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Vlasov–Poisson equations for a two-component plasma in a homogeneous magnetic field

A. L. Skubachevskii

Abstract. This paper is concerned with the first mixed problem for the Vlasov–Poisson equations in an infinite cylinder, a problem describing the evolution of the density distribution of ions and electrons in a high temperature plasma under an external magnetic field. A stationary solution is constructed for which the charged-particle density distributions are supported in a strictly interior cylinder. A classical solution for which the supports of the charged-particle density distributions are at a distance from the cylindrical boundary is shown to exist and to be unique in some neighbourhood of the stationary solution.

Bibliography: 127 titles.

Keywords: Vlasov–Poisson equations, mixed problem, classical solutions, homogeneous magnetic field.

Contents

Introduction

The Vlasov equations (or the kinetic equations with a self-consistent field) were first obtained in [117] and are now regarded as one of the best-known mathematical models in the kinetic theory of gases. The study of these equations has made it possible to theoretically predict a number of new and unexpected physical phenomena such as the Landau damping effect describing a collisionless damping of waves in a plasma [70]. There is an extensive literature on the Vlasov equations

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in physics (see [72], [73], [85], [118]–[120] and the references given there). In mathematics the interest in these equations developed later, but in recent years the Vlasov equations have received much attention (see [1]–[18], [20]–[36], [38]–[64], [66]–[69], [71], [74]–[84], [86], [88]–[108], [110], [111], [113]–[116], [121]–[127]). These equations are popular most of all because of their numerous applications, including the kinetic theory of particles in electric, magnetic, and gravitational fields, waves in a collisionless plasma, and so on. Depending on the initial physical models, one distinguishes the Vlasov–Poisson equations, the Vlasov–Maxwell equations, the Vlasov–Einstein equations, the generalized Vlasov equations, and so on. A key reason for the increasing interest in the Vlasov equations is probably their applications in the study of high-temperature rarefied plasmas and, most of all, in control processes of thermonuclear fusion.

The Vlasov–Poisson equations, which describe a high-temperature rarefied plasma in the coordinate space $\mathbb{R}^3$ and the velocity space $\mathbb{R}^3$, are as follows:

$$-\Delta\varphi(x,t) = 4\pi e \int_{\mathbb{R}^3} \sum_{\beta=\pm 1} \beta f^\beta(x,v,t)\,dv \qquad (x \in \mathbb{R}^3,\ 0<t<T), \tag{1}$$

$$\frac{\partial f^\beta}{\partial t} + (v, \nabla_x f^\beta) + \frac{\beta e}{m_\beta}\left(-\nabla_x\varphi + \frac{1}{c}[v,B], \nabla_v f^\beta\right) = 0 \tag{2}$$
$$(x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ 0<t<T,\ \beta=\pm 1).$$

These equations are augmented with the initial conditions

$$f^\beta(x,v,t)\big|_{t=0} = f_0^\beta(x,v) \qquad (x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ \beta=\pm 1). \tag{3}$$

Here $f^\beta = f^\beta(x,v,t)$ is the density distribution function of positively charged ions (for $\beta=+1$) or of electrons (for $\beta=-1$) at a point x with velocity v at a time t; $\varphi=\varphi(x,t)$ is the potential of the self-consistent electric field; ∇_x and ∇_v are, respectively, the gradients with respect to x and v; m_{+1} and m_{-1} are the ion and electron masses; e is the electron charge; c is the velocity of light; B is the external magnetic field induction; $(\,\cdot\,,\cdot\,)$ is the inner product in $\mathbb{R}^3$; $[\,\cdot\,,\cdot\,]$ is the vector product in $\mathbb{R}^3$.

Equation (1), in which the right-hand side is the density of the total electric charge at a time t at a point x, is an equivalent statement of the Coulomb law. The Vlasov equations (2) are obtained from the Boltzmann equations in which the collision integral is neglected. For a rarefied plasma, this assumption is justified in [117], [73]. Despite the absence of the collision integral, the interaction of charged particles is taken into account by the self-consistent electric field, which is calculated from the charged-particle density distributions according to equation (1). If we include the self-consistent magnetic field generated by the motion of the charged particles, then, in addition to equation (1), equations (2) will need to be augmented with three more Maxwell equations. The resulting system is known as the Vlasov–Maxwell system.

In the present paper we shall be concerned with the solvability of the Vlasov–Poisson system. To give a brief survey in this topic, we formally reduce the system of three equations (1), (2) to two integro-differential equations. With the use of the

Newtonian potential the solution of (1) can be written as follows:

$$\varphi(x,t) = e\int_{\mathbb{R}^3} \frac{dy}{|x-y|} \int_{\mathbb{R}^3} \sum_{\beta=\pm 1} \beta f^\beta(y,v,t)\,dv. \tag{4}$$

Substituting (4) into (2), this gives

$$\begin{gathered} \frac{\partial f^\beta}{\partial t} + (v, \nabla_x f^\beta) \\ + \frac{\beta e}{m_\beta}\biggl(\int_{\mathbb{R}^3} K(x,y)\,dy \int_{\mathbb{R}^3} \sum_{\beta=\pm 1} \beta f^\beta(y,v,t)\,dv + \frac{1}{c}[v,B], \nabla_v f^\beta\biggr) = 0 \\ (x \in \mathbb{R}^3,\ v \in \mathbb{R}^3,\ 0 < t < T,\ \beta = \pm 1). \end{gathered} \tag{5}$$

Here the kernel

$$K(x,y) = e\frac{x-y}{|x-y|^3} \tag{6}$$

has a weak singularity. Note that equations (5) are non-linear and non-local.

In the multidimensional case the global solvability of equations of the form (5) with 'smoothed' kernel in the absence of a magnetic field B was examined by Braun and Hepp [23], Maslov [82], and Dobrushin [35]. In [35] a global generalized solution of the Cauchy problem for equations of the form (5) was shown to exist and to be unique if $K(x,y)$ is a continuously differentiable function of x and y. If, moreover, $K(x,y)$ is twice continuously differentiable, then a global classical solution exists and is unique. By using the method of characteristics and taking into account the smoothness of the kernel $K(x,y)$ it proved possible to reduce this problem to a system of ordinary differential equations with an integral term and apply the machinery of dynamical systems to prove solvability.

However, proof of the solvability of equations (5) with kernel $K(x,y)$ of the form (6) and with initial conditions (3) is a more involved problem. This is so because, first of all, the original system (1), (2) involves equations of various types. The Poisson equation (1) is an elliptic second-order equation, while the Vlasov equations (2) are first-order partial differential equations. As is known, the method of characteristics enables us to reduce first-order partial differential equations to a system of ordinary differential equations. Hence, a classical solution of these equations should be sought in spaces of continuously differentiable functions. On the other hand, the Poisson equation is investigated by methods of potential theory, and hence it is more natural to search for its classical solution in the corresponding Hölder space.

The existence of a global generalized solution of the Cauchy problem for the Vlasov–Poisson equations (1), (2) was proved by Arsen'ev [2]. His proof depended on regularization of the Laplacian, solvability of the corresponding regularized problem, and taking the weak limit in the integral relation for the generalized solution. There is a certain analogy between the study of this regularized problem and the investigation of the problem (5), (3) with a smoothed kernel. The existence of a global generalized solution and its weak stability in the case of the Cauchy problem for the Vlasov–Poisson and the Vlasov–Maxwell equations were studied by DiPerna and Lions [33], [34], Horst and Hunze [58], and others.

In the one-dimensional case the existence of a global classical solution of the Cauchy problem for the Vlasov equations was proved by Iordanskii [63]. The corresponding result in the two-dimensional setting is due to Ukai and Okabe [113]. In the three-dimensional setting, which is most important in physical applications, the problem is considerably more involved. Batt [9] has shown the existence and uniqueness of a global classical solution for spherically symmetric initial distribution functions with compact supports with respect to v. In addition, Batt [9] and Horst [55] have shown that in the three-dimensional case a sufficient condition for a global classical solution of the Cauchy problem for the Vlasov–Poisson system to exist for any sufficiently smooth initial distribution functions with compact supports with respect to v is that the supports of the distribution functions with respect to the velocities v remain compact for all $t \in [0, \infty)$. In other words, to prove the existence of a global classical solution it sufficed to show that the diameters of the supports of the distribution functions with respect to v can grow only with finite velocity as $t \to \infty$. This result is now known as the velocity lemma, and its various interpretations have been useful in many studies. Bardos and Degond [7] proved the existence and uniqueness of a classical solution of the Cauchy problem for the Vlasov–Poisson system with small initial data. The existence of a global classical solution of the Cauchy problem for the Vlasov–Poisson equations with arbitrary initial distribution functions is due to Pfaffelmoser [90]. Later, a simpler proof of this result was presented by Schäffer [104]. In the four-dimensional case, Horst [56] has shown that the Cauchy problem for this system may fail to have a global classical solution. Classical solutions of the initial-value problem for the Vlasov–Poisson equations were also studied in [3], [18], [44], [45], [57], [74], [75], [97], [98], and elsewhere.

Stationary solutions of the Vlasov–Poisson equations have been the subject of papers by Vedenyapin [114], [115], Batt and Fabian [12], Batt, Faltenbacher, and Horst [13], Pokhozhaev [91], Greengard and Raviart [48], and Rein [95]. The papers [79], [80], [100], [101], [105]–[108] were concerned with stationary solutions of the Vlasov–Maxwell equations and their bifurcation.

A considerable number of interesting studies have been devoted to the investigation of both the linearized model [28], [83] and the non-linear model of the Landau damping effect (see, for example, [42], [60], [64]). An extensive survey of the corresponding literature is given by Mouhot and Villani in the paper [86] devoted to the general case of non-linear Landau damping.

However, much less attention has been paid to the existence of solutions of the Vlasov equations in domains with boundary. The studies here have been mostly focused on generalized solutions of mixed problems for the Vlasov–Poisson equations and the Vlasov–Maxwell equations (see Arsen'ev [5], Alexandre [1], Ben Abdallah [16], Guo [49], and Weckler [122]). The stability of generalized solutions of initial-value and mixed problems for the Vlasov equations has been studied by Kozlov [66], [67], DiPerna and Lions, [33], [34], Rein [96], Wan [121], and Weckler [122]. It is worth noting that for the Vlasov equations there are no exhaustive results on an increase in smoothness of the generalized solutions of mixed problems (as in the case of classical second-order partial differential equations). Consequently, the study of the existence of classical and strong solutions of mixed problems for the Vlasov equations has great value. The existence of classical and strong solutions of mixed problems in the general setting is still an open problem (see Kozlov [66], Samarskii [102], and Weckler [122]). This problem is relevant to the design of a controlled thermonuclear fusion reactor, a mathematical model of which is described by mixed problems for the Vlasov system with respect to the density distributions of charged particles of opposite signs in a bounded domain. Tokamaks are now the best-known devices for the production of thermonuclear fusion. The word 'tokamak', which is an acronym developed from the Russian words 'TOroidalnaya KAmera i ee MAgnitaya Katushka' meaning a 'toroidal chamber with magnetic coils', was introduced by I. N. Golovin (see [85], the editor's comments on the Russian translation on p. 277). The vacuum chamber of a tokamak reactor is a torus whose cross-section looks like the roman capital letter 'D' (see Fig. 1).

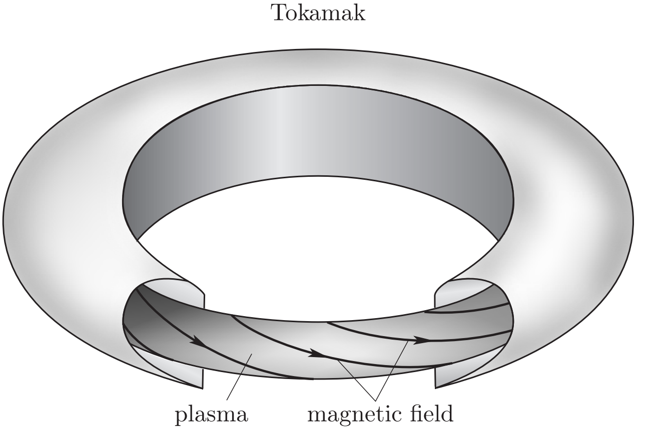

Figure 1

One of the alternative devices for thermonuclear fusion is the mirror trap, which can be visualized as a long cylinder tapered at the ends (see Fig. 2).

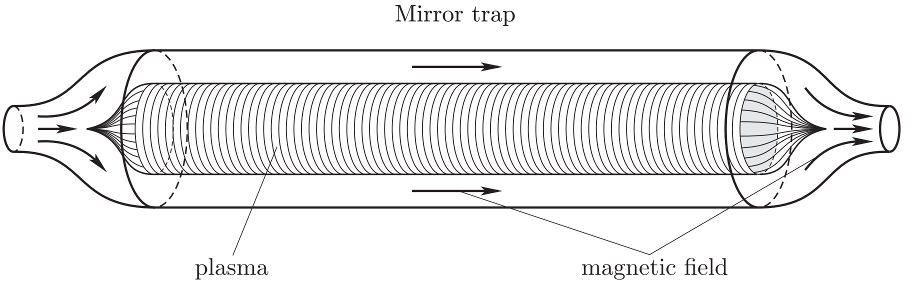

Figure 2

Note that Figs. 1, 2 provide only a very coarse picture of these devices. More detailed schematics of mirror traps for the confinement of high-temperature plasmas and a description of their operating principles can be found in [109].

The production of a stable high-temperature plasma in a reactor requires that the so-called plasma column be strictly inside the domain during some time interval in order to keep it away from the vacuum container wall ([85], the introduction to Chap. 6). In most models of thermonuclear fusion reactors an external magnetic field [72], [85] is used as a control ensuring the existence of a plasma in the reactor. From the point of view of differential equations this means that one has to ensure the existence of solutions of the Vlasov–Poisson equations for which the supports of the charged-particle density distributions do not intersect the boundary, which can be achieved by the influence of the external magnetic field.

We now give a brief survey of some of the most important papers on the existence of classical solutions of mixed problems for the Vlasov–Poisson equations in domains with boundary. The global existence of classical solutions of mixed problems for these equations in a half-space with Neumann or Dirichlet boundary conditions for the electric-field potential and the conditions of elastic reflection for charged-particle density distributions on the boundary was proved by Guo [50] and Hwang and Velázquez [61]. Hwang [59] proved that the classical solutions of the Vlasov–Poisson equations in a ball with spherically symmetric initial data and conditions of elastic reflection for the density distributions can have singularities only at the centre of the ball. The main difficulties in the study of classical solutions of mixed problems for these equations have to do with the behaviour of the characteristics near the boundary. We note that the effect of the magnetic field on the trajectories of the particles was not taken into account.

In the present paper we shall be concerned with classical solutions of the mixed problem for the Vlasov–Poisson system in $Q \times \mathbb{R}^3 \times (0,T)$ with Dirichlet boundary condition for the electric-field potential on $\partial Q \times (0,T)$, where $Q = G \times \mathbb{R}$ is an infinite cylinder and $G \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial G \in C^\infty$. A distinctive feature of this paper is that, for the solution obtained, the supports of the charged-particle density distributions lie at some distance from the cylindrical surface ∂Q with respect to the spatial variable x and are compact with respect to v. To produce such a solution we assume first that the external magnetic field B is directed along the axis of the cylinder and is sufficiently strong, and second that the initial density distributions $f_0^\beta(x,v)$ have supports lying at some distance from the boundary ∂Q with respect to x and are compact with respect to v. These assumptions imply that the characteristics do not intersect ∂Q. This phenomenon can be interpreted physically as follows: the charged particles do not reach the walls of the vacuum chamber of the thermonuclear fusion reactor because they move along trajectories close to the Larmor trajectories. Consequently, this problem can be used in a certain sense as a mathematical model of the cylindrical part of the mirror trap. According to [85], the presence of a considerable number of particles on the boundary can result in either destruction of the reactor walls or in cooling of the high-temperature plasma due to its contact with the reactor walls. As distinct from other papers (see, for example, [61]) which have dealt with the Vlasov–Poisson equations for particles of the same sign, we are concerned here with those equations for a two-component plasma, since the word ‘plasma’ is used in physics to designate this high-temperature state of an ionized gas with charge neutrality [85]. This creates additional difficulties in the (physical and mathematical) analysis.

