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Reduction theorems for weighted integral inequalities on the cone of monotone functions

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Abstract. This paper surveys results related to the reduction of integral inequalities involving positive operators in weighted Lebesgue spaces on the real semi-axis and valid on the cone of monotone functions, to certain more easily manageable inequalities valid on the cone of non-negative functions. The case of monotone operators is new. As an application, a complete characterization for all possible integrability parameters is obtained for a number of Volterra operators.

Bibliography: 118 titles.

Keywords: weighted Lebesgue space, cone of monotone functions, duality principle, weighted integral inequality, bounded operators, reduction theorem.

Contents

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Introduction

Our aim in this paper is to give a survey of weighted integral inequalities in Lebesgue spaces on cones of monotone functions on the real semi-axis.

Let $\mathbb{R}_+ := [0,\infty)$. We denote by $\mathfrak{M}$ the space of all Lebesgue-measurable functions on $\mathbb{R}_+$. For $0 < p \leqslant \infty$ let

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}_+} |f(x)|^p\,dx\right)^{1/p}, & 0<p<\infty, \\ \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)|, & p=\infty, \end{cases}$$

and define the Lebesgue space L_p as the set of all $f \in \mathfrak{M}$ for which $\|f\|_p < \infty$.

A well-known problem in functional analysis is to find necessary and/or sufficient conditions for the validity of integral inequalities of the form

$$\left(\int_{\mathbb{R}_+} \left| w(x) \int_{\mathbb{R}_+} k(x,y) f(y) u(y)\,dy \right|^q dx\right)^{1/q} \leqslant C \left(\int_{\mathbb{R}_+} |f(x)|^p\,dx\right)^{1/p}, \quad f \in \mathfrak{M}, \tag{1}$$

for fixed parameters $0 < p, q \leqslant \infty$, where $k(x,y)$ is a jointly measurable kernel, $u(x)$ and $w(x)$ are locally integrable weight functions, and the constant $C \geqslant 0$ is independent of f and is the smallest possible, that is, C equals the norm $\|\mathscr{K}\|_{L_p\to L_q}$ of the operator

$$\mathscr{K} f(x) := w(x) \int_{\mathbb{R}_+} k(x,y) f(y) u(y)\,dy \tag{2}$$

acting from L_p to L_q.

It is known (see Theorem 2 in [86]) that $\|\mathscr{K}\|_{L_p\to L_q} = 0$ if $0 < p < 1$ and $\|\mathscr{K}\|_{L_p\to L_q} < \infty$. Consequently, the study of the inequality (1) and the properties of the operator $\mathscr{K}\colon L_p \to L_q$ is restricted to the case $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$. There is an extensive literature devoted to the above boundedness problem for integral operators and to the study of their general properties (see, for example, the monographs [38], Chap. 11 in [48], [49], [50], [72], [108]).

It should be noted that for signed kernels $k(x,y)$ the inequality (1) may hold because the kernel oscillates as it grows unboundedly (see, for example, [46], [108]). For this reason operators with such kernels are studied in important particular cases. The successes of the general theory in the last twenty years are mainly connected with the study of operators (2) with non-negative kernels $k(x,y) \geqslant 0$. In spite of this essential restriction, a deep and substantive theory has been constructed for some classes of Volterra operators

$$K f(x) := w(x) \int_0^x k(x,y) f(y) u(y)\,dy \tag{3}$$

and their duals (see the books [15], [52], [56]). A first step is to find a two-sided estimate $\|K\|_{L_p\to L_q} \approx F(k,u,w)$, where the functional $F(k,u,w)$ is explicitly expressed in terms of the kernel and weight functions, and the equivalence constants depend only on p and q. This would enable one to find explicit compactness criteria, and in certain cases to carry out a comprehensive study, including a study of the behaviour of characteristic numbers (approximation numbers, entropy numbers, and so on). The aims of this programme were achieved, in greater or lesser generality, for operators with kernels satisfying *Oinarov's condition*: there exists a constant $D \geqslant 1$ independent of x, y, z such that

$$D^{-1}k(x,y) \leqslant k(x,z) + k(z,y) \leqslant Dk(x,y), \qquad x \geqslant z \geqslant y \geqslant 0. \tag{4}$$

Typical examples of such kernels include $(x-y)_+^{\alpha-1}$ with $\alpha \geqslant 1$, $\log^\beta(x/y)$ with $\beta \geqslant 0$, $\bigl(\int_y^x h(z)\,dz\bigr)^\gamma$ with $\gamma \geqslant 0$ and $h(z) \geqslant 0$, and various combinations of these. For brevity we call measurable functions $k(x,y) \geqslant 0$ satisfying the condition (4) for $x \geqslant y \geqslant 0$ *Oinarov kernels*.

In § 1 we give a short survey of the results in this direction.

There has been considerable progress in Λ-*analysis* — the circle of problems concerning characterization of boundedness of classical operators acting in weighted Lorentz spaces — since the beginning of the 1990s. Let us recall the definition of a Lorentz space. The non-increasing rearrangement f^* of a measurable function f on $\mathbb{R}^n$ is defined as the function

$$f^*(t) := \inf\bigl\{s > 0\colon \operatorname{meas}\{x \in \mathbb{R}^n\colon |f(x)| > s\} \leqslant t\bigr\}, \qquad t \in \mathbb{R}_+.$$

For a fixed $p \in (0,\infty)$ and a fixed non-negative measurable function v on $\mathbb{R}_+$, the *Lorentz* Λ-*space* $\Lambda^p(v)$ is defined as the set of all measurable functions f on $\mathbb{R}^n$ satisfying $\|f\|_{\Lambda^p(v)} < \infty$, where

$$\|f\|_{\Lambda^p(v)} := \biggl(\int_0^\infty [f^*(t)]^p v(t)\,dt\biggr)^{1/p}.$$

Let $\mathfrak{M}^+$ denote the cone of all non-negative functions in $\mathfrak{M}$, and let $\mathfrak{M}^\downarrow$ (respectively, $\mathfrak{M}^\uparrow$) denote the cone of all non-increasing (non-decreasing) functions in $\mathfrak{M}^+$. We define

$$L^\downarrow_{p,v} := \biggl\{f \in \mathfrak{M}^\downarrow\colon \|f\|_{p,v} := \biggl(\int_{\mathbb{R}_+} |f(x)|^p v(x)\,dx\biggr)^{1/p} < \infty\biggr\},$$

and similarly for $L^\uparrow_{p,v}$ and $L^+_{p,v}$. If for some operator $T\colon \Lambda^p(v) \to \Lambda^q(w)$ one can get an estimate

$$[Tf]^*(t) \ll [Sf^*](t), \qquad t > 0,$$

with a positive operator S, then boundedness of $S\colon L^\downarrow_{p,v} \to L^+_{q,w}$ implies that of $T\colon \Lambda^p(v) \to \Lambda^q(w)$, with the norm inequality

$$\|T\|_{\Lambda^p(v)\to\Lambda^q(w)} \ll \|S\|_{L^\downarrow_{p,v}\to L^+_{q,w}}.$$

The reverse inequality is usually sought from an analysis of the operator on radial decreasing functions (see, for example, [23]).

If the two-sided estimate

$$[Tf]^*(t) \approx \int_0^\infty k(t,s)u(s)f^*(s)\,ds \tag{5}$$

holds, where $k(t,s)\geqslant 0$ and $u(s)\geqslant 0$, then the boundedness of an operator T from $\Lambda^p(v)$ to $\Lambda^q(w)$ is characterized by the inequality

$$\biggl(\int_0^\infty\biggl(\int_0^\infty k(t,s)f(s)u(s)\,ds\biggr)^q w(t)\,dt\biggr)^{1/q}\leqslant C\biggl(\int_0^\infty [f(t)]^p v(t)\,dt\biggr)^{1/p} \tag{6}$$

for all $f\in\mathfrak{M}^{\downarrow}$. For example, it is known that the Hardy–Littlewood maximal operator

$$Mf(x):=\sup_B\frac{1}{\operatorname{meas}B}\int_B|f(y)|\,dy,$$

where the supremum is taken over all balls B about the point $x\in\mathbb{R}^n$, satisfies the two-sided estimate

$$[Mf]^*(t)\approx\frac{1}{t}\int_0^t f^*(s)\,ds.$$

Therefore, its boundedness from $\Lambda^p(v)$ to $\Lambda^q(u)$ is characterized by Hardy's inequality

$$\biggl(\int_0^\infty\biggl(\frac{1}{t}\int_0^t f(s)\,ds\biggr)^q w(t)\,dt\biggr)^{1/q}\leqslant C\biggl(\int_0^\infty [f(t)]^p v(t)\,dt\biggr)^{1/p},\qquad f\in\mathfrak{M}^{\downarrow}. \tag{7}$$

Similarly, for the Riesz potentials

$$I_\alpha f(x):=\int_{\mathbb{R}^n}\frac{f(y)\,dy}{|x-y|^{n-\alpha}},\qquad 0<\alpha<n,$$

and the Hilbert transform

$$\mathscr{H}f(x):=\lim_{\varepsilon\to 0}\int_{|y|>\varepsilon}\frac{f(x-y)\,dy}{y}$$

the following two-sided estimates hold:

$$(I_\alpha f)^*(t)\ll t^{\alpha/n-1}\int_0^t f^*(z)\,dz+\int_t^\infty t^{\alpha/n-1}f^*(z)\,dz\ll (I_\alpha\widetilde{f})^*(t),\qquad t>0,$$

where $\widetilde{f}(y)=f^*(A|y|^n)$, and respectively,

$$(\mathscr{H}f)^*(t)\ll C_1\biggl[t^{-1}\int_0^t f^*(z)\,dz+\int_t^\infty z^{-1}f^*(z)\,dz\biggr]\ll(\mathscr{H}f^*)^*(t).$$

Thus, their boundedness from $\Lambda^p(v)$ to $\Lambda^q(w)$ is also characterized by an inequality of type (6).

We present another example in which the operator S is non-linear. For $\gamma\in(0,n)$ the fractional maximal operator M_γ is defined by

$$M_\gamma f(x):=\sup_B|B|^{\gamma/n-1}\int_B|f(y)|\,dy.$$

Then

$$(M_\gamma f)^*(t)\lesssim\sup_{\tau\geqslant t}\tau^{\gamma/n-1}\int_0^\tau f^*(s)\,ds, \tag{8}$$

and for any function $\varphi \in \mathfrak{M}^{\downarrow}$ there exists a function f on $\mathbb{R}^n$ such that $f^* = \varphi$ almost everywhere and

$$(M_\gamma f)^*(t) \gtrsim \sup_{\tau \geqslant t} \tau^{\gamma/n-1} \int_0^\tau f^*(s)\,ds =: S_\gamma f^*(t). \tag{9}$$

In this case, $\|M_\gamma\|_{\Lambda^p(v)\to\Lambda^q(w)} \approx \|S_\gamma\|_{L^{\downarrow}_{p,v}\to L^{+}_{q,w}}$. The operators M_γ play an important role in the theory of embeddings of Sobolev spaces.

