigr)^{1/p}$, where $p\in[1,+\infty)$. For $p=\infty$ the space $\ell^p(\mathcal{T})$ consists of all bounded complex functions $\alpha\colon t\mapsto\alpha_t$ on $\mathcal{T}$, and $\|\alpha\|_\infty=\sup_{t\in\mathcal{T}}|\alpha_t|$. In those cases when we have to specify the set $\mathcal{T}$ on which the family α is defined, we will write $\|\alpha\|_{\ell^p(\mathcal{T})}$ instead of $\|\alpha\|_p$. We denote by $c_0(\mathcal{T})$ the subspace of $\ell^\infty(\mathcal{T})$ consisting of the functions α tending to zero at infinity.

Let $\mathcal{S}$ and $\mathcal{T}$ be arbitrary non-empty sets. With each bounded operator $A\colon\ell^2(\mathcal{T})\to\ell^2(\mathcal{S})$ one can associate a unique matrix $\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ such that $(Ax)_s=\sum_{t\in\mathcal{T}}a(s,t)x_t$ for all $x=\{x_t\}_{t\in\mathcal{T}}$ in $\ell^2(\mathcal{T})$. In this case we say that *the matrix $\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ induces a bounded operator $A\colon\ell^2(\mathcal{T})\to\ell^2(\mathcal{S})$*. Let

$$\|\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}\|\stackrel{\text{def}}{=}\|A\|\qquad\text{and}\qquad\|\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}\|_{\boldsymbol{S}_1}\stackrel{\text{def}}{=}\|A\|_{\boldsymbol{S}_1}.$$

When $A \notin \boldsymbol{S}_1(\ell^2(\mathcal{T}), \ell^2(\mathcal{S}))$, we assume that the last norm equals ∞. If the matrix $\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ does not induce a bounded operator from $\ell^2(\mathcal{T})$ to $\ell^2(\mathcal{S})$, then we assume that its operator norm (as well as its trace norm) equals ∞. Let $\mathcal{B}(\mathcal{S}\times\mathcal{T})$ be *the set of matrices* $\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ *inducing bounded operators from* $\ell^2(\mathcal{T})$ *to* $\ell^2(\mathcal{S})$. Sometimes we write $\|\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}\|_{\mathcal{B}(\mathcal{S}\times\mathcal{T})}$ instead of $\|\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}\|$ and $\|\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}\|_{\boldsymbol{S}_1(\mathcal{S}\times\mathcal{T})}$ instead of $\|\{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}\|_{\boldsymbol{S}_1}$.

A matrix $\Phi = \{\Phi(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ is called a *Schur multiplier* of the space $\mathcal{B}(\mathcal{S}\times\mathcal{T})$ if for every matrix $A = \{a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ in $\mathcal{B}(\mathcal{S}\times\mathcal{T})$ the matrix $\Phi\star A \stackrel{\text{def}}{=} \{\Phi(s,t)a(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ also belongs to $\mathcal{B}(\mathcal{S}\times\mathcal{T})$.

We denote by $\mathfrak{M}(\mathcal{S}\times\mathcal{T})$ *the set of Schur multipliers of* $\mathcal{B}(\mathcal{S}\times\mathcal{T})$. It is easy to deduce from the closed graph theorem that the Schur multipliers induce bounded operators on $\mathcal{B}(\mathcal{S}\times\mathcal{T})$. Let

$$\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})} \stackrel{\text{def}}{=} \sup\{\|\Phi\star A\|\colon A\in\mathcal{B}(\mathcal{S}\times\mathcal{T}),\ \|A\|_{\mathcal{B}}\leqslant 1\}.$$

Hence by duality

$$\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})} = \sup\{\|\Phi\star A\|_{\boldsymbol{S}_1}\colon A\in\mathcal{B}(\mathcal{S}\times\mathcal{T}),\ \|A\|_{\boldsymbol{S}_1}\leqslant 1\}. \tag{2.1.1}$$

It is easy to see that

$$\|A\|_{\mathcal{B}(\mathcal{S}\times\mathcal{T})} = \sup\|A\|_{\mathcal{B}(\mathcal{S}_0\times\mathcal{T}_0)}, \qquad \|A\|_{\boldsymbol{S}_1(\mathcal{S}\times\mathcal{T})} = \sup\|A\|_{\boldsymbol{S}_1(\mathcal{S}_0\times\mathcal{T}_0)},$$
$$\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})} = \sup\|\varphi\|_{\mathfrak{M}(\mathcal{S}_0\times\mathcal{T}_0)},$$

where the suprema are taken over all finite subsets $\mathcal{S}_0$ and $\mathcal{T}_0$ of the sets $\mathcal{S}$ and $\mathcal{T}$.

Note also that $\|\Phi\|_{\ell^\infty(\mathcal{S}\times\mathcal{T})}\leqslant\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$. It is easy to see that the inequality turns into an equality for every matrix $\{\Phi(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ of rank 1. There are other classes of matrices for which this inequality turns into an equality. For example, if each row (or each column) of Φ has at most one non-zero entry, then $\|\Phi\|_{\ell^\infty(\mathcal{S}\times\mathcal{T})} = \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$.

We need one more characteristic of the matrix Φ. Let

$$\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})} \stackrel{\text{def}}{=} \sup\{\|\Phi\star A\|\colon A\in\mathcal{B}(\mathcal{S}\times\mathcal{T}),\ \|A\|\leqslant 1,\ a(t,t)=0 \text{ for } t\in\mathcal{S}\cap\mathcal{T}\}.$$

We denote by $\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})$ the set of matrices $\Phi = \{\Phi(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ such that $\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}<\infty$. Obviously, $\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}\leqslant\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$. It is easy to see that

$$\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})} = \sup\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}_0\times\mathcal{T}_0)},$$

where the supremum is taken over all finite subsets $\mathcal{S}_0$ and $\mathcal{T}_0$ of $\mathcal{S}$ and $\mathcal{T}$.

We also observe that if the matrices $\Phi = \{\Phi(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ and $\Psi = \{\Psi(s,t)\}_{(s,t)\in\mathcal{S}\times\mathcal{T}}$ coincide off the 'diagonal' $\{(t,t)\colon t\in\mathcal{S}\cap\mathcal{T}\}$, then

$$\|\Phi-\Psi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})} = 0 \quad\text{and}\quad \|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})} = \|\Psi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}.$$

We note that $\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})} = \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ if $\mathcal{S}\cap\mathcal{T}=\varnothing$.

Lemma 2.1.1. *Let* $\Phi\in\ell^\infty(\mathcal{S}\times\mathcal{T})$, *where* $\mathcal{S}$ *and* $\mathcal{T}$ *are arbitrary sets such that* $\mathcal{S}\cap\mathcal{T}\neq\varnothing$. *Then*

$$\max\bigl\{\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}, \|\Phi(t,t)\|_{\ell^\infty(\mathcal{S}\cap\mathcal{T})}\bigr\} \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$$
$$\leqslant 2\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})} + \|\Phi(t,t)\|_{\ell^\infty(\mathcal{S}\cap\mathcal{T})}.$$

Proof. The first inequality is obvious. Let us prove the second. Denote by χ the characteristic function of the set $\{(s,t)\in\mathcal{S}\times\mathcal{T}\colon s=t\}$. It is easy to see that $\|\chi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}=1$, whence $\|1-\chi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant\|1\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}+\|\chi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}=2$. Let $A\in\mathcal{B}(\mathcal{S}\times\mathcal{T})$ and $\|A\|\leqslant 1$. Then $\Phi\star A=\Phi\star(1-\chi)\star A+\Phi\star\chi\star A$. It remains to observe that

$$\|\Phi\star(1-\chi)\star A\|\leqslant\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}\|(1-\chi)\star A\|\leqslant 2\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})},$$
$$\|\Phi\star\chi\star A\|\leqslant\|\Phi\star\chi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}=\|\Phi(t,t)\|_{\ell^\infty(\mathcal{S}\cap\mathcal{T})}. \qquad \square$$

Corollary 2.1.2. *If $\Phi(t,t)=0$ for all $t\in\mathcal{S}\cap\mathcal{T}$, then*

$$\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}\leqslant\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant 2\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}.$$

Lemma 2.1.3. *Let $\mathcal{S}$ be a Hausdorff topological space. Suppose that the set $\mathcal{S}\cap\mathcal{T}$ has no isolated points in $\mathcal{S}$. Then $\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}=\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ for any function $\Phi\in\ell^\infty(\mathcal{S}\times\mathcal{T})$ continuous in the variable $s\in\mathcal{S}$.*

Proof. It suffices to prove that $\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}$, or, what is the same, $\|\Phi\|_{\mathfrak{M}(\mathcal{S}_0\times\mathcal{T}_0)}\leqslant\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}$ for all finite subsets $\mathcal{S}_0$ and $\mathcal{T}_0$ of the sets $\mathcal{S}$ and $\mathcal{T}$. Fix finite subsets $\mathcal{S}_0$ and $\mathcal{T}_0$ of $\mathcal{S}$ and $\mathcal{T}$. Obviously, for any $\varepsilon>0$ there exists a perturbation $\widetilde{\mathcal{S}}_0$ of the set $\mathcal{S}_0$ such that $\widetilde{\mathcal{S}}_0\cap\mathcal{T}_0=\varnothing$ and $\|\Phi\|_{\mathfrak{M}(\mathcal{S}_0\times\mathcal{T}_0)}<\varepsilon+\|\Phi\|_{\mathfrak{M}(\widetilde{\mathcal{S}}_0\times\mathcal{T}_0)}$. Consequently,

$$\|\Phi\|_{\mathfrak{M}(\mathcal{S}_0\times\mathcal{T}_0)}<\varepsilon+\|\Phi\|_{\mathfrak{M}(\widetilde{\mathcal{S}}_0\times\mathcal{T}_0)}=\varepsilon+\|\Phi\|_{\mathfrak{M}_0(\widetilde{\mathcal{S}}_0\times\mathcal{T}_0)}\leqslant\varepsilon+\|\Phi\|_{\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})}$$

for any $\varepsilon>0$. $\square$

We are going to consider an analogue of the space $\mathfrak{M}_0(\mathcal{S}\times\mathcal{T})$ that is defined in terms of the $\boldsymbol{S}_1$ norm instead of the operator norm. To this end we set

$$\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}\stackrel{\text{def}}{=}\sup\{\|\Phi\star A\|_{\boldsymbol{S}_1}: A\in\mathcal{B}(\mathcal{S}\times\mathcal{T}),\ \|A\|_{\boldsymbol{S}_1}\leqslant 1,\ a(t,t)=0 \text{ for } t\in\mathcal{S}\cap\mathcal{T}\}$$

and $\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})\stackrel{\text{def}}{=}\{\Phi\colon \|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}<+\infty\}$. It should be noted that there is no need to define a corresponding analogue of $\mathfrak{M}(\mathcal{S}\times\mathcal{T})$, because it coincides with the same space $\mathfrak{M}(\mathcal{S}\times\mathcal{T})$ in view of (2.1.1).

One can prove the following facts in the same way as for the operator norm.

Lemma 2.1.4. *Let $\Phi\in\ell^\infty(\mathcal{S}\times\mathcal{T})$, where $\mathcal{S}$ and $\mathcal{T}$ are arbitrary sets such that $\mathcal{S}\cap\mathcal{T}\neq\varnothing$. Then*

$$\max\bigl\{\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})},\|\Phi(t,t)\|_{\ell^\infty(\mathcal{S}\cap\mathcal{T})}\bigr\}\leqslant\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant 2\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}+\|\Phi(t,t)\|_{\ell^\infty(\mathcal{S}\cap\mathcal{T})}.$$

If $\Phi(t,t)=0$ for any $t\in\mathcal{S}\cap\mathcal{T}$, then

$$\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}\leqslant\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant 2\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}.$$

Corollary 2.1.5. *If $\Phi(t,t)=0$ for all $t\in\mathcal{S}\cap\mathcal{T}$, then*

$$\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}\leqslant\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant 2\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}.$$

Lemma 2.1.6. *Let $\mathcal{S}$ be a Hausdorff topological space. Suppose that the set $\mathcal{S}\cap\mathcal{T}$ has no isolated points in $\mathcal{S}$. Then $\|\Phi\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathcal{S}\times\mathcal{T})}=\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ for any function Φ in $\ell^\infty(\mathcal{S}\times\mathcal{T})$ that is continuous in the variable $s\in\mathcal{S}$.*

2.2. A description of discrete Schur multipliers

Theorem 2.2.1. *Let* $\{u_s\}_{s\in\mathcal{S}}$ *and* $\{v_t\}_{t\in\mathcal{T}}$ *be families of vectors in a (not necessarily separable) Hilbert space* $\mathscr{H}$ *such that* $\|u_s\|\cdot\|v_t\|\leqslant 1$ *for all* s *in* $\mathcal{S}$ *and* t *in* $\mathcal{T}$. *Put* $\Phi(s,t)\stackrel{\text{def}}{=}(u_s,v_t)$ *for* $s\in\mathcal{S}$ *and* $t\in\mathcal{T}$. *Then* $\Phi\in\mathfrak{M}(\mathcal{S}\times\mathcal{T})$ *and* $\|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}\leqslant 1$.

Proof. By (2.1.1), it suffices to prove that

$$\|\{a(s,t)(u_s,v_t)\}\|_{\boldsymbol{S}_1}\leqslant\|\{a(s,t)\}\|_{\boldsymbol{S}_1}$$

for any matrix $\{a(s,t)\}$ that induces a trace class operator. Clearly, it suffices to consider the case when $\operatorname{rank}\{a(s,t)\}=1$. Moreover, one can assume that $\|a(s,t)\|_{\boldsymbol{S}_1}=1$. Then $a(s,t)=\alpha_s\beta_t$ for some $\alpha\in\ell^2(\mathcal{S})$ and $\beta\in\ell^2(\mathcal{T})$ such that $\|\alpha\|_{\ell^2(\mathcal{S})}=\|\beta\|_{\ell^2(\mathcal{T})}=1$. Let $\{e_j\}_{j\in J}$ be an orthonormal basis in $\mathscr{H}$, and put $\widehat{x}(j)\stackrel{\text{def}}{=}(x,e_j)$ for $j\in J$. Then

$$\begin{aligned}\|\{\alpha_s\beta_t(u_s,v_t)\}\|_{\boldsymbol{S}_1}&\leqslant\sum_{j\in J}\|\{\alpha_s\beta_t\widehat{u}_s(j)\overline{\widehat{v}_t(j)}\}\|_{\boldsymbol{S}_1}\\&=\sum_{j\in J}\|\{\alpha_s\widehat{u}_s(j)\}\|_{\ell^2(\mathcal{S})}\|\{\beta_t\overline{\widehat{v}_t(j)}\}\|_{\ell^2(\mathcal{T})}\\&\leqslant\left(\sum_{j\in J}\|\{\alpha_s\widehat{u}_s(j)\}\|^2_{\ell^2(\mathcal{S})}\right)^{1/2}\left(\sum_{j\in J}\|\{\beta_t\overline{\widehat{v}_t(j)}\}\|^2_{\ell^2(\mathcal{T})}\right)^{1/2}.\end{aligned}$$

Obviously,

$$\begin{aligned}\sum_{j\in J}\|\{\alpha_s\widehat{u}_s(j)\}\|^2_{\ell^2(\mathcal{S})}&=\sum_{j\in J}\sum_{s\in\mathcal{S}}|\alpha_s|^2|\widehat{u}_s(j)|^2\\&=\sum_{s\in\mathcal{S}}|\alpha_s|^2\sum_{j\in J}|\widehat{u}_s(j)|^2=\sum_{s\in\mathcal{S}}|\alpha_s|^2\|u_s\|^2\leqslant\sup_{s\in\mathcal{S}}\|u_s\|^2.\end{aligned}$$

Similarly, $\sum_{j\in J}\|\{\beta_t\overline{\widehat{v}_t(j)}\}\|^2_{\ell^2(\mathcal{T})}\leqslant\sup_{t\in\mathcal{T}}\|v_t\|^2$. Hence

$$\|\{\alpha_s\beta_t(u_s,v_t)\}\|_{\boldsymbol{S}_1}\leqslant\sup_{s\in\mathcal{S}}\|u_s\|\sup_{t\in\mathcal{T}}\|u_t\|\leqslant 1.\qquad\square$$

It is very non-trivial that the converse also holds (see Theorem 5.1 of the monograph [65], and also [66]). We state this result without a proof.

Theorem 2.2.2. *Let* $\Phi\stackrel{\text{def}}{=}\{\Phi(s,t)\}$ *be a Schur multiplier of* $\mathcal{B}(\mathcal{S}\times\mathcal{T})$, *and let* $\|\Phi\|_{\mathfrak{M}}\leqslant 1$. *Then there are two families* $\{u_s\}_{s\in\mathcal{S}}$ *and* $\{v_t\}_{t\in\mathcal{T}}$ *of vectors in a (not necessarily separable) Hilbert space* $\mathscr{H}$ *such that* $\|u_s\|\leqslant 1$ *for all* $s\in\mathcal{S}$, $\|v_t\|\leqslant 1$ *for all* $t\in\mathcal{T}$, *and*

$$\Phi(s,t)=(u_s,v_t),\qquad s\in\mathcal{S},\quad t\in\mathcal{T}.$$

Remark on Theorem 2.2.2. In this theorem we can additionally require that the linear spans of both the family $\{u_s\}_{s\in\mathcal{S}}$ *and the family* $\{v_t\}_{t\in\mathcal{T}}$ *are dense in* $\mathscr{H}$. Indeed, let $\mathscr{H}_1$ be the closed linear span of the family $\{v_t\}_{t\in\mathcal{T}}$ and let P_1 be the orthogonal projection onto $\mathscr{H}_1$. Then $\{P_1 u_s\}_{s\in\mathcal{S}}$ and $\{v_t\}_{t\in\mathcal{T}}$ are families in the Hilbert space $\mathscr{H}_1$ such that $\Phi(s,t) = (P_1 u_s, v_t)$ for all $(s,t) \in \mathcal{S}\times\mathcal{T}$. Now let $\mathscr{H}_2$ be the closed linear span of the family $\{P_1 u_s\}_{s\in\mathcal{S}}$ and let P_2 be the orthogonal projection onto $\mathscr{H}_2$. Then $\{P_1 u_s\}_{s\in\mathcal{S}}$ and $\{P_2 v_t\}_{t\in\mathcal{T}}$ are families in $\mathscr{H}_2$ such that $\Phi(s,t) = (P_1 u_s, P_2 v_t)$ for $(s,t) \in \mathcal{S}\times\mathcal{T}$. It is clear that the linear spans of the families $\{P_1 u_s\}_{s\in\mathcal{S}}$ and $\{P_2 v_t\}_{t\in\mathcal{T}}$ are dense in $\mathscr{H}_2$.

The following theorem is contained in the results of [39] and [5].

Theorem 2.2.3. *Let $\Phi \in \mathfrak{M}(\mathcal{S}\times\mathcal{T})$, where $\mathcal{S}$ and $\mathcal{T}$ are topological spaces. If Φ is continuous in each variable, then there are two families $\{u_s\}_{s\in\mathcal{S}}$ and $\{v_t\}_{t\in\mathcal{T}}$ in a (not necessarily separable) Hilbert space $\mathscr{H}$ such that:*

(a) *the linear span of $\{u_s\}_{s\in\mathcal{S}}$ is dense in $\mathscr{H}$;*
(b) *the linear span of $\{v_t\}_{t\in\mathcal{T}}$ is dense in $\mathscr{H}$;*
(c) *the map $s \mapsto u_s$ is weakly continuous;*
(d) *the map $t \mapsto v_t$ is weakly continuous;*
(e) $\|u_s\|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ *for all* $s \in \mathcal{S}$;
(f) $\|v_t\|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ *for all* $t \in \mathcal{T}$;
(g) $\Phi(s,t) = (u_s, v_t)$ *for all* $(s,t) \in \mathcal{S}\times\mathcal{T}$.

Proof. By Theorem 2.2.2 and the subsequent remark, there are families $\{u_s\}_{s\in\mathcal{S}}$ and $\{v_t\}_{t\in\mathcal{T}}$ satisfying the conditions (a), (b), (e), (f), and (g). The function $s \mapsto (u_s, h)$ is clearly continuous for $h = v_t$, where $t \in \mathcal{T}$. Thus, the function $s \mapsto (u_s, h)$ is continuous for all $h \in \mathscr{H}$ by (b). The map $s \mapsto u_s$ is then weakly continuous. Similarly, from (a) one can deduce weak continuity of the map $t \mapsto v_t$. $\square$

Remark. If at least one of the spaces $\mathcal{S}$ and $\mathcal{T}$ is separable, then the space $\mathscr{H}$ is also separable. Indeed, it suffices to observe that if, for example, $\mathcal{S}$ is separable, then the closed linear span of the family $\{u_s\}_{s\in\mathcal{S}}$ is separable.

This remark leads us to the following assertion.

Theorem 2.2.4. *Let $\Phi \in \mathfrak{M}(\mathcal{S}\times\mathcal{T})$, where $\mathcal{S}$ and $\mathcal{T}$ are topological spaces at least one of which is separable. Suppose that Φ is continuous in each variable. Then there exist a sequence $\{\varphi_n\}_{n\geqslant 1}$ of continuous functions on $\mathcal{S}$ and a sequence $\{\psi_n\}_{n\geqslant 1}$ of continuous functions on $\mathcal{T}$ such that*

$$\sum_{n=1}^{\infty} |\varphi_n(s)|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}, \qquad \sum_{n=1}^{\infty} |\psi_n(t)|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})},$$
$$\sum_{n=1}^{\infty} \varphi_n(s)\psi_n(t) = \Phi(s,t)$$

for all $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

Proof. Let $\{u_s\}_{s\in\mathcal{S}}$ and $\{v_t\}_{t\in\mathcal{T}}$ be two families in a Hilbert space $\mathscr{H}$ whose existence is guaranteed by Theorem 2.2.3. It follows from the remark after that theorem that the space $\mathscr{H}$ is separable. Let $\{e_n\}_{n=1}^N$ be an orthonormal basis in $\mathscr{H}$, where $0 \leqslant N \leqslant \infty$. It remains to define $\varphi_n(s) \stackrel{\text{def}}{=} (u_s, e_n)$ and $\psi_n(t) \stackrel{\text{def}}{=} (e_n, v_t)$; if $N < \infty$, then $\varphi_n(s) \stackrel{\text{def}}{=} \psi_n(t) \stackrel{\text{def}}{=} 0$ for $n > N$. $\square$

Definition. A map g from a topological space $\mathcal{T}$ to a Hilbert space $\mathscr{H}$ is said to be *weakly Borel measurable* if the function $t \mapsto (g(t), u)$ is Borel measurable on T for any u in $\mathscr{H}$.

It is easy to see that we need only verify the Borel measurability of the function $t \mapsto (g(t), u)$ for vectors u in a subset of $\mathscr{H}$ whose linear span is dense in $\mathscr{H}$.

Theorem 2.2.5. *Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces, and let $\Phi \in \mathfrak{M}(\mathcal{S}\times\mathcal{T})$. Suppose that Φ is Borel measurable in each variable. Then there exist two families $\{u_s\}_{s\in\mathcal{S}}$ and $\{v_t\}_{t\in\mathcal{T}}$ in a (not necessarily separable) Hilbert space $\mathscr{H}$ such that:*
- (a) *the linear span of $\{u_s\}_{s\in\mathcal{S}}$ is dense in $\mathscr{H}$;*
- (b) *the linear span of $\{v_t\}_{t\in\mathcal{T}}$ is dense in $\mathscr{H}$;*
- (c) *the map $s \mapsto u_s$ is weakly Borel measurable;*
- (d) *the map $t \mapsto v_t$ is weakly Borel measurable;*
- (e) *$\|u_s\|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ for all $s \in \mathcal{S}$;*
- (f) *$\|v_t\|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ for all $t \in \mathcal{T}$;*
- (g) *$\Phi(s,t) = (u_s, v_t)$ for all $(s,t) \in \mathcal{S}\times\mathcal{T}$.*

The proof of Theorem 2.2.5 repeats almost word-for-word the proof of Theorem 2.2.3.

Theorem 2.2.6. *Let $\mathcal{S}$ and $\mathcal{T}$ be topological spaces, and let Φ be a Borel function in $\mathfrak{M}(\mathcal{S}\times\mathcal{T})$. Suppose that μ and ν are σ-finite Borel measures on $\mathcal{S}$ and $\mathcal{T}$. Then there exist sequences $\{\varphi_k\}_{k\geqslant 1}$ and $\{\psi_k\}_{k\geqslant 1}$ such that:*
- (a) *$\varphi_k \in L^\infty(\mu)$ and $\psi_k \in L^\infty(\nu)$ for all $k \geqslant 1$;*
- (b) *$\sum_{k=1}^\infty |\varphi_k(s)|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ for μ-almost all $s \in \mathcal{S}$;*
- (c) *$\sum_{k=1}^\infty |\psi_k(t)|^2 \leqslant \|\Phi\|_{\mathfrak{M}(\mathcal{S}\times\mathcal{T})}$ for ν-almost all $t \in \mathcal{T}$;*
- (d) *$\Phi(s,t) = \sum_{k=1}^\infty \varphi_k(s)\psi_k(t)$ for $\mu\otimes\nu$-almost all $(s,t) \in \mathcal{S}\times\mathcal{T}$.*

Proof. Clearly, we can assume that $\|\Phi\|_{\mathfrak{M}(S\times T)} = 1$. Let $\{u_s\}_{s\in\mathcal{S}}$ and $\{v_t\}_{t\in\mathcal{T}}$ be families in a Hilbert space $\mathscr{H}$ whose existence is guaranteed by Theorem 2.2.5. Let $\{e_j\}_{j\in J}$ be an orthonormal basis in $\mathscr{H}$. Put $\varphi_j(s) \stackrel{\text{def}}{=} (u_s, e_j)$ and $\psi_j(t) \stackrel{\text{def}}{=} (e_j, v_t)$. Then the functions φ_j and ψ_j are Borel measurable,

$$\sum_{j\in J} |\varphi_j(s)|^2 \leqslant 1 \quad \text{for all } s \in \mathcal{S}, \qquad \sum_{j\in J} |\psi_j(t)|^2 \leqslant 1 \quad \text{for all } t \in \mathcal{T},$$

and

$$\Phi(s,t) = \sum_{j\in J} \varphi_j(s)\psi_j(t) \quad \text{for all } (s,t) \in \mathcal{S}\times\mathcal{T}.$$

This immediately completes the proof of the theorem in the case when J is at most countable. To consider the case of an arbitrary set J, we set $\Psi(s,t) \stackrel{\text{def}}{=} \sum_{j\in J} |\varphi_j(s)|\cdot|\psi_j(t)|$. By the Cauchy–Schwarz inequality,

$$\Psi(s,t) \leqslant \left(\sum_{j\in J} |\varphi_j(s)|^2\right)^{1/2} \left(\sum_{j\in J} |\psi_j(t)|^2\right)^{1/2} \leqslant 1.$$

We can assume that μ and ν are probability measures. Let $J_s \stackrel{\text{def}}{=} \{j \in J\colon \varphi_j(s) \neq 0\}$, where $s \in \mathcal{S}$. Note that J_s is at most countable for each $s \in \mathcal{S}$, because

$\sum_{j\in J}|\varphi_j(s)|^2 \leqslant 1$. It is easy to see that for all s in $\mathcal{S}$,

$$\sum_{j\in J}|\varphi_j(s)|\int_{\mathcal{T}}|\psi_j(t)|\,d\nu(t) = \sum_{j\in J_s}|\varphi_j(s)|\int_{\mathcal{T}}|\psi_j(t)|\,d\nu(t)$$
$$= \int_{\mathcal{T}}\Bigl(\sum_{j\in J_s}|\varphi_j(s)|\cdot|\psi_j(t)|\Bigr)\,d\nu(t) = \int_{\mathcal{T}}\Psi(s,t)\,d\nu(t).$$

To integrate with respect to s, we now consider the at most countable set

$$J_\flat \stackrel{\text{def}}{=} \Bigl\{j\in J\colon \int_{\mathcal{T}}|\psi_j(t)|\,d\nu(t)\neq 0\Bigr\}.$$

Then

$$\sum_{j\in J}\int_{\mathcal{S}}|\varphi_j(s)|\,d\mu(s)\int_{\mathcal{T}}|\psi_j(t)|\,d\nu(t) = \sum_{j\in J_\flat}\int_{\mathcal{S}}|\varphi_j(s)|\,d\mu(s)\int_{\mathcal{T}}|\psi_j(t)|\,d\nu(t)$$
$$= \int_{\mathcal{S}}\Bigl(\int_{\mathcal{T}}\Psi(s,t)\,d\nu(t)\Bigr)\,d\mu(s).$$

It is now clear that $\int_{\mathcal{S}}\bigl(\int_{\mathcal{T}}\sum_{j\in J\setminus J_\flat}|\varphi_j(s)|\cdot|\psi_j(t)|\,d\nu(t)\bigr)\,d\mu(s)=0$. This, together with the inequality $\bigl|\Phi(s,t)-\sum_{j\in J_\flat}\varphi_j(s)\psi_j(t)\bigr| \leqslant \sum_{j\in J\setminus J_\flat}|\varphi_j(s)|\cdot|\psi_j(t)|$, implies that

$$\sum_{j\in J_\flat}\varphi_j(s)\psi_j(t)=\Phi(s,t)$$

for $\mu\otimes\nu$-almost all $(s,t)\in\mathcal{S}\times\mathcal{T}$. □

We consider some examples of Schur multipliers. Let $\mathcal{M}(\mathbb{T}^2)$ be the space of all complex Borel measures on the 2-torus $\mathbb{T}^2$ with the norm $\|\mu\|_{\mathcal{M}(\mathbb{T}^2)} \stackrel{\text{def}}{=} |\mu|(\mathbb{T})$.

Example. Let $\mu\in\mathcal{M}(\mathbb{T}^2)$. Then $\{\widehat{\mu}(m,n)\}\in\mathfrak{M}(\mathbb{Z}^2)$ and $\|\widehat{\mu}\|_{\mathfrak{M}(\mathbb{Z}\times\mathbb{Z})}\leqslant\|\mu\|$.

This fact is an obvious consequence of Theorem 2.2.1. It is clear that not every Schur multiplier $\boldsymbol{a}\in\mathfrak{M}(\mathbb{Z}\times\mathbb{Z})$ can be represented as $\boldsymbol{a}=\widehat{\mu}$, where $\mu\in\mathcal{M}(\mathbb{T}^2)$. Consider, for example, the case when the matrix $\boldsymbol{a}=\{a_{mn}\}_{m,n\in\mathbb{Z}}$ consists of the same columns (or rows). To be definite, suppose that $a_{mn}=t_n$ for all $m,n\in\mathbb{Z}$. Then $\boldsymbol{a}\in\mathfrak{M}(\mathbb{Z}\times\mathbb{Z})$ if and only if $\boldsymbol{a}\in\ell^\infty(\mathbb{Z}\times\mathbb{Z})$ and $\|\boldsymbol{a}\|_{\mathfrak{M}(\mathbb{Z}\times\mathbb{Z})}=\|\boldsymbol{a}\|_{\ell^\infty(\mathbb{Z}\times\mathbb{Z})}=\|\{t_n\}\|_{\ell^\infty}$. Of course, by no means are all such matrices $\boldsymbol{a}$ with bounded entries representable as $\boldsymbol{a}=\widehat{\mu}$, where $\mu\in\mathcal{M}(\mathbb{T}^2)$.

On the other hand, if we assume that a matrix $\boldsymbol{a}=\{a_{mn}\}_{m,n\in\mathbb{Z}}$ is a *Laurent* matrix, that is, $a_{mn}=t_{m-n}$, then the situation changes considerably.

Theorem 2.2.7. *Let $A=\{a_{mn}\}_{m,n\in\mathbb{Z}}$ be a Laurent matrix. Then $A\in\mathfrak{M}(\mathbb{Z}^2)$ if and only if $a_{mn}=\widehat{\mu}(m-n)$ for some measure μ in $\mathcal{M}(\mathbb{T})$, and $\|\boldsymbol{a}\|_{\mathfrak{M}(\mathbb{Z}^2)}=\|\mu\|_{\mathcal{M}(\mathbb{T})}$.*

All these results can be generalized to locally compact Abelian groups. In the case of a non-discrete Abelian group G one has to assume that the corresponding functions are continuous in the statement of the analogue of Theorem 2.2.7.

Theorem 2.2.8. *Let h be a continuous function on $\mathbb{R}$. Then the matrix $A=\{h(s-t)\}_{s,t\in\mathbb{R}}$ belongs to $\mathfrak{M}(\mathbb{R}\times\mathbb{R})$ if and only if there exists a complex Borel measure μ such that $h=\mathscr{F}\mu$. Moreover, $\|A\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})}=\|\mu\|_{\mathcal{M}(\mathbb{R})}$.*

2.3. Double operator integrals

Double operator integrals are expressions of the form

$$\int_{\mathcal{S}}\int_{\mathcal{T}}\Phi(s,t)\,dE_1(s)T\,dE_2(t), \tag{2.3.1}$$

where E_1 and E_2 are spectral measures on a separable Hilbert space $\mathscr{H}$, Φ is a bounded measurable function, and T is a bounded linear operator on $\mathscr{H}$.

