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Distribution of the zeros of Padé polynomials and analytic continuation

S. P. Suetin

Abstract. The problem of analytic continuation of a multivalued analytic function with finitely many branch points on the Riemann sphere is discussed. The focus is on Padé approximants: classical (one-point) Padé approximants, multipoint Padé approximants, and Hermite–Padé approximants. The main result is a theorem on the distribution of zeros and the convergence of the Hermite–Padé approximants for a system $[1, f, f^2]$, where f is a multivalued function in the so-called Laguerre class $\mathscr{L}$.

Bibliography: 128 titles.

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Contents

1. Introduction

The problem of analytic continuation is a classical problem in complex analysis. Various approaches to its solution are known (see first of all [18], and also [15], [19], [69], [123]). In this paper we will discuss one of the classical methods for

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solving the problem of analytic continuation, the one based on the construction of Padé approximants. In our treatment of Padé approximants (PA) here we will follow the terminology of [27]–[30], where Padé approximants are understood to refer to classical (one-point) PA, multipoint PA, and Hermite–Padé approximants. We will take a similar approach in our discussion of Padé polynomials, by which we will mean Padé polynomials proper, and also polynomials corresponding to multipoint PA and Hermite–Padé approximants.

Let f be a holomorphic function at the point at infinity $z=\infty$: $f\in\mathscr{H}(\infty)$. Throughout what follows we assume that f is a multivalued analytic function on the Riemann sphere $\overline{\mathbb{C}}$, with a finite set $\Sigma\subset\mathbb{C}$ of singular points, $\operatorname{Card}\Sigma<\infty$, at least one of which is a branch point of f. Thus, f is a multivalued analytic function in the domain $\overline{\mathbb{C}}\setminus\Sigma$. For fixed Σ we denote the class of such functions f by $\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma)$:

$$\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma):=\mathscr{A}(\overline{\mathbb{C}}\setminus\Sigma)\setminus\mathscr{H}(\overline{\mathbb{C}}\setminus\Sigma)$$

(see § 2.1 for details).

Assume that the analytic function f has the explicit representation

$$f(z)=\prod_{j=1}^{p}(z-a_j)^{\alpha_j},\qquad \alpha_j\in\mathbb{C}\setminus\mathbb{Z},\quad \sum_{j=1}^{p}\alpha_j=0, \tag{1}$$

where all the points $a_j\in\mathbb{C}$ are distinct: $a_j\ne a_k$ for $j\ne k$. In what follows we will impose various additional conditions on the branch points a_j and the parameters α_j in (1), depending on the particular problem under consideration (see, for example, (41)). The function f is a multivalued analytic function in the extended complex plane $\overline{\mathbb{C}}$ with a finite set $\Sigma=\{a_1,\dots,a_p\}$ of branch points, so that $f\in\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma)$. Since the a_j and α_j are parameters in (1), here we are actually considering the whole class $\mathscr{L}$ of multivalued analytic functions given by (1). We call $\mathscr{L}$ the *Laguerre class*.

A function f of the form (1) satisfies the differential equation

$$A_p(z)w'+B_{p-2}(z)w=0, \tag{2}$$

where $A_p(z)=\prod_{j=1}^{p}(z-a_j)$ and $B_{p-2}\in\mathbb{C}_{p-2}[z]$ is the polynomial of degree $p-2$ defined by $B_{p-2}(z)=-A_p(z)\sum_{j=1}^{p}\alpha_j(z-a_j)^{-1}$. Therefore, we could initially define the function f of the form (1) in a neighbourhood of a point $z=z_1\in\mathbb{C}\setminus\Sigma$ as a solution $w=f_1$ of the differential equation (2). Then it is natural to regard any other function $w=f_2$ satisfying (2) in a neighbourhood of another point $z=z_2\ne z_1$ as an '*analytic continuation*' of the original function $f_1=f$. This approach to the concept of analytic continuation was essentially proposed in [55] and [27], [28] in the case of a differential equation of arbitrary order with coefficients in $\mathbb{C}(z)$. The differential equation (2), like the representation (1), defines a multivalued analytic function $f(z)=w(z)$ using *finitely many* complex parameters: the coefficients of the polynomials A_p and B_{p-2}. The explicit representation (1) does not uniquely define the multivalued analytic function f: we must fix some branch of this function at a given point $z_0\in\overline{\mathbb{C}}$. For instance, since $\sum_{j=1}^{p}\alpha_j=0$, we can set $f(\infty)=1$ at the point $z=\infty$. The differential equation (2) leaves room for even more arbitrariness: any function of the form $w=\operatorname{const} f$ with $\operatorname{const}\ne 0$ and f given by (1) is a non-trivial solution of it. Nevertheless, we can very well start with a differential equation of type (2) in *defining* an analytic function. It is then natural to pose the problem of studying various properties of the analytic function directly on the basis of an equation of type (2) or, more generally, of a linear differential equation with rational coefficients.

This approach to the notion of an analytic function is the basis for the so-called 'Kolchin theory'. In the process of developing a theory of integrability[1] for linear homogeneous differential equations with rational coefficients in the field $\mathbb{C}(z)$ Kolchin [55] first posed in 1959 the problem of *best rational approximation* of multivalued analytic functions which satisfy a differential equation of the form

$$L[w] \equiv 0, \tag{3}$$

where

$$L[w] := w^{(n)} + a_{n-1}w^{(n-1)} + \cdots + a_1 w' + a_0 w \tag{4}$$

is a linear differential operator of order $n \in \mathbb{N}$ with coefficients a_k in the field $\mathbb{C}(z)$. We remark that Kolchin addressed there the more general problem in which the coefficients $a_k \in \Bbbk(z)$ of an equation of the form (3) are rational functions of the formal variable z with coefficients in a field $\Bbbk$ of characteristic zero. He considered solutions of (3) in the class $\Bbbk[[z]]$ of formal power series and posed the problem of the best rational approximation of such solutions of (3) by rational functions in the class $\Bbbk(z)$. The 'best' approximation was understood in the local sense: the order of tangency[2] $\nu_n(f) := \max\{\operatorname{ord}_{z=z_0}(f-r)\colon r \in \Bbbk_n(z)\}$ to the given formal series $f \in \Bbbk[[z]]$ by a rational function in the class $\Bbbk_n(z) := \{r(z) = p(z)/q(z),\ p, q \in \Bbbk[z],\ \deg p,\ \deg q \leqslant n\}$ of rational functions of fixed order n was to be the *maximium possible*. In the present paper $\Bbbk = \mathbb{C}$, so that we are looking at the problem of rational approximation (in the class $\mathbb{C}_n(z)$) of analytic solutions (with respect to the variable z) of a differential equation of the form (3) whose coefficients a_k are rational functions in the field $\mathbb{C}(z)$. Kolchin's paper [55] turned out to be closely related to so-called functional analogues of the classical Thue–Siegel–Roth theorem and its subsequent generalizations in number theory. There has been interest in this range of problems for quite a few decades and it shows no signs of declining (see first and foremost [28], [55], [64], [83], [85], [86], [100], and also § 2.1 below).

Thus, it is quite natural to assume that an analytic function f is given as a solution of a linear differential equation with polynomial coefficients in $\mathbb{C}[z]$, and to pose the problem of investigating various properties of the analytic function so defined in terms of this differential equation, without having the possibility of solving the equation explicitly (see [54]). This is what was done in [28], [85], [87], where the *approximability* of analytic functions given as solutions of differential equations by rational functions was studied. More precisely, following Kolchin [55], the authors of [28] and [49] studied the *local* approximation properties of the *locally best* rational approximations of multivalued analytic functions given as solutions of an equation of the form $L[w] \equiv 0$, where L is an operator (4) with coefficients in the field $\mathbb{C}(z)$. In [28] the term 'locally' was understood in the sense of the maximum

[1] Or more precisely, a theory of non-integrability of such differential equations (see Khovanskii's monograph [54] and its bibliography for details).

[2] Here and below, $\operatorname{ord}_{x=x_0} \varphi(x)$ is the order of the zero of a function φ at a point x_0.

possible order of tangency between a rational function of fixed order and the given function at *one* or *several* points.

It is well known that the Padé approximants[3] are *locally* the best rational approximations of a given analytic function $f \in \mathbb{C}[[z]]$ in the corresponding class of rational functions.

In this paper we are concerned with just one question related to the study of properties of an analytic function given as a solution of a differential equation of the form $L[w] \equiv 0$: the problem of *representing* such an analytic function by a continued J-fraction, or more generally, by a T-fraction. More precisely, we mean the following problem: *where* (*in what region or union of regions*) *does such a representation hold*, *and in what sense does it hold*? Clearly, here we are concerned with the analysis of various *global* approximation properties of the *locally* best rational approximations of analytic functions, that is, *global* approximation properties of Padé approximants. This is where there is a fundamental distinction between results obtained in the framework of Kolchin's theory and results that follow from the theorems of Stahl (for classical one-point PA; see § 2.2) and Buslaev (for multipoint PA; see § 2.3).

To understand how important such a problem is and how strongly it differs from the problems solved in the framework of Kolchin's theory it is natural to consider the class of special functions with representations of the form (1). On the one hand, such functions are given by an explicit representation, and on the other hand, they solve a differential equation of the form (2), which is a very special case of the class of differential equations $L[w] \equiv 0$ addressed in [28], [49], and [55] (see also the monograph [54] and [77]). We remark that the approach to an analytic function as a solution of a differential equation of the form (2) (which is a very special case of the general homogeneous algebraic equation (4)) is particularly important in the theory of Hermite–Padé approximants (see our § 2.4), that is, in the part of the general theory of PA where, regarding the distribution of the zeros of the Hermite–Padé polynomials for the class of multivalued analytic functions with a finite set of branch points, there are so far no general results which could be used as a kind of substitute for Stahl's and Buslaev's theorems (see [5], [81], [112] and also [75] and [76]).

Similarly, we can assume that an analytic function f is an algebraic function, that is, is given by an algebraic equation with polynomial coefficients in $\mathbb{C}[z]$. And accordingly, we can pose the problem of studying various properties of such an f on the basis of this algebraic equation (see [28], [114], and also [54]).

We note that defining an analytic function by means of a differential equation is not restricted to linear algebraic differential equations of the form $L[w] \equiv 0$. For a corresponding example of a non-linear differential equation we can take the well-known free van der Pol equation

$$\frac{d^2U}{dt^2} + \varepsilon(U^2 - 1)\frac{dU}{dt} + U = 0, \tag{5}$$

where $U = U(t;\varepsilon)$ is a quantity connected with the current strength in an electric circuit, t is the time, and the physical characteristics of the physical device ('oscillator') itself are described by the single 'small' parameter ε. In this case the frequency

[3] Recall that here PA can be classical (one-point) PA, multipoint PA, or Hermite–Padé approximants.

and the amplitude corresponding to a limit cycle of this equation are analytic functions of ε (more precisely, of ε^2; see details in [2], [31], [118]; see also [3], [16], [17], [34], [52], [53], [80], [95], [101]–[105] for other possible applications of Hermite–Padé polynomials).

2. Best rational approximations

Throughout this paper, by best rational approximations we only mean locally best approximations. Regarding the best rational Chebyshev approximations and the corresponding use of the general GRS-method, see [92] and the references there.

2.1. Kolchin's theory and functional analogues of the Thue–Siegel–Roth theorem. Interest in various functional analogues of the classical Thue–Siegel–Roth theorem and its subsequent generalizations in number theory has been steady for quite a few decades and continues today (see first of all [55], [85], and also [28], [64], [86], [100], [127]). In recent years such results have been associated with G. and D. Chudnovsky [28], Osgood [83]–[87], Schmidt [97]–[99] and Vojta [125], [126]. Such analogues of the classical Thue–Siegel–Roth theorem and the Schmidt theorem have as a rule been connected mainly with Kolchin's paper [55] and are regarded as contributions to 'Kolchin's theory', which Kolchin himself developed for rather general differential fields over an arbitrary field $\Bbbk$ (see [28], [55], [85]). Here we look at such a theory from the standpoint of Padé approximants, taking the field of rational functions $\mathbb{C}(z)$ over the complex number field $\mathbb{C}$ as the ground field $\Bbbk(z)$ and taking multivalued analytic functions with a finite set of singular points (for instance, algebraic functions or functions satisfying linear homogeneous algebraic differential equations) as the function space. We remark that the range of applications of the Gonchar–Rakhmanov–Stahl method (GRS-method; see details in § 3), which we discuss here, is much broader than the class of multivalued analytic functions with finitely many singularities. The close relationship between Kolchin's theory and the theory of Padé approximants was very well understood by Gonchar: in both cases one considers the *locally* best rational approximation of a formal power series or a finite set of such series. The difference lies with the problems that are posed and, by implication, with the methods for their solution. Starting from problems arising in a natural way in the theory of AP, Gonchar stated several conjectures in 1978 [39] (see also [6], Chap. 1, § 6.3), which he thought could be regarded as the natural functional analogues of the Thue–Siegel–Roth theorem. Most of these conjectures have been proved by now (see primarily [28], [110], [111]), but at least one is still open (see [6], Chap. 1, § 6.3, Conjecture 6.10). It deals with the strong asymptotics of the best Chebyshev rational approximants for multivalued algebraic functions (see also [4], [34], [92]).

As Gonchar viewed it (see [39] and also [6], Chap. 1, § 6.3), functional analogues of the Thue–Siegel–Roth theorem are essentially results on the *structure of the possible pattern of gaps* in the sequence of orders of tangency $\nu_n(f)$, $n=1,2,\dots$, $\nu_n(f):=\max\{\operatorname{ord}_{z=z_0}(f-r)\colon r\in\mathbb{C}_n(z)\}$. Equivalently, these are results stating that the indices in a diagonal sequence of PA are normal (or perhaps unnormal). Thus, these are all results about some important but nevertheless auxiliary properties of PA, and the theory of best rational approximations of analytic functions cannot be reduced to a study of them. The remarkable results [28] of the Chudnovskys state in essence that for solutions of differential equations the size of the gaps (and therefore the size of blocks in the Padé table) is bounded by an effective constant.

In this paper we will show, in particular, how some functional analogues of the Thue–Siegel–Roth theorem can be deduced as immediate consequences of deep results due to Stahl and Buslaev on the limiting distribution of the zeros and poles of the PA for multivalued analytic functions.

In [55] Kolchin stated the following conjecture (see [28]). Let f be a solution of an algebraic differential equation (3) which is analytic[4] at some point $z = z_0 \in \overline{\mathbb{C}}$. He conjectured that for any $\varepsilon > 0$ there exists a constant $C = C(f, z_0, \varepsilon) > 0$ such that for *any* polynomials $P, Q \in \mathbb{C}[z]$

$$\operatorname*{ord}_{z=z_0}\left(f(z) - \frac{P(z)}{Q(z)}\right) < (2+\varepsilon)\max\{\deg P, \deg Q\} + C, \tag{6}$$

where $\operatorname{ord}_{x=x_0}\varphi(x)$ denotes the order of the zero of the function φ at x_0. Besides (6), authors subsequently began also looking at the following more general relation (see [28] and [86]):

$$\sum_{j\in J}\operatorname*{ord}_{z=z_j}\left(f(z) - \frac{P(z)}{Q(z)}\right) < (2+\varepsilon)\max\{\deg P, \deg Q\} + C, \tag{7}$$

where J is a finite index set, $\operatorname{card} J < \infty$. Here it is assumed that the function f is holomorphic at each point z_j, $j \in J$, and that $L[f](z) \equiv 0$ in some neighbourhood U_j of each point z_j. Kolchin's conjecture was also understood later in a stronger sense, with ε possibly vanishing in (6) and (7) with an 'effective' constant C. It is clear that (6) is related to some properties of the classical Padé approximants for certain classes of analytic functions, while (7) is related to properties of multipoint Padé approximants (see §§ 2.2 and 2.3). Similar relations were also considered by the Chudnovskys in [28] for Hermite–Padé approximants. In [28] they proved that (6) and (7) hold in the class consisting of the solutions of algebraic differential equations of arbitrary order and of algebraic functions, and moreover they hold in their strongest version: *for* $\varepsilon = 0$ *and with a certain effective constant* C. It is one of our aims here to deduce (6) and (7) in a quite simple way, with $\varepsilon > 0$ and an ineffective constant C though, but in a much broader class of functions than in [28], [55], and [86], namely, in the *class of multivalued analytic functions with finitely many branch points in* $\overline{\mathbb{C}}$ (for more details, see §§ 2.2 and 2.3 below). With our approach, (6) becomes a direct consequence of Stahl's theorem [110], and (7) is a direct consequence of Buslaev's theorem [21], or more specifically, of the fact that, in the context of these theorems, the corresponding Padé polynomials P_n and Q_n of degree n have a limiting distribution of zeros, and

$$\frac{1}{n}\chi(P_n), \frac{1}{n}\chi(Q_n) \xrightarrow{*} \lambda^{\mathrm{eq}}, \qquad n \to \infty,$$

where λ^{eq} is the corresponding equilibrium measure (for the trivial exterior field $\psi(z) \equiv 0$ in Stahl's case and the exterior field $\psi(z) = V^{-\nu}(z)$ in Buslaev's case,

[4]As usual, we say that a function is analytic at a point if it is analytic in a neighbourhood of this point.

with ν the unit positive measure concentrated on the finite set of points $z_1, \dots, z_m$ of an m-point interpolation of the original function f). Here $\chi(Q)$ is the measure 'counting' the zeros of a polynomial $Q \in \mathbb{C}[z]$ with multiplicities taken into account; see (25).

In what follows we will only consider the case when $\deg P = \deg Q$ in (6) and (7), that is, we will only discuss diagonal Padé approximants.

Remark 1. We can also consider the more general situation of a given normed differential field $\mathscr{T}$ in which an operation δ with the properties $\delta(ab) = \delta(a)b + a\delta(b)$ and $\delta(a+b) = \delta(a) + \delta(b)$ for any $a, b \in \mathscr{T}$ is defined which is also compatible with the norm $|\cdot| \geqslant 0$, $|ab| = |a|\,|b|$, in the sense that there exist positive numbers c_1 and c_2 such that $c_1|a| \leqslant |\delta a| \leqslant c_2|a|$ for all $|a| < 1$, $a \in \mathscr{T}$. In this connection see [64], [85] and also [46], [47].