An essential part of Λ-analysis is thus the **problem** of characterizing $(L^{\downarrow}_{p,v} \to L^{+}_{q,w})$-inequalities for positive operators, and this is the concern of the present paper. In passing, we solve the analogous problem for $(L^{\uparrow}_{p,v} - L^{+}_{q,w})$-inequalities.

We mention two more peculiarities. The first is that satisfaction of an inequality on monotone functions does not guarantee its satisfaction for all non-negative functions. A simple example of this is the following [1]: assume that the weight function $v(x)$ is

$$v(x) = \begin{cases} 0, & 1 < x < 2, \\ x^{-1/2}, & 0 < x \leqslant 1,\ x \geqslant 2. \end{cases}$$

Then the inequality

$$\biggl(\int_0^\infty \biggl(\frac{1}{t}\int_0^t f(s)\,ds\biggr)^2 v(t)\,dt\biggr)^{1/2} \leqslant C\biggl(\int_0^\infty [f(t)]^2 v(t)\,dt\biggr)^{1/2}, \qquad f \in \mathfrak{M}^+,$$

may fail to hold. For example, if $f = \chi_{(1,2)}$ (the characteristic function of the interval $(1,2)$), then the right-hand side is zero, while the left-hand side is positive. At the same time, this inequality is satisfied for $f \in \mathfrak{M}^{\downarrow}$. Since $v(x) \leqslant x^{-1/2}$, it follows from the classical Hardy inequality (Theorem 327 in [40]) that

$$\begin{aligned}\biggl(\int_0^\infty \biggl(\frac{1}{t}\int_0^t f(s)\,ds\biggr)^2 v(t)\,dt\biggr)^{1/2} &\leqslant \biggl(\int_0^\infty \biggl(\frac{1}{t}\int_0^t f(s)\,ds\biggr)^2 t^{-1/2}\,dt\biggr)^{1/2} \\ &\leqslant \frac{4}{3}\biggl(\int_0^\infty [f(t)]^2 t^{-1/2}\,dt\biggr)^{1/2} \leqslant \frac{2^{5/2}}{3}\biggl(\int_0^\infty [f(t)]^2 v(t)\,dt\biggr)^{1/2},\end{aligned}$$

since $\int_1^2 [f(t)]^2 t^{-1/2}\,dt \leqslant \int_0^1 [f(t)]^2 v(t)\,dt$ for $f \in \mathfrak{M}^{\downarrow}$.

Thus, the inequality (6) may be satisfied for all $f \in \mathfrak{M}^{\downarrow}$ but fail to hold for some $f \in \mathfrak{M}^+$. Nevertheless, in the one particular case (6) it is possible by using Halperin's lemma [39], [102] to guarantee the equivalence of the inequalities.

Theorem 1 [91], [76]. *Let $1 < p < \infty$ and $0 < q < \infty$. Then the inequality*

$$\biggl(\int_0^\infty \biggl(\int_0^t f(s)v(s)\,ds\biggr)^q w(t)\,dt\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty [f(t)]^p v(t)\,dt\biggr)^{1/p} \tag{10}$$

holds for all $f \in \mathfrak{M}^+$ if it holds for all $f \in \mathfrak{M}^{\downarrow}$.

The second peculiarity is that the inequality (6) on all $f \in \mathfrak{M}^+$ is easily reducible to a two-weighted inequality by the change $fu \to f$. This effect is absent in general for inequalities on monotone functions. Hence, it is essential that the inequality

(6) involves three weight functions. In addition, we point out that $(L_p \to L_q)$-inequalities on monotone functions make sense for all values of the parameters $0 < p,\, q \leqslant \infty$.

Almost from the beginning of Λ-analysis, the *method of reduction* has been a fundamental tool in the study of $(L^{\downarrow}_{p,v} \to L^{+}_{q,w})$-inequalities. In this approach a given inequality on monotone functions is reduced to some inequality on non-negative functions which is more easily characterized than the original one. The *Sawyer duality principle* [87], which applies for $0 < p \leqslant \infty$ and $1 \leqslant q \leqslant \infty$, is one of the universal tools in the method of reduction for positive *linear* operators. It is described in § 2.

The main results of our paper are given in § 3, where we propose a new method of reduction of $(L^{\downarrow}_{p,v} \to L^{+}_{q,w})$- and $(L^{\uparrow}_{p,v} \to L^{+}_{q,w})$-inequalities for positive *monotone* operators when $1 \leqslant p \leqslant \infty$ and $0 < q \leqslant \infty$. For $0 < p \leqslant q < \infty$ we also present a general theorem for monotone operators with an additional restriction, and in the case $0 < q < p \leqslant 1$ we solve the problem for the integral Volterra operator (3) and its dual with a kernel satisfying Oinarov's condition (4).

Throughout the paper, expressions like $0 \cdot \infty$ are taken to be 0. The notation $A \ll B$ means the inequality $A \leqslant cB$ with a constant c depending only on insignificant parameters. We shall write $A \approx B$ in place of $A \ll B \ll A$ or $A = cB$. We let $\mathbb{Z}$ denote the set of all integers and let χ_E denote the characteristic function (indicator) of a subset E of $\mathbb{R}_+$. New quantities are defined using the symbols $:=$ and $=:$. We also set $p' := p/(p-1)$ for $1 < p < \infty$, $p' := 1$ for $p = \infty$, $p' := \infty$ for $p = 1$, and $r := p\,q/(p-q)$ for $0 < q < p < \infty$. By letters A, B, C with indices (say, $C_1, C_2, \dots$) we denote constants, which may differ in different assertions even if they have the same indices.

1. Integral operators

Let u and w be non-negative weight functions that are locally integrable on $\mathbb{R}_+$, and let $k(x,y)$ be a jointly measurable kernel. Progress in the theory of integral operators over the last two decades is mainly related to the study of Volterra operators

$$Kf(x) := w(x) \int_0^x k(x,y) f(y) u(y)\, dy, \qquad x > 0, \tag{1.1}$$

and inequalities

$$\|Kf\|_q \leqslant C \|f\|_p, \tag{1.2}$$

which are important in the theory of integral and differential equations, spectral theory, embedding theorems for Sobolev spaces, and so on (see, for example, the books [10], [15], [52], [56], [64], the papers [3], [4], [118], [16]–[18], [57]–[60], [70], [71], [80]–[86], [92]–[109], [113], and more).

The study of the operators (1.1) begins with the determination of criteria for their boundedness on Lebesgue spaces — in other words, necessary and/or sufficient conditions for inequalities (1.2), with the constant C chosen to be the smallest possible, that is, to be the norm $\|K\|_{L_p \to L_q}$ of the operator K. Here the quality of a criterion plays a key role in solving the problem of compactness of the operator K and in more subtle investigations of the behaviour of its characteristic numbers. For example, the simplest case $k(x,y) = \rho(x) \geqslant 0$ was studied in the framework of Sturm–Liouville equations in [19], where a criterion for the discreteness of the spectrum was obtained. This and other cases derive from inequalities of Hardy (Theorems 327 and 330 in [40]) which have been generalized by many authors (see, for example, the historical survey [51]). As a typical example (see Theorems 329, 383, and 402 in [40]) we mention the inequality (1.2) involving the Riemann–Liouville operator

$$R_\alpha f(x) := w(x)\int_0^x (x-y)^{\alpha-1} f(y)u(y)\,dy, \qquad x>0, \tag{1.3}$$

where $\alpha > 0$. Here the case $0<\alpha<1$ differs completely from the case $\alpha \geqslant 1$. For $0<\alpha<1$ the boundedness of $R_\alpha\colon L_p \to L_q$ has been characterized only in special cases of the integrability parameters and weight functions [69], [61], [65], [79], [86]. The most general criterion was obtained in [61] for $1<p\leqslant q<\infty$, but it is difficult to verify. For $\alpha\geqslant 1$ this problem was solved [95]–[98] and it has been generalized [63], [4], [71], [100], [102], [106], [107], [115] for kernels $k(x,y)\geqslant 0$ satisfying Oinarov's condition (4).

As in the case of the operators (1.1), an analogous theory is valid for operators of the form

$$K^* g(y) := u(y)\int_x^\infty k(x,y)g(x)w(x)\,dx, \tag{1.4}$$

which are dual to the operators (1.1) with respect to the bilinear form $(f,g) = \int_{\mathbb{R}_+} fg$. Where needed, we will also mention results for this class of operators.

For the extreme values $p=1,\infty$ or $q=1,\infty$ of the integrability parameters, the exact value of the norm $\|K\|_{L_p\to L_q}$ is found from the following general theorem (see [48], Chap. XI, § 1.5, Theorem 4).

Theorem 1.1. *The following equalities hold for the operator* (1.1):

$$\|K\|_{L_1\to L_q} = \operatorname*{ess\,sup}_{t>0} u(t)\|\chi_{[t,\infty)}(\cdot)k(\cdot\,,t)w(\cdot)\|_q, \qquad 1\leqslant q\leqslant\infty, \tag{1.5}$$

$$\|K\|_{L_\infty\to L_q} = \biggl(\int_0^\infty \biggl|w(x)\int_0^x k(x,y)u(y)\,dy\biggr|^q dx\biggr)^{1/q}, \qquad 1\leqslant q<\infty, \tag{1.6}$$

$$\|K\|_{L_p\to L_1} = \biggl(\int_0^\infty \biggl|u(y)\int_y^\infty k(x,y)w(x)\,dx\biggr|^{p'} dy\biggr)^{1/p'}, \qquad 1<p<\infty, \tag{1.7}$$

$$\|K\|_{L_p\to L_\infty} = \operatorname*{ess\,sup}_{t>0} w(t)\|\chi_{[0,t]}(\cdot)k(t,\cdot)u(\cdot)\|_{p'}, \qquad 1<p\leqslant\infty. \tag{1.8}$$

In order to formulate results on boundedness of operators (1.1) with Oinarov kernels (4), we shall need the following functionals and quantities:

$$\mathbf{U}(x) := \int_0^x [u(y)]^{p'}\,dy, \qquad \mathbf{W}(y) := \int_y^\infty [w(x)]^q\,dx,$$

$$U_1(x) := \int_0^x k(x,y)[u(y)]^{p'}\,dy, \qquad U_p(x) := \int_0^x [k(x,y)]^{p'}[v(y)]^{p'}\,dy,$$

$$W_1(y) := \int_y^\infty k(x,y)[w(x)]^q\,dx, \qquad W_q(y) := \int_y^\infty [k(x,y)]^q[w(x)]^q\,dx,$$

$$\mathbf{A} := \max(\mathbf{A}_0,\mathbf{A}_1),$$

where

$$\mathbf{A}_0 := \sup_{t>0} \mathbf{A}_0(t) := \sup_{t>0} [W_q(t)]^{1/q} [\mathbf{U}(t)]^{1/p'},$$
$$\mathbf{A}_1 := \sup_{t>0} \mathbf{A}_1(t) := \sup_{t>0} [\mathbf{W}(t)]^{1/q} [U_p(t)]^{1/p'}.$$