Double operator integrals appeared in the paper [23]. In the papers [19]–[21] Birman and Solomyak created a nice theory of double operator integrals. Their idea was first to define double operator integrals of the form (2.3.1) for arbitrary bounded measurable functions Φ and operators T of Hilbert–Schmidt class $\boldsymbol{S}_2$. For this purpose they introduced a spectral measure $\mathscr{E}$ that takes values in the set of orthogonal projections on the Hilbert space $\boldsymbol{S}_2$ and is defined by

$$\mathscr{E}(\Lambda\times\Delta)T=E_1(\Lambda)TE_2(\Delta),\qquad T\in\boldsymbol{S}_2,$$

where Λ and Δ are measurable subsets of $\mathcal{S}$ and $\mathcal{T}$. It is clear that left multiplication by $E_1(\Lambda)$ commutes with right multiplication by $E_2(\Delta)$. In [22] it was shown that $\mathscr{E}$ extends to a spectral measure on $\mathcal{S}\times\mathcal{T}$. In this situation the double operator integral (2.3.1) is defined by

$$\int_{\mathcal{S}}\int_{\mathcal{T}}\Phi(s,t)\,dE_1(s)T\,dE_2(t)\stackrel{\text{def}}{=}\biggl(\int_{\mathcal{S}\times\mathcal{T}}\Phi\,d\mathscr{E}\biggr)T.$$

It follows immediately from this definition that

$$\biggl\|\int_{\mathcal{S}}\int_{\mathcal{T}}\Phi(s,t)\,dE_1(s)T\,dE_2(t)\biggr\|_{\boldsymbol{S}_2}\leqslant\|\Phi\|_{L^\infty}\|T\|_{\boldsymbol{S}_2}.$$

If a function Φ possesses the property that

$$T\in\boldsymbol{S}_1\quad\Rightarrow\quad\int_{\mathcal{S}}\int_{\mathcal{T}}\Phi(s,t)\,dE_1(s)T\,dE_2(t)\in\boldsymbol{S}_1,$$

then Φ is called a *Schur multiplier of the space $\boldsymbol{S}_1$ with respect to the spectral measures E_1 and E_2*.

To define double operator integrals (2.3.1) for bounded operators T, we consider the transformer

$$Q\mapsto\int_{\mathcal{T}}\int_{\mathcal{S}}\Phi(t,s)\,dE_2(t)\,Q\,dE_1(s),\qquad Q\in\boldsymbol{S}_1,$$

and assume that the function $(y,x)\mapsto\Phi(y,x)$ is a Schur multiplier of $\boldsymbol{S}_1$ with respect to E_2 and E_1. In this case the transformer

$$T\mapsto\int_{\mathcal{S}}\int_{\mathcal{T}}\Phi(s,t)\,dE_1(s)T\,dE_2(t),\qquad T\in\boldsymbol{S}_2, \tag{2.3.2}$$

extends by duality to a bounded linear transformer on the space of bounded linear operators on $\mathscr{H}$. In this case Φ is said to be a *Schur multiplier* (*with respect to* E_1 *and* E_2) *of the space of bounded linear operators.* We denote the space of such Schur multipliers by $\mathfrak{M}(E_1,E_2)$. The norm of Φ in $\mathfrak{M}(E_1,E_2)$ is defined as the norm of the transformer (2.3.2) on the space of bounded linear operators.

It is easy to see that if a function Φ on $\mathcal{S}\times\mathcal{T}$ belongs to the *projective tensor product* $L^\infty(E_1)\widehat{\otimes}L^\infty(E_2)$ of the spaces $L^\infty(E_1)$ and $L^\infty(E_2)$ (that is, Φ admits a representation $\Phi(s,t)=\sum_{n\geqslant0}\varphi_n(s)\psi_n(t)$, where $\sum_{n\geqslant0}\|\varphi_n\|_{L^\infty(E_1)}\|\psi_n\|_{L^\infty(E_2)}<\infty$), then $\Phi\in\mathfrak{M}(E_1,E_2)$. For such functions Φ,

$$\int_{\mathcal{S}}\int_{\mathcal{T}}\Phi(s,t)\,dE_1(s)T\,dE_2(t)=\sum_{n\geqslant0}\biggl(\int_{\mathcal{S}}\varphi_n\,dE_1\biggr)T\biggl(\int_{\mathcal{T}}\psi_n\,dE_2\biggr).$$

More generally, $\Phi\in\mathfrak{M}(E_1,E_2)$ if Φ belongs to the *integral projective tensor product* $L^\infty(E_1)\,\widehat{\otimes}_{\rm i}\,L^\infty(E_2)$ of the spaces $L^\infty(E_1)$ and $L^\infty(E_2)$, that is, Φ admits a representation

$$\Phi(s,t)=\int_\Omega\varphi(s,w)\psi(t,w)\,d\sigma(w),\tag{2.3.3}$$

where (Ω,σ) is a space with a σ-finite measure, φ is a measurable function on $\mathcal{S}\times\Omega$, ψ is a measurable function on $\mathcal{T}\times\Omega$, and

$$\int_\Omega\|\varphi(\,\cdot\,,w)\|_{L^\infty(E_1)}\|\psi(\,\cdot\,,w)\|_{L^\infty(E_2)}\,d\sigma(w)<\infty.$$

It turns out that all Schur multipliers can be obtained in this way (see Theorem 2.3.1 below).

Another sufficient condition for a function to be a Schur multiplier can be stated in terms of the Haagerup tensor product $L^\infty(E_1)\,\widehat{\otimes}_{\rm h}\,L^\infty(E_2)$, which is defined as the space of functions Φ of the form

$$\Phi(s,t)=\sum_{n\geqslant0}\varphi_n(s)\psi_n(t),\tag{2.3.4}$$

where $\{\varphi_n\}_{n\geqslant0}\in L^\infty_{E_1}(\ell^2)$ and $\{\psi_n\}_{n\geqslant0}\in L^\infty_{E_2}(\ell^2)$. Let

$$\|\Phi\|_{L^\infty(E_1)\,\widehat{\otimes}_{\rm h}\,L^\infty(E_2)}\stackrel{\rm def}{=}\inf\biggl\|\sum_{n\geqslant0}|\varphi_n|^2\biggr\|^{1/2}_{L^\infty(E_1)}\biggl\|\sum_{n\geqslant0}|\psi_n|^2\biggr\|^{1/2}_{L^\infty(E_2)},$$

where the infimum is taken over all representations of Φ in the form (2.3.4). It is easy to verify that if $\Phi\in L^\infty(E_1)\,\widehat{\otimes}_{\rm h}\,L^\infty(E_2)$, then $\Phi\in\mathfrak{M}(E_1,E_2)$ and

$$\iint\Phi(s,t)\,dE_1(s)T\,dE_2(t)=\sum_{n\geqslant0}\biggl(\int\varphi_n\,dE_1\biggr)T\biggl(\int\psi_n\,dE_2\biggr),\tag{2.3.5}$$

where the series on the right-hand side is convergent in the weak operator topology, and

$$\|\Phi\|_{\mathfrak{M}(E_1,E_2)}\leqslant\|\Phi\|_{L^\infty(E_1)\,\otimes_{\rm h}\,L^\infty(E_2)}.$$

As is clear from the next theorem, the condition $\Phi\in L^\infty(E_1)\,\widehat{\otimes}_{\rm h}\,L^\infty(E_2)$ is not only sufficient, but also necessary.

Theorem 2.3.1. *Let Φ be a measurable function on $\mathcal{S} \times \mathcal{T}$, and let μ and ν be positive σ-finite measures on $\mathcal{S}$ and $\mathcal{T}$ which are mutually absolutely continuous with respect to E_1 and E_2. Then the following conditions are equivalent*:

(a) $\Phi \in \mathfrak{M}(E_1, E_2)$;

(b) $\Phi \in L^\infty(E_1) \widehat{\otimes}_{\mathrm{i}} L^\infty(E_2)$;

(c) $\Phi \in L^\infty(E_1) \widehat{\otimes}_{\mathrm{h}} L^\infty(E_2)$;

(d) *there exist a σ-finite measure σ on a set Ω and measurable functions φ on $\mathcal{S} \times \Omega$ and ψ on $\mathcal{T} \times \Omega$ such that* (2.3.3) *holds and*

$$\left\| \left(\int_\Omega |\varphi(\,\cdot\,, w)|^2 \, d\sigma(w) \right)^{1/2} \right\|_{L^\infty(E_1)} \left\| \left(\int_\Omega |\psi(\,\cdot\,, w)|^2 \, d\sigma(w) \right)^{1/2} \right\|_{L^\infty(E_2)} < \infty; \tag{2.3.6}$$

(e) *if an integral operator $f \mapsto \int k(x,y) f(y) \, d\nu(y)$ from $L^2(\nu)$ to $L^2(\mu)$ belongs to $\boldsymbol{S}_1$, then the integral operator $f \mapsto \int \Phi(x,y) k(x,y) f(y) \, d\nu(y)$ belongs to the same class.*

The implications (d) $\Rightarrow$ (a) $\Leftrightarrow$ (e) were established in [21]. In the case of matrix Schur multipliers the implication (a) $\Rightarrow$ (b) was proved in [16]. We refer the reader to [56] for the proof of the equivalence of (a), (b), and (d). The proof of the equivalence of (c) and (d) is elementary.

It is easy to see that the conditions (a)–(e) are also equivalent to the condition that Φ is a Schur multiplier of $\boldsymbol{S}_1$.

Note that one can also define double operator integrals of the form (2.3.1) in the case when E_1 and E_2 are spectral measures on different Hilbert spaces and T is an operator from one Hilbert space to another.

Remark. It follows easily from Theorems 2.2.6 and 2.3.1 that if $\mathcal{S}$ and $\mathcal{T}$ are topological spaces and Φ is a Borel function on $\mathcal{S} \times \mathcal{T}$ of class $\mathfrak{M}(\mathcal{S} \times \mathcal{T})$ (that is, Φ is a discrete Schur multiplier), then $\Phi \in \mathfrak{M}(E_1, E_2)$ for any Borel spectral measures E_1 and E_2 on $\mathcal{S}$ and $\mathcal{T}$.

Double operator integrals can also be defined with respect to semispectral measures. We recall that a *semispectral measure* $\mathcal{E}$ on a measurable space $(\mathscr{X}, \mathfrak{B})$ is a map defined on the σ-algebra $\mathfrak{B}$, with values in the set of bounded linear operators on a Hilbert space $\mathscr{H}$, countably additive in the strong operator topology, and such that

$$\mathcal{E}(\Delta) \geqslant \mathbf{0}, \quad \Delta \in \mathfrak{B}, \quad \mathcal{E}(\varnothing) = \mathbf{0} \quad \text{and} \quad \mathcal{E}(\mathscr{X}) = I.$$

By Naimark's theorem [46], each semispectral measure $\mathcal{E}$ has a *spectral dilation*, that is, a spectral measure E defined on the same measurable space $(\mathscr{X}, \mathfrak{B})$, taking values in the set of orthogonal projections on a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$, and such that

$$\mathcal{E}(\Delta) = P_{\mathscr{H}} E(\Delta) \big| \mathscr{H}, \qquad \Delta \in \mathfrak{B},$$

where $P_{\mathscr{H}}$ is the orthogonal projection from $\mathscr{K}$ onto $\mathscr{H}$. Such a spectral dilation can be chosen to be *minimal* in the sense that

$$\mathscr{K} = \operatorname{clos\,span}\{E(\Delta)\mathscr{H} \colon \Delta \in \mathfrak{B}\}.$$

It was shown in [44] that if E is a minimal spectral dilation of a semispectral measure $\mathscr{E}$, then E and $\mathscr{E}$ are mutually absolutely continuous.

Integrals with respect to semispectral measures are defined as follows:

$$\int_{\mathscr{X}} \varphi(x)\,d\mathcal{E}(x) = P_{\mathscr{H}}\left(\int_{\mathscr{X}} \varphi(x)\,dE(x)\right)\Big|\mathscr{H}, \qquad \varphi \in L^\infty(\mathcal{E}) \stackrel{\text{def}}{=} L^\infty(E).$$

If $\mathscr{E}_1$ and $\mathscr{E}_2$ are semispectral measures on $(\mathscr{X}_1, \mathfrak{B}_1)$ and $(\mathscr{X}_2, \mathfrak{B}_2)$, E_1 and E_2 are spectral dilations of them on Hilbert spaces $\mathscr{K}_1$ and $\mathscr{K}_2$, and a function Φ on $\mathscr{X}_1 \times \mathscr{X}_2$ satisfies the equivalent conditions of Theorem 2.3.1, then the double operator integral with respect to $\mathcal{E}_1$ and $\mathcal{E}_2$ is defined by

$$\int_{\mathscr{X}_1}\int_{\mathscr{X}_2} \Phi(x_1,x_2)\,d\mathcal{E}_1(x_1)Q\,d\mathcal{E}_2(x_2) = P_{\mathscr{H}}^{[1]}\int_{\mathscr{X}_1}\int_{\mathscr{X}_2} \Phi(x_1,x_2)\,dE_1(x_1)\big(QP_{\mathscr{H}}^{[2]}\big)\,dE_2(x_2)\Big|\mathscr{H}$$

for an arbitrary bounded linear operator Q on $\mathscr{H}$. Here $P_{\mathscr{H}}^{[1]}$ and $P_{\mathscr{H}}^{[2]}$ are the orthogonal projections from $\mathscr{K}_1$ and $\mathscr{K}_2$ onto $\mathscr{H}$. If $\Phi \in L^\infty(E_1) \otimes_{\mathrm{h}} L^\infty(E_2)$, then

$$\iint \Phi(x_1,x_2)\,d\mathcal{E}_1(x_1)T\,d\mathcal{E}_2(x_2) = \sum_{n\geqslant 0}\left(\int \varphi_n\,d\mathcal{E}_1\right)T\left(\int \psi_n\,d\mathcal{E}_2\right), \qquad (2.3.7)$$

where $T \in \mathcal{B}(\mathscr{H})$, and φ_n and ψ_n are functions in the representation (2.3.4).

Double operator integrals with respect to semispectral measures were introduced in [57] (see also [62]).

Chapter III. Operator Lipschitz function on subsets of the plane

In this chapter we study operator Lipschitz and commutator Lipschitz functions on closed subsets of the complex plane. A significant role will be played by Schur multipliers. We offer two methods for obtaining difference and commutator estimates. The first method uses discrete Schur multipliers and approximation by operators with finite spectrum. The second method is based on double operator integrals.

3.1. Operator Lipschitz and commutator Lipschitz functions on closed subsets of the plane

We define here the classes of operator Lipschitz functions and commutator Lipschitz functions on closed subsets of the plane. We will see that, unlike the case of functions on the line and the circle, these two classes by no means have to coincide. When defining them, we consider only bounded operators. In the next section we will see that if we admit not necessarily bounded operators, then we obtain the same classes of functions.

Let $\mathfrak{F}$ be a non-empty subset of the complex plane $\mathbb{C}$. We denote by $\mathrm{Lip}(\mathfrak{F})$ the space of functions $f\colon \mathfrak{F} \to \mathbb{C}$ satisfying the *Lipschitz condition*:

$$|f(z) - f(w)| \leqslant C|z-w|, \qquad z, w \in \mathbb{C}. \qquad (3.1.1)$$

The smallest constant $C \geqslant 0$ satisfying (3.1.1) is denoted by $\|f\|_{\mathrm{Lip}(\mathfrak{F})} = \|f\|_{\mathrm{Lip}}$. Let $\|f\|_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} \infty$ if $f \notin \mathrm{Lip}(\mathfrak{F})$.

Usually we require that the set $\mathfrak{F}$ be closed.

It follows easily from the spectral theorem for pairs of *commuting* normal operators that the inequality

$$\|f(N_1) - f(N_2)\| \leqslant \|f\|_{\mathrm{Lip}(\mathfrak{F})}\|N_1 - N_2\| \tag{3.1.2}$$

holds for any commuting normal operators N_1 and N_2 whose spectra are contained in $\mathfrak{F}$.

A complex continuous function f on a non-empty closed set $\mathfrak{F} \subset \mathbb{C}$ will be said to be *operator Lipschitz* if there exists a positive number C such that

$$\|f(N_1) - f(N_2)\| \leqslant C\|N_1 - N_2\| \tag{3.1.3}$$

for any normal operators N_1 and N_2 with spectra in $\mathfrak{F}$. We denote the space of operator Lipschitz functions on $\mathfrak{F}$ by $\mathrm{OL}(\mathfrak{F})$. The smallest constant C satisfying (3.1.3) is denoted by $\|f\|_{\mathrm{OL}(\mathfrak{F})} = \|f\|_{\mathrm{OL}}$. Let $\|f\|_{\mathrm{OL}} = \infty$ if $f \notin \mathrm{OL}(\mathfrak{F})$.

If a function f is defined on a bigger set $\mathfrak{G} \supset \mathfrak{F}$, then we will usually write for brevity $f \in \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}(\mathfrak{F})}$ instead of $f|\mathfrak{F} \in \mathrm{OL}(\mathfrak{F})$ and $\|f|\mathfrak{F}\|_{\mathrm{OL}(\mathfrak{F})}$. We will also use the same convention for other function spaces.

It is easy to see that $\mathrm{OL}(\mathfrak{F}) \subset \mathrm{Lip}(\mathfrak{F})$ and $\|f\|_{\mathrm{Lip}(\mathfrak{F})} \leqslant \|f\|_{\mathrm{OL}(\mathfrak{F})}$ for any $f \in \mathrm{OL}(\mathfrak{F})$. We will see in § 3.14 that the equality $\mathrm{OL}(\mathfrak{F}) = \mathrm{Lip}(\mathfrak{F})$ holds only for finite sets $\mathfrak{F}$.

If $f \in \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}} \leqslant 1$, then

$$\|f(N_1)U - Uf(N_2)\| \leqslant \|N_1U - UN_2\| \tag{3.1.4}$$

for all unitary operators U and all normal operators N_1 and N_2 such that $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}$. To see this, it suffices to apply the inequality (3.1.3) with $C = 1$ to the normal operators U^*N_1U and N_2. Conversely, if (3.1.4) holds for all unitary operators U and all normal operators N_1 and N_2 such that $N_1 = N_2$ and $\sigma(N_1) = \sigma(N_2) \subset \mathfrak{F}$, then $f \in \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant 1$. Indeed, applying the inequality (3.1.4) to the operators $\mathcal{N}_1 = \mathcal{N}_2 = \begin{pmatrix} N_1 & \mathbf{0} \\ \mathbf{0} & N_2 \end{pmatrix}$ and $\mathcal{U} = \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix}$, we find that $\|f(N_1) - f(N_2)\| \leqslant \|N_1 - N_2\|$. In this argument we have dealt only with self-adjoint unitary operators $\mathcal{U}$, that is, normal operators $\mathcal{U}$ such that $\mathcal{U}^2 = I$, or, what is the same, unitary operators with spectra in $\{-1, 1\}$.

Theorem 3.1.1. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. Then the following conditions are equivalent*:

(a) $\|f(N_1) - f(N_2)\| \leqslant \|N_1 - N_2\|$ *for all normal operators N_1 and N_2 such that $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}$*;

(b) $\|f(N_1)U - Uf(N_2)\| \leqslant \|N_1U - UN_2\|$ *for all unitary operators U and all normal operators N_1 and N_2 such that $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}$*;

(c) $\|f(N)U - Uf(N)\| \leqslant \|NU - UN\|$ *for all self-adjoint unitary operators U and all normal operators N such that $\sigma(N) \subset \mathfrak{F}$*;

(d) $\|f(N)A - Af(N)\| \leqslant \|NA - AN\|$ *for all self-adjoint operators A and all normal operators N such that $\sigma(N) \subset \mathfrak{F}$.*

Proof. The equivalence of (a), (b), and (c) was proved in essence before the statement of the theorem. The implication (d) ⇒ (c) is obvious. It remains to prove that (c) implies (d). Denote by $\mathfrak{X}$ the set of operators R such that $\|f(N)R-Rf(N)\|\leqslant\|NR-RN\|$ for all normal operators N with spectrum in $\mathfrak{F}$. It is clear that the set $\mathfrak{X}$ is closed in the norm and $\alpha U+\beta I\in\mathfrak{X}$ for any unitary operator U and all $\alpha,\beta\in\mathbb{C}$. To prove that an arbitrary self-adjoint operator A belongs to $\mathfrak{X}$, it suffices to observe that the operator $(I-\varepsilon\mathrm{i}A)(I+\varepsilon\mathrm{i}A)^{-1}$ is unitary for all ε in $(-\|A\|^{-1},\|A\|^{-1})$ and

$$A=\lim_{\varepsilon\to0}\frac{1}{2\varepsilon\mathrm{i}}\big(I-(I-\varepsilon\mathrm{i}A)(I+\varepsilon\mathrm{i}A)^{-1}\big).\qquad\square$$

Remark. A unitary operator U is self-adjoint if and only if it can be represented in the form $U=2P-I$, where P is an orthogonal projection. The condition (c) in Theorem 3.1.1 can be rewritten as follows: $\|f(N)P-Pf(N)\|\leqslant\|NP-PN\|$ *for all orthogonal projections P and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$.*

Theorem 3.1.2. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. Then the following conditions are equivalent*:

(a) $\|f(N_1)-f(N_2)\|\leqslant\|N_1-N_2\|$ *for all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$;*

(b) $\|f(N)R-Rf(N)\|\leqslant\max\{\|NR-RN\|,\|N^*R-RN^*\|\}$ *for all operators $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$;*

(c) $\|f(N_1)R-Rf(N_2)\|\leqslant\max\{\|N_1R-RN_2\|,\|N_1^*R-RN_2^*\|\}$ *for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$.*

Proof. Let us first prove the implication (a) ⇒ (b). Suppose that (a) holds. Then it follows from Theorem 3.1.1 that $\|f(N)A-Af(N)\|\leqslant\|NA-AN\|$ for all self-adjoint operators A and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$. Applying this assertion to the normal operator $\begin{pmatrix}N&\mathbf{0}\\\mathbf{0}&N\end{pmatrix}$ and the self-adjoint operator $\begin{pmatrix}\mathbf{0}&R\\R^*&\mathbf{0}\end{pmatrix}$, we obtain

$$\max\{\|f(N)R-Rf(N)\|,\|f(N)R^*-R^*f(N)\|\}\\ \leqslant\max\{\|NR-RN)\|,\|NR^*-R^*N\|\},$$

which implies (b).

Applying (b) to the normal operator $\begin{pmatrix}N_1&\mathbf{0}\\\mathbf{0}&N_2\end{pmatrix}$ and the operator $\begin{pmatrix}\mathbf{0}&R\\\mathbf{0}&\mathbf{0}\end{pmatrix}$, we get (c). The implication (c) ⇒ (a) is obvious. $\square$

In the proofs of Theorems 3.1.1 and 3.1.2 we have used the standard technique of passing to block matrix operators, which in certain cases allows one to pass from one operator to a pair of operators. This technique will be useful in what follows. Kittaneh [40] calls it the Berberian trick, apparently having in mind the paper [17] by Berberian.

Theorems 3.1.1 and 3.1.2 are contained in Theorem 3.1 of [13], but in a certain form they can in essence be extracted from the paper [38], where arbitrary symmetric norms are also considered together with the operator norm.

We note that the equality $\|N_1^*R - RN_2^*\| = \|N_1R - RN_2\|$, and hence also the equality

$$\max\{\|N_1R - RN_2\|, \|N_1^*R - RN_2^*\|\} = \|N_1R - RN_2\|,$$

holds in each of the following special cases:

1) the operators N_1 and N_2 are self-adjoint (this is the case if $\mathfrak{F} \subset \mathbb{R}$);
2) the operators N_1 and N_2 are unitary (this is the case if $\mathfrak{F} \subset \mathbb{T}$);
3) R is self-adjoint and $N_1 = N_2$;
4) R is a unitary operator.

A complex function f continuous on a closed set $\mathfrak{F} \subset \mathbb{C}$ is said to be *commutator Lipschitz* if there is a number $C \geqslant 0$ such that

$$\|f(N)R - Rf(N)\| \leqslant C\|NR - RN\| \tag{3.1.5}$$

for any $R \in \mathcal{B}(\mathscr{H})$ and any normal operator N with spectrum in $\mathfrak{F}$. We denote the set of commutator Lipschitz functions on $\mathfrak{F}$ by $\mathrm{CL}(\mathfrak{F})$. The smallest constant C satisfying (3.1.5) is denoted by $\|f\|_{\mathrm{CL}(\mathfrak{F})} = \|f\|_{\mathrm{CL}}$. Let $\|f\|_{\mathrm{CL}(\mathfrak{F})} = \infty$ if $f \notin \mathrm{CL}(\mathfrak{F})$.

Theorem 3.1.3. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. The following three conditions are equivalent*:

(a) $\|f(N)R - Rf(N)\| \leqslant \|NR - RN\|$ *for any $R \in \mathcal{B}(\mathscr{H})$ and any normal operator N such that $\sigma(N) \subset \mathfrak{F}$*;
(b) $\|f(N_1)R - Rf(N_2)\| \leqslant \|N_1R - RN_2\|$ *for any $R \in \mathcal{B}(\mathscr{H})$ and any normal operators N_1 and N_2 such that $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}$*;
(c) $\|f(N_1)A - Af(N_2)\| \leqslant \|N_1A - AN_2\|$ *for any self-adjoint operator A and any normal operators N_1 and N_2 such that $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}$.*

Proof. To prove the implication (a) ⇒ (b), it suffices to apply (a) to the normal operator $\begin{pmatrix} N_1 & \mathbf{0} \\ \mathbf{0} & N_2 \end{pmatrix}$ and the operator $\begin{pmatrix} \mathbf{0} & R \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. The implication (b) ⇒ (c) is obvious. It remains to prove that (c) implies (a). Applying (c) to $N_1 = U^*NU$ and $N_2 = N$, where U is a unitary operator, we get that

$$\|f(N)UA - UAf(N)\| = \|f(U^*NU)A - Af(N)\| \leqslant \|NUA - UAN\|$$

for any self-adjoint operator A, any unitary operator U, and any normal operator N such that $\sigma(N) \subset \mathfrak{F}$. Note that if (a) is satisfied for an operator $R \in \mathcal{B}(\mathscr{H})$, then it is also satisfied for the operator $R + \lambda I$, where $\lambda \in \mathbb{C}$. Thus, we can assume that R is invertible. Then applying the polar decomposition to the invertible operator R, we obtain $R = UA$, where U is a unitary operator and A is a (positive) self-adjoint operator. □

It follows immediately from Theorems 3.1.1 and 3.1.3 that $\mathrm{CL}(\mathfrak{F}) \subset \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant \|f\|_{\mathrm{CL}(\mathfrak{F})}$ for all f in $\mathrm{CL}(\mathfrak{F})$.

Remark. In the conditions (b) of Theorem 3.1.1, (c) of Theorem 3.1.2, and (b) of Theorem 3.1.3) we can assume that the normal operators N_1 and N_2 act in different Hilbert spaces (herewith the unitary operator U can act from one Hilbert space to another). This can be seen from the proofs. As an illustration, we give here a relevant reformulation of the condition (b) in Theorem 3.1.3: $\|f(N_1)R - Rf(N_2)\| \leqslant \|N_1R - RN_2\|$ *for all operators $R \in \mathcal{B}(\mathscr{H}_2, \mathscr{H}_1)$ and all normal operators N_1 and N_2 acting in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and satisfying the condition $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}$.*

Analogues of Theorems 3.1.1–3.1.3 with appropriate modifications hold for symmetrically normed ideals with practically the same proofs. We consider here only the trace class ideal $\boldsymbol{S}_1$. With each closed set $\mathfrak{F}\subset\mathbb{C}$ we associate the space $\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})$ of *trace class Lipschitz* (or $\boldsymbol{S}_1$-*Lipschitz*) functions and the space $\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$ of *trace class commutator Lipschitz* (or $\boldsymbol{S}_1$-*commutator Lipschitz*) functions. To define the spaces $\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})$ and $\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$ we only have to replace the operator norm by the trace norm in (3.1.3) and (3.1.5).

The corresponding 'united' analogue of Theorems 3.1.1 and 3.1.2 for the trace norm can be stated as follows.

Theorem 3.1.4. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of the complex plane $\mathbb{C}$. Then the following conditions are equivalent*:

(a) $\|f(N_1)-f(N_2)\|_{\boldsymbol{S}_1}\leqslant\|N_1-N_2\|_{\boldsymbol{S}_1}$ *for all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$*;

(b) $\|f(N_1)U-Uf(N_2)\|_{\boldsymbol{S}_1}\leqslant\|N_1U-UN_2\|_{\boldsymbol{S}_1}$ *for all unitary operators U and all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$*;

(c) $\|f(N)U-Uf(N)\|_{\boldsymbol{S}_1}\leqslant\|NU-UN\|_{\boldsymbol{S}_1}$ *for all self-adjoint unitary operators U and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$*;

(d) $\|f(N)A-Af(N)\|_{\boldsymbol{S}_1}\leqslant\|NA-AN\|_{\boldsymbol{S}_1}$ *for all self-adjoint operators A and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$*;

(e) $\|f(N)R-Rf(N)\|_{\boldsymbol{S}_1}+\|\overline{f}(N)R-R\overline{f}(N)\|_{\boldsymbol{S}_1}\leqslant\|NR-RN\|_{\boldsymbol{S}_1}+\|N^*R-RN^*\|_{\boldsymbol{S}_1}$ *for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$*;

(f) $\|f(N_1)R-Rf(N_2)\|_{\boldsymbol{S}_1}+\|\overline{f}(N_1)R-R\overline{f}(N_2)\|_{\boldsymbol{S}_1}\leqslant\|N_1R-RN_2\|_{\boldsymbol{S}_1}+\|N_1^*R-RN_2^*\|_{\boldsymbol{S}_1}$ *for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$*.

We now state the analogue of Theorem 3.1.3.

Theorem 3.1.5. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. Then the following conditions are equivalent*:

(a) $\|f(N)R-Rf(N)\|_{\boldsymbol{S}_1}\leqslant\|NR-RN\|_{\boldsymbol{S}_1}$ *for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N such that $\sigma(N)\subset\mathfrak{F}$*;

(b) $\|f(N_1)R-Rf(N_2)\|_{\boldsymbol{S}_1}\leqslant\|N_1R-RN_2\|_{\boldsymbol{S}_1}$ *for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$*;

(c) $\|f(N_1)A-Af(N_2)\|_{\boldsymbol{S}_1}\leqslant\|N_1A-AN_2\|_{\boldsymbol{S}_1}$ *for all self-adjoint operators A and all normal operators N_1 and N_2 such that $\sigma(N_1),\sigma(N_2)\subset\mathfrak{F}$*.

We note that one can reformulate Theorem 3.1.4 for self-adjoint operators as follows.

Theorem 3.1.6. *Let f be a real continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Then the following conditions are equivalent*:

(a) $\|f(A)-f(B)\|_{\boldsymbol{S}_1}\leqslant\|A-B\|_{\boldsymbol{S}_1}$ *for all self-adjoint operators A and B such that $\sigma(A),\sigma(B)\subset\mathfrak{F}$*;

(b) $\|f(A)U-Uf(B)\|_{\boldsymbol{S}_1}\leqslant\|AU-UB\|_{\boldsymbol{S}_1}$ *for all unitary operators U and all self-adjoint operators A and B such that $\sigma(A),\sigma(B)\subset\mathfrak{F}$*;

(c) $\|f(A)U-Uf(A)\|_{\boldsymbol{S}_1}\leqslant\|AU-UA\|_{\boldsymbol{S}_1}$ *for all self-adjoint unitary operators U and all self-adjoint operators A such that $\sigma(A)\subset\mathfrak{F}$*;

(d) $\|f(A)R-Rf(A)\|_{\boldsymbol{S}_1}\leqslant\|AR-RA\|_{\boldsymbol{S}_1}$ *for all self-adjoint operators A and R such that $\sigma(A)\subset\mathfrak{F}$*;

(e) $\|f(A)R - Rf(A)\|_{\boldsymbol{S}_1} \leqslant \|AR - RA\|_{\boldsymbol{S}_1}$ *for all* $R \in \mathcal{B}(\mathscr{H})$ *and all self-adjoint operators* A *such that* $\sigma(A) \subset \mathfrak{F}$;

(f) $\|f(A)R - Rf(B)\|_{\boldsymbol{S}_1} \leqslant \|AR - RB\|_{\boldsymbol{S}_1}$ *for all* $R \in \mathcal{B}(\mathscr{H})$ *and all self-adjoint operators* A *and* B *such that* $\sigma(A), \sigma(B) \subset \mathfrak{F}$.

Corollary 3.1.7. *If* f *is a continuous real function on a closed subset* $\mathfrak{F}$ *of the real line* $\mathbb{R}$, *then* $\|f\|_{\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})} = \|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})}$.

It follows that $\|f\|_{\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})} \leqslant \|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})} \leqslant 2\|f\|_{\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})}$ for a complex continuous function f.

The same can be said also in the case of unitary operators N_1 and N_2, that is, in the case when $\mathfrak{F}$ is contained in the unit circle $\mathbb{T}$.

Obviously, $\overline{z} \in \mathrm{OL}(\mathfrak{F})$ for any closed set $\mathfrak{F}$ in $\mathbb{C}$, and $\|\overline{z}\|_{\mathrm{OL}(\mathfrak{F})} = 1$ if $\mathfrak{F}$ has at least two points.

Definition. A closed subset $\mathfrak{F}$ of $\mathbb{C}$ is called a *Fuglede set* if $\mathrm{CL}(\mathfrak{F}) = \mathrm{OL}(\mathfrak{F})$.

This notion was introduced by Kissin and Shulman in [38].

Johnson and Williams [32] proved that each function $f \in \mathrm{CL}(\mathfrak{F})$ is differentiable in the complex sense at each non-isolated point of $\mathfrak{F}$ (see Theorems 3.3.2 and 3.3.3 below). We note that $\overline{z} \in \mathrm{OL}(\mathfrak{F})$. Therefore, a Fuglede set cannot have interior points and even cannot contain two intersecting intervals not contained in the same straight line. Kissin and Shulman proved in [38] that each compact curve of class C^2 is a Fuglede set.

The following theorem is essentially contained in Proposition 4.5 of [38].

Theorem 3.1.8. *A closed subset* $\mathfrak{F}$ *of* $\mathbb{C}$ *is a Fuglede set if and only if* $\overline{z} \in \mathrm{CL}(\mathfrak{F})$. *If* $\overline{z} \in \mathrm{CL}(\mathfrak{F})$, *then* $\|f\|_{\mathrm{CL}(\mathfrak{F})} \leqslant \|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})}\|f\|_{\mathrm{OL}(\mathfrak{F})}$ *for all* $f \in \mathrm{OL}(\mathfrak{F})$.

This theorem is a straightforward consequence of Theorems 3.1.2 and 3.1.3.

Corollary 3.1.9. *Let* $\mathfrak{F}$ *be a closed subset of* $\mathbb{C}$. *Then the equality* $\mathrm{CL}(\mathfrak{F}) = \mathrm{OL}(\mathfrak{F})$ *holds together with the seminorm equality* $\|\cdot\|_{\mathrm{CL}(\mathfrak{F})} = \|\cdot\|_{\mathrm{OL}(\mathfrak{F})}$ *if and only if* $\|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})} \leqslant 1$.