The paper is organized as follows. In § 1 we introduce the notation, pose the problem, and formulate the main result (Theorem 1.1). This result guarantees the existence of a stationary solution of the Vlasov–Poisson equations in $Q \times \mathbb{R}^3 \times (0,T)$

when the charged-particle density distributions are supported with respect to x in a strictly interior cylinder. Moreover, in some neighbourhood of this stationary solution there is a unique classical solution for which the supports of the density distributions with respect to x are disjoint from the boundary ∂Q. In § 2 we study the characteristics of the system (2) in $Q\times\mathbb{R}^3\times(0,T)$ for a fixed potential φ. In the absence of an electric field ($\varphi=0$), the strong magnetic field B along the cylinder axis makes the particles move along circular or helical paths with Larmor frequency $e|B|/(m_\beta c)$ in the cylinder Q without reaching ∂Q. This phenomenon is very well known in plasma physics. For sufficiently small potentials φ, the characteristics emerging from some strictly interior cylinder in Q also fail to reach ∂Q. In § 3 the characteristics examined above are employed for constructing a solution of the initial-value problem (2), (3) in $Q\times\mathbb{R}^3\times(0,T)$ for a fixed potential φ. Since the characteristics do not reach ∂Q, it follows that for initial densities $f_0^\beta(x,v)$ supported in the domain $Q\times\mathbb{R}^3$ the supports of the solutions $f_\varphi^\beta(x,v,t)$ of the problem (2), (3) remain in $Q\times\mathbb{R}^3$ for all $0<t<T$. Substituting the solutions $f_\varphi^\beta(x,v,t)$ in equation (1), we obtain Hölder estimates for the right-hand side of (1) with $f^\beta(x,v,t)=f_\varphi^\beta(x,v,t)$. In § 4 we build a stationary solution of the problem (1), (2) with the above properties. Further, using the Hölder estimates in § 3, taking into account the unique solvability of the Dirichlet problem for the Poisson equation in Hölder spaces, and employing the Banach contraction principle, we prove the existence and uniqueness of a classical solution in some neighbourhood of the stationary solution thus constructed. In § 5 we extend Theorem 1.1 to abstract Vlasov equations and mention some unsolved problems.

It should be noted that the classical solutions of mixed problems for the Vlasov–Poisson equations in a half-space for sufficiently small compactly supported initial densities and an external magnetic field of high intensity were examined in [110] and [111]. The presence of an external magnetic field was also assumed in a number of other papers on the Vlasov–Poisson equations in domains with boundary (see, for example, [5], [50]). However, the effect of this field on the nucleation of Larmor trajectories and problems of plasma confinement at some distance from the boundary was not considered.

1. Statement of the problem. The main result

1.1. We consider the Vlasov–Poisson system in an infinite cylinder:

$$-\Delta\varphi(x,t)=4\pi e\int_{\mathbb{R}^3}\sum_\beta \beta f^\beta(x,v,t)\,dv \qquad (x\in Q,\ 0<t<T), \tag{1.1}$$

$$\frac{\partial f^\beta}{\partial t}+(v,\nabla_x f^\beta)+\frac{\beta e}{m_\beta}\biggl(-\nabla_x\varphi+\frac{1}{c}[v,B],\nabla_v f^\beta\biggr)=0 \tag{1.2}$$

$$(x\in Q,\ v\in\mathbb{R}^3,\ 0<t<T,\ \beta=\pm1)$$

with the initial conditions

$$f^\beta(x,v,t)\big|_{t=0}=f_0^\beta(x,v) \qquad (x\in\overline{Q},\ v\in\mathbb{R}^3,\ \beta=\pm1) \tag{1.3}$$

and the Dirichlet boundary condition

$$\varphi(x,t)=0 \qquad (x\in\partial Q,\ 0\leqslant t<T). \tag{1.4}$$

Here $Q = G \times \mathbb{R}$, $G \subset \mathbb{R}^2$ is a bounded domain with boundary $\partial G \in C^\infty$, $\partial Q = \partial G \times \mathbb{R}$, the functions $\varphi(x,t)$ and $f^\beta(x,v,t)$ are unknowns, the vector B is given, and the constants $m_{\pm 1}$, e, and c have the same meaning as in the Introduction.

1.2. To find the classical solution of the problem (1.1)–(1.4) we introduce some function spaces.

We let $C^s(\mathbb{R}^n)$ (respectively, $C^s(\overline{\Omega})$) with $s \geqslant 0$ and $n \in \mathbb{N}$ denote the Hölder space of continuous functions on $\mathbb{R}^n$ (on $\overline{\Omega}$) that have continuous derivatives in $\mathbb{R}^n$ (in $\overline{\Omega}$) up to and including total order $k = [s]$, equipped with the finite norm

$$\begin{aligned} \|u\|_s &= \max_{|\alpha|\leqslant k} \sup_x |\mathscr{D}^\alpha u(x)| && \text{for } s = k \in \mathbb{Z}, \quad 0 \leqslant k, \\ \|u\|_s &= \|u\|_k + |u|_\sigma && \text{for } s = k+\sigma, \quad 0 \leqslant k \in \mathbb{Z}, \quad 0 < \sigma < 1, \end{aligned} \tag{1.5}$$

where $\Omega \subset \mathbb{R}^n$ is a domain with C^∞-boundary $\partial\Omega$ or the cylinder $Q = G \times \mathbb{R} \subset \mathbb{R}^3$,

$$|u|_\sigma = \max_{|\alpha|=k} \sup_{x\neq y} |x-y|^{-\sigma} |\mathscr{D}^\alpha u(x) - \mathscr{D}^\alpha u(y)|, \tag{1.6}$$

$$\mathscr{D}^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}, \qquad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \alpha_1 + \dots + \alpha_n.$$

Let $C(\mathbb{R}^n) = C^0(\mathbb{R}^n)$ and $C(\overline{\Omega}) = C^0(\overline{\Omega})$.

Similarly, we introduce the space $C^1(\overline{Q} \times \mathbb{R}^3 \times [0,T])$ of bounded continuous functions with bounded continuous first-order derivatives in $\overline{Q} \times \mathbb{R}^3 \times [0,T]$.

Remark 1.1. If $s = k+\sigma$, $0 \leqslant k \in \mathbb{Z}$, and $0 < \sigma < 1$, then in view of Theorem 2 in § 4.5.2 of [112], we can endow $C^s(\mathbb{R}^n)$ (respectively, $C^s(\overline{\Omega})$) with the equivalent norm

$$\|u\|'_s = \|u\|_k + |u|_{\sigma,b}, \tag{1.7}$$

where

$$|u|_{\sigma,b} = \max_{|\alpha|=k} \sup_{\substack{x\neq y,\\ 0<|x-y|<b}} |x-y|^{-\sigma} |\mathscr{D}^\alpha u(x) - \mathscr{D}^\alpha u(y)|, \qquad 0 < b \leqslant 1. \tag{1.8}$$

Remark 1.2. For any $s \geqslant 0$ the spaces $C^s(\mathbb{R}^n)$ and $C^s(\overline{\Omega})$ are Banach spaces. If $s = k+\sigma$, $0 \leqslant k \in \mathbb{Z}$, and $0 < \sigma < 1$, then the space $C^s(\mathbb{R}^n)$ ($C^s(\overline{\Omega})$) is not separable, and the set of functions infinitely differentiable in $\mathbb{R}^n$ ($\overline{\Omega}$) with finite norm $\|\cdot\|_s$ is not dense in $C^s(\mathbb{R}^n)$ ($C^s(\overline{\Omega})$) (see [19]).

Let $\dot{C}^k(\mathbb{R}^n)$ with $k, n \in \mathbb{N}$ denote the space of k-times continuously differentiable functions on $\mathbb{R}^n$ having compact support.

Also, let $C^s_0(\overline{Q})$ with $s \geqslant 0$ denote the closure of the set of functions in $C^s(\overline{Q})$ with compact support in $\overline{Q}$.

We consider the Banach space $C([0,T], C^s(\overline{\Omega}))$, $s > 0$, of continuous functions $[0,T] \ni t \mapsto \varphi(\,\cdot\,,t) \in C^s(\overline{\Omega})$ with norm

$$\|\varphi\|_{s,T} = \sup_{0\leqslant t\leqslant T} \|\varphi(\,\cdot\,,t)\|_s\,. \tag{1.9}$$

The space $C([0,T], C^s_0(\overline{Q}))$ is defined similarly.

Let $B_\rho(x_0) = \{x \in \mathbb{R}^3 \colon |x - x_0| < \rho\}$, $B_\rho = B_\rho(0)$, and $|B_\rho| = 4\pi\rho^3/3$.

In what follows, k_i, c_j, $\widehat{k}_i$, $\widehat{c}_j$ are positive constants.

1.3. To state the theorem on the unique solvability of the problem (1.1)–(1.4) near a stationary solution, we first give the definition of a classical solution of this problem and the definition of a stationary solution of the problem (1.1), (1.2), (1.4).

Definition 1.1. A vector function $\{\varphi, f^\beta\}$ with $\varphi \in C([0,T], C_0^{2+\sigma}(\overline{Q}))$ and $f^\beta \in C^1(\overline{Q}\times\mathbb{R}^3\times[0,T])$ is called a classical solution of the problem (1.1)–(1.4) if $\{\varphi, f^\beta\}$ satisfies equations (1.1), (1.2), the initial conditions (1.3), and the boundary condition (1.4).

In the study of the Vlasov equations an important role is played by stationary solutions.

Definition 1.2. A vector function $\{\mathring{\varphi}, \mathring{f}^\beta\}$ with $\mathring{\varphi} \in C_0^{2+\sigma}(\overline{Q})$ and $\mathring{f}^\beta \in C^1(\overline{Q}\times\mathbb{R}^3)$ is called a stationary solution of equations (1.1), (1.2) with the boundary condition (1.4) if $\{\mathring{\varphi}, \mathring{f}^\beta\}$ satisfies the equations

$$-\Delta\mathring{\varphi}(x) = 4\pi e\int_{\mathbb{R}^3}\sum_\beta \beta\mathring{f}^\beta(x,v)\,dv \qquad (x \in Q), \tag{1.10}$$

$$(v, \nabla_x\mathring{f}^\beta) + \frac{\beta e}{m_\beta}\left(-\nabla_x\mathring{\varphi} + \frac{1}{c}[v,B], \nabla_v\mathring{f}^\beta\right) = 0 \qquad (x \in Q,\ v \in \mathbb{R}^3,\ \beta = \pm 1) \tag{1.11}$$

and the boundary condition (1.4).

We now formulate the conditions which the magnetic field B and the initial charged-particle density distributions $f_0^\beta(x,v)$ must satisfy.

Let $G_\delta = \{x' \in G \colon \operatorname{dist}(x', \partial G) > \delta\}$ and $Q_\delta = \{x \in Q \colon \operatorname{dist}(x, \partial Q) > \delta\}$, where $\delta > 0$. Assuming that $G_{2\delta} \neq \varnothing$, we let $\delta_0 = \delta_0(\delta) > 0$ denote the radius of a largest circle inscribed in $G_{2\delta}$.

Condition 1.1. Let $B = (0,0,h)$ for $x \in \overline{Q}$, where $h > 0$ is independent of x and

$$32\,\frac{c\rho m_{+1}}{e\delta} < h. \tag{1.12}$$

Condition 1.2. Let $f_0^\beta \in C^\infty(\overline{Q}\times\mathbb{R}^3)$ be non-negative functions and let $\operatorname{supp} f_0^\beta \subset Q_{2\delta}\times B_{\rho/4}$, where $\rho > 0$, $\delta_0 > \delta$.

Theorem 1.1. *Let $\delta > 0$ be such that $G_{2\delta} \neq \varnothing$ and let $\delta_0 > \delta$. Assume that Condition 1.1 is satisfied for this δ and some h, $\rho > 0$. Then for any $\alpha > 0$ there is a stationary solution $\{0, \mathring{f}^\beta\}$ of equations (1.1), (1.2) with the boundary condition (1.4) such that $\mathring{f}^\beta \in C^\infty(\overline{Q}\times\mathbb{R}^3)$, $\operatorname{supp}\mathring{f}^\beta \subset Q_{2\delta}\times B_{\rho/4}$, and $\sup_{x,v}\mathring{f}^\beta(x,v) > \alpha$.*

If, moreover, Condition 1.2 holds, then for any $T > 0$ there exists an $\varepsilon = \varepsilon(T,\delta,\rho,h,\sigma) > 0$ such that, for all stationary solutions $\{0, \mathring{f}^\beta\}$ with the indicated properties and for all initial functions f_0^β such that

$$\begin{aligned}&\operatorname{supp}(f_0^\beta - \mathring{f}^\beta) \subset (Q_{2\delta}\cap Q'_N)\times B_{\rho/4},\\ &\|\mathring{f}^\beta_{v_i}\|_2 < \varepsilon \quad (i=1,2,3), \qquad \|f_0^\beta - \mathring{f}^\beta\|_2 < \varepsilon\end{aligned} \qquad (\beta = \pm 1), \tag{1.13}$$

where $Q'_N = \{x \in Q\colon |x_3| < N\}$ *for a number* $N > 0$, *there is a unique classical solution of* (1.1)–(1.4). *Furthermore,*

$$\operatorname{supp} f^\beta(\,\cdot\,,\cdot\,,t) \subset Q_{5\delta/4} \times B_\rho \quad \textit{for all } t \in [0,T].$$

The proof of Theorem 1.1 will be given in § 4.

2. Trajectories of charged particles in an infinite cylinder

2.1. Assume that Conditions 1.1 and 1.2 are satisfied and that there is a stationary solution $\{0, \mathring{f}^\beta\}$ of the problem (1.1), (1.2), (1.4) with the properties in the first assertion of Theorem 1.1. We shall regard a solution $\{\varphi, f^\beta\}$ of (1.1)–(1.4) as a perturbed stationary solution $\{0, \mathring{f}^\beta\}$ of the problem (1.1), (1.2), (1.4). Also, let $g^\beta(x,v,t) = f^\beta(x,v,t) - \mathring{f}^\beta(x,v)$ and $g_0^\beta(x,v) = f_0^\beta(x,v) - \mathring{f}^\beta(x,v)$. Then by (1.1)–(1.4) and (1.10), (1.11),

$$-\Delta\varphi(x,t) = 4\pi e \int_{\mathbb{R}^3} \sum_{\beta=\pm1} \beta g^\beta(x,v,t)\,dv \qquad (x \in Q,\ 0 < t < T), \tag{2.1}$$

$$\frac{\partial g^\beta}{\partial t} + (v, \nabla_x g^\beta) + \frac{\beta e}{m_\beta}\Bigl(-\nabla_x\varphi + \frac{1}{c}[v,B], \nabla_v g^\beta\Bigr) = \frac{\beta e}{m_\beta}(\nabla_x\varphi, \nabla_v \mathring{f}^\beta) \qquad (x \in Q,\ v \in \mathbb{R}^3,\ \beta = \pm1), \tag{2.2}$$

$$g^\beta(x,v,t)\big|_{t=0} = g_0^\beta(x,v) \qquad (x \in \overline{Q},\ v \in \mathbb{R}^3), \tag{2.3}$$

$$\varphi(x,t)\big|_{t=0} = 0 \qquad (x \in \partial Q,\ 0 \leqslant t < T). \tag{2.4}$$

We shall assume that $\operatorname{supp} g_0^\beta \subset (Q_{2\delta} \cap Q'_N) \times B_{\rho/4}$, where $N > 0$ is some number.

A classical solution of (2.1)–(2.4) is defined like that of the problem (1.1)–(1.4). However, unlike equations (1.2), equations (2.2) are inhomogeneous with respect to g^β.

Given a fixed function $\varphi \in C([0,T], C_0^{2+\sigma}(\overline{Q}))$, equation (2.2) with the initial condition (2.3) can be solved using the method of characteristics. To this end, we consider the following system of ordinary differential equations:

$$\frac{dX_\varphi^\beta}{d\tau} = V_\varphi^\beta \qquad (0 < \tau < T,\ \beta = \pm1), \tag{2.5}$$

$$\frac{dV_\varphi^\beta}{d\tau} = -\frac{\beta e}{m_\beta}\nabla_x\varphi(X_\varphi^\beta, \tau) + \frac{\beta e}{m_\beta c}[V_\varphi^\beta, B] \qquad (0 < \tau < T,\ \beta = \pm1) \tag{2.6}$$

with the initial conditions

$$X_\varphi^\beta\big|_{\tau=0} = x \qquad (\beta = \pm1), \tag{2.7}$$

$$V_\varphi^\beta\big|_{\tau=0} = v \qquad (\beta = \pm1), \tag{2.8}$$

where $x \in Q$ and $v \in \mathbb{R}^3$.

We have $\varphi \in C\big([0,T], C_0^{2+\sigma}(\overline{Q})\big)$, and hence from the theorem on non-continuable solutions it follows that for any $x \in Q$ and $v \in \mathbb{R}^3$ there is a unique non-continuable solution of (2.5)–(2.8) on some half-open interval $[0, T_\varphi^\beta(x,v))$ with $T_\varphi^\beta(x,v) \leqslant T$. We denote this solution by $\{X_\varphi^\beta(x,v,\tau), V_\varphi^\beta(x,v,\tau)\}$.