Remark 2. Now we briefly discuss the connections between the asymptotic properties of PA and the so-called Schmidt approximation spectrum (see [64]). Let $z_0 = \infty$. For the convergents[5] P_n/Q_n we have

$$\left| f - \frac{P_n}{Q_n} \right| = \left| \frac{1}{Q_n} \right|^{1+\deg Q_{n+1}/\deg Q_n}. \tag{8}$$

The Schmidt approximation spectrum ([64], § 4) is defined by[6]

$$\operatorname{Spec}(f) = \left\{ 1 + \frac{\deg Q_{n+1}}{\deg Q_n},\ n = 1, 2, \dots \right\}'.$$

Let $f \in \mathscr{A}^0(\overline{\mathbb{C}} \backslash \Sigma)$. Then it follows immediately from Stahl's theorem (see § 2.2) that $\operatorname{Spec}(f) = \{2\}$. On the other hand, if $\mathfrak{f} = \{(f_1, z_1), \dots, (f_m, z_m)\}$ is a multigerm with $f_j \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$ and $f_j \in \mathscr{H}(z_j)$, then Buslaev's theorem (see § 2.3) gives us that $\operatorname{Spec}(\mathfrak{f}) = \{2\}$.

2.2. Padé approximants: Stahl's theory. Let $\Sigma = \{a_1, \dots, a_p\}$ be a finite point set in the complex plane $\mathbb{C}$. Then we let $\mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma)$ denote the set of analytic functions on the domain $\mathbb{C} \setminus \Sigma$, which means that each $f \in \mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma)$ is holomorphic at each point $z \notin \Sigma$ and can be analytically continued from this point along any path disjoint from Σ. We denote the class of functions which are analytic but not holomorphic in $\overline{\mathbb{C}} \setminus \Sigma$ by $\mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$, so that $\mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma) := \mathscr{A}(\overline{\mathbb{C}} \setminus \Sigma) \setminus \mathscr{H}(\overline{\mathbb{C}} \setminus \Sigma)$. Thus, for any function $f \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$ at least one point $a_j \in \Sigma$ is a branch point.

At present, the only known way to prove theorems on representing analytic functions of the form (1) by continued J-fractions or, more generally, T-fractions is to deduce them as consequences of the general theorems of Stahl (for J-fractions) and Buslaev (for T-fractions and more general continued fractions) on convergence of the corresponding diagonal PA in the class of *all* multivalued analytic functions with a finite set of branch points. Since functions of the form (1) satisfy a differential equation of the form (2), the corresponding Padé polynomials also satisfy a homogeneous algebraic differential equation of the second order (see (14) below). However, although this equation has polynomial coefficients of fixed degree, they

[5] Here we mean both J-fractions and also T-fractions and multipoint PA.

[6] The prime means the set of limit points.

depend on the index of the corresponding PA. More precisely, these polynomials contain so-called *accessory parameters*, which depend on the index n of the corresponding Padé polynomial. Their asymptotic behaviour as $n \to \infty$ is not known in advance. For both classical PA and two-point PA we can only study their behaviour on the basis of Stahl's or Buslaev's theory, respectively. As a result, in some cases we can also find asymptotic formulae for the corresponding Padé polynomials (see [57], [74], [82], and cf. also [70] and [71]).

Recall that, since the a_j and α_j, $j = 1, \dots, p$, are parameters, in fact we are dealing here with a whole class $\mathscr{L}$ of multivalued analytic functions given by a representation of the form (1). All the functions in the Laguerre class $\mathscr{L}$ satisfy a differential equation of the form (2).

The question of Laurent series representing these functions is quite easy. Namely, if $f \in \mathscr{L}$, then f is holomorphic at the point at infinity $z = \infty$, so we can expand it in a convergent Laurent series at $z = \infty$:

$$f(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^k}, \qquad |z| > \max_{j=1,\dots,p} |a_j|. \tag{9}$$

We fix a branch of f by setting $f(\infty) = 1 = c_0$. Then starting from (1) we can uniquely recover the coefficients c_k of the Laurent series (9) using recurrence formulae which can easily be expressed explicitly.

The problem of expanding an f with $f(\infty) = 1$ in a continued J-fraction has proved to be much more complicated.

In 1885 Laguerre [62] turned to the problem of expanding a function of the form (1) in a continued J-fraction and thereby came naturally to the question of the asymptotic behaviour of the denominators Q_n of the nth convergents $J_n = P_n/Q_n$ of the continued J-fraction, which he saw to be non-Hermitian orthogonal polynomials. In particular, he considered this problem (see also [30], [67], [81]) for functions of the form

$$f(z) = \prod_{j=1}^{3} (z - a_j)^{\alpha_j}, \tag{10}$$

where $\prod_{j=1}^{3} \alpha_j = 0$, $\alpha_j \in \mathbb{C} \setminus \mathbb{Z}$, $f(\infty) = 1$, and the three points a_1, a_2, and a_3 are in general position, so do not lie on a straight line. It turned out that the denominators Q_n of the convergents $J_n = P_n/Q_n$ corresponding to the continued J-fraction defined by the relations

$$\begin{aligned} f(z) &= \prod_{j=1}^{3} (z - a_j)^{\alpha_j} = 1 + \sum_{k=1}^{\infty} \frac{c_k}{z^k} = 1 + \cfrac{c_1}{z - \widehat{b}_1 + f_1(z)} \\ &= 1 + \cfrac{\widehat{a}_1^2}{z - \widehat{b}_1 - \cfrac{\widehat{a}_2^2}{z - \widehat{b}_2 + f_2(z)}} \simeq 1 + \cfrac{\widehat{a}_1^2}{z - \widehat{b}_1 - \cfrac{\widehat{a}_2^2}{z - \widehat{b}_2 - \cdots}} =: J(z), \end{aligned} \tag{11}$$

are non-Hermitian orthogonal. More precisely,

$$\oint_{\Gamma} Q_n(\zeta) \zeta^k f(\zeta) \, d\zeta = 0, \qquad k = 0, \dots, n-1, \tag{12}$$

where Γ is an *arbitrary* closed contour separating a_1, a_2, and a_3 from the point at infinity $z = \infty$. We note that the monic orthogonal polynomials[7] $Q_n(z) = z^n + \cdots$ satisfy the three-term recurrence relations

$$Q_n(z) = (z - \widehat{b}_n)Q_{n-1}(z) - \widehat{a}_n^2 Q_{n-2}(z), \qquad n = 1, 2, \dots, \tag{13}$$

$Q_{-1}(z) \equiv 0$, $Q_0(z) \equiv 1$, $Q_1(z) = z - \widehat{b}_1$. In connection with the problem of *representing* f by a continued J-fraction (11), that is, the problem of the asymptotic behaviour of the corresponding convergents J_n, there arises the natural question of whether we can obtain a description of the asymptotic behaviour of the polynomials Q_n or, equivalently, of the denominators of the rational functions J_n directly from the orthogonality relations (12). Were the answer affirmative, Laguerre would have solved the problem of representing the function (9) by a continued J-fraction, that is, the *problem of the equality* $f(z) = J(z)$. More precisely, he would have answered the question of the region in the complex plane where the function (9) is represented by a continued J-fraction. We remark that in [62] (see also [76], [82], [89]) Laguerre also derived the following second-order differential equation which is satisfied by the polynomials P_n and the functions $Q_n f$ and $Q_n f - P_n$ (cf. (15)):

$$A_3(z)\Pi_{n,1}(z)w'' + \Pi_{n,3}(z)w' + \Pi_{n,2}(z)w = 0, \tag{14}$$

where $A_3(z) = \prod_{j=1}^{3}(z - a_j)$ and the $\Pi_{n,k} \in \mathbb{C}_k[z]$ for $k = 1, 2, 3$ are polynomials of degree precisely k. More specifically,

$$\Pi_{n,1}(z) = z - z_n, \qquad \Pi_{n,2}(z) = -n(n+1)(z - b_n)(z - v_n),$$
$$\Pi_{n,3}(z) = (z - z_n)B_2(z) - A_3(z), \qquad B_2(z) = A_3'(z)\frac{f'(z)}{f(z)}\,.$$

The differential equation (14) is an algebraic differential equation of the second order with polynomial coefficients of fixed degree. However, these coefficients depend on n. More precisely, the coefficients of (14) contain the three accessory parameters z_n, b_n, and v_n, whose asymptotic behaviour as $n \to \infty$ is not known in advance, and also the large parameter $n(n+1)$ multiplying the free term. Although he obtained the orthogonality relations (12) and the differential equation (14), Laguerre could not solve the seemingly simple problem of the asymptotic behaviour of the polynomials Q_n. It was solved by Nuttall [82] in 1986 (see also [74] and [76]), but only after Stahl [107]–[111] (see also [6] and [122]) had completely solved the problem of the *limiting distribution of the zeros* of the Padé polynomials corresponding to an (arbitrary) multivalued analytic function with finitely many branch points on the Riemann sphere.

It is well known (see [81], [110], [111]) that orthogonality relations of the form (12) appear naturally in the theory of Padé approximants. This is quite understandable because the convergents J_n of the continued J-fraction corresponding to an *arbitrary* (generic) Laurent series $f \in \mathscr{H}(\infty)$ coincide with the nth diagonal Padé approximant, which can be constructed as follows from the Laurent series. For any $n \in \mathbb{N}$ we determine Padé polynomials $P_{n,0}, P_{n,1} \in \mathbb{C}_n[z]$ with $\deg P_{n,0}, \deg P_{n,1} \leqslant n$

[7]For points in general position, $\deg Q_n = n$ for all n.

and $P_{n,1} \not\equiv 0$ from the relation

$$H_n(z) := (P_{n,0} + P_{n,1}f)(z) = O\biggl(\frac{1}{z^{n+1}}\biggr), \qquad z \to \infty. \tag{15}$$

Such polynomials always exist, but they are not uniquely defined by (15). However, the rational function $P_{n,0}/P_{n,1}$ of order $\leqslant n$ is uniquely defined; the function $[n/n]_f := -P_{n,0}/P_{n,1}$ is called the diagonal Padé approximant of order n for the function $f \in \mathscr{H}(\infty)$ (or the nth diagonal Padé approximant). Furthermore, $J_n = [n/n]_f$. In particular, for functions of the form (10) the polynomials $P_{n,1}$ satisfy the same non-Hermitian orthogonality relations (12). It is well known that *general* orthogonal polynomials were discovered by Chebyshev [26] in 1855 in precisely the framework of the theory of continued J-fractions. This fact is reflected in Szegő's monograph [120] (§ 3.5): "*Historically, the orthogonal polynomials… originated in the theory of continued fractions. This relationship is of great importance and is one of the possible starting points of the treatment of orthogonal polynomials… .*"

We remark that in this approach the continued fraction expansion of a function $f \in \mathscr{L}$ of the form

$$f(z) = \biggl(\frac{z-1}{z+1}\biggr)^{\alpha}, \quad f(\infty) = 1, \qquad \alpha \in \biggl(-\frac{1}{2}, \frac{1}{2}\biggr), \quad \alpha \neq 0, \tag{16}$$

leads naturally to consideration of the Jacobi polynomials $P^{(\alpha,\beta)}(z)$ with parameters $\alpha \in (-1/2, 1/2) \setminus \{0\}$ and $\beta = -\alpha$, which are orthogonal on $\Delta := [-1, 1]$ with the weight $((1-x)/(1+x))^{\alpha}$ (see [120]).

Although the Laguerre class $\mathscr{L}$ consists of multivalued analytic functions of a rather special form, which are in fact given by the explicit formulae (1), the problem of the asymptotic properties of the corresponding Padé polynomials (or equivalently, of polynomials satisfying the orthogonality relations (12)) turned out to be very characteristic from the standpoint of the general theory of PA. Namely, even for a function f of the form (10) Laguerre could not solve the problem of the asymptotic behaviour of the corresponding orthogonal polynomials (the denominators of the diagonal PA). As mentioned above, Nuttall [82] was able to solve this problem in 1986 (see also [74]), but only after Stahl had completely solved the problem of the limiting distribution of the zeros of the Padé polynomials corresponding to an arbitrary multivalued analytic function with finitely many branch points on the Riemann sphere (see [114]). The cornerstone of Stahl's theory was his result stating that for each multivalued analytic function $f \in \mathscr{H}(\infty)$ with finitely many branch points on the Riemann sphere[8] there exists a unique (up to sets with capacity zero) compact set $S = S(f)$ with a certain 'symmetry' property (called the S-property; see (18)) such that S is made up of finitely many analytic arcs, the set $D := \overline{\mathbb{C}} \setminus S$ is a domain, and the original function extends holomorphically to D, that is, $f \in \mathscr{H}(D)$. On the basis of this result it is proved in Stahl's theory that there is a limiting distribution of the zeros of the Padé polynomials, and it coincides with the Robin equilibrium measure $\lambda = \lambda_S^{\mathrm{rob}}$ for S, so that $-\int_S \log|z-\zeta|\,d\lambda(\zeta) \equiv \mathrm{const} = \gamma_S$, $z \in S$, where γ_S is the Robin constant

[8] Stahl's results are in fact much more general and hold in the class of multivalued analytic functions whose singular set has capacity zero.

of S. Stahl's compact set S is uniquely characterized[9] by the property of *minimum capacity* in the class of compact sets Γ such that $\Gamma = \partial G$, where G is a domain, $G \ni \infty$, and $f \in \mathscr{H}(G)$; in other words,

$$\operatorname{cap} S = \min_{\Gamma=\partial G} \operatorname{cap} \Gamma, \tag{17}$$

and the indicated symmetry property means that for any $z \in S^\circ$

$$\frac{\partial g_D(z,\infty)}{\partial n^+} = \frac{\partial g_D(z,\infty)}{\partial n^-}, \tag{18}$$

where S° is the union of the open analytic arcs whose closures form S, $g_D(z,\infty)$ is the Green's function of the Stahl domain $D \ni \infty$ with pole at infinity, and $\partial/\partial n^\pm$ are the normal derivatives to S at $z \in S^\circ$ from opposite sides of S. Since $g_D(z,\infty) = \gamma_S - V^\lambda(z)$, (18) is equivalent to the equality

$$\frac{\partial V^\lambda}{\partial n^+}(z) = \frac{\partial V^\lambda}{\partial n^-}(z), \qquad z \in S^\circ. \tag{19}$$

In the case of the classical Jacobi polynomials, which correspond to the function (16), the Stahl compact set S coincides with the unit interval: $S = \Delta = [-1,1]$. For the generalized Jacobi polynomials, which correspond to the function (10) (recall that a_1, a_2, a_3 are three points in general position and therefore do not lie on a straight line), the Stahl compact set S coincides with the Chebotarev *continuum* $\mathrm{C}(a_1,a_2,a_3)$ (see [61], and also Figures 1 and 2). Here S is formed by the critical trajectories of the quadratic differential

$$-\frac{z-v}{A_3(z)}\,dz^2 > 0, \qquad A_3(z) := \prod_{j=1}^{3}(z-a_j), \tag{20}$$

that join the branch points a_j (which are simple poles of the quadratic differential in (20)) with the so-called *Chebotarev point* $z = v$, a simple zero of the differential in (20). The Chebotarev point v is a *transcendental parameter* of this problem and is uniquely determined from the condition that all the periods of the Abelian integral

$$\int^z \sqrt{\frac{\zeta-v}{A_3(\zeta)}}\,d\zeta \tag{21}$$

are purely imaginary. Therefore,

$$\operatorname{Re}\int_{a_1}^{z} \sqrt{\frac{\zeta-v}{A_3(\zeta)}}\,d\zeta \tag{22}$$

is a single-valued harmonic function on the two-sheeted elliptic Riemann surface $\mathfrak{R}_2\colon w^2 = (z-v)A_3(z)$, the Chebotarev–Stahl compact set is defined by the relation

$$S = \left\{z \in \mathbb{C}\colon\ \operatorname{Re}\int_{a_1}^{z} \sqrt{\frac{\zeta-v}{A_3(\zeta)}}\,d\zeta = 0\right\}, \tag{23}$$

[9] Up to compact sets with capacity zero.

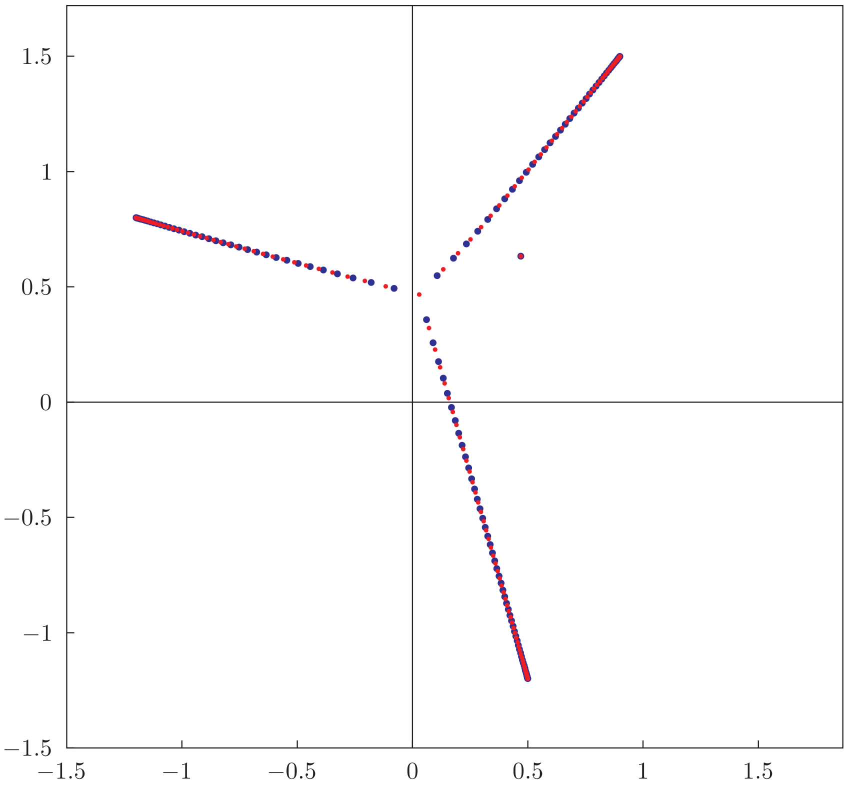

Figure 1. The zeros and poles of the diagonal Padé approximant $[130/130]_f$ of the function $f(z) = \big[(z+1.2-i\cdot 0.8)(z-0.9-i\cdot 1.5)(z-0.5+i\cdot 1.2)\big]^{-1/3}$. For $n = 130$ the distribution of the zeros and poles corresponds to Rakhmanov's electrostatic model [91]. One Froissart doublet can be seen in the figure. Since the corresponding (Stahl) Riemann surface has genus 1, there can be at most one doublet. The behaviour of this Froissart doublet as $n \to \infty$ is governed by an equation in [82].

and

$$g(z) := \operatorname{Re} \int_{a_1}^{z} \sqrt{\frac{\zeta - v}{A_3(\zeta)}}\, d\zeta \tag{24}$$

is the Green's function $g_D(z, \infty)$ of the Stahl domain $D = \overline{\mathbb{C}} \setminus S$.