Similarly,

$$\mathbb{A} := \max(\mathbb{A}_0, \mathbb{A}_1),$$
$$\mathbb{A}_0 := \sup_{t>0} \mathbb{A}_0(t) := \sup_{t>0} [\mathbf{U}(t)]^{-1/p} \left(\int_0^t [U_1(x)]^q [w(x)]^q \, dx \right)^{1/q},$$
$$\mathbb{A}_1 := \sup_{t>0} \mathbb{A}_1(t) := \sup_{t>0} [U_p(t)]^{-1/p} \left(\int_0^t [U_p(x)]^q [w(x)]^q \, dx \right)^{1/q};$$
$$\mathscr{A} := \max(\mathscr{A}_0, \mathscr{A}_1),$$
$$\mathscr{A}_0 := \sup_{t>0} \mathscr{A}_0(t) := \sup_{t>0} [W_q(t)]^{-1/q'} \left(\int_t^\infty [W_q(y)]^{p'} [u(y)]^{p'} \, dy \right)^{1/p'},$$
$$\mathscr{A}_1 := \sup_{t>0} \mathscr{A}_1(t) := \sup_{t>0} [\mathbf{W}(t)]^{-1/q'} \left(\int_t^\infty [W_1(y)]^{p'} [u(y)]^{p'} \, dy \right)^{1/p'};$$
$$\mathbf{B} := \max(\mathbf{B}_0, \mathbf{B}_1),$$
$$\mathbf{B}_0 := \left(\int_0^\infty [W_q(t)]^{r/q} \, d[\mathbf{U}(t)]^{r/p'} \right)^{1/r},$$
$$\mathbf{B}_1 := \left(\int_0^\infty [U_p(t)]^{r/p'} \, d(-[\mathbf{W}(t)]^{r/q}) \right)^{1/r};$$
$$\mathbb{B} := \max(\mathbb{B}_0, \mathbb{B}_1),$$
$$\mathbb{B}_0 := \left(\int_0^\infty [\mathbb{U}(t)]^{-r/p} \, d\left(\int_a^t [U_1(x)]^q [w(x)]^q \, dx \right)^{r/q} \right)^{1/r},$$
$$\mathbb{B}_1 := \left(\int_a^b [U_p(t)]^{-r/p} \, d\left(\int_a^t [U_p(x)]^q [w(x)]^q \, dx \right)^{r/q} \right)^{1/r};$$
$$\mathscr{B} := \max(\mathscr{B}_0, \mathscr{B}_1),$$
$$\mathscr{B}_0 := \left(\int_0^\infty [W_q(t)]^{-r/q'} \, d\left(-\left(\int_t^\infty [W_q(y)]^{p'} [v(y)]^{p'} \, dy \right)^{r/p'} \right) \right)^{1/r},$$
$$\mathscr{B}_1 := \left(\int_0^\infty [\mathbf{W}(t)]^{-r/q'} \, d\left(-\left(\int_0^\infty [W_1(y)]^{p'} [v(y)]^{p'} \, dy \right)^{r/p'} \right) \right)^{1/r}.$$

With the use of these functionals we give three alternative criteria for the boundedness of operators (1.1) with Oinarov kernels (4) in each of the following two cases of mutual relationships between the integrability parameters: $1 < p \leqslant q < \infty$ and $1 < q < p < \infty$.

Theorem 1.2. *The following relations hold for an operator* (1.1) *with kernel* $k(x,y)$ *satisfying the condition* (4). *If* $1<p\leqslant q<\infty$, *then*

$$\|K\|_{L_p\to L_q}\approx \mathbf{A}\approx \mathbb{A}\approx \mathscr{A}. \tag{1.9}$$

If $1<q<p<\infty$, *then*

$$\|K\|_{L_p\to L_q}\approx \mathbf{B}\approx \mathbb{B}\approx \mathscr{B}. \tag{1.10}$$

The equivalence coefficients in (1.9) *and* (1.10) *depend only on* p, q, *and the constant* D *in the condition* (4).

Remark 1.1. 1) For $1<p\leqslant q<\infty$ a characterization of the inequality (1.2) in terms of the condition $\mathscr{A}<\infty$ was obtained in [4] (respectively, in terms of the condition $\mathbf{A}<\infty$ in [71], and in terms of the conditions $\mathbf{A}<\infty$ or $\mathbb{A}<\infty$ in [106]). Here the kernel $k(x,y)$ was subjected to some additional assumptions of monotonicity or continuity type, but these assumptions were later removed [58]. In fact, it can be assumed without loss of generality that $k(x,y)$ is non-increasing in x and non-decreasing in y, for otherwise one can replace it by the equivalent kernel

$$k_0(x,y):=\sup_{y\leqslant z\leqslant x}\ \sup_{z\leqslant t\leqslant x} k(t,z)$$

having these properties [58] and the property that $k(x,y)\leqslant k_0(x,y)\leqslant D^2k(x,y)$.

2) For $1<q<p<\infty$ the equivalence (1.10) with the constant $\mathbf{B}$ was obtained in [71] and [106]. The remaining criteria, (1.9) with the constant $\mathscr{A}$ and (1.10) with the constants $\mathbb{B}$ and $\mathscr{B}$, were obtained in [115].

3) Compactness criteria for an operator (1.1) with kernel $k(x,y)$ satisfying the condition (4) were obtained in [106].

4) The components of each of the constants of type A and B in Theorem 1.2 are in general independent (see, for example, [63], § 4 in [97], [115]), and we have the relations

(i)	$\mathbf{A}_0<\infty\Longleftrightarrow\mathscr{A}_0<\infty$,	(ii)	$\mathbf{A}_1<\infty\Longleftrightarrow\mathbb{A}_1<\infty$,
(iii)	$\mathbf{A}_0<\infty\Longrightarrow\mathbb{A}_0<\infty$,	(iv)	$\mathbf{A}_1<\infty\Longrightarrow\mathscr{A}_1<\infty$,
(v)	$\mathscr{A}_0<\infty\Longrightarrow\mathbb{A}_0<\infty$,	(vi)	$\mathbb{A}_1<\infty\Longrightarrow\mathscr{A}_1<\infty$.

Similarly, for the remaining constants

(i)	$\mathbf{B}_0<\infty\Longleftrightarrow\mathscr{B}_0<\infty$,	(ii)	$\mathbf{B}_1<\infty\Longleftrightarrow\mathbb{B}_1<\infty$,
(iii)	$\mathbf{B}_0<\infty\Longrightarrow\mathbb{B}_0<\infty$,	(iv)	$\mathbf{B}_1<\infty\Longrightarrow\mathscr{B}_1<\infty$,
(v)	$\mathscr{B}_0<\infty\Longrightarrow\mathbb{B}_0<\infty$,	(vi)	$\mathbb{B}_1<\infty\Longrightarrow\mathscr{B}_1<\infty$.

In each group of relations the reverse implications in (iii)–(vi) may fail to hold in general.

For $0<q<1\leqslant p<\infty$ the following discrete criterion for the inequality (1.2) is well known [54].

Theorem 1.3 (Theorem 5 in [54]). *Let* $0<q<1\leqslant p<\infty$. *Then the relation*

$$\|K\|_{L_p\to L_q}\approx \mathfrak{B}_0+\mathfrak{B}_1 \tag{1.11}$$

holds for an operator (1.1) *with kernel* $k(x,y)$ *satisfying the condition* (4)*, where*

$$\mathfrak{B}_0^r := \sup_{x_k} \sum_k \left(\int_{x_k}^{x_{k+1}} [k(y,x_k)]^q [w(y)]^q \, dy \right)^{r/q} \left(\int_{x_{k-1}}^{x_k} u^{p'} \right)^{r/p'}, \tag{1.12}$$

$$\mathfrak{B}_1^r := \sup_{x_k} \sum_k \left(\int_{x_k}^{x_{k+1}} w^q \right)^{r/q} \left(\int_{x_{k-1}}^{x_k} [k(x_k,y)]^{p'} [u(y)]^{p'} \, dy \right)^{r/p'}, \tag{1.13}$$

with the supremum taken over all increasing sequences $\{x_k\} \subset \mathbb{R}_+$*, and with the corresponding integrals replaced by the suprema of the integrand functions in the case* $p = 1$*.*

Prokhorov [85] recently found an integral criterion for the inequality (1.2).

Let $\alpha := D^q + 1$ and assume that $\int_t^\infty w^q < \infty$ for all $t \in (0,\infty)$. Let the function $\zeta\colon [0,\infty) \to [0,\infty)$ be defined by

$$\zeta(x) := \sup\left\{ y \in (0,\infty) \colon \int_y^\infty w^q \geqslant \alpha \int_x^\infty w^q \right\}, \qquad x \in [0,\infty)$$

(here $\sup \varnothing = 0$). We also let ζ_m, $m \in \mathbb{N}$, denote the superposition of m copies of ζ.

Theorem 1.4. *Let* $1 \leqslant p < \infty$ *and* $0 < q < p < \infty$*. Assume that the kernel* $k(x,y)$ *satisfies the condition* (4)*. Then*

$$\|K\|_{L_p \to L_q} \approx \mathbb{B} + C_\zeta, \tag{1.14}$$

where

$$\mathbb{B} := \left(\int_0^\infty w^q(x) \left[\int_x^\infty w^q \right]^{r/p} \|\chi_{(\zeta_3(x),x)}(\cdot) k(x,\cdot) u(\cdot)\|_{p'}^r \, dx \right)^{1/r} \tag{1.15}$$

and

$$C_\zeta \approx \left(\int_0^\infty (w(x) k(x,\zeta_2(x)))^q \|\chi_{(0,\zeta_2(x))} u\|_{p'}^r \left(\int_x^\infty (w(s) k(s,\zeta_2(s)))^q \, ds \right)^{r/p} dx \right)^{1/r}.$$

Remark 1.2. 1) In the case $k(x,y) \equiv 1$ we set

$$Hf(x) := w(x) \int_0^x f(y) u(y) \, dy, \qquad x > 0. \tag{1.16}$$

The inequality

$$\|Hf\|_q \leqslant C \|f\|_p \tag{1.17}$$

is known as the generalized Hardy inequality. In this case, $\mathbf{A} = \mathbf{A}_0 = \mathbf{A}_1$, $\mathbb{A} = \mathbb{A}_0 = \mathbb{A}_1$, $\mathscr{A} = \mathscr{A}_0 = \mathscr{A}_1$, $\mathbf{B}_0 \approx \mathbf{B}_1$, $\mathbb{B} = \mathbb{B}_0 = \mathbb{B}_1$, and $\mathscr{B} = \mathscr{B}_0 = \mathscr{B}_1$. Moreover, the conclusion of Theorem 1.2 also holds for $0 < q < p$, $p > 1$, and the following result holds for $0 < q < 1 = p$ (see [92]):

$$\|H\|_{L_1 \to L_q} \approx \left(\int_0^\infty \left[\operatorname*{ess\,sup}_{0<t<x} u(t) \right]^{\frac{q}{1-q}} \left(\int_x^\infty w^q \right)^{\frac{q}{1-q}} [w(x)]^q \, dx \right)^{\frac{1-q}{q}}. \tag{1.18}$$

A similar equivalence also holds for the dual integration operator and for its simplest modifications.