Lemma 2.1. *Let $\varphi \in C([0,T], C_0^{2+\sigma}(\overline{Q}))$ and let $\|\varphi\|_{1,T} \leqslant R_0$. Then for any $x \in Q$, $|v| < \rho$, and $0 < t < T_\varphi^\beta(x,v)$,*

$$\big|V_\varphi^\beta(x,v,t)\big| < \rho_t \qquad (\beta = \pm 1), \tag{2.9}$$

where $\rho_t = \sqrt{2(\rho^2 + 6t^2e^2R_0^2/m_{-1}^2)}$ with $\rho > 0$.

Proof. Multiplying both sides of (2.6) by V_φ^β, integrating with respect to τ from 0 to t, and employing the identity $(V_\varphi^\beta, [V_\varphi^\beta, B]) = 0$, we get that

$$\frac{1}{2}|V_\varphi^\beta(x,v,t)|^2 - \frac{1}{2}|v|^2 = -\frac{\beta e}{m_\beta}\int_0^t \big(\nabla_x\varphi(X_\varphi^\beta,\tau), V_\varphi^\beta(x,v,\tau)\big)\,d\tau. \tag{2.10}$$

Clearly,

$$ab \leqslant \frac{\varepsilon^{-1}a^2}{2} + \frac{\varepsilon b^2}{2} \qquad (a, b \in \mathbb{R},\ \varepsilon > 0).$$

Hence,

$$|V_\varphi^\beta(x,v,t)|^2 \leqslant |v|^2 + 3\varepsilon^{-1}eR_0^2tm_\beta^{-1} + \varepsilon e \max_{\tau\in[0,t]} |V_\varphi^\beta(x,v,\tau)|^2 tm_\beta^{-1}.$$

We choose $\tau_0 \in [0,t]$ so that $|V_\varphi^\beta(x,v,\tau_0)| = \max_{\tau\in[0,t]} |V_\varphi^\beta(x,v,\tau)|$. Letting $t = \tau_0$ in the last inequality, we obtain

$$|V_\varphi^\beta(x,v,\tau_0)|^2 \leqslant |v|^2 + 3\varepsilon^{-1}eR_0^2\tau_0 m_\beta^{-1} + \varepsilon e\tau_0|V_\varphi^\beta(x,v,\tau_0)|^2 m_\beta^{-1}.$$

Next, putting $\varepsilon = m_\beta(2\tau_0 e)^{-1}$, we have the estimate

$$|V_\varphi^\beta(x,v,t)|^2 \leqslant |V_\varphi^\beta(x,v,\tau_0)|^2 \leqslant \rho_{\tau_0}^2 \leqslant \rho_t^2. \qquad \square$$

2.2. We consider the trajectories of the system (2.5), (2.6) with $\varphi = 0$. Then the system of equations (2.5), (2.6) assumes the form

$$\frac{dX_0^\beta}{d\tau} = V_0^\beta \qquad (0 < \tau), \tag{2.11}$$

$$\frac{dV_0^\beta}{d\tau} = \frac{\beta e}{m_\beta c}[V_0^\beta, B] \qquad (0 < \tau). \tag{2.12}$$

Let $x' = (x_1, x_2)$ and $X_\varphi^{\beta'}(x,v,\tau) = \big\{X_{\varphi 1}^\beta(x,v,\tau), X_{\varphi 2}^\beta(x,v,\tau)\big\}$.

Lemma 2.2. *Assume that Condition 1.1 is satisfied for some $\delta, \rho > 0$. Let δ', ρ' be such that $G_{2\delta'} \neq \varnothing$, $\delta' \geqslant \delta/2$, and $0 < \rho' \leqslant 2\rho$. Also, let $\{X_0^\beta(x,v,\tau), V_0^\beta(x,v,\tau)\}$ be a non-continuable solution of the problem (2.11), (2.12), (2.7), (2.8) on the half-open interval $[0, T_0^\beta(x,v))$. Then the following assertions hold.*

(a) *$|V_0^\beta(x,v,\tau)| = |v|$ for all $\tau \in [0, T_0^\beta(x,v))$.*

(b) *If $x \in \partial Q_{\delta'}$ and $|v| = \rho'$, then $T_0^\beta(x,v) = \infty$ and $|x' - X_0^{\beta'}(x,v,\tau)| < \delta/8$ for all $\tau \in [0,\infty)$.*

Proof. The first assertion follows from (2.10) for $\varphi = 0$.

To prove the second assertion we note that, in view of Condition 1.1, the system (2.12) can be written as

$$\begin{aligned} \frac{dV_{01}^{\beta}}{d\tau} &= \frac{\beta eh}{m_{\beta}c} V_{02}^{\beta}, \\ \frac{dV_{02}^{\beta}}{d\tau} &= -\frac{\beta eh}{m_{\beta}c} V_{01}^{\beta}, \\ \frac{dV_{03}^{\beta}}{d\tau} &= 0. \end{aligned} \tag{2.13}$$

The spatial coordinates of the solution of the problem (2.11), (2.13), (2.7), (2.8) are

$$\begin{aligned} X_{01}^{\beta}(\tau) &= -\frac{\beta c r m_{\beta}}{eh} \cos\biggl(\frac{\beta eh}{m_{\beta}c}\tau + k_0\biggr) + k_1, \\ X_{02}^{\beta}(\tau) &= \frac{\beta c r m_{\beta}}{eh} \sin\biggl(\frac{\beta eh}{m_{\beta}c}\tau + k_0\biggr) + k_2, \\ X_{03}^{\beta}(\tau) &= v_3\tau + k_3, \end{aligned} \tag{2.14}$$

where $(V_{01}^{\beta})^2 + (V_{02}^{\beta})^2 = v_1^2 + v_2^2 = r^2 = \mathrm{const}$, $V_{03}^{\beta} = v_3 = \mathrm{const}$,

$$\begin{aligned} k_1 &= x_1 + \frac{\beta c r m_{\beta}}{eh} \cos k_0, \\ k_2 &= x_2 - \frac{\beta c r m_{\beta}}{eh} \sin k_0, \\ k_3 &= x_3, \\ \sin k_0 = v_1 r^{-1}&, \qquad \cos k_0 = v_2 r^{-1}. \end{aligned} \tag{2.15}$$

For $v_3 = 0$ the trajectories are circles, and for $v_3 \neq 0$ they take the form of helical curves. The number $\omega_{\beta} = eh/(m_{\beta}c)$ is called the Larmor frequency, and $r_{\beta} = crm_{\beta}/(eh)$ is called the Larmor radius.

If $x \in \partial Q_{\delta'}$ and $|v| = \rho'$, then (1.12) and the inequality $r \leqslant \rho' \leqslant 2\rho$ imply that $r_{\beta} = crm_{\beta}/(eh) < \delta/16$. Consequently, $|x' - X_0^{\beta'}(x, v, \tau)| < \delta/8$ for any $\tau > 0$; that is, the non-continuable solution of the problem (2.11), (2.13), (2.7), (2.8) exists for all $\tau \in [0, \infty)$. $\square$

2.3. We now consider the trajectories of the system (2.5), (2.6) with a sufficiently small electric-field potential φ.

Let $R_1 > 0$ be such that

$$\frac{2eTR_1}{m_{-1}} \exp(a_0 T) < \min\biggl\{\frac{\delta}{8}, \frac{\rho}{4}\biggr\}, \tag{2.16}$$

where $a_0 = 1 + eh/(cm_{-1})$.

Lemma 2.3. *Assume that Condition 1.1 is satisfied for some* $\delta, \rho > 0$ *and let* δ', ρ' *be such that* $G_{2\delta'} \neq \varnothing$, $\delta' \geqslant \delta/2$, *and* $0 < \rho' \leqslant 2\rho$. *Then for all* $\varphi \in C\big([0,T], C_0^{2+\sigma}(\overline{Q})\big)$ *with* $\|\varphi\|_{1,T} \leqslant R_1$ *the non-continuable solution* $\big\{X_\varphi^\beta(x,v,\tau),$ $V_\varphi^\beta(x,v,\tau)\big\}$ *of* (2.5)–(2.8) *has the following property: if* $x \in \partial Q_{\delta'}$ *and* $|v| = \rho'$, *then*

$$T_\varphi^\beta(x,v) = T, \qquad |x' - X_\varphi^{\beta\prime}(x,v,\tau)| < \frac{\delta}{4}, \qquad \rho' - \frac{\rho}{4} < |V_\varphi^\beta(x,v,\tau)| < \rho' + \frac{\rho}{4}$$

for all $\tau \in [0,T]$.

Proof. Let $x \in \partial Q_{\delta'}$ and $|v| = \rho'$. Consider the system of equations (2.5), (2.6). We now subtract (2.11) from (2.5) and (2.12) from (2.6) and integrate the resulting equalities with respect to τ from 0 to q, taking into account the initial conditions (2.7) and (2.8). Introducing the new variables $\tau = q$ and $s = \tau$, we obtain

$$X_\varphi^\beta(\tau) - X_0^\beta(\tau) = \int_0^\tau \big(V_\varphi^\beta(s) - V_0^\beta(s)\big)\,ds \qquad (0 < \tau < T_\varphi^\beta(x,v)), \tag{2.17}$$

$$\begin{aligned} V_\varphi^\beta(\tau) - V_0^\beta(\tau) = {} & \frac{\beta e}{m_\beta c}\int_0^\tau [V_\varphi^\beta(s) - V_0^\beta(s), B]\,ds \\ & - \frac{\beta e}{m_\beta}\int_0^\tau \nabla_x\varphi\big(X_\varphi^\beta(s), s\big)\,ds \qquad (0 < \tau < T_\varphi^\beta(x,v)). \end{aligned} \tag{2.18}$$

By (2.17) and (2.18),

$$|X_\varphi^\beta(\tau) - X_0^\beta(\tau)| \leqslant \int_0^\tau |V_\varphi^\beta(s) - V_0^\beta(s)|\,ds \qquad (0 < \tau < T_\varphi^\beta(x,v)), \tag{2.19}$$

$$|V_\varphi^\beta(\tau) - V_0^\beta(\tau)| \leqslant \frac{2e\tau}{m_\beta}\|\varphi\|_{1,T} + \frac{eh}{m_\beta c}\int_0^\tau |V_\varphi^\beta(s) - V_0^\beta(s)|\,ds \quad (0 < \tau < T_\varphi^\beta(x,v)). \tag{2.20}$$

Next, by (2.19), (2.20), and Gronwall's lemma,

$$|X_\varphi^\beta(\tau) - X_0^\beta(\tau)| + |V_\varphi^\beta(\tau) - V_0^\beta(\tau)| \leqslant \frac{2e\tau}{m_{-1}} R_1 e^{a_0\tau} \quad (0 < \tau < T_\varphi^\beta(x,v)). \tag{2.21}$$

As a result, by the second assertion of Lemma 2.2, the properties of non-continuable solutions, and the inequality (2.16), we see that

$$T_\varphi^\beta(x,v) = T, \qquad |x' - X_\varphi^{\beta\prime}(x,v,\tau)| < \frac{\delta}{4}, \qquad \rho' - \frac{\rho}{4} < |V_\varphi^\beta(x,v,\tau)| < \rho' + \frac{\rho}{4}$$

for all $\tau \in [0,T)$. $\square$

2.4. Let us consider the system of equations (2.5), (2.6) on the interval $(0,t)$, $0 < t \leqslant T$, with the initial conditions

$$X_\varphi^\beta\big|_{\tau=t} = y, \tag{2.22}$$

$$V_\varphi^\beta\big|_{\tau=t} = w. \tag{2.23}$$

By the theorem on non-continuable solutions, the problem (2.22), (2.23), (2.5), (2.6) has a unique non-continuable solution $\{X^\beta_\varphi(y,w,t,\tau),\ V^\beta_\varphi(y,w,t,\tau)\}$ ($\tau \in (T^\beta_\varphi(y,w,t),t]$, $0 \leqslant T^\beta_\varphi(y,w,t) < t$) for all $y \in Q$ and $w \in \mathbb{R}^3$. For $\varphi = 0$ the system (2.5), (2.6) takes the form (2.11), (2.12). Let $\{X^\beta_0(y,w,t,\tau), V^\beta_0(y,w,t,\tau)\}$ ($\tau \in (T^\beta_0(y,w,t),t]$) be the solution of the problem (2.11), (2.12), (2.22), (2.23).

Lemma 2.4. *Assume that Condition 1.1 is satisfied for some $\delta,\rho>0$ and let δ', ρ' be such that $G_{2\delta'} \neq \varnothing$, $\delta' \geqslant \delta/2$, and $0<\rho' \leqslant 2\rho$. Next, let $\{X^\beta_0(y,w,t,\tau)$, $V^\beta_0(y,w,t,\tau)\}$ be a non-continuable solution of the problem (2.11), (2.12), (2.22), (2.23) on the half-open interval $(T^\beta_0(y,w,t),t]$. Then the following assertions hold.*

(a) *$|V^\beta_0(y,w,t,\tau)| = |w|$ for all $\tau \in (T^\beta_0(y,w,t),t]$.*

(b) *If $y \in \partial Q_{\delta'}$ and $|w| = \rho'$, then $T^\beta_0(y,w,t) = 0$ and $|x' - X^{\beta'}_0(y,w,t,\tau)| < \delta/8$ for all $\tau \in (0,t]$.*

The proof is similar to that of Lemma 2.2.

Lemma 2.5. *Assume that Condition 1.1 holds for some $\delta,\rho>0$ and let δ', ρ' be such that $G_{2\delta'} \neq \varnothing$, $\delta' \geqslant \delta/2$, and $0<\rho' \leqslant 2\rho$. Then for any function $\varphi \in C([0,T],C^{2+\sigma}_0(\overline{Q}))$ with $\|\varphi\|_{1,T} \leqslant R_1$ the non-continuable solution $\{X^\beta_\varphi(y,w,t,\tau)$, $V^\beta_\varphi(y,w,t,\tau)\}$ ($\tau \in (T^\beta_\varphi(y,w,t),t]$) of the problem (2.5), (2.6), (2.22), (2.23) has the following property*:

if $y \in \partial Q_{\delta'}$, $|w| = \rho'$, and $0 < t \leqslant T$, then $T^\beta_\varphi(y,w,t) = 0$, $|y' - X^{\beta'}_\varphi(y,w,t,\tau)| < \delta/4$, and $\rho' - \rho/4 < |V^\beta_\varphi(y,w,t,\tau)| < \rho' + \rho/4$ for $\tau \in (0,T]$.

The proof depends on Lemma 2.4 and is similar to that of Lemma 2.3.

3. Hölder estimates for the electric-charge density

3.1. We set $\Omega_0 = Q_\delta \times B_\rho$ and

$$\Omega^\beta_{\varphi,t} = \{(y,w) \in \mathbb{R}^6 \colon y = X^\beta_\varphi(x,v,t),\ w = V^\beta_\varphi(x,v,t),\ (x,v) \in \Omega_0\},$$

where $0 \leqslant t < T$, $\varphi \in C([0,T],C^{2+\sigma}_0(\overline{Q}))$, and $\|\varphi\|_{2,T} \leqslant R_1$. Clearly, $\Omega^\beta_{\varphi,0} = \Omega_0$.

Given $0 \leqslant t < T$, consider the map $S^\beta_{\varphi,t} \colon \Omega_0 \to \Omega^\beta_{\varphi,t}$ defined by $S^\beta_{\varphi,t}(x,v) = \bigl(X^\beta_\varphi(x,v,t), V^\beta_\varphi(x,v,t)\bigr)$.

Since $\varphi \in C([0,T],C^{2+\sigma}_0(\overline{Q}))$, if we use Condition 1.1, Lemma 2.3 with $\delta' = \delta$ and $\rho' = \rho$, and the fact that the solutions of the differential equations are continuously differentiable with respect to the initial data for any $0 \leqslant t < T$, we see that the map $S^\beta_{\varphi,t} \colon \Omega_0 \to \Omega^\beta_{\varphi,t}$ is continuously differentiable with respect to x and v on Ω_0. Moreover,

$$\Omega^\beta_{\varphi,t} \subset Q_{3\delta/4} \times B_{5\rho/4}. \tag{3.1}$$

It is obvious that $S^\beta_{\varphi,0}(x,v) = (x,v)$. We extend the map $S^\beta_{\varphi,t}$ by continuity at $t = T$.

Consider the system (2.5), (2.6) on the interval $(0,t)$, $0 < t \leqslant T$, with the initial conditions (2.22), (2.23). By (3.1) and Lemma 2.5, the problem (2.5), (2.6), (2.22), (2.23) has a unique non-continuable solution $\{X^\beta_\varphi(y,w,t,\tau), V^\beta_\varphi(y,w,t,\tau)\}$ on the half-open interval $(0,t]$ for any $(y,w) \in Q_{3\delta/4} \times B_{5\rho/4}$ and $0 < t \leqslant T$.