Therefore, we can solve the problem of the distribution of the zeros of the Padé polynomials for functions having the very special form (10) only in the framework of the general Stahl theorem, which he proved for an (arbitrary) multivalued analytic function with finitely many branch points on the Riemann sphere.

In view of the foregoing, it is natural to regard the Padé polynomials for functions of the form (1) as a generalization of the classical Jacobi polynomials (see [74] and [82]). Another possible class of generalized Jacobi polynomials is connected with the so-called *two-point* Padé polynomials (see [56], [57], and also Figure 3).

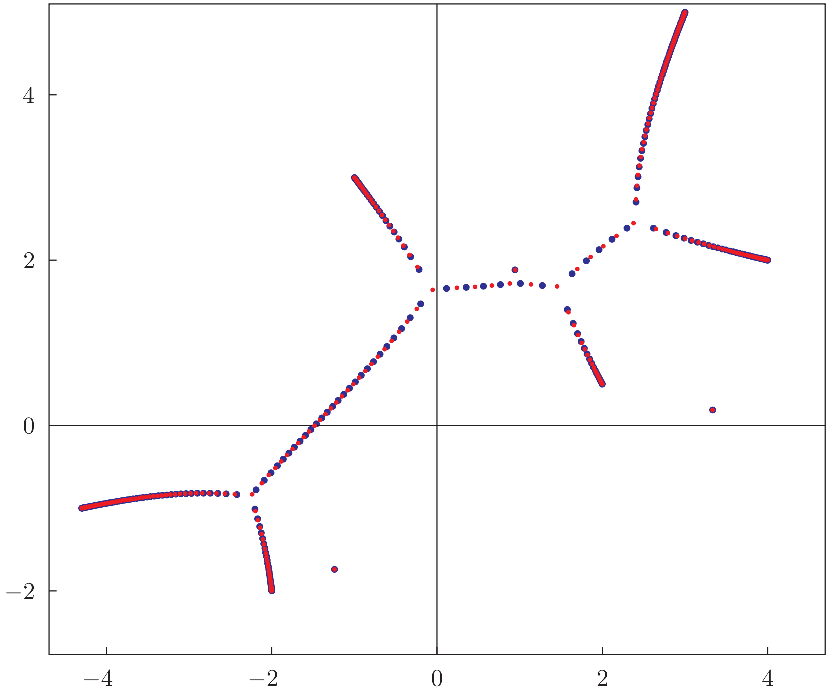

Figure 2. The zeros and poles of the diagonal Padé approximant $[266/266]_f$ of the function $f(z) = \bigl[(z+4.3+i)(z-2-i\cdot 0.5)(z+2+i\cdot 2)(z+1-i\cdot 3)(z-4-i\cdot 2)(z-3-i\cdot 5)\bigr]^{-1/6}$. In the limit as $n\to\infty$ the zeros and poles of the diagonal Padé approximants $[n/n]_f$ must be distributed according to Stahl's theorem [114]. For the given $n=266$ these zeros and poles are distributed according to Rakhmanov's electrostatic model [91]. Since for this function f the corresponding (Stahl) hyperelliptic surface has genus 4, there can be at most 4 Froissart doublets for each n. For $n=266$ three Froissart doublets can be seen in the picture.

The analysis of the corresponding asymptotic properties of such polynomials is based on Buslaev's theorem, which is the two-point analogue of Stahl's theorem (see [21], [22], and also § 2.3). We note that Buslaev's result in [21] is more general: it holds for m-point PA in the class of all multivalued analytic functions with finitely many singular points on the Riemann sphere (see § 2.3). Finally, yet another possible generalization of the Jacobi polynomials is connected with the Hermite–Padé polynomials of the first kind for a system $[1, f, f^2]$ of three functions, where f has a representation (16) with $\alpha\in(-1/2,1/2)$, $\alpha\neq 0$ (see [75] and [76] for more details).

For an arbitrary measure μ with $\operatorname{supp}\mu \Subset \mathbb{C}$ let $V^{\mu}(z)$ denote the logarithmic potential of μ:

$$V^{\mu}(z) := -\int \log|z-t|\,d\mu(t),$$

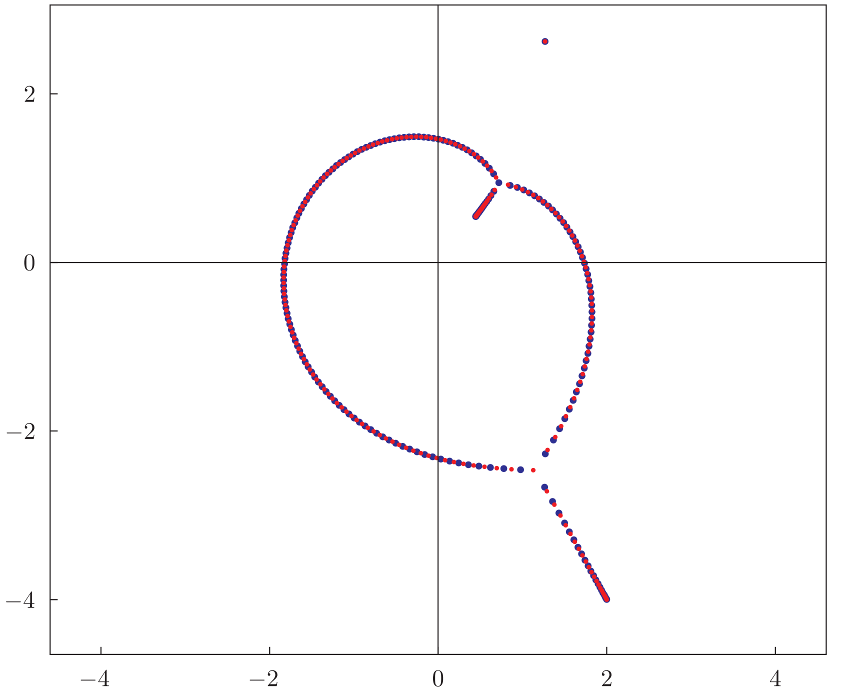

Figure 3. The numerical picture of the distribution of the zeros and poles of the two-point Padé approximant $[199/199]_f$ of the multivalued function $f(z) = \left(\dfrac{z-a_1}{z-a_2}\right)^{1/4}$ with $a_1 = 0.9 - i \cdot 1.1$ and $a_2 = 0.1 + i \cdot 0.2$ in the case when the two 'significantly different' branches $f_0 = \left(\dfrac{z-a_1}{z-a_2}\right)^{1/4}$ and $f_\infty = -\left(\dfrac{z-a_1}{z-a_2}\right)^{1/4}$ of f are taken. In aggregate, almost all the zeros (blue points) and poles (red points) simulate numerically the Buslaev compact set. In addition, there is a Froissart doublet, whose behaviour as $n \to \infty$ is governed by an equation in [57].

and let $V_*^\mu(z)$ denote the corresponding spherically normalized potential

$$V_*^\mu(z) := \int_{|\zeta|\leqslant 1} \log\frac{1}{|z-\zeta|}\,d\mu(\zeta) + \int_{|\zeta|>1} \log\frac{1}{|1-z/\zeta|}\,d\mu(\zeta).$$

For an arbitrary polynomial $Q \in \mathbb{C}[z]$ with $Q \not\equiv 0$ we let

$$\chi(Q) = \sum_{\zeta\colon Q(\zeta)=0} \delta_\zeta \tag{25}$$

denote the associated measure which 'counts' the zeros of the polynomial Q with multiplicities, and we let Q^* denote the corresponding spherically normalized potential

$$Q^*(z)=\prod_{\substack{\zeta:\,|\zeta|\leqslant 1\\ Q(\zeta)=0}}(z-\zeta)\cdot\prod_{\substack{\zeta:\,|\zeta|>1\\ Q(\zeta)=0}}\biggl(1-\frac{z}{\zeta}\biggr).$$

Stahl's Theorem (see [110]). *Suppose that* $f\in\mathscr{H}(\infty)$ *and* $f\in\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma)$, *where* $\operatorname{Card}\Sigma<\infty$. *Let* $D=D(f)$ *be the Stahl domain for* f, $S=S(f)=\partial D$ *the corresponding Stahl compact set, and* $[n/n]_f=-P_{n,0}/P_{n,1}$ *the nth diagonal Padé approximant of* f *(at infinity). Then the following statements hold*:

1) *the Padé polynomials* $P_{n,j}$, $j=0,1$, *have a limiting distribution of zeros, and*

$$\frac{1}{n}\chi(P_{n,j})\xrightarrow{*}\lambda_S^{\mathrm{rob}},\qquad n\to\infty,\quad j=0,1, \tag{26}$$

where λ_S^{rob} *is the Robin equilibrium measure on the Stahl compact set* S, *that is,* $V^{\lambda_S^{\mathrm{rob}}}(z)\equiv\mathrm{const}=\gamma_S$ *for* $z\in S$, *where* γ_S *is the Robin constant of* S;

2) *the diagonal PA converge to* f *in capacity*[10] *on compact subsets of* D, *that is,*

$$[n/n]_f(z)\xrightarrow{\mathrm{cap}}f(z),\qquad n\to\infty,\quad z\in D, \tag{27}$$

and the rate of convergence in (27) *is characterized by*

$$|(f-[n/n]_f)(z)|^{1/n}\xrightarrow{\mathrm{cap}}e^{-2g_D(z,\infty)},\qquad n\to\infty,\quad z\in D, \tag{28}$$

where $g_D(z,\infty)$ *is the Green's function for the domain* D.

It follows immediately from Stahl's theorem that

$$\frac{\deg P_{n,j}}{n}\to 1,\qquad n\to\infty,\quad j=0,1. \tag{29}$$

The relations (29) and (28) imply Kolchin's conjecture (6) for an *arbitrary function* $f\in\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma)$.

2.3. Multipoint Padé approximants: Buslaev's theory. Let $f_j\in\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma)$, where $\operatorname{Card}\Sigma<\infty$, $j=1,\dots,m$. Let $z_1,\dots,z_m\in\mathbb{C}\setminus\Sigma$ be pairwise distinct points and $f_j\in\mathscr{H}(z_j)$, $j=1,\dots,m$. Let $P_n,Q_n\in\mathbb{C}_n[z]$, $P_n,Q_n\not\equiv 0$, be polynomials of degree $\leqslant n$ such that

$$(Q_nf_j-P_n)(z_j)=O\bigl((z-z_j)^{n_j}\bigr),\qquad z\to z_j,\quad j=1,\dots,m, \tag{30}$$

where $\sum_{j=1}^m n_j=2n+1$, $n_j\in\mathbb{Z}_+$, $j=1,\dots,m$. The relation (30) does not uniquely define P_n and Q_n, but it does uniquely define the rational function $B_n=P_n/Q_n$, which is called a *multipoint* (or *m-point*) Padé approximant of the set $\mathfrak{f}=\{f_1,\dots,f_m\}$ of analytic functions $f_j\in\mathscr{A}^0(\overline{\mathbb{C}}\setminus\Sigma)$, at the corresponding points z_j, or briefly, of the *m-germ*[11] $\{(f_1,z_1),\dots,(f_m,z_m)\}$, of m analytic functions f_j holomorphic at the points z_j, $j=1,\dots,m$. If $z_j=\infty$ for some $j\in\{1,\dots,m\}$, then the corresponding relations (30) and (32) must be modified (see [21] and [22]). For

[10] See the definition of convergence in capacity in [44], [110], or [114].

[11] As usual, we use the same letter f for the germ of a multivalued analytic function f as for the function itself.

fixed m we will occasionally call an m-germ $\mathfrak{f} = \{(f_1, z_1), \dots, (f_m, z_m)\}$ a *multigerm* for short.

The functions $f_1, \dots, f_m$ in (30) are, generally speaking, distinct analytic functions, where no f_j can be obtained by continuing another function $f_k \in \mathfrak{f}$ with $k \neq j$ analytically along a path $\gamma \subset \overline{\mathbb{C}} \setminus \Sigma$. However, if there exists an algebraic differential operator of finite order $L[w]$ of the form (4) such that $L[f_j](z) \equiv 0$ for $z \in U_j(z_j)$ for each $j = 1, \dots, m$, then according to the approach in [27]–[30], [49], [55], it is natural to view all the functions $f_j \in \mathfrak{f}$ as branches of the same multivalued analytic function f such that $L[f](z) \equiv 0$, $z \notin \Sigma$. We remark that this interpretation of an analytic function as a solution of a differential equation of the form (2) (which is a very special case of the general homogeneous algebraic equation of the form (4)) is particularly important in the theory of Hermite–Padé approximants (see § 2.4 below), which is a part of the theory of PA where there is not yet known any kind of general result on the distribution of the zeros of Hermite–Padé polynomials in the class of multivalued analytic functions with finitely many branch points that is to any extent comparable to the theorems of Stahl and Buslaev (see [5], [81], [112], and also [75] and [76]).

In the case of general position the relations (30) are equivalent to the relations

$$(f_j - B_n)(z) = O\bigl((z - z_j)^{n_j}\bigr), \qquad z \to z_j, \quad j = 1, \dots, m. \tag{31}$$

Assume that in (30) we have $\dfrac{n_j}{n} \to 2p_j$ as $n \to \infty$ and $\sum_{j=1}^m p_j = 1$, $p_j \geqslant 0$, $j = 1, \dots, m$. Buslaev's theory asserts (see [21]–[23], and also [20]) that (in the non-degenerate case) there exists a unique[12] compact set $F = F_{\mathrm{Bus}}$ which is an S-curve[13] weighted in the external field generated by a unit negative measure $-\nu$ with $\nu = \sum_{j=1}^m p_j \delta_{z_j}$ concentrated at the points $z_1, \dots, z_m$. This compact set F has the following properties: F consists of finitely many analytic arcs; the complement $\overline{\mathbb{C}} \setminus F$ is a union $\bigcup_{j=1}^m D_j$ of finitely many domains $D_j \ni z_j$; each function $f_j \in \mathscr{H}(z_j)$ is holomorphic (that is, single-valued analytic) in the corresponding domain D_j, $f_j \in \mathscr{H}(D_j)$; if $D_j = D_k$ for some $k \neq j$, then also $f_k = f_j$; F has the property of 'symmetry' in the external field $V_*^{-\nu}$, namely,

$$\frac{\partial (V^{\beta_F} - V_*^{\nu})}{\partial n^+}(z) = \frac{\partial (V^{\beta_F} - V_*^{\nu})}{\partial n^-}(z), \qquad z \in F^{\circ}, \tag{32}$$

where $\beta_F \in M_1(F)$ is the unique unit equilibrium measure on the compact set F in the external field $V_*^{-\nu}$ (that is, the identity $V^{\beta_F}(z) - V_*^{\nu}(z) \equiv \mathrm{const} = w_F$ holds for $z \in F$), F° is the union of the open arcs whose closures form F, and $\partial/\partial n^{\pm}$ are the normal derivatives to F at a point $z \in F^{\circ}$ from opposite sides of F. We note that for a fixed multigerm $\mathfrak{f} = \{(f_1, z_1), \dots, (f_m, z_m)\}$ the set F depends to an essential degree on the choice of the numbers ('weights') $p_j \geqslant 0$ with $\sum_{j=1}^m p_j = 1$. Thus, the 'optimal' partition of the Riemann sphere into the domains D_j also depends on the choice of the numbers p_j.

We underscore that, just as in Stahl's theory, the existence of an S-curve F weighted in the external field is a cornerstone of Buslaev's theory. Once we have

[12] Up to arbitrary sets with capacity zero.

[13] See [44], [60], [91], [114] regarding this notion.

established its existence, the question of the corresponding equilibrium measure β_F is easy to resolve: $\beta_F = \widetilde{\nu} = \mathfrak{b}_F(\nu)$ is the balayage of the measure ν from the open set $D = \bigcup_{j=1}^m D_j$ to its boundary $\partial D = F$. The boundary ∂D_j of each domain D_j contains an open arc $\gamma_j^\circ \subset \partial D_j$ such that $\gamma_j^\circ \cap \partial D_k = \varnothing$ for $k \neq j$. As in the case of Stahl's theory, the S-property (32) of the weighted S-curve F can be equivalently written as

$$\frac{\partial\bigl(\sum_{j=1}^m p_j g_{D_j}(z, z_j)\bigr)}{\partial n^+} = \frac{\partial\bigl(\sum_{j=1}^m p_j g_{D_j}(z, z_j)\bigr)}{\partial n^-}, \qquad z \in F^\circ \tag{33}$$

(cf. (18) and (19)), where $g_{D_j}(z, z_j)$ is the Green's function of the domain D_j (as usual, for $z \in D_k \neq D_j$ we set $g_{D_j}(z, z_j) \equiv 0$).