2) It was recently discovered [21], [77] that in parallel with the aforementioned criteria there exist families (scales) of equivalent conditions dependent on a continuous parameter and equivalent to the $(L_p - L_q)$-boundedness of the operator H.

3) Various forms of the criteria (1.9) and (1.10) broaden the range of applicability. For example, by using the constants $\mathbb{A}$ and $\mathbb{B}$ for the generalized Hardy inequality one can find a stable functional that characterizes the weighted inequality for the geometric mean operator [75]. For the Hardy–Steklov operator this problem is addressed in [114], [74], [116].

2. Duality principles

The converse to Hölder's inequality,

$$\sup_{f\in\mathfrak{M}} \frac{\int_0^\infty |f(s)g(s)|\,ds}{\left(\int_0^\infty |f(t)|^p\,dt\right)^{1/p}} = \|g\|_{p'}, \tag{2.1}$$

where $1 < p < \infty$, is known from functional analysis textbooks. This equality, sometimes known as the duality principle in Lebesgue spaces, lets us reduce inequalities $\|Tf\|_q \leqslant C\|f\|_p$ with a linear operator T to inequalities $\|T^*g\|_{p'} \leqslant C\|g\|_{q'}$ with the dual (adjoint) operator T^*.

In 1990 Sawyer [87] generalized a result of Ariño and Muckenhoupt [1] characterizing the inequality (7) with $1 < p = q < \infty$ and $w = v$ and discovered the duality principle for the L_p-cone of non-negative non-increasing functions, thereby making it possible to reduce inequalities of type (6) on $\mathfrak{M}^\downarrow$ to inequalities on the cone $\mathfrak{M}^+$ of non-negative functions, and there are many more criteria available to characterize the latter inequalities. This provided the impetus for a flourishing field of research in weighted inequalities on cones of monotone functions (see, for example, [2], [5]–[14], [20], [22]–[37], [41]–[45], [47], [53], [55], [62], [66]–[68], [73], [76], [78], [88]–[91], [94], [99], [103], [104], [110], [111], [112], [117], and others).

In the next subsection we present a number of results generalizing to some extent Sawyer's original result, and we illustrate the ranges of their applicability by providing a complete description of Hardy's inequality of type (7) for all $0 < p, q < \infty$.

2.1. Sawyer duality principle. Given functions $u, w \in \mathfrak{M}^+$ locally integrable on $\mathbb{R}_+$, we define $V(t) := \int_0^t v(s)\,ds$ and $U(t) := \int_0^t u(s)\,ds$ and assume that $V(t) < \infty$ and $U(t) < \infty$ for all $t > 0$. We also let $V(\infty) := \lim_{t\to\infty} V(t)$ when this limit exists, and similarly for $U(\infty)$. We shall require the following result.

Proposition 2.1. *Let $f \in \mathfrak{M}^\downarrow$. Then there exists a sequence $\{g_n\} \subset \mathfrak{M}^+$ of compactly supported functions such that each of the functions $f_n(x) := \int_x^\infty g_n(s)\,ds$ increases with respect to n for any $x > 0$ and $f(x) = \lim_{n\to\infty} \int_x^\infty g_n(y)\,dy$ for almost all $x > 0$.*

Proof. The functions

$$f_n(x) := n \int_x^{x+1/n} \chi_{(0,n)}(s) f(s)\,ds =: \int_x^\infty g_n(s)\,ds,$$

where

$$g_n(s) = n \begin{cases} f(s) - f(s+1/n), & s \in (0, n-1/n], \\ f(s), & s \in (n-1/n, n], \\ 0, & s > n, \end{cases}$$

are easily seen to satisfy all the requirements. □

Lemma 2.1. *Let $0 < p \leqslant q < \infty$. Then*

$$J_{p,q} := \sup_{f \in \mathfrak{M}^{\downarrow}} \frac{\left(\int_0^\infty [f(s)]^q u(s)\,ds\right)^{1/q}}{\left(\int_0^\infty [f(t)]^p v(t)\,dt\right)^{1/p}} = \sup_{t>0} \frac{U^{1/q}(t)}{V^{1/p}(t)} =: A_{p,q}. \tag{2.2}$$

Proof. The estimate $J_{p,q} \geqslant A_{p,q}$ follows if one puts $f = \chi_{[0,t]}$ under the supremum sign. We have $A_{p,q} = A_{1,q/p}^{1/p}$, and hence the substitution $f^p \to f$ reduces matters to the special case $1 = p \leqslant q < \infty$. Using Proposition 2.1, the monotone convergence theorem, and Minkowski's inequality, we get that

$$\begin{aligned}
\left(\int_0^\infty [f(s)]^q u(s)\,ds\right)^{1/q} &= \lim_{n\to\infty} \left(\int_0^\infty \left(\int_s^\infty g_n(t)\,dt\right)^q u(s)\,ds\right)^{1/q} \\
&\leqslant \lim_{n\to\infty} \int_0^\infty U^{1/q}(t) g_n(t)\,dt \leqslant A_{1,q} \lim_{n\to\infty} \int_0^\infty V(t) g_n(t)\,dt \\
&= A_{1,q} \lim_{n\to\infty} \int_0^\infty \left(\int_0^t v(s)\,ds\right) g_n(t)\,dt = A_{1,q} \lim_{n\to\infty} \int_0^\infty \left(\int_s^\infty g_n(t)\,dt\right) v(s)\,ds \\
&= A_{1,q} \int_0^\infty f(s) v(s)\,ds. \qquad \square
\end{aligned}$$

Lemma 2.2. *Let $0 < q < p < \infty$ and $1/r := 1/q - 1/p$. Then*

$$J_{p,q} \approx \left(\int_0^\infty \left(\int_t^\infty \frac{u(s)\,ds}{V(s)}\right)^{r/q} v(t)\,dt\right)^{1/r} =: B_{p,q}. \tag{2.3}$$

Proof. Since $B_{p,q} = B_{p/q,1}^{1/q}$, the substitution $f^q \to f$ reduces matters to the special case $1 = q < p < \infty$. In other words, it suffices to show that

$$J_{p,1} \approx \left(\int_0^\infty \left(\int_t^\infty \frac{u(s)\,ds}{V(s)}\right)^{p'} v(t)\,dt\right)^{1/p'} = B_{p,1}. \tag{2.4}$$

Let $f \in \mathfrak{M}^{\downarrow}$. By Hölder's inequality,

$$\begin{aligned}
\int_0^\infty f(s) u(s)\,ds &= \int_0^\infty f(s) \frac{u(s)}{V(s)} \left(\int_0^s v(t)\,dt\right) ds \\
&= \int_0^\infty v(t) \left(\int_t^\infty \frac{f(s) u(s)\,ds}{V(s)}\right) dt \leqslant \int_0^\infty v(t) f(t) \left(\int_t^\infty \frac{u(s)\,ds}{V(s)}\right) dt \\
&\leqslant B_{p,1} \left(\int_0^\infty [f(t)]^p v(t)\,dt\right)^{1/p},
\end{aligned} \tag{2.5}$$

and hence $J_{p,1} \leqslant B_{p,1}$. To prove the reverse inequality, we consider the test function

$$f_0(t) := \left(\int_t^\infty \frac{u(s)\,ds}{V(s)}\right)^{1/(p-1)}.$$

Integrating by parts, we get that

$$\begin{aligned}\int_0^\infty [f_0(t)]^p v(t)\,dt &= \int_0^\infty \left(\int_t^\infty \frac{u(s)\,ds}{V(s)}\right)^{p/(p-1)} v(t)\,dt \\ &= p' \int_0^\infty \left(\int_t^\infty \frac{u(s)\,ds}{V(s)}\right)^{1/(p-1)} u(t)\,dt = p' \int_0^\infty f_0(t)u(t)\,dt.\end{aligned}$$

Thus, by the equality of the second and last expressions,

$$J_{p,1} \geqslant \frac{\int_0^\infty f_0(t)u(t)\,dt}{\left(\int_0^\infty [f_0(t)]^p v(t)\,dt\right)^{1/p}} = (p')^{-1/p}\left(\int_0^\infty f_0(t)u(t)\,dt\right)^{1/p'} = \frac{1}{p'}B_{p,1}. \qquad \square$$

Remark 2.1. It can be shown by examples that the inequalities

$$\frac{1}{p'}B_{p,1} \leqslant J_{p,1} \leqslant B_{p,1} \tag{2.6}$$

obtained in the proof of Lemma 2.2 are sharp (see [76]).

The following result is a useful alternative variant of Lemma 2.2

Lemma 2.3. *Let $0 < q < p < \infty$ and $1/r := 1/q - 1/p$. Then*

$$J_{p,q} \approx \left(\int_0^\infty \left(\frac{U(t)}{V(t)}\right)^{r/q} v(t)\,dt\right)^{1/r} + \frac{U^{1/q}(\infty)}{V^{1/p}(\infty)} =: \mathbf{B}_{p,q}. \tag{2.7}$$

Proof. Since $\mathbf{B}_{p,q} = \mathbf{B}_{p/q,1}^{1/q}$, the substitution $f^q \to f$ reduces matters to the special case $1 = q < p < \infty$. In other words, it suffices to show that

$$J_{p,1} \approx \left(\int_0^\infty \left(\frac{U(t)}{V(t)}\right)^{p'} v(t)\,dt\right)^{1/p'} + \frac{U(\infty)}{V^{1/p}(\infty)} = \mathbf{B}_{p,1}. \tag{2.8}$$

Restricting the supremum in the expression for $J_{p,1}$ to constant functions, we see that

$$J_{p,1} \geqslant \frac{U(\infty)}{V^{1/p}(\infty)}. \tag{2.9}$$

Similarly, restricting the supremum in the expression for $J_{p,1}$ to functions of the form $f(s) = \int_s^\infty h$, we have

$$J_{p,1} \geqslant \sup_{h\in\mathfrak{M}^+} \frac{\int_0^\infty \left(\int_s^\infty h(t)\,dt\right)u(s)\,ds}{\left(\int_0^\infty \left(\int_s^\infty h(t)\,dt\right)^p v(t)\,dt\right)^{1/p}} = \sup_{h\in\mathfrak{M}^+} \frac{\int_0^\infty h(t)U(t)\,dt}{\left(\int_0^\infty \left(\int_s^\infty h(t)\,dt\right)^p v(t)\,dt\right)^{1/p}}.$$