We note that $\|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})} \geqslant \|\overline{z}\|_{\mathrm{OL}(\mathfrak{F})} = 1$ if $\mathfrak{F}$ has at least two points. Thus, the condition that $\|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})} \leqslant 1$ can be replaced by the condition that $\|\overline{z}\|_{\mathrm{CL}(\mathfrak{F})} = 1$ if $\mathfrak{F}$ has at least two points.

Theorem 3.1.10. *If a closed subset* $\mathfrak{F}$ *of* $\mathbb{C}$ *is contained in a straight line or in a circle, then* $\mathfrak{F}$ *is a Fuglede set and* $\|\cdot\|_{\mathrm{CL}(\mathfrak{F})} = \|\cdot\|_{\mathrm{OL}(\mathfrak{F})}$.

Proof. It is easy to see that the (semi)norms in $\mathrm{CL}(\mathfrak{F})$ and $\mathrm{OL}(\mathfrak{F})$ coincide if and only if

$$\|N^*R - RN^*\| = \|NR - RN\| \tag{3.1.6}$$

for every normal operator N with spectrum in $\mathfrak{F}$ and every bounded operator R. As we observed above (see the special cases 1) and 2) before Theorem 3.1.3), this equality obviously holds for both self-adjoint and unitary operators N. This proves the theorem in the two special cases $\mathfrak{F} \subset \mathbb{R}$ and $\mathfrak{F} \subset \mathbb{T}$. The general case can be reduced to these special cases with the help of affine transformations of the complex plane. $\square$

Kamowitz [33] proved that for a given operator N the equality (3.1.6) holds for all bounded operators R if and only if N is a normal operator whose spectrum is contained in a circle or a line. It follows from this result of Kamowitz that Theorem 3.1.10 has a converse. In other words, the equality $\|\cdot\|_{\mathrm{CL}(\mathfrak{F})}=\|\cdot\|_{\mathrm{OL}(\mathfrak{F})}$ holds if and only if $\mathfrak{F}$ is contained in a circle or a line.

Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be non-empty closed subsets of $\mathbb{C}$. Denote by $\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)$ the space of continuous functions f on $\mathfrak{F}=\mathfrak{F}_1\cup\mathfrak{F}_2$, for which there exists a constant $C\geqslant 0$ such that

$$\|f(N_1)R-Rf(N_2)\|\leqslant C\|N_1R-RN_2\| \tag{3.1.7}$$

for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N_1 and N_2 with spectra in $\mathfrak{F}_1$ and $\mathfrak{F}_2$. We denote by $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)}$ the smallest constant C satisfying (3.1.7). Let $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)}=\infty$ if $f\notin\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)$.

Passing to the adjoint operators, we see that (3.1.7) is equivalent to the condition that $\|R^*\overline{f}(N_1)-\overline{f}(N_2)R^*\|\leqslant C\|R^*N_1^*-N_2^*R^*\|$. It follows that $f\in\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)$ if and only if $\overline{f}(\overline{z})\in\mathrm{CL}(\overline{\mathfrak{F}}_2,\overline{\mathfrak{F}}_1)$ and $\|\overline{f}(\overline{z})\|_{\mathrm{CL}(\overline{\mathfrak{F}}_2,\overline{\mathfrak{F}}_1)}=\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)}$, where for a subset $\mathfrak{F}$ of $\mathbb{C}$ we denote by $\overline{\mathfrak{F}}$ the set $\{\overline{\zeta}\colon\zeta\in\mathfrak{F}\}$.

If we rewrite (3.1.7) in terms of matrices and then consider the transposed matrices, then we find that $\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)=\mathrm{CL}(\mathfrak{F}_2,\mathfrak{F}_1)$ and $\|\cdot\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)}=\|\cdot\|_{\mathrm{CL}(\mathfrak{F}_2,\mathfrak{F}_1)}$.

Theorem 3.1.11. *Let f be a continuous function on a union $\mathfrak{F}_1\cup\mathfrak{F}_2$ of closed subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\mathbb{C}$. Then the following conditions are equivalent*:

(a) $\|f(N_1)R-Rf(N_2)\|\leqslant\|N_1R-RN_2\|$ *for all $R\in\mathcal{B}(\mathscr{H})$ and all normal operators N_1 and N_2 such that $\sigma(N_1)\subset\mathfrak{F}_1$ and $\sigma(N_2)\subset\mathfrak{F}_2$*;

(b) $\|f(N_1)R-Rf(N_2)\|\leqslant\|N_1R-RN_2\|$ *for any operator R from a Hilbert space $\mathscr{H}_2$ to a Hilbert space $\mathscr{H}_1$ and any normal operators N_1 and N_2 on $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $\sigma(N_1)\subset\mathfrak{F}_1$ and $\sigma(N_2)\subset\mathfrak{F}_2$*;

(c) *the condition* (b) *holds under the additional assumption that the normal operators N_1 and N_2 have simple spectra*;

(d) $\|f(N_1)A-Af(N_2)\|\leqslant\|N_1A-AN_2\|$ *for all self-adjoint operators A and all normal operators N_1 and N_2 such that $\sigma(N_1)\subset\mathfrak{F}_1$ and $\sigma(N_2)\subset\mathfrak{F}_2$.*

Proof. The implications (b) ⇒ (a) and (b) ⇒ (c) are trivial. Let us prove that (a) ⇒ (b). If the spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ are isomorphic, then there exists a unitary operator $U\colon\mathscr{H}_1\to\mathscr{H}_2$. The operator RU and the normal operators N_1 and U^*N_2U are operators on the same Hilbert space $\mathscr{H}_1$, and hence

$$\|f(N_1)(RU)-(RU)f(U^*N_2U)\|\leqslant\|N_1(RU)-(RU)U^*N_2U\|$$

by (a), which immediately implies the desired estimate. To reduce the general case to the special case considered above, we introduce the operators

$$\mathcal{R}\stackrel{\mathrm{def}}{=}\bigoplus_{j\geqslant1}R,\quad \mathcal{N}_1\stackrel{\mathrm{def}}{=}\bigoplus_{j\geqslant1}N_1,\quad\text{and}\quad \mathcal{N}_2\stackrel{\mathrm{def}}{=}\bigoplus_{j\geqslant1}N_2.$$

It is easy to see that the inequality $\|f(N_1)R-Rf(N_2)\|\leqslant\|N_1R-RN_2\|$ is equivalent to the inequality $\|f(\mathcal{N}_1)\mathcal{R}-\mathcal{R}f(\mathcal{N}_2)\|\leqslant\|\mathcal{N}_1\mathcal{R}-\mathcal{R}\mathcal{N}_2\|$.

We now prove that (c) implies (b). Assume the contrary. Then there exist $R\in\mathcal{B}(\mathscr{H}_2,\mathscr{H}_1)$ and normal operators N_1 and N_2 acting in $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $\sigma(N_1) \subset \mathfrak{F}_1$, $\sigma(N_2) \subset \mathfrak{F}_2$, $\|N_1R - RN_2\| = 1$, and $\|f(N_1)R - Rf(N_2)\| > 1$. Thus, there are vectors $u_0 \in \mathscr{H}_2$ and $v_0 \in \mathscr{H}_1$ such that $\|u_0\| = 1$, $\|v_0\| = 1$, and $\big|\big((f(N_1)R - Rf(N_2))u_0, v_0\big)\big| > 1$. Let $\mathscr{H}_1^0$ and $\mathscr{H}_2^0$ be the smallest reducing subspaces of N_1 and N_2 containing v_0 and u_0, respectively, and let P and Q be the orthogonal projections onto these subspaces. Note that $\|f(N_1)PRQ - PRQf(N_2)\| > 1$ since $\big((f(N_1)PRQ - PRQf(N_2))u_0, v_0\big) = \big((f(N_1)R - Rf(N_2)\big)u_0, v_0\big)$. Moreover, $\|N_1PRQ - PRQN_2\| = \|P(N_1R - RN_2)Q\| \leqslant 1$. Let $N_1^0 \stackrel{\text{def}}{=} N|\mathscr{H}_1^0$ and $N_2^0 \stackrel{\text{def}}{=} N|\mathscr{H}_2^0$. Then N_1^0 and N_2^0 can be regarded as normal operators acting in $\mathscr{H}_1^0$ and $\mathscr{H}_2^0$. Clearly, N_1^0 and N_2^0 are normal operators with simple spectra. To get a contradiction, it suffices to observe that $\|f(N_1^0)PRQ - PRQf(N_2^0)\| > 1$ and $\|N_1^0PRQ - PRQN_2^0\| \leqslant 1$.

The implication (a) $\Rightarrow$ (d) is trivial. It remains to prove that (d) implies (a). Applying (d) to the normal operators U^*N_1U and N_2, where U is a unitary operator, we find that

$$\|f(N_1)UA - UAf(N_2)\| \leqslant \|N_1UA - UAN_2\|$$

for any self-adjoint operator A, any unitary operator U, and any normal operators N_1 and N_2 such that $\sigma(N_1) \subset \mathfrak{F}_1$ and $\sigma(N_2) \subset \mathfrak{F}_2$. With the help of polar decomposition this implies (a) for invertible operators R. Therefore, (a) holds for operators R that belong to the closure of the set of invertible operators in the operator norm. It remains to observe that in the general case the block operator $\mathcal{R} = \begin{pmatrix} R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$ on $\mathscr{H} \oplus \mathscr{H}$ can obviously be approximated with arbitrary accuracy in the operator norm by invertible operators in $\mathcal{B}(\mathscr{H} \oplus \mathscr{H})$. One can proceed from the operator $\mathcal{R}$ to the operator R by using the Berberian trick discussed after the proof of Theorem 3.1.2. $\square$

We need the following well-known elementary result.

Lemma 3.1.12. *Let N be a bounded normal operator. Suppose that the subset Λ of $\mathbb{C}$ is an ε-net of the spectrum $\sigma(N)$ of N, that is, for each $\zeta \in \sigma(N)$ there is a $\lambda \in \Lambda$ such that $|\lambda - \zeta| < \varepsilon$. Then there exists a normal operator N_0 such that $NN_0 = N_0N$, $\|N - N_0\| < \varepsilon$, and $\sigma(N_0)$ is a finite subset of Λ.*

Proof. Since the spectrum of N is compact, there exists a finite ε-net Λ_0 of $\sigma(N)$ such that $\Lambda_0 \subset \Lambda$. Then we can find a Borel function $\eta\colon \sigma(N) \to \Lambda_0$ such that $\sup\{|z - \eta(z)|\colon z \in \sigma(N)\} < \varepsilon$. It remains to put $N_0 \stackrel{\text{def}}{=} \eta(N)$. $\square$

It follows easily from this lemma and the inequality (3.1.2) that if the inequality (3.1.3) holds for all normal operators N_1 and N_2 with finite spectra in $\mathfrak{F}$, then $f \in \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant C$.

In other words, the following equality holds for every continuous function f on a closed set $\mathfrak{F} \subset \mathbb{C}$:

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} = \sup\big\{\|f\|_{\mathrm{OL}(\Lambda)}\colon \Lambda \subset \mathfrak{F},\ \Lambda \text{ is finite}\big\}. \tag{3.1.8}$$

Moreover,

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} = \sup\big\{\|f\|_{\mathrm{OL}(\Lambda)}\colon \Lambda \subset \mathfrak{F}_0,\ \Lambda \text{ is finite}\big\}, \tag{3.1.9}$$

where $\mathfrak{F}_0$ is a dense subset of $\mathfrak{F}$.

Similar equalities hold also for commutator Lipschitz seminorms.

Hence we would obtain nothing essentially new if we tried to define the spaces $\mathrm{OL}(\mathfrak{F})$ and $\mathrm{CL}(\mathfrak{F})$ for an arbitrary subset $\mathfrak{F}$ of $\mathbb{C}$.

To be definite, we dwell on the space $\mathrm{OL}(\mathfrak{F})$ ($\mathrm{CL}(\mathfrak{F})$ can be treated similarly). We say that an arbitrary function $f\colon \mathfrak{F}\to\mathbb{C}$ belongs to $\mathrm{OL}(\mathfrak{F})$ if there is a constant $C\geqslant 0$ such that (3.1.3) holds for all normal operators N_1 and N_2 with finite spectra in $\mathfrak{F}$. Note that since the spectra are finite, we can define $f(N_1)$ and $f(N_2)$ for any function f. Obviously, $\mathrm{OL}(\mathfrak{F})\subset\mathrm{Lip}(\mathfrak{F})$. Thus, each such function f admits a Lipschitz extension to the closure $\operatorname{clos}\mathfrak{F}$ of $\mathfrak{F}$. It is easy to see from (3.1.9) that this extension belongs to $\mathrm{OL}(\operatorname{clos}\mathfrak{F})$ and its OL-seminorm does not change. Therefore, the space $\mathrm{OL}(\mathfrak{F})$ can be identified in a natural way with the space $\mathrm{OL}(\operatorname{clos}\mathfrak{F})$.

Taking this remark into account, for brevity we write $\mathrm{OL}(\mathbb{D})$, $\mathrm{OL}(\mathbb{C}_+)$, $\mathrm{CL}(\mathbb{D})$, and $\mathrm{CL}(\mathbb{C}_+)$ instead of $\mathrm{OL}(\operatorname{clos}\mathbb{D})$, $\mathrm{OL}(\operatorname{clos}\mathbb{C}_+)$, $\mathrm{CL}(\operatorname{clos}\mathbb{D})$, and $\mathrm{CL}(\operatorname{clos}\mathbb{C}_+)$.

3.2. Bounded and unbounded normal operators

We prove in this section certain auxiliary results giving us that in the definitions of operator and commutator Lipschitz (as well as operator Hölder) functions we can either consider only bounded normal operators or admit unbounded ones. In either case we get the same classes of functions with the same norms.

Let N_1 and N_2 be not necessarily bounded normal operators acting in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, with domains $\mathcal{D}_{N_1}$ and $\mathcal{D}_{N_2}$. Let R be a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_1$. We say that N_1R-RN_2 is a *bounded operator* if $R(\mathcal{D}_{N_2})\subset\mathcal{D}_{N_1}$ and $\|N_1Ru-RN_2u\|\leqslant C\|u\|$ for all $u\in\mathcal{D}_{N_2}$. Then there exists a unique bounded operator K such that $Ku=N_1Ru-RN_2u$ for all $u\in\mathscr{D}_{N_2}$. In this case we write $K=N_1R-RN_2$. Thus, N_1R-RN_2 is a bounded operator if and only if

$$\big|(Ru,{N_1}^*v)-(N_2u,R^*v)\big|\leqslant C\|u\|\cdot\|v\| \tag{3.2.1}$$

for all $u\in\mathcal{D}_{N_2}$ and $v\in\mathcal{D}_{N_1^*}=\mathcal{D}_{N_1}$. It is easy to see that N_1R-RN_2 is a bounded operator if and only if $N_2^*R^*-R^*N_1^*$ is a bounded operator. Furthermore, $(N_1R-RN_2)^*=-(N_2^*R^*-R^*N_1^*)$. In particular, we write $N_1R=RN_2$ if $R(\mathcal{D}_{N_2})\subset\mathcal{D}_{N_1}$ and $N_1Ru=RN_2u$ for all $u\in\mathcal{D}_{N_2}$. We say that $\|N_1R-RN_2\|=\infty$ if N_1R-RN_2 is not a bounded operator.

Remark. Let N_1 and N_2 be normal operators. Suppose that N_1^* is the closure of $N_{1\flat}$ and N_2 is the closure of $N_{2\sharp}$. If (3.2.1) holds for all $u\in\mathcal{D}_{N_{2\sharp}}$ and $v\in\mathcal{D}_{N_{1\flat}}$, then it holds for all $u\in\mathcal{D}_{N_2}$ and $v\in\mathcal{D}_{N_1}$.

Theorem 3.2.1. *Let N_1 and N_2 be normal operators acting in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, and let R be a bounded operator from $\mathscr{H}_2$ to $\mathscr{H}_1$. Then there exist sequences $\{N_{1,n}\}_{n\geqslant1}$ and $\{N_{2,n}\}_{n\geqslant1}$ of bounded normal operators on Hilbert spaces $\mathscr{H}_{1,n}$ and $\mathscr{H}_{2,n}$ and a sequence of bounded operators $\{R_n\}_{n\geqslant1}$ from $\mathscr{H}_{2,n}$ to $\mathscr{H}_{1,n}$ such that:*

(a) *the sequence $\{\|R_n\|\}_{n\geqslant1}$ is non-decreasing and $\lim_{n\to\infty}\|R_n\|=\|R\|$;*

(b) *$\sigma(N_{1,n})\subset\sigma(N_1)$ and $\sigma(N_{2,n})\subset\sigma(N_2)$ for all $n\geqslant1$;*

(c) *for any continuous function f on $\sigma(N_1)\cup\sigma(N_2)$ the sequence $\big\{\|f(N_{1,n})R_n-R_nf(N_{2,n})\|\big\}_{n\geqslant1}$ is non-decreasing and*

$$\lim_{n\to\infty}\|f(N_{1,n})R_n-R_nf(N_{2,n})\|=\|f(N_1)R-Rf(N_2)\|;$$

(d) *for any continuous function* f *on* $\sigma(N_1)\cup\sigma(N_2)$ *with* $\|f(N_1)R-Rf(N_2)\|<\infty$ *and for any natural number* j *the sequence* $\bigl\{s_j\bigl(f(N_{1,n})R_n-R_nf(N_{2,n})\bigr)\bigr\}_{n\geqslant 0}$ *of singular values is non-decreasing and*

$$\lim_{n\to\infty} s_j\bigl(f(N_{1,n})R_n - R_nf(N_{2,n})\bigr) = s_j\bigl(f(N_1)R - Rf(N_2)\bigr).$$

Proof. Without loss of generality we can assume that $0 \in \sigma(N_1)\cup\sigma(N_2)$. Let $P_{1,n} \stackrel{\text{def}}{=} E_{N_1}\bigl(\{|\lambda|\leqslant n\}\bigr)$ and $P_{2,n} \stackrel{\text{def}}{=} E_{N_2}\bigl(\{|\lambda|\leqslant n\}\bigr)$, where E_{N_1} and E_{N_2} are the spectral measures of the normal operators N_1 and N_2, and let

$$\widetilde N_{1,n} \stackrel{\text{def}}{=} P_{1,n}N_1 = N_1P_{1,n} = P_{1,n}N_1P_{1,n},$$
$$\widetilde N_{2,n} \stackrel{\text{def}}{=} P_{2,n}N_2 = N_2P_{2,n} = P_{2,n}N_2P_{2,n},$$

$$\mathscr H_{1,n} \stackrel{\text{def}}{=} P_{1,n}\mathscr H_1, \quad \text{and} \quad \mathscr H_{2,n} \stackrel{\text{def}}{=} P_{2,n}\mathscr H_2.$$

Clearly, $\widetilde N_{1,n}$ and $\widetilde N_{2,n}$ are bounded normal operators on $\mathscr H_1$ and $\mathscr H_2$, and $\mathscr H_{1,n}$ and $\mathscr H_{2,n}$ are reducing subspaces of $\widetilde N_{1,n}$ and $\widetilde N_{2,n}$.

Let $N_{1,n} \stackrel{\text{def}}{=} \widetilde N_{1,n}\big|\mathscr H_{1,n}$ and $N_{2,n} \stackrel{\text{def}}{=} \widetilde N_{2,n}\big|\mathscr H_{2,n}$. Then $N_{1,n}$ and $N_{2,n}$ are normal operators acting in $\mathscr H_{1,n}$ and $\mathscr H_{2,n}$. The operator R_n in $\mathcal B(\mathscr H_{2,n},\mathscr H_{1,n})$ is defined by $R_nu \stackrel{\text{def}}{=} P_{1,n}Ru = P_{1,n}RP_{2,n}u$ for $u\in\mathscr H_{2,n}$. Then (a) and (b) are obvious. To prove the remaining assertions, it suffices to observe that

$$P_{1,n}\bigl(f(N_1)R - Rf(N_2)\bigr)P_{2,n}u = \bigl(f(N_{1,n})R_n - R_nf(N_{2,n})\bigr)u$$

for all u in $\mathscr H_{2,n}$. $\square$

3.3. Divided differences and commutator Lipschitzness

With each function f on a closed set $\mathfrak F\subset\mathbb C$ we associate the function $\mathfrak D_0f\colon \mathfrak F\times\mathfrak F\to\mathbb C$ given by

$$(\mathfrak D_0f)(z,w) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z\ne w,\\ 0 & \text{if } z=w. \end{cases} \tag{3.3.1}$$

If $\mathfrak F$ has no isolated points and for each point z in $\mathfrak F$ there exists a finite derivative $f'(z)$ in the complex sense, then we can define the *divided difference* $\mathfrak Df\colon \mathfrak F\times\mathfrak F \to\mathbb C$ by

$$(\mathfrak Df)(z,w) \stackrel{\text{def}}{=} \begin{cases} \dfrac{f(z)-f(w)}{z-w} & \text{if } z\ne w,\\ f'(z) & \text{if } z=w. \end{cases}$$

Theorem 3.3.1. *Let* f *be a continuous function on the union* $\mathfrak F_1\cup\mathfrak F_2$ *of closed subsets* $\mathfrak F_1$ *and* $\mathfrak F_2$ *of* $\mathbb C$. *Then* $f\in \mathrm{CL}(\mathfrak F_1,\mathfrak F_2)$ *if and only if* $\mathfrak D_0f\in\mathfrak M(\mathfrak F_1\times\mathfrak F_2)$. *Moreover,*

$$\|f\|_{\mathrm{CL}(\mathfrak F_1,\mathfrak F_2)} = \|\mathfrak D_0f\|_{\mathfrak M_0(\mathfrak F_1\times\mathfrak F_2)} \leqslant \|\mathfrak D_0f\|_{\mathfrak M(\mathfrak F_1\times\mathfrak F_2)} \leqslant 2\|f\|_{\mathrm{CL}(\mathfrak F_1,\mathfrak F_2)}.$$

Proof. We prove only the equality $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)} = \|\mathfrak{D}_0 f\|_{\mathfrak{M}_0(\mathfrak{F}_1\times\mathfrak{F}_2)}$, because everything else follows from Corollary 2.1.2. Consider first the case of finite sets $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Let N_1 and N_2 be normal operators such that $\sigma(N_1)\subset\mathfrak{F}_1$ and $\sigma(N_2)\subset\mathfrak{F}_2$. By Theorem 3.1.11, we can assume that N_1 and N_2 have simple spectra. Then there exist orthonormal bases $\{u_\lambda\}_{\lambda\in\sigma(N_1)}$ and $\{v_\mu\}_{\mu\in\sigma(N_2)}$ in $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $N_1u_\lambda=\lambda u_\lambda$ for all $\lambda\in\sigma(N_1)$ and $N_2v_\mu=\mu v_\mu$ for all $\mu\in\sigma(N_2)$. With each operator $X\colon\mathscr{H}_2\to\mathscr{H}_1$ we associate the matrix $\{(Xv_\mu,u_\lambda)\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)}$. We have

$$\big((N_1R-RN_2)v_\mu,u_\lambda\big)=\big(Rv_\mu,N_1^*u_\lambda\big)-(RN_2v_\mu,u_\lambda)=(\lambda-\mu)(Rv_\mu,u_\lambda).$$

Similarly, $\big((f(N_1)R-Rf(N_2))v_\mu,u_\lambda\big)=(f(\lambda)-f(\mu))(Rv_\mu,u_\lambda)$. Obviously,

$$\{(f(\lambda)-f(\mu))(Rv_\mu,u_\lambda)\}=\{(\mathfrak{D}_0f)(\lambda,\mu)\}\star\{(\lambda-\mu)(Rv_\mu,u_\lambda)\}.$$

Note that the matrix $\{a_{\lambda\mu}\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)}$ can be represented in the form

$$\{a_{\lambda\mu}\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)}=\{(\lambda-\mu)(Rv_\mu,u_\lambda)\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)},$$

where R is an operator from $\mathscr{H}_2$ to $\mathscr{H}_1$, if and only if $a_{\lambda\mu}=0$ for $\lambda=\mu$. The equality $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)} = \|\mathfrak{D}_0 f\|_{\mathfrak{M}_0(\mathfrak{F}_1\times\mathfrak{F}_2)}$ for finite sets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ is now obvious.

The inequality $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)} \geqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_0(\mathfrak{F}_1\times\mathfrak{F}_2)}$ easily reduces to the case of finite sets $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

Let us proceed to the inequality $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)} \leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_0(\mathfrak{F}_1\times\mathfrak{F}_2)}$, which means that $\|f(N_1)R-Rf(N_2)\|\leqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}_0(\mathfrak{F}_1\times\mathfrak{F}_2)}\|N_1R-RN_2\|$ for any bounded operator R and any bounded normal operators N_1 and N_2 such that $\sigma(N_1)\subset\mathfrak{F}_1$ and $\sigma(N_2)\subset\mathfrak{F}_1$. It follows from the special case treated above that this inequality certainly holds in the case when the normal operators N_1 and N_2 have finite spectra. The case of arbitrary normal operators N_1 and N_2 with spectra in $\mathfrak{F}_1$ and $\mathfrak{F}_2$ can be reduced to this special case with the help of Lemma 3.1.12. □

Remark. The inequality $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)} \leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}_1\times\mathfrak{F}_2)}$ can also be proved with the help of double operator integrals (see the remark after Theorem 3.5.2).

In the case when $\mathfrak{F}_1=\mathfrak{F}_2$, Theorem 3.3.1 reduces to the following result.

Theorem 3.3.2. *Let f be a function on a non-empty closed subset $\mathfrak{F}$ of $\mathbb{C}$. Then $f\in\mathrm{CL}(\mathfrak{F})$ if and only if $\mathfrak{D}_0f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$. Furthermore,*

$$\|f\|_{\mathrm{CL}(\mathfrak{F})}=\|\mathfrak{D}_0f\|_{\mathfrak{M}_0(\mathfrak{F}\times\mathfrak{F})}\leqslant\|\mathfrak{D}_0f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}\leqslant 2\|f\|_{\mathrm{CL}(\mathfrak{F})}.$$

If $\mathfrak{D}_0f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$ for a function f on $\mathscr{F}$, then f is continuous and even satisfies the Lipschitz condition. Indeed, if $\zeta,\tau\in\mathfrak{F}$, then $|(\mathfrak{D}_0f)(\tau,\zeta)|\leqslant\|\mathfrak{D}_0f\|_{\mathfrak{M}_0(\mathfrak{F}\times\mathfrak{F})}$, whence $|f(\zeta)-f(\tau)|\leqslant\|\mathfrak{D}_0f\|_{\mathfrak{M}_0(\mathfrak{F}\times\mathfrak{F})}|\zeta-\tau|$.

The following assertion was obtained in [32].

Theorem 3.3.3. *Let f be a function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$ such that $\mathfrak{D}_0f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$. Then f is differentiable in the complex sense at each non-isolated point of $\mathfrak{F}$. Moreover, if $\mathscr{F}$ is unbounded, then there exists a finite limit $\lim_{|z|\to\infty}z^{-1}f(z)$.*

We will need an elementary lemma, which we give without proof.

Lemma 3.3.4. *Let S and T be arbitrary sets. Suppose that a sequence $\{\varphi_n\}$ of functions on $S\times T$ converges pointwise to a function φ. Then $\|\varphi\|_{\mathfrak{M}(S\times T)}\leqslant \underline{\lim}_{n\to\infty}\|\varphi_n\|_{\mathfrak{M}(S\times T)}$.*

Proof of Theorem 3.3.3. We first prove differentiability at each non-isolated point a of $\mathfrak{F}$. Without loss of generality we can assume that $a=0$ and $f(0)=0$. We have to show that the function $z^{-1}f(z)$ has a finite limit as $z\to 0$. Suppose that this function has at least two finite (because f is Lipschitz) limit values as $z\to 0$. Clearly, we can assume that these limit values are 1 and -1. Thus, there exist two sequences $\{\lambda_n\}_{n\geqslant 1}$ and $\{\mu_n\}_{n\geqslant 1}$ of points of $\mathfrak{F}\setminus\{0\}$ that tend to zero and are such that $\lim_{n\to\infty}\lambda_n^{-1}f(\lambda_n)=1$ and $\lim_{n\to\infty}\mu_n^{-1}f(\mu_n)=-1$. Passing to subsequences if necessary, we can achieve the following conditions:

(a) $|\lambda_n|>|\mu_n|>|\lambda_{n+1}|$ for all $n\geqslant 1$;

(b) $\lim_{n\to\infty}\mu_n^{-1}\lambda_n=0$ and $\lim_{n\to\infty}\lambda_{n+1}^{-1}\mu_n=0$.

Obviously, $\|\{(\mathfrak{D}_0 f)(\lambda_m,\mu_n)\}\|_{\mathfrak{M}(\mathbb{N}\times\mathbb{N})}\leqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}$. Note that the sequence $\bigl\{\|\{(\mathfrak{D}_0 f)(\lambda_{m+k},\mu_{n+k})\}\|_{\mathfrak{M}(\mathbb{N}\times\mathbb{N})}\bigr\}_{k\geqslant 1}$ is non-increasing and

$$\lim_{k\to\infty}(\mathfrak{D}_0 f)(\lambda_{m+k},\mu_{n+k})=\operatorname{sgn}\biggl(m-n+\frac{1}{2}\biggr).$$

It now follows from Lemma 3.3.4 that $\|\{\operatorname{sgn}(m-n+1/2)\}\|_{\mathfrak{M}(\mathbb{N}\times\mathbb{N})}<+\infty$, which contradicts Theorem 2.2.7.

The existence of a finite limit $\lim_{|z|\to\infty}z^{-1}f(z)$ in the case of an unbounded set $\mathfrak{F}$ can be proved in a similar way, with the only difference that we should now choose sequences $\{\lambda_n\}_{n\geqslant 1}$ and $\{\mu_n\}_{n\geqslant 1}$ that tend to infinity. □

Corollary 3.3.5. *The space* $\mathrm{CL}(\mathbb{C})$ *coincides with the set of linear functions $az+b$ with $a,b\in\mathbb{C}$.*

Proof. Every function of the form $az+b$ with $a,b\in\mathbb{C}$ clearly belongs to $\mathrm{CL}(\mathbb{C})$. Conversely, it follows from Theorem 3.3.3 that f is an entire function. Obviously, f' is bounded, because $\mathrm{CL}(\mathbb{C})\subset\mathrm{OL}(\mathbb{C})\subset\mathrm{Lip}(\mathbb{C})$. It remains to use Liouville's theorem. □

Theorem 3.3.6. *Let f be a continuous function on a perfect set $\mathfrak{F}$ in $\mathbb{C}$. Then $f\in\mathrm{CL}(\mathfrak{F})$ if and only if f is differentiable in the complex sense at each point of the set $\mathfrak{F}$ and $\mathfrak{D}f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$. Moreover, $\|f\|_{\mathrm{CL}(\mathfrak{F})}=\|\mathfrak{D}f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}$.*

Proof. If $f\in\mathrm{CL}(\mathfrak{F})$, then $\mathfrak{D}_0 f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$ by Theorem 3.3.2 and Corollary 2.1.2. The differentiability of f follows from Theorem 3.3.3. Conversely, if $\mathfrak{D}f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$, then $\mathfrak{D}_0 f\in\mathfrak{M}_0(\mathfrak{F}\times\mathfrak{F})$, and we can apply Theorem 3.3.2. The equality $\|f\|_{\mathrm{CL}(\mathfrak{F})}=\|\mathfrak{D}f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}$ follows from Theorem 3.3.2, Lemma 2.1.3, and the obvious equality $\|\mathfrak{D}f\|_{\mathfrak{M}_0(\mathfrak{F}\times\mathfrak{F})}=\|\mathfrak{D}_0 f\|_{\mathfrak{M}_0(\mathfrak{F}\times\mathfrak{F})}$. □

The following theorem shows that to estimate quasi-commutator norms, there is no need to consider all normal operators N_1 and N_2, but rather it suffices to consider only one pair of normal operators N_1 and N_2 such that $\sigma(N_1)=\mathfrak{F}_1$ and $\sigma(N_2)=\mathfrak{F}_2$. In particular, when considering the space $\mathrm{CL}(\mathfrak{F})$ we can assume that $N_1=N_2$, that is, we can consider only one normal operator $N=N_1=N_2$ such that $\sigma(N)=\mathfrak{F}$.

Theorem 3.3.7. *Let N_1 and N_2 be normal operators acting in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. Suppose that a continuous function f on $\sigma(N_1)\cup\sigma(N_2)$ has the property that*

$$\|f(N_1)R - Rf(N_2)\| \leqslant \|N_1R - RN_2\| \tag{3.3.2}$$

for all $R\in\mathcal{B}(\mathscr{H}_2,\mathscr{H}_1)$. Then $f\in\mathrm{CL}(\sigma(N_1),\sigma(N_2))$ and $\|f\|_{\mathrm{CL}(\sigma(N_1),\sigma(N_2))}\leqslant 1$.

Let f be a continuous function on a subset of the complex plane. Suppose that N_1 and N_2 are normal operators acting in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ and that the union of their spectra is contained in the domain of f. We say that the pair (N_1,N_2) is *f-regular* if the inequality (3.3.2) holds for all $R\in\mathcal{B}(\mathscr{H}_2,\mathscr{H}_1)$.

Theorem 3.3.7 can be reformulated as follows.

If an ordered pair (N_1,N_2) of normal operators is f-regular, then any pair (M_1,M_2) of normal operators with $\sigma(M_1)\subset\sigma(N_1)$ and $\sigma(M_2)\subset\sigma(N_2)$ is also f-regular.

We first prove a lemma.