Buslaev's Theorem (see [21], [22]). *Let $\mathfrak{f} = \{(f_1, z_1), \dots, (f_m, z_m)\}$ be a multigerm of m analytic functions f_j such that $f_j \in \mathscr{H}(z_j)$ and $f_j \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$ for $j = 1, \dots, m$, where $\operatorname{Card}\Sigma < \infty$ and the points $z_1, \dots, z_m \in \mathbb{C}$ are pairwise distinct. Let $p_j \geqslant 0$ with $\sum_{j=1}^m p_j = 1$, let $F = F(\mathfrak{f}; p_1, \dots, p_m)$ be the corresponding Buslaev compact set with the S-property in the external field $V_*^{-\nu}(z)$, where $\nu = \sum_{j=1}^m \delta_{z_j}$, and let $\bigcup_{j=1}^m D_j = \overline{\mathbb{C}} \setminus F$ be a corresponding optimal partition of the Riemann sphere with $D_j \ni z_j$ and $f_j \in \mathscr{H}(D_j)$. Let $B_n = P_n/Q_n$, $B_n(z) = B_n(z; \mathfrak{f}; p_1, \dots, p_m)$, be the corresponding diagonal m-point Padé approximant of the multigerm $\mathfrak{f}$. Then the following assertions hold*:

1) *the zeros and poles of the m-point PA $B_n = P_n/Q_n$ have a limiting distribution, namely,*

$$\frac{1}{n}\chi(P_n), \frac{1}{n}\chi(Q_n) \xrightarrow{*} \beta_F, \qquad n \to \infty; \tag{34}$$

2) *the m-point PA converge to f_j in capacity on compact subsets of D_j, $j = 1, \dots, m$,*

$$B_n(z) \xrightarrow{\text{cap}} f_j(z), \qquad n \to \infty, \quad z \in D_j, \quad j = 1, \dots, m, \tag{35}$$

and for $j = 1, \dots, m$ the rate of convergence in (35) is characterized by

$$|f_j(z) - B_n(z)|^{1/n} \xrightarrow{\text{cap}} \exp\biggl\{-2\sum_{k=1}^m p_k g_{D_k}(z, z_k)\biggr\}, \qquad n \to \infty, \quad z \in D_j. \tag{36}$$

An immediate consequence of Buslaev's theorem is Kolchin's conjecture (7) for an *arbitrary m-germ* $\mathfrak{f} = \{f_1, \dots, f_m\}$, where $f_j \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$ and $\operatorname{Card}\Sigma < \infty$.

2.4. Hermite–Padé approximants. Suppose that $f_1, f_2 \in \mathscr{A}^0(\overline{\mathbb{C}} \setminus \Sigma)$ with $\operatorname{Card}\Sigma < \infty$, and $f_1, f_2 \in \mathscr{H}(\infty)$. Throughout what follows, the functions f_1, f_2, and $f_0 \equiv 1$ are rationally independent. Let $\mathbb{P}_n := \mathbb{C}_n[z]$.

For any $n \in \mathbb{N}$ the Hermite–Padé polynomials of the first kind $Q_{n,0}, Q_{n,1}, Q_{n,2} \in \mathbb{P}_n$ with $\deg Q_{n,j} \leqslant n$ and $Q_{n,j} \not\equiv 0$ are defined by the relation

$$R_n(z) := (Q_{n,0} \cdot 1 + Q_{n,1} f_1 + Q_{n,2} f_2)(z) = O\biggl(\frac{1}{z^{2n+2}}\biggr), \qquad z \to \infty, \tag{37}$$

where R_n is called the *remainder function*. The relation (37) does not uniquely define the polynomials $Q_{n,j}$, but their ratios, for example, $Q_{n,1}/Q_{n,2}$ are uniquely defined. Since 1, f_1, and f_2 are rationally independent, the remainder function does not vanish identically: $R_n \not\equiv 0$. The reader can find a more extended account of the properties of Hermite–Padé polynomials and the corresponding Hermite–Padé approximants principally in [17], [68], [79], [81], [112], and also in [5], [7], [10], [41], [121]. We note that here we are only dealing with Hermite–Padé polynomials of the first kind, as defined by (37). The so-called Hermite–Padé polynomials of the second kind, that is, multiple orthogonal polynomials, are connected with Hermite–Padé polynomials of the first kind by means of some formal relations (see [81], (2.1.3) in § 2, [37], and [122]). However, in this paper we will neither discuss this connection nor consider Hermite–Padé polynomials of the second kind.

The question of the asymptotic properties of the Hermite–Padé polynomials of the first kind,[14] even for a system $[1, f_1, f_2]$ of three multivalued analytic functions, has been much less studied than for classical and multipoint PA. Here we encounter fundamental difficulties already for the simplest analytic functions (from the point of view of the theory of classical or two-point PA). This can be seen in the example of three functions $[1, f, f^2]$, where f has the form

$$f(z) = \prod_{j=1}^{3} (z - a_j)^{\alpha_j}, \qquad 2\alpha_j \in \mathbb{C} \setminus \mathbb{Z}, \quad \sum_{j=1}^{3} \alpha_j = 0, \tag{38}$$

with the points a_1, a_2, and a_3 in general position and, in particular, not on a straight line. Apart from the symmetric case where a_1, a_2, and a_3 are the vertices of an equilateral triangle (see [81]), it has not yet been proved that the zeros of the corresponding Hermite–Padé polynomials have a limiting distribution; see [11], [12], [119] for details. Under these conditions the pair of functions f, f^2 forms a Nikishin system (for the corresponding definition, the convergence in the real case, and the main properties of such systems, see Nikishin's original paper [78] first of all, and also [9], [36], [45], [65]).

Our next example is no less characteristic in that it displays the difficulties encountered in the general theory of the distribution of the zeros of the Hermite–Padé polynomials for multivalued analytic functions with a finite set of branch points. We consider the system of three functions $[1, f_1, f_2]$, where f_1 and f_2 are given by the representations

$$f_1(z) = \left(\frac{z-a}{z+1}\right)^{1/2} = \frac{1}{\pi} \int_{-1}^{a} \sqrt{\frac{a-x}{x+1}} \, \frac{dx}{x-z} + 1, \quad z \notin \Delta_1 := [-1, a], \tag{39}$$

$$f_2(z) = \left(\frac{z-1}{z+a}\right)^{1/2} = \frac{1}{\pi} \int_{-a}^{1} \sqrt{\frac{1-x}{x+a}} \, \frac{dx}{x-z} + 1, \quad z \notin \Delta_2 := [-a, 1], \tag{40}$$

and $a \in (0,1)$ is a parameter. In (39) and (40) we have taken the branch of the square root $(\,\cdot\,)^{1/2}$ such that $f_1(z), f_2(z) \to 1$ as $z \to \infty$; here and in what follows, $\sqrt{\cdot}$ always denotes the arithmetical value of the root function, that is, $\sqrt{x^2} = x$ for $x \in \mathbb{R}_+$. Clearly, f_1 and f_2 are Markov functions with supports the respective intervals $\Delta_1 = [-1, a]$ and $\Delta_2 = [-a, 1]$. Each of f_1 and f_2 has a pair of singular points, $-1, a$ and $-a, 1$, respectively, which are second-order branch points, that is,

[14] As well as of the Hermite–Padé polynomials of the second kind.

their singular sets $\{-1, a\}$ and $\{-a, 1\}$ are disjoint, in contrast to the previous case of $[1, f, f^2]$. Since $a \in (0,1)$, it follows that $\Delta_1 \cap \Delta_2 = [-a, a] \neq \varnothing$, but $\Delta_1 \not\subset \Delta_2$ and $\Delta_2 \not\subset \Delta_1$. Thus, the pair of functions f_1, f_2 is neither an Angelesco nor a Nikishin system. Numerical experiments in [51] (see also [50]) showed that in this seemingly quite simple case, with all initial data purely real, the description of the limiting distribution of the corresponding Hermite–Padé polynomials necessarily involves S-curves which now lie in the complex plane (but of course, are mirror-symmetric with respect to the real axis; see the details in [51]). So far the theory of the distribution of the zeros of Hermite–Padé polynomials does not have any general results which could be used to explain the numerical results obtained experimentally in [51] on the distribution of the zeros of the Hermite–Padé polynomials. We mention the recent paper [13], where a similar problem has been considered for Hermite–Padé polynomials of the second kind.

Thus, even for the simplest (from the point of view of the general theory of classical or multipoint PA) systems $[1, f, f^2]$ and $[1, f_1, f_2]$ of three multivalued analytic functions the problem of characterizing the corresponding S-curves in terms of an associated (but not yet known, not even regarding its formal statement) potential-theoretic equilibrium problem is still open.

Now assume that the points a_j and parameters α_j in the representation (1) satisfy the following additional conditions:

$$p = 2q, \quad a_j = e_j \in \mathbb{R}, \quad e_1 < \cdots < e_{2q}, \quad \alpha_j = (-1)^j \alpha,$$

where $2\alpha \in \mathbb{R} \setminus \mathbb{Z}$. That is, the analytic function f has a representation

$$f(z) = \prod_{j=1}^{q} \left(\frac{z - e_{2j-1}}{z - e_{2j}} \right)^{\alpha}, \tag{41}$$

where the points e_j and the parameter α satisfy the above conditions. Obviously, f continues to satisfy a differential equation of the form (2).

In this paper we investigate the problem of the limiting distribution of the zeros of the Hermite–Padé polynomials of the first kind for a system of three functions $[1, f, f^2]$, where f is given by a representation (41) with the points e_j and the parameter α satisfying the above conditions. Throughout, we denote the class of such functions by $\mathscr{L}_E$, $E := \bigsqcup_{j=1}^{q} [e_{2j-1}, e_{2j}]$. It is explained below (see Remark 3) why we choose such an ostensibly very special set of branch points e_j and the same values of the parameter α. The case $q = 1$ was considered in [75] and [76]. Then f has the form $f(z) = ((z+1)/(z-1))^{\alpha}$, and it is natural to regard the corresponding Hermite–Padé polynomials $Q_{n,j}$ as analogues of the classical Jacobi polynomials. For any $q > 1$ the corresponding Hermite–Padé polynomials $Q_{n,j}$ can be treated in a natural way as a generalization of the Akhiezer polynomials [1], which are orthogonal on several intervals (see (69), and also [81]).

In view of the above, the problem in this paper of the limiting distribution of the zeros of the Hermite–Padé polynomials of the first kind for a system of three functions $[1, f, f^2]$, where f has the representation (41), is a quite topical problem. In his original paper [78] (see also [45], [101]–[105]) Nikishin mainly considered a more general statement of the problem. Namely, he investigated a system of $m \geqslant 3$ Markov functions $[1, f_1, \dots, f_{m-1}]$ such that $f_1, \dots, f_{m-1}$ form a Nikishin system.

We recall that this means, in particular, that all the Markov functions $f_j = \widehat{\mu}_j$ have the same support, which is a union of finitely many segments of the real line. The limiting distribution of the zeros of the corresponding Hermite–Padé polynomials of the first kind was characterized in [78] in terms of a certain potential-theoretic equilibrium problem very similar to the problem considered first by Gonchar and Rakhmanov [41] in 1981 in the case of functions $f_1, \dots, f_{m-1}$ forming an Angelesco system (see also [40], [43]). The potential-theoretic equilibrium problem that Nikishin introduced in [78] can be fully characterized by the so-called *Nikishin* interaction matrix. For $m = 3$ and for the system $[1, f, f^2]$ this matrix has the form $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. By contrast, the potential-theoretic equilibrium problem under consideration here, in terms of which we characterize the limiting distribution of the zeros of the Hermite–Padé polynomials $Q_{n,j}$ for the system of functions $[1, f, f^2]$, is *scalar* (see (49) below, and also Remark 3).

We remark that the transition to complex branch points e_j and distinct complex parameters α_j in place of the single α in (41) presented a fundamental difficulty even for classical and two-point PA and had resisted analysis until Stahl's and Buslaev's general theorems.

Now we fix the germ of a function $f \in \mathscr{L}_E$ by setting $f(\infty) = 1$. We note that, instead of the explicit representation (41), we could start directly from the differential equation (2). Since $f(\infty) = 1$, we get from (41) that

$$f(z) = \sum_{k=0}^{\infty} \frac{c_k}{z^k}, \quad \text{where } c_k \in \mathbb{R}.$$

Hence, we can refine the standard definition (37) of the Hermite–Padé polynomials of the first kind for the system $[1, f, f^2]$ of three functions as follows. Let $Q_{n,0}, Q_{n,1} Q_{n,2} \in \mathbb{R}_n[z]$ with $Q_{n,2} \not\equiv 0$ be polynomials of degree $\leqslant n$ such that

$$R_n(z) := (Q_{n,0} \cdot 1 + Q_{n,1} f + Q_{n,2} f^2)(z) = O\biggl(\frac{1}{z^{2n+2}}\biggr), \qquad z \to \infty. \tag{42}$$

As before, such polynomials always exist and are not uniquely defined, but their ratios, for example, $Q_{n,1}/Q_{n,2}$, are uniquely defined rational functions of order $\leqslant n$. The function R_n is the remainder function. It follows from the conditions $e_1 < \dots < e_{2q}$ and $2\alpha \in \mathbb{R} \setminus \mathbb{Z}$ that f and f^2 are Markov functions which form a generalized *Nikishin system* (see [33], [78]). Therefore, $[1, f, f^2]$ is a non-degenerate system (see [33] and [65]), and therefore $R_n(z) \not\equiv 0$. Moreover, on the right-hand side of (42) the order of a zero can exceed $2n + 2$ only by a fixed quantity depending on q, e_j, and α, and the degrees of the polynomials $Q_{n,j}$ can differ from n only by a certain quantity independent of n, that is, $\deg Q_{n,j} \geqslant n - m$, where $m \in \mathbb{N}$ is independent of n.

Since f, f^2 is a Nikishin system, it follows from [65] that 'almost all' zeros of the polynomials $Q_{n,j}$, apart from $o(n)$ zeros, lie in the set $F := \overline{\mathbb{R}} \setminus E$, where $E = \bigsqcup_{j=1}^{q} [e_{2j-1}, e_{2j}]$. Moreover, the rational functions $Q_{n,j}/Q_{n,2}$, $j = 0, 1$, have certain interpolation properties and converge in the Hausdorff σ_1-measure on compact subsets of the domain $G := \overline{\mathbb{C}} \setminus F$. However, [65] says nothing about the rate of this convergence. It is one of our aims in this paper to characterize the rate of such convergence.

The germ $f \in \mathscr{H}(\infty)$ of the function f given by (41) and fixed by setting $f(\infty) = 1$ defines f as a holomorphic (that is, single-valued analytic) function on the Stahl domain $D := \overline{\mathbb{C}} \setminus E$ (defined with respect to the point at infinity: $D = D_\infty(f)$). Furthermore, on the set $\bigsqcup_{j=1}^q (e_{2j-1}, e_{2j})$ this germ generates in the natural way a *family of germs* $\widetilde{f}_j(x)$, $x \in (e_{2j-1}, e_{2j}) =: E_j^\circ$, $j = 1, \dots, q$, of multivalued analytic functions by the formulae

$$\widetilde{f}_j(x) := f^+(x) + f^-(x), \qquad x \in (e_{2j-1}, e_{2j}), \tag{43}$$

where $f^+(x)$ for $x \in (e_{2j-1}, e_{2j})$ denotes the limit value of $f(z)$, $f \in \mathscr{H}(D)$, as $z \to x$ in the upper half-plane, $f^+(x) := f(x + i \cdot 0)$, and $f^-(x)$ has a similar meaning, $f^-(x) := f(x - i \cdot 0)$. Obviously, each function $\widetilde{f}_j$, $j = 1, \dots, q$, satisfies the differential equation (2). Thus, all the functions $\widetilde{f}_j$ are branches of the same multivalued analytic function f defined by (2) using a finite number of complex parameters, the polynomial coefficients of the equation (2). We note that, up to a non-trivial multiplicative constant, all the functions $\widetilde{f}_j$ are branches of the multivalued analytic function f given by the representation (41). Therefore, each of these branches can be obtained from the germ fixed at $z = \infty$ by means of analytic continuation of it along some paths avoiding the points $e_1, \dots, e_{2q}$ and by subsequent multiplication by a non-trivial constant. It is easy to see that, by (41), for $2\alpha \in \mathbb{R} \setminus \mathbb{Z}$ each function $\widetilde{f}_j$ makes a non-trivial jump on the gaps (e_{2j-2}, e_{2j-1}) and (e_{2j}, e_{2j+1}) adjoining the starting interval (e_{2j-1}, e_{2j}). In addition, any two 'adjacent' functions, say $\widetilde{f}_j$ and $\widetilde{f}_{j+1}$, are immediate analytic continuations of each other across both the upper and the lower half-plane. Thus, a function $\widetilde{f}$ holomorphic in $G := \overline{\mathbb{C}} \setminus F \not\ni \infty$ and such that $\widetilde{f}|_{E_j^\circ} = \widetilde{f}_j$, $j = 1, \dots, m$, arises naturally in this situation.

This function also satisfies (2) and so is an analytic continuation of the original function f, $f \in \mathscr{H}(D)$, $f(\infty) = 1$, in the sense specified in § 1.

The construction (42), which in fact defines the Hermite–Padé polynomials, is a natural generalization of the construction (15) defining Padé polynomials. Therefore, Padé polynomials $P_{n,0}$ and $P_{n,1}$ for f are Hermite–Padé polynomials for the system $[1, f]$. The rational function $[n/n]_f := -P_{n,0}/P_{n,1}$ has the maximum possible order of tangency to the given function f at $z = \infty$. It follows from Stahl's general theorem that if f has the form (1), then the Padé polynomials $P_{n,j}$ have a limiting distribution of zeros equal to the (Robin) equilibrium measure of the compact set $E := \bigsqcup_{j=1}^q [e_{2j-1}, e_{2j}]$:

$$\frac{1}{n}\chi(P_{n,j}) \to \lambda_E^{\mathrm{rob}}, \qquad V_E^{\lambda^{\mathrm{rob}}}(x) \equiv \mathrm{const}, \quad x \in E. \tag{44}$$

We recall that for $E = [-1, 1]$

$$d\lambda_E^{\mathrm{rob}} = d\lambda^{\mathrm{cheb}} = \frac{1}{\pi}\frac{dx}{\sqrt{1 - x^2}}\,.$$

By Stahl's theorem (see § 2.2) the diagonal PA $[n/n]_f$ converge to the function $f \in \mathscr{H}(D)$ in capacity on compact subsets of the Stahl domain D:

$$[n/n]_f(z) \xrightarrow{\text{cap}} f(z), \qquad z \in D, \quad n \to \infty. \tag{45}$$

The fact that, in principle, the diagonal PA do not converge uniformly in the Stahl domain D follows already from Dumas' classical result [35] of 1908. Such convergence fails even in the case of *a single gap*. This is due to the presence of the so-called Froissart doublets [38] (see Figures 1, 2, and 3), or in other words, pairs of so-called 'spurious' zeros and 'spurious' poles of the diagonal PA, which are not associated with a zero, a pole, or any other singularity of the original function f. The number of them is primarily related to the genus of the corresponding two-sheeted Stahl Riemann surface (see [59], [113]), and they do not have a limit as $n \to \infty$, but in each such pair the zero and pole are infinitesimally close in the limit. When passing from n to $n+1$ such a pair shifts by an 'almost' fixed distance (in the corresponding metric), and typically they are dense on the Riemann sphere. For example, for the elliptic function considered by Dumas (which is a function of genus 1) the asymptotic behaviour of the pair 'spurious zero–spurious pole' corresponds generically to a dense winding on the torus. By Stahl's theorem, as $n \to \infty$ 'almost all' the zeros and poles of the diagonal PA, with the possible exception of $o(n)$ of them, are attracted to the Stahl compact set, hence the question of the so-called *strong asymptotics* of the Padé polynomials is directly connected with the problem of a complete description of *all* the zeros and poles of such PA. In fact, this problem splits into two parts (see [81] and [113]). First, we must show that for a given multivalued function f there are only a finite number (depending on f) of such Froissart doublets. Next, once we know their number, must describe the asymptotic behaviour of the duplets in suitable terms. The second problem is in fact equivalent to the *Jacobi inversion problem* [106], [128], posed on the canonical Stahl hyperelliptic surface associated with f (see [14], [57], [74], [81], [82], [116], [117]).