By Hardy's inequality,

$$\left(\int_0^\infty \left(\int_t^\infty h\right)^p v(t)\,dt\right)^{1/p} \ll \left(\int_0^\infty [hV]^p [v]^{1-p}\right)^{1/p}. \tag{2.10}$$

Therefore,

$$J_{p,1} \gg \sup_{h\in\mathfrak{M}^+} \frac{\int_0^\infty h(t)U(t)\,dt}{\left(\int_0^\infty [h(t)V(t)]^p[v(t)]^{1-p}\,dt\right)^{1/p}},$$

and now, by the duality principle for L_p-spaces,

$$J_{p,1} \gg \left(\int_0^\infty \left(\frac{U(t)}{V(t)}\right)^{p'} v(t)\,dt\right)^{1/p'}. \tag{2.11}$$

The lower bound $J_{p,1} \gg \mathbf{B}_{p,1}$ follows from (2.9) and (2.11). To prove the upper bound, we integrate by parts and write

$$\int_x^\infty \frac{u(s)\,ds}{V(s)} = \int_x^\infty \frac{dU(s)}{V(s)} \leqslant \frac{U(\infty)}{V(\infty)} + \int_t^\infty \frac{U(s)v(s)\,ds}{V^2(s)}.$$

Let $f \in \mathfrak{M}^\downarrow$. Hence, continuing the chain in (2.5), we get that

$$\int_0^\infty f(t)u(t)\,dt \leqslant \left(\int_0^\infty f(x)v(x)\,dx\right)\frac{U(\infty)}{V(\infty)} + \int_0^\infty f(x)v(x)\left(\int_x^\infty \frac{U(s)v(s)\,ds}{V^2(s)}\right)dx.$$

By Hölder's inequality, this gives

$$\int_0^\infty f(t)u(t)\,dt \leqslant \left(\int_0^\infty f^p v\right)^{1/p}\left(\frac{U(\infty)}{V^{1/p}(\infty)} + \left(\int_0^\infty v(x)\left(\int_x^\infty \frac{U(s)v(s)\,ds}{V^2(s)}\right)^{p'} dx\right)^{1/p'}\right).$$

By Hardy's inequality,

$$\left(\int_0^\infty v(x)\left(\int_x^\infty \frac{U(s)v(s)\,ds}{V^2(s)}\right)^{p'} dx\right)^{1/p'} \ll \left(\int_0^\infty \frac{U^{p'}(s)v(s)\,ds}{V^{p'}(s)}\right)^{1/p'},$$

and hence

$$\frac{\int_0^\infty fu}{\left(\int_0^\infty f^p v\right)^{1/p}} \ll \left(\int_0^\infty \frac{U^{p'}(s)v(s)\,ds}{V^{p'}(s)}\right)^{1/p'} + \frac{U(\infty)}{V^{1/p}(\infty)}.$$

Since $f \in \mathfrak{M}^\downarrow$ is arbitrary, we have $J_{p,1} \ll \mathbf{B}_{p,1}$. □

Remark 2.2. Lemmas 2.1–2.3 with $q = 1$ constitute the duality principle for the cone of decreasing functions. This principle lets us characterize the inequality (6) with $q \geqslant 1$.

We first formulate the case $q = 1$.

Theorem 2.1. *Let $0 < p < \infty$ and let C be the smallest possible constant in the inequality*

$$\int_0^\infty u(x)\int_0^\infty k(x,y)f(y)w(y)\,dy\,dx \leqslant C\left(\int_0^\infty [f(x)]^p v(x)\,dx\right)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow. \tag{2.12}$$

Then for $0 < p \leqslant 1$

$$C = \sup_{t>0} \frac{\int_0^t w(y) \int_0^\infty k(x,y)u(x)\,dx\,dy}{V^{1/p}(t)}, \tag{2.13}$$

while for $1 < p < \infty$ *the following two-sided estimates hold*:

$$C \approx \left(\int_0^\infty \left(\int_t^\infty \frac{w(y)\int_0^\infty k(x,y)u(x)\,dx\,dy}{V(y)}\right)^{p'} v(t)\,dt\right)^{1/p'}, \tag{2.14}$$

$$C \approx \left(\int_0^\infty \left(\frac{\int_0^t w(y)\int_0^\infty k(x,y)u(x)\,dx\,dy}{V(t)}\right)^{p'} v(t)\,dt\right)^{1/p'} + \frac{\int_0^\infty w(y)\int_0^\infty k(x,y)u(x)\,dx\,dy}{V^{1/p}(\infty)}. \tag{2.15}$$

Proof. Just use Lemmas 2.1–2.3 with $q = 1$. $\square$

For $1 < q < \infty$, the inequality (6) on functions in $\mathfrak{M}^\downarrow$ reduces to inequalities on functions in $\mathfrak{M}^+$. The following theorem holds.

Theorem 2.2. *Let* $0 < p < \infty$ *and* $1 < q < \infty$, *and let* C *be the smallest possible constant in the inequality*

$$\left(\int_0^\infty \left(\int_0^\infty k(x,y)f(y)w(y)\,dy\right)^q u(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty [f(x)]^p v(x)\,dx\right)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow. \tag{2.16}$$

Then for $0 < p \leqslant 1$

$$C = \sup_{t>0} \frac{\left(\int_0^\infty \left(\int_0^t k(x,y)w(y)\,dy\right)^q u(x)\,dx\right)^{1/q}}{V^{1/p}(t)}, \tag{2.17}$$

while for $1 < p < \infty$ *the following two-sided estimates hold*:

$$C \approx \sup_{g\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_0^\infty \left(\int_t^\infty \frac{k(x,y)w(y)\,dy}{V(y)}\right) g(t)v(t)\,dt\right)^q u(x)\,dx\right)^{1/q}}{\left(\int_0^\infty g^p v\right)^{1/p}}, \tag{2.18}$$

$$C \approx \frac{\left(\int_0^\infty \left(\int_0^\infty k(x,y)w(y)\,dy\right)^q u(x)\,dx\right)^{1/q}}{V^{1/p}(\infty)} + \sup_{g\in\mathfrak{M}^+} \frac{\left(\int_0^\infty \left(\int_0^\infty \left(\int_0^t k(x,y)w(y)\,dy\right) \frac{g(t)v(t)\,dt}{V(t)}\right)^q u(x)\,dx\right)^{1/q}}{\left(\int_0^\infty g^p v\right)^{1/p}}. \tag{2.19}$$

The proof depends on Lemmas 2.1–2.3 and the duality principle (2.1) for L_p.

Remark 2.3. The formula (2.17) is also valid for $0 < p \leqslant q < \infty$ and $0 < p \leqslant 1$ (see [99], [104], [53], [66]). Moreover, a sort of dual result holds.

Theorem 2.3. *Let $1 \leqslant p \leqslant q < \infty$. Then for the smallest possible constant C in the inequality*

$$\left(\int_0^\infty f^q u\right)^{1/q} \leqslant C\left(\int_0^\infty \left(\int_0^\infty k(x,y)f(y)w(y)\,dy\right)^p v(x)\,dx\right)^{1/p}, \qquad f \in \mathfrak{M}^{\downarrow}, \tag{2.20}$$

the following equality holds:

$$C = \sup_{t>0} \frac{U^{1/q}(t)}{\left(\int_0^\infty \left(\int_0^t k(x,y)w(y)\,dy\right)^p v(x)\,dx\right)^{1/p}} =: \mathscr{A}_{p,q}. \tag{2.21}$$

Proof. Substituting the test function $f_t := \chi_{[0,t]}$ in the inequality (2.20) and taking the supremum over $t > 0$, we get that $C \geqslant \mathscr{A}_{p,q}$. To prove the reverse inequality we employ Proposition 2.1. By Minkowski's inequality,

$$\begin{aligned}
\left(\int_0^\infty f^q u\right)^{1/q} &= \lim_{n\to\infty}\left(\int_0^\infty \left(\int_x^\infty g_n\right)^q u(x)\,dx\right)^{1/q} \\
&= \lim_{n\to\infty}\left(\int_0^\infty \left(p\int_x^\infty \left(\int_t^\infty g_n\right)^{p-1} g_n(t)\,dt\right)^{q/p} u(x)\,dx\right)^{1/q} \\
&\leqslant \lim_{n\to\infty}\left(\int_0^\infty p\left(\int_t^\infty g_n\right)^{p-1} g_n(t)U^{p/q}(t)\,dt\right)^{1/p} \\
&\leqslant \mathscr{A}_{p,q}\lim_{n\to\infty}\left(\int_0^\infty p\left(\int_t^\infty g_n\right)^{p-1} g_n(t)\int_0^\infty \left(\int_0^t k(x,y)w(y)\,dy\right)^p v(x)\,dx\,dt\right)^{1/p} \\
&= \mathscr{A}_{p,q}\lim_{n\to\infty}\left(p\int_0^\infty v(x)\int_0^\infty \left(\int_0^t k(x,y)w(y)\,dy\right)^p \left(\int_t^\infty g_n\right)^{p-1} g_n(t)\,dt\,dx\right)^{1/p} \\
&\leqslant \mathscr{A}_{p,q}\lim_{n\to\infty}\left(\int_0^\infty \left(\int_0^\infty k(x,y)w(y)\left(\int_y^\infty g_n\right)dy\right)^p v(x)\,dx\right)^{1/p} \\
&= \mathscr{A}_{p,q}\left(\int_0^\infty \left(\int_0^\infty k(x,y)f(y)w(y)\,dy\right)^p v(x)\,dx\right)^{1/p}. \qquad \square
\end{aligned}$$

Theorem 2.4. *Let $0 < p \leqslant 1 \leqslant q < \infty$ and $k_i(x,y) \geqslant 0$, $i = 1,2$. Then the smallest possible constant C in the inequality*

$$\begin{aligned}
&\left(\int_0^\infty \left(\int_0^\infty k_1(x,y)f(y)\,dy\right)^q u(x)\,dx\right)^{1/q} \\
&\qquad\leqslant C\left(\int_0^\infty \left(\int_0^\infty k_2(x,y)f(y)\,dy\right)^p v(x)\,dx\right)^{1/p}, \qquad f \in \mathfrak{M}^{\downarrow},
\end{aligned}$$

is $C = \mathbf{A}_{p,q}$, where

$$\mathbf{A}_{p,q} := \sup_{t>0}\left(\int_0^\infty \left(\int_0^t k_1(x,y)\,dy\right)^q u(x)\,dx\right)^{1/q}\left(\int_0^\infty \left(\int_0^t k_2(x,y)\,dy\right)^p v(x)\,dx\right)^{-1/p}.$$