Lemma 3.3.8. *Let (N_1,N_2) be an f-regular pair of bounded normal operators on $\mathscr{H}_1$ and $\mathscr{H}_2$, and let $\mathscr{K}_1$ and $\mathscr{K}_2$ be reducing subspaces of these operators. If M_1 is unitarily equivalent to $N_1\big|\mathscr{K}_1$ and M_2 is unitarily equivalent to $N_2\big|\mathscr{K}_2$, then the pair (M_1,M_2) is f-regular.*

Proof. Let $M_1\in\mathcal{B}(\widetilde{\mathscr{H}_1})$ and $M_2\in\mathcal{B}(\widetilde{\mathscr{H}_2})$. It suffices to consider the following two special cases:

1. $\mathscr{K}_1=\mathscr{H}_1$ and $\mathscr{K}_2=\mathscr{H}_2$. Then $M_1=U_1^*N_1U_1$ and $M_2=U_2^*N_2U_2$ for some unitary operators U_1 and U_2. We have

$$\begin{aligned}\|f(M_1)R-Rf(M_2)\| &= \|U_1^*f(N_1)U_1R-RU_2^*f(N_2)U_2\|\\ &= \|f(N_1)U_1RU_2^*-U_1RU_2^*f(N_2)\|\\ &\leqslant \|N_1U_1RU_2^*-U_1RU_2^*N_2\| = \|M_1R-RM_2\|\end{aligned}$$

for any R in $\mathcal{B}(\widetilde{\mathscr{H}_2},\widetilde{\mathscr{H}_1})$.

2. $M_1=N_1\big|\mathscr{K}_1$ and $M_2=N_2\big|\mathscr{K}_2$. Let P_1 be the orthogonal projection from $\mathscr{H}_1$ onto $\mathscr{K}_1$ and let P_2 be the orthogonal projection from $\mathscr{H}_2$ onto $\mathscr{K}_2$. If $R\in\mathcal{B}(\mathscr{K}_2,\mathscr{K}_1)$, then

$$\begin{aligned}\|f(M_1)R-Rf(M_2)\| &= \|P_1(f(M_1)R-Rf(M_2))P_2\|\\ &= \|P_1(f(N_1)R-Rf(N_2))P_2\|\\ &= \|f(N_1)P_1RP_2-P_1RP_2f(N_2)\| \leqslant \|N_1P_1RP_2-P_1RP_2N_2\|\\ &= \|M_1R-RM_2\|. \qquad \square\end{aligned}$$

Proof of Theorem 3.3.7. By Theorem 3.2.1, it suffices to consider the case of bounded operators N_1 and N_2. In the case when the spectra of N_1 and N_2 are finite, Theorem 3.3.7 follows immediately from Lemma 3.3.8 and Theorem 3.1.11. As follows from Lemma 3.1.12, it remains to prove that for any finite subsets Δ_1 and Δ_2 of $\sigma(N_1)$ and $\sigma(N_2)$ there are normal operators $M_1\in\mathcal{B}(\mathscr{K}_1)$ and $M_2\in\mathcal{B}(\mathscr{K}_2)$ such that $\sigma(M_1)=\Delta_1$, $\sigma(M_2)=\Delta_2$, and $\|f(M_1)R-Rf(M_2)\|\leqslant\|M_1R-RM_2\|$ for all R in $\mathcal{B}(\mathscr{K}_2,\mathscr{K}_1)$. With each normal operator N we associate the function α_N

such that $\alpha_N(\zeta)$ is the spectral multiplicity of N at an isolated point ζ of its spectrum $\sigma(N)$, and $\alpha_N(\zeta)=\infty$ at each non-isolated point ζ of its spectrum.

We can take the operators M_1 and M_2 to be normal operators acting in the Hilbert spaces $\mathscr{K}_1$ and $\mathscr{K}_2$ and having the following properties:

1) $\sigma(M_1)=\Delta_1$ and $\sigma(M_2)=\Delta_2$;

2) the functions α_{M_1} and α_{M_2} are the restrictions of α_{N_1} and α_{N_2}.

Let $\Delta_1^{(\varepsilon)}$ and $\Delta_2^{(\varepsilon)}$ be the closed ε-neighbourhoods of Δ_1 and Δ_2. Let $N_1^{(\varepsilon)}$ be the restriction of N_1 to the subspace $E_{N_1}\bigl(\Delta_1^{(\varepsilon)}\cap\sigma(N_1)\bigr)$ and let $N_2^{(\varepsilon)}$ be the restriction of N_2 to the subspace $E_{N_2}\bigl(\Delta_2^{(\varepsilon)}\cap\sigma(N_2)\bigr)$, where E_{N_1} and E_{N_2} are the spectral measures of N_1 and N_2. It is easy to see that there exist operators $M_1^{(\varepsilon)}$ in $\mathcal{B}(\mathscr{K}_1)$ and $M_2^{(\varepsilon)}$ in $\mathcal{B}(\mathscr{K}_2)$ such that $M_1^{(\varepsilon)}$ is unitarily equivalent to $N_1^{(\varepsilon)}$, $M_2^{(\varepsilon)}$ is unitarily equivalent to $N_2^{(\varepsilon)}$, $\|M_1-M_1^{(\varepsilon)}\|\leqslant\varepsilon$, and $\|M_2-M_2^{(\varepsilon)}\|\leqslant\varepsilon$. Then for every $R\in\mathcal{B}(\mathscr{K}_2,\mathscr{K}_1)$,

$$\begin{aligned}\|f(M_1)R-Rf(M_2)\| &\leqslant \|R\|\cdot\|f(M_1)-f(M_1^{(\varepsilon)})\|+\|R\|\cdot\|f(M_2)-f(M_2^{(\varepsilon)})\|\\ &\qquad+\|f(M_1^{(\varepsilon)})R-Rf(M_2^{(\varepsilon)})\|\\ &\leqslant \|R\|\cdot\|f(M_1)-f(M_1^{(\varepsilon)})\|+\|R\|\cdot\|f(M_2)-f(M_2^{(\varepsilon)})\|\\ &\qquad+\|M_1^{(\varepsilon)}R-RM_2^{(\varepsilon)}\|\\ &\leqslant \|R\|\cdot\|f(M_1)-f(M_1^{(\varepsilon)})\|+\|R\|\cdot\|f(M_2)-f(M_2^{(\varepsilon)})\|\\ &\qquad+2\varepsilon\|R\|+\|M_1R-RM_2\|.\end{aligned}$$

It remains to pass to the limit as $\varepsilon\to0$. $\square$

The following theorem is contained in [32].

Theorem 3.3.9. *Let M and N be operators acting in a Hilbert space $\mathscr{H}$, with N normal. Then the following conditions are equivalent*:

(a) *$M=f(N)$ for some f in* $\mathrm{CL}(\sigma(N))$;

(b) *there exists a constant c such that $\|MR-RM\|\leqslant c\|NR-RN\|$ for every bounded operator R*;

(c) *there exists a constant c such that $\|MR-RM\|_{\boldsymbol{S}_1}\leqslant c\|NR-RN\|_{\boldsymbol{S}_1}$ for every bounded operator R*;

(d) *for each bounded operator T there exists a bounded operator S such that $SN-NS=TM-MT$*;

(e) *for each compact operator T there exists a bounded operator S such that $SN-NS=TM-MT$*;

(f) *for each T in $\boldsymbol{S}_1(\mathscr{H})$ there exists an operator S in $\boldsymbol{S}_1(\mathscr{H})$ such that $SN-NS=TM-MT$.*

3.4. Schur multipliers and operator Lipschitzness

If a closed set $\mathfrak{F}$ is a Fuglede set, then $\mathrm{OL}(\mathfrak{F})=\mathrm{CL}(\mathfrak{F})$ by Theorem 3.1.8. Therefore, in this case Theorem 3.3.2 gives a complete description of the space $\mathrm{OL}(\mathfrak{F})$ in terms of Schur multipliers.

In particular, for subsets $\mathfrak{F}$ of a line or a circle we have a complete description of $\mathrm{OL}(\mathfrak{F})$ in terms of Schur multipliers. Moreover, in the last case the seminorm of an operator Lipschitz function can be expressed in terms of the norm of the corresponding Schur multiplier.

In the case when $\mathfrak{F}$ is not a Fuglede set we are not aware of a complete description of operator Lipschitz functions on $\mathfrak{F}$ in terms of Schur multipliers.

In this case we can offer the following sufficient condition for operator Lipschitzness.

Theorem 3.4.1. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$. Suppose that there are Schur multipliers $\Phi_1, \Phi_2 \in \mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$ such that*

$$f(z)-f(w)=(z-w)\Phi_1(z,w)+(\overline{z}-\overline{w})\Phi_2(z,w).$$

Then $f \in \mathrm{OL}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant \|\Phi_1\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})} + \|\Phi_2\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}$.

This theorem can be proved with the help of approximation by operators with finite spectra as was done in the proof of Theorem 3.3.1. We omit this proof and instead give a proof based on double operator integrals (see Theorem 3.5.5 and the remark after it).

Remark. Sometimes it is more convenient to use Theorem 3.4.1 in terms of the real variables $z = x_1 + \mathrm{i}y_1$ and $w = x_2 + \mathrm{i}y_2$. Suppose that there are Schur multipliers $F_1, F_2 \in \mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$ such that

$$f(z)-f(w)=(x_1-x_2)F_1(z,w)+(y_1-y_2)F_2(z,w).$$

Then $f \in \mathrm{OL}(\mathfrak{F})$ and

$$\|f\|_{\mathrm{OL}(\mathfrak{F})} \leqslant \frac{1}{2}\|F_1+\mathrm{i}F_2\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}+\frac{1}{2}\|F_1-\mathrm{i}F_2\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})} \leqslant \|F_1\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}+\|F_2\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}.$$

Theorem 3.4.2. *Let $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed set in $\mathbb{C}$. Then for any line l the restriction $f\big|l\cap\mathfrak{F}$ is differentiable at each non-isolated point of $l\cap\mathfrak{F}$, and at ∞ if the set $l\cap\mathfrak{F}$ is unbounded.*

Proof. Clearly, $f\big|l\cap\mathfrak{F} \in \mathrm{OL}(l\cap\mathfrak{F})$. It remains to observe that $\mathrm{CL}(l\cap\mathfrak{F}) = \mathrm{OL}(l\cap\mathfrak{F})$ by Theorem 3.1.10, and to apply Theorem 3.3.6 to the function $f\big|l\cap\mathfrak{F}$. □

Corollary 3.4.3. *Let $f \in \mathrm{OL}(\mathfrak{F})$, where $\mathfrak{F}$ is a closed subset of the complex plane. Then f is differentiable in an arbitrary direction at each interior point of $\mathfrak{F}$.*

Remark. A function f in $\mathrm{OL}(\mathfrak{F})$ does not have to be differentiable as a function of two real variables. For example, it is easy to verify that the function f defined in polar coordinates by $f(r,\theta) = re^{3\mathrm{i}\theta}$ belongs to $\mathrm{OL}(\mathbb{C})$, but it is not differentiable at the origin as a function of the two real variables x and y. This was observed in [13] (see also [2]).

3.5. The role of double operator integrals

In this section we demonstrate the role of double operator integrals in estimates of operator differences and (quasi-)commutators. We start with estimates of operator differences under a perturbation of a self-adjoint operator by a Hilbert–Schmidt operator, and we discuss the Birman–Solomyak formula.

Next, we return to the results of the two previous sections where we obtained conditions for commutator Lipschitzness and operator Lipschitzness in terms of the membership of certain functions in the space of discrete Schur multipliers. In this section we give another proof of the sufficiency of these conditions with the help of double operator integrals. We obtain useful formulae that express operator differences and commutators in terms of double operator integrals.

Finally, we obtain formulae for operator derivatives in terms of double operator integrals.

The following theorem was obtained by Birman and Solomyak in [21].

Theorem 3.5.1. *Let f be a Lipschitz function on $\mathbb{R}$, and let A and B be self-adjoint operators acting in a Hilbert space and with difference $A-B$ in the Hilbert–Schmidt class $\boldsymbol{S}_2$. Then*

$$f(A)-f(B)=\int_{\mathbb{R}}\int_{\mathbb{R}}(\mathfrak{D}_0 f)(x,y)\,dE_A(x)(A-B)\,dE_B(y). \tag{3.5.1}$$

Note that the last formula directly implies the inequality

$$\|f(A)-f(B)\|_{\boldsymbol{S}_2}\leqslant\|f\|_{\mathrm{Lip}}\|A-B\|_{\boldsymbol{S}_2}.$$

In other words, Lipschitz functions are $\boldsymbol{S}_2$-Lipschitz. It turns out that Lipschitz functions are also $\boldsymbol{S}_p$-Lipschitz for $p\in(1,\infty)$. This was recently proved in [67]. We recall that for $p=1$ the corresponding statement is false. This was first proved in [26]. Moreover, the class of $\boldsymbol{S}_1$-Lipschitz functions coincides with the class of operator Lipschitz functions (see Theorem 3.6.5).

We now proceed to commutator Lipschitzness.

Theorem 3.5.2. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be closed subsets of $\mathbb{C}$. Suppose that f is a continuous function on $\mathfrak{F}_1\cup\mathfrak{F}_2$ such that the function $\mathfrak{D}_0 f$ defined by (3.3.1) belongs to the class $\mathfrak{M}(\mathfrak{F}_1\times\mathfrak{F}_2)$ of Schur multipliers. If N_1 and N_2 are normal operators such that $\sigma(N_j)\subset\mathfrak{F}_j$ for $j=1,2$ and R is a bounded linear operator, then*

$$f(N_1)R-Rf(N_2)=\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1R-RN_2)\,dE_2(\zeta_2), \tag{3.5.2}$$

where E_j is the spectral measure of N_j.

Remark. It follows immediately from (3.5.2) that

$$\begin{aligned}\|f(N_1)R-Rf(N_2)\|&\leqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}(E_1,E_2)}\|N_1R-RN_2\|\\&\leqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}_1\times\mathfrak{F}_2)},\|N_1R-RN_2\|\end{aligned}$$

and in particular, f is a commutator Lipschitz function.

In the special case when R is the identity operator we obtain the following result.

Theorem 3.5.3. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{C}$ and let f be a continuous function on $\mathfrak{F}$ such that $\mathfrak{D}_0 f\in\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$. If N_1 and N_2 are normal operators with spectra in $\mathfrak{F}$, then*

$$f(N_1)-f(N_2)=\int_{\mathfrak{F}}\int_{\mathfrak{F}}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1-N_2)\,dE_2(\zeta_2). \tag{3.5.3}$$

Proof of Theorem 3.5.2. Suppose first that N_1 and N_2 are bounded. Then

$$\begin{aligned}\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}&(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1R-RN_2)\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)N_1R\,dE_2(\zeta_2)\\&\quad-\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)RN_2\,dE_2(\zeta_2).\end{aligned}$$

It follows from the definition of double operator integrals that

$$\begin{aligned}\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}&(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)N_1R\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}\zeta_1(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)R\,dE_2(\zeta_2),\\\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}&(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)RN_2\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}\zeta_2(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)R\,dE_2(\zeta_2).\end{aligned}$$

Since $(\zeta_1-\zeta_2)(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)=f(\zeta_1)-f(\zeta_2)$ for $\zeta_1\in\mathfrak{F}_1$ and $\zeta_2\in\mathfrak{F}_2$, we get that

$$\begin{aligned}\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}&(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1R-RN_2)\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}f(\zeta_1)\,dE_1(\zeta_1)R\,dE_2(\zeta_2)-\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}f(\zeta_2)\,dE_1(\zeta_1)R\,dE_2(\zeta_2).\end{aligned}$$

Again, it is easy to see from the definition of double operator integrals that

$$\begin{aligned}\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}f(\zeta_1)\,dE_1(\zeta_1)R\,dE_2(\zeta_2)&=\biggl(\int_{\mathfrak{F}_1}f(\zeta_1)\,dE_1(\zeta_1)\biggr)R=f(N_1)R,\\\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}f(\zeta_2)\,dE_1(\zeta_1)R\,dE_2(\zeta_2)&=R\int_{\mathfrak{F}_1}f(\zeta_1)\,dE_1(\zeta_1)=Rf(N_2),\end{aligned}$$

which implies (3.5.2).

We now suppose that N_1 and N_2 are not necessarily bounded normal operators. The special case of Theorem 3.5.2 proved above and Theorem 3.2.1 imply the commutator Lipschitz estimate, and hence the operator $f(N_1)R-Rf(N_2)$ is bounded.

Let

$$P_k\stackrel{\text{def}}{=}E_1\bigl(\{\zeta\in\mathbb{C}\colon|\zeta|\leqslant k\}\bigr)\quad\text{and}\quad Q_k\stackrel{\text{def}}{=}E_2\bigl(\{\zeta\in\mathbb{C}\colon|\zeta|\leqslant k\}\bigr),\qquad k>0.$$

Then $N_{1,k}\stackrel{\text{def}}{=}P_kN_1$ and $N_{2,k}\stackrel{\text{def}}{=}Q_kN_2$ are bounded normal operators. Let $E_{j,k}$ be the spectral measure of $N_{j,k}$, $j=1,2$. We have

$$\begin{aligned}&P_k\biggl(\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1R-RN_2)\,dE_2(\zeta_2)\biggr)Q_k\\&\quad=P_k\biggl(\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_{1,k}(\zeta_1)\bigl(P_kf(N_1)R-Rf(N_2)Q_k\bigr)\,dE_{2,k}(\zeta_2)\biggr)Q_k.\end{aligned}$$

Applying (3.5.2) to the bounded normal operators $N_{1,k}$ and $N_{2,k}$, we obtain

$$P_k\big(f(N_{1,k})R - Rf(N_{2,k})\big)Q_k = P_k\left(\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_{1,k}(\zeta_1)(P_kN_1R - RN_2Q_k)\,dE_{2,k}(\zeta_2)\right)Q_k.$$

Since $P_k\big(f(N_{1,k})R - Rf(N_{2,k})\big)Q_k = P_k\big(f(N_1)R - Rf(N_2)\big)Q_k$,

$$P_k\big(f(N_1)R - Rf(N_2)\big)Q_k = P_k\left(\int_{\mathfrak{F}_1}\int_{\mathfrak{F}_2}(\mathfrak{D}_0 f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1R - RN_2)\,dE_2(\zeta_2)\right)Q_k.$$

It remains to pass to the limit in the strong operator topology. □

It is easy to verify that in all the formulae in this section one can replace the function $\mathfrak{D}_0 f(\zeta_1,\zeta_2)$ under the sign of a double operator integral by an arbitrary bounded measurable function $F(\zeta_1,\zeta_2)$ that coincides with $\mathfrak{D}_0 f(\zeta_1,\zeta_2)$ for all ζ_1 and ζ_2 with $\zeta_1 \ne \zeta_2$.

In particular, in the case when $\mathfrak{F}_1 = \mathfrak{F}_2$ and the set $\mathfrak{F}_1$ is perfect, Theorem 3.3.6 lets us replace $\mathfrak{D}_0 f$ in (3.5.2) by the divided difference $\mathfrak{D} f$.

Theorem 3.5.4. *Let $\mathfrak{F}$ be a closed perfect subset of $\mathbb{C}$ and let $f \in \mathrm{CL}(\mathfrak{F})$. If N_1 and N_2 are normal operators such that $\sigma(N_j) \subset \mathfrak{F}$ for $j = 1,2$ and R is a bounded linear operator, then the following formula holds:*

$$f(N_1)R - Rf(N_2) = \int_{\mathfrak{F}}\int_{\mathfrak{F}}(\mathfrak{D} f)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1R - RN_2)\,dE_2(\zeta_2), \qquad (3.5.4)$$

where E_j is the spectral measure of N_j.

Let us now interpret the results of § 3.4 in terms of double operator integrals.

Theorem 3.5.5. *Let f be a continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{C}$ and suppose that there exist Schur multipliers $\Phi_1, \Phi_2 \in \mathfrak{M}(\mathfrak{F}\times\mathfrak{F})$ such that*

$$f(\zeta_1) - f(\zeta_2) = (\zeta_1 - \zeta_2)\Phi_1(\zeta_1,\zeta_2) + (\overline{\zeta}_1 - \overline{\zeta}_2)\Phi_2(\zeta_1,\zeta_2), \qquad \zeta_1,\zeta_2 \in \mathfrak{F}.$$

Let N_1 and N_2 be normal operators whose spectra are contained in $\mathfrak{F}$. Then

$$f(N_1) - f(N_2) = \int_{\mathfrak{F}}\int_{\mathfrak{F}}\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1 - N_2)\,dE_2(\zeta_2) + \int_{\mathfrak{F}}\int_{\mathfrak{F}}\Phi_2(\zeta_1,\zeta_2)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1^* - N_2^*)\,dE_2(\zeta_2). \quad (3.5.5)$$

Remark. It follows easily from (3.5.5) that $\|f(N_1) - f(N_2)\| \leqslant \big(\|\Phi_1\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})} + \|\Phi_2\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}\big)\|N_1 - N_2\|$, and in particular, f is an operator Lipschitz function.

Proof of Theorem 3.5.5. As in the proof of Theorem 3.5.2, we assume first that N_1 and N_2 are bounded. Then

$$\begin{aligned}\int_{\mathfrak{F}}\int_{\mathfrak{F}}&\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1-N_2)\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}}\int_{\mathfrak{F}}\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)N_1\,dE_2(\zeta_2)-\int_{\mathfrak{F}}\int_{\mathfrak{F}}\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)N_2\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}}\int_{\mathfrak{F}}\zeta_1\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)\,dE_2(\zeta_2)-\int_{\mathfrak{F}}\int_{\mathfrak{F}}\zeta_2\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}}\int_{\mathfrak{F}}(\zeta_1-\zeta_2)\Phi_1(\zeta_1,\zeta_2)\,dE_1(\zeta_1)\,dE_2(\zeta_2).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\mathfrak{F}}\int_{\mathfrak{F}}&\Phi_2(\zeta_1,\zeta_2)(\zeta_1,\zeta_2)\,dE_1(\zeta_1)(N_1^*-N_2^*)\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}}\int_{\mathfrak{F}}(\overline{\zeta}_1-\overline{\zeta}_2)\Phi_2(\zeta_1,\zeta_2)\,dE_1(\zeta_1)\,dE_2(\zeta_2).\end{aligned}$$

Therefore, the right-hand side of (3.5.5) is equal to

$$\begin{aligned}\int_{\mathfrak{F}}\int_{\mathfrak{F}}&(f(\zeta_1)-f(\zeta_2))\,dE_1(\zeta_1)\,dE_2(\zeta_2)\\&=\int_{\mathfrak{F}}f(\zeta_1)\,dE_1(\zeta_1)-\int_{\mathfrak{F}}f(\zeta_2)\,dE_2(\zeta_2)=f(N_1)-f(N_2).\end{aligned}$$

The passage from bounded to unbounded operators can be done just as in the proof of Theorem 3.5.2. □

We now consider applications of double operator integrals in problems of operator differentiability.

Theorem 3.5.6. *Let f be an operator Lipschitz function on $\mathbb{R}$, and let A and K be self-adjoint operators with K bounded. Then*

$$\lim_{t\to 0}\frac{1}{t}(f(A+tK)-f(A))=\int_{\mathbb{R}}\int_{\mathbb{R}}(\mathfrak{D}f)(x,y)\,dE_A(x)K\,dE_A(y),$$

where the limit is taken in the strong operator topology.

We need several auxiliary results. Let $\widehat{\mathbb{R}}\stackrel{\text{def}}{=}\mathbb{R}\cup\{\infty\}$ denote the *one-point compactification of the real line* $\mathbb{R}$. We recall that any function $f\in\mathrm{OL}(\mathbb{R})$ is everywhere differentiable on $\widehat{\mathbb{R}}$ (see Theorem 3.3.3).

Lemma 3.5.7. *If $f\in\mathrm{OL}(\mathbb{R})$, then there are two sequences $\{\varphi_n\}_{n\geqslant 0}$ and $\{\psi_n\}_{n\geqslant 0}$ of continuous functions on $\widehat{\mathbb{R}}$ such that*:

(a) $\sum_{n\geqslant 0}|\varphi_n|^2\leqslant\|f\|_{\mathrm{OL}(\mathbb{R})}$ *everywhere on* $\widehat{\mathbb{R}}$;
(b) $\sum_{n\geqslant 0}|\psi_n|^2\leqslant\|f\|_{\mathrm{OL}(\mathbb{R})}$ *everywhere on* $\widehat{\mathbb{R}}$;
(c) $(\mathfrak{D}f)(x,y)=\sum_{n\geqslant 0}\varphi_n(x)\psi_n(y)$ *for all* $x,y\in\mathbb{R}$.

Proof. By Theorem 3.3.6, $\mathfrak{D}f \in \mathfrak{M}(\mathbb{R}\times\mathbb{R})$ and $\|\mathfrak{D}f\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})} = \|f\|_{\mathrm{OL}(\mathbb{R})}$. We extend the function $\mathfrak{D}f$ to the set $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ by putting $(\mathfrak{D}f)(x,y) = f'(\infty) = \lim_{t\to\infty} t^{-1}f(t)$ in the case when $|x|+|y| = \infty$. Clearly, this extended function $\mathfrak{D}f$ on $\widehat{\mathbb{R}}\times\widehat{\mathbb{R}}$ is continuous in each variable. Hence

$$\begin{aligned}\|\mathfrak{D}f\|_{\mathfrak{M}(\widehat{\mathbb{R}}\times\widehat{\mathbb{R}})} &= \sup\{\|\mathfrak{D}f\|_{\mathfrak{M}(\Lambda_1\times\Lambda_2)}\colon \Lambda_1,\Lambda_2\subset\widehat{\mathbb{R}},\ \Lambda_1 \text{ and } \Lambda_2 \text{ are finite}\}\\ &= \sup\{\|\mathfrak{D}f\|_{\mathfrak{M}(\Lambda_1\times\Lambda_2)}\colon \Lambda_1,\Lambda_2\subset\mathbb{R},\ \Lambda_1 \text{ and } \Lambda_2 \text{ are finite}\}\\ &= \|\mathfrak{D}f\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})}.\end{aligned}$$

It remains to apply Theorem 2.2.4 to the function $\mathfrak{D}f\colon \widehat{\mathbb{R}}\times\widehat{\mathbb{R}}\to\mathbb{C}$. □

Lemma 3.5.8. *Let A and K be self-adjoint operators, with K bounded. Then for every function f in $C(\widehat{\mathbb{R}})$ the function $H(t) \stackrel{\text{def}}{=} f(A+tK)$ acts continuously from $\mathbb{R}$ to the space $\mathcal{B}(\mathscr{H})$ with the norm topology.*

We remark that a considerably stronger result was obtained in [8].

Proof. We can assume that $f(\infty) = 0$. Then we can construct a sequence $\{f_n\}_{n\geqslant0}$ of functions of class C^∞ with compact support such that $f_n \to f$ uniformly. Each function $H_n(t) \stackrel{\text{def}}{=} f_n(A+tK)$ is continuous, because $f_n \in \mathrm{OL}(\mathbb{R})$ for $n \geqslant 0$. It remains to observe that $H_n \to H$ uniformly. □

Lemma 3.5.9. *Let $\{X_n\}_{n\geqslant0}$ be a sequence in $\mathcal{B}(\mathscr{H})$ and $\{u_n\}_{n\geqslant0}$ a sequence in $\mathscr{H}$. Assume that $\sum_{n\geqslant0} X_nX_n^* \leqslant a^2I$ and $\sum_{n\geqslant0}\|u_n\|^2 \leqslant b^2$ for some non-negative numbers a and b. Then the series $\sum_{n\geqslant0} X_nu_n$ converges weakly, and*

$$\Big\|\sum_{n\geqslant0} X_nu_n\Big\| \leqslant ab.$$

Proof. Let $v\in\mathscr{H}$ and $\|v\| = 1$. Then

$$\sum_{n\geqslant0}|(X_nu_n,v)| = \sum_{n\geqslant0}|(u_n,X_n^*v)| \leqslant \Big(\sum_{n\geqslant0}\|u_n\|^2\Big)^{1/2}\Big(\sum_{n\geqslant0}\|X_n^*v\|^2\Big)^{1/2} \leqslant ab,$$

which implies the result. □

Proof of Theorem 3.5.6. By the formulae (3.5.4) and (2.3.5), Theorem 3.5.6 can be reformulated as follows:

$$\lim_{t\to0}\sum_{n\geqslant0}\varphi_n(A+tK)K\psi_n(A) = \sum_{n\geqslant0}\varphi_n(A)K\psi_n(A)$$

in the strong operator topology, where φ_n and ψ_n are functions from the conclusion of Lemma 3.5.7. In other words, we have to prove that for any $u\in\mathscr{H}$

$$\lim_{t\to0}\sum_{n\geqslant0}(\varphi_n(A+tK)-\varphi_n(A))K\psi_n(A)u = \mathbf{0},$$

where the series is understood in the weak topology of $\mathscr{H}$, while the limit is taken in the norm of $\mathscr{H}$. Assume that $\|u\| = 1$ and $\|f\|_{\mathrm{OL}(\mathbb{R})} = 1$. Then $\sum_{n\geqslant0}|\varphi_n|^2 \leqslant 1$ and $\sum_{n\geqslant0}|\psi_n|^2 \leqslant 1$ everywhere on $\mathbb{R}$.

Let $u_n \stackrel{\text{def}}{=} K\psi_n(A)u$. We have

$$\sum_{n\geqslant 0}\|u_n\|^2 \leqslant \|K\|^2\sum_{n\geqslant 0}\|\psi_n(A)u\|^2 = \|K\|^2\sum_{n\geqslant 0}(|\psi_n|^2(A)u,u) \leqslant \|K\|^2 < +\infty.$$

Let $\varepsilon > 0$ and choose a natural number N such that $\sum_{n>N}\|u_n\|^2 < \varepsilon^2$. Then it follows from Lemma 3.5.9 that

$$\left\|\sum_{n>N}(\varphi_n(A+tK)-\varphi_n(A))u_n\right\| \leqslant 2\varepsilon$$

for all $t\in\mathbb{R}$. By Lemma 3.5.8,

$$\left\|\sum_{n=0}^{N}(\varphi_n(A+tK)-\varphi_n(A))u_n\right\| \leqslant \|K\|\sum_{n=0}^{N}\|\varphi_n(A+tK)-\varphi_n(A)\| < \varepsilon$$

for all t sufficiently close to zero. Thus, $\left\|\sum_{n\geqslant 0}(\varphi_n(A+tK)-\varphi_n(A))u_n\right\| < 3\varepsilon$ for all t sufficiently close to zero. □

By analogy with Theorem 3.5.6 we can prove the following theorem.

Theorem 3.5.10. *Let A and K be bounded self-adjoint operators. Then*

$$\lim_{t\to 0}\frac{1}{t}(f(A+tK)-f(A)) = \int_{\mathbb{R}}\int_{\mathbb{R}}(\mathfrak{D}f)(x,y)\,dE_A(x)K\,dE_A(y)$$

for all f in $\mathrm{OL}_{\mathrm{loc}}(\mathbb{R})$, where the limit is taken in the strong operator topology.

Theorem 3.5.6 implies the following result.

Theorem 3.5.11. *Let f be an operator differentiable function on $\mathbb{R}$, and let A and K be self-adjoint operators with K bounded. Then the derivative of the function $t\mapsto f(A+tK)-f(A)$ in the operator norm is*

$$\frac{d}{dt}\big(f(A+tK)-f(A)\big)\Big|_{t=0} = \int_{\mathbb{R}}\int_{\mathbb{R}}\frac{f(x)-f(y)}{x-y}\,dE_A(x)K\,dE_A(y). \qquad (3.5.6)$$

In particular, the last formula holds for any function f of Besov class $B^1_{\infty,1}(\mathbb{R})$ (see Theorem 1.6.4).

Similar results hold for functions on the unit circle.

3.6. Trace class Lipschitzness and trace class commutator Lipschitzness

The purpose of this section is to prove that the classes $\mathrm{CL}(\mathfrak{F})$ and $\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$ coincide for an arbitrary closed set $\mathfrak{F}$ in the plane. In particular, if $\mathfrak{F}\subset\mathbb{R}$, then $\mathrm{OL}(\mathfrak{F}) = \mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})$ (see § 3.1, where the classes $\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$ and $\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})$ are defined).

Note that the definition of the class $\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$ can be extended naturally to the definition of the class $\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F}_1,\mathfrak{F}_2)$, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are non-empty closed subsets of $\mathbb{C}$.

Lemma 3.6.1. *Let f be a continuous function on a union $\mathfrak{F}_1\cup\mathfrak{F}_2$ of closed subsets $\mathfrak{F}_1$ and $\mathfrak{F}_2$ of $\mathbb{C}$. Then*

$$\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F}_1,\mathfrak{F}_2)}\geqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathfrak{F}_1\times\mathfrak{F}_2)}\geqslant\frac{1}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}_1\times\mathfrak{F}_2)}.$$

Proof. The second inequality follows from Corollary 2.1.5. Let us prove the first. It suffices to consider the case of finite sets $\mathfrak{F}_1$ and $\mathfrak{F}_2$. Let N_1 and N_2 be normal operators with simple spectra such that $\sigma(N_1)=\mathfrak{F}_1$ and $\sigma(N_2)=\mathfrak{F}_2$. Then there exist orthonormal bases $\{u_\lambda\}_{\lambda\in\sigma(N_1)}$ and $\{v_\mu\}_{\mu\in\sigma(N_2)}$ in $\mathscr{H}_1$ and $\mathscr{H}_2$ such that $N_1u_\lambda=\lambda u_\lambda$ for $\lambda\in\sigma(N_1)$ and $N_2v_\mu=\mu v_\mu$ for $\mu\in\sigma(N_2)$. With each operator $X\colon\mathscr{H}_2\to\mathscr{H}_1$ we associate the matrix $\{(Xv_\mu,u_\lambda)\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)}$. We have

$$\big((N_1R-RN_2)v_\mu,u_\lambda\big)=(Rv_\mu,N_1^*u_\lambda)-(RN_2v_\mu,u_\lambda)=(\lambda-\mu)(Rv_\mu,u_\lambda).$$

Similarly, $\big((f(N_1)R-Rf(N_2))v_\mu,u_\lambda\big)=(f(\lambda)-f(\mu))(Rv_\mu,u_\lambda)$. It is easy to see that $\{(f(\lambda)-f(\mu))(Rv_\mu,u_\lambda)\}=\{(\mathfrak{D}_0 f)(\lambda,\mu)\}\star\{(\lambda-\mu)(Rv_\mu,u_\lambda)\}$. We note that the matrix $\{a_{\lambda\mu}\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)}$ can be represented in the form

$$\{a_{\lambda\mu}\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)}=\{(\lambda-\mu)(Rv_\mu,u_\lambda)\}_{(\lambda,\mu)\in\sigma(N_1)\times\sigma(N_2)},$$

where R is an operator from $\mathscr{H}_2$ to $\mathscr{H}_1$, if and only if $a_{\lambda\mu}=0$ for $\lambda=\mu$. The inequality $\|f\|_{\mathrm{CL}(\mathfrak{F}_1,\mathfrak{F}_2)}\geqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathfrak{F}_1\times\mathfrak{F}_2)}$ is now obvious. □

Corollary 3.6.2. *Let f be a real continuous function on a closed subset $\mathfrak{F}$ of $\mathbb{R}$. Then*

$$\|f\|_{\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})}=\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})}\geqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathfrak{F}\times\mathfrak{F})}\geqslant\frac{1}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}.\tag{3.6.1}$$

If f is not necessarily real, then

$$\|f\|_{\mathrm{OL}_{\boldsymbol{S}_1}(\mathfrak{F})}\geqslant\frac{1}{2}\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})}\geqslant\frac{1}{2}\|\mathfrak{D}_0 f\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathfrak{F}\times\mathfrak{F})}\geqslant\frac{1}{4}\|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}.\tag{3.6.2}$$

Proof. The equality in (3.6.1) follows from Corollary 3.1.7. All the inequalities in (3.6.1) have already been proved above. Obviously, (3.6.2) follows from (3.6.1). □

We now proceed to the main results of this section.