The transition to complex branch points e_j in (41) and to distinct complex parameters α_j in place of a single α, even for classical PA required the development of essentially new methods of investigation for solving the problem of the limiting distribution of the zeros of the Padé polynomials (see § 2.2).

Let $f_1(z) := f(z)$ for $z \in D_1$, where $D_1 := D = \overline{\mathbb{C}} \setminus E$, $E := \bigsqcup_{j=1}^{q} [e_{2j-1}, e_{2j}]$. Then $f_1 \in \mathscr{H}(D_1)$ and $f_1(\infty) = 1$. Let $e_1 = -1$ and $e_{2q} = 1$. Thus, $\widehat{E} = \operatorname{conv} E = [-1, 1]$.

We also set $D_2 = G := \overline{\mathbb{C}} \setminus F$, $F := \overline{\mathbb{R}} \setminus E$. Let f_2 be a function defined on E° by

$$f_2(x) := -2\cos(\alpha\pi) \prod_{j=1}^{k-1} \left(\frac{x - e_{2j-1}}{x - e_{2j}} \right)^{\alpha} \cdot \left(\frac{x - e_{2k-1}}{e_{2k} - x} \right)^{\alpha} \cdot \prod_{j=k+1}^{q} \left(\frac{e_{2j-1} - x}{e_{2j} - x} \right)^{\alpha}, \tag{46}$$

$x \in (e_{2k-1}, e_{2k})$, $k = 1, \dots, q$. Then f_2 extends from E° to $\overline{\mathbb{C}} \setminus \{e_1, \dots, e_{2q}\}$ as a multivalued analytic function. The relation (46) fixes a holomorphic branch of f_2 on D_2, $f_2 \in \mathscr{H}(D_2)$. Obviously, f_2 also satisfies the differential equation (2). Thus, according to the approach of § 1, f_2 is another branch of f_1, which is holomorphic in another domain $D_2 \neq D_1$.

Let $g_E(\zeta,z)$, $z,\zeta\in D_1$, be the Green's function of D_1 with pole at $\zeta=z$, and let $g_F(\zeta,z)$, $z,\zeta\in D_2$, be the Green's function of D_2 with pole at $\zeta=z$. Then we define

$$G^\mu_F(z):=\int_E g_F(x,z)\,d\mu(x),\qquad z\in D_2,\tag{47}$$

to be the Green potential (with respect to D_2) of the unit (positive Borel) measure μ with support on E, $\mu\in M_1(E)$.

The following theorem is the main result of this paper.

Theorem 1. *Let $\alpha\in(-1/2,1/2)$, $\alpha\ne 0$, let $e_1<\dots<e_{2q}$, let $f_1=f$ be a function given by (41), $f_1\in\mathscr{H}(D_1)$, and let f_2 be a function given by (46), $f_2\in\mathscr{H}(D_2)$. Let $Q_{n,j}=Q_{n,j}(z;f)$ be the Hermite–Padé polynomials for the system $[1,f,f^2]$. Then the following assertions hold.*

1) *All the zeros of the polynomials $Q_{n,0}$, $Q_{n,1}$ and $Q_{n,2}$, with the possible exception of a fixed number of zeros which is independent of n, lie in the set F; the zeros of the polynomials $Q_{n,j}$ have a limiting distribution which coincides with the (unique) unit measure $\eta_F\in M_1(F)$ supported on F that is the equilibrium measure for the mixed potential $3V^\mu_*(z)+G^\mu_E(z)$ in the external field $\psi(z):=3g_E(z,\infty)$,*

$$\frac1n\chi(Q_{n,j})\xrightarrow{*}\eta_F,\qquad n\to\infty,\tag{48}$$

where

$$3V^{\eta_F}_*(y)+G^{\eta_F}_E(y)+3g_E(y,\infty)\equiv\text{const},\qquad y\in F.\tag{49}$$

2) *The rational function $r_n:=Q_{n,1}/Q_{n,2}$ interpolates the function $f_2(z)$ at (no fewer than) $2n-m$ distinct points $x_{n,j}$ in the open set $E^\circ:=\bigsqcup_{j=1}^q(e_{2j-1},e_{2j})$, where $m\in\mathbb N$ is fixed and is independent of n; the free interpolation points $x_{n,j}$ have a limiting distribution which coincides with the (unique) unit measure $\eta_E\in M_1(E)$ supported on E that is an equilibrium measure with respect to the mixed Green-logarithmic potential $3V^\nu_*(z)+G^\nu_F(z)$,*

$$\frac1{2n}\sum_{j=1}^{2n-m}\delta_{x_{n,j}}\xrightarrow{*}\eta_E,\qquad n\to\infty,\tag{50}$$

where

$$3V^{\eta_E}_*(x)+G^{\eta_E}_F(x)\equiv\text{const},\qquad x\in E.\tag{51}$$

3) *The following relation holds in D_2 (cf. [93], Theorem 1, and [76], Theorem 1.6):*

$$\frac{Q_{n,1}}{Q_{n,2}}\xrightarrow{\text{cap}}f_2(z),\qquad z\in D_2,\quad n\to\infty,\tag{52}$$

and the rate of convergence in (52) is characterized by the relations (cf. (28) and (32))

$$\left|f_2(z)-\frac{Q_{n,1}}{Q_{n,2}}(z)\right|^{1/n}\xrightarrow{\text{cap}}e^{-2G^{\eta_E}_F(z)}<1,\qquad z\in D_2\setminus E,\quad n\to\infty,\tag{53}$$

$$\varlimsup_{n\to\infty}\left|f_2(x)-\frac{Q_{n,1}}{Q_{n,2}}(x)\right|^{1/n}\leqslant e^{-2G^{\eta_E}_F(x)}<1,\qquad x\in E^\circ,\tag{54}$$

where η_E is the measure solving the problem (51), *and moreover*, *the upper regularization of the function on the left-hand side of the inequalities* (54) *is equal to the right-hand side for all $x\in E^\circ$.*

Thus, it is shown in Theorem 1 that the rational function $r_n := Q_{n,1}/Q_{n,2}$ constructed from the $3n+1$ Laurent coefficients (at the point at infinity) of the original function $f_1=f$ satisfying (2) provides a multipoint rational interpolation (at $2n-m$ points) of the function f_2 (see (46)), which also satisfies the differential equation (2). We underscore that the rational function r_n of order n just constructed has a priori not just free zeros and poles, but also free points of interpolation (their number is asymptotically equal to $2n$). Furthermore, both the points of interpolation and the zeros and poles of r_n behave 'optimally'. Namely, their limiting distribution corresponds to solutions of the extremal problems (97) and (121). Such rational functions are quite similar in their properties to the best Chebyshev rational approximants (see [40] and [92]), but in contrast to the latter are constructed from *finitely* many Laurent coefficients. The rational function r_n constructed from a function $f\in\mathscr{H}(D)$ given by the differential equation (2) approximates, in some domain $G\neq D$, *another branch* $\widetilde{f}\in\mathscr{H}(G)$ of this multivalued analytic function. Actually, r_n approximates the given multivalued analytic function f on *another sheet* of the corresponding Riemann surface.

Remark 3. The case when $q=1$, $f(z)=((z-1)/(z+1))^\alpha$, and $f(\infty)=1$, where $2\alpha\in\mathbb{C}\setminus\mathbb{Z}$, was investigated in [76] (see also [75]), and for $2\alpha\in\mathbb{R}\setminus\mathbb{Z}$ an analogue of Theorem 1 was established. The problem of the distribution of the zeros of the polynomials $Q_{n,j}$ for such f was solved in [76] for $2\alpha\in\mathbb{C}\setminus\mathbb{Z}$, and explicit representations for the equilibrium measures η_E and η_F were given:

$$\frac{d\eta_F}{dx}(x)=\frac{\sqrt{3}}{2\pi}\,\frac{1}{\sqrt[3]{x^2-1}}\biggl(\frac{1}{\sqrt[3]{|x|-1}}-\frac{1}{\sqrt[3]{|x|+1}}\biggr),\qquad x\in\overline{\mathbb{R}}\setminus[-1,1],$$
$$\frac{d\eta_E}{dx}(x)=\frac{\sqrt{3}}{4\pi}\,\frac{1}{\sqrt[3]{1-x^2}}\biggl(\frac{1}{\sqrt[3]{1-x}}+\frac{1}{\sqrt{1+x}}\biggr),\qquad x\in(-1,1).$$

For $q=1$ and $\alpha=1/3$, that is, for $f(z)=((z-1)/(z+1))^{1/3}$, the relation (54) in Theorem 1 can be improved significantly. Namely, the rational function $Q_{n,1}/Q_{n,2}$ has the property of an 'almost Chebyshev alternance' on $(-1,1)$ in the following sense. For each positive θ, which can be arbitrarily close to zero, there exist at least $N_n=[2n(1-\theta)]$ consecutive points x_j on the interval $(-1,1)$, $-1<x_1<\dots<x_{N_n}$, at which the difference under the absolute value sign on the let-hand side of (54) takes extremal values with alternating signs:

$$f_2(x_j)-\frac{Q_{n,1}}{Q_{n,2}}(x_j)=(-1)^j\frac{2}{3}\sqrt[3]{\frac{1+x_j}{1-x_j}}\,e^{-2nG_F^{\eta_E}(x_j)}(1+\varepsilon_n(x_j)),\tag{55}$$

where $\varepsilon_n(x)\to0$ as $n\to\infty$ with the rate of a geometric progression locally uniformly on $(-1,1)$ (in (55) we mean by $\sqrt[3]{\cdot}$ the arithmetical cube root: $\sqrt[3]{a^3}=a$ for $a>0$).

Remark 4. Theorem 1 solves the problem of the distribution of the zeros of the Hermite–Padé polynomials $Q_{n,j}$ in terms of the scalar equilibrium problem (49).

The fact that this equilibrium problem can be used to characterized the limiting distribution of the zeros is based on the orthogonality relations (63) (see also (113)). All our further arguments will rely on (63). The same orthogonality relations can also very well be used in the framework of the classical scheme first proposed by Gonchar and Rakhmanov [41] and based on the observation that, in the real case, the relations (63) yield (cf. [76]) interpolation conditions which hold for the function $Q_{n,1}/Q_{n,2}$. The corresponding vector equilibrium problem is defined by the 2×2 matrix $\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$ and is easily seen to be equivalent to the pair of problems (49), (51). However, here we prefer to demonstrate a new approach to the problem of the limiting distribution of the zeros, first proposed by Rakhmanov and Suetin in [94] (see also [115]). In the final analysis this approach is also based on the general GRS-method. However, in its framework we do not need an interpolation property to solve the problem of the limiting distribution of the zeros. Thus, this approach extends the applicability range of the GRS-method by allowing one to use it also in the *complex* case, that is, when such an interpolation property does not hold a priori (see [94], [119], and also Remark 7). It could be a rather difficult task to establish such a property even, for instance, when just one branch point e_j is moved away from the real line into the complex plane. We remark that in his original paper [78] Nikishin solved the problem of the asymptotic behaviour as $n\to\infty$ of the function $|R_n(z)|^{1/n}$ for the system $[1,f_1,\dots,f_{m-1}]$, where $m\in\mathbb{N}$ is arbitrary and the functions $f_1,\dots,f_{m-1}$ form a Nikishin system. He did not consider there the problem of the limiting distribution of the zeros of the polynomials $Q_{n,j}$ (see also [101]–[105]).

Remark 5. As before, assume that all the branch points e_j are real, $e_j\in\mathbb{R}$, and pairwise distinct, but in (41) allow distinct parameters α_j with $2\alpha_j\in\mathbb{R}\setminus\mathbb{Z}$ instead of just a single α. We are therefore looking at a broader function class than (41), in which functions have representations of the form

$$f(z)=\prod_{j=1}^{q}\biggl(\frac{z-e_{2j-1}}{z-e_{2j}}\biggr)^{\alpha_j} \tag{56}$$

with the e_j and α_j satisfying the conditions above. In addition, we fix the branch of f by the same condition $f(\infty)=1$ as before. For each f with $f(\infty)=1$ in this broader class we define the family of functions $\widetilde{f}_j(x):=f^+(x)+f^-(x)$, $x\in(e_{2j-1},e_{2j})$, in a similar way as above, each of which is an 'analytic continuation' of the original function f (in the sense of it being a solution of the differential equation (2)). However, if $\alpha_j\neq\alpha_{j+1}$ for some $j\in\{1,\dots,q-1\}$, then two 'adjacent' functions $\widetilde{f}_j$ and $\widetilde{f}_{j+1}$ can no longer be analytically continued one to the other, neither across the upper nor across the lower half-plane. Correspondingly, the domains of holomorphy of the two analytic elements $(\widetilde{f}_j,E_j^\circ)$ and $(\widetilde{f}_{j+1},E_{j+1}^\circ)$ are distinct and must be separated by a '*membrane*' (cf. [11], [12], [119]). General theoretical observations (see [81], [112]), supported by numerical experiments (see [51] and also Figure 4), imply that then the limiting distribution of the zeros of the Hermite–Padé polynomials is characterized in terms of an equilibrium measure concentrated on the union of all the gaps (e_{2j},e_{2j+1}) and all such membranes, that is, its support no

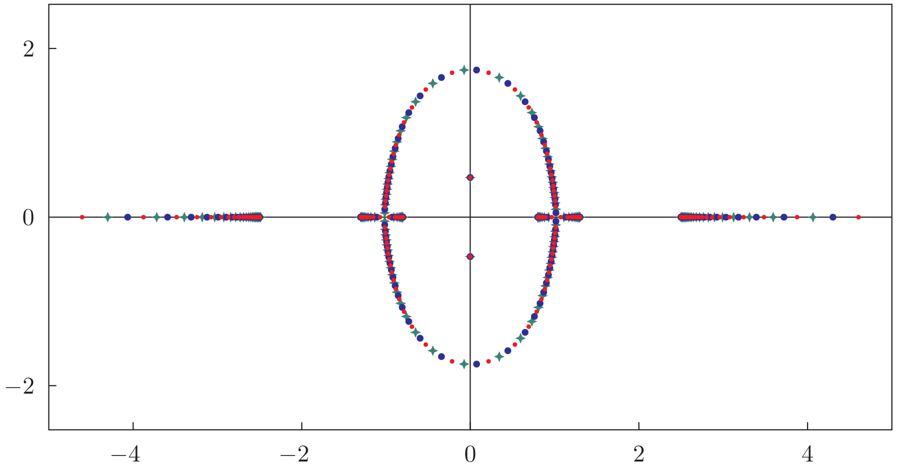

Figure 4. The zeros of the Hermite–Padé polynomials $Q_{200,0}$ (blue points), $Q_{200,1}$ (red points), and $Q_{200,2}$ (black points) for the system of functions $[1, f, f^2]$, where $f(z) = \left(\dfrac{z+2.5}{z+1.3}\right)^{1/3}\left(\dfrac{z+0.8}{z-0.8}\right)^{-1/3}\left(\dfrac{z-1.3}{z-2.5}\right)^{1/3}$. The zeros of the polynomials $Q_{200,j}$ form a membrane connecting the segments $[-1.3, -0.8]$ and $[0.8, 1.3]$. The two points where it intersects the real line segments correspond to the two Chebotarev points. Two Froissart triplets can be seen in the picture (cf. [13], [50], [51]).

longer lies on the real line but rather in the complex plane (mirror-symmetric with respect to the real line of course, as follows directly from the condition $f(\infty) = 1$ and the fact that all the branch points e_j and the parameters α_k are real).

The complexity of the general problem of the distribution of the zeros of the Hermite–Padé polynomials can already be seen in this simple case of a system of three functions $[1, f, f^2]$. Even here, when f is given by an explicit representation (56) with $e_1 < \dots < e_{2q}$ but with $\alpha_j \neq \alpha_k$ in general for $j \neq k$, we have not been able to characterize the limiting behaviour of the zeros of the Padé polynomials of the first kind $Q_{n,j}$ in terms of a corresponding potential-theoretic equilibrium problem. In particular, we have not been able to describe the structure and geometric properties of the membranes appearing. In other words, so far there is no sufficiently general result in the theory of Hermite–Padé polynomials which would give us a framework for solving the problem under consideration for a function of the form (56).

In contrast to the general case of arbitrary α_j with $2\alpha_j \in \mathbb{R} \setminus \mathbb{Z}$, in the special case when all the α_j are equal to some α with $2\alpha \in \mathbb{R} \setminus \mathbb{Z}$, as we have already mentioned, all the germs $(\widetilde{f}_j, E_j^\circ)$ are the 'traces' (on the distinct intervals E_j°) of the same branch of the multivalued analytic function $\widetilde{f}$ (the solution of the differential equation (2)), which is holomorphic in $G = \overline{\mathbb{C}} \setminus F$, that is, in the complex plane cut along finitely many segments of the real line.