Proof. The inequality $\mathbf{A}_{p,q} \leqslant C$ follows by substituting the test function $f = \chi_{[0,t]}$. To prove the reverse inequality, we use the representation $f(y) = \int_y^\infty h(s)\,ds$ and twice apply Minkowski's inequality:

$$\begin{aligned}
&\biggl(\int_0^\infty \biggl(\int_0^\infty k_1(x,y)f(y)\,dy\biggr)^q u(x)\,dx\biggr)^{1/q} \\
&\quad= \biggl(\int_0^\infty \biggl(\int_0^\infty k_1(x,y)\biggl(\int_y^\infty h(t)\,dt\biggr)\,dy\biggr)^q u(x)\,dx\biggr)^{1/q} \\
&\quad= \biggl(\int_0^\infty \biggl(\int_0^\infty h(t)\int_0^t k_1(x,y)\,dy\,dt\biggr)^q u(x)\,dx\biggr)^{1/q} \\
&\quad\leqslant \int_0^\infty h(t)\biggl(\int_0^\infty \biggl(\int_0^t k_1(x,y)\,dy\biggr)^q u(x)\,dx\biggr)^{1/q} dt \\
&\quad\leqslant \mathbf{A}_{p,q}\int_0^\infty h(t)\biggl(\int_0^\infty \biggl(\int_0^t k_2(x,y)\,dy\biggr)^p v(x)\,dx\biggr)^{1/p} dt \\
&\quad\leqslant \mathbf{A}_{p,q}\biggl(\int_0^\infty \biggl(\int_0^\infty \biggl(\int_0^t k_2(x,y)\,dy\biggr)h(t)\,dt\biggr)^p v(x)\,dx\biggr)^{1/p} \\
&\quad= \mathbf{A}_{p,q}\biggl(\int_0^\infty \biggl(\int_0^\infty k_2(x,y)\biggl(\int_y^\infty h(t)\,dt\biggr)\,dy\biggr)^p v(x)\,dx\biggr)^{1/p} \\
&\quad= \mathbf{A}_{p,q}\biggl(\int_0^\infty \biggl(\int_0^\infty k_2(x,y)f(y)\,dy\biggr)^p v(x)\,dx\biggr)^{1/p}.
\end{aligned}$$

The required result now follows from Proposition 2.1. □

2.2. Hardy's inequality on monotone functions. Letting $k(x,y) = \chi_{[0,x]}(y)$ in (6), we obtain the three-weighted Hardy inequality

$$\biggl(\int_0^\infty \biggl(\int_0^x f(y)u(y)\,dy\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty [f(x)]^p v(x)\,dx\biggr)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow, \tag{2.22}$$

which has numerous applications.

Recall the notation $U(t) := \int_0^t u$, $V(t) := \int_0^t v$, $W(t) := \int_0^t w$.

Theorem 2.5. *The smallest possible constant C in the inequality (2.22) satisfies the following two-sided estimates*:

(a) *if $1 < p \leqslant q < \infty$, then $C \approx A_0 + A_1$, where*

$$A_0 = \sup_{t>0} A_0(t) := \sup_{t>0}\biggl(\int_0^t U^q w\biggr)^{1/q} V^{-1/p}(t),$$

$$A_1 := \sup_{t>0}\biggl(\int_t^\infty w\biggr)^{1/q}\biggl(\int_0^t \biggl(\frac{U}{V}\biggr)^{p'} v\biggr)^{1/p'};$$

(b) *if* $0<q<p<\infty$ *and* $1<p<\infty$, *then* $C\approx B_0+B_1$, *where*

$$B_0=B_0(p,q):=\left(\int_0^\infty V^{-r/p}(t)\left(\int_0^t U^q w\right)^{r/p}U^q(t)w(t)\,dt\right)^{1/r},$$
$$B_1=B_1(p,q):=\left(\int_0^\infty\left(\int_t^\infty w\right)^{r/p}\left(\int_0^t\left(\frac{U}{V}\right)^{p'}v\right)^{r/p'}w(t)\,dt\right)^{1/r};$$

(c) *if* $0<q<p\leqslant 1$, *then* $C\approx B_0+\mathscr{B}_1$, *where*

$$\mathscr{B}_1=\mathscr{B}_1(p,q):=\left(\int_0^\infty\left(\operatorname*{ess\,sup}_{s\in[0,t]}\frac{U^p(s)}{V(s)}\right)^{r/p}\left(\int_t^\infty w\right)^{r/p}w(t)\,dt\right)^{1/r};$$

(d) *if* $0<p\leqslant q<\infty$ *and* $0<p\leqslant 1$, *then* $C=\mathscr{A}_1$, *where*

$$\mathscr{A}_1:=\sup_{t>0}V^{-1/p}(t)\left(\int_0^\infty U^q(\min\{s,t\})w(s)\,ds\right)^{1/q}.$$

Proof. The assertion (d) follows from (2.17) in view of Remark 2.3.

Since f is monotone, the inequality

$$\left(\int_0^\infty[fU]^q w\right)^{1/q}\leqslant C\left(\int_0^\infty[f]^p v\right)^{1/p},\qquad f\in\mathfrak{M}^{\downarrow},$$

follows from (2.22), and thus in view of Lemmas 2.1 and 2.3, the following lower estimates hold: $A_0\leqslant C$ for $0<p\leqslant q<\infty$, and

$$\mathscr{B}_0:=\left(\int_0^\infty\left(\int_0^t U^q w\right)^{r/q}V^{-r/q}(t)v(t)\,dt\right)^{1/r}\ll C,\quad A_0(\infty)\leqslant C$$

for $0<q<p<\infty$. Integration by parts gives

$$B_0=\left(\frac{q}{r}\frac{\left(\int_0^\infty U^q w\right)^{r/q}}{V^{r/p}(\infty)}+\frac{q}{p}\mathscr{B}_0^r\right)^{1/r}\approx A_0(\infty)+\mathscr{B}_0,$$

from which the inequality $B_0\leqslant C$ easily follows. We note that for any $t>0$

$$B_0^r\geqslant\int_0^t V^{-r/p}(s)\left(\int_0^s U^q w\right)^{r/p}U^q(s)w(s)\,ds=\frac{q}{r}\int_0^t V^{-r/p}(s)\,d\left(\int_0^s U^q w\right)^{r/q}$$
$$\geqslant\frac{q}{r}V^{-r/p}(t)\int_0^t d\left(\int_0^s U^q w\right)^{r/q}=\frac{q}{r}V^{-r/p}(t)\left(\int_0^t U^q w\right)^{r/q}=\frac{q}{r}A_0^r(t),$$

and hence

$$B_0\gg A_0\geqslant A_0(\infty).\tag{2.23}$$

To prove the remaining claims in the assertion (a) and the assertion (b) for $q\geqslant 1$, we use Theorems 2.1 and 2.2 and criteria for the validity of Hardy's inequalities for non-negative functions. We now turn to the details. By (2.19),

$$
\begin{aligned}
C &\approx A_0(\infty) + \sup_{g\in\mathfrak{M}^+} \frac{\bigl(\int_0^\infty \bigl(\int_0^x (Ugv/V) + U(x)\int_x^\infty (gv/V)\bigr)^q w(x)\,dx\bigr)^{1/q}}{\bigl(\int_0^\infty g^p v\bigr)^{1/p}} \\
&\approx A_0(\infty) + \sup_{g\in\mathfrak{M}^+} \frac{\bigl(\int_0^\infty \bigl(\int_0^x (Ugv/V)\bigr)^q w(x)\,dx\bigr)^{1/q}}{\bigl(\int_0^\infty g^p v\bigr)^{1/p}} \\
&\quad + \sup_{g\in\mathfrak{M}^+} \frac{\bigl(\int_0^\infty \bigl(U(x)\int_x^\infty (gv/V)\bigr)^q w(x)\,dx\bigr)^{1/q}}{\bigl(\int_0^\infty g^p v\bigr)^{1/p}}. \qquad (2.24)
\end{aligned}
$$

Consequently, for $1 < p \leqslant q < \infty$ we can use criteria for the validity of Hardy's inequalities to show that

$$
C \approx A_0(\infty) + A_1 + \sup_{t>0}\biggl(\int_0^t U^q w\biggr)^{1/q}\biggl(\int_t^\infty \frac{v}{V^{p'}}\biggr)^{1/p'},
$$

whence follow the estimates $C \gg A_1$ and $C \ll A_0 + A_1$, verifying the assertion (a).

For $1 < q < p < \infty$ we have, by (2.24) and criteria for Hardy's inequalities,

$$
C \approx A_0(\infty) + B_1 + \biggl(\int_0^\infty \biggl(\int_0^t U^q w\biggr)^{r/p}\biggl(\int_t^\infty \frac{v}{V^{p'}}\biggr)^{r/p'} U^q(t)w(t)\,dt\biggr)^{1/r},
$$

and hence $C \gg B_1$ and $C \ll B_0 + B_1$. For $1 = q < p < \infty$ we have by (2.15)

$$
\begin{aligned}
C &\approx A_0(\infty) + \biggl(\int_0^\infty V^{-p'}(t)\biggl(\int_0^t u(y)\biggl(\int_y^\infty w\biggr)\,dy\biggr)^{p'} v(t)\,dt\biggr)^{1/p'} \\
&\approx A_0(\infty) + \biggl(\int_0^\infty V^{-p'}(t)\biggl(\int_0^t Uw\biggr)^{p'} v(t)\,dt\biggr)^{1/p'} \\
&\quad + \biggl(\int_0^\infty \biggl(\frac{U(t)}{V(t)}\biggr)^{p'}\biggl(\int_t^\infty w\biggr)^{p'} v(t)\,dt\biggr)^{1/p'} \\
&= A_0(\infty) + (p')^{1/p'}B_1 + \biggl(\int_0^\infty V^{-p'}(t)\biggl(\int_0^t Uw\biggr)^{p'} v(t)\,dt\biggr)^{1/p'}.
\end{aligned}
$$

From this the inequalities $C \gg B_1$ and $C \ll B_0 + B_1$ follow, thereby verifying the assertion (b) for $q \geqslant 1$.

Let us consider the assertion (b) for $0 < q < 1 < p < \infty$. Note that the lower estimate $C \gg B_0$ is already established. By Theorem 3.1 below, the inequality (2.22) is equivalent to the inequality

$$
\biggl(\int_0^\infty \biggl(\int_0^x \biggl(\int_y^\infty h(z)u(z)\,dz\biggr)\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty h^p V^p v^{1-p}\biggr)^{1/p}, \quad h \in \mathfrak{M}^+, \qquad (2.25)
$$

which, in turn, splits into the two Hardy inequalities

$$
\biggl(\int_0^\infty \biggl(\int_x^\infty h\biggr)^q U^q(x)w(x)\,dx\biggr)^{1/q} \leqslant C_1\biggl(\int_0^\infty h^p V^p v^{1-p}\biggr)^{1/p}, \qquad h \in \mathfrak{M}^+, \qquad (2.26)
$$

and

$$\left(\int_0^\infty\left(\int_0^x hU\right)^q w(x)\,dx\right)^{1/q} \leqslant C_2\left(\int_0^\infty h^pV^pv^{1-p}\right)^{1/p}, \qquad h\in\mathfrak{M}^+, \qquad (2.27)$$

and thus $C\approx C_1+C_2$. From criteria for Hardy's inequalities it follows that $C_1\ll B_0$ and $C_2\approx B_1$, and hence $C\approx B_0+B_1$.