Theorem 3.6.3. *Let $\mathfrak{F}$ be a closed set in $\mathbb{C}$. Then $\mathrm{CL}(\mathfrak{F})=\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$ and*

$$\frac{1}{2}\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})}\leqslant\|f\|_{\mathrm{CL}(\mathfrak{F})}\leqslant 2\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})},\qquad f\in\mathrm{CL}(\mathfrak{F}).$$

Proof. Let $f\in\mathrm{CL}(\mathfrak{F})$, and let N_1 and N_2 be normal operators with spectra in $\mathfrak{F}$ such that $N_1R-RN_2\in\boldsymbol{S}_1$, where R is a bounded operator. Then in view of the remark after Theorem 3.5.2,

$$\begin{aligned}\|f(N_1)R-Rf(N_2)\|_{\boldsymbol{S}_1}&\leqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}(E_1,E_2)}\|N_1R-RN_2\|_{\boldsymbol{S}_1}\\&\leqslant\|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}\|N_1R-RN_2\|_{\boldsymbol{S}_1}\\&\leqslant 2\|f\|_{\mathrm{CL}(\mathfrak{F})}\|N_1R-RN_2\|_{\boldsymbol{S}_1}\end{aligned}$$

by Theorem 3.3.2. This implies the inequality $\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})} \leqslant 2\|f\|_{\mathrm{CL}(\mathfrak{F})}$. On the other hand, we get from Lemma 3.6.1 and Theorem 3.3.2 that

$$\|f\|_{\mathrm{CL}(\mathfrak{F})} \leqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})} \leqslant 2\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})}. \qquad \square$$

If $\mathfrak{F}$ is a perfect set, then we can improve the result.

Theorem 3.6.4. *Let $\mathfrak{F}$ be a closed perfect set in $\mathbb{C}$. Then $\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})} = \|f\|_{\mathrm{CL}(\mathfrak{F})}$ for all f in $\mathrm{CL}(\mathfrak{F}) = \mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})$.*

Proof. By (3.5.4), for $f \in \mathrm{CL}(\mathfrak{F})$

$$\begin{aligned}\|f(N_1)R - Rf(N_2)\|_{\boldsymbol{S}_1} &\leqslant \|\mathfrak{D}f\|_{\mathfrak{M}(E_1,E_2)}\|N_1R - RN_2\|_{\boldsymbol{S}_1}\\ &\leqslant \|\mathfrak{D}f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})}\|N_1R - RN_2\|_{\boldsymbol{S}_1}\\ &= \|f\|_{\mathrm{CL}(\mathfrak{F})}\|N_1R - RN_2\|_{\boldsymbol{S}_1}.\end{aligned}$$

The last equality is guaranteed by Theorem 3.3.6. Thus, we have proved that $\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})} \leqslant \|f\|_{\mathrm{CL}(\mathfrak{F})}$. Using Lemmas 3.6.1 and 2.1.6 along with Theorem 3.3.6, we obtain

$$\|f\|_{\mathrm{CL}_{\boldsymbol{S}_1}(\mathfrak{F})} \geqslant \|\mathfrak{D}_0 f\|_{\mathfrak{M}_{0,\boldsymbol{S}_1}(\mathfrak{F}\times\mathfrak{F})} = \|\mathfrak{D}f\|_{\mathfrak{M}(\mathfrak{F}\times\mathfrak{F})} = \|f\|_{\mathrm{CL}(\mathfrak{F})}. \qquad \square$$

It is time to proceed to the central result in this section.

Theorem 3.6.5. *Let f be a continuous function on $\mathbb{R}$. Then the following conditions are equivalent*:

(a) *f is operator Lipschitz*;

(b) *f is trace class Lipschitz*;

(c) *$f(A) - f(B) \in \boldsymbol{S}_1$ if A and B are self-adjoint operators with $A - B$ in $\boldsymbol{S}_1$.*

In (c) *one must consider not only bounded operators A and B.*

Proof. The equivalence of (a) and (b) is established in Corollary 3.6.2. The implication (b) $\Rightarrow$ (c) is trivial. Let us show that (c) $\Rightarrow$ (b). Suppose that $f \notin \mathrm{CL}_{\boldsymbol{S}_1}(\mathbb{R})$. Then we can find sequences A_n and B_n of self-adjoint operators such that $A_n - B_n \in \boldsymbol{S}_1$ and $\|A_n - B_n\|_{\boldsymbol{S}_1}^{-1}\|f(A_n) - f(B_n)\|_{\boldsymbol{S}_1} \to \infty$ as $n \to \infty$. Without loss of generality we can assume that $\lim_{n\to\infty}\|A_n - B_n\|_{\boldsymbol{S}_1} = 0$. Indeed, consider the increment $A_n \mapsto A_n + K_n$, where $K_n \stackrel{\mathrm{def}}{=} B_n - A_n$. And consider now the following increments: $A_n + (j/M_n)K_n \mapsto A_n + ((j+1)/M_n)K_n$ for $0 \leqslant j \leqslant M_n - 1$, where $\{M_n\}$ is a sequence of natural numbers such that $\lim_{n\to\infty}\|A_n - B_n\|_{\boldsymbol{S}_1}/M_n = 0$. We then choose a j that maximizes the number

$$\left\|f\left(A_n + \frac{j+1}{M_n}K_n\right) - f\left(A_n + \frac{j}{M_n}K_n\right)\right\|_{\boldsymbol{S}_1},$$

and we replace the pair (A_n, B_n) by the pair $(A_n + (j/M_n)K_n, A_n + ((j+1)/M_n)K_n)$. Then

$$\lim_{n\to\infty}\|A_n - B_n\|_{\boldsymbol{S}_1}^{-1}\|f(A_n) - f(B_n)\|_{\boldsymbol{S}_1} = \infty \quad \text{and} \quad \lim_{n\to\infty}\|A_n - B_n\|_{\boldsymbol{S}_1} = 0.$$

It now suffices, if necessary, to choose a subsequence of the sequence (A_n, B_n) or to repeat certain terms of this sequence in order to achieve the condition that

$$\sum_n \|B_n - A_n\|_{\boldsymbol{S}_1} < \infty, \quad \text{but} \quad \sum_n \|f(B_n) - f(A_n)\|_{\boldsymbol{S}_1} = \infty.$$

Let A be the orthogonal sum of the operators A_n and let B be the orthogonal sum of the B_n. Then $B - A \in \boldsymbol{S}_1$ but $f(B) - f(A) \notin \boldsymbol{S}_1$. $\square$

Remark. A similar result holds for functions on the unit circle and unitary operators.

3.7. Operator Lipschitz functions on the plane. Sufficient conditions

In this section we obtain a sufficient condition for operator Lipschitzness in terms of the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$. It is similar to Theorem 1.6.1 for functions on the real line. The results in this section were obtained in [14].

Recall (see (3.5.3)) that in the case of functions on the line, the operator Lipschitzness of f can be obtained from the formula

$$f(A) - f(B) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(s) - f(t)}{s - t} \, dE_A(s)(A - B) \, dE_B(t).$$

Here A and B are self-adjoint operators. This is the way the operator Lipschitzness of functions of the class $B^1_{\infty,1}(\mathbb{R})$ was established in [56] and [58].

It would be natural to try the same approach also for functions on the plane. However, (see Corollary 3.3.5) if the divided difference is a Schur multiplier for arbitrary Borel spectral measures on $\mathbb{C}$, then the function must be linear.

In [14] another method was used: for normal operators N_1 and N_2, the difference $f(N_1) - f(N_2)$ is represented as a sum of double operator integrals, the integrands being the divided differences with respect to each variable.

We introduce the following notation. Let N_1 and N_2 be normal operators acting in a Hilbert space. Put $A_j \stackrel{\text{def}}{=} \operatorname{Re} N_j$ and $B_j \stackrel{\text{def}}{=} \operatorname{Im} N_j$ for $j = 1, 2$, and let E_j be the spectral measure of N_j. In other words, $N_j = A_j + \mathrm{i} B_j$ for $j = 1, 2$, where A_j and B_j are commuting self-adjoint operators.

If f is a function on $\mathbb{R}^2$ with partial derivatives with respect to each variable, then we consider the divided differences with respect to each variable:

$$(\mathfrak{D}_x f)(z_1, z_2) \stackrel{\text{def}}{=} \frac{f(x_1, y_2) - f(x_2, y_2)}{x_1 - x_2}, \qquad z_1, z_2 \in \mathbb{C},$$

$$(\mathfrak{D}_y f)(z_1, z_2) \stackrel{\text{def}}{=} \frac{f(x_1, y_1) - f(x_1, y_2)}{y_1 - y_2}, \qquad z_1, z_2 \in \mathbb{C},$$

where $x_j \stackrel{\text{def}}{=} \operatorname{Re} z_j$ and $y_j \stackrel{\text{def}}{=} \operatorname{Im} z_j$ for $j = 1, 2$. On the sets $\{(z_1, z_2) \colon x_1 = x_2\}$ and $\{(z_1, z_2) \colon y_1 = y_2\}$ the divided differences are understood as the corresponding partial derivatives of f.

The following result gives us a key estimate.

Theorem 3.7.1. *Let f be a bounded continuous function on $\mathbb{R}^2$ whose Fourier transform $\mathscr{F}f$ has compact support. Then $\mathfrak{D}_x f$ and $\mathfrak{D}_y f$ are Schur multipliers of class $\mathfrak{M}(\mathbb{C}\times\mathbb{C})$. Moreover, if* $\operatorname{supp}\mathscr{F}f\subset\{\zeta\in\mathbb{C}\colon|\zeta|\leqslant\sigma\}$, $\sigma>0$, *then*

$$\|\mathfrak{D}_x f\|_{\mathfrak{M}(\mathbb{C}\times\mathbb{C})}\leqslant\operatorname{const}\sigma\|f\|_{L^\infty}\quad\textit{and}\quad\|\mathfrak{D}_y f\|_{\mathfrak{M}(\mathbb{C}\times\mathbb{C})}\leqslant\operatorname{const}\sigma\|f\|_{L^\infty}. \tag{3.7.1}$$

It follows from the definition of the Besov class $B^1_{\infty,1}(\mathbb{R}^2)$ and from Theorem 3.7.1 that for every $f\in B^1_{\infty,1}(\mathbb{R}^2)$ the divided differences $\mathfrak{D}_x f$ and $\mathfrak{D}_y f$ are Schur multipliers and

$$\|\mathfrak{D}_x f\|_{\mathfrak{M}(\mathbb{C}\times\mathbb{C})}\leqslant\operatorname{const}\|f\|_{B^1_{\infty,1}}\quad\text{and}\quad\|\mathfrak{D}_y f\|_{\mathfrak{M}(\mathbb{C}\times\mathbb{C})}\leqslant\operatorname{const}\|f\|_{B^1_{\infty,1}}. \tag{3.7.2}$$

The inequalities (3.7.2) together with Theorem 3.5.5 imply the following central result in this section, obtained in [14].

Theorem 3.7.2. *Let f be a function in $B^1_{\infty,1}(\mathbb{R}^2)$. Suppose that N_1 and N_2 are normal operators such that N_1-N_2 is bounded. Then*

$$\begin{aligned}f(N_1)-f(N_2)&=\iint_{\mathbb{C}^2}(\mathfrak{D}_y f)(z_1,z_2)\,dE_1(z_1)(B_1-B_2)\,dE_2(z_2)\\&\qquad+\iint_{\mathbb{C}^2}(\mathfrak{D}_x f)(z_1,z_2)\,dE_1(z_1)(A_1-A_2)\,dE_2(z_2)\end{aligned}$$

and $\|f(N_1)-f(N_2)\|\leqslant\operatorname{const}\|f\|_{B^1_{\infty,1}}\|N_1-N_2\|$, *that is, f is an operator Lipschitz function on $\mathbb{C}$.*

To prove Theorem 3.7.1 we use a formula for a representation of the divided difference as an element of the Haagerup tensor product. Recall that $\mathscr{E}_\sigma$ denotes the set of entire functions (of one complex variable) of exponential type at most σ.

Lemma 3.7.3. *Let $\varphi\in\mathscr{E}_\sigma\cap L^\infty(\mathbb{R})$. Then*

$$\frac{\varphi(x)-\varphi(y)}{x-y}=\sum_{n\in\mathbb{Z}}\sigma\,\frac{\varphi(x)-\varphi\bigl(\pi n\sigma^{-1}\bigr)}{\sigma x-\pi n}\,\frac{\sin(\sigma y-\pi n)}{\sigma y-\pi n}\,. \tag{3.7.3}$$

Moreover,

$$\sum_{n\in\mathbb{Z}}\frac{|\varphi(x)-\varphi(\pi n\sigma^{-1})|^2}{(\sigma x-\pi n)^2}\leqslant 3\|\varphi\|^2_{L^\infty(\mathbb{R})},\qquad x\in\mathbb{R}, \tag{3.7.4}$$

$$\sum_{n\in\mathbb{Z}}\frac{\sin^2(\sigma y-\pi n)}{(\sigma y-\pi n)^2}=1,\qquad y\in\mathbb{R}. \tag{3.7.5}$$

We refer the reader to [14], where § 5 contains a proof of Lemma 3.7.3 based on the Kotel'nikov–Shannon formula, which in turn is based on the fact that the family of functions

$$\{(z-\pi n)^{-1}\sin(z-\pi n)\}_{n\in\mathbb{Z}}$$

forms an orthonormal basis in $\mathscr{E}_1\cap L^2(\mathbb{R})$ (see [42], § 3, Lecture 20).

Proof of Theorem 3.7.1. Obviously, f is the restriction to $\mathbb{R}^2$ of an entire function of two complex variables. Moreover, $f(\,\cdot\,,a)$, $f(a,\,\cdot\,)\in\mathscr{E}_\sigma\cap L^\infty(\mathbb{R})$ for all $a\in\mathbb{R}$. Without loss of generality we can assume that $\sigma=1$. By Lemma 3.7.3

$$\begin{aligned}(\mathfrak{D}_x f)(z_1,z_2)&=\frac{f(x_1,y_2)-f(x_2,y_2)}{x_1-x_2}\\&=\sum_{n\in\mathbb{Z}}(-1)^n\,\frac{f(\pi n,y_2)-f(x_2,y_2)}{\pi n-x_2}\,\frac{\sin(x_1-\pi n)}{x_1-\pi n}\,,\\(\mathfrak{D}_y f)(z_1,z_2)&=\frac{f(x_1,y_1)-f(x_1,y_2)}{y_1-y_2}\\&=\sum_{n\in\mathbb{Z}}(-1)^n\,\frac{f(x_1,y_1)-f(x_1,\pi n)}{y_1-\pi n}\,\frac{\sin(y_2-\pi n)}{y_2-\pi n}\,.\end{aligned}$$

Note that the expressions $\dfrac{\sin(x_1-\pi n)}{x_1-\pi n}$ and $\dfrac{f(x_1,y_1)-f(x_1,\pi n)}{y_1-\pi n}$ depend on $z_1=(x_1,y_1)$ but not on $z_2=(x_2,y_2)$, while the expressions $\dfrac{f(\pi n,y_2)-f(x_2,y_2)}{\pi n-x_2}$ and $\dfrac{\sin(y_2-\pi n)}{y_2-\pi n}$ depend on $z_2=(x_2,y_2)$ but not on $z_1=(x_1,y_1)$. Moreover, by Lemma 3.7.3

$$\sum_{n\in\mathbb{Z}}\frac{|f(x_1,y_1)-f(x_1,\pi n)|^2}{(y_1-\pi n)^2}\leqslant 3\|f(x_1,\,\cdot\,)\|^2_{L^\infty(\mathbb{R})}\leqslant 3\|f\|^2_{L^\infty(\mathbb{C})},$$

$$\sum_{n\in\mathbb{Z}}\frac{|f(\pi n,y_2)-f(x_2,y_2)|^2}{(\pi n-x_2)^2}\leqslant 3\|f(\,\cdot\,,y_2)\|^2_{L^\infty(\mathbb{R})}\leqslant 3\|f\|^2_{L^\infty(\mathbb{C})}$$

and

$$\sum_{n\in\mathbb{Z}}\frac{\sin^2(x_1-\pi n)}{(x_1-\pi n)^2}=\sum_{n\in\mathbb{Z}}\frac{\sin^2(y_2-\pi n)}{(y_2-\pi n)^2}=1,$$

which proves (3.7.1). $\square$

The inequalities (3.7.1) play the role of operator Bernstein inequalities (see § 1.4). One can prove the following assertions just as in the case of functions of self-adjoint operators.

Theorem 3.7.4. *Let $0<\alpha<1$ and let f be a function of Hölder class $\Lambda_\alpha(\mathbb{R}^2)$. Then*

$$\|f(N_1)-f(N_2)\|\leqslant c(1-\alpha)^{-1}\|f\|_{\Lambda_\alpha}\|N_1-N_2\|^\alpha$$

for some constant $c>0$ and for any normal operators N_1 and N_2 with bounded difference N_1-N_2.

One can generalize Theorem 3.7.4 to the case of arbitrary moduli of continuity and thereby obtain an analogue of Theorem 1.7.3.

Theorem 3.7.5. *Let $0<\alpha<1$ and $p>1$, and let f be a function of Hölder class $\Lambda_\alpha(\mathbb{R}^2)$. Then there exists a positive number c such that*

$$\|f(N_1)-f(N_2)\|_{\boldsymbol{S}_{p/\alpha}}\leqslant c\|f\|_{\Lambda_\alpha}\|N_1-N_2\|^\alpha_{\boldsymbol{S}_p}$$

for any normal operators N_1 and N_2 with difference in the Schatten–von Neumann class $\boldsymbol{S}_p$.

We refer the reader to [14], where there are proofs of these results as well as other related results.

3.8. A sufficient condition for commutator Lipschitzness in terms of Cauchy integrals

In this section we give a sufficient condition obtained in [3] for commutator Lipschitzness.

Let $\mathfrak{F}$ be a non-empty closed subset of $\mathbb{C}$ such that $\mathfrak{F} \neq \mathbb{C}$. We denote by $\mathscr{M}(\mathbb{C}\backslash\mathfrak{F})$ the space of complex Radon measures μ on $\mathbb{C} \setminus \mathfrak{F}$ such that

$$\|\mu\|_{\mathscr{M}(\mathbb{C}\backslash\mathfrak{F})} \stackrel{\text{def}}{=} \sup_{z\in\mathfrak{F}} \int_{\mathbb{C}\backslash\mathfrak{F}} \frac{d|\mu|(\zeta)}{|\zeta - z|^2} < +\infty. \tag{3.8.1}$$

For $\mu \in \mathscr{M}(\mathbb{C} \setminus \mathfrak{F})$ the Cauchy integral

$$\widehat{\mu}(z) = \int_{\mathbb{C}\backslash\mathfrak{F}} \frac{d\mu(\zeta)}{\zeta - z}$$

is not defined in general even for $z \in \mathfrak{F}$, because the function $\zeta \mapsto (\zeta - z)^{-1}$ does not have to be integrable with respect to the measure $|\mu|$. With each fixed point $z_0 \in \mathfrak{F}$ we associate the modified Cauchy integral

$$\widehat{\mu}_{z_0}(z) \stackrel{\text{def}}{=} \int_{\mathbb{C}\backslash\mathfrak{F}} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - z_0}\right) d\mu(\zeta).$$

It follows from the Cauchy–Schwarz inequality that $\widehat{\mu}_{z_0}(z)$ is well defined for $z \in \mathfrak{F}$ and $|\widehat{\mu}_{z_0}(z)| \leqslant \|\mu\|_{\mathscr{M}(\mathbb{C}\backslash\mathfrak{F})}|z - z_0|$. Moreover, $\widehat{\mu}_{z_0}(z_1) = -\widehat{\mu}_{z_1}(z_0)$ and

$$|\widehat{\mu}_{z_0}(z_1) - \widehat{\mu}_{z_0}(z_2)| = |\widehat{\mu}_{z_1}(z_1) - \widehat{\mu}_{z_1}(z_2)| = |\widehat{\mu}_{z_1}(z_2)| \leqslant \|\mu\|_{\mathscr{M}(\mathbb{C}\backslash\mathfrak{F}))}|z_1 - z_2|$$

for all $z_1, z_2 \in \mathfrak{F}$.

Note that $z \mapsto (\zeta - z)^{-1}$ is a continuous map from $\mathfrak{F}$ to the Hilbert space $L^2(|\mu|)$ endowed with the weak topology. This lets us easily verify that the function $\widehat{\mu}_{z_0}(z)$ is differentiable as a function of the complex variable at every non-isolated point of $\mathfrak{F}$. In particular, $\widehat{\mu}_{z_0}(z)$ is analytic in the interior of $\mathfrak{F}$.

We denote by $\widehat{\mathscr{M}}(\mathfrak{F})$ the set of functions f on $\mathfrak{F}$ that can be represented in the form $f = c + \widehat{\mu}_{z_0}$, where c is a constant. Let

$$\|f\|_{\widehat{\mathscr{M}}(\mathfrak{F})} \stackrel{\text{def}}{=} \inf\{\|\mu\|_{\mathscr{M}(\mathbb{C}\backslash\mathfrak{F})} \colon \mu \in \mathscr{M}(\mathbb{C} \setminus \mathfrak{F}),\ f - \widehat{\mu}_{z_0} = \text{const on } \mathfrak{F}\}.$$

It is easy to see that the definition of the space $\widehat{\mathscr{M}}(\mathfrak{F})$ and the seminorm $\|f\|_{\widehat{\mathscr{M}}(\mathfrak{F})}$ do not depend on the choice of $z_0 \in \mathfrak{F}$.

Theorem 3.8.1. *Let $\mathfrak{F}$ be a proper closed subset of $\mathbb{C}$. Then $\widehat{\mathscr{M}}(\mathfrak{F}) \subset \mathrm{CL}(\mathfrak{F})$, and $\|f\|_{\mathrm{CL}(\mathfrak{F})} \leqslant \|f\|_{\widehat{\mathscr{M}}(\mathfrak{F})}$ for all f in $\widehat{\mathscr{M}}(\mathfrak{F})$.*

Proof. Let $\mu \in \mathscr{M}(\mathbb{C}\setminus\mathfrak{F})$ and $f = \widehat{\mu}_{z_0}$, and consider the divided difference

$$\frac{f(z)-f(w)}{z-w} = \frac{1}{z-w}\int_{\mathbb{C}\setminus\mathfrak{F}}\left(\frac{1}{\zeta-z}-\frac{1}{\zeta-w}\right)d\mu(\zeta) = \int_{\mathbb{C}\setminus\mathfrak{F}}\frac{d\mu(\zeta)}{(\zeta-z)(\zeta-w)}.$$

The inequality (3.8.1) means that this divided difference satisfies the condition (d) of Theorem 2.3.1. Thus, it is a Schur multiplier for arbitrary Borel spectral measures on $\mathfrak{F}$, and its multiplier norm is at most $\|\mu\|_{\mathscr{M}(\mathbb{C}\setminus\mathfrak{F})}$. It remains to refer to Theorem 3.3.2. □

Let $\widehat{\mathscr{M}_\infty}(\mathfrak{F})$ denote the space of functions of the form $f + az$, where $f \in \widehat{\mathscr{M}}(\mathfrak{F})$ and $a \in \mathbb{C}$. It is easy to see that the linear function az belongs to $\widehat{\mathscr{M}}(\mathfrak{F})$ if $\mathfrak{F}$ is compact. Therefore, $\widehat{\mathscr{M}_\infty}(\mathfrak{F}) = \widehat{\mathscr{M}}(\mathfrak{F})$ for compact $\mathfrak{F}$. In the case of an unbounded set $\mathfrak{F}$ it is easy to verify that $f'(\infty) = 0$ for all $f \in \widehat{\mathscr{M}}(\mathfrak{F})$. Thus, $\widehat{\mathscr{M}_\infty}(\mathfrak{F}) \neq \widehat{\mathscr{M}}(\mathfrak{F})$ for non-compact sets $\mathfrak{F}$. It follows from Theorem 3.8.1 that $\widehat{\mathscr{M}_\infty}(\mathfrak{F}) \subset \mathrm{CL}(\mathfrak{F})$.

The authors do not know whether the equality $\widehat{\mathscr{M}_\infty}(\mathfrak{F}) = \mathrm{CL}(\mathfrak{F})$ holds, even for such simple sets $\mathfrak{F}$ as the circle or the line.

3.9. Commutator Lipschitz functions on the disk and the half-plane

We consider here the spaces of commutator Lipschitz functions on the unit disk $\mathbb{D}$ and on the upper half-plane $\mathbb{C}_+ \stackrel{\text{def}}{=} \{\zeta \in \mathbb{C}\colon \operatorname{Re}\zeta > 0\}$. In particular, we present the results by Kissin and Shulman [39] and their analogues for the upper half-plane.

Let $\mathrm{C_A}$ denote the disk algebra, that is, the space of functions f analytic in the open disk $\mathbb{D}$ and continuous in its closure. It was proved in [39] that $\mathrm{CL}(\mathbb{D}) = \{f \in \mathrm{C_A}\colon f \in \mathrm{OL}(\mathbb{T})\}$. The next theorem shows that this equality is isometric.

Theorem 3.9.1. *Let $f \in \mathrm{CL}(\mathbb{D})$. Then $f \in \mathrm{C_A}$ and $\|f\|_{\mathrm{CL}(\mathbb{D})} = \|f\|_{\mathrm{CL}(\mathbb{T})} = \|f\|_{\mathrm{OL}(\mathbb{T})}$. If $f \in \mathrm{C_A}$, then $f \in \mathrm{CL}(\mathbb{D})$ if and only if $f \in \mathrm{OL}(\mathbb{T})$.*

Proof. The equality $\|f\|_{\mathrm{CL}(\mathbb{T})} = \|f\|_{\mathrm{OL}(\mathbb{T})}$ follows from Theorem 3.1.10. The inequality $\|f\|_{\mathrm{CL}(\mathbb{T})} \leqslant \|f\|_{\mathrm{CL}(\mathbb{D})}$ is obvious. It remains to prove that $\|f\|_{\mathrm{CL}(\mathbb{D})} \leqslant \|f\|_{\mathrm{CL}(\mathbb{T})}$. We can assume that $\|f\|_{\mathrm{CL}(\mathbb{T})} = 1$. Then $\|\mathfrak{D}f\|_{\mathfrak{M}(\mathbb{T}\times\mathbb{T})} = 1$ by Theorem 3.3.6. Let us apply Theorem 2.2.3. We obtain two families $\{u_\zeta\}_{\zeta\in\mathbb{T}}$ and $\{v_\tau\}_{\tau\in\mathbb{T}}$ in a Hilbert space $\mathscr{H}$ that depend on parameters continuously in the weak topology and are such that $\|u_\zeta\| \leqslant 1$, $\|v_\tau\| \leqslant 1$, and $(\mathfrak{D}f)(\zeta,\tau) = (u_\zeta, v_\tau)$ for all $\zeta,\tau \in \mathbb{T}$. Consider the harmonic extensions of the functions $\zeta \mapsto u_\zeta$ and $\tau \mapsto v_\tau$ to the unit disk by putting

$$u_z \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{1-|z|^2}{|z-\zeta|^2} u_\zeta \, d\boldsymbol{m}(\zeta) \quad \text{and} \quad v_w \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{1-|w|^2}{|w-\tau|^2} v_\tau \, d\boldsymbol{m}(\tau)$$

for $z, w \in \mathbb{D}$. The integrals are understood as integrals of $\mathscr{H}$-valued functions continuous in the weak topology. Applying the Poisson integral with respect to the variable ζ to both sides of the equality $(\mathfrak{D}f)(\zeta,\tau) = (u_\zeta, v_\tau)$, we get that $(\mathfrak{D}f)(z,\tau) = (u_z, v_\tau)$ for all $z \in \operatorname{clos}\mathbb{D}$ and $\tau \in \mathbb{T}$. Applying the Poisson integral now to the last equality, we now find that $(\mathfrak{D}f)(z,w) = (u_z, v_w)$ for all $z \in \operatorname{clos}\mathbb{D}$

and $w \in \operatorname{clos}\mathbb{D}$. It is clear that

$$\|f\|_{\mathrm{CL}(\mathbb{D})} = \|\mathfrak{D}f\|_{\mathfrak{M}(\operatorname{clos}\mathbb{D}\times\operatorname{clos}\mathbb{D})} \leqslant \sup_{z\in\operatorname{clos}\mathbb{D}}\|u_z\| \sup_{w\in\operatorname{clos}\mathbb{D}}\|v_w\|$$
$$= \sup_{\zeta\in\mathbb{T}}\|u_\zeta\| \sup_{\tau\in\mathbb{T}}\|v_\tau\| = 1. \qquad \square$$

We give an analogue of Theorem 2.2.4 for functions on the unit disk.

Theorem 3.9.2. *Let $f \in \mathrm{CL}(\mathbb{D})$. Then there are sequences $\{\varphi_n\}_{n\geqslant1}$ and $\{\psi_n\}_{n\geqslant1}$ in the disk algebra $\mathrm{C_A}$ such that*

$$\left(\sup_{z\in\mathbb{D}}\sum_{n=1}^\infty|\varphi_n(z)|^2\right)\left(\sup_{w\in\mathbb{D}}\sum_{n=1}^\infty|\psi_n(w)|^2\right) = \|f\|^2_{\mathrm{CL}(\mathbb{D})},$$
$$(\mathfrak{D}f)(z,w) = \sum_{n=1}^\infty\varphi_n(z)\psi_n(w),$$

where all the series converge uniformly with respect to z and w in a compact subset of the unit disk.

Proof. We can assume that $\|f\|_{\mathrm{CL}(\mathbb{D})} = 1$. To prove the first equality, it suffices to prove the inequality $\leqslant$, because the inequality $\geqslant$ follows from Theorems 3.3.6 and 2.2.1. Let $\mathscr{H}$, u_z and v_w denote the same as in the proof of Theorem 3.9.1. Consider an orthonormal basis $\{e_n\}_{n=1}^\infty$ in $\mathscr{H}$, and put $\varphi_n(z) \stackrel{\mathrm{def}}{=} (u_z, e_n)$ and $\psi_n(w) \stackrel{\mathrm{def}}{=} (e_n, v_w)$ for $n \geqslant 1$. Let us prove that $\varphi_n \in \mathrm{C_A}$ and $\psi_n \in \mathrm{C_A}$. Denote by X the set of vectors $e \in \mathscr{H}$ such that $(u_z, e) \in \mathrm{C_A}$. Clearly, X is a closed subspace of $\mathscr{H}$. We note that $v_\tau \in X$ for all $\tau \in \mathbb{T}$, because $(\mathfrak{D}f)(\,\cdot\,,\tau) \in \mathrm{C_A}$ for all $\tau \in \mathbb{T}$. Thus, $X = \mathscr{H}$, since the linear span of $\{v_\tau\}_{\tau\in\mathbb{T}}$ is dense in $\mathscr{H}$. Therefore, $(u_z, e) \in \mathrm{C_A}$ for all $e \in \mathscr{H}$. Similarly, one can prove that $(e, v_w) \in \mathrm{C_A}$ for all $e \in \mathscr{H}$. It remains to prove uniform convergence on compacta. Note that

$$\left|\sum_{n=N}^\infty\varphi_n(z)\psi_n(w)\right| \leqslant \left(\sum_{n=N}^\infty|\varphi_n(z)|^2\right)^{1/2}\left(\sum_{n=N}^\infty|\psi_n(w)|^2\right)^{1/2}.$$

Thus, it suffices to establish uniform convergence on compact subsets of $\mathbb{D}$ for the series $\sum_{n=1}^\infty|\varphi_n(z)|^2$ and $\sum_{n=1}^\infty|\psi_n(z)|^2$. This is a consequence of the following elementary lemma. $\square$

Lemma 3.9.3. *Let $\{h_k\}_{k=1}^\infty$ be a sequence of analytic functions on $\mathbb{D}$. Suppose that the function $\sum_{k=1}^\infty|h_k(z)|$ is bounded in $\mathbb{D}$. Then the series $\sum_{k=1}^\infty|h_k(z)|$ converges uniformly on compact subsets of the open unit disk.*

We denote by $(\mathrm{OL})_+(\mathbb{T})$ the space of functions f in $\mathrm{OL}(\mathbb{T})$ that admit an analytic extension to the unit disk $\mathbb{D}$ that is continuous up to the boundary. It follows from Theorem 3.3.3 that every function $f \in \mathrm{CL}(\mathbb{D})$ is analytic in $\mathbb{D}$. Thus, Theorem 3.9.1 implies the following result from [39].

Theorem 3.9.4. *The restriction operator $f \mapsto f\big|\mathbb{T}$ is a linear isometry of* $\mathrm{CL}(\mathbb{D})$ *onto* $(\mathrm{OL})_+(\mathbb{T})$.