We see that already in the case of distinct real parameters α_j with $2\alpha_j \in \mathbb{R}\setminus\mathbb{Z}$, even if we keep the same assumption $e_1 < \cdots < e_{2q}$ about the branch points in the representation (41), the potential-theoretic problem of an equilibrium measure η_E supported on E, which corresponds to the problem of the distribution of the zeros of the Hermite–Padé polynomials, is *complex* in the following sense. It keeps the form (49), but the corresponding compact set F does not in general coincide with the set $\overline{\mathbb{C}}\setminus E$. In this more general setting F is not known in advance: it is itself an *unknown parameter* of the problem. It follows from general results on the asymptotic behaviour of non-Hermitian orthogonal polynomials (see first of all [44], [110], [111], and also [8]) that this a priori unknown compact set F must have a certain characteristic 'symmetry' property connected in a natural way with the class of potentials under consideration and the family of *admissible* compact sets. In other words, F is an S-curve (see [60] and [91] for this notion) associated with a given class of potentials. Proving that such an S-curve exists and describing its characteristic properties is usually a very difficult problem, which up to now has been solved in only a few cases, namely:

1) for the logarithmic potential $V^\mu(z)$ and an arbitrary multivalued analytic function with a finite set of branch points (classical PA, Stahl [107]–[109]; see § 2.2);

2) for the logarithmic potential $V^\mu(z)$ with an external field $V_*^{-\nu}(z)$ equal to the potential of a unit negative measure concentrated at a finite set of points $\{z_1,\dots,z_m\}$ in the complex plane, and for an arbitrary multigerm with a finite set of branch points (m-point PA; Buslaev [21]–[23]; see § 2.3);

3) for the mixed Green-logarithmic potential $3V^\mu(z)+G^\mu_E(z)$ with external field $\psi(z)=3g_E(z,\infty)$, where E is a union of finitely many segments of the real line and $\mu\in M_1(K)$ with K in the corresponding family $\mathfrak{K}_f^{(3)}$ of admissible compact sets (Rakhmanov and Suetin [93] and [94]; see also [23] and [72]).

All three problems were solved using the variational method proposed in [88] and based on varying the energy functional corresponding to the problem of the limiting distribution of the zeros of the Padé polynomials. (Below, $\mathfrak{K}_f^{(1)}$, $\mathfrak{K}_f^{(2)}$, and $\mathfrak{K}_f^{(3)}$ are different classes of admissible compact sets for the multivalued function f).

In the classical case we have

$$\inf_{K\in\mathfrak{K}_f^{(1)}} \operatorname{cap} K = \exp\Bigl\{-\sup_{K\in\mathfrak{K}_f^{(1)}} J(K;\lambda_K)\Bigr\}, \tag{57}$$

where λ_K is the equilibrium measure for $K\in\mathfrak{K}_f^{(1)}$, and for $\mu\in M_1(K)$

$$J(K;\mu)=\iint \log\frac{1}{|z-\zeta|}\,d\mu(\zeta)\,d\mu(z)$$

is the corresponding energy functional.

In the case corresponding to multipoint PA, for the energy functional with an external field we have

$$\begin{aligned} I_\psi(K;\mu) &= \iint \biggl(\log\frac{1}{|z-\zeta|}+\psi(z)+\psi(\zeta)\biggr)\,d\mu(z)\,d\mu(\zeta) \\ &= \int_K \bigl(V^\mu(z)+2\psi(z)\bigr)\,d\mu(z), \end{aligned} \tag{58}$$

where $\psi(z) = V_*^{-\nu}(z)$ is the external field generated by the unit negative measure $-\nu$ with $\nu = \sum_{j=1}^m p_j\delta_{z_j}$ concentrated on the points $z_1,\dots,z_m$ of interpolation and $K \in \mathfrak{K}_f^{(2)}$.

Finally, in the case of Hermite–Padé approximants for a system $[1, f_1, f_2]$ with the pair f_1, f_2 forming a generalized Nikishin system, the corresponding energy functional has the form

$$J_3(K;\mu) = \iint\biggl(3\log\frac{1}{|x-y|} + g_K(x,y)\biggr)\,d\mu(x)\,d\mu(y) = \int_E \bigl(3V^\mu(x) + G_K^\mu(x)\bigr)\,d\mu(x) \tag{59}$$

and is considered in the class of measures $\mu \in M_1(E)$, where E is a union of finitely many segments of the real line and $g_K(x,y)$ is the Green's function for the admissible compact set $K \in \mathfrak{K}_f^{(3)}$.

Correspondingly, in each of the three cases an admissible S-compact set $F \in \mathfrak{K}_f^{(j)}$, $j \in \{1,2,3\}$, exists (and is unique) and is completely characterized by the following S-property (symmetry property):

1) in the first case $F = S \in \mathfrak{K}_f^{(1)}$ is the Stahl compact set, that is,

$$\frac{\partial g_D(\zeta,\infty)}{\partial n^+} = \frac{\partial g_D(\zeta,\infty)}{\partial n^-}, \qquad \zeta \in S^\circ, \tag{60}$$

where S° is the union of the open analytic arcs in S and $\partial/\partial n^\pm$ are the normal derivatives to the opposite sides of S° (this result is due to Stahl, but it can also be proved following the scheme proposed in [88]);

2) in the second case $F \in \mathfrak{K}_f^{(2)}$ has the S-property if

$$\frac{\partial(V^\beta + \psi)}{\partial n^+}(\zeta) = \frac{\partial(V^\beta + \psi)}{\partial n^-}(\zeta), \qquad \zeta \in F^\circ, \tag{61}$$

where $\beta \in M_1(F)$ is the corresponding equilibrium measure, $\psi(z) = V_*^{-\nu}(z)$ is the external field, F° is the union of all the open arcs in F, and $\partial/\partial n^\pm$ are the normal derivatives to the opposite sides of F° (Buslaev [21]);

3) finally, in the third case $F = F(3) \in \mathfrak{K}_f^{(3)}$ has the S-*property* (or is an S-curve) if

$$\frac{\partial G_F^{\eta_E}}{\partial n^+}(\zeta) = \frac{\partial G_F^{\eta_E}}{\partial n^-}(\zeta), \qquad \zeta \in F^\circ, \tag{62}$$

where $\eta_E \in M_1(E)$ is the corresponding equilibrium measure, F° is the union of all the open arcs in F, $\partial/\partial n^\pm$ are the normal derivatives to the opposite sides of F°, and G_F^μ is the Green potential (this result was proved in [73]).

The relations (60)–(62) may be different in appearance, but all the three define the S-curves corresponding to quite concrete problems.

3. Proof of Theorem 1

Theorem 1 is proved using the GRS-method. For brevity, in this paper we just give the scheme of the proof and only in the case when (101) holds. The general case is reduced to this case by means of an analogue of Lemma 5 in [94] (cf. [44], Lemma 9). The basics of the GRS-method were developed in 1981–1984 by Gonchar and Rakhmanov [41], [42], who first investigated the problem of the limiting distribution of the zeros of polynomials orthogonal with respect to a varying (that is, depending on the index of the polynomial) weight on the real line. This problem was solved in terms of *logarithmic* potentials with external fields (see also the 1997 monograph [96]). In 1985–1986 Stahl [107]–[111] investigated the problem of the limiting distribution of the zeros of polynomials orthogonal on an S-curve in connection with the problem of the convergence of the PA for multivalued analytic functions, although he did not consider varying weights. In 1987 Gonchar and Rakhmanov [44], in connection with the solution of the 'one-ninth conjecture' (see [124], and also `Wolfram MathWorld: ‘‘One-ninth constant’’`[15]) first formulated and investigated the problem of the distribution of the zeros of polynomials orthogonal on weighted S-curves.

In full accordance with the GRS-method, the proof of Theorem 1 is based on the orthogonality relations for the Hermite–Padé polynomials $Q_{n,2}$, that is, on the relations (94) (see also [94], cf. [66]). These are typical orthogonality relations with varying weight (see [41], [40], [42], [58], [94]). However, since the varying weight Ψ_n itself depends on an (arbitrary) polynomial ω_n (see (93)), the results of Theorem 1 cannot be deduced directly from the general results in [42]. Furthermore, in [42] the limiting distribution of the zeros of orthogonal polynomials is characterized in terms of equilibrium measures for logarithmic potentials with external fields. In this paper we show that the limiting distribution of the zeros of the Hermite–Padé polynomials coincides with the equilibrium measure for a *mixed* (Green-logarithmic) potential with an external field.

3.1. Proof of assertion 1) of Theorem 1. Let f be a function given by (41) with $\alpha\in(-1/2,1/2)$, $\alpha\neq 0$, and assume that the condition $f(\infty)=1$ fixes a branch of this function at $z=\infty$. Let $Q_{n,j}\in\mathbb{R}_n[z]$, $Q_{n,j}\not\equiv 0$, be the corresponding Hermite–Padé polynomials for the multi-index (n,n,n), so that (42) holds. It follows immediately from this relation that for an arbitrary polynomial $q\in\mathbb{P}_{2n}$ we have

$$\int_\Gamma (Q_{n,1}f+Q_{n,2}f^2)(z)q(z)\,dz=0 \tag{63}$$

(cf. (12)), where Γ is an arbitrary closed contour separating E from the point $z=\infty$. Let $P_{n,0}$ and $P_{n,1}$ be the Padé polynomials for f at the point at infinity, so that $P_{n,0},P_{n,1}\in\mathbb{R}_n[z]$, $P_{n,j}\not\equiv 0$, and

$$H_n(z):=(P_{n,0}+P_{n,1}f)(z)=O\biggl(\frac{1}{z^{n+1}}\biggr),\qquad z\to\infty, \tag{64}$$

where H_n is the remainder function. It follows from (64) that for any polynomial $p\in\mathbb{P}_{n-1}$

$$\int_\Gamma P_{n,1}(\zeta)f(\zeta)p(\zeta)\,d\zeta=0. \tag{65}$$

[15]The corresponding constant is known as the Halphen constant [44], [48] and also as the Varga constant; see `Wolfram MathWorld: ‘‘Varga’s constant’’`.

Now in (63) we set

$$q = P_{n+k,1}, \qquad k = 1, \dots, n. \tag{66}$$

Then by (65) we obtain from (63) the relation

$$\int_\Gamma (Q_{n,2} f^2)(\zeta) P_{n+k}(\zeta)\, d\zeta = 0, \qquad k = 1, \dots, n, \tag{67}$$

where we have denoted $P_{n+k,1}$ by P_{n+k}; we will use this notation throughout. Let

$$\widetilde{f}(x) = (f^+ + f^-)(x), \qquad x \in E^\circ. \tag{68}$$

Since $|\alpha| \in (0, 1/2)$, the function $\Delta(f^2)(x) = \widetilde{f}(x)\Delta f(x)$ is integrable on E. Hence, we can rewrite (67) equivalently as

$$\int_E Q_{n,2}(x)\widetilde{f}(x) P_{n+k}(x) \Delta f(x)\, dx = 0, \qquad k = 1, \dots, n. \tag{69}$$

Since $\alpha \in \mathbb{R}$, it follows from (41) that $\operatorname{const}\widetilde{f}(x)\Delta f(x) > 0$ for $x \in E^\circ$, where $\operatorname{const} \neq 0$. Below we will derive from the orthogonality relations (69) that all but perhaps finitely many of the zeros of the polynomials $Q_{n,2}$ lie in F°.

Now let $\gamma = \bigsqcup_{j=1}^q \gamma_j$ be the union of q pairwise disjoint closed analytic contours γ_j, $\gamma_j \cap \gamma_k = \varnothing$ for $j \neq k$, where each γ_j is symmetric with respect to the real line, passes through the points e_{2j-1} and e_{2j}, and encloses the open interval (e_{2j-1}, e_{2j}). We assume that all the contours γ_j are clockwise oriented. Since by the representation (41) all the functions $\widetilde{f}_j(x) := (f^+ + f^-)(x)$, $x \in E_j^\circ$, $j = 1, \dots, q$, are traces of one and the same function $\widetilde{f}$ which is holomorphic (single-valued analytic) in the domain $G = \overline{\mathbb{C}} \setminus F$, it follows that each $\widetilde{f}_j$ is holomorphic in the domain $G_j := \operatorname{int}\gamma_j \supset E_j^\circ$ and integrable on γ_j, $\widetilde{f}_j = \widetilde{f}\big|_{G_j}$. By (64) the jump ΔH_n of the remainder function H_n on E° is equal to $\Delta H_n(x) = P_n(x)\Delta f(x)$, $x \in E^\circ$. Thus, ΔH_n is an integrable function on E. Since $|\alpha| \in (0, 1/2)$, it also follows that H_n is integrable on $\gamma = \bigsqcup_{j=1}^q \gamma_j$. In the theory of orthogonal polynomials H_n is called a function of the second kind corresponding to the orthogonal polynomials P_n, and this is the term we use in what follows to avoid confusion with the remainder functions R_n (see (42)). It follows immediately from (64) that the function of the second kind H_n has the representation

$$H_n(z) = \frac{1}{p(z)} \frac{1}{2\pi i} \int_\Gamma \frac{P_n(\zeta) p(\zeta)}{\zeta - z} f(\zeta)\, d\zeta, \tag{70}$$

where $z \in \operatorname{ext}\Gamma$ (that is, z lies in the connected component of the complement of Γ containing the point at infinity), the contour Γ is clockwise oriented and encloses γ, and $p \in \mathbb{P}_n$ is an arbitrary polynomial of degree $\leqslant n$. The following properties of functions of the second kind are direct consequences of (64): a function of the second kind H_n has a zero of order $n+1$ at infinity; it is holomorphic outside E and makes an integrable jump ΔH_n on E; $\Delta H_n(x) = P_n(x)\Delta f(x)$ for $x \in E^\circ$; each of the q functions $\Delta H_n(x)$ with $x \in (e_{2j-1}, e_{2j})$ can be holomorphically extended from the interval E_j° to the domain G_j and is the trace in G_j of one and the same function $P_n\widehat{f}$ which is holomorphic in a domain $G \supset G_j$, $j=1,\dots,q$, $G=\overline{\mathbb{C}}\setminus F$; $\widehat{f}\,\big|_{E_j^\circ} = \Delta f(x) = f^+(x) - f^-(x)$ for $x \in E_j^\circ$.

By the hypotheses of Theorem 1 the function $H_n\widetilde{f}_j$ is integrable on γ_j. Hence, the integrals

$$\int_{\gamma_j} Q_{n,2}(\zeta)H_{n+k}(\zeta)\widetilde{f}_j(\zeta)\,d\zeta, \qquad \widetilde{f}_j \in \mathscr{H}(G_j), \quad \gamma_j = \partial G_j, \quad j=1,\dots,q, \tag{71}$$

are defined, and therefore so is the corresponding integral over $\gamma = \bigsqcup_{j=1}^q \gamma_j$,

$$\int_{\gamma} Q_{n,2}(\zeta)H_{n+k}(\zeta)\widetilde{f}(\zeta)\,d\zeta, \qquad k=1,2,\dots .$$

It is now easy to see that

$$\int_{\gamma} Q_{n,2}(\zeta)H_{n+k}(\zeta)\widetilde{f}(\zeta)\,d\zeta = \int_E Q_{n,2}(x)P_{n+k}(x)\widetilde{f}(x)\Delta f(x)\,dx.$$

By (69) the last integral vanishes for $k=1,\dots,n$. Hence,

$$\int_{\gamma} Q_{n,2}(\zeta)H_{n+k}(\zeta)\widetilde{f}(\zeta)\,d\zeta = 0, \qquad k=1,\dots,n, \tag{72}$$

where $\widetilde{f}\,\big|_{G_j} = \widetilde{f}_j$. Since H_{n+k} has a zero of order $n+k+1 \geqslant n+2$ at infinity, $Q_{n,2} \in \mathbb{P}_n$, and the function $\widetilde{f}$ is bounded in a neighbourhood of infinity, it follows from (72) that

$$\int_F Q_{n,2}(y)H_{n+k}(y)\Delta\widetilde{f}(y)\,dy = 0, \qquad k=1,\dots,n \tag{73}$$

(by the above conditions all the integrals in (73) exist). The collection of equalities (73) can be rewritten equivalently as

$$\int_F Q_{n,2}(y)\biggl(\sum_{k=1}^n c_kH_{n+k}(y)\biggr)\Delta\widetilde{f}(y)\,dy = 0, \tag{74}$$

where the c_k, $k=1,\dots,n$, are arbitrary complex numbers. We now use the well-known fact [120] that the functions of the second kind H_n, like the monic orthogonal polynomials $P_n(z)=z^n+\cdots$, satisfy the following three-term recurrence relations (cf. (13)):

$$H_n(z) = (z-\widehat{b}_n)H_{n-1}(z) - \widehat{a}_n^2H_{n-2}(z), \qquad n=1,2,\dots, \tag{75}$$

where $H_{-1}(z)\equiv 1$ and $H_0(z)\equiv f(z)$; furthermore, the conditions on f ensure that all the $\widehat{a}_n^2$ are positive. Let $n=2m$ be an even integer (the case of odd n is treated similarly). Then using (75), we get for arbitrary complex constants $c_1,\dots,c_n$ that

$$\sum_{k=1}^n c_kH_{n+k}(z) = q_{n,1}(z)H_{n+m}(z) + q_{n,2}(z)H_{n+m+1}(z), \tag{76}$$

where the polynomials $q_{n,1},q_{n,2}\in\mathbb{P}_{m-1}$ are arbitrary, since the constants $c_1,\dots,c_n$ are arbitrary. Thus, we can rewrite (74) equivalently as

$$\int_F Q_{n,2}(y)\{q_{n,1}(y)H_{n+m}(y)+q_{n,2}(y)H_{n+m+1}(y)\}\Delta\widetilde{f}(y)\,dy=0. \tag{77}$$

We underscore that, since the coefficients $c_1,\dots,c_n$ in (74) can be arbitrary, the polynomials $q_{n,1},q_{n,2}\in\mathbb{P}_{m-1}$ in (77) can also be chosen arbitrarily. We now rewrite (77) as

$$\int_F Q_{n,2}(y)\biggl\{q_{n,2}(y)\frac{H_{n+m+1}(y)}{H_{n+m}(y)}-q_{n,1}(y)\biggr\}H_{n+m}(y)\Delta\widetilde{f}(y)\,dy=0 \tag{78}$$

(in passing from (77) to (78) we have changed the sign of the *arbitrary* polynomial $q_{n,1}\in\mathbb{P}_{m-1}$).