To prove the assertion (c) we shall require the following modification of the formula (1.18): for $0<q<1$ the smallest possible constant C in the inequality

$$\left(\int_0^\infty\left(\int_0^x h(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C\int_0^\infty hv, \qquad h\in\mathfrak{M}^+,$$

has the two-sided estimate

$$C\approx\left(\int_0^\infty\left(\operatorname*{ess\,sup}_{s\in[0,t]}\frac{1}{v(s)}\right)^{\frac{q}{1-q}}\left(\int_t^\infty w\right)^{\frac{q}{1-q}} w(t)\,dt\right)^{\frac{1-q}{q}} =: \mathscr{B}. \qquad (2.28)$$

We first examine the case $0<q<1=p$. Assume that the inequality (2.22) holds. Then it also holds, with the same constant C, for all functions f of the form $f(x)=\int_x^\infty h$, where $h\in\mathfrak{M}^+$. We therefore have

$$\left(\int_0^\infty\left(\int_0^x\left(\int_y^\infty h\right)u(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C\int_0^\infty\left(\int_x^\infty h\right)v(x)\,dx, \qquad h\in\mathfrak{M}^+.$$

Hence,

$$\left(\int_0^\infty\left(\int_0^x hU\right)^q w(x)\,dx\right)^{1/q} \leqslant C\int_0^\infty hV, \qquad h\in\mathfrak{M}^+.$$

Now $C\gg\mathscr{B}_1(1,q)$ by (2.28). To prove the reverse estimate we use Proposition 2.1, Lemma 2.3, and the estimates (2.23) and (2.28). It follows that

$$\begin{aligned}
\left(\int_0^\infty\left(\int_0^x fu\right)^q w(x)\,dx\right)^{1/q} &= \lim_{n\to\infty}\left(\int_0^\infty\left(\int_0^x\left(\int_y^\infty h_n\right)u(y)\,dy\right)^q w(x)\,dx\right)^{1/q}\\
&\ll \lim_{n\to\infty}\left(\int_0^\infty\left(\int_0^x h_nU\right)^q w(x)\,dx\right)^{1/q} + \lim_{n\to\infty}\left(\int_0^\infty\left(\int_x^\infty h_n\right)^q U^q(x)w(x)\,dx\right)^{1/q}\\
&\ll \mathscr{B}_1(1,q)\lim_{n\to\infty}\int_0^\infty h_nV + B_0\lim_{n\to\infty}\int_0^\infty\left(\int_x^\infty h_n\right)v(x)\,dx\\
&= (\mathscr{B}_1(1,q)+B_0)\int_0^\infty fv.
\end{aligned}$$

Let us now take the case $0<q<p<1$. The substitution $f^p\to f$ in (2.22) gives the equivalent inequality

$$\left(\int_0^\infty\left(\int_0^x f^{1/p}u\right)^q w(x)\,dx\right)^{p/q} \leqslant C^p\int_0^\infty fv, \qquad f\in\mathfrak{M}^\downarrow. \qquad (2.29)$$

Arguing as in the previous paragraph and using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\int_0^\infty\left(\int_0^x f^{1/p}u\right)^q w(x)\,dx\right)^{p/q}\\
&\qquad=\lim_{n\to\infty}\left(\int_0^\infty\left(\int_0^x\left(\int_y^\infty h_n\right)^{1/p}u(y)\,dy\right)^q w(x)\,dx\right)^{p/q}\\
&\qquad\ll\lim_{n\to\infty}\left(\int_0^\infty\left(\int_0^x\left(\int_y^x h_n\right)^{1/p}u(y)\,dy\right)^q w(x)\,dx\right)^{p/q}\\
&\qquad\quad+\lim_{n\to\infty}\left(\int_0^\infty\left(\int_x^\infty h_n\right)^{q/p}U^q(x)w(x)\,dx\right)^{p/q}\\
&\qquad\ll\lim_{n\to\infty}\left(\int_0^\infty\left(\int_0^x h_nU^p\right)^{q/p}w(x)\,dx\right)^{p/q}\\
&\qquad\quad+B_0\left(1,\frac{q}{p}\right)\lim_{n\to\infty}\int_0^\infty\left(\int_x^\infty h_n\right)v(x)\,dx\\
&\qquad\ll\left(\mathscr{B}_1\left(1,\frac{q}{p}\right)+B_0\left(1,\frac{q}{p}\right)\right)\int_0^\infty fv=(\mathscr{B}_1^p(p,q)+B_0^p(p,q))\int_0^\infty fv,
\end{aligned}$$

thereby establishing the upper estimate $C\ll B_0+\mathscr{B}_1$. It remains to prove the lower estimate $C\gg\mathscr{B}_1$, since it has been shown that $C\gg B_0$. We use an idea from [2]. Setting $\mathscr{V}(t):=\operatorname{ess\,sup}_{s\in[0,t]}(U^p(s)/V(s))$, we assume first that $\mathscr{V}(0)=0$. Note that $\mathscr{V}(t)$ is a continuous non-decreasing function. We let

$$g(t):=\max\{2^m,m\in\mathbb{Z}\colon 2^m\leqslant\mathscr{V}^{r/p}(t)\},\qquad \tau_m:=\inf\{y\in[0,\infty)\colon 2^m\leqslant\mathscr{V}^{r/p}(y)\}.$$

Since $\mathscr{V}(t)$ is continuous and non-decreasing, the quantities τ_m exist for all $m\in\mathbb{Z}$. It is clear that they are increasing. From their definitions we have

$$\frac{U^r(\tau_m)}{V^{r/p}(\tau_m)}=2^m=\mathscr{V}^{r/p}(\tau_m)\leqslant\mathscr{V}_p^{r/p}(t)\leqslant 2^{m+1},\qquad t\in[\tau_m,\tau_{m+1}],$$
$$g(\tau_m)=2^m,\qquad g(s)\leqslant 2^{m-1}\quad\text{for all } s\in[0,\tau_m).$$

Observing that

$$g(t)=\sum_{m\in\mathbb{Z}}2^m\chi_{[\tau_m,\tau_{m+1})}(t)\leqslant\mathscr{V}^{r/p}(t),\tag{2.30}$$

we define

$$f(t):=\int_t^\infty\frac{\left(\int_x^\infty w\right)^{r/q}}{V(x)}\,dg(x),\tag{2.31}$$

the outer integral being understood in the Stieltjes sense. Thus, $f\in\mathfrak{M}^\downarrow$, and hence by (2.30),

$$\begin{aligned}
\int_0^\infty fv&=\int_0^\infty\left(\int_x^\infty w\right)^{r/q}dg(x)=\frac{q}{r}\int_0^\infty g(x)\left(\int_x^\infty w\right)^{r/p}w(x)\,dx\\
&\leqslant\frac{q}{r}\int_0^\infty\mathscr{V}^{r/p}(x)\left(\int_x^\infty w\right)^{r/p}w(x)\,dx=:\frac{q}{r}\mathscr{B}_1^r.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\bigg(\int_0^\infty\bigg(\int_0^x f^{1/p}(y)\,dU(y)\bigg)^q w(x)\,dx\bigg)^{1/q}\\
&\qquad\geqslant\bigg(\sum_m\int_{\tau_m}^{\tau_{m+1}} w(x)\bigg(\int_0^{\tau_m}\bigg(\int_y^{\tau_m}\frac{\big(\int_s^\infty w\big)^{r/q}}{V(s)}\,dg(s)\bigg)^{1/p}dU(y)\bigg)^q dx\bigg)^{1/q}\\
&\qquad\geqslant\bigg(\sum_m\bigg(\int_{\tau_m}^{\tau_{m+1}} w\bigg)\bigg(\int_{\tau_m}^\infty w\bigg)^{r/p}\\
&\qquad\qquad\times\bigg(V^{-1/p}(\tau_m)\int_0^{\tau_m}(g(\tau_m)-g(y))^{1/p}\,dU(y)\bigg)^q\bigg)^{1/q}\\
&\qquad\gg\bigg(\sum_m\bigg(\int_{\tau_m}^{\tau_{m+1}} w\bigg)\bigg(\int_{\tau_m}^\infty w\bigg)^{r/p}\bigg(\frac{2^{m/p}U(\tau_m)}{V(\tau_m)^{1/p}}\bigg)^q\bigg)^{1/q}\\
&\qquad\geqslant\bigg(\sum_m 2^m\int_{\tau_m}^{\tau_{m+1}}\bigg(\int_s^\infty w\bigg)^{r/p} w(s)\,ds\bigg)^{1/q}\\
&\qquad\gg\bigg(\int_0^\infty \mathscr{V}^{r/p}(s)\bigg(\int_s^\infty w\bigg)^{r/p} w(s)\,ds\bigg)^{1/q}=\mathscr{B}_1^{r/q}.
\end{aligned}$$

The inequality (2.29) with the function f from (2.31) implies that $C^p\mathscr{B}_1^r\gg\mathscr{B}_1^{pr/q}$. Thus, $C\gg\mathscr{B}_1$. Now let us show how to get rid of the restriction $\mathscr{V}(0)=0$. Let $\varepsilon>0$ be a small number, and let $u_\varepsilon(x):=u(x)\chi_{[\varepsilon,\infty]}(x)$. Hence, the inequality (2.29) will be satisfied with u_ε instead of u and with the same constant C. Letting $U_\varepsilon(t)=\int_0^t u_\varepsilon$ and $\mathscr{V}_\varepsilon:=\operatorname{ess\,sup}_{s\in[0,t]}(U_\varepsilon^p(s)/V(s))$, one clearly has $\mathscr{V}_\varepsilon(0)=0$. Let

$$\mathscr{B}_{1,\varepsilon}:=\bigg(\int_0^\infty\bigg(\operatorname*{ess\,sup}_{s\in[0,t]}\frac{U_\varepsilon^p(s)}{V(s)}\bigg)^{r/p}\bigg(\int_t^\infty w\bigg)^{r/p} w(t)\,dt\bigg)^{1/r}.$$

By the preceding argument, $\mathscr{B}_{1,\varepsilon}\ll C$. The estimate $\mathscr{B}_1\ll C$ now follows by the monotone convergence theorem. This completes the proof of Theorem 2.5. □

3. Reduction theorems

In this section we present a new method of reduction of $(L^{\downarrow}_{p,v}\to L^{+}_{q,w})$- and $(L^{\uparrow}_{p,v}\to L^{+}_{q,w})$-inequalities for positive *monotone* operators. In particular, we substantially supplement some results in [30].