Similar results also hold for the space $\mathrm{CL}(\mathbb{C}_+)$.

Theorem 3.9.5. *Let f be a continuous function on the closed upper half-plane* $\operatorname{clos}\mathbb{C}_+$. *Suppose that f is analytic in the open half-plane $\mathbb{C}_+$. Then* $\|f\|_{\mathrm{CL}(\mathbb{C}_+)} = \|f\|_{\mathrm{CL}(\mathbb{R})} = \|f\|_{\mathrm{OL}(\mathbb{R})}$. *In particular, $f \in \mathrm{CL}(\mathbb{C}_+)$ if and only if $f \in \mathrm{OL}(\mathbb{R})$.*

Denote by $\mathrm{C_A}(\mathbb{C}_+)$ the set of functions analytic in $\mathbb{C}_+$ and continuous in $\operatorname{clos}\mathbb{C}_+$ and having a finite limit at infinity.

Theorem 3.9.6. *If $f \in \mathrm{CL}(\mathbb{C}_+)$, then there are sequences $\{\varphi_n\}_{n=1}^\infty$ and $\{\psi_n\}_{n=1}^\infty$ in $\mathrm{C_A}(\mathbb{C}_+)$ such that*

$$\left(\sup_{z\in\mathbb{C}_+}\sum_{n=1}^\infty |\varphi_n(z)|^2\right)\left(\sup_{w\in\mathbb{C}_+}\sum_{n=1}^\infty |\psi_n(w)|^2\right) = \|f\|^2_{\mathrm{CL}(\mathbb{C}_+)},$$

$$(\mathfrak{D}f)(z,w) = \sum_{n=1}^\infty \varphi_n(z)\psi_n(w).$$

Furthermore, these series converge uniformly with respect to z and w in a compact subset of the open upper half-plane.

We skip here the proofs of Theorems 3.9.5 and 3.9.6. They are similar to the proofs of the corresponding results for functions on the unit disk.

We also state the following analogue of Theorem 3.9.4 for the real line.

Theorem 3.9.7. *The restriction operator $f \mapsto f\big|\mathbb{R}$ is a linear isometry of* $\mathrm{CL}(\mathbb{C}_+)$ *onto* $(\mathrm{OL})_+(\mathbb{R})$.

In [5] the following result was obtained, containing in essence both Theorem 3.9.1 and Theorem 3.9.5.

Theorem 3.9.8. *Let $\mathfrak{F}_0$ and $\mathfrak{F}$ be non-empty perfect subsets of $\mathbb{C}$ such that $\mathfrak{F}_0 \subset \mathfrak{F}$ and the set $\Omega \stackrel{\mathrm{def}}{=} \mathfrak{F}\setminus\mathfrak{F}_0$ is open. Suppose that a function $f_0 \in \mathrm{CL}(\mathfrak{F}_0)$ admits a continuous extension f to $\mathfrak{F}$ such that f is analytic in Ω and $|f(z)z^{-2}| \to 0$ as $z \to \infty$ in each unbounded*[3] *connected component of Ω. Then $f \in \mathrm{CL}(\mathfrak{F})$ and $\|f\|_{\mathrm{CL}(\mathfrak{F})} = \|f_0\|_{\mathrm{CL}(\mathfrak{F}_0)}$.*

The authors do not know an answer to the following question. Let f be a continuous function on the closed unit disk that is harmonic inside the disk. Suppose that $f \in \mathrm{OL}(\mathbb{T})$. Does it follow that $f \in \mathrm{OL}(\mathbb{D})$? The analogous question can be posed for the half-plane as well as for other domains.

Recall that if T is a contraction on a Hilbert space $\mathscr{H}$, then by the Szőkefalvi-Nagy theorem (see [72], Chap. I, § 5) T has a *unitary dilation*, that is, a unitary operator U on a Hilbert space $\mathscr{K}$ with $\mathscr{H} \subset \mathscr{K}$ such that $T^n = P_{\mathscr{H}}U^n\big|\mathscr{H}$ for $n \geqslant 0$. A dilation can always be chosen to be minimal. This lets us define a linear and multiplicative functional calculus: $\varphi \mapsto \varphi(T) \stackrel{\mathrm{def}}{=} P_{\mathscr{H}}\varphi(U)\big|\mathscr{H}$, $\varphi \in \mathrm{C_A}$. The *semispectral measure $\mathcal{E}_T$ of the contraction T* is defined by $\mathcal{E}_T(\Delta) \stackrel{\mathrm{def}}{=} P_{\mathscr{H}}E_U(\Delta)\big|h$, where E_U is the spectral measure of U and Δ is a Borel subset of $\mathbb{T}$. It is easy to see that

$$\varphi(T) = \int_{\mathbb{T}} \varphi(\zeta)\,d\mathcal{E}(\zeta), \qquad \varphi \in \mathrm{C_A}.$$

[3]The last condition holds automatically if Ω is bounded.

Theorem 3.9.9. *Let $f \in \mathrm{CL}(\mathbb{D})$, let T_1 and T_2 be contractions on a Hilbert space $\mathscr{H}$, and let $R \in \mathcal{B}(\mathscr{H})$. Then*

$$f(T_1)R - Rf(T_2) = \int_{\mathbb{T}}\int_{\mathbb{T}} (\mathfrak{D}f)(\zeta,\tau)\, d\mathcal{E}_1(\zeta)(T_1R - RT_2)\, d\mathcal{E}_2(\tau), \qquad (3.9.1)$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are the semispectral measures of T_1 and T_2, and the following inequality holds:

$$\|f(T_1)R - Rf(T_2)\| \leqslant \|f\|_{\mathrm{CL}(\mathbb{D})}\|T_1R - RT_2\|. \qquad (3.9.2)$$

Proof. Let $\{\varphi_n\}_{n\geqslant 1}$ and $\{\psi_n\}_{n\geqslant 1}$ be sequences of functions in the disk algebra that satisfy the conclusion of Theorem 3.9.2. By (2.3.7) we have

$$\begin{aligned}
\int_{\mathbb{T}}\int_{\mathbb{T}} (\mathfrak{D}f)(\zeta,\tau)\, d\mathcal{E}_1(\zeta)(T_1R - RT_2)\, d\mathcal{E}_2(\tau) &= \sum_{n\geqslant 1}\varphi_n(T_1)(T_1R - RT_2)\psi_n(T_2)\\
&= \sum_{n\geqslant 1} T_1\varphi_n(T_1)R\psi_n(T_2) - \sum_{n\geqslant 1}\varphi_n(T_1)R\psi_n(T_2)T_2\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} \zeta(\mathfrak{D}f)(\zeta,\tau)\, d\mathcal{E}_1(\zeta)R\, d\mathcal{E}_2(\tau) - \int_{\mathbb{T}}\int_{\mathbb{T}} \tau(\mathfrak{D}f)(\zeta,\tau)\, d\mathcal{E}_1(\zeta)R\, d\mathcal{E}_2(\tau)\\
&= \int_{\mathbb{T}}\int_{\mathbb{T}} (f(\zeta) - f(\tau))\, d\mathcal{E}_1(\zeta)R\, d\mathcal{E}_2(\tau) = f(T_1)R - Rf(T_2),
\end{aligned}$$

which proves the formula (3.9.1), which in turn immediately implies the inequality (3.9.2). □

The inequality (3.9.2) was proved by Kissin and Shulman in [39] by a different method. The proof given here is similar to the proof of Theorem 4.1 in [62] (see also [57]). In the case when $f \in B^1_{\infty,1}(\mathbb{T}) \cap \mathrm{C_A}$ and $R = I$, Theorem 3.9.9 was proved in [57].

A similar result can be proved also for dissipative operators (see [12] for perturbations of functions of dissipative operators).

3.10. Operator Lipschitz functions and linear-fractional transformations

Let $\mathrm{Aut}(\widehat{\mathbb{C}})$ denote the Möbius group of linear-fractional transformations of the extended complex plane $\widehat{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$. In other words,

$$\mathrm{Aut}(\widehat{\mathbb{C}}) = \left\{\varphi\colon \varphi(z) = \frac{az+b}{cz+d},\ a,b,c,d \in \mathbb{C},\ ad - bc \neq 0\right\}.$$

The set of linear-fractional transformations of the complex plane is denoted by $\mathrm{Aut}(\mathbb{C})$, that is,

$$\mathrm{Aut}(\mathbb{C}) = \{\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})\colon \varphi(\infty) = \infty\} = \{\varphi\colon \varphi(z) = az + b,\ a,b \in \mathbb{C},\ a \neq 0\}.$$

Let $\widehat{\mathbb{R}}$ denote the one-point compactification of $\mathbb{R}$: $\widehat{\mathbb{R}} \stackrel{\text{def}}{=} \mathbb{R} \cup \{\infty\}$. Put

$$\mathrm{Aut}(\widehat{\mathbb{R}}) \stackrel{\text{def}}{=} \{\varphi \in \mathrm{Aut}(\widehat{\mathbb{C}})\colon \varphi(\widehat{\mathbb{R}}) = \widehat{\mathbb{R}}\} \quad\text{and}\quad \mathrm{Aut}(\mathbb{R}) \stackrel{\text{def}}{=} \{\varphi \in \mathrm{Aut}(\mathbb{C})\colon \varphi(\mathbb{R}) = \mathbb{R}\}.$$

With each linear-fractional transformation φ and each function f on a closed set $\mathfrak{F}\subset\mathbb{C}$, we associate the function $\mathcal{Q}_\varphi f$ defined on the set $\mathfrak{F}_\varphi\stackrel{\text{def}}{=}\mathbb{C}\cap\varphi^{-1}(\mathfrak{F}\cup\{\infty\})$ by

$$(\mathcal{Q}_\varphi f)(z)\stackrel{\text{def}}{=}\begin{cases}\dfrac{f(\varphi(z))}{\varphi'(z)} & \text{if } z\in\mathbb{C},\ \varphi(z)\in\mathfrak{F},\ \text{and } \varphi(z)\ne\infty,\\ 0 & \text{if } z\in\mathbb{C} \text{ and } \varphi(z)=\infty.\end{cases}$$

It is easy to see that if $\varphi\in\operatorname{Aut}(\mathbb{C})$, then $\mathfrak{F}_\varphi=\varphi^{-1}(\mathfrak{F})$, $\mathcal{Q}_\varphi f=(\varphi'(0))^{-1}(f\circ\varphi)$, $\mathcal{Q}_\varphi(\mathrm{OL}(\mathfrak{F}))=\mathrm{OL}(\mathfrak{F}_\varphi)$, $\mathcal{Q}_\varphi(\mathrm{CL}(\mathfrak{F}))=\mathrm{CL}(\mathfrak{F}_\varphi)$, $\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)}=\|f\|_{\mathrm{OL}(\mathfrak{F})}$ for all f in $\mathrm{OL}(\mathfrak{F})$, and $\|\mathcal{Q}_\varphi f\|_{\mathrm{CL}(\mathfrak{F}_\varphi)}=\|f\|_{\mathrm{CL}(\mathfrak{F})}$ for all f in $\mathrm{CL}(\mathfrak{F})$. Therefore, we will be mostly interested in the case when $\varphi\notin\operatorname{Aut}(\mathbb{C})$. Note that if $\mathfrak{F}=\mathbb{C}$, then $\mathfrak{F}_\varphi=\mathbb{C}$ for all φ in $\operatorname{Aut}(\widehat{\mathbb{C}})$. And if $\mathfrak{F}=\mathbb{R}$, then $\mathfrak{F}_\varphi=\mathbb{R}$ for all φ in $\operatorname{Aut}(\widehat{\mathbb{R}})$.

Let $a\in\mathfrak{F}$, where $\mathfrak{F}$ is a closed subset of $\mathbb{C}$, and let

$$\mathrm{OL}_a(\mathfrak{F})\stackrel{\text{def}}{=}\{f\in\mathrm{OL}(\mathfrak{F})\colon f(a)=0\}\quad\text{and}\quad\mathrm{CL}_a(\mathfrak{F})\stackrel{\text{def}}{=}\{f\in\mathrm{CL}(\mathfrak{F})\colon f(a)=0\}.\tag{3.10.1}$$

Obviously, $\mathrm{OL}_a(\mathfrak{F})$ and $\mathrm{CL}_a(\mathfrak{F})$ are Banach spaces.

Theorem 3.10.1. *Let $\mathfrak{F}$ be a closed subset of $\mathbb{C}$, let $a\in\mathfrak{F}$, and let φ be an automorphism in $\operatorname{Aut}(\widehat{\mathbb{C}})$ such that $a\stackrel{\text{def}}{=}\varphi(\infty)$. Then $\mathcal{Q}_\varphi(\mathrm{OL}_a(\mathfrak{F}))\subset\mathrm{OL}(\mathfrak{F}_\varphi)$, and*

$$\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)}\leqslant 3\|f\|_{\mathrm{OL}(\mathfrak{F})}\quad\textit{for all } f \textit{ in } \mathrm{OL}_a(\mathfrak{F}).$$

Proof. Consider first the case $\varphi(z)=\phi(z)\stackrel{\text{def}}{=}z^{-1}$. Then $a=0$, and we have to prove that $\mathcal{Q}_\phi(\mathrm{OL}_0(\mathfrak{F}))\subset\mathrm{OL}(\mathfrak{F}_\varphi)$ and $\|\mathcal{Q}_\phi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)}\leqslant 3\|f\|_{\mathrm{OL}(\mathfrak{F})}$. Let $f\in\mathrm{OL}_0(\mathfrak{F})$. We can assume that $\|f\|_{\mathrm{OL}(\mathfrak{F})}=1$. Then everything reduces to the inequality

$$\|(\mathcal{Q}_\varphi f)(N)R-R(\mathcal{Q}_\varphi f)(N)\|\leqslant 3\max\{\|NR-RN\|,\|N^*R-RN^*\|\}$$

for any bounded operators N and R such that N is normal and $\sigma(N)\subset\mathfrak{F}_\phi$. We define the function h by $h(z)=zf(z^{-1})$ for $z\ne0$ and $h(0)=0$. It is easy to see that $\sup|h|\leqslant\|f\|_{\mathrm{Lip}(\mathfrak{F})}\leqslant\|f\|_{\mathrm{OL}(\mathbb{R})}=1$ since $f(0)=0$. Note that $(\mathcal{Q}_\phi f)(N)=-Nh(N)$. Thus, we have to prove that

$$\|Nh(N)R-RNh(N)\|\leqslant 3\max\{\|NR-RN\|,\|N^*R-RN^*\|\}.$$

We use the elementary identity

$$Nh(N)R-RNh(N)=h(N)(NR-RN)+h(N)RN-NRh(N)+(NR-RN)h(N).\tag{3.10.2}$$

Note that

$$\|h(N)(NR-RN)\|\leqslant\|NR-RN\|\leqslant\max\{\|NR-RN\|,\|N^*R-RN^*\|\}.$$

We can estimate the norm of $(NR-RN)h(N)$ similarly.

It remains to prove that

$$\|h(N)RN-NRh(N)\|\leqslant\max\{\|NR-RN\|,\|N^*R-RN^*\|\}.$$

If N is invertible, then

$$\begin{aligned}\|h(N)RN-NRh(N)\| &= \|f(N^{-1})NRN-NRNf(N^{-1})\|\\ &\leqslant \max\{\|RN-NR\|,\|(N^*)^{-1}NRN-NRN(N^*)^{-1}\|\}\\ &= \max\{\|NR-RN\|,\|(N^*)^{-1}N(RN^*-N^*R)N(N^*)^{-1}\|\}\\ &= \max\{\|NR-RN\|,\|N^*R-RN^*\|\}.\end{aligned}$$

If 0 is a limit point of $\mathfrak{F}_\phi$ (that is, the set $\mathfrak{F}$ is unbounded), then the proof can be concluded, for in this case each normal operator N with spectrum in $\mathfrak{F}_\phi$ can be approximated arbitrarily well by a normal operator M such that $MN = NM$ and $\sigma(M)\subset\mathfrak{F}_\varphi\setminus\{0\}$. This follows from Lemma 3.1.12.

Now suppose that 0 is an isolated point of $\mathfrak{F}_\phi$. Consider a non-invertible normal operator N with spectrum $\mathfrak{F}_\phi$. Then N can be represented as $N=\mathbf{0}\oplus N_0$, where N_0 is an invertible normal operator. We note that $\mathcal{Q}_\phi(N)=\mathbf{0}\oplus N_0^2f(N_0^{-1})$. Let P be the orthogonal projection onto the subspace on which N_0 is defined. It is easy to see that

$$\begin{aligned}&\|h(N)RN-NRh(N)\| = \|P(h(N)RN-NRh(N))P\|\\ &\quad= \|h(N)PRPN-NPRPh(N)\| = \|h(N_0)(PRP)N_0-N_0(PRP)h(N_0)\|\\ &\quad\leqslant \max\{\|N_0(PRP)-(PRP)N_0\|,\|N_0^*(PRP)-(PRP)N_0^*\|\}\\ &\quad\leqslant \max\{\|NR-RN\|,\|N^*R-RN^*\|\}.\end{aligned}$$

Let us proceed to the general case. Put $b=\varphi^{-1}(\infty)$. Clearly, $\varphi(z)=a+c\phi(z-b)$, where $c\in\mathbb{C}\setminus\{0\}$. Thus, everything reduces to the case $a=b=0$, that is, $\varphi=c\phi$, because translations preserve the operator Lipschitz norm. Finally, the case $\varphi=c\phi$ reduces easily to the case $\varphi=\phi$ already treated. □

Example. Let $\varphi(z)=z^{-1}$, $\mathfrak{F}=\mathbb{C}$, and $f=\overline{z}$. Then $f\in\mathrm{OL}_0(\mathbb{C})$ and $\|f\|_{\mathrm{OL}(\mathbb{C})}=1$. Moreover, $(\mathcal{Q}_\varphi f)(z)=-\overline{z}^{\,-1}z^2$ and

$$3=\|\mathcal{Q}_\varphi f\|_{\mathrm{Lip}(\mathbb{T})}\leqslant\|\mathcal{Q}_\varphi f\|_{\mathrm{Lip}(\mathbb{C})}\leqslant\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathbb{C})}\leqslant 3\|f\|_{\mathrm{OL}(\mathbb{C})}=3.$$

This example shows that $\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathbb{C})}=\|\mathcal{Q}_\varphi f\|_{\mathrm{Lip}(\mathbb{C})}=3$, and the constant 3 in Theorem 3.10.1 is best possible.

Theorem 3.10.1 easily implies the following result.

Theorem 3.10.2. *Let $\varphi\in\mathrm{Aut}(\widehat{\mathbb{C}})$, $a=\varphi(\infty)$, and $b=\varphi^{-1}(\infty)$. Suppose that $\mathfrak{F}$ is a closed set in $\mathbb{C}$ that contains a. Then $\mathcal{Q}_\varphi(\mathrm{OL}_a(\mathfrak{F}))=\mathrm{OL}_b(\mathfrak{F}_\varphi)$, and*

$$\frac{1}{3}\|f\|_{\mathrm{OL}(\mathfrak{F})}\leqslant\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)}\leqslant 3\|f\|_{\mathrm{OL}(\mathfrak{F})}\quad\textit{for all } f \textit{ in } \mathrm{OL}_a(\mathfrak{F}).$$

Proof. Note that $(\mathcal{Q}_\varphi(\mathrm{OL}_a(\mathfrak{F})))(b)=0$. Thus, it follows from Theorem 3.10.1 that $\mathcal{Q}_\varphi(\mathrm{OL}_a(\mathfrak{F}))\subset\mathrm{OL}_b(\mathfrak{F}_\varphi)$ and $\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)}\leqslant 3\|f\|_{\mathrm{OL}(\mathfrak{F})}$. To prove that $\mathcal{Q}_\varphi(\mathrm{OL}_a(\mathfrak{F}))\supset\mathrm{OL}_b(\mathfrak{F}_\varphi)$ and obtain the desired lower estimate for $\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)}$, it suffices to apply Theorem 3.10.1 to the closed set $\mathfrak{F}_\varphi$ and the linear-fractional transformation φ^{-1}. □

We present one more related result.

Theorem 3.10.3. *Let $\varphi \in \operatorname{Aut}(\widehat{\mathbb{C}}) \setminus \operatorname{Aut}(\mathbb{C})$ and let $a = \varphi(\infty)$. Suppose that $\mathfrak{F}$ is a closed subset of $\mathbb{C}$ such that $a \notin \mathfrak{F}$. If z_0 is one of the closest points of $\mathfrak{F}$ to a, then $\mathcal{Q}_\varphi(\mathrm{OL}_{z_0}(\mathfrak{F})) \subset \mathrm{OL}(\mathfrak{F}_\varphi)$ and*

$$\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathfrak{F}_\varphi)} \leqslant 5\|f\|_{\mathrm{OL}(\mathfrak{F})} \quad \textit{for all } f \in \mathrm{OL}_{z_0}(\mathfrak{F}).$$

Proof. As in the proof of Theorem 3.10.1, it suffices to consider the case $\varphi(z) = \phi(z) \stackrel{\text{def}}{=} z^{-1}$. Let $f \in \mathrm{OL}_{z_0}(\mathfrak{F})$ and $\|f\|_{\mathrm{OL}_{z_0}(\mathfrak{F})} = 1$. We have to prove that

$$\|(\mathcal{Q}_\varphi f)(N)R - R(\mathcal{Q}_\varphi f)(N)\| \leqslant 5\max\{\|NR - RN\|, \|N^*R - RN^*\|\}$$

for any normal operators N_1 and N_2 such that $\sigma(N_1), \sigma(N_2) \subset \mathfrak{F}_\phi$. Let h denote the same function as in the proof of Theorem 3.10.1. However, we cannot now say that $\sup|h| \leqslant 1$. We have

$$\begin{aligned}\sup_{z\in\mathfrak{F}_\phi} |h(z)| &\leqslant \sup\{|zf(z^{-1})| \colon z \in \phi^{-1}(\mathfrak{F})\} = \sup\{|z|^{-1}|f(z) - f(z_0)| \colon z \in \mathfrak{F}\} \\ &\leqslant \sup\{|z|^{-1}|z - z_0| \colon z \in \mathfrak{F}\} \leqslant \sup\{1 + |z|^{-1}|z_0| \colon z \in \mathfrak{F}\} = 2.\end{aligned}$$

Repeating the reasoning in the proof of Theorem 3.10.1, we obtain

$$\begin{aligned}&\|(\mathcal{Q}_\varphi f)(N)R - R(\mathcal{Q}_\varphi f)(N)\| \\ &\qquad \leqslant (1 + 2\sup|h(z)|)\max\{\|NR - RN\|, \|N^*R - RN^*\|\} \\ &\qquad \leqslant 5\max\{\|NR - RN\|, \|N^*R - RN^*\|\}. \qquad \square\end{aligned}$$

Example. Let $\varphi(z) = z^{-1}$, $\mathfrak{F} = \mathbb{T}$, $z_0 = 1$, and $f = 1 - \overline{z}$. Then $f \in \mathrm{OL}_{z_0}(\mathbb{T})$ and $\|f\|_{\mathrm{OL}(\mathbb{T})} = 1$. It is easy to verify that $(\mathcal{Q}_\varphi f)(z) = z^3 - z^2$ and $\|\mathcal{Q}_\varphi f\|_{\mathrm{Lip}(\mathbb{T})} \geqslant 5$. Then

$$5 \leqslant \|\mathcal{Q}_\varphi f\|_{\mathrm{Lip}(\mathbb{T})} \leqslant \|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathbb{T})} \leqslant \|z^3\|_{\mathrm{OL}(\mathbb{T})} + \|z^2\|_{\mathrm{OL}(\mathbb{T})} = 5.$$

This example shows that the constant 5 in Theorem 3.10.3 is best possible.

Remark 1. We can introduce the following generalization of $\mathcal{Q}_\varphi$:

$$(\mathcal{Q}_{n,\varphi} f)(z) \stackrel{\text{def}}{=} \begin{cases} \dfrac{|\varphi'(z)|^n f(\varphi(z))}{(\varphi'(z))^{n+1}} & \text{if } z \in \mathbb{C},\ \varphi(z) \in \mathfrak{F},\ \text{and } \varphi(z) \neq \infty, \\ 0 & \text{if } z \in \mathbb{C} \text{ and } \varphi(z) = \infty, \end{cases}$$

where $n \in \mathbb{Z}$. Then analogues of Theorems 3.10.1–3.10.3 for the operators $\mathcal{Q}_{n,\varphi}$ hold with constants depending on n. Analogues of Theorems 3.10.1 and 3.10.2 can be found in [2]. An analogue of Theorem 3.10.3 can be obtained in the same way.

Remark 2. The proofs of Theorems 3.10.1–3.10.3 also work for the spaces of commutator Lipschitz functions. The case of Theorems 3.10.1 and 3.10.2 is treated in [2]. Clearly, in the case of the spaces $\mathrm{CL}(\mathfrak{F})$ we can speak about generalizations to the operators $\mathcal{Q}_{n,\varphi}$ (see Remark 1) only for 'sparse' sets $\mathfrak{F}$. For example, if $\mathfrak{F}$ has interior points, then such generalizations are impossible, because the functions in $\mathrm{CL}(\mathfrak{F})$ are analytic in the interior of $\mathfrak{F}$.

Later we will be mostly interested in the case when $\mathfrak{F} = \mathbb{R}$ and $\mathfrak{F} = \mathbb{T}$. In these cases the isometric equality $\mathrm{CL}(\mathfrak{F}) = \mathrm{OL}(\mathfrak{F})$ holds.

Theorem 3.10.1 implies the following result.

Theorem 3.10.4. *Let* $\varphi \in \operatorname{Aut}(\widehat{\mathbb{C}})$, *and suppose that* $\varphi(\widehat{\mathbb{R}}) = \mathbb{T}$. *Then*

$$\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathbb{R})} \leqslant 3\|f\|_{\mathrm{OL}(\mathbb{T})}$$

for all $f \in \mathrm{OL}_a(\mathbb{T})$, *where* $a = \varphi(\infty)$.

Proof. We apply Theorem 3.10.2 to $\mathfrak{F} = \mathbb{T}$. Then $\mathfrak{F}_\varphi = \mathbb{R} \cup \{\varphi^{-1}(\infty)\}$ and

$$\|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|\mathcal{Q}_\varphi f\|_{\mathrm{OL}(\mathbb{R}\cup\{\varphi^{-1}(\infty)\})} \leqslant 3\|f\|_{\mathrm{OL}(\mathbb{T})}$$

for all $f \in \mathrm{OL}_a(\mathbb{T})$. □

Let

$$(\mathrm{OL})'(\mathbb{R}) \stackrel{\mathrm{def}}{=} \{f' \colon f \in \mathrm{OL}(\mathbb{R})\} \quad \text{and} \quad \|f'\|_{(\mathrm{OL})'(\mathbb{R})} \stackrel{\mathrm{def}}{=} \|f\|_{\mathrm{OL}(\mathbb{R})}. \tag{3.10.3}$$

Then $\mathrm{OL}'(\mathbb{R})$ is a Banach space of functions on $\widehat{\mathbb{R}}$.

Theorem 3.10.5. *Let* $f \in \mathrm{OL}(\mathbb{R})$. *Then* $(x-a)^{-1}(f(x)-f(a)) \in (\mathrm{OL})'(\mathbb{R})$ *and*

$$\big\|(x-a)^{-1}(f(x)-f(a))\big\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})} \quad \textit{for all } a \in \mathbb{R}.$$

Proof. It suffices to consider the case $a = 0$ and $f(0) = 0$. Let

$$F(x) = \int_0^x \frac{f(t)}{t}\,dt = \int_0^1 \frac{f(tx)}{t}\,dt.$$

We have to prove that $F \in \mathrm{OL}(\mathbb{R})$ and $\|F\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{R})}$. For every t in $(0,1]$ the function $x \mapsto t^{-1}f(tx)$ belongs to $\mathrm{OL}_0(\mathbb{R})$ (see (3.10.1)), and $\|t^{-1}f(tx)\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{OL}(\mathbb{R})}$ for all t in $(0,1]$. Consequently,

$$\|F\|_{\mathrm{OL}(\mathbb{R})} \leqslant \int_0^1 \|t^{-1}f(tx)\|_{\mathrm{OL}(\mathbb{R})}\,dt = \|f\|_{\mathrm{OL}(\mathbb{R})}. \qquad \square$$

Remark. One can prove in a similar way that for any closed non-degenerate interval J and any function f in $\mathrm{OL}(J)$

$$\big\|(x-a)^{-1}(f(x)-f(a))\big\|_{(\mathrm{OL})'(J)} \leqslant \|f\|_{\mathrm{OL}(J)} \quad \text{for all } a \in J,$$

where $\mathrm{OL}'(J) \stackrel{\mathrm{def}}{=} \{g' \colon g \in \mathrm{OL}(J)\}$ and $\|g'\|_{\mathrm{OL}'(J)} \stackrel{\mathrm{def}}{=} \|g\|_{\mathrm{OL}(J)}$.

Theorem 3.10.6. *If* $f \in \mathrm{OL}(\mathbb{R})$, *then* $(x-a-b\mathrm{i})^{-1}(f(x)-f(a)) \in (\mathrm{OL})'(\mathbb{R})$ *and*

$$\big\|(x-a-b\mathrm{i})^{-1}(f(x)-f(a))\big\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant 2\|f\|_{\mathrm{OL}(\mathbb{R})} \quad \textit{for all } a,b \in \mathbb{R}.$$

Proof. It suffices to consider the case when $a = 0$, $b = 1$, $f(0) = 0$, and $\|f\|_{\mathrm{OL}(\mathbb{R})} = 1$. It follows from Theorem 3.10.5 that

$$\|(x-\mathrm{i})^{-1}f(x)\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant \|(x-\mathrm{i})^{-1}xf(x)\|_{\mathrm{OL}(\mathbb{R})}.$$

It remains to prove that $\|(x-\mathrm{i})^{-1}xf(x)\|_{\mathrm{OL}(\mathbb{R})} \leqslant 2$. Let A and B be self-adjoint operators. We have

$$A(A-\mathrm{i}I)^{-1}f(A) - B(B-\mathrm{i}I)^{-1}f(B) = A(A-\mathrm{i}I)^{-1}(f(A)-f(B)) \\ + \mathrm{i}(A-\mathrm{i}I)^{-1}(B-A)(B-\mathrm{i}I)^{-1}f(B),$$

whence

$$\|A(A-\mathrm{i}I)^{-1}f(A) - B(B-\mathrm{i}I)^{-1}f(B)\| \leqslant \|f(A)-f(B)\| + \|A-B\|\cdot\|g(B)\| \\ \leqslant 2\|A-B\|,$$

where $g(t) = (t-\mathrm{i})^{-1}f(t)$. $\square$

Corollary 3.10.7. *Let $f \in \mathrm{OL}(\mathbb{R})$. Then $(x-a-b\mathrm{i})^{-1}f(x) \in (\mathrm{OL})'(\mathbb{R})$ and*

$$\|(x-a-b\mathrm{i})^{-1}f(x)\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant \left(2 + \frac{|f(a)|}{|b|}\right)\|f\|_{\mathrm{OL}(\mathbb{R})} \quad \textit{for } a,b \in \mathbb{R},\ b \neq 0.$$

Proof. We can assume that $a=0$, $b=1$, and $\|f\|_{\mathrm{OL}(\mathbb{R})} = 1$. Using Theorem 3.10.6 and Example 2 in § 1.1, we obtain

$$\left\|\frac{f(x)}{x-\mathrm{i}}\right\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant \left\|\frac{f(x)-f(0)}{x-\mathrm{i}}\right\|_{(\mathrm{OL})'(\mathbb{R})} + |f(0)|\cdot\left\|\frac{1}{x-\mathrm{i}}\right\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant 2+|f(0)|. \qquad \square$$

Theorem 3.10.8. *Let $h \in \mathrm{OL}'(\mathbb{R})$. Then $h\circ\varphi \in \mathrm{OL}'(\mathbb{R})$ for all linear-fractional $\varphi \in \mathrm{Aut}(\widehat{\mathbb{R}})$, and $\|h\|_{\mathrm{OL}'(\mathbb{R})}/9 \leqslant \|h\circ\varphi\|_{\mathrm{OL}'(\mathbb{R})} \leqslant 9\|h\|_{\mathrm{OL}'(\mathbb{R})}$.*

Proof. The result is obvious if $\varphi \in \mathrm{Aut}(\mathbb{R})$. In this case $\|h\|_{\mathrm{OL}'(\mathbb{R})} = \|h\circ\varphi\|_{\mathrm{OL}'(\mathbb{R})} = \|h\|_{\mathrm{OL}'(\mathbb{R})}$. Thus, everything reduces to the case $\varphi(t) = \phi(t) \overset{\mathrm{def}}{=} t^{-1}$. Let $h = f'$ for some function $f \in \mathrm{OL}(\mathbb{R})$ such that $f(0)=0$ and $\|f\|_{\mathrm{OL}(\mathbb{R})} = \|h\|_{\mathrm{OL}'(\mathbb{R})}$. It follows from Theorem 3.10.2 that $\|x^2f(x^{-1})\|_{\mathrm{OL}(\mathbb{R})} \leqslant 3\|h\|_{\mathrm{OL}'(\mathbb{R})}$, whence

$$\|(x^2f(x^{-1}))'\|_{\mathrm{OL}'(\mathbb{R})} = \|2xf(x^{-1}) - h(x^{-1})\|_{\mathrm{OL}'(\mathbb{R})} \leqslant 3\|h\|_{\mathrm{OL}'(\mathbb{R})}.$$

Theorem 3.10.5 implies that

$$\|xf(x^{-1})\|_{\mathrm{OL}'(\mathbb{R})} \leqslant \|x^2f(x^{-1})\|_{\mathrm{OL}(\mathbb{R})} \leqslant 3\|h\|_{\mathrm{OL}'(\mathbb{R})}.$$

Hence

$$\|h(x^{-1})\|_{\mathrm{OL}'(\mathbb{R})} \leqslant \|(x^2f(x^{-1}))'\|_{\mathrm{OL}'(\mathbb{R})} + 2\|xf(x^{-1})\|_{\mathrm{OL}'(\mathbb{R})} \leqslant 9\|h\|_{\mathrm{OL}'(\mathbb{R})}.$$

Applying this inequality to $h(x^{-1})$, we get that $(1/9)\|h(x^{-1})\|_{\mathrm{OL}'(\mathbb{R})} \leqslant \|h\|_{\mathrm{OL}'(\mathbb{R})}$. $\square$

3.11. The spaces OL($\mathbb{R}$) and OL($\mathbb{T}$)

The main purpose of this section and the next is to 'transplant' Theorem 3.10.8 from the line to the circle.