In addition, $X^\beta_\varphi(y,w,t,\tau)\in Q_{\delta/2}$ and $V^\beta_\varphi(y,w,t,\tau)\in B_{3\rho/2}$ for $\tau\in(0,t]$. Extending the functions $X^\beta_\varphi(y,w,t,\tau)$ and $V^\beta_\varphi(y,w,t,\tau)$ by continuity at $\tau=0$, we set $\widehat X^\beta_\varphi(y,w,t)=X^\beta_\varphi(y,w,t,0)$ and $\widehat V^\beta_\varphi(y,w,t)=V^\beta_\varphi(y,w,t,0)$.

Clearly, for any t with $0<t\leqslant T$ the map $\widehat S^\beta_{\varphi,t}\colon \Omega^\beta_{\varphi,t}\to\Omega_0$ defined by

$$\widehat S^\beta_{\varphi,t}(y,w)=\big(\widehat X^\beta_\varphi(y,w,t),\widehat V^\beta_\varphi(y,w,t)\big)$$

is the inverse of the map $S^\beta_{\varphi,t}$; that is,

$$\widehat S^\beta_{\varphi,t}\big(S^\beta_{\varphi,t}(x,v)\big)=(x,v)\qquad((x,v)\in\Omega_0). \tag{3.2}$$

Let $\widehat S^\beta_{\varphi,0}(x,v)=(x,v)$.

We have $\varphi\in C([0,T],C^{2+\sigma}_0(\overline Q))$, and hence, by the theorem on differentiability of the solutions with respect to the initial data, the function $\widehat S^\beta_{\varphi,t}(y,w)$ is continuously differentiable with respect to y and w on the set $\Omega^\beta_{\varphi,t}$. The function $\widehat S^\beta_{\varphi,t}(y,w)$ $((y,w)\in\Omega^\beta_{\varphi,t})$ is continuously differentiable with respect to y and w, and $S^\beta_{\varphi,t}(x,v)$ $((x,v)\in\Omega_0)$ is continuously differentiable with respect to t, therefore it follows from (3.2) that the function $\widehat S^\beta_{\varphi,t}(y,w)$ $((y,w)\in\Omega^\beta_{\varphi,t})$ is continuously differentiable with respect to t.

Lemma 3.1. *Assume that Condition 1.1 is satisfied for some $\delta,\rho>0$ such that $G_{2\delta}\neq\varnothing$, and let $\varphi\in C([0,T],C^{2+\sigma}_0(\overline Q))$ and $\|\varphi\|_{2,T}\leqslant R_1$. Then there is a constant $\widehat c_0=\widehat c_0(T,\delta,\rho,h)>0$ such that*

$$\sum_{i=1}^3\big(|\mathscr D\widehat X^\beta_{\varphi i}(x,v,t)|+|\mathscr D\widehat V^\beta_{\varphi i}(x,v,t)|\big)\leqslant\widehat c_0\qquad((x,v)\in\Omega^\beta_{\varphi,t},\ 0<t<T), \tag{3.3}$$

where $\mathscr D=\dfrac{\partial}{\partial x_j}$ *or* $\dfrac{\partial}{\partial v_j}$ $(j=1,2,3)$.

Proof. Let $(x,v)\in\Omega^\beta_{\varphi,t}$. The variational equations for the system (2.5), (2.6) are

$$\frac{d}{d\tau}\bigg(\frac{\partial X^\beta_{\varphi i}}{\partial x_j}\bigg)=\frac{\partial V^\beta_{\varphi i}}{\partial x_j}\qquad(0<\tau<t,\ 1\leqslant i\leqslant 3), \tag{3.4}$$

$$\frac{d}{d\tau}\bigg(\frac{\partial V^\beta_{\varphi i}}{\partial x_j}\bigg)=-\frac{\beta e}{m_\beta}\sum_{k=1}^3\frac{\partial^2\varphi(X^\beta_\varphi,\tau)}{\partial X^\beta_{\varphi i}\,\partial X^\beta_{\varphi k}}\,\frac{\partial X^\beta_{\varphi k}}{\partial x_j}+\frac{\beta e}{m_\beta c}\bigg[\frac{\partial V^\beta_\varphi}{\partial x_j},B\bigg]_i\qquad(0<\tau<t,\ 1\leqslant i\leqslant 3). \tag{3.5}$$

In view of (2.22) and (2.23) the initial conditions for the system (3.4), (3.5) are

$$\frac{\partial X^\beta_{\varphi i}}{\partial x_j}\Big|_{\tau=t}=\delta_{ij}\qquad(1\leqslant i\leqslant 3), \tag{3.6}$$

$$\frac{\partial V^\beta_{\varphi i}}{\partial x_j}\Big|_{\tau=t}=0\qquad(1\leqslant i\leqslant 3). \tag{3.7}$$

We change the variable to $\xi = \tau$ in (3.4), (3.5) and integrate the resulting equations with respect to ξ from τ to t, taking into account the initial conditions (3.6), (3.7). Then we change to the new variable $s = t - \xi$. Let $\widetilde{X}^\beta_\varphi(s) = X^\beta_\varphi(t-s)$ and $\widetilde{V}^\beta_\varphi(s) = V^\beta_\varphi(t-s)$, and define $\tau_1 = t - \tau$. Then

$$\left|\frac{\partial \widetilde{X}^\beta_{\varphi i}(\tau_1)}{\partial x_j}\right| \leqslant \delta_{ij} + \int_0^{\tau_1} \left|\frac{\partial \widetilde{V}^\beta_{\varphi i}(s)}{\partial x_j}\right| ds \qquad (0 < \tau_1 < t,\ 1 \leqslant i \leqslant 3),$$

$$\left|\frac{\partial \widetilde{V}^\beta_{\beta i}(\tau_1)}{\partial x_j}\right| \leqslant \frac{e\|\varphi\|_{2,T}}{m_\beta} \sum_{k=1}^3 \int_0^{\tau_1} \left|\frac{\partial \widetilde{X}^\beta_{\varphi k}(s)}{\partial x_j}\right| ds + \frac{eh}{m_\beta c} \sum_{k=1}^3 \int_0^{\tau_1} \left|\frac{\partial \widetilde{V}^\beta_{\varphi k}}{\partial x_j}\right| ds \qquad (0 < \tau_1 < t,\ 1 \leqslant i \leqslant 3).$$

From these inequalities and Gronwall's lemma,

$$\sum_{i=1}^3 \left(\left|\frac{\partial X^\beta_{\varphi i}(\tau)}{\partial x_j}\right| + \left|\frac{\partial V_{\varphi i}(\tau)}{\partial x_j}\right|\right) = \sum_{i=1}^3 \left(\left|\frac{\partial \widetilde{X}^\beta_{\varphi i}(\tau_1)}{\partial x_j}\right| + \left|\frac{\partial \widetilde{V}^\beta_{\varphi i}(\tau_1)}{\partial x_j}\right|\right) \leqslant \widehat{c}_0 \qquad (0 < \tau < T). \tag{3.8}$$

Letting $\tau = 0$ in the functions

$$\frac{\partial X^\beta_\varphi(\tau)}{\partial x_j} = \frac{\partial X^\beta_\varphi(x, v, t, \tau)}{\partial x_j} \qquad \text{and} \qquad \frac{\partial V^\beta_\varphi(\tau)}{\partial x_j} = \frac{\partial V^\beta_\varphi(x, v, t, \tau)}{\partial x_j},$$

we obtain (3.3) from (3.8). The case $\mathscr{D} = \partial/\partial v_j$ is dealt with similarly. □

Lemma 3.2. *Assume that Condition 1.1 is satisfied for some $\delta, \rho > 0$, $G_{2\delta} \neq \varnothing$, $\varphi \in C([0,T], C_0^{2+\sigma}(\overline{Q}))$, and $\|\varphi\|_{2,T} \leqslant R_1$. Then there is a constant $c_0 = c_0(T, \delta, \rho, h) > 0$ such that*

$$\sum_{i=1}^3 \bigl(|\mathscr{D} X^\beta_{\varphi i}(x, v, \tau)| + |\mathscr{D} V^\beta_{\varphi i}(x, v, \tau)|\bigr) \leqslant c_0 \qquad ((x, v) \in \Omega_0,\ 0 < t < T). \tag{3.9}$$

The proof is similar to that of Lemma 3.1.

Given a fixed function $\varphi \in C\bigl([0,T], C_0^{2+\sigma}(\overline{Q})\bigr)$ with $\|\varphi\|_{2,T} \leqslant R_1$, let $\{g^\beta_\varphi\}_{\beta=\pm 1}$ denote the solution of the problem (2.2), (2.3). Also, let

$$p^\beta_\varphi(y, w, \tau) = g^\beta_\varphi\bigl(S^\beta_{\varphi,\tau}(y, w), \tau\bigr) \qquad ((y, w) \in \Omega_0,\ 0 \leqslant \tau \leqslant T).$$

Clearly, the function $p^\beta_\varphi(y, w, \tau)$ satisfies the differential equation

$$\frac{\partial p^\beta_\varphi(y, w, \tau)}{\partial \tau} = \Psi^\beta_\varphi\bigl(S^\beta_{\varphi,\tau}(y, w), \tau\bigr) \qquad ((y, w) \in \Omega_0,\ 0 < \tau < T),$$

where

$$\Psi^\beta_\varphi(z, t) = \frac{\beta e}{m_\beta}\bigl(\nabla_{z'} \varphi(z', t), \nabla_{z''} \mathring{f}^\beta(z', z'')\bigr), \qquad z = (z', z''), \quad z', z'' \in \mathbb{R}^3.$$

Integrating this equation with respect to τ from 0 to t, we get that

$$p^\beta_\varphi(y,w,t)=p^\beta_\varphi(y,w,0)+\int_0^t \Psi^\beta_\varphi\big(S^\beta_{\varphi,\tau}(y,w),\tau\big)\,d\tau \qquad ((y,w)\in\Omega_0,\ 0<t<T). \tag{3.10}$$

Let $\mathscr{D}_0=Q_{3\delta/2}\times B_{\rho/2}$, $\mathscr{D}^\beta_{\varphi,t}=\big\{\eta\in\mathbb{R}^6 \colon \eta=S^\beta_{\varphi,t}(y,w),\ (y,w)\in\mathscr{D}_0\big\}$, and $(x,v)=S^\beta_{\varphi,t}(y,w)$.

From (3.10) we get that

$$g^\beta_\varphi(x,v,t)=g^\beta_0\big(\widehat{S}^\beta_{\varphi,t}(x,v)\big)+\int_0^t \Psi^\beta_\varphi\big(S^\beta_{\varphi,\tau}\big(\widehat{S}^\beta_{\varphi,t}(x,v)\big),\tau\big)\,d\tau \tag{3.11}$$
$$((x,v)\in\mathscr{D}^\beta_{\varphi,t},\ 0\leqslant t\leqslant T).$$

We set the function $g^\beta_\varphi(x,v,t)$ equal to zero outside $\mathscr{D}^\beta_{\varphi,t}$:

$$g^\beta_\varphi(x,v,t)=0 \qquad ((x,v)\in(\overline{Q}\times\mathbb{R}^3)\setminus\mathscr{D}^\beta_{\varphi,t},\ 0\leqslant t\leqslant T). \tag{3.12}$$

To show that $g^\beta_\varphi\in C^1(\overline{Q}\times\mathbb{R}^3\times[0,T])$ we recall that

$$\operatorname{supp} g^\beta_0\subset\mathscr{D}_0=Q_{2\delta}\times B_{\rho/4}$$

by the hypotheses of Theorem 1.1. Next, $\widehat{S}^\beta_{\varphi,t}(\mathscr{D}^\beta_{\varphi,t})=\mathscr{D}_0$ by the definition of the map $\widehat{S}^\beta_{\varphi,t}$. Hence, since $\widehat{S}^\beta_{\varphi,t}$ is continuously differentiable with respect to x, v, t, the function $g^\beta_0(\widehat{S}^\beta_{\varphi,t}(x,v))$, extended by zero outside $\mathscr{D}^\beta_{\varphi,t}$, belongs to $C^1(\overline{Q}\times\mathbb{R}^3\times[0,T])$. By the definition of the function Ψ^β_φ and since $S^\beta_{\varphi,\tau}$ and $\widehat{S}^\beta_{\varphi,t}$ are continuously differentiable with respect to x, v, τ, and t, it remains to verify that $\operatorname{supp}\mathring{f}^\beta\subset\mathscr{D}^\beta_{\varphi,\tau}$ $(0\leqslant\tau\leqslant T)$. Applying Lemma 2.3 with $\delta'=3\delta/2$ and $\rho'=\rho/2$, we have

$$\partial\mathscr{D}^\beta_{\varphi,\tau}\subset(Q_{5\delta/4}\setminus\overline{Q}_{7\delta/4})\times(B_{3\rho/4}\setminus\overline{B}_{\rho/4}). \tag{3.13}$$

Therefore, $\operatorname{supp}\mathring{f}^\beta\subset\mathscr{D}^\beta_{\varphi,\tau}$ $(0\leqslant\tau\leqslant T)$, because $\operatorname{supp}\mathring{f}^\beta\subset Q_{2\delta}\times B_{\rho/4}$.

Since the function g^β_φ defined by (3.11) and (3.12) belongs to $C^1(\overline{Q}\times\mathbb{R}^3\times[0,T])$, we see by the method of characteristics that it is a classical solution of the problem (2.2), (2.3), and it is unique.

Let

$$F_\varphi(x,t)=\int_{\mathbb{R}^3}\sum_\beta \beta g^\beta_\varphi(x,v,t)\,dv \qquad (x\in\overline{Q},\ 0\leqslant t\leqslant T). \tag{3.14}$$

Remark 3.1. Assume that Condition 1.1 is satisfied. Then by (3.13) and (3.1) we have $|v|<\rho$ $(|v|<5\rho/4)$ if $(x,v)\in\mathscr{D}^\beta_{\varphi,t}$ (respectively, $(x,v)\in\Omega^\beta_{\varphi,t}$). Thus, in (3.14) we integrate over B_ρ $(B_{5\rho/4})$; that is, the integral in (3.14) exists.

Let

$$m_k=\max_\beta\|g^\beta_0\|_k=\max_\beta\max_{|\alpha|\leqslant k}\sup_{\eta\in Q\times\mathbb{R}^3}|\mathscr{D}^\alpha g^\beta_0(\eta)|$$

and define

$$\delta_1=\min_\beta\inf_{t\in[0,T]}\operatorname{dist}(\mathscr{D}^\beta_{0,t},\partial\Omega^\beta_{0,t}). \tag{3.15}$$

Since $\mathscr{D}_0 = Q_{3\delta/2} \times B_{\rho/2}$ and $\Omega_0 = Q_\delta \times B_\rho$, it follows from Lemma 2.2 that

$$\delta_1 \geqslant \min\left\{\frac{\delta}{4}, \frac{\rho}{2}\right\}. \tag{3.16}$$

Also, let

$$R = \min\{R_1, R_2\}, \tag{3.17}$$

where R_1 satisfies the inequality (2.16) and $R_2 > 0$ satisfies the condition

$$\frac{2eTR_2}{m_{-1}} \exp(a_0 T) < \frac{\delta_1}{8}. \tag{3.18}$$

Next we define $M_s = \{\varphi \in C([0,T], C_0^s(\overline{Q}))\colon \|\varphi\|_{s,T} \leqslant R\}$, where $s > 0$.

From (3.15), the inequality (2.21) with R instead of R_1, and the inequality (3.16) it follows that

$$\delta_2 = \min_\beta \inf_{\varphi\in M_{2+\sigma}} \inf_{t\in[0,T]} \operatorname{dist}(\mathscr{D}^\beta_{\varphi,t}, \partial\Omega^\beta_{\varphi,t}) \geqslant \frac{3\delta_1}{4}. \tag{3.19}$$

Lemma 3.3. *Let $\delta > 0$ be such that $G_{2\delta} \neq \varnothing$. Assume that Conditions* 1.1 and 1.2 *hold and there is a stationary solution $\{0, \mathring{f}^\beta\}$ of the problem* (1.1), (1.2), (1.4) *as in the first assertion of Theorem* 1.1. *Assume also that* $\operatorname{supp}(f_0^\beta - \mathring{f}^\beta) \subset (Q_{2\delta} \cap Q'_N) \times B_{\rho/4}$ *for some $N > 0$. Then $F_\varphi \in C([0,T], C_0^\sigma(\overline{Q}))$ for any $\varphi \in M_{2+\sigma}$.*

Proof. In the first two steps of the proof it will be assumed that $\varphi \in M_{2+\delta}$ and that for each $t \in [0,T]$ the function $\varphi(x,t)$ has compact support with respect to x.