Now we will use the fact that a function f given by (41) has a representation of the form $f(z)=1+\operatorname{const}\widehat{\sigma}(z)$, where $\operatorname{const}\neq 0$ and $\widehat{\sigma}$ is the Markov function corresponding to a positive measure σ with support in E. Hence, the ratio of the two remainder functions H_{n+1} and H_n for $n\geqslant 1$ has the representation

$$\frac{H_{n+1}}{H_n}(z)=\cfrac{\widehat{a}_{n+2}^2}{z-\widehat{b}_{n+2}-\cfrac{\widehat{a}_{n+3}^2}{z-\widehat{b}_{n+3}-\cdots}} \tag{79}$$

(see [120], Ya. L. Geronimus's supplement to the Russian edition, Chap. IV, formula (IV.5)), where all the $\widehat{a}_k$ and $\widehat{b}_k$ are real, $\widehat{a}_k\neq 0$, and $\{\widehat{b}_{n+2},\widehat{b}_{n+3},\dots\}\Subset\mathbb{R}$. Then Favard's theorem [120] implies that

$$\frac{H_{n+1}}{H_n}(z)=\widehat{\sigma}_n(z),$$

where σ_n is a positive measure with $\operatorname{supp}\sigma_n\Subset\mathbb{R}$. Let $\widehat{H}_n(z)$ be the functions of the second kind corresponding to the orthogonal polynomials $\widehat{P}_n(z)=\widehat{P}_{n,1}(z)$. The following auxiliary result refines the properties of the measure σ_n.

Lemma 1. *Under the conditions of Theorem 1 the ratio H_{n+1}/H_n has the representation*

$$\frac{H_{n+1}}{H_n}(z)=\widehat{\rho}_n(z)+\sum_{j=1}^{q-1}\frac{c_{n,j}}{z-x_{n,j}}=\int_E\frac{\rho_n(x)\,dx}{z-x}+\sum_{j=1}^{q-1}\frac{c_{n,j}}{z-x_{n,j}}\,,\qquad z\notin E, \tag{80}$$

where all the $c_{n,j}$ are non-negative, the points $x_{n,j}$ lie on the segment $[e_{2j},e_{2j+1}]$, and

$$\rho_n(x)=-\frac{1}{2\pi i}\frac{\Delta f(x)}{\widehat{H}_n^+(x)\widehat{H}_n^-(x)}\,,\qquad x\in E^\circ. \tag{81}$$

Proof. Writing the recurrence relations (75) at distinct points z and ζ as

$$H_{n+1}(z)=(z-\widehat{b}_{n+1})H_n(z)-\widehat{a}_{n+1}^2H_{n-1}(z),$$
$$H_{n+1}(\zeta)=(\zeta-\widehat{b}_{n+1})H_n(\zeta)-\widehat{a}_{n+1}^2H_{n-1}(\zeta),$$

we find that

$$
\begin{aligned}
H_{n+1}(z)H_n(\zeta) - H_{n+1}(\zeta)H_n(z) &= (z-\zeta)H_n(z)H_n(\zeta) \\
&\quad - \widehat{a}_{n+1}^2\{H_n(\zeta)H_{n-1}(z) - H_n(z)H_{n-1}(\zeta)\}.
\end{aligned}
\tag{82}
$$

It follows directly from (82) that for $x \in E^\circ$ the corresponding limit values of the functions of the second kind are related by

$$
\begin{aligned}
H_{n+1}^+(x)H_n^-(x) - H_{n+1}^-(x)H_n^+(x) &= \prod_{k=1}^{n+1} \widehat{a}_k^2 \cdot \Delta H_0(x) \\
&= \prod_{k=1}^{n+1} \widehat{a}_k^2 \cdot \Delta f(x) = \frac{\Delta f(x)}{k_n^2}, \qquad x \in E^\circ;
\end{aligned}
\tag{83}
$$

here $k_n > 0$ is the leading coefficient of the corresponding polynomial $\widehat{P}_n(x) = k_n x^n + \dots = k_n P_n(x)$ which is orthonormal with respect to a positive measure $d\sigma(x) = \operatorname{const}\Delta f(x)\,dx$, $x \in E$. We get immediately from (83) that

$$
\frac{H_{n+1}^+}{H_n^+}(x) - \frac{H_{n+1}^-}{H_n^-}(x) = \frac{\Delta f(x)}{\widehat{H}_n^+(x)\widehat{H}_n^-(x)}\sigma_n'(x), \qquad x \in E^\circ, \tag{84}
$$

where $\widehat{H}_n$ is the function of the second kind corresponding to the orthonormal polynomial $\widehat{P}_n$. The polynomials P_n are orthogonal on E with respect to the positive measure $\sigma(x) = \operatorname{const}\Delta f(x)\,dx$. Hence, all the zeros of P_n lie in the convex hull $\widehat{E}$ of E. Furthermore, each gap (e_{2j}, e_{2j+1}), $j = 1, \dots, q-1$, contains at most one zero of P_n. It now follows directly from (70) that the corresponding function of the second kind H_n, apart from the zero of order $n+1$ at infinity, can have at most $q-1$ additional zeros (which are additional points of interpolation of the function f by the Padé approximant $[n/n]_f$), which lie in the $q-1$ gaps between the segments E_j, at most one in each gap. It follows from the above and (84) that

$$
\frac{H_{n+1}}{H_n}(z) = \widehat{\rho}_n(z) + \sum_{j=1}^{q-1} \frac{c_{n,j}}{z - x_{n,j}}, \tag{85}
$$

where all the $c_{n,j}$ are non-negative (because σ_n is a positive measure), the points $x_{n,j}$ lie in $[e_{2j}, e_{2j+1}]$, and

$$
\rho_n(x) = \sigma_n'(x) = -\frac{1}{2\pi i}\,\frac{\Delta f(x)}{\widehat{H}_n^+(x)\widehat{H}_n^-(x)}, \qquad x \in E^\circ. \tag{86}
$$

The proof is complete. □

It follows from [74] that if $f \in \mathscr{L}_E$, then

$$
\frac{\widehat{H}_n^+(x)\widehat{H}_n^-(x)}{\prod_{j=1}^{q-1}(x - x_{n,j})} \rightrightarrows 1, \qquad n \to \infty, \quad x \in K \Subset E^\circ. \tag{87}
$$

Therefore, $d\sigma_n(x)=\rho_n(x)\,dx+\sum_{j=1}^{q-1}\delta_{x_{n,j}}$ by Lemma 1, and by (78)

$$\int_F Q_{n,2}(y)\bigl\{q_{n,2}(y)\widehat{\rho}_n(y)-q_{n,1}(y)\bigr\}H_{n+m}(y)\tau_n(y)\Delta\widetilde{f}(y)\,dy=0, \tag{88}$$

where

$$\tau_n(z):=\prod_{j=1}^{q-1}(z-x_{n,j})$$

and $q_{n,1},q_{n,2}\in\mathbb{P}_{m-q}$ are arbitrary polynomials of degree $\leqslant m-q$. Although we lost something with regard to the degrees of the arbitrary polynomials $q_{n,1}$ and $q_{n,2}$ in going over from (78) to (88), we did get rid of the rational term in (80).

Now let $\omega_n\in\mathbb{P}_{2m-2q+1}$ be a monic polynomial of degree $\leqslant 2m-2q+1$ with only simple zeros which lie in F°, and let ω_n^* be the corresponding spherically normalized polynomial. We select polynomials $q_{n,1},q_{n,2}\in\mathbb{P}_{m-q}$ so that the following function is holomorphic on F°:

$$\frac{q_{n,2}(z)\widehat{\rho}_n(z)-q_{n,1}(z)}{\omega_n^*(z)}\,; \tag{89}$$

obviously, this condition is equivalent to certain interpolation relations, and in view of the number of free parameters we can always satisfy these relations by our choice of the polynomials $q_{n,1},q_{n,2}\in\mathbb{P}_{m-q}$. It follows from (89) that the polynomials $q_{n,2}$ are orthogonal on E with respect to the varying measure $d\rho_n(x)/\omega_n^*(x)$. Namely,

$$\int_E q_{n,2}(x)x^s\,\frac{d\rho_n(x)}{\omega_n^*(x)}=0,\qquad s=0,1,\dots,m-q-1. \tag{90}$$

It follows from (90) that all the zeros of $q_{n,2}$ lie in $\operatorname{conv}E$; more precisely, all but possibly at most $q-1$ of the zeros of this polynomial lie in E, and we have the representation

$$\frac{q_{n,2}(z)\widehat{\rho}_n(z)-q_{n,1}(z)}{\omega_n^*(z)}=\frac{1}{p(z)}\int_E\frac{q_{n,2}(x)p(x)\,d\rho_n(x)}{\omega_n^*(x)(z-x)}\,,\qquad z\notin E, \tag{91}$$

where $p(z)\in\mathbb{P}_{m-q}$ is an arbitrary polynomial of degree $\leqslant m-q$. In view of (91), we can rewrite (88) as

$$0=\int_F Q^*_{n,2}(y)\frac{\omega_n^*(y)}{q_{n,2}(y)}\biggl\{\int_E\frac{q^2_{n,2}(x)\,d\rho_n(x)}{\omega_n^*(x)(y-x)}\biggr\}H_{n+m+1}(y)\tau_n(y)\Delta\widetilde{f}(y)\,dy. \tag{92}$$

We stress that ω_n^* in (92) is an arbitrary polynomial of degree $\leqslant 2m-2q+1$. Let

$$\begin{aligned}\Psi_n(z)&:=\frac{1}{q_{n,2}(z)}\biggl\{\int_E\frac{q^2_{n,2}(x)\,d\rho_n(x)}{\omega_n^*(x)(z-x)}\biggr\}H_{n+m+1}(z)\tau_n(z)\Delta\widetilde{f}(z)\\&=\frac{1}{\widetilde{q}_{n,2}(z)}\biggl\{\int_E\frac{q_{n,2}(x)\widetilde{q}_{n,2}(x)\,d\rho_n(x)}{\omega_n^*(x)(z-x)}\biggr\}H_{n+m+1}(z)\tau_n(z)\Delta\widetilde{f}(z),\qquad z\notin E,\end{aligned} \tag{93}$$

where $\widetilde{q}_{n,2} \in \mathbb{P}_{m-q}$. Then the relations (92) take the form

$$\int_F Q^*_{n,2}(y)\omega^*_n(y)\Psi_n(y)\,dy = 0 \tag{94}$$

for an arbitrary $\omega^*_n \in \mathbb{P}_{2m-2q+1}$, where the function Ψ_n takes the part of a varying weight (depending on n). In view of the definition (93), the function Ψ_n in (94) depends itself on ω^*_n. It is easy to see that if the polynomials ω^*_n and $q_{n,2}$ in (93) are real, then Ψ_n is also a real function, and it can change sign on F only finitely many times, with number independent of n. From this it now follows easily that all the zeros of $Q_{n,2}$, apart from some number of zeros which does not depend on n, lie in F, and the degree of $Q_{n,2}$ can be less than n only by some number that is bounded for $n \to \infty$.

Remark 6. The problem treated in this paper possesses a certain symmetry from the outset, namely, a real symmetry. Therefore, the relations (94) can be viewed as quasi-orthogonality relations with varying weight Ψ_n which depends on the index of the polynomial $Q_{n,2}$. These relations can well be taken as the starting point in the use of the general method developed by Gonchar and Rakhmanov in [41]–[44] in connection with their solution of the 'one-ninth conjecture', and intended for an investigation of the limiting distribution of the zeros of (non-Hermitian) orthogonal polynomials with varying weight. It is well known (see, first of all, the original paper [44], and also [32] and [40]) that this method is based on the existence of a compact set which has a certain symmetry in the context of the problem under consideration, that is, the existence of a so-called S-curve associated with this problem. We are looking at a problem in which the existence of an S-curve (or more accurately, a pair E, F of S-curves) follows from the very formulation of the problem, namely, from its real symmetry. Specifically, since the problem to be solved possesses a real symmetry, there exists a pair of S-curves E and F which form a so-called Nuttall condenser (E,F) with mirror-symmetric plates E and F (regarding this concept, see first of all [93], where it was first introduced, and also [57] and [94]). The fact that in (94) the varying weight Ψ_n itself depends on the polynomial ω^*_n, does not in general restrict the applicability of the GRS-method (in this connection see [94], and also [92], Conjecture 2).

We now consider the following equilibrium problem (see [24], [93], [94]).

Equilibrium problem 1. *Let $\mu \in M_1(F)$ be an arbitrary unit (positive Borel) measure μ with support on F. Let $V^\mu_*(z)$, $z \notin F$, denote the spherically normalized logarithmic potential and let*

$$G^\mu_E(z) := \int_F g_E(x,z)\,d\mu(x), \qquad z \notin F,$$

be the Green potential of the measure μ (with respect to D), where $g_E(x,z)$ is the Green's function of D. Here and in what follows we assume that all these functions exist, and without loss of generality we assume that $\operatorname{conv} E = [-1,1]$.

We consider the following extremal problem for the mixed (Green-logarithmic) potential $3V^\mu_*(z) + G^\mu_E(z)$ of the measure $\mu \in M_1(F)$ with external field $\psi(z) = 3g_E(z,\infty)$:

$$\inf_{\mu \in M_1(F)} J_\psi(\mu), \tag{95}$$

where

$$J_\psi(\mu)=\int_F\bigl(3V^\mu_*(y)+G^\mu_E(y)\bigr)\,d\mu(y)+6\int_F g_E(y,\infty)\,d\mu(y) \tag{96}$$

is (double) the energy of the measure μ with potential $3V^\mu_*(z)+G^\mu_E(z)$ in the external field $\psi(z)=3g_E(z,\infty)$. Using standard methods in potential theory (see [42], [44], [63], [96]) we can prove that in the problem (95) there exists a unique extremal measure $\eta_F\in M_1(F)$ with support on F, that is, a measure such that

$$J_\psi(\eta_F)=\inf_{\mu\in M_1(F)}J_\psi(\mu). \tag{97}$$

This extremal measure η_F is also the unique equilibrium measure for the above potential with external field ψ (see [42]), that is, the following equilibrium relations hold:

$$3V^{\eta_F}_*(y)+G^{\eta_F}_E(y)+3g_E(y,\infty)\equiv\text{const}=w_F,\qquad y\in F, \tag{98}$$

where w_F is an equilibrium constant.

We assert that

$$\frac1n\chi(Q_{n,2})\xrightarrow{*}\eta_F,\qquad n\to\infty. \tag{99}$$

According to the GRS-method, assume that this fails. Since all but finitely many of the zeros of the polynomials $Q_{n,2}$ lie in F and $\deg Q_{n,2}\geqslant n-\text{const}$ with const independent of n, for some infinite subsequence $\Lambda\subset\mathbb N$ we have

$$\frac1n\chi(Q_{n,2})\xrightarrow{*}\mu^Q=\mu\ne\eta_F,\qquad n\to\infty,\quad n\in\Lambda, \tag{100}$$

where $|\mu^Q|=1$ and $\operatorname{supp}\mu^Q\subset F$. And since $\mu\ne\eta_F$, $\mu\in M_1(F)$, and the equilibrium measure η_F is unique (in the class $M_1(F)$), it follows from (100) that the equilibrium relations (98) cannot hold for μ, so that $3V^\mu_*(y)+G^\mu_E(y)+3g_E(y,\infty)\not\equiv$ const for $y\in F$. The function $u(z):=3V^\mu_*(z)+G^\mu_E(z)+3g_E(z,\infty)$ is lower semicontinuous on F, hence it attains its minimum on F at some point $y_0\in F$: $u(y_0)=\min_{y\in F}u(y)=\mathfrak m$. By the foregoing, $u(y)\not\equiv\mathfrak m=u(y_0)$, $y\in F$. We assume that y_0 lies in the interior of F, $y_0\in F^\circ$, and is the unique minimum point of $u(y)$ on F, that is,

$$\mathfrak m=\min_{t\in F}u(t)=u(y_0)<u(y),\qquad y\in F\setminus\{y_0\}. \tag{101}$$

We underscore that this assumption involves no loss of generality in our further arguments, because in the framework of the GRS-method the general case when $u(y)$ attains its minimum on F at several points or when the unique minimum point is an endpoint e_j of F can be reduced to this case in a standard way, with the help of analogues of Lemma 9 in [44] (see [94], Lemma 5). Now we take a polynomial ω^*_n in (89) such that

$$\frac1n\chi(\omega^*_n)\xrightarrow{*}\mu^Q=\mu,\qquad n\to\infty. \tag{102}$$

Then using the GRS-method in the standard fashion, we get from the orthogonality relations (90) for the polynomial $q_{n,2}$ that (recall that $n=2m$)

$$\frac2n\chi(q_{n,2})\to\nu,\qquad n\to\infty, \tag{103}$$

where $\nu \in M_1(E)$ and $\nu = \widetilde{\mu} = \mathfrak{b}_E(\mu)$ is the balayage of $\mu \in M_1(F)$ from D to the boundary $\partial D = E$. Thus, we have $V^\nu_*(z) \equiv V^\mu_*(z) - G^\mu_E(z) + \text{const}$, $z \in \overline{\mathbb{C}}$. Due to the GRS-method

$$\biggl|\int_E \frac{q^2_{n,2}(x)\,d\rho_n(x)}{\omega^*_n(x)(y-x)}\biggr|^{1/n} \xrightarrow{\text{cap}} e^{\text{const}} \neq 0, \infty, \qquad y \in F^0, \tag{104}$$

and therefore application of the GRS-method to the expression (93) defining the varying weight Ψ_n gives us that

$$|\Psi_n(y)|^{2/n} \xrightarrow{\text{cap}} e^{V^\nu_*(y) - 3g_E(z,\infty)} = e^{V^\mu_*(z) - G^\mu_E(z) - 3g_E(z,\infty) + \text{const}}. \tag{105}$$

Consequently, in view of (102), (103), and the identity $V^\nu_*(z) \equiv V^\mu_*(z) - G^\mu_E(z) + \text{const}$ we deduce from (92) the limit relation

$$\begin{aligned} \bigl|Q^*_{n,2}(y)\omega^*_n(y)\Psi_n(y)\bigr|^{2/n} &\xrightarrow{\text{cap}} e^{-4V^\mu_*(y) + V^\nu_*(y) - 3g_E(y,\infty)} \\ &= e^{-4V^\mu_*(y) + V^\mu_*(y) - G^\mu_E(y) - 3g_E(y,\infty) + \text{const}} \\ &= e^{-3V^\mu_*(y) - G^\mu_E(y) - 3g_E(y,\infty) + \text{const}} \\ &= e^{-u(y) + \text{const}}. \end{aligned} \tag{106}$$

Hence, since $u(y_0) < u(y)$ for $y \in F \setminus \{y_0\}$, in the limit as $n \to \infty$ the absolute value of the integrand in (92), raised to the power $2/n$, has a unique strict maximum on F at the point $y_0 \in F^\circ$. Using standard methods of potential theory (see [44], Lemma 7, and [63]), we find that

$$\begin{aligned} \biggl(\int_F \bigl|Q^*_{n,2}(y)\omega^*_n(y)\Psi_n(y)\bigr|\,|dy|\biggr)^{2/n} &\to e^{-3V^\mu_*(y_0) - G^\mu_E(y_0) - 3g_E(y_0,\infty) + \text{const}} \\ &= e^{-u(y_0) + \text{const}} \neq 0. \end{aligned} \tag{107}$$

Let us now return to (94). It is our next goal to show that there exists a sequence of polynomials ω^*_n which still has the properties (102) and (107), but is such that the absolute value of the integral on the left-hand side of (94), raised to the power $2/n$, behaves asymptotically like the left-hand side of (107), that is,

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n\in\Lambda}} \biggl|\int_F Q^*_{n,2}(y)\omega^*_n(y)\Psi_n(y)\,dy\biggr|^{2/n} &= \lim_{\substack{n\to\infty \\ n\in\Lambda}} \biggl(\int_F \bigl|Q^*_{n,2}(y)\omega^*_n(y)\Psi_n(y)\bigr|\,|dy|\biggr)^{2/n} \\ &= e^{-u(y_0) + \text{const}} \neq 0. \end{aligned} \tag{108}$$

Obviously, once we have deduced (108), we will have arrived at a contradiction to (94).