It is easy to see that if $f \in \mathrm{OL}(\mathbb{T})$, then $f(e^{\mathrm{i}t}) \in \mathrm{OL}(\mathbb{R})$ and $\|f(e^{\mathrm{i}t})\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{T})}$. We show here that the converse also holds, that is, each 2π-periodic function F in $\mathrm{OL}(\mathbb{R})$ can be represented in the form $F = f(e^{\mathrm{i}t})$, where $f \in \mathrm{OL}(\mathbb{T})$ and $\|f\|_{\mathrm{OL}(\mathbb{T})} \leqslant \mathrm{const}\,\|F\|_{\mathrm{OL}(\mathbb{R})}$. This can easily be deduced from Lemma 9.8 of [11] (see also Lemma 5.7 in [2]).

Lemma 3.11.1. *Let $h(x,y) = (x-y)/(e^{\mathrm{i}x} - e^{\mathrm{i}y})$. Then $\|h\|_{\mathfrak{M}(J_1\times J_2)} \leqslant 3\sqrt{2}\,\pi/4$ for all intervals J_1 and J_2 such that $J_1 - J_2 \subset [-3\pi/2, 3\pi/2]$.*

Proof. Consider the 3π-periodic function ξ given by $\xi(t) = t(2\sin(t/2))^{-1}$ for $t \in [-3\pi/2, 3\pi/2]$. Then

$$\|h\|_{\mathfrak{M}(J_1\times J_2)} = \|e^{\mathrm{i}x/2}h(x,y)e^{\mathrm{i}y/2}\|_{\mathfrak{M}(J_1\times J_2)} = \|\xi(x-y)\|_{\mathfrak{M}(J_1\times J_2)},$$

since $x - y \in [-3\pi/2, 3\pi/2]$ for $x \in J_1$ and $y \in J_2$. Let us expand the function ξ in a Fourier series:

$$\xi(t) = \sum_{n\in\mathbb{Z}} a_n e^{2nit/3} = a_0 + 2\sum_{n=1}^{\infty} a_n \cos\frac{2}{3}nt,$$

because $a_n = a_{-n}$ for all $n \in \mathbb{Z}$. Obviously, $a_0 > 0$. Note that the function ξ is convex on $[-3\pi/2, 3\pi/2]$. It follows that $(-1)^n a_n \geqslant 0$ for all n (see Theorem 35 in [29]). It remains to observe that

$$\|\xi(x-y)\|_{\mathfrak{M}(J_1\times J_2)} \leqslant \|\xi(x-y)\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})} \leqslant \sum_{n\in\mathbb{Z}} |a_n| \cdot \|e^{2nix/3}e^{-2niy/3}\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})}$$
$$= \sum_{n\in\mathbb{Z}} |a_n| = \xi\left(\frac{3\pi}{2}\right) = \frac{3\sqrt{2}\,\pi}{4}. \qquad \square$$

Theorem 3.11.2. *Let f be a continuous function on $\mathbb{T}$. Then*

$$\|f(e^{\mathrm{i}x})\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{T})} \leqslant \frac{3\sqrt{2}\,\pi}{2}\|f(e^{\mathrm{i}x})\|_{\mathrm{OL}(\mathbb{R})}.$$

Proof. As observed above, the first inequality is obvious. We prove the second. Let $g(x) \stackrel{\mathrm{def}}{=} f(e^{\mathrm{i}x})$. We can assume that $\|g\|_{\mathrm{OL}(\mathbb{R})} < \infty$. Then g is differentiable everywhere on $\mathbb{R}$. It follows that f is differentiable everywhere on $\mathbb{T}$. By Theorems 3.1.10 and 3.3.6,

$$\|g\|_{\mathrm{OL}(\mathbb{R})} = \|\mathfrak{D}g\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})},$$
$$\|f\|_{\mathrm{OL}(\mathbb{T})} = \|\mathfrak{D}f\|_{\mathfrak{M}(\mathbb{T}\times\mathbb{T})} = \|(\mathfrak{D}f)(e^{\mathrm{i}x}, e^{\mathrm{i}y})\|_{\mathfrak{M}([0,2\pi)\times[-\pi/2,3\pi/2))}.$$

Therefore, we have to prove that

$$\|(\mathfrak{D}f)(e^{\mathrm{i}x}, e^{\mathrm{i}y})\|_{\mathfrak{M}([0,2\pi)\times[-\pi/2,3\pi/2))} \leqslant \frac{3\sqrt{2}\,\pi}{2}\|\mathfrak{D}g\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})}.$$

Denote by χ_{jk} the characteristic function of $\boldsymbol{J}_{j,k} \stackrel{\mathrm{def}}{=} [j\pi, (j+1)\pi) \times [k\pi - \pi/2, k\pi + \pi/2)$, where $j, k \in \mathbb{Z}$. We note that

$$\chi_{jk}(x,y)(\mathfrak{D}f)(e^{\mathrm{i}x}, e^{\mathrm{i}y}) = \chi_{jk}(x,y)h(x,y)(\mathfrak{D}g)(x,y),$$

where h denotes the same function as in Lemma 3.11.1. With Lemma 3.11.1, this implies that

$$\big\|(\mathfrak{D}f)(e^{\mathrm{i}x}, e^{\mathrm{i}y})\big\|_{\mathfrak{M}(\boldsymbol{J}_{j,k})} \leqslant \frac{3\sqrt{2}\,\pi}{4}\|\mathfrak{D}g\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})} \tag{3.11.1}$$

for $(j,k) \in \{0,1\}$ with $(j,k) \neq (1,0)$.

The case when $j=1$ and $k=0$ should be considered separately, for in this case $J_1-J_2\not\subset[-3\pi/2,3\pi/2]$ and we cannot apply Lemma 3.11.1 directly.

For $j=1$ and $k=0$ we have

$$\chi_{10}(x+2\pi,y)(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})=\chi_{10}(x+2\pi,y)h(x,y)(\mathfrak{D}g)(x,y).$$

Using Lemma 3.11.1, we get that

$$\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J}_{1,0})}=\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J}_{-1,0})}\leqslant\frac{3\sqrt{2}\,\pi}{4}\|\mathfrak{D}g\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})}.$$

Also, let $\boldsymbol{J}\overset{\text{def}}{=}[0,2\pi)\times[-\pi/2,3\pi/2)$. Then

$$\begin{aligned}\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J})}&\leqslant\big\|(\chi_{00}(x,y)+\chi_{11}(x,y))(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\big\|_{\mathfrak{M}(\boldsymbol{J})}\\&\qquad+\big\|(\chi_{01}(x,y)+\chi_{10}(x,y))(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\big\|_{\mathfrak{M}(\boldsymbol{J})}\\&\leqslant\max\big\{\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J}_{0,0})},\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J}_{1,1})}\big\}\\&\qquad+\max\big\{\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J}_{0,1})},\|(\mathfrak{D}f)(e^{\mathrm{i}x},e^{\mathrm{i}y})\|_{\mathfrak{M}(\boldsymbol{J}_{1,0})}\big\}\\&\leqslant\frac{3\sqrt{2}\,\pi}{2}\|\mathfrak{D}g\|_{\mathfrak{M}(\mathbb{R}\times\mathbb{R})}.\qquad\square\end{aligned}$$

Remark. It follows from the proof of the theorem that

$$\|f(e^{\mathrm{i}x})\|_{\mathrm{OL}(\mathbb{R})}\leqslant\|f\|_{\mathrm{OL}(\mathbb{T})}\leqslant\frac{3\sqrt{2}\,\pi}{2}\|f(e^{\mathrm{i}x})\|_{\mathrm{OL}(J)}$$

for any f in $C(\mathbb{T})$, where J is an interval of length 3π.

3.12. The spaces $(\mathrm{OL})'(\mathbb{R})$ and $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$

The space $(\mathrm{OL})'(\mathbb{R})$ was defined in § 3.10 (see (3.10.3)). We define the space $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ by

$$(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})\overset{\text{def}}{=}\{f\colon f(e^{\mathrm{i}t})\in(\mathrm{OL})'(\mathbb{R})\}\quad\text{and}\quad\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}\overset{\text{def}}{=}\|f(e^{\mathrm{i}t})\|_{(\mathrm{OL})'(\mathbb{R})}.$$

Note that $\|f\|_{L^\infty(\mathbb{T})}=\|f(e^{\mathrm{i}t})\|_{L^\infty(\mathbb{R})}\leqslant\|f(e^{\mathrm{i}t})\|_{(\mathrm{OL})'(\mathbb{R})}=\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}$.

We need the following elementary lemma.

Lemma 3.12.1. *Let* $f\in\mathrm{Lip}(\mathbb{T})$. *Then* $f\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ *and*

$$\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}\leqslant|\widehat{f}(0)|+\frac{\pi}{\sqrt{3}}\|f\|_{\mathrm{Lip}(\mathbb{T})}.$$

Proof. Note that $\|f'\|_{L^2(\mathbb{T})}\leqslant\|f'\|_{L^\infty(\mathbb{T})}\leqslant\|f\|_{\mathrm{Lip}(\mathbb{T})}$ and $\|z^n\|_{(\mathrm{OL})'(\mathbb{T})}=1$ for $n\in\mathbb{Z}$. Consequently,

$$\begin{aligned}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}&\leqslant\sum_{n\in\mathbb{Z}}|\widehat{f}(n)|\leqslant|\widehat{f}(0)|+\bigg(\sum_{n\neq0}n^2|\widehat{f}(n)|^2\bigg)^{1/2}\bigg(\sum_{n\neq0}\frac{1}{n^2}\bigg)^{1/2}\\&=|\widehat{f}(0)|+\frac{\pi}{\sqrt{3}}\|f'\|_{L^2(\mathbb{T})}\leqslant|\widehat{f}(0)|+\frac{\pi}{\sqrt{3}}\|f'\|_{L^\infty(\mathbb{T})}\\&\leqslant|\widehat{f}(0)|+\frac{\pi}{\sqrt{3}}\|f\|_{\mathrm{Lip}(\mathbb{T})}.\qquad\square\end{aligned}$$

Corollary 3.12.2. *The space* $\mathrm{OL}(\mathbb{T})$ *is contained in* $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$, *and*

$$\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant |\widehat{f}(0)| + \frac{\pi}{\sqrt{3}}\|f\|_{\mathrm{OL}(\mathbb{T})}.$$

Remark. One can see from the proof of Lemma 3.12.1 that

$$\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant |\widehat{f}(0)| + \frac{\pi}{\sqrt{3}}\|f'\|_{L^2(\mathbb{T})}.$$

Theorem 3.12.3. *If* $f \in \mathrm{OL}(\mathbb{T})$, *then* $zf'(z) \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ *and* $\|zf'(z)\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{T})}$. *If* $f \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ *and* $\int_{\mathbb{T}} f(z)\,d\boldsymbol{m}(z) = 0$, *then there exists a function* F *in* $\mathrm{OL}(\mathbb{T})$ *such that* $zF'(z) = f$ *and* $\|F\|_{\mathrm{OL}(\mathbb{T})} \leqslant \mathrm{const}\,\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}$.

Proof. The first statement is obvious because if $f \in \mathrm{OL}(\mathbb{T})$, then

$$\int_0^x e^{\mathrm{i}t} f'(e^{\mathrm{i}t})\,dt = \mathrm{i}f(1) - \mathrm{i}f(e^{\mathrm{i}x}) \quad \text{and} \quad \|f'\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} = \|f(e^{\mathrm{i}x})\|_{\mathrm{OL}(\mathbb{R})} \leqslant \|f\|_{\mathrm{OL}(\mathbb{T})}.$$

Let us prove the second statement. Put $F(e^{\mathrm{i}x}) \stackrel{\mathrm{def}}{=} \mathrm{i}\int_0^x f(e^{\mathrm{i}t})\,dt$. F is well defined because $\int_0^{2\pi} f(e^{\mathrm{i}t})\,dt = 2\pi\int_{\mathbb{T}} f(z)\,d\boldsymbol{m}(z) = 0$. Clearly, $zF'(z) = f(z)$. It remains to observe that $\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} = \|F(e^{\mathrm{i}x})\|_{\mathrm{OL}(\mathbb{R})}$ and to apply Theorem 3.11.2. □

Corollary 3.12.4. *A function* f *on* $\mathbb{T}$ *belongs to* $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ *if and only if it can be represented in the form* $f = \widehat{f}(0) + zF'(z)$, *where* $F \in \mathrm{OL}(\mathbb{T})$. *Furthermore,*

$$\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant |\widehat{f}(0)| + \|F\|_{\mathrm{OL}(\mathbb{T})} \leqslant \mathrm{const}\,\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.$$

Proof. It is easy to see that $\|1\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} = 1$. This together with Theorem 3.12.3 implies that if $f = \widehat{f}(0) + zF'(z)$ for some function F in $\mathrm{OL}(\mathbb{T})$, then $f \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ and

$$\begin{aligned}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} &\leqslant \|\widehat{f}(0) + zF'(z)\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant |\widehat{f}(0)| + \|zF'(z)\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \\ &\leqslant \|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} + \|F\|_{\mathrm{OL}(\mathbb{T})} \leqslant c\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.\end{aligned}$$

Let $f \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$. Then by Theorem 3.12.3 the function $f - \widehat{f}(0)$ can be represented in the form $f - \widehat{f}(0) = zF'(z)$, where $F \in \mathrm{OL}(\mathbb{T})$. □

Corollary 3.12.5. *If* $f \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$, *then* $z^n f(z) \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ *for all* n *in* $\mathbb{Z}$.

Proof. It suffices to consider the case when $f = zF'(z)$, where $F \in \mathrm{OL}(\mathbb{T})$. Then $z^n f(z) = z^{n+1}F'(z) = z(z^nF(z))' - nz^nF(z) \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$, because $z^nF(z) \in \mathrm{OL}(\mathbb{T})$ and $\mathrm{OL}(\mathbb{T}) \subset (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ by Corollary 3.12.2. □

Corollary 3.12.6. *A function* f *on* $\mathbb{T}$ *belongs to* $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ *if and only if it can be represented in the form* $f = \widehat{f}(-1)z^{-1} + F'(z)$, *where* $F \in \mathrm{OL}(\mathbb{T})$ *and*

$$\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant |\widehat{f}(-1)| + \|F\|_{\mathrm{OL}(\mathbb{T})} \leqslant \mathrm{const}\,\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.$$

Proof. Let $g(z) \stackrel{\mathrm{def}}{=} zf(z)$. Then $\widehat{f}(-1) = \widehat{g}(0)$ and $g(z) = \widehat{g}(0) + zF'(z)$. It remains to refer to Corollaries 3.12.4 and 3.12.5. □

The following assertion is obvious.

Lemma 3.12.7. *Let $f,g\in\mathrm{OL}(J)$, where J is a bounded closed interval in $\mathbb{R}$. Then $fg\in\mathrm{OL}(J)$ and*

$$\|fg\|_{\mathrm{OL}(J)}\leqslant\big(\boldsymbol{m}(J)\|g\|_{\mathrm{OL}(J)}+\max_J|g|\big)\|f\|_{\mathrm{OL}(J)}.$$

Lemma 3.12.8. *Let $f\in\mathrm{OL}(\mathbb{T})$ and $\zeta\in\mathbb{T}$. Then*

$$\frac{f(z)-f(\zeta)}{z-\zeta}\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})\qquad\text{and}\qquad\left\|\frac{f(z)-f(\zeta)}{z-\zeta}\right\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}\leqslant\mathrm{const}\,\|f\|_{\mathrm{OL}(\mathbb{T})}.$$

Proof. It suffices to consider the case $\zeta=1$. We can assume that $f(1)=0$. We have to estimate the $\mathrm{OL}(\mathbb{R})$-seminorm of the function Φ given by

$$\Phi(x)\stackrel{\mathrm{def}}{=}\int_0^x\frac{f(e^{\mathrm{i}t})}{e^{\mathrm{i}t}-1}\,dt.$$

Clearly, Φ can be represented in the form $\Phi(x)=\lambda x+\Phi_0(x)$, where Φ_0 is a 2π-periodic function. We have

$$|\lambda|=\left|\frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{\mathrm{i}t})}{e^{\mathrm{i}t}-1}\,dt\right|\leqslant\frac{1}{2\pi}\int_0^{2\pi}\frac{|f(e^{\mathrm{i}t})-f(1)|}{|e^{\mathrm{i}t}-1|}\,dt\leqslant\|f\|_{\mathrm{Lip}(\mathbb{T})}\leqslant\|f\|_{\mathrm{OL}(\mathbb{T})}.$$

Therefore, it remains to estimate the $\mathrm{OL}(\mathbb{R})$-seminorm of Φ_0.

We first estimate $\|\Phi\|_{\mathrm{OL}([-3\pi/2,3\pi/2])}$. By the remark after Theorem 3.10.5 and by Lemma 3.12.7,

$$\begin{aligned}\|\Phi\|_{\mathrm{OL}([-3\pi/2,3\pi/2])}&\leqslant\left\|\frac{tf(e^{\mathrm{i}t})}{e^{\mathrm{i}t}-1}\right\|_{\mathrm{OL}([-3\pi/2,3\pi/2])}\leqslant\mathrm{const}\,\|f(e^{\mathrm{i}t})\|_{\mathrm{OL}([-3\pi/2,3\pi/2])}\\&\leqslant\mathrm{const}\,\|f\|_{\mathrm{OL}(\mathbb{T})},\end{aligned}$$

because the function $t\mapsto t/(e^{\mathrm{i}t}-1)$ is infinitely differentiable on $[-3\pi/2,3\pi/2]$. Hence $\|\Phi_0\|_{\mathrm{OL}([-3\pi/2,3\pi/2])}\leqslant\mathrm{const}\,\|f\|_{\mathrm{OL}(\mathbb{T})}$. The remark after Theorem 3.11.2 tells us that

$$\|\Phi_0\|_{\mathrm{OL}(\mathbb{R})}\leqslant\frac{3\sqrt{2}\,\pi}{2}\|\Phi_0\|_{\mathrm{OL}([-3\pi/2,3\pi/2])}\leqslant\mathrm{const}\,\|f\|_{\mathrm{OL}(\mathbb{T})}.\qquad\square$$

Theorem 3.12.9. *Let f be a function on $\mathbb{T}$, and let ψ be a linear-fractional transformation such that $\psi(\widehat{\mathbb{R}})=\mathbb{T}$. Then $f\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ if and only if $f\circ\psi\in(\mathrm{OL})'(\mathbb{R})$. Moreover,*

$$c_1\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}\leqslant\|f\circ\psi\|_{(\mathrm{OL})'(\mathbb{R})}\leqslant c_2\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})},\tag{3.12.1}$$

where c_1 and c_2 are absolute positive constants.

Proof. Let $a=\psi^{-1}(0)$. It is easy to see that $a\in\mathbb{C}\setminus\mathbb{R}$ and $\psi(z)=\zeta(z-\overline{a})^{-1}(z-a)$ for all $z\in\widehat{\mathbb{C}}$, where $|\zeta|=1$. Without loss of generality it can be assumed that $\zeta=1$. We first prove the second inequality. Let $f\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$. By Corollary 3.12.6,

f can be represented in the form $f(z) = \widehat{f}(-1)z^{-1} + F'(z)$, where $F \in \mathrm{OL}(\mathbb{T})$ and $|\widehat{f}(-1)| + \|F\|_{\mathrm{OL}(\mathbb{T})} \leqslant c\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}$. We have

$$\begin{aligned}\|f\circ\psi\|_{(\mathrm{OL})'(\mathbb{R})} &= \left\|\widehat{f}(-1)\frac{1}{\psi} + F'\circ\psi\right\|_{(\mathrm{OL})'(\mathbb{R})}\\ &\leqslant c\left\|\frac{1}{\psi}\right\|_{(\mathrm{OL})'(\mathbb{R})}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} + \|F'\circ\psi\|_{(\mathrm{OL})'(\mathbb{R})}.\end{aligned}$$

Note that

$$\left\|\frac{1}{\psi}\right\|_{(\mathrm{OL})'(\mathbb{R})} = \|(t-a)^{-1}(t-\overline{a})\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant 1 + 2|\operatorname{Im} a|\cdot\|(t-a)^{-1}\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant 3,$$

as follows easily from Example 2 in § 1.1. We now estimate $\|F'\circ\psi\|_{(\mathrm{OL})'(\mathbb{R})}$. Choose F so that $F(1) = F(\psi(\infty)) = 0$. It follows from Theorem 3.10.4 that

$$\left\|\frac{F\circ\psi}{\psi'}\right\|_{\mathrm{OL}(\mathbb{R})} = \|\mathcal{Q}_\psi F\|_{\mathrm{OL}(\mathbb{R})} \leqslant 3\|F\|_{\mathrm{OL}(\mathbb{T})} \leqslant \operatorname{const}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.$$

Consequently,

$$\left\|F'\circ\psi - \frac{(F\circ\psi)\psi''}{(\psi')^2}\right\|_{(\mathrm{OL})'(\mathbb{R})} = \left\|\left(\frac{F\circ\psi}{\psi'}\right)'\right\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant \operatorname{const}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.$$

It remains to estimate

$$\left\|\frac{(F\circ\psi)\psi''}{(\psi')^2}\right\|_{(\mathrm{OL})'(\mathbb{R})} = \left\|\frac{(\mathcal{Q}_\psi F)\psi''}{\psi'}\right\|_{(\mathrm{OL})'(\mathbb{R})} = 2\big\|(z-\overline{a})^{-1}\mathcal{Q}_\psi F\big\|_{(\mathrm{OL})'(\mathbb{R})}.$$

Using Theorem 3.10.6, we find that

$$\begin{aligned}\big\|(z-\overline{a})^{-1}\mathcal{Q}_\psi F\big\|_{(\mathrm{OL})'(\mathbb{R})} &\leqslant \big\|(z-\overline{a})^{-1}(\mathcal{Q}_\psi F - (\mathcal{Q}_\psi F)(\operatorname{Re} a))\big\|_{(\mathrm{OL})'(\mathbb{R})}\\ &\quad + |(\mathcal{Q}_\psi F)(\operatorname{Re} a)|\cdot\|(z-\overline{a})^{-1}\|_{(\mathrm{OL})'(\mathbb{R})}\\ &\leqslant 2\|\mathcal{Q}_\psi F\|_{\mathrm{OL}(\mathbb{R})} + 2|F(-1)|\cdot|\operatorname{Im} a|\cdot\|(z-\overline{a})^{-1}\|_{(\mathrm{OL})'(\mathbb{R})}\\ &\leqslant 6\|F\|_{\mathrm{OL}(\mathbb{T})} + 2|F(-1)-F(1)| \leqslant 10\|F\|_{\mathrm{OL}(\mathbb{T})}\\ &\leqslant \operatorname{const}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}.\end{aligned}$$

Let us now prove the first inequality. We can select a function $g \in \mathrm{OL}(\mathbb{R})$ such that $g'(t) \stackrel{\mathrm{def}}{=} f(\psi(t)) \in (\mathrm{OL})'(\mathbb{R})$ and $g(\operatorname{Re} a) = 0$. Let $\varkappa$ denote the linear-fractional transformation which is the inverse of ψ, that is, $\varkappa(z) = (1-z)^{-1}(a-\overline{a}z)$. It follows from Theorem 3.10.3 that

$$\big\|(2\operatorname{Im} a)^{-1}(1-z)^2 g(\varkappa(z))\big\|_{\mathrm{OL}(\mathbb{T})} \leqslant 5\|g\|_{\mathrm{OL}(\mathbb{R})}. \tag{3.12.2}$$

Therefore, $\big\|(\operatorname{Im} a)^{-1}(z-1)g(\varkappa(z)) + f(z)\big\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant 5\|g\|_{\mathrm{OL}(\mathbb{R})} = 5\|f\circ\psi\|_{(\mathrm{OL})'(\mathbb{R})}$ by Corollary 3.12.6. It remains to prove that

$$\big\|(\operatorname{Im} a)^{-1}(z-1)g(\varkappa(z))\big\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant \operatorname{const}\|f\circ\psi\|_{(\mathrm{OL})'(\mathbb{R})}.$$

This follows immediately from (3.12.2) and Lemma 3.12.8. □

Theorem 3.12.10. *Let $f \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ and let φ be a linear-fractional transformation such that $\varphi(\mathbb{T}) = \mathbb{T}$. Then $f \circ \varphi \in (\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ and $c^{-1}\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant \|f \circ \varphi\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})} \leqslant c\|f\|_{(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})}$ for some positive number c.*

Proof. This theorem is essentially the analogue for the circle $\mathbb{T}$ of Theorem 3.10.8, which concerns the line $\mathbb{R}$. Theorem 3.12.9 enables us to 'transplant' Theorem 3.10.8 from $\mathbb{R}$ to $\mathbb{T}$. □

3.13. Concerning the Arazy–Barton–Friedman sufficient condition

We consider in this section a sufficient condition for operator Lipschitzness on the circle $\mathbb{T}$ that was found by Arazy, Barton, and Friedman [15], as well as its analogue for the line $\mathbb{R}$. Following [3], we show how to deduce these sufficient conditions from Theorem 3.8.1. Then we introduce the notion of a Carleson measure in the strong sense and reformulate these sufficient conditions in terms of Carleson measures in the strong sense. We also show how to deduce from them the sufficient conditions in terms of Besov classes (see § 1.6). We start with the case of the line.

Let $(\mathrm{CL})'(\mathbb{C}_+) \stackrel{\mathrm{def}}{=} \{g' : g \in \mathrm{CL}(\mathbb{C}_+)\}$ and $\|g'\|_{(\mathrm{CL})'(\mathbb{C}_+)} = \|g\|_{\mathrm{CL}(\mathbb{C}_+)}$. Obviously, $(\mathrm{CL})'(\mathbb{C}_+)$ is a Banach space. The functions of class $(\mathrm{CL})'(\mathbb{C}_+)$ are defined everywhere on $\operatorname{clos}\mathbb{C}_+ \cup \{\infty\}$. It is easy to see that for any g in $\mathrm{CL}(\mathbb{C}_+)$ the Poisson integral of $g'\big|\mathbb{R}$ coincides with $g'\big|\mathbb{C}_+$. Indeed, it suffices to observe that for all $t > 0$ the Poisson integral of $t^{-1}(g_t - g)\big|\mathbb{R}$ coincides with $t^{-1}(g_t - g)\big|\mathbb{C}_+$, where $g_t(z) \stackrel{\mathrm{def}}{=} g(z+t)$, and to pass to the limit as $t \to 0^+$. We denote by $(\mathrm{OL})_+(\mathbb{R})$ the space of all functions $f \in \mathrm{OL}(\mathbb{R})$ having an analytic extension to the upper half-plane $\mathbb{C}_+$ that is continuous up to the boundary. Put $(\mathrm{OL})'_+(\mathbb{R}) \stackrel{\mathrm{def}}{=} \{g' : g \in \mathrm{OL}_+(\mathbb{R})\}$.

It follows from Theorem 3.9.7 that the space $(\mathrm{OL})'_+(\mathbb{R})$ can be identified in a natural way with the space $(\mathrm{CL})'(\mathbb{C}_+)$. Moreover,

$$\|f'\|_{(\mathrm{OL})'(\mathbb{R})} = \|f\|_{\mathrm{OL}(\mathbb{R})} = \|f\|_{\mathrm{CL}(\mathbb{C}_+)} = \|f'\|_{(\mathrm{CL})'(\mathbb{C}_+)} \quad \text{for all } f \in (\mathrm{CL})(\mathbb{C}_+).$$

The analogue of the Arazy–Barton–Friedman theorem for the half-plane can be stated as follows.

Theorem 3.13.1. *Let f be a function analytic in $\mathbb{C}_+$ and such that*

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)|f'(w)|\, d\boldsymbol{m}_2(w)}{|t-w|^2} < +\infty.$$

Then f has finite angular boundary values[4] *(which will be denoted using the same letter f) everywhere on $\widehat{\mathbb{R}}$, $f \in (\mathrm{CL})'(\mathbb{C}_+)$, and*

$$\|f - f(\infty)\|_{(\mathrm{CL})'(\mathbb{C}_+)} \leqslant \frac{2}{\pi} \sup_{t\in\mathbb{R}} \int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)|f'(w)|\, d\boldsymbol{m}_2(w)}{|t-w|^2}\,.$$

Lemma 3.13.2. *Let f be a function analytic in $\mathbb{C}_+$ and such that*

$$\int_{\mathbb{C}_+} (\operatorname{Im} w)(1+|w|^2)^{-1}|f'(w)|\, d\boldsymbol{m}_2(w) < +\infty.$$

[4] By $f(\infty)$ we mean the limit of $f(z)$ as $|z| \to \infty$ while remaining in any closed angle with vertex in $\mathbb{R}$ and all its points except for the vertex lying in $\mathbb{C}_+$.

Then f has a finite angular value $f(\infty)$ at infinity, and

$$f(z)-f(\infty)=\frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)f'(w)\,d\boldsymbol{m}_2(w)}{(\overline{w}-z)^2}\quad \textit{for all } z\in\mathbb{C}_+.$$

Proof. Let

$$g(z)\stackrel{\mathrm{def}}{=}\frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)f'(w)\,d\boldsymbol{m}_2(w)}{(\overline{w}-z)^2}\quad \text{for } z\in\mathbb{C}_+.$$

Clearly, g is analytic in $\mathbb{C}_+$ and $g'(z)=\frac{4\mathrm{i}}{\pi}\int_{\mathbb{C}_+}\frac{f'(w)\,d\boldsymbol{m}_2(w)}{(\overline{w}-z)^3}=f'(z)$ for all $z\in\mathbb{C}_+$. The last equality follows from the fact that $4\mathrm{i}(\pi)^{-1}(\overline{w}-z)^{-3}$ is the reproducing kernel for the Bergman space consisting of the functions that are analytic in $\mathbb{C}_+$ and belong to $L^2(\mathbb{C}_+,y\,d\boldsymbol{m}_2(x+\mathrm{i}y))$. This is well known and easily verifiable. It remains to prove that the non-tangential limit of g at infinity is zero. It follows from the equality

$$g(z)=\frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+}\left(\frac{\overline{w}-\mathrm{i}}{\overline{w}-z}\right)^2\frac{f'(w)\,d\boldsymbol{m}_2(w)}{(\overline{w}-\mathrm{i})^2}$$

and the Lebesgue dominated convergence theorem that the restriction of $g(z)$ to any half-plane $\varepsilon\mathrm{i}+\mathbb{C}_+$ with $\varepsilon>0$ tends to zero as $|z|\to\infty$. □

Proof of Theorem 3.13.1. Let

$$F(z)\stackrel{\mathrm{def}}{=}\frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+}(\operatorname{Im} w)f'(w)\left(\frac{1}{\overline{w}-z}-\frac{1}{\overline{w}}\right)d\boldsymbol{m}_2(w)=\frac{2\mathrm{i}z}{\pi}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)f'(w)\,d\boldsymbol{m}_2(w)}{\overline{w}(\overline{w}-z)}$$

for all $z\in\mathbb{C}$ with $\operatorname{Im} z\geqslant 0$. The convergence of the integrals for real z follows from the Cauchy–Schwarz inequality if we take into account the obvious inequality

$$\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)|f'(w)|\,d\boldsymbol{m}_2(w)}{|z-\overline{w}|^2}\leqslant\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)|f'(w)|\,d\boldsymbol{m}_2(w)}{|\operatorname{Re} z-w|^2}.$$

We note that by Lemma 3.13.2

$$F'(z)=\frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)f'(w)\,d\boldsymbol{m}_2(w)}{(\overline{w}-z)^2}=f(z)-f(\infty).\tag{3.13.1}$$

Consider the Radon measure μ given in the lower half-plane $\mathbb{C}_-$ by

$$d\mu(w)\stackrel{\mathrm{def}}{=}\frac{2\mathrm{i}}{\pi}(\operatorname{Im}\overline{w})f'(\overline{w})\,d\boldsymbol{m}_2(w).$$

Then $F(z)=\widehat{\mu}_0(z)$ if $\operatorname{Im} z\geqslant 0$, and

$$\begin{aligned}\|\mu\|_{\mathscr{M}(\mathbb{C}_-)}&=\frac{2}{\pi}\sup_{z\in\mathbb{C}_+}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)|f'(w)|\,d\boldsymbol{m}_2(w)}{|\overline{z}-w|^2}\\&=\frac{2}{\pi}\sup_{t\in\mathbb{R}}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)|f'(w)|\,d\boldsymbol{m}_2(w)}{|t-w|^2}.\end{aligned}$$

It now follows from Theorem 3.8.1 that

$$\|f-f(\infty)\|_{(\mathrm{CL})'(\mathbb{C}_+)}=\|F\|_{\mathrm{CL}(\mathbb{C}_+)}\leqslant\frac{2}{\pi}\sup_{t\in\mathbb{R}}\int_{\mathbb{C}_+}\frac{(\operatorname{Im} w)|f'(w)|\,d\boldsymbol{m}_2(w)}{|t-w|^2}.$$ □

We denote by $\mathcal{P}M(\mathbb{C}_+)$ the space of complex harmonic functions u defined in the upper half-plane $\mathbb{C}_+$ and such that

$$\|u\|_{\mathcal{P}M(\mathbb{C}_+)} \stackrel{\text{def}}{=} \sup_{y>0} \int_{\mathbb{R}} |u(x+\mathrm{i}y)|\,dx < +\infty.$$

It is well known (see, for example, [71], Chap. II, Theorems 2.3 and 2.5) that $\mathcal{P}M(\mathbb{C}_+)$ coincides with the set of functions u that can be represented in the form

$$u(z) = (\mathcal{P}\nu)(z) \stackrel{\text{def}}{=} \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z\,d\nu(t)}{|z-t|^2}, \qquad z \in \mathbb{C}_+,$$

where ν is a complex Borel measure on $\mathbb{R}$ and $\|u\|_{\mathcal{P}M(\mathbb{C}_+)} = \|\nu\|_{M(\mathbb{R})} \stackrel{\text{def}}{=} |\nu|(\mathbb{R})$.