I. Let

$$\begin{aligned} F_{1\varphi}(x,t) &= \int_{\mathbb{R}^3} \sum_\beta \beta g^\beta_{1\varphi}(x,v,t)\,dv, \\ F_{2\varphi}(x,t) &= F_\varphi(x,t) - F_{1\varphi}(x,t) \qquad (x \in \overline{Q}, \quad 0 \leqslant t \leqslant T), \end{aligned} \tag{3.20}$$

where

$$g^\beta_{1\varphi}(x,v,t) = \begin{cases} g_0^\beta\bigl(\widehat{S}^\beta_{\varphi,t}(x,v)\bigr), & (x,v) \in \mathscr{D}^\beta_{\varphi,t},\ 0 \leqslant t \leqslant T, \\ 0, & (x,v) \in (\overline{Q} \times \mathbb{R}^3) \setminus \mathscr{D}^\beta_{\varphi,t},\ 0 \leqslant t \leqslant T. \end{cases}$$

We first show that $F_{1\varphi} \in C([0,T], C(\overline{Q}))$. Using (3.20), Remark 3.1, and Taylor's formula, we get that

$$\begin{aligned} |F_{1\varphi}(x,t+\Delta t) - F_{1\varphi}(x,t)| \leqslant 2|B_\rho| m_1 \sum_\beta \bigl\{ &\sup_{x,v} |\widehat{X}_\varphi(x,v,t+\Delta t) - \widehat{X}^\beta_\varphi(x,v,t)| \\ &+ \sup_{x,v} |\widehat{V}^\beta_\varphi(x,v,t+\Delta t) - \widehat{V}^\beta_\varphi(x,v,t)| \bigr\} \end{aligned}$$

for all $x \in Q$ such that $\{v\colon (x,v) \in \mathscr{D}^\beta_{\varphi,t} \cup \mathscr{D}^\beta_{\varphi,t+\Delta t}\} \neq \varnothing$ and t, $t + \Delta t \in [0,T]$. Here the suprema are taken over the set $\mathscr{D}^\beta_{\varphi,t} \cup \mathscr{D}^\beta_{\varphi,t+\Delta t}$.

The functions $\widehat{X}^\beta_\varphi(x,v,t)$ and $\widehat{V}^\beta_\varphi(x,v,t)$ are continuous on the compact set $\{(x,v,t)\colon (x,v)\in S^\beta_{\varphi,t}(\overline{\mathscr{D}_{0N}}),\ t\in[0,T]\}$, and hence for any $\varepsilon>0$ there exists a $b_0>0$ such that

$$\sup_{x\in Q}|F_{1\varphi}(x,t+\Delta t)-F_{1\varphi}(x,t)|\leqslant\frac{\varepsilon}{5}\tag{3.21}$$

for t, $t+\Delta t\in[0,T]$, $|\Delta t|<b_0$, where $\mathscr{D}_{0N}=\{(y,w)\in\mathscr{D}_0\colon |y_3|<N\}$.

II. We now show that $F_\varphi\in C([0,T],C^\sigma(\overline{Q}))$. Let

$$(\delta_{\Delta x}f)(x)=f(x+\Delta x)-f(x),\qquad \delta_3=\min\{\delta_2,1\},$$

where $\delta_2>0$ is given by (3.19).

From (3.20) it follows that $(\delta_{\Delta x}F_{1\varphi})(x,t)=0$ for all $x\in Q$ such that $\{v\colon (x,v)\in(\mathscr{D}^\beta_{\varphi,t})^{\delta_3}\}=\varnothing$, where $0\leqslant t\leqslant T$, $|\Delta x|\leqslant\delta_3$, and $(\mathscr{D}^\beta_{\varphi,t})^{\delta_3}=\{(x,v)\colon \operatorname{dist}((x,v),\mathscr{D}^\beta_{\varphi,t})<\delta_3\}$. Therefore by Remark 3.1 and Taylor's formula,

$$\begin{aligned}
&|\delta_{\Delta x}F_{1\varphi}(x,t+\Delta t)-\delta_{\Delta x}F_{1\varphi}(x,t)|\\
&\qquad\leqslant|B_{2\rho}|\sum_\beta\sup_{x,\Delta x,v}\bigl|\delta_{\Delta x}g^\beta_0(\widehat{S}^\beta_{\varphi,t+\Delta t}(x,v))-\delta_{\Delta x}g^\beta_0(\widehat{S}^\beta_{\varphi,t}(x,v))\bigr|\\
&\qquad\leqslant\sum_\beta\sum_{l=1,2}\sup_{x,\Delta x,v}\Phi^\beta_l,
\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}
\Phi^\beta_1={}&|B_{2\rho}|\int_0^1\sum_j\Bigl\{\bigl|\delta_{\Delta x}g^\beta_{0X_j}\bigl(\widehat{S}^\beta_{\varphi,t}+\theta(\widehat{S}^\beta_{\varphi,t+\Delta t}-\widehat{S}^\beta_{\varphi,t})\bigr)\bigl(\widehat{X}^\beta_{\varphi j}(x+\Delta x,v,t+\Delta t)\\
&-\widehat{X}^\beta_{\varphi j}(x+\Delta x,v,t)\bigr)\bigr|+\bigl|\delta_{\Delta x}g^\beta_{0V_j}\bigl(\widehat{S}^\beta_{\varphi,t}+\theta(\widehat{S}^\beta_{\varphi,t+\Delta t}-\widehat{S}^\beta_{\varphi,t})\bigr)\\
&\times\bigl(\widehat{V}^\beta_{\varphi j}(x+\Delta x,v,t+\Delta t)-\widehat{V}^\beta_{\varphi j}(x+\Delta x,v,t)\bigr)\bigr|\Bigr\}\,d\theta,\\
\Phi^\beta_2={}&|B_{2\rho}|\int_0^1\sum_j\Bigl\{\bigl|g^\beta_{0X_j}\bigl(\widehat{S}^\beta_{\varphi,t}+\theta(\widehat{S}^\beta_{\varphi,t+\Delta t}-\widehat{S}^\beta_{\varphi,t})\bigr)\delta_{\Delta x}\bigl(\widehat{X}^\beta_{\varphi j}(x,v,t+\Delta t)\\
&-\widehat{X}^\beta_{\varphi j}(x,v,t)\bigr)\bigr|+\bigl|g^\beta_{0V_j}\bigl(\widehat{S}^\beta_{\varphi,t}+\theta(\widehat{S}^\beta_{\varphi,t+\Delta t}-\widehat{S}^\beta_{\varphi,t})\bigr)\\
&\times\delta_{\Delta x}\bigl(\widehat{V}^\beta_{\varphi j}(x,v,t+\Delta t)-\widehat{V}^\beta_{\varphi j}(x,v,t)\bigr)\bigr|\Bigr\}\,d\theta;
\end{aligned}$$

in (3.22) and below, the suprema are taken over $(x,v)\in(\mathscr{D}^\beta_{\varphi,t})^{\delta_3}\cup(\mathscr{D}^\beta_{\varphi,t+\Delta t})^{\delta_3}$, $0<|\Delta x|<\delta_3$, assuming that t, $t+\Delta t\in[0,T]$.

Again by Taylor's formula, Lemma 3.1, and the continuity of the functions $\widehat{X}^\beta_\varphi(x,v,t)$ and $\widehat{V}^\beta_\varphi(x,v,t)$ on the compact set $\{(x,v,t)\colon (x,v)\in\overline{(S^\beta_{\varphi,t}(\mathscr{D}_{0N}))^{\delta_3}},\ t\in[0,T]\}$, it follows that for any $\varepsilon>0$ there exists a $b_1>0$ such that

$$\begin{aligned}
\varphi^\beta_1\leqslant{}&k_1(\rho)m_2\widehat{c}_0\bigl(|\widehat{X}^\beta_\varphi(x+\Delta x,v,t+\Delta t)-\widehat{X}^\beta_\varphi(x+\Delta x,v,t)|\\
&+|\widehat{V}^\beta_\varphi(x+\Delta x,v,t+\Delta t)-\widehat{V}^\beta_\varphi(x+\Delta x,v,t)|\bigr)|\Delta x|<\frac{|\Delta x|\varepsilon}{5}
\end{aligned}$$

for all $(x,v) \in (\mathscr{D}^{\beta}_{\varphi,t})^{\delta_3} \cup (\mathscr{D}^{\beta}_{\varphi,t+\Delta t})^{\delta_3}$, $0 < |\Delta x| < \delta_3$, and $t, t+\Delta t \in [0,T]$ such that $|\Delta t| < b_1$.

Consequently,

$$\sup_{x,x+\Delta x,v} \frac{\Phi_1^{\beta}}{|\Delta x|^{\sigma}} \leqslant \frac{\varepsilon}{5} \quad \text{for} \quad t, t+\Delta t \in [0,T], \quad |\Delta t| < b_1. \tag{3.23}$$

By Taylor's formula and the continuity of the functions $\widehat{X}^{\beta}_{\varphi x_j}(x,v,t)$ and $\widehat{V}^{\beta}_{\varphi x_j}(x,v,t)$ on the compact set $\{(x,v,t)\colon (x,v) \in \overline{(S^{\beta}_{\varphi,t}(\mathscr{D}_{0N}))^{\delta_3}},\ t \in [0,T]\}$, we get that for any $\varepsilon > 0$ there exists a $b_2 > 0$ such that

$$\begin{aligned} \Phi_2^{\beta} \leqslant k_2(\rho) m_1 \sum_j \int_0^1 \bigl\{ |\widehat{X}^{\beta}_{\varphi x_j}(x+s\Delta x, v, t+\Delta t) - \widehat{X}^{\beta}_{\varphi x_j}(x+s\Delta x, v, t)| \\ + |\widehat{V}^{\beta}_{\varphi x_j}(x+s\Delta x, v, t+\Delta t) - \widehat{V}^{\beta}_{\varphi x_j}(x+s\Delta x, v, t)| \bigr\}\, ds\, |\Delta x| < \frac{|\Delta x| \varepsilon}{5} \end{aligned}$$

for all $(x,v) \in (\mathscr{D}^{\beta}_{\varphi,t})^{\delta_3} \cup (\mathscr{D}^{\beta}_{\varphi,t+\Delta t})^{\delta_3}$, $0 < |\Delta x| < \delta_3$, and t, $t+\Delta t \in [0,T]$ such that $|\Delta t| < b_2$.

Hence,

$$\sup_{x,x+\Delta x,v} \frac{\Phi_2^{\beta}}{|\Delta x|^{\sigma}} \leqslant \frac{\varepsilon}{5} \quad \text{for} \quad t, t+\Delta t \in [0,T], \quad |\Delta t| < b_2. \tag{3.24}$$

Let $b_3 = \min\{b_1, b_2, b_0\}$. Then by (3.21)–(3.24) and Remark 1.1 with $b = \delta_3$ we have for any $\varepsilon > 0$

$$\|F_{1\varphi}(\,\cdot\,, t+\Delta t) - F_{1\varphi}(\,\cdot\,, t)\|_{\sigma} \leqslant \varepsilon \quad \text{for} \quad t, t+\Delta t \in [0,T], \quad |\Delta t| < b.$$

Thus, the map $[0,T] \ni t \mapsto F_{1\varphi}(\,\cdot\,, t) \in C^{\sigma}(\overline{Q})$ is continuous on $[0,T]$.

Similarly, using Remark 3.1, Lemmas 3.1 and 3.2, and Taylor's formula, we prove that the map $[0,T] \ni t \mapsto F_{2\varphi}(\,\cdot\,, t) \in C^{\sigma}(\overline{Q})$ is continuous on the interval $[0,T]$.

III. It remains to show that for $\varphi \in C([0,T], C_0^{2+\sigma}(\overline{Q}))$ the map

$$[0,T] \ni t \mapsto F_{\varphi}(\,\cdot\,, t) \in C^{\sigma}(\overline{Q})$$

is continuous on $[0,T]$, and $F_{\varphi}(\,\cdot\,, t) \in C_0^{\sigma}(\overline{Q})$. Indeed, there is a sequence of functions $\varphi_p \in C([0,T], C^{2+\sigma}(\overline{Q}))$ with compact supports in $\overline{Q}$ for each $t \in [0,T]$ and such that $\varphi_p \to \varphi$ in $C([0,T], C^{2+\sigma}(\overline{Q}))$. From the estimate (3.28), which will be proved later, it follows that $F_{\varphi_p} \to F_{\varphi}$ in the norm of the space $C([0,T], C^{\sigma}(\overline{Q}))$. By the above, $F_{\varphi_p} \in C([0,T], C^{\sigma}(\overline{Q}))$. Hence $F_{\varphi} \in C([0,T], C^{\sigma}(\overline{Q}))$. Moreover, since $g_0^{\beta} \subset Q'_N$ and $\varphi_p(\,\cdot\,, t)$ have compact supports for $t \in [0,T]$, the functions $F_{\varphi_p}(\,\cdot\,, t)$ are also compactly supported for $t \in [0,T]$. Consequently, $F_{\varphi}(\,\cdot\,, t) \in C_0^{\sigma}(\overline{Q})$ for all $t \in [0,T]$ by definition. $\square$

We set $n_{k+1} = \max_{\beta,i} \|\mathring{f}_{v_i}\|_k$.

Lemma 3.4. *Let the hypotheses of Lemma 3.3 hold. Then for any $\varphi \in M_{2+\sigma}$*

$$\|F_{\varphi}\|_{\sigma,T} \leqslant c_1(m_1 + n_2), \tag{3.25}$$

where $c_1 = c_1(T, \delta, \rho, h, \sigma) > 0$ is independent of φ.

Proof. I. From (3.20) and Remark 3.1 it follows that

$$|F_{1\varphi}(x,t)| \leqslant \sum_{\beta} \int_{|v|<\rho} \bigl|g_0^\beta\bigl(\widehat{S}_{\varphi,t}(x,v)\bigr)\bigr|\,dv \leqslant 2|B_\rho|m_0 \quad (x\in Q,\ 0\leqslant t\leqslant T). \tag{3.26}$$

II. In view of (3.19) we have $\delta_3>0$. Clearly, $(\delta_{\Delta x}F_\varphi)(x,t)=0$ for $x\in Q$ such that $\{v\colon (x,v)\in(\mathscr{D}^\beta_{\varphi,t})^{\delta_3}\}=\varnothing$, where $0\leqslant t\leqslant T$ and $|\Delta x|\leqslant\delta_3$. Hence by Remark 3.1, Taylor's formula, and Lemma 3.1,

$$\begin{aligned}|\delta_{\Delta x}F_{1\varphi}(x,t)| &\leqslant \sum_\beta \int_{|v|<2\rho} \bigl|\delta_{\Delta x}g_0^\beta\bigl(\widehat{S}_{\varphi,t}(x,v)\bigr)\bigr|\,dv\\ &\leqslant m_1\sum_\beta\int_{|v|<2\rho}\sum_i\bigl\{|\delta_{\Delta x}\widehat{X}^\beta_{\varphi i}(x,v,t)|+|\delta_{\Delta x}\widehat{V}^\beta_{\varphi i}(x,v,t)|\bigr\}\,dv\\ &\leqslant 2m_1\widehat{c}_0|B_{2\rho}|\,|\Delta x| \end{aligned}\tag{3.27}$$

for all $x\in Q$ such that $\{v\colon (x,v)\in(\mathscr{D}^\beta_{\varphi,t})^{\delta_3}\}\neq\varnothing$, and for $|\Delta x|\leqslant\delta_3$ and $0\leqslant t\leqslant T$.

Note that $\delta_3=\delta_3(T,\delta,\rho,h)$. Therefore, $\|F_{1\varphi}\|_{\sigma,T}\leqslant k_1m_1$ by (3.26), (3.27), the condition $|\Delta x|\leqslant\delta_3\leqslant 1$, and Remark 1.1 with $b=\delta_3$. Similarly, $\|F_{2\varphi}\|_{\sigma,T}\leqslant k_2n_2$. Here $k_i=k_i(T,\delta,\rho,h,\sigma)>0$. This gives us the inequality (3.25). ☐

Lemma 3.5. *Let the hypotheses of Lemma 3.3 hold. Then for any $\varphi_1,\varphi_2\in M_{2+\sigma}$*

$$\|F_{\varphi_1}-F_{\varphi_2}\|_{\sigma,T}\leqslant c_2(m_2+n_3)\|\varphi_1-\varphi_2\|_{2,T}, \tag{3.28}$$

where $c_2=c_2(T,\delta,\rho,h,\sigma)>0$ is independent of φ_1 and φ_2.