The limiting distribution of the zeros of the polynomials $\omega^*_n \in \mathbb{P}_{n-\text{const}}$, where the constant const depends on q but not on n, is the same as that of the polynomials $\widetilde{\omega}_n$, which differ from the polynomials ω^*_n by the absence of some $k_n \in \mathbb{N}$ factors, where $k_n = o(n)$ as $n \to \infty$. Since the function u has a strict minimum at y_0, we obviously have $\mu(\{y_0\}) = 0$. Thus, there exists a sufficiently small positive ε such that the number of zeros of $Q_{n,2}$ outside the ε-neighbourhood $U_\varepsilon(y_0)$ of y_0 grows unboundedly with n. We fix this $\varepsilon > 0$. Then for each $n \in \Lambda$ there exists a monic polynomial g_n of fixed degree independent of n which has the following properties: g_n divides $Q^*_{n,2}$, $\omega^*_n = Q^*_{n,2}/g_n \in \mathbb{R}_{n-l}[z]$, and g_n has constant sign in $U_\varepsilon(y_0)$. The function $\Psi_n(z)$ is independent of the choice of $\widetilde{q}_{n,2} \in \mathbb{P}_{m-q}$ in (93). Now that we have found the limiting distribution of the zeros of the polynomials $q_{n,2}$ as $n \to \infty$ and have shown that, with the possible exception of $o(n)$ zeros, all of them lie in E, we can remove the corresponding factors from $\widetilde{q}_{n,2}$, thus transforming it into a polynomial with constant sign on F. Then (94) takes the form

$$\int_F Q^{*2}_{n,2}(y)\frac{\Psi_n(y)}{g_n(y)}\,dy = 0, \tag{109}$$

where the integrand has constant sign in $U_\varepsilon(y_0)$. Without loss of generality we assume below that it is positive in $U_\varepsilon(y_0)$. We rewrite (109) as

$$\int_{U_\varepsilon(y_0)} Q^{*2}_{n,2}(y)\frac{\Psi_n(y)}{g_n(y)}\,dy = -\int_{F\setminus\overline{U}_\varepsilon(y_0)} Q^{*2}_{n,2}(y)\frac{\Psi_n(y)}{g_n(y)}\,dy. \tag{110}$$

Denote the integral on the left-hand side of (110) by I_1 and the one on the right-hand side by I_2. Since $u(y_0) < u(y)$ for $y \in F \setminus \{y_0\}$,

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}} |I_1|^{2/n} = \lim_{\substack{n\to\infty\\ n\in\Lambda}}\biggl(\int_{U_\varepsilon(y_0)}\biggl|Q^{*2}_{n,2}(y)\frac{\Psi_n(y)}{g_n(y)}\biggr|\,|dy|\biggr)^{2/n} = e^{-u(y_0)} = e^{-\mathfrak{m}}. \tag{111}$$

And since u has a unique minimum at y_0, we get for I_2 that

$$\varlimsup_{\substack{n\to\infty\\ n\in\Lambda}} |I_2|^{2/n} < e^{-\mathfrak{m}}. \tag{112}$$

In combination, the relations (111) and (112) contradict (110). Therefore, our assumption that

$$\frac{1}{n}\chi(Q_{n,2}) \xrightarrow{*} \mu \neq \eta_F, \qquad n \to \infty, \quad n \in \Lambda,$$

has brought us to a contradiction and thus is false. That is, we have proved that

$$\lim_{n\to\infty} \frac{1}{n}\chi(Q_{n,2}) = \eta_F.$$

The solution of the problem of the limiting distribution of the zeros of the polynomials $Q_{n,2}$ is complete.

Since the polynomials $Q_{n,0}$ correspond to the polynomials $Q_{n,2}$ but for the function $1/f$, that is, $Q_{n,0}(z; f) = Q_{n,2}(z; 1/f)$, we have also solved the problem of the distribution of the zeros of the former.

3.2. Proof of assertions 2) and 3) of Theorem 1. Next we look at the problem of the limiting distribution of the zeros of the polynomials $Q_{n,1}$. We solve it in tandem with the problem of the interpolation of $f_2 = -\widetilde{f}$ by the rational function $Q_{n,1}/Q_{n,2}$ and of the convergence of the corresponding sequence of rational functions.

We now investigate the limiting distribution of the points of interpolation. Recall that $\widetilde{f}_j = \widetilde{f}|_{E_j^\circ} = f^+ + f^-$, $j=1,\dots,q$, $\widetilde{f}\in\mathscr{H}(G)$. Let us return to (63) and rewrite it as

$$\int_E (Q_{n,1}+Q_{n,2}\widetilde{f})(x)q(x)\Delta f(x)\,dx = 0, \tag{113}$$

where $q\in\mathbb{P}_{2n}$ is an arbitrary polynomial. It follows from the representation (56) for f and the condition $f(\infty)=1$ that $\widetilde{f}$ is a real-valued function on E, and for some non-trivial constant const $\neq 0$ we have $\operatorname{const}\Delta f(x)\geqslant 0$ for $x\in E$. Since $Q_{n,j}\in\mathbb{R}[x]$, it follows directly from the orthogonality relations (113) that $Q_{n,1}+Q_{n,2}\widetilde{f}$ has at least $2n+1$ simple zeros on $\widehat{E}=\operatorname{conv}E$, with at most one zero in each gap. Therefore, this function has at least $2n+2-q$ simple zeros on E. Let $\Omega_n(z)=z^{\deg\Omega_n}+\cdots$ be the corresponding monic polynomial with simple zeros at these points; then $2n+2-q\leqslant\deg\Omega_n\leqslant 2n+1$. By the definition of Ω_n the function $(Q_{n,1}+Q_{n,2}\widetilde{f})/\Omega_n$ is holomorphic in E° and therefore also in $G:=\overline{\mathbb{C}}\setminus F$. Hence, for any polynomial $q\in\mathbb{P}_{n-q}$ we have

$$0=\int_\gamma \frac{(Q_{n,1}+Q_{n,2}\widetilde{f})(t)q(t)\,dt}{\Omega_n(t)} = \int_\gamma \frac{Q_{n,2}(t)q(t)\widetilde{f}(t)\,dt}{\Omega_n(t)}, \tag{114}$$

and the following representation holds:

$$\frac{(Q_{n,1}+Q_{n,2}\widetilde{f})(z)}{\Omega_n(z)} = \frac{1}{2\pi i\widetilde{Q}_{n,2}(z)}\int_\gamma \frac{Q_{n,2}(t)\widetilde{Q}_{n,2}(t)\widetilde{f}(t)\,dt}{\Omega_n(t)(t-z)}, \qquad z\in\operatorname{int}\gamma, \tag{115}$$

where the monic polynomial $\widetilde{Q}_{n,2}$ differs from $Q_{n,2}$ only by the removal of finitely many simple factors, namely, of those with zeros outside F. The integrals in (114) and (115) are taken over a curve γ consisting of a finite number q of curves γ_j separating the zeros of Ω_n and the point z from the point at infinity. In view of the properties of $\widetilde{f}$, the relations (114) and (115) can be rewritten equivalently in the respective forms (cf. [90], (15) and (16))

$$\int_F \frac{Q^*_{n,2}(y)q(y)\Delta\widetilde{f}(y)\,dy}{\Omega_n(y)} = 0 \quad \forall\, q\in\mathbb{P}_{n-q}, \tag{116}$$

$$\frac{(Q_{n,1}+Q_{n,2}\widetilde{f})(z)}{\Omega_n(z)} = \frac{1}{2\pi i\widetilde{Q}^*_{n,2}(z)}\int_F \frac{Q_{n,2}(y)\widetilde{Q}^*_{n,2}(y)\Delta\widetilde{f}(y)\,dy}{\Omega_n(y)(y-z)}, \tag{117}$$

$$\widetilde{Q}_{n,2}\in\mathbb{P}_{n-q}, \qquad z\notin F$$

(we note that $\Delta\widetilde{f}(z)=\operatorname{const}f(z)$, $z\in D$, where const $\neq 0$). Assume that as $n\to\infty$, $n\in\Lambda$,

$$\frac{1}{2n}\chi(\Omega_n)\to\nu, \qquad \operatorname{supp}\nu\subset E, \quad |\nu|=1.$$

Then according to the GRS-method the orthogonality relations (113) and (116) (see also (118) below) imply that $n^{-1}\chi(Q_{n,2})\to\widetilde{\nu}\in M_1(F)$, where $\widetilde{\nu}=\mathfrak{b}_F(\nu)$ is the balayage of ν from the domain G to F. But we know already that $n^{-1}\chi(Q_{n,2})\to\eta_F$, so $\widetilde{\nu}=\eta_F$. Using (117), we can rewrite the orthogonality relations (113) as

$$\int_E \Omega_n(x)q(x)\Psi_n(x)\Delta f(x)\,dx = 0 \qquad \forall\, q\in\mathbb{P}_{2n-q}, \tag{118}$$

where

$$\Psi_n(z) := \frac{1}{\widetilde{Q}^*_{n,2}(z)} \int_F \frac{Q^*_{n,2}(y)\widetilde{Q}^*_{n,2}(y)\Delta\widetilde{f}(y)\,dy}{\Omega_n(y)(y-z)}, \qquad z \notin F, \tag{119}$$

is a varying weight which depends on the polynomial Ω_n itself. However, this does not hinder our use of the GRS-method, since we know that all the zeros of Ω_n lie in E and $2n+2-q \leqslant \deg\Omega_n \leqslant 2n+1$, and if $(2n)^{-1}\chi(\Omega_n) \to \nu$ as $n \to \infty$ for $n \in \Lambda$, then $\operatorname{supp}\nu \subset E$ and $\widetilde{\nu} = \eta_F$. Hence, the GRS-method shows that

$$|\Psi_n(z)|^{1/n} \xrightarrow{\text{cap}} e^{V_*^{\eta_F}(z)+\text{const}}, \qquad z \notin F. \tag{120}$$

We now prove that if $(2n)^{-1}\chi(\Omega_n) \to \nu$, then $\nu = \eta_E$, where $\eta_E \in M_1(E)$ is the unique extremal measure for the following potential-theoretic problem.

Equilibrium problem 2. *Let $3V^\nu(z)+G^\nu_F(z)$ be a mixed Green-logarithmic potential of an arbitrary (positive Borel) unit measure ν with support on E, $\nu \in M_1(E)$, where $G^\nu_F(z) := \int_E g_F(x,z)\,d\nu(x)$ is the Green potential (with respect to F) of ν and $g_F(x,z)$ is the Green's function of the domain $G = \overline{\mathbb{C}} \setminus F$. Let*

$$\begin{aligned} J(\nu) &:= \int_{E\times E}\biggl(3\log\frac{1}{|x-u|} + g_F(x,u)\biggr)\,d\nu(x)\,d\nu(u) \\ &= \int_E (3V^\nu(x) + G^\nu_F(x))\,d\nu(x) \end{aligned}$$

be the corresponding energy functional. Consider the following extremal problem:

$$\mathfrak{m} = \inf_{\nu\in M_1(E)} J(\nu). \tag{121}$$

Using standard methods of potential theory, one can prove as in [42] (see also [63], [96]) that there exists a unique (in $M_1(E)$) extremal measure $\eta_E \in M_1(E)$ which delivers the minimum in (121):

$$J(\eta_E) = \min_{\nu\in M_1(E)} J(\nu).$$

The measure η_E is also the unique equilibrium measure for the above potential, that is, the following equilibrium conditions hold for this measure:

$$3V^{\eta_E}(x) + G^{\eta_E}_F(x) \equiv \text{const} = w_E, \qquad x \in E, \tag{122}$$

where w_E is an equilibrium constant. It was shown in [25] (see also [23], [72]) that $\widetilde{\eta}_E = \mathfrak{b}_F(\eta_E) = \eta_F$ (from which it follows, in particular, that $\operatorname{supp}\eta_F = F$). Consequently, $V^{\eta_F}(z) \equiv V^{\eta_E}_*(z) - G^{\eta_E}_F(z) + \text{const}$, $z \in \overline{\mathbb{C}}$.

Since the problem under consideration in this paper has real symmetry, $\operatorname{supp}\eta_E \subset E$, and $\operatorname{supp}\eta_F \subset F$, it follows that

$$\frac{\partial(V^{\eta_E} - V^{\eta_F}_*)}{\partial n^+}(x) = \frac{\partial(V^{\eta_E} - V^{\eta_F}_*)}{\partial n^-}(x), \qquad x \in E^\circ,$$

that is, the compact set E is an S-curve in the external field generated by the potential of the signed measure $-\eta_F$. The polynomials Ω_n are quasi-orthogonal on E

(see (118)) with respect to the varying weight Ψ_n, which satisfies the asymptotic relation (120). Since $\deg\Omega_n/n \to 2$ as $n\to\infty$, we immediately get from the Gonchar–Rakhmanov theorem ([44], Theorem 3), which is a particular case of the general GRS-method, that there is a limiting distribution of the zeros of the polynomials Ω_n, and it coincides with the equilibrium measure $\mu^{\mathrm{eq}}\in M_1(E)$ on E in the external field $\psi(z)=-(1/4)V_*^{\eta_F}(z)$, that is,

$$\frac{1}{2n}\chi(\Omega_n)\to\mu^{\mathrm{eq}}\in M_1(E),\qquad n\to\infty,$$

where $\mu^{\mathrm{eq}}\in M_1(E)$ is the unique measure such that

$$V_*^{\mu^{\mathrm{eq}}}(x)-\frac{1}{4}V_*^{\eta_F}(x)\equiv\mathrm{const},\qquad x\in E. \tag{123}$$

In view of the equality $\eta_F=\widetilde{\eta}_E$, which means that $V^{\eta_F}(z)=V_*^{\eta_E}(z)-G_F^{\eta_E}(z)+\mathrm{const}$, (123) is clearly equivalent to the following equilibrium relation:

$$4V_*^{\mu^{\mathrm{eq}}}(x)-V^{\eta_E}(x)+G_F^{\eta_E}(x)\equiv\mathrm{const},\quad x\in E. \tag{124}$$

Since the solution of the equilibrium problem (123) is unique, (124) is also uniquely solvable. It follows immediately from (122) and (123) that $\mu^{\mathrm{eq}}=\eta_E$. Thus, we have proved that

$$\frac{1}{2n}\chi(\Omega_n)\xrightarrow{*}\eta_E,\qquad n\to\infty.$$

We see that there is a limiting distribution of the points of interpolation of $\widetilde{f}$ by the rational function $Q_{n,1}/Q_{n,2}$ on E, and it coincides with the equilibrium measure η_E for the problem (122).

We now use the GRS-method to get directly from (117) that

$$\biggl|\widetilde{f}(z)+\frac{Q_{n,1}}{Q_{n,2}}(z)\biggr|^{1/n}\xrightarrow{\mathrm{cap}}e^{2(V^{\eta_F}(z)-V_*^{\eta_E}(z)-\mathrm{const})}=e^{-2G_F^{\eta_E}(z)}<1,\qquad z\notin\mathbb{R} \tag{125}$$

(it is easy to see that the const in (125) is the same as in the identity $V_*^{\eta_F}(z)=V^{\eta_E}(z)-G_F^{\eta_E}(z)+\mathrm{const}$). The inequality (54) follows from (117) and the principle of descent for the logarithmic potential (see [63]). Theorem 1 is proved.

Remark 7. If $\alpha\in\mathbb{C}$, $|\alpha|\in(0,1/2)$, then all the above results on the asymptotic behaviour of the expression $|Q_{n,2}^*(y)\omega_n^*(y)\Psi_n(y)|^{1/n}$ still hold. Our flexibility in choosing the second factor $\omega_{n,2}$ of small degree $k_n=o(n)$ in ω_n^* turns out to be quite sufficient to ensure that, while remaining in the framework of the GRS-method, we can correct the argument of the product $Q_{n,2}^*(y)\omega_n^*(y)\Psi_n(y)$ in an arbitrarily small (but fixed) neighbourhood of the point of asymptotic maximum $y_0\in F^\circ$ in such a way that as $n\to\infty$ the integrals

$$\biggl(\int_F\bigl|Q_{n,2}^*(y)\omega_n^*(y)\Psi_n(y)\bigr|\,|dy|\biggr)^{2/n}\quad\text{and}\quad\biggl|\int_F Q_{n,2}^*(y)\omega_n^*(y)\Psi_n(y)\,dy\biggr|^{2/n} \tag{126}$$

have the same asymptotic behaviour (see details in [21], [44], Lemma 9, and [94], Lemma 5). Once we have established this equality, we will have proved the existence of the following limit:

$$\lim_{\substack{n\to\infty\\ n\in\Lambda}}\biggl|\int_F Q^*_{n,2}(y)\omega^*_n(y)\Psi_n(y)\,dy\biggr|^{2/n}=e^{-3V^{\mu}_*(y_0)-G^{\mu}_E(y_0)-3g_E(y_0,\infty)+\mathrm{const}}\neq 0. \qquad (127)$$

As before, (127) will contradict the orthogonality relations (94). Thus, the assumption that $\mu^Q=\mu\neq\eta_F$ turns out to lead to a contradiction. Hence,

$$\frac{1}{n}\chi(Q^*_{n,2})\to\eta_F, \qquad n\to\infty.$$

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