We denote by $\mathcal{P}L(\mathbb{C}_+)$ the subspace of $\mathcal{P}M(\mathbb{C}_+)$ consisting of the functions $u \in \mathcal{P}L(\mathbb{C}_+)$ that correspond to absolutely continuous measures ν.

A positive measure μ on $\mathbb{C}_+$ is called a *Carleson measure in the strong sense* if $\int_{\mathbb{C}_+} |u(z)|\,d\mu(z) < +\infty$ for any $u \in \mathcal{P}M(\mathbb{C}_+)$. Note that $\mathcal{P}M(\mathbb{C}_+)$ contains the Hardy class H^1 on the upper half-plane $\mathbb{C}_+$. It follows that a Carleson measure in the strong sense must be a Carleson measure in the usual sense. *We denote by* $\mathrm{CM_s}(\mathbb{C}_+)$ *the space of all Radon measures* μ *on* $\mathbb{C}_+$ *such that* $|\mu|$ *is a Carleson measure in the strong sense.* Let

$$\|\mu\|_{\mathrm{CM_s}(\mathbb{C}_+)} \stackrel{\text{def}}{=} \sup\left\{\int_{\mathbb{C}_+} |u(z)|\,d\mu(z)\colon u \in \mathcal{P}M(\mathbb{C}_+),\ \|u\|_{\mathcal{P}M(\mathbb{C}_+)} \leqslant 1\right\}.$$

It is easy to see that

$$\|\mu\|_{\mathrm{CM_s}(\mathbb{C}_+)} = \sup\left\{\int_{\mathbb{C}_+} |u(z)|\,d\mu(z)\colon u \in \mathcal{P}L(\mathbb{C}_+),\ \|u\|_{\mathcal{P}M(\mathbb{C}_+)} \leqslant 1\right\}$$

and

$$\|\mu\|_{\mathrm{CM_s}(\mathbb{C}_+)} = \frac{1}{\pi}\sup_{t\in\mathbb{R}} \int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)\,d\mu(w)}{|t-w|^2} = \frac{1}{\pi}\sup_{z\in\mathbb{C}_+} \int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)\,d\mu(w)}{|\overline{z}-w|^2}.$$

We can now reformulate the analogue of the Arazy–Barton–Friedman theorem for the half-plane as follows.

Theorem 3.13.3. *Let* f *be a function analytic in* $\mathbb{C}_+$. *Suppose that* $|f'|\,d\boldsymbol{m}_2 \in \mathrm{CM_s}(\mathbb{C}_+)$. *Then* f *has finite non-tangential boundary values everywhere on* $\widehat{\mathbb{R}}$, $f \in (\mathrm{CL})'(\mathbb{C}_+)$, *and*

$$\|f - f(\infty)\|_{(\mathrm{CL})'(\mathbb{C}_+)} \leqslant 2\|f'\,d\boldsymbol{m}_2\|_{\mathrm{CM_s}(\mathbb{C}_+)},$$

where the same symbol f *is used for the corresponding boundary-value function.*

In a similar way we can obtain one more version of the Arazy–Barton–Friedman theorem. In the next theorem as well as in the whole section, $|\!|\!|(\nabla u)(a)|\!|\!|$ denotes the operator norm of the differential $d_a u$ of a function u at a point a.

Theorem 3.13.4. *Let u be a (complex) harmonic function on $\mathbb{C}_+$. Suppose that $\|\nabla u\|\, d\boldsymbol{m}_2 \in \mathrm{CM_s}(\mathbb{C}_+)$. Then u has non-tangential boundary values everywhere on $\widehat{\mathbb{R}}$, $u|\mathbb{R} \in (\mathrm{OL})'(\mathbb{R})$, and*

$$\|u - u(\infty)\|_{(\mathrm{OL})'(\mathbb{R})} \leqslant 2\| \,\|\nabla u\|\, d\boldsymbol{m}_2\|_{\mathrm{CM_s}(\mathbb{C}_+)}.$$

Proof. Consider functions f and g analytic in $\mathbb{C}_+$ and such that $f + \overline{g} = u$. Then $f' = \dfrac{\partial u}{\partial z} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} - \mathrm{i}\dfrac{\partial u}{\partial y}\right)$ and $\overline{g'} = \dfrac{\partial u}{\partial \overline{z}} = \dfrac{1}{2}\left(\dfrac{\partial u}{\partial x} + \mathrm{i}\dfrac{\partial u}{\partial y}\right)$. Let

$$\begin{aligned}
F(z) &\stackrel{\text{def}}{=} \frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+} (\operatorname{Im} w) f'(w)\left(\frac{1}{\overline{w} - z} - \frac{1}{\overline{w}}\right) d\boldsymbol{m}_2(w)\\
&= \frac{2\mathrm{i}z}{\pi}\int_{\mathbb{C}_+} \frac{(\operatorname{Im} w) f'(w)\, d\boldsymbol{m}_2(w)}{\overline{w}(\overline{w} - z)},\\
G(z) &\stackrel{\text{def}}{=} \frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+} (\operatorname{Im} w) g'(w)\left(\frac{1}{\overline{w} - z} - \frac{1}{\overline{w}}\right) d\boldsymbol{m}_2(w)\\
&= \frac{2\mathrm{i}z}{\pi}\int_{\mathbb{C}_+} \frac{(\operatorname{Im} w) g'(w)\, d\boldsymbol{m}_2(w)}{\overline{w}(\overline{w} - z)}
\end{aligned}$$

for all $z \in \mathbb{C}$ with $\operatorname{Im} z \geqslant 0$. Using the identity (3.13.1) and the same identity for G, we obtain

$$\begin{aligned}
u(x) - u(\infty) &= F'(x) + \overline{G}'(x)\\
&= \frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+} \frac{(\operatorname{Im} w) f'(w)\, d\boldsymbol{m}_2(w)}{(\overline{w} - x)^2} + \frac{2\mathrm{i}}{\pi}\int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)\overline{g'(w)}\, d\boldsymbol{m}_2(w)}{(w - x)^2}.
\end{aligned}$$

By Theorem 3.8.1,

$$\begin{aligned}
&\|u - u(\infty)\|_{(\mathrm{OL})'(\mathbb{R})}\\
&\quad\leqslant \frac{2}{\pi}\sup_{x\in\mathbb{R}}\left(\int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)|f'(w)|\, d\boldsymbol{m}_2(w)}{|x - \overline{w}|^2} + \int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)|g'(w)|\, d\boldsymbol{m}_2(w)}{|x - w|^2}\right)\\
&\quad= \sup_{x\in\mathbb{R}}\int_{\mathbb{C}_+} \frac{(\operatorname{Im} w)(|f'(w)| + |g'(w)|)\, d\boldsymbol{m}_2(w)}{|x - w|^2}.
\end{aligned}$$

It remains to observe that $|f'(w)| + |g'(w)| = \|(\nabla u)(w)\|$ for all w in $\mathbb{C}_+$, because the operator norm of the linear map $h \mapsto \alpha h + \beta\overline{h}$ equals $|\alpha| + |\beta|$. ☐

Corollary 3.13.5. *Let $f \in \mathrm{Lip}(\mathbb{R})$, and suppose that $\|\operatorname{Hess}\mathcal{P}f\|\, d\boldsymbol{m}_2 \in \mathrm{CM_s}(\mathbb{C}_+)$. Then $f \in \mathrm{OL}(\mathbb{R})$.*

We now show that the Arazy–Barton–Friedman sufficient condition implies the sufficient condition obtained in [56] and [58] for operator Lipschitzness (see Theorem 1.6.1 above).

To obtain this sufficient condition, we need the elementary inequality

$$\|\varphi\, d\boldsymbol{m}_2\|_{\mathrm{CM_s}(\mathbb{C}_+)} \leqslant \int_0^\infty \operatorname{ess\,sup}\{\varphi(x + \mathrm{i}y) \colon x \in \mathbb{R}\}\, dy \tag{3.13.2}$$

for an arbitrary non-negative measurable function φ on $\mathbb{C}_+$.

We now proceed to an alternative proof of the sufficient condition obtained in [56], [58].

Theorem 3.13.6. *Let $f \in B^1_{\infty,1}(\mathbb{R})$. Then $f \in \mathrm{OL}(\mathbb{R})$.*

Proof. Clearly, $f' \in L^\infty(\mathbb{R})$. Let u be the Poisson integral of f'. It is well known (see § 2) that the membership $f \in B^1_{\infty,1}(\mathbb{R})$ is equivalent to the condition that $\int_0^\infty \sup_{x\in\mathbb{R}} \|\nabla u(x+\mathrm{i}y)\|\,dy < +\infty$. It remains to use the inequality (3.13.2) and refer to Theorem 3.13.4. □

Now consider the case of the disk. Let $(\mathrm{CL})'(\mathbb{D}) \stackrel{\mathrm{def}}{=} \{g' \colon g \in \mathrm{CL}(\mathbb{C}_+)\}$ and let $\|g'\|_{(\mathrm{CL})'(\mathbb{D})} = \|g\|_{\mathrm{CL}(\mathbb{D})}$.

We denote by $\mathcal{P}M(\mathbb{D})$ the space of complex harmonic functions u defined in $\mathbb{D}$ such that

$$\|u\|_{\mathcal{P}M(\mathbb{D})} \stackrel{\mathrm{def}}{=} \sup_{0\leqslant r<1} \int_{\mathbb{T}} |u(r\zeta)|\,|d\zeta| < +\infty.$$

It is well known (see, for example, [31], Chap. 3) that the space $\mathcal{P}M(\mathbb{D})$ coincides with the set of functions u representable in the form

$$u(z) = (\mathcal{P}\nu)(z) \stackrel{\mathrm{def}}{=} \frac{1}{2\pi}\int_{\mathbb{T}} \frac{(1-|z|^2)\,d\nu(\zeta)}{|z-\zeta|^2}, \qquad z \in \mathbb{D},$$

where ν is a complex Borel measure on $\mathbb{T}$, and $\|u\|_{\mathcal{P}M(\mathbb{D})} = \|\nu\|_{M(\mathbb{T})} \stackrel{\mathrm{def}}{=} |\nu|(\mathbb{T})$.

We denote by $\mathcal{P}L(\mathbb{D})$ the subspace of $\mathcal{P}M(\mathbb{D})$ that consists of the functions $u \in \mathcal{P}L(\mathbb{D})$ corresponding to absolutely continuous measures ν.

A positive measure μ on $\mathbb{D}$ is called a *Carleson measure in the strong sense* if $\int_{\mathbb{D}} |u(z)|\,d\mu(z) < \infty$ for any $u \in \mathcal{P}M(\mathbb{D})$. The space $\mathcal{P}M(\mathbb{D})$ contains the Hardy class H^1 on the unit disk $\mathbb{D}$, and it follows that a Carleson measure in the strong sense is a Carleson measure in the usual sense. *We denote by* $\mathrm{CM_s}(\mathbb{D})$ *the space of Radon measures* μ *on* $\mathbb{D}$ *such that* $|\mu|$ *a Carleson measure in the strong sense.* Let

$$\|\mu\|_{\mathrm{CM_s}(\mathbb{D})} \stackrel{\mathrm{def}}{=} \sup\biggl\{\int_{\mathbb{D}} |u(z)|\,d\mu(z) \colon u \in \mathcal{P}M(\mathbb{D}),\ \|u\|_{\mathcal{P}M(\mathbb{D})} \leqslant 1\biggr\}.$$

It is easy to see that

$$\|\mu\|_{\mathrm{CM_s}(\mathbb{D})} = \sup\biggl\{\int_{\mathbb{D}} |u(z)|\,d\mu(z) \colon u \in \mathcal{P}L(\mathbb{D}),\ \|u\|_{\mathcal{P}M(\mathbb{D})} \leqslant 1\biggr\}$$

and

$$\|\mu\|_{\mathrm{CM_s}(\mathbb{D})} = \frac{1}{2\pi}\sup_{\zeta\in\mathbb{T}} \int_{\mathbb{D}} \frac{(1-|w|^2)\,d\mu(w)}{|\zeta-w|^2}.$$

Note that

$$\sup_{\zeta\in\mathbb{T}} \int_{\mathbb{D}} \frac{(1-|w|^2)\,d\mu(w)}{|\zeta-w|^2} = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)\,d\mu(w)}{|1-z\overline{w}|^2}. \tag{3.13.3}$$

This follows from the maximum principle for L^2-valued analytic functions on $\mathbb{D}$.

We now use our notation to state the Arazy–Barton–Friedman sufficient condition in the case of the circle (see [15]).

Theorem 3.13.7. *If f is analytic on $\mathbb{D}$ and $\zeta^{-1}f'(\zeta)\,d\boldsymbol{m}_2(\zeta)\in \mathrm{CM_s}(\mathbb{D})$, then f has finite non-tangential boundary values everywhere on $\mathbb{T}$, $f\in(\mathrm{CL})'(\mathbb{D})$, and*

$$\|f-f(0)\|_{(\mathrm{CL})'(\mathbb{D})}\leqslant 2\big\|\zeta^{-1}f'(\zeta)\,d\boldsymbol{m}_2(\zeta)\big\|_{\mathrm{CM_s}(\mathbb{D})},$$

where the same symbol f is used for the corresponding boundary-value function.

We need an analogue of Lemma 3.13.2.

Lemma 3.13.8. *If f is analytic on $\mathbb{D}$ and $\int_{\mathbb{D}}(1-|w|^2)|f'(w)|\,d\boldsymbol{m}_2(w)<+\infty$, then*

$$f(z)-f(0)=\frac{1}{\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)f'(w)\,d\boldsymbol{m}_2(w)}{(1-z\overline{w})^2\overline{w}}=\frac{1}{\pi}\int_{\mathbb{C}\setminus\mathbb{D}}\frac{(|w|^2-1)f'(\overline{w}^{\,-1})\,d\boldsymbol{m}_2(w)}{(w-z)^2\overline{w}^3}$$

for all $z\in\mathbb{D}$.

Proof. We prove only the first equality, because the second can be obtained from the first by the change of variable $w\mapsto\overline{w}^{\,-1}$. For $z=0$ the desired equality follows from the mean value theorem. It remains to observe that

$$f'(z)=\frac{2}{\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)f'(w)\,d\boldsymbol{m}_2(w)}{(1-z\overline{w})^3}$$

for all $z\in\mathbb{D}$ (see, for example, Corollary 1.5 in the monograph [30]). □

Note also that the first equality in the lemma can be obtained by differentiating the equality (4.3) in [15] with respect to z.

Proof of Theorem 3.13.7. Let

$$F(z)\stackrel{\mathrm{def}}{=}\frac{1}{\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)f'(w)}{\overline{w}^2}\left(\frac{1}{1-z\overline{w}}-1\right)d\boldsymbol{m}_2(w)$$
$$=\frac{z}{\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)f'(w)\,d\boldsymbol{m}_2(w)}{\overline{w}(1-z\overline{w})}$$

for all $z\in\mathbb{C}$ with $|z|\leqslant 1$. The convergence of the integrals for $z\in\mathbb{T}$ is a consequence of the Cauchy–Schwarz inequality if we take into account the identity (3.13.3). Note that $F'(z)=\frac{1}{\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)f'(w)\,d\boldsymbol{m}_2(w)}{(1-z\overline{w})^2\overline{w}}=f(z)-f(0)$ by Lemma 3.13.2. Consider the Radon measure μ given in $\mathbb{C}\setminus\overline{\mathbb{D}}$ by

$$d\mu(w)\stackrel{\mathrm{def}}{=}\frac{1}{\pi}\overline{w}^{\,-3}(|w|^2-1)f'(\overline{w}^{\,-1})\,d\boldsymbol{m}_2(w).$$

Then

$$\widehat{\mu}_0(z)=\frac{1}{\pi}\int_{\mathbb{C}\setminus\mathbb{D}}\frac{(|w|^2-1)f'(\overline{w}^{\,-1})}{\overline{w}^3}\left(\frac{1}{w-z}-\frac{1}{w}\right)d\boldsymbol{m}_2(w)$$
$$=\frac{1}{\pi}\int_{\mathbb{D}}\frac{(1-|w|^2)f'(w)}{\overline{w}^2}\left(\frac{1}{1-z\overline{w}}-1\right)d\boldsymbol{m}_2(w)=F(z).$$

We note that

$$\begin{aligned}\|\mu\|_{\mathscr{M}(\mathbb{C}\setminus\operatorname{clos}\mathbb{D})} &= \sup_{z\in\operatorname{clos}\mathbb{D}}\int_{\mathbb{C}\setminus\operatorname{clos}\mathbb{D}}\frac{d|\mu|(w)}{|w-z|^2}\\ &= \frac{1}{\pi}\sup_{z\in\overline{\mathbb{D}}}\int_{\mathbb{C}\setminus\operatorname{clos}\mathbb{D}}\frac{(|w|^2-1)|f'(\overline{w}^{-1})|}{|w-z|^2|w|^3}\,d\boldsymbol{m}_2(w)\\ &= \frac{1}{\pi}\sup_{z\in\operatorname{clos}\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|w|^2)|f'(w)|}{|1-z\overline{w}|^2|w|}\,d\boldsymbol{m}_2(w)\\ &= \frac{1}{\pi}\sup_{\zeta\in\mathbb{T}}\int_{\mathbb{D}}\frac{(1-|w|^2)|f'(w)|}{|\zeta-w|^2|w|}\,d\boldsymbol{m}_2(w) = 2\big\|\zeta^{-1}f'(\zeta)\,d\boldsymbol{m}_2(\zeta)\big\|_{\mathrm{CM_s}(\mathbb{D})}.\end{aligned}$$

It follows from Theorem 3.8.1 that

$$\|f-f(0)\|_{(\mathrm{CL})'(\mathbb{D})} = \|F\|_{\mathrm{CL}(\mathbb{D})} \leqslant \|\mu\|_{\mathscr{M}(\mathbb{C}\setminus\operatorname{clos}\mathbb{D})} = 2\big\|\zeta^{-1}f'(\zeta)\,d\boldsymbol{m}_2(\zeta)\big\|_{\mathrm{CM_s}(\mathbb{D})}. \qquad \square$$

Corollary 3.13.9. *Let f be an analytic function on $\mathbb{D}$ with $f'\,d\boldsymbol{m}_2\in\mathrm{CM_s}(\mathbb{D})$. Then f has finite non-tangential boundary values everywhere on $\mathbb{T}$, $f\in(\mathrm{CL})'(\mathbb{D})$, and $\|f-f(0)\|_{(\mathrm{CL})'(\mathbb{D})}\leqslant\mathrm{const}\,\|f'\,d\boldsymbol{m}_2\|_{\mathrm{CM_s}(\mathbb{D})}$, where the same symbol f is used for the corresponding boundary-value function.*

Proof. It suffices to observe that for every continuous function h on $\mathbb{D}$, the condition that $h\,d\boldsymbol{m}_2\in\mathrm{CM_s}(\mathbb{D})$ implies that $\zeta^{-1}h(\zeta)\,d\boldsymbol{m}_2(\zeta)\in\mathrm{CM_s}(\mathbb{D})$. It remains to apply the closed graph theorem. $\square$

Remark. One can obtain without the closed graph theorem the explicit estimate

$$C_s\big(|\zeta|^{-1}h(\zeta)\,d\boldsymbol{m}_2(\zeta)\big)\leqslant\frac{8}{3}C_s\big(h\,d\boldsymbol{m}_2\big)$$

for any function h subharmonic in $\mathbb{D}$, but we will not need this.

Theorem 3.13.10. *If $\|\nabla u\|\,d\boldsymbol{m}_2\in\mathrm{CM_s}(\mathbb{D})$ for some harmonic function u on $\mathbb{D}$, then u has non-tangential boundary values everywhere on $\mathbb{T}$ and $u\in(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$.*

Proof. The function u can be represented as $u=f+\overline{g}$, where f and g are analytic functions on $\mathbb{D}$. It follows from Corollary 3.13.9 that $f,g\in(\mathrm{OL})'(\mathbb{T})$. It remains to observe that the definition of the space $(\mathrm{OL})'_{\mathrm{loc}}(\mathbb{T})$ tells us immediately that it is invariant under complex conjugation. $\square$

Corollary 3.13.11. *If $\|\operatorname{Hess}u\|\,d\boldsymbol{m}_2\in\mathrm{CM_s}(\mathbb{D})$ for some harmonic function u on $\mathbb{D}$, then u extends to a continuous function on $\mathbb{D}\cup\mathbb{T}$ and $u\in\mathrm{OL}(\mathbb{T})$.*

This easily implies the following result from [56], whose proof was given in § 1.6 above (see Theorem 1.6.2).

Theorem 3.13.12. *Let $f\in B^1_{\infty,1}(\mathbb{T})$. Then $f\in\mathrm{OL}(\mathbb{T})$.*

We have deduced the Arazy–Barton–Friedman sufficient condition from Theorem 3.8.1. It can be shown that Theorem 3.8.1 provides examples of operator Lipschitz functions that do not satisfy the analogue of the Arazy–Barton–Friedman sufficient condition for $\mathbb{C}_+$. In [3] an example is given of a function f in $\widehat{\mathscr{M}}(\operatorname{clos}\mathbb{C}_+)$ such that $f''\,d\boldsymbol{m}_2\notin\mathrm{CM_s}(\mathbb{C}_+)$. It follows from Theorem 3.8.1 that such a function f belongs to $(\mathrm{CL})'(\mathbb{C}_+)$, though the Arazy–Barton–Friedman condition fails for this function. A similar assertion is also true for functions in $\mathbb{D}$.

Remark. In the Arazy–Barton–Friedman paper [15] it was mentioned that their sufficient condition for the operator Lipschitzness of a function on the unit circle implies the sufficient condition obtained in [56] (see also § 1.6 above). It follows from the results of [3] that the Arazy–Barton–Friedman sufficient condition can work even if f' is not continuous. On the other hand, it is easy to see that if $f \in B^1_{\infty,1}(\mathbb{T})$, then $f' \in C(\mathbb{T})$. The same can be said about functions in $B^1_{\infty,1}(\mathbb{R})$ (see Theorem 1.6.4). Indeed, it is easy to verify that the function $f(z) = \exp(-\mathrm{i}z^{-1})$ satisfies the conditions of Theorem 3.13.1, though its restriction to the real line is discontinuous at 0. We note also that in [3] it is proved in essence that a subset of the real line is the set of discontinuity points of $f\big|\mathbb{R}$ for a function f satisfying the conditions of Theorem 3.13.1 if and only if it is an F_σ set and has no interior points. The analogous statements hold for functions on $\mathbb{T}$ and $\mathbb{D}$.

It is interesting to compare the sufficient condition for operator Lipschitzness given in this section with the necessary condition in § 1.5. A combination of these conditions is given in the following theorem.

Theorem 3.13.13. *If* $f \in \mathrm{Lip}(\mathbb{R})$ *and* $\|\operatorname{Hess} \mathcal{P} f\|\, d\boldsymbol{m}_2$ *is a Carleson measure in the strong sense, then* $f \in \mathrm{OL}(\mathbb{R})$. *And if* $f \in \mathrm{OL}(\mathbb{R})$, *then* $\|\operatorname{Hess} \mathcal{P} f\|\, d\boldsymbol{m}_2$ *is a Carleson measure.*

The analogous assertion holds also for functions on the circle $\mathbb{T}$.

3.14. In which cases does the equality $\mathrm{OL}(\mathfrak{F}) = \mathrm{Lip}(\mathfrak{F})$ hold?

Theorem 3.14.1. *Suppose that* $\mathrm{OL}(\mathfrak{F}) = \mathrm{Lip}(\mathfrak{F})$ *for some closed subset* $\mathfrak{F}$ *of* $\mathbb{C}$. *Then* $\mathfrak{F}$ *is finite.*

Proof. Suppose that $\mathfrak{F}$ is infinite. Then $\mathfrak{F}$ has a limit point $a \in \widehat{\mathbb{C}} \stackrel{\text{def}}{=} \mathbb{C} \cup \{\infty\}$. If $a \in \mathbb{C}$, then we can assume that $a = 0$. The case $a = \infty$ can be treated similarly, and in fact it can be reduced to the case $a = 0$ with the help of linear-fractional transformations.

Assume first that $\mathfrak{F} \subset \mathbb{R}$. Then it is easy to construct a function $f \in \mathrm{Lip}(\mathfrak{F})$ that has no derivative at 0. Obviously, $f \notin \mathrm{OL}(\mathfrak{F})$.

To get rid of the assumption that $\mathfrak{F} \subset \mathbb{R}$, we need the following lemma.

Lemma 3.14.2. *Let* $0 < q < 1$ *and let* $\{a_n\}_{n \geqslant 1}$ *be a sequence of positive numbers such that* $a_{n+1} \leqslant q a_n$ *for all* $n \geqslant 1$. *Then for any numerical sequence* b_n *satisfying the condition that* $\sum_{n \geqslant 1} |b_n| a_n^{-1} < +\infty$ *there exists a function* $v \in \mathrm{OL}(\mathbb{R})$ *such that* $v(a_n) = b_n$ *for all* $n \geqslant 1$.

Proof. Fix a function φ of class $C^\infty(\mathbb{R})$ such that $\varphi(0) = 1$ and $\operatorname{supp} \varphi \subset [-\delta, \delta]$, where δ will be chosen at the end of the proof. Let

$$v(t) \stackrel{\text{def}}{=} \sum_{n \geqslant 1} b_n \varphi(a_n^{-1}(t - a_n)).$$

Then

$$\|v\|_{\mathrm{OL}(\mathbb{R})} \leqslant \sum_{n \geqslant 1} |b_n| \cdot \|\varphi(a_n^{-1}(t - a_n))\|_{\mathrm{OL}(\mathbb{R})} = \|\varphi\|_{\mathrm{OL}(\mathbb{R})} \sum_{n \geqslant 1} |b_n| a_n^{-1} < +\infty$$

and $v(a_n) = b_n$ for all $n \geqslant 1$ if δ is sufficiently small. $\square$

Let us continue the proof of Theorem 3.14.1. It is well known that each Lipschitz function on a subset of $\mathbb{C}$ can be extended to a Lipschitz function on the whole complex plane $\mathbb{C}$ (see, for example, [70], Chap. VI, § 2, Theorem 3). Thus, it suffices to consider the case when $\mathfrak{F} \setminus \{0\}$ consists of the terms of a sequence $\{\lambda_n\}_{n\geqslant 1}$ tending to 0 arbitrarily rapidly. Let $\lambda_n = a_n + \mathrm{i}b_n$. We can assume that $\lim_{n\to\infty} \lambda_n/|\lambda_n| = 1$ and the real sequences $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ satisfy the conditions of Lemma 3.14.2. Let $h(t) \stackrel{\text{def}}{=} t + \mathrm{i}v(t)$, where v means the same as in Lemma 3.14.2. Now the case of the set $\mathfrak{F}$ reduces to the case of the set $\operatorname{Re}\mathfrak{F}$, which has already been treated, because

$$\|A - B\| \leqslant \|h(A) - h(B)\| \leqslant (1 + \|v\|_{\mathrm{OL}(\mathbb{R})})\|A - B\|$$

for any self-adjoint operators A and B such that $\sigma(A), \sigma(B) \subset \operatorname{Re}\mathfrak{F}$. $\square$

Concluding remarks

In this section we briefly discuss certain results that were not covered in the main part of the paper.

1. Operator moduli of continuity. For a continuous function f on $\mathbb{R}$ the operator modulus of continuity Ω_f is defined by

$$\Omega_f(\delta) \stackrel{\text{def}}{=} \sup\{\|f(A) - f(B)\| \colon A,\ B \text{ are self-adjoint operators, } \|A - B\| < \delta\}.$$

Operator moduli of continuity were introduced in [8] and were studied in detail in [11]. Theorem 1.7.3 stated above means that if $f \in \Lambda_\omega(\mathbb{R})$, where ω is a modulus of continuity, then

$$\Omega_f(\delta) \leqslant \operatorname{const} \omega_*(\delta), \quad \text{where } \omega_*(\delta) \stackrel{\text{def}}{=} \delta \int_\delta^\infty \frac{\omega(t)}{t^2}\, dt.$$

In [11] the sharpness of such estimates was discussed, and considerably sharper estimates were obtained for continuous 'piecewise convex-concave' functions f. In particular, the following best possible estimate was obtained:

$$\big\|\, |A| - |B| \,\big\| \leqslant C\|A - B\| \log\left(2 + \log \frac{\|A\| + \|B\|}{\|A - B\|}\right)$$

for bounded self-adjoint operators A and B. This inequality significantly improves the estimate of Kato obtained in [34].

2. Commutator estimates for functions of normal operators. Lemma 3.7.3 enables us to obtain the following quasi-commutator estimate:

$$\|f(N_1)R - Rf(N_2)\| \leqslant \operatorname{const} \omega_*\big(\max\{\|N_1R - RN_2\|, \|N_1^*R - RN_2^*\|\}\big)$$

for any modulus of continuity ω, any function f of class $\Lambda_\omega(\mathbb{R})$, any linear operator R of norm 1, and any normal operators N_1 and N_2 (see [14]). In [13] the quasi-commutator norm on the left-hand side of the inequality was estimated solely in terms of the norm $\|N_1R - RN_2\|$. However, on the right-hand side ω_* has to be replaced by $\omega_{**} \stackrel{\text{def}}{=} (\omega_*)_*$:

$$\|f(N_1)R - Rf(N_2)\| \leqslant \text{const}\, \omega_{**}\bigl(\{\|N_1R - RN_2\|\}\bigr).$$

In the case of Hölder classes, that is, $\omega(t) = t^\alpha$ with $0 < \alpha < 1$, we have $\omega_{**}(t) \leqslant \text{const}(1-\alpha)^{-2}t^\alpha$. In other words, we obtain a commutator Hölder estimate.

3. The Nikolskaya–Farforovskaya approach to operator Hölder functions. The authors of [49] present an alternative approach to operator Hölder functions, based on the following assertion.

Let $0 < \alpha < 1$. *Then* $\Lambda_\alpha(\mathbb{Z}) \subset \mathrm{OL}(\mathbb{Z})$. *Moreover, there exists a* c_α *such that* $\|f\|_{\mathrm{OL}(\mathbb{Z})} \leqslant c_\alpha \|f\|_{\Lambda_\alpha(\mathbb{Z})}$.

Theorem 1.7.2 can be deduced from this result with the help of the easily verified inequality $\Omega_f(\delta) \leqslant 2\omega_f(\delta/2) + 2\|f(\delta x)\|_{\mathrm{OL}(\mathbb{Z})}$, which can be proved by interpolating a function of class $\Lambda_\alpha(\mathbb{Z})$ by a function of class $B^1_{\infty,1}(\mathbb{R})$ and by using Theorem 1.6.1 (though it was proved in [49] in a quite different way).

4. Functions of collections of commuting self-adjoint operators. The study of functions of normal operators is equivalent to the study of functions of pairs of commuting self-adjoint operators. In [48] results in [14] (see § 3.7 above) were generalized to the case of functions of an arbitrary number of commuting self-adjoint operators. Furthermore, completely new methods were used.

In [4] some results in [2] on linear-fractional substitutions (see § 3.10 above) were generalized to operator Lipschitz functions of several variables. In the multidimensional situation the role of linear-fractional transformations is played by Möbius transformations, that is, compositions of finitely many inversions.

5. Lipschitz functions of collections of commuting self-adjoint operators. In [35] the results of [67] were generalized to the case of functions of n commuting self-adjoint operators, and Lipschitz-type estimates in the norm of $\boldsymbol{S}_p$ $(1 < p < \infty)$ were obtained for Lipschitz functions on $\mathbb{R}^n$.

6. Functions of pairs of non-commuting self-adjoint operators. The paper [6] is devoted to the study of functions $f(A,B)$ of a pair (A,B) of not necessarily commuting self-adjoint operators. The functions are defined in terms of double operator integrals, and in [6] their behaviour under perturbations of the pair was studied. It turned out that in contrast to the case of functions of commuting operators, Lipschitz-type estimates in the operator norm and in the trace norm differ strongly. In particular, it was shown there that for $f \in B^1_{\infty,1}(\mathbb{R}^2)$

$$\|f(A_1,B_1) - f(A_2,B_2)\|_{\boldsymbol{S}_p} \leqslant \text{const}\,\|f\|_{B^1_{\infty,1}} \max\{\|A_1 - A_2\|_{\boldsymbol{S}_p}, \|B_1 - B_2\|_{\boldsymbol{S}_p}\}$$

for $p \in [1,2]$. Such an inequality was obtained previously in [14] for functions of commuting operators for $p \geqslant 1$. However, in the case of functions of non-commuting operators *this inequality is false for* $p > 2$ *and for the operator norm* (see [6]).

The main tools used in [6] were triple operator integrals and certain *modified* Haagerup tensor products of L^∞ spaces that were introduced there.

7. Operator Lipschitz functions and the Lifshits–Krein trace formula. Let A and B be self-adjoint operators with trace class difference $A-B$. For each such pair there is a unique real function ξ in $L^1(\mathbb{R})$, called the *spectral shift function*, such that for sufficiently nice functions f on $\mathbb{R}$, the following Lifshits–Krein trace formula holds:

$$\operatorname{trace}(f(A)-f(B))=\int_{\mathbb{R}} f'(t)\xi(t)\,dt$$

(see [43] and [41]). M. G. Krein showed in [41] that this formula holds for functions f whose derivative is the Fourier transform of a complex measure. In [58] the trace formula was extended to functions f of Besov class $B^1_{\infty,1}(\mathbb{R})$. Theorem 3.6.5 above says that for the operator $f(A)-f(B)$ to be in the trace class under the assumption that $A-B$ is in the trace class, it is necessary and sufficient that f be operator Lipschitz. Finally, in the recent paper [64] it was shown that for operator Lipschitz functions, the left-hand side of the Lifshits–Krein trace formula not only makes sense, but also coincides with its right-hand side. In other words, *the Lifshits–Krein trace formula holds for any self-adjoint operators A and B with trace class difference if and only if the function f is operator Lipschitz.*

To conclude the paper, we mention the recent survey [63], in which applications of multiple operator integrals in various problems of perturbation theory are considered.

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Received 02/MAY/16
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