Proof. I. By definition, $\operatorname{supp}F^\beta_{1\varphi_j}(x,v,t)\subset\mathscr{D}^\beta_{\varphi_j,t}$ $(j=1,2, 0\leqslant t\leqslant T)$. Using (3.15), (3.18), and the inequality (2.21) with R instead of R_1, we get that

$$\min_\beta\ \inf_{\varphi_1,\varphi_2\in M_{2+\sigma}}\ \inf_{t\in[0,T]}\operatorname{dist}(\mathscr{D}^\beta_{\varphi_k,t},\partial\Omega_{\varphi_j,t})\geqslant\frac{3\delta_1}{4};$$

that is, $\mathscr{D}^\beta_{\varphi_1,t}\cup\mathscr{D}^\beta_{\varphi_2,t}\subset\Omega_{\varphi_j,t}$ $(j=1,2)$. Hence, the maps $\widehat{S}^\beta_{\varphi_j,t}(x,v)$ are defined for all $(x,v)\in\mathscr{D}^\beta_{\varphi_1,t}\cup\mathscr{D}^\beta_{\varphi_2,t}$ and $0\leqslant t\leqslant T$.

On the other hand, by (3.20) we have $F_{1\varphi_1}(x,t)=F_{1\varphi_2}(x,t)=0$ for all $x\in Q$ such that $\{v\colon (x,v)\in\bigcup_\beta(\mathscr{D}^\beta_{\varphi_1,t}\cup\mathscr{D}^\beta_{\varphi_2,t})\}=\varnothing$, where $0\leqslant t\leqslant T$. Therefore by Remark 3.1 and Taylor's formula,

$$\begin{aligned}|F_{1\varphi_1}(x,t)-F_{1\varphi_2}(x,t)| &\leqslant\sum_\beta\int_{|v|<\rho}\bigl|g_0^\beta\bigl(\widehat{S}^\beta_{\varphi_1,t}(x,v)\bigr)-g_0^\beta\bigl(\widehat{S}^\beta_{\varphi_2,t}(x,v)\bigr)\bigr|\,dv\\ \leqslant 2m_1\sum_\beta\int_{|v|<\rho}&\{|\widehat{X}^\beta_{\varphi_1}(x,v,t)-\widehat{X}^\beta_{\varphi_2}(x,v,t)|+|\widehat{V}^\beta_{\varphi_1}(x,v,t)-\widehat{V}^\beta_{\varphi_2}(x,v,t)|\}\,dv\end{aligned}\tag{3.29}$$

for all $x\in Q$ such that $\bigl\{v\colon (x,v)\in\bigcup_\beta(\mathscr{D}^\beta_{\varphi_1,t}\cup\mathscr{D}^\beta_{\varphi_2,t})\bigr\}\neq\varnothing$, where $0\leqslant t\leqslant T$.

We now estimate the right-hand side of (3.29).

Let $\{X^\beta_{\varphi_l}(\tau), V^\beta_{\varphi_l}(\tau)\} = \{X^\beta_{\varphi_l}(x,v,t,\tau), V^\beta_{\varphi_l}(x,v,t,\tau)\}$, $\tau \in (0,t]$, be the solution of the system (2.5), (2.6) for $\varphi = \varphi_l$ $(l = 1,2)$ with the initial conditions

$$X_{\varphi_l}(x,v,t,\tau)\big|_{\tau=t} = x, \qquad V_{\varphi_l}(x,v,t,\tau)\big|_{\tau=t} = v, \tag{3.30}$$

where $(x,v) \in \mathscr{D}^\beta_{\varphi_1,t} \cup \mathscr{D}^\beta_{\varphi_2,t}$.

By Taylor's formula,

$$\begin{aligned} &\frac{d}{d\tau}(X^\beta_{\varphi_1} - X^\beta_{\varphi_2}) = (V^\beta_{\varphi_1} - V^\beta_{\varphi_2}) \qquad (0 < \tau < t),\\ &\frac{d}{d\tau}(V^\beta_{\varphi_1} - V^\beta_{\varphi_2}) = -\frac{\beta e}{m_\beta}\sum_{j=1}^{3}\int_0^1 \Big(\frac{\partial}{\partial X_j}\nabla_X\varphi_1\Big)\big(X^\beta_{\varphi_2} + \theta(X^\beta_{\varphi_1} - X^\beta_{\varphi_2}),\tau\big)\,d\theta\\ &\qquad\qquad \times (X^\beta_{\varphi_1,j} - X^\beta_{\varphi_2,j}) - \frac{\beta e}{m_\beta}\big(\nabla_X\varphi_1(X^\beta_{\varphi_2},\tau) - \nabla_X\varphi_2(X^\beta_{\varphi_2},\tau)\big)\\ &\qquad\qquad + \frac{\beta e}{m_\beta c}[V^\beta_{\varphi_1} - V^\beta_{\varphi_2}, B] \qquad (0 < \tau < t). \end{aligned} \tag{3.31}$$

Changing the variable to $s = \tau$, integrating the system (3.31) with respect to s from τ to t, $0 < \tau < t$, and taking into account the initial conditions (3.30), we find that

$$\begin{aligned} &|X^\beta_{\varphi_1}(\tau) - X^\beta_{\varphi_2}(\tau)| \leqslant \int_\tau^t |V^\beta_{\varphi_1}(s) - V^\beta_{\varphi_2}(s)|\,ds,\\ &|V^\beta_{\varphi_1}(\tau) - V^\beta_{\varphi_2}(\tau)| \leqslant \frac{2e}{m_\beta}\|\varphi_1\|_{2,T}\int_\tau^t |X^\beta_{\varphi_1}(s) - X^\beta_{\varphi_2}(s)|\,ds\\ &\qquad\qquad + \frac{2Te}{m_\beta}\|\varphi_1 - \varphi_2\|_{1,T} + \frac{e}{m_\beta c}h\int_\tau^t |V^\beta_{\varphi_1}(s) - V^\beta_{\varphi_2}(s)|\,ds. \end{aligned} \tag{3.32}$$

We introduce the new variables $s_1 = t - s$, $\tau_1 = t - \tau$ and define $\widetilde{X}^\beta_{\varphi_j}(\tau_1) = X^\beta_{\varphi_j}(\tau)$ and $\widetilde{V}^\beta_{\varphi_j}(\tau_1) = V^\beta_{\varphi_j}(\tau)$ $(j = 1,2)$. An appeal to the inequalities (3.32) and Gronwall's lemma shows that

$$\begin{aligned} &|X^\beta_{\varphi_1}(\tau) - X^\beta_{\varphi_2}(\tau)| + |V^\beta_{\varphi_1}(\tau) - V^\beta_{\varphi_2}(\tau)|\\ &\qquad = |\widetilde{X}^\beta_{\varphi_1}(\tau_1) - \widetilde{X}^\beta_{\varphi_2}(\tau_1)| + |\widetilde{V}^\beta_{\varphi_1}(\tau_1) - \widetilde{V}^\beta_{\varphi_2}(\tau_1)| \leqslant k_0\|\varphi_1 - \varphi_2\|_{1,T}, \end{aligned} \tag{3.33}$$

where $k_0 = k_0(T,\delta,\rho,h) > 0$ is independent of φ_1 and φ_2.

Putting $\tau = 0$ in (3.33), we have

$$\begin{aligned} &|X^\beta_{\varphi_1}(0) - X^\beta_{\varphi_2}(0)| + |V^\beta_{\varphi_1}(0) - V^\beta_{\varphi_2}(0)|\\ &\qquad = |\widehat{X}^\beta_{\varphi_1}(x,v,t) - \widehat{X}^\beta_{\varphi_2}(x,v,t)| + |\widehat{V}^\beta_{\varphi_1}(x,v,t) - \widehat{V}^\beta_{\varphi_2}(x,v,t)|\\ &\qquad \leqslant k_0\|\varphi_1 - \varphi_2\|_{1,T}. \end{aligned} \tag{3.34}$$

It now follows from (3.29) and (3.34) that

$$\sup_{x\in Q}|F_{1\varphi_1}(x,t) - F_{1\varphi_2}(x,t)| \leqslant 4m_1k_0|B_\rho|\,\|\varphi_1 - \varphi_2\|_{1,T}. \tag{3.35}$$

II. By (3.20) we have $\delta_{\Delta x}F_{1\varphi_j}(x,t)=0$, $j=1,2$, for $x\in Q$ such that $\{v\colon (x,v)\in (\mathscr{D}^\beta_{\varphi_1,t})^{\delta_3}\cup(\mathscr{D}^\beta_{\varphi_2,t})^{\delta_3}\}\neq\varnothing$, where $0\leqslant t\leqslant T$ and $|\Delta x|\leqslant\delta_3$.

Hence, by Remark 3.1 and Taylor's formula,

$$\begin{aligned}
&\bigl|\delta_{\Delta x}\bigl(F_{1\varphi_1}(x,t)-F_{1\varphi_2}(x,t)\bigr)\bigr| \\
&\qquad\leqslant\sum_\beta\int_{|v|<2\rho}\bigl|\delta_{\Delta x}\bigl(g_0^\beta(\widehat S^\beta_{\varphi_1,t}(x,v))-g_0^\beta(\widehat S^\beta_{\varphi_2,t}(x,v))\bigr)\bigr|\,dv \\
&\qquad\leqslant\sum_\beta\sum_{j=1,2}I_j^\beta,
\end{aligned}\tag{3.36}$$

where

$$\begin{aligned}
I_1^\beta=\int_{|v|<2\rho}dv\int_0^1\sum_{j=1}^3\Bigl\{&\bigl|\delta_{\Delta x}g^\beta_{0X_j}\bigl(\widehat S^\beta_{\varphi_2,t}+\theta\bigl(\widehat S^\beta_{\varphi_1,t}-\widehat S^\beta_{\varphi_2,t}\bigr)\bigr) \\
&\times\bigl(\widehat X^\beta_{\varphi_1,j}(x+\Delta x,v,t)-\widehat X^\beta_{\varphi_2,j}(x+\Delta x,v,t)\bigr)\bigr| \\
&+\bigl|\delta_{\Delta x}g^\beta_{0V_j}\bigl(\widehat S^\beta_{\varphi_2,t}+\theta\bigl(\widehat S^\beta_{\varphi_1,t}-\widehat S^\beta_{\varphi_2,t}\bigr)\bigr)\bigl(\widehat V^\beta_{\varphi_1,j}(x+\Delta x,v,t) \\
&-\widehat V^\beta_{\varphi_2,j}(x+\Delta x,v,t)\bigr)\bigr|\Bigr\}\,d\theta, \\
I_2^\beta=\int_{|v|<2\rho}dv\int_0^1\sum_{j=1}^3\Bigl\{&\bigl|g^\beta_{0X_j}\bigl(\widehat S^\beta_{\varphi_2,t}+\theta\bigl(\widehat S^\beta_{\varphi_1,t}-\widehat S^\beta_{\varphi_2,t}\bigr)\bigr)\delta_{\Delta x}\bigl(\widehat X^\beta_{\varphi_1,j}-\widehat X^\beta_{\varphi_2,j}\bigr)\bigr| \\
&+\bigl|g^\beta_{0V_j}\bigl(\widehat S^\beta_{\varphi_2,t}+\theta\bigl(\widehat S^\beta_{\varphi_1,t}-\widehat S^\beta_{\varphi_2,t}\bigr)\bigr)\delta_{\Delta x}\bigl(\widehat V^\beta_{\varphi_1,j}-\widehat V^\beta_{\varphi_2,j}\bigr)\bigr|\Bigr\}\,d\theta.
\end{aligned}$$

Let us estimate I_1^β. Clearly, the inequality (3.34), which was obtained for $(x,v)\in\bigcup_\beta(\mathscr{D}^\beta_{\varphi_1,t}\cup\mathscr{D}^\beta_{\varphi_2,t})$ and $0\leqslant t\leqslant T$, is also true for $(x,v)\in\bigcup_\beta((\mathscr{D}^\beta_{\varphi_1,t})^{\delta_3}\cup(\mathscr{D}^\beta_{\varphi_2,t})^{\delta_3})$ and $0\leqslant t\leqslant T$. Therefore by Lemma 3.1,

$$I_1^\beta\leqslant\widehat k_0\widehat c_0 m_2|B_{2\rho}|\,\|\varphi_1-\varphi_2\|_{1,T}|\Delta x|.\tag{3.37}$$

It is readily checked that

$$I_2^\beta\leqslant 2m_1|B_{2\rho}|\{|\delta_{\Delta x}(\widehat X^\beta_{\varphi_1}-\widehat X^\beta_{\varphi_2})|+|\delta_{\Delta x}(\widehat V^\beta_{\varphi_1}-\widehat V^\beta_{\varphi_2})|\}.\tag{3.38}$$

To estimate the right-hand side of (3.38), we apply the operator $\delta_{\Delta x}$ to both parts of the system (3.31). As a result,

$$\frac{d}{d\tau}\,\delta_{\Delta x}(X^\beta_{\varphi_1}-X^\beta_{\varphi_2})=\delta_{\Delta x}(V^\beta_{\varphi_1}-V^\beta_{\varphi_2})\qquad(0<\tau<t),\tag{3.39}$$

$$\frac{d}{d\tau}\,\delta_{\Delta x}(V^\beta_{\varphi_1}-V^\beta_{\varphi_2})=-\frac{\beta e}{m_\beta}\sum_{\mu=1}^4 J^\beta_\mu\qquad(0<\tau<t),\tag{3.40}$$

where

$$J_1^\beta = \sum_{j=1}^3 \int_0^1 \Big(\delta_{\Delta x}\frac{\partial}{\partial X_j}\nabla_X\varphi_1\Big)\big(X^\beta_{\varphi_2}+\theta(X^\beta_{\varphi_1}-X^\beta_{\varphi_2}),\tau\big)\,d\theta \\ \times\big(X^\beta_{\varphi_1,j}(x+\Delta x,v,t,\tau)-X^\beta_{\varphi_2,j}(x+\Delta x,v,t,\tau)\big),$$

$$J_2^\beta = \sum_{j=1}^3 \int_0^1 \Big(\frac{\partial}{\partial X_j}\nabla_X\varphi_1\Big)\big(X^\beta_{\varphi_2}+\theta(X^\beta_{\varphi_1}-X^\beta_{\varphi_2}),\tau\big)\,d\theta\,\delta_{\Delta x}(X^\beta_{\varphi_1,j}-X^\beta_{\varphi_2,j}),$$

$$J_3^\beta = \delta_{\Delta x}\big(\nabla_X\varphi_1(X^\beta_{\varphi_2},\tau)-\nabla_X\varphi_2(X^\beta_{\varphi_2},\tau)\big),$$

$$J_4^\beta = -\frac{1}{c}[\delta_{\Delta x}(V^\beta_{\varphi_1}-V^\beta_{\varphi_2}),B].$$

Using Taylor's formula, Lemma 2.5, and the inequalities (3.8) and (3.33), we see that

$$\begin{aligned} &|J_1^\beta|\leqslant k_1\|\varphi_1-\varphi_2\|_{1,T}|\Delta x|^\sigma, \qquad |J_2^\beta|\leqslant k_2|\delta_{\Delta x}(X^\beta_{\varphi_1}-X^\beta_{\varphi_2})|,\\ &|J_3^\beta|\leqslant k_3\|\varphi_1-\varphi_2\|_{2,T}|\Delta x|, \qquad |J_4^\beta|\leqslant k_4|\delta_{\Delta x}(V_{\varphi_1}-V_{\varphi_2})|,\end{aligned} \tag{3.41}$$

where the $k_j=k_j(T,\delta,\rho,h)>0$ with $j=1,\dots,4$ are independent of φ_1 and φ_2.

By (3.30) the initial conditions for the system of differential equations (3.39), (3.40) are

$$\delta_{\Delta x}(X^\beta_{\varphi_1}-X^\beta_{\varphi_2})\big|_{\tau=t}=0, \tag{3.42}$$

$$\delta_{\Delta x}(V^\beta_{\varphi_1}-V^\beta_{\varphi_2})\big|_{\tau=t}=0. \tag{3.43}$$

Integrating the system (3.39), (3.40) from τ to t, $0<\tau<t$, with the initial conditions (3.42) and (3.43), we get by (3.41) that

$$\big|\delta_{\Delta x}\big(X^\beta_{\varphi_1}(\tau)-X^\beta_{\varphi_2}(\tau)\big)\big| \leqslant \int_\tau^t \big|\delta_{\Delta x}\big(V^\beta_{\varphi_1}(s)-V^\beta_{\varphi_2}(s)\big)\big|\,ds, \tag{3.44}$$

$$\big|\delta_{\Delta x}\big(V^\beta_{\varphi_1}(\tau)-V^\beta_{\varphi_2}(\tau)\big)\big| \leqslant \frac{Te}{m_\beta}(k_1+k_3)\|\varphi_1-\varphi_2\|_{2,T}|\Delta x|^\sigma \\ + \frac{e}{m_\beta}\int_\tau^t\big\{k_2\big|\delta_{\Delta x}\big(X^\beta_{\varphi_1}(s)-X^\beta_{\varphi_2}(s)\big)\big| \\ + k_4\big|\delta_{\Delta x}\big(V^\beta_{\varphi_1}(s)-V^\beta_{\varphi_2}(s)\big)\big|\big\}\,ds. \tag{3.45}$$

Making a change of variables in (3.44), (3.45) and using Gronwall's lemma, we have, as in the case of (3.34),

$$\big|\delta_{\Delta x}\big(X^\beta_{\varphi_1}(0)-X^\beta_{\varphi_2}(0)\big)\big|+\big|\delta_{\Delta x}\big(V^\beta_{\varphi_1}(0)-V^\beta_{\varphi_2}(0)\big)\big| \\ = \big|\delta_{\Delta x}\big(\widehat{X}_{\varphi_1}(x,v,t)-\widehat{X}_{\varphi_2}(x,v,t)\big)\big| \\ + \big|\delta_{\Delta x}\big(\widehat{V}_{\varphi_1}(x,v,t)-\widehat{V}_{\varphi_2}(x,v,t)\big)\big| \leqslant k_5\|\varphi_1-\varphi_2\|_{2,T}|\Delta x|^\sigma,$$

where $k_5=k_5(T,\delta,\rho,h)>0$ is independent of φ_1 and φ_2.

Thus, from (3.36)–(3.38) we deduce the estimate

$$\sup_{\substack{x,x+\Delta x\in Q,\\ 0<|\Delta x|<\delta_3}} \frac{|\delta_{\Delta x}(F_{1\varphi_1}(x,t)-F_{1\varphi_2}(x,t))|}{|\Delta x|^\sigma} \leqslant k_6 m_2 \|\varphi_1-\varphi_2\|_{2,T}, \tag{3.46}$$

where $k_6=k_6(T,\delta,\rho,h)>0$ is independent of φ_1 and φ_2.

Next, by (3.35), (3.46), and Remark 1.1,

$$\|F_{1\varphi_1}-F_{1\varphi_2}\|_{\sigma,T} \leqslant k_7 m_2 \|\varphi_1-\varphi_2\|_{2,T}.$$

Similarly,

$$\|F_{2\varphi_1}-F_{2\varphi_2}\|_{\sigma,T} \leqslant k_8 n_3 \|\varphi_1-\varphi_2\|_{2,T}.$$

Here $k_i=k_i(T,\delta,\rho,h,\sigma)>0$ for $i=7,8$. This proves the inequality (3.28). $\square$

4. Proof of Theorem 1.1

4.1. To prove Theorem 1.1 we shall need an auxiliary result on the unique solvability of the Poisson equation with Dirichlet condition in an infinite cylinder. A closely related result can be found in [87], but for the reader's convenience we present here a fairly simple independent proof of this fact.

Consider the Poisson equation

$$-\Delta u(x)=f(x) \qquad (x\in Q) \tag{4.1}$$

with the Dirichlet boundary condition

$$u(x)=0 \qquad (x\in\partial Q). \tag{4.2}$$

We define $C_0(\overline{Q})=\{w\in C(\overline{Q})\colon w(x)\to 0$ as $|x_3|\to\infty$ uniformly with respect to $x'\in G\}$. Obviously, $C_0^s(\overline{Q})\subset C_0(\overline{Q})$.

In what follows, the norm in the Hölder space $C^s(\overline{\mathscr{D}})$ will be denoted by $\|\cdot\|_{C^s(\overline{\mathscr{D}})}$ to emphasize the domain $\mathscr{D}$ in question.

Lemma 4.1. *For any function $f\in C_0^\sigma(\overline{Q})$ there is a unique solution $u\in C_0^{2+\sigma}(\overline{Q})$ of the problem (4.1), (4.2). Moreover,*

$$\|u\|_{C^{2+\sigma}(\overline{Q})} \leqslant c_3 \|f\|_{C^\sigma(\overline{Q})}, \tag{4.3}$$

where $c_3>0$ is independent of f.

Proof. Let us first assume that $f\in C^\sigma(\overline{Q})$ has compact support.

I. We claim that

$$\|u\|_{C(\overline{Q})} \leqslant k_1 \|f\|_{C(\overline{Q})} \tag{4.4}$$

for any solution $u\in C^2(Q)\cap C_0(\overline{Q})$ of the problem (4.1), (4.2), where $k_1>0$ is independent of f.

By the maximum principle, for $f(x)\equiv 0$ $(x\in\overline{Q})$ there is a unique trivial solution of (4.1), (4.2) in $C^2(Q)\cap C_0(\overline{Q})$, which clearly satisfies the inequality (4.4). So below we assume that $f(x)\not\equiv 0$ $(x\in\overline{Q})$.

Consider the auxiliary problem

$$\Delta_{x'}\psi(x') = -1 \qquad (x' \in G), \tag{4.5}$$

$$\psi(x') = 0 \qquad (x' \in \partial G). \tag{4.6}$$

This problem has a unique solution $\psi \in C^\infty(\overline{G})$, and $\psi(x') \geqslant 0$ $(x' \in \overline{G})$. Obviously, the function $v(x) = A\psi(x')$ is a solution of the problem

$$\Delta v(x) = -A \qquad (x \in Q), \tag{4.7}$$

$$v(x) = 0 \qquad (x \in \partial Q), \tag{4.8}$$

where $A \in \mathbb{R}$.

Let $A = 2\|f\|_{C(\overline{Q})}$. We claim that $u(x) \leqslant A\psi(x')$ for all $x \in \overline{Q}$. Assume on the contrary that there exists an $x^0 \in Q$ such that $u(x^0) > v(x^0)$. Since $u \in C_0(\overline{Q})$, there is a number $N > 0$ such that $u(x) < u(x^0) - v(x^0)$ for $|x_3| \geqslant N$, $x' \in \overline{G}$, and $x^0 \in G \times (-N, N)$. Then $u(x) - v(x) < u(x^0) - v(x^0)$ for $|x_3| \geqslant N$ and $x' \in \overline{G}$, because $v(x) \geqslant 0$. Let $Q_{N,f} = \{x \in Q \colon |x_3| < N,\ u(x) > v(x)\}$ and $w(x) = v(x) - u(x)$. By construction, $x^0 \in Q_{N,f}$. Hence $Q_{N,f} \neq \varnothing$. Clearly, $w(x) = 0$ for $x \in \partial G \times [-N, N]$, and $w(x) > w(x^0)$ for $x \in (G \times \{-N\}) \cup (G \times \{N\})$. Since $w(x^0) < 0$, the function $w(x)$ takes its minimum negative value on the set $\overline{Q}_{N,f}$ at a point $x^1 \in Q_{N,f}$. Therefore, $\Delta w(x^1) \geqslant 0$. On the other hand, $\Delta w(x^1) < 0$ since $A > \|f\|_{C(\overline{Q})}$. This contradiction shows that $u(x) \leqslant A\psi(x')$. A similar argument gives us that $u(x) \geqslant -A\psi(x')$. Consequently, $|u(x)| \leqslant A\psi(x')$, and the inequality (4.4) now follows.

II. We claim that if $u \in C^2(Q) \cap C_0(\overline{Q})$ is a solution of the problem (4.1), (4.2), then $u \in C^{2+\sigma}(\overline{Q})$ and (4.3) holds.

Let $Q'_N = \{x \in Q \colon |x_3| < N\}$. By Lemma 6.18 in Chap. 6 of [37], $u \in C^{2+\sigma}(\overline{Q'_N})$ for any $N > 0$. This together with Lemma 6.5 in the same chapter gives us that for any $N > 0$

$$\|u\|_{C^{2+\sigma}(\overline{Q'_N})} \leqslant k_2\big(\|u\|_{C(\overline{Q'_{N+1}})} + \|f\|_{C^\sigma(\overline{Q'_{N+1}})}\big), \tag{4.9}$$

where $k_2 > 0$ is independent of N and f.

Using (4.9) and (4.4), we get that

$$\|u\|_{C^{2+\sigma}(\overline{Q'_N})} \leqslant k_2\big(\|u\|_{C(\overline{Q})} + \|f\|_{C^\sigma(\overline{Q})}\big) \leqslant k_3\|f\|_{C^\sigma(\overline{Q})}. \tag{4.10}$$

This gives us the estimate (4.3).

III. We now assert that the problem (4.1), (4.2) has a solution $u \in C^2(Q) \cap C_0(\overline{Q})$ for any compactly supported function $f \in C^\sigma(\overline{Q})$.

We define the weight space $W^k_{2,\beta}(Q)$ as the completion of the space $C_0^\infty(\overline{Q})$ with respect to the norm

$$\|u\|_{W^k_{2,\beta}(Q)} = \left(\sum_{|\alpha| \leqslant k} \int_Q e^{2\beta x_3} |\mathscr{D}^\alpha u(x)|^2\, dx\right)^{1/2},$$

where $C_0^\infty(\overline{Q})$ is the space of compactly supported infinitely differentiable functions on $\overline{Q}$, $k \geqslant 0$ is an integer, and $\beta \in \mathbb{R}$. For $\beta = 0$ the space $W^k_{2,\beta}(Q)$ coincides with the Sobolev space $W^k_2(Q)$.

Together with problem (4.1), (4.2), we consider the auxiliary eigenvalue/eigenfunction problem

$$-\Delta_{x'} e(x') + \lambda^2 e(x') = 0 \qquad (x' \in G), \tag{4.11}$$
$$e(x') = 0 \qquad (x' \in \partial G). \tag{4.12}$$

It is known that all the eigenvalues of this problem are purely imaginary and isolated, and have finite multiplicity. Also, zero is not an eigenvalue of the problem. From Theorem 1.1 in [65] it follows that if the line $\operatorname{Im}\lambda = \beta$ does not contain eigenvalues of the problem (4.11), (4.12), then the boundary-value problem (4.1), (4.2) has a unique solution $u \in W^2_{2,\beta}(Q)$ for any right-hand side $f \in W^0_{2,\beta}(Q)$.

Therefore, (4.1), (4.2) has a unique solution $u \in W^2_{2,0}(Q) = W^2_2(Q)$ for any compactly supported function $f \in C^\sigma(\overline{Q})$. Since $f \in C^\sigma(\overline{Q})$, it follows from Theorem 9.19 in Chap. 9 of [37] that $u \in C^{2+\sigma}(\overline{Q'_N})$ for any $N > 0$. Moreover, the relation $u \in W^2_2(Q)$ implies that $\|u\|_{W^2_2(Q\setminus\overline{Q'_N})} \to 0$ as $N \to \infty$. By the Sobolev embedding theorem, $u \in C(\overline{Q})$ and

$$\|u\|_{C(\overline{Q\setminus Q'_N})} \to 0 \quad \text{as } N \to \infty; \tag{4.13}$$

that is, $u \in C_0(\overline{Q})$. Consequently, for any compactly supported function $f \in C^\sigma(\overline{Q})$ there is a solution $u \in C^2(Q) \cap C_0(\overline{Q})$ of the problem (4.1), (4.2). Hence, from parts I and II of the proof it follows that for any compactly supported function $f \in C^\sigma(\overline{Q})$ there is a unique solution $u \in C^{2+\sigma}(\overline{Q}) \cap C_0(\overline{Q})$ of the problem (4.1), (4.2), and moreover, the estimate (4.3) holds.

IV. We now prove that $u \in C_0^{2+\sigma}(\overline{Q})$. It suffices to check that $\xi_N u \to u$ in $C^{2+\sigma}(\overline{Q})$ as $N \to \infty$, where $\xi_N = \xi_N(x_3) \in \dot{C}^\infty(\mathbb{R})$ is an even function, $0 \leqslant \xi_N(x_3) \leqslant 1$ for $x_3 \in \mathbb{R}$, $\xi_N(x_3) = 1$ for $|x_3| \leqslant N$, $\xi_N(x_3) = 0$ for $|x_3| \geqslant N+1$, and $|\xi_N^{(i)}(x_3)| \leqslant k'_3$ for $x_3 \in \mathbb{R}$ $(i = 1, 2, 3)$, with $k'_3 > 0$ independent of x_3 and N.

Using the estimate for the norm of the product of two functions in the Hölder space together with an inequality similar to (4.9), we obtain

$$\begin{aligned}
\|(1-\xi_N)u\|_{C^{2+\sigma}(\overline{Q})} &= \|(1-\xi_N)u\|_{C^{2+\sigma}(\overline{Q\setminus Q'_N})} \\
&\leqslant k_4 \|1-\xi_N\|_{C^{2+\sigma}(\overline{Q\setminus Q'_N})} \|u\|_{C^{2+\sigma}(\overline{Q\setminus Q'_N})} \\
&\leqslant k_5 \|u\|_{C^{2+\sigma}(\overline{Q\setminus Q'_N})} \\
&\leqslant k_6 \big(\|u\|_{C(\overline{Q\setminus Q'_{N-1}})} + \|f\|_{C^\sigma(\overline{Q\setminus Q'_{N-1}})}\big) \\
&= k_6 \|u\|_{C(\overline{Q\setminus Q'_{N-1}})}.
\end{aligned} \tag{4.14}$$

Here N is such that $f(x) \equiv 0$ for $x \in \overline{Q \setminus Q'_{N-1}}$, and $k_4, k_5, k_6 > 0$ are independent of N and f.

From (4.13) and (4.14) it follows that

$$\|u - \xi_N u\|_{C^{2+\sigma}(\overline{Q})} \to 0 \quad \text{as } N \to \infty.$$

To conclude the proof it remains to note that by definition the space of compactly supported functions in $C^\sigma(\overline{Q})$ is dense in $C_0^\sigma(\overline{Q})$. □

4.2. We prove the first assertion of Theorem 1.1. Let $\delta > 0$ be such that $G_{2\delta} \neq \varnothing$ and $\delta < \delta_0$. Assume that Condition 1.1 is satisfied for this δ and some h, $\rho > 0$. We now build a stationary solution $\{0, \mathring{f}^\beta\}$ of equations (1.1), (1.2) such that

$$\mathring{f}^\beta \in C^\infty(\overline{Q} \times \mathbb{R}^3), \quad \operatorname{supp} \mathring{f}^\beta \subset Q_{2\delta} \times B_{\rho/4}, \quad \sup_{x,v} \mathring{f}^\beta(x,v) > \alpha.$$

4.2a. Let $\mathring{\varphi}(x) \equiv 0$ $(x \in \overline{Q})$. Then the system (1.11) assumes the form

$$(v, \nabla_x \mathring{f}^\beta) + \frac{\beta e}{c m_\beta}\big([v, B], \nabla_v \mathring{f}^\beta\big) = 0 \qquad (x \in Q,\ v \in \mathbb{R}^3,\ \beta = \pm 1). \tag{4.15}$$

We shall find a solution of equation (4.15) as a product of two cut-off functions whose arguments are first integrals of the system (2.5), (2.6). Different particular solutions of equation (4.15) will be denoted by $\mathring{f}_i^\beta$ $(i = 1, \dots, 4)$.

Clearly, the function $\mathring{f}_1^\beta(x,v) = |v|^2$ is a solution of (4.15) for any $x \in Q$, $v \in \mathbb{R}^3$, and $\beta = \pm 1$. We consider even functions $\psi_1^\beta \in \dot{C}^\infty(\mathbb{R})$ such that $\psi_1^\beta(0) = 2\alpha > 0$, $\psi_1^\beta(\tau) \geqslant 0$,

$$\frac{1}{m_{+1}^{3/2}}\, \psi_1^{+1}\bigg(\frac{\tau}{m_{+1}^2}\bigg) = \frac{1}{m_{-1}^{3/2}}\, \psi_1^{-1}\bigg(\frac{\tau}{m_{-1}^2}\bigg) \qquad (\tau \in \mathbb{R}),$$

and $\operatorname{supp} \psi_1^{-1} \subset (-\rho_1^2/16, \rho_1^2/16)$, where $0 < \rho_1 < \rho$.

Since $m_{+1} > m_{-1}$, $\operatorname{supp} \psi_1^{+1} \subset (-\rho_1^2/16, \rho_1^2/16)$. The function $\mathring{f}_2^\beta(x,v) = \psi_1^\beta(|v|^2)$ is a solution of (4.15).

We now look for a solution of (4.15) as a quadratic form with undetermined coefficients:

$$\mathring{f}_3^\beta(x,v) = \sum_{i,j=1}^{3} (\alpha_{ij} x_i x_j + \beta_{ij} x_i v_j + \gamma_{ij} v_i v_j). \tag{4.16}$$

Substituting (4.16) in (4.15) and equating the coefficients of like terms, we get that

$$\mathring{f}_3^\beta(x,v) = \bigg(\frac{eh}{m_\beta c} x_1 + \beta v_2\bigg)^2 + \bigg(\frac{eh}{m_\beta c} x_2 - \beta v_1\bigg)^2.$$

We consider even functions $\psi_2^\beta \in \dot{C}^\infty(\mathbb{R})$ such that $\psi_2^\beta(0) = 1$, $\psi_2^\beta(\tau) \geqslant 0$,

$$\frac{1}{m_{+1}^{3/2}} \psi_2^{+1}\bigg(\frac{\tau}{m_{+1}^2}\bigg) = \frac{1}{m_{-1}^{3/2}} \psi_2^{-1}\bigg(\frac{\tau}{m_{-1}^2}\bigg) \qquad (\tau \in \mathbb{R}),$$

and $\operatorname{supp} \psi_2^{-1} \subset (-\rho_0^2, \rho_0^2)$, where $\rho_0 = 15\rho\delta_0/\delta$. Since $m_{+1} > m_{-1}$, $\operatorname{supp} \psi_2^{+1} \subset (-\rho_0^2, \rho_0^2)$. The function

$$\mathring{f}_4^\beta(x,v) = \psi_2^\beta\Bigg(\bigg(\frac{eh}{m_\beta c} x_1 + \beta v_2\bigg)^2 + \bigg(\frac{eh}{m_\beta c} x_2 - \beta v_1\bigg)^2\Bigg)$$

is a solution of (4.15).

4.2b. We prove that the vector function $\{0, \mathring{f}_2^\beta \mathring{f}_4^\beta\}$ is a stationary solution of the problem (1.1), (1.2), (1.4) satisfying the hypotheses of Theorem 1.1.

By construction, the function $\mathring{f}^\beta(x,v) = \mathring{f}_2^\beta(x,v)\mathring{f}_4^\beta(x,v)$ satisfies equation (4.15) and $\sup_{x,v} \mathring{f}^\beta(x,v) \geqslant \mathring{f}^\beta(0,0) = 2\alpha > 0$. By Lemma 4.1 it suffices to show that the right-hand side of (1.10) is identically zero and $\operatorname{supp} \mathring{f}^\beta \subset Q_{2\delta} \times B_{\rho/4}$.

Let us show that

$$\int_{\mathbb{R}^3} \mathring{f}^{+1}(x,v)\,dv = \int_{\mathbb{R}^3} \mathring{f}^{-1}(x,v)\,dv.$$

We make the change of variables $y = (eh/c)x$, $w = m_{+1}(v_2, -v_1, v_3)$, and define $y' = (y_1, y_2)$ and $w' = (w_1, w_2)$. Then using the equalities

$$\frac{1}{m_{+1}^{3/2}}\psi_j^{+1}\left(\frac{\tau}{m_{+1}^2}\right) = \frac{1}{m_{-1}^{3/2}}\psi_j^{-1}\left(\frac{\tau}{m_{-1}^2}\right) \qquad (j = 1,2;\ \tau \in \mathbb{R})$$

and introducing the variables $x = (c/(eh))y$ and $v = (1/m_{-1})(w_2, -w_1, w_3)$, we find that

$$\begin{aligned}
\int_{\mathbb{R}^3} \mathring{f}^{+1}(x,v)\,dv &= \int_{\mathbb{R}^3} \psi_1^{+1}(|v|^2)\psi_2^{+1}\left(\left(\frac{eh}{m_{+1}c}x_1 + v_2\right)^2 + \left(\frac{eh}{m_{+1}c}x_2 - v_1\right)^2\right) dv \\
&= \int_{\mathbb{R}^3} \frac{1}{m_{+1}^3}\psi_1^{+1}\left(\frac{|w|^2}{m_{+1}^2}\right)\psi_2^{+1}\left(\frac{|y'+w'|^2}{m_{+1}^2}\right) dw \\
&= \int_{\mathbb{R}^3} \frac{1}{m_{-1}^3}\psi_1^{-1}\left(\frac{|w|^2}{m_{-1}^2}\right)\psi_2^{-1}\left(\frac{|y'+w'|^2}{m_{-1}^2}\right) dw \\
&= \int_{\mathbb{R}^3} \psi_1^{-1}(|v|^2)\psi_2^{-1}\left(\left(\frac{eh}{m_{-1}c}x_1 - v_2\right)^2 + \left(\frac{eh}{m_{-1}c}x_2 + v_1\right)^2\right) dv \\
&= \int_{\mathbb{R}^3} \mathring{f}^{-1}(x,v)\,dv.
\end{aligned}$$

Consequently, the right-hand side of (1.10) is identically zero.

We now assert that $\operatorname{supp} \mathring{f}^\beta \subset Q_{2\delta} \times B_{\rho/4}$. Indeed, if $|v| > \rho_1/4$, then $f_2^\beta(x,v) = \psi_1^\beta(|v|^2) = 0$ by construction. Hence, $\mathring{f}^\beta(x,v) = 0$ for $|v| > \rho_1/4$. Let $B_{\delta_0}(g)$ be the circle of greatest radius inscribed in $G_{2\delta}$. Without loss of generality we assume that $g = 0$. If $|x'| > \delta_0/2$ and $|v| \leqslant \rho_1/4$, then Condition 1.1 and the inequality $\delta_0/\delta > 1$ imply that

$$\left|\frac{eh}{m_\beta c}\,x' + \beta z'\right| \geqslant \frac{eh}{m_\beta c}|x'| - |z'| > \frac{16c\rho}{e\delta}\,\frac{m_\beta e\delta_0}{m_\beta c} - \rho > \frac{15\rho\delta_0}{\delta}\,,$$

where $z' = (v_2, -v_1)$.

Therefore,

$$f_4^\beta(x,v) = \psi_2^\beta\left(\left|\frac{eh}{m_\beta c}x' + \beta z'\right|^2\right) = 0.$$

Consequently, $\mathring{f}^\beta(x,v) = 0$ for $|x'| > \delta_0/2$ and $|v| \leqslant \rho/4$.

4.3. To prove Theorem 1.1 it suffices to show that under Conditions 1.1 and 1.2, for any $T > 0$ and all $\mathring{f}^\beta$ and g_0^β such that

$$\operatorname{supp} g_0^\beta \subset (Q_{2\delta} \cap Q'_N) \times B_{\rho/4}, \tag{4.17}$$

$$\|g_0^\beta\|_1 + \max_i \|\mathring{f}_{v_i}^\beta\|_1 < R(4\pi e c_1 c_3)^{-1}, \tag{4.18}$$

$$\|g_0^\beta\|_2 + \max_i \|\mathring{f}_{v_i}^\beta\|_2 < (4\pi e c_2 c_3)^{-1}, \tag{4.19}$$

there is a unique classical solution of (2.1)–(2.4), and moreover,

$$\operatorname{supp} g^\beta(\,\cdot\,,\cdot\,,t) \subset Q_{5\delta/4} \times B_\rho \quad \text{for all } t \in [0,T].$$

Here c_1, c_2, $c_3 > 0$ are the constants from Lemmas 3.4, 3.5, and 4.1.

Let $\varphi \in M_{2+\sigma}$ be an arbitrary function. Then by Lemma 3.3

$$F_\varphi \in C([0,T], C_0^\sigma(\overline{Q})).$$

Next, according to Lemma 4.1 there is a unique classical solution $u_\varphi(x,t)$ of the problem (4.1), (4.2) with $f(x) = 4\pi e F_\varphi(x,t)$ for each $t \in [0,T]$, and

$$u_\varphi \in C\bigl([0,T], C_0^{2+\sigma}(\overline{Q})\bigr) \quad \text{and} \quad u_\varphi\big|_{\partial Q \times (0,T)} = 0.$$

Let $A\varphi = u_\varphi$.

It follows from (3.25), (4.3), and (4.18) that

$$\|A\varphi\|_{2+\sigma,T} \leqslant R \qquad (\varphi \in M_{2+\sigma}). \tag{4.20}$$

By virtue of (4.19), (3.28), and (4.3),

$$\|A\varphi_1 - A\varphi_2\|_{2+\sigma,T} \leqslant \theta \|\varphi_1 - \varphi_2\|_{2+\sigma,T} \qquad (\varphi_1, \varphi_2 \in M_{2+\sigma}), \tag{4.21}$$

where $\theta = 4\pi e(m_2 + n_3)c_2 c_3 < 1$.

Thus, the operator A maps the complete metric space $M_{2+\sigma}$ into itself and is a contraction operator. By the Banach contraction principle A has a unique fixed point in $M_{2+\sigma}$. Hence, the problem (2.1)–(2.4) has a unique classical solution $\{\varphi, g_\varphi^\beta\}$, where φ is the fixed point of the operator A and the functions g_φ^β are defined by (3.11) and (3.12). The proof of Theorem 1.1 is complete.

5. Some generalizations

In the proof of Theorem 1.1 we have only used the boundedness of the inverse of the Laplacian operator with homogeneous Dirichlet condition, acting from $C_0^\sigma(\overline{Q})$ to $C_0^{2+\sigma}(\overline{Q})$, while the explicit form of the operator and the boundary conditions did not play a role. This suggests considering the following generalization of the problem (1.1)–(1.4):

$$\frac{\partial f^\beta}{\partial t} + (v, \nabla_x f^\beta) + \frac{\beta e}{m_\beta}\biggl(-\nabla_x P\biggl(\int_{\mathbb{R}^3} \sum_\beta \beta f^\beta(x,v,t)\,dv\biggr) + \frac{1}{c}[v,B], \nabla_v f^\beta\biggr) = 0 \tag{5.1}$$

$$(x \in Q,\ v \in \mathbb{R}^3,\ 0 < t < T,\ \beta = \pm 1),$$

$$f^\beta(x,v,t)\big|_{t=0} = f_0^\beta(x,v) \qquad (x \in Q,\ v \in \mathbb{R}^3,\ \beta = \pm 1). \tag{5.2}$$

Here P is a bounded linear operator mapping $C_0^\sigma(\overline{Q})$ to $C_0^{2+\sigma}(\overline{Q})$.

Remark 5.1. Since the operator $P\colon C_0^{\sigma}(\overline{Q}) \to C_0^{2+\sigma}(\overline{Q})$ is bounded, we have $(PF)(x,t) \in C([0,T],\, C_0^{2+\sigma}(\overline{Q}))$ and

$$\|PF\|_{2+\sigma,T} \leqslant c_4 \|F\|_{\sigma,T} \tag{5.3}$$

for any function $F(x,t) \in C([0,T],\, C_0^{\sigma}(\overline{Q}))$.

Definition 5.1. A vector function $\{f^{\beta}\}$ with $f^{\beta} \in C^1(\overline{Q} \times \mathbb{R}^3 \times [0,T])$ for $\beta = \pm 1$ is called a classical solution of the problem (5.1), (5.2) if

$$\int_{\mathbb{R}^3} \sum_{\beta} \beta f^{\beta}(\,\cdot\,, v, t)\, dv \in C([0,T], C_0^{\sigma}(\overline{Q}))$$

and $\{f^{\beta}\}$ satisfies equation (5.1) and the initial condition (5.2).

Definition 5.2. A vector function $\{\mathring{f}^{\beta}\}$ with $\mathring{f}^{\beta} \in C^1(\overline{Q}\times\mathbb{R}^3)$ is called a stationary solution of equation (5.1) if

$$\int_{\mathbb{R}^3} \sum_{\beta} f^{\beta}(\,\cdot\,, v)\, dv \in C_0^{\sigma}(\overline{Q})$$

and $\{\mathring{f}^{\beta}\}$ satisfies the equation

$$(v, \nabla_x \mathring{f}^{\beta}) + \frac{\beta e}{m_{\beta}} \biggl(-\nabla_x P\biggl(\int_{\mathbb{R}^3} \sum_{\beta} \beta \mathring{f}^{\beta}(x,v)\, dv \biggr) + \frac{1}{c}[v, B],\ \nabla_v f^{\beta} \biggr) = 0 \qquad (x \in Q,\ v \in \mathbb{R}^3,\ \beta \pm 1). \tag{5.4}$$

Repeating the proof of Theorem 1.1 and taking into account Remark 5.1, we arrive at the following result.

Theorem 5.1. *Let $\delta > 0$ be such that $G_{2\delta} \neq \varnothing$ and $\delta_0 > \delta$. Let Condition 1.1 be satisfied for this δ and some h, $\rho > 0$. Then for any $\alpha > 0$ there is a stationary solution $\{\mathring{f}^{\beta}\}$ of (5.1) with the following properties*:

$$\mathring{f}^{\beta} \in C^{\infty}(\overline{Q} \times \mathbb{R}^3), \qquad \operatorname{supp} \mathring{f}^{\beta} \subset Q_{2\delta} \times B_{\rho/4},$$
$$\int_{\mathbb{R}^3} \sum_{\beta} \beta \mathring{f}^{\beta}(x,v)\, dv = 0 \quad (x \in \overline{Q}), \qquad \sup_{x,v} \mathring{f}^{\beta}(x,v) > \alpha.$$

If, in addition, Condition 1.2 is satisfied, then for any $T > 0$ and all stationary solutions $\mathring{f}^{\beta}$ with the above properties and initial functions f_0^{β} satisfying

$$\operatorname{supp}(f_0^{\beta} - \mathring{f}^{\beta}) \subset (Q_{2\delta} \cap Q'_N) \times B_{\rho/4}, \tag{5.5}$$

$$\|f_0^{\beta} - \mathring{f}^{\beta}\|_1 + \max_{i=1,2,3} \|\mathring{f}^{\beta}_{v_i}\|_1 < R(4\pi e c_1 c_4)^{-1}, \tag{5.6}$$

$$\|f_0^{\beta} - \mathring{f}^{\beta}\|_2 + \max_{i=1,2,3} \|\mathring{f}^{\beta}_{v_i}\|_2 < (4\pi e c_2 c_4)^{-1} \tag{5.7}$$

$$(\beta = \pm 1)$$

for some $N > 0$, there is a unique classical solution of the problem (5.1), (5.2), *and*

$$\operatorname{supp} f^{\beta}(\cdot,\cdot,t) \subset Q_{5\delta/4} \times B_{\rho} \quad \textit{for all } t \in [0,T].$$

From the results of § 4 it follows that the problem (1.1)–(1.4) can be written in the form (5.1), (5.2).

Remark 5.2. Theorems 1.1 and 5.1 can be extended to the case when the magnetic field $B = B(x)$ is a sufficiently smooth vector function of the form $B = (0, 0, h)$ on the set $\overline{Q} \setminus Q_{2\delta}$. The solvability of the Vlasov–Poisson equations in the half-space $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 \colon x_1 > 0\}$ with external magnetic field of this form on the set $\{x \in \mathbb{R}^3 \colon 0 \leqslant x_1 \leqslant 2\delta\}$ was studied in [110] and [111].

In conclusion we mention some unsolved problems.

1. Construction of stationary solutions with compactly supported charged-particle density distributions f^β and non-zero potential φ in a half-space and in an infinite cylinder. Such a solution would be more natural from the viewpoint of plasma physics, because if the charged-particle density distributions in the infinite cylinder are independent of x_3, then the total charge is infinite.

2. Study of the existence and stability of global classical solutions of mixed problems for the Vlasov–Poisson equations with density distributions supported strictly interior to a domain in the cases of a half-space and an infinite cylinder for arbitrary initial data.

3. Construction of stationary solutions of the Vlasov–Poisson equations in arbitrary bounded domains, study of global classical solutions with compact supports in a domain, and investigation of their stability. Here the torus is of special importance from a practical point of view (see Fig. 1). Since plasma control in a thermonuclear fusion reactor depends on the external magnetic field, some domains — for example, a ball — are not used in modeling reactor chambers. This is so because, first of all, there is no non-trivial vector field on a sphere. On the other hand, the 'drift' of charged particles in the case when the magnetic field on the torus is circumferentially directed is very well known in plasma physics. Therefore, in the study of classical solutions of the Vlasov–Poisson equations in bounded domains one must at the same time investigate the form of the external magnetic field. In general, the external magnetic field must depend on the electric-field potential φ and the charged-particle density distributions f^β. We thus arrive at the question of investigating the system of equations describing controlled thermonuclear fusion.

4. In a more refined mathematical model of thermonuclear fusion one needs to take into account the magnetic field produced by the moving particles. This brings us to the Vlasov–Maxwell equations. There also arises the question of the existence of global classical solutions (with compact supports in a domain) of mixed problems for these equations and of their stability in a half-space, an infinite cylinder, and a bounded domain.

5. The study of conditions ensuring that a generalized solution is classical has great value in the theory of boundary-value problems for equations of mathematical physics. This question for mixed problems for the Vlasov equations is virtually unexplored.

6. There is also a certain interest in the study of the solvability, the asymptotic behaviour of solutions, and the form of supports of solutions of the Vlasov equations in domains with singularities. Such singularities are exhibited by the vacuum chambers of the tokamak and mirror-trap types of reactors (correspondingly, edge and zero-angle types) (see Figs. 1 and 2).

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