The following conditions will be used below:

(i) $T(\lambda f)=\lambda Tf$ for all $\lambda\geqslant 0$ and $f\in\mathfrak{M}^+$;

(ii) $Tf(x)\leqslant cTg(x)$ for almost all $x\in\mathbb{R}_+$ if $f(x)\leqslant g(x)$ for almost all $x\in\mathbb{R}_+$, with a constant $c>0$ independent of f and g;

(iii) $T(f+\lambda\mathbf{1})\leqslant c(Tf+\lambda T\mathbf{1})$ for all $f\in\mathfrak{M}^+$ and $\lambda\geqslant 0$, with a constant $c>0$ independent of f and λ, where $\mathbf{1}$ is the constant function 1 on $\mathbb{R}_+$.

In § 3.1 we obtain reduction theorems for the inequality

$$\bigg(\int_0^\infty (Tf(t))^q w(t)\,dt\bigg)^{1/q}\leqslant C\bigg(\int_0^\infty (f(t))^p v(t)\,dt\bigg)^{1/p}\tag{3.1}$$

on the cones $\mathfrak{M}^{\downarrow}$ and $\mathfrak{M}^{\uparrow}$ with $0 < q < \infty$ and $1 \leqslant p < \infty$ (see Theorems 3.1–3.4). In contrast to the reduction theorems based on the Sawyer duality principle, in this range of variation of the integrability parameters one can investigate the inequalities in question for $0 < q < 1 \leqslant p < \infty$. In § 3.2 we study the case $0 < p \leqslant q < \infty$ under some additional restrictions on a monotone operator.

We do not present a general theory in the remaining most difficult case $0 < q < p \leqslant 1$. However, in §§ 3.3 and 3.4 we give a complete characterization of the inequalities

$$\biggl(\int_0^\infty\biggl(\int_x^\infty k(t,x)f(t)u(t)\,dt\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty (f(t))^p v(t)\,dt\biggr)^{1/p} \tag{3.2}$$

and

$$\biggl(\int_0^\infty\biggl(\int_0^x k(x,t)f(t)u(t)\,dt\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty (f(t))^p v(t)\,dt\biggr)^{1/p} \tag{3.3}$$

with Oinarov kernels on the cones $\mathfrak{M}^{\downarrow}$ and $\mathfrak{M}^{\uparrow}$.

3.1. Reduction theorems for monotone operators when $0 < q \leqslant \infty$ and $1 \leqslant p < \infty$. We set $V(t) := \int_0^t v$.

Theorem 3.1. *Let $0 < q \leqslant \infty$ and $1 < p < \infty$, and let $T\colon \mathfrak{M}^+ \to \mathfrak{M}^+$ be a positive operator. Then the condition* (3.1) *implies the inequality*

$$\biggl(\int_0^\infty\biggl(T\biggl(\int_x^\infty h\biggr)\biggr)^q w\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty h^p V^p v^{1-p}\biggr)^{1/p}, \qquad h \in \mathfrak{M}^+. \tag{3.4}$$

If $V(\infty) = \infty$ and if T is a monotone operator satisfying the conditions (i) *and* (ii), *then the condition* (3.4) *is sufficient for the inequality* (3.1) *to hold on the cone $\mathfrak{M}^{\downarrow}$. Further, if $0 < V(\infty) < \infty$, then a sufficient condition for* (3.1) *to hold on $\mathfrak{M}^{\downarrow}$ is that both* (3.4) *and*

$$\biggl(\int_0^\infty (T\mathbf{1})^q w\biggr)^{1/q} \leqslant C\biggl(\int_0^\infty v\biggr)^{1/p} \tag{3.5}$$

hold in the case when T satisfies the conditions (i)–(iii).

Proof. Let $0 < q < \infty$.

Necessity. Assume that a function $h \in \mathfrak{M}^+$ is integrable on $[x, \infty)$ for any $x > 0$. Then $f(x) = \int_x^\infty h \in \mathfrak{M}^{\downarrow}$, and hence by (3.1), Hardy's inequality (2.10) with $p > 1$, and Fubini's theorem with $p = 1$ we have

$$\begin{aligned}\biggl(\int_0^\infty\biggl(T\biggl(\int_x^\infty h\biggr)\biggr)^q w\biggr)^{1/q} &\leqslant C\biggl(\int_0^\infty\biggl(\int_x^\infty h\biggr)^p v(x)\,dx\biggr)^{1/p} \\ &\ll C\biggl(\int_0^\infty h^p V^p v^{1-p}\biggr)^{1/p},\end{aligned}$$

whence (3.4) follows.

Sufficiency. Assuming first that $V(\infty)=\infty$ and $f\in\mathfrak{M}^{\downarrow}$, we have

$$f(x)=\frac{f(x)V^2(x)}{V^2(x)}=2\biggl(\int_x^\infty\frac{v}{V^3}\biggr)f(x)V^2(x)$$
$$\leqslant\biggl(\int_x^\infty\frac{v}{V^3}\biggr)\int_0^x fVv\leqslant\int_x^\infty\biggl(\int_0^t fVv\biggr)\frac{v(t)\,dt}{V^3(t)}.$$

By (ii) and (3.4) with the function

$$h(t)=\biggl(\int_0^t fVv\biggr)\frac{v(t)}{V^3(t)},$$

this gives

$$\biggl(\int_0^\infty (Tf)^q w\biggr)^{1/q}\leqslant C\biggl(\int_0^\infty\biggl(\int_0^t fVv\biggr)^p\frac{v(t)\,dt}{V^{2p}(t)}\biggr)^{1/p}$$
$$\ll C\biggl(\int_0^\infty f^p v\biggr)^{1/p}$$

by Hardy's inequality (2.10) with $p>1$ and Fubini's theorem with $p=1$. Next, if $0<V(\infty)<\infty$, then

$$f(x)=\biggl[\frac{1}{V^2(x)}-\frac{1}{V^2(\infty)}\biggr]f(x)V^2(x)+\frac{V^2(x)}{V^2(\infty)}f(x)$$
$$\leqslant 4\biggl(\int_x^\infty\frac{v}{V^3}\biggr)\int_0^x fVv+\frac{V^{1/p'}(x)V^{1/p}(x)}{V(\infty)}f(x)$$
$$\leqslant 4\int_x^\infty\biggl(\int_0^t fVv\biggr)\frac{v(t)\,dt}{V^3(t)}+\frac{\bigl(\int_0^\infty f^p v\bigr)^{1/p}}{V^{1/p}(\infty)}=:\int_x^\infty h+\lambda\mathbf{1}.$$

Further, by (i)–(iii), (3.4), (3.5), Hardy's inequality with $p>1$, and Fubini's theorem with $p=1$, we have

$$\biggl(\int_0^\infty (Tf)^q w\biggr)^{1/q}\ll\biggl(\int_0^\infty\biggl(T\biggl(\int_x^\infty h\biggr)\biggr)^q w\biggr)^{1/q}+\lambda\biggl(\int_0^\infty (T\mathbf{1})^q w\biggr)^{1/q}$$
$$\ll C\biggl(\biggl(\int_0^\infty\biggl(\int_0^t fVv\biggr)^p\frac{v(t)\,dt}{V^{2p}(t)}\biggr)^{1/p}+\biggl(\int_0^\infty f^p v\biggr)^{1/p}\biggr)$$
$$\ll C\biggl(\int_0^\infty f^p v\biggr)^{1/p}.$$

The case $q=\infty$ is treated similarly. □

The following reduction theorem is a useful alternative.

Theorem 3.2. *Let* $0<q\leqslant\infty$ *and* $1\leqslant p<\infty$, *and assume that the operator* $T\colon\mathfrak{M}^+\to\mathfrak{M}^+$ *satisfies the conditions* (i) *and* (ii). *Then a sufficient condition for the inequality* (3.1) *to hold on the cone* $\mathfrak{M}^{\downarrow}$ *is that*

$$\biggl(\int_0^\infty\biggl(T\biggl(\frac{1}{V^2(x)}\int_0^x hV\biggr)\biggr)^q w\biggr)^{1/q}\leqslant C\biggl(\int_0^\infty h^p v^{1-p}\biggr)^{1/p},\qquad h\in\mathfrak{M}^+.\quad(3.6)$$

This condition is necessary if $V(\infty) = \infty$. If $0 < V(\infty) < \infty$, then (3.1) is necessary for (3.6) to hold if the conditions (i)–(iii) and the inequality (3.5) are all satisfied.

Proof. Let $0 < q < \infty$.

Sufficiency. If $f \in \mathfrak{M}^{\downarrow}$, then $f(x) \leqslant \dfrac{2}{V^2(x)} \displaystyle\int_0^x fvV)$. Using (ii) and (3.6) with $h = 2fv$, we get that

$$\left(\int_0^\infty (Tf)^q w\right)^{1/q} \ll \left(\int_0^\infty \left(T\left(\frac{1}{V^2(x)}\int_0^x hV\right)\right)^q w(x)\,dx\right)^{1/q}$$
$$\leqslant C\left(\int_0^\infty h^p v^{1-p}\right)^{1/p} = 2C\left(\int_0^\infty f^p v\right)^{1/p}.$$

Necessity. Assume first that $V(\infty) = \infty$ and that $h \in \mathfrak{M}^+$ is integrable on $[0, x]$ for all $x > 0$. Then

$$\frac{1}{V^2(x)}\int_0^x hV = 2\int_x^\infty \frac{v}{V^3}\int_0^x hV \leqslant 2\int_x^\infty \frac{v(s)\,ds}{V^3(s)}\int_0^s hV =: f(x) \in \mathfrak{M}^{\downarrow}.$$

By the condition (ii) and the inequality (3.1),

$$J := \left(\int_0^\infty \left(T\left(\frac{1}{V^2(x)}\int_0^x hV\right)\right)^q w\right)^{1/q} \ll \left(\int_0^\infty (Tf)^q w\right)^{1/q} \leqslant C\left(\int_0^\infty f^p v\right)^{1/p}$$
$$= 2C\left(\int_0^\infty \left(\int_x^\infty \frac{v(s)\,ds}{V^3(s)}\int_0^s hV\right)^p v(x)\,dx\right)^{1/p}.$$

If $p = 1$, then

$$J \ll C\int_0^\infty \left(\int_x^\infty \frac{v(s)\,ds}{V^3(s)}\int_0^s hV\right) v(x)\,dx = C\int_0^\infty \frac{v(s)\,ds}{V^2(s)}\int_0^s hV = C\int_0^\infty h.$$

Now let $p > 1$. We have

$$2\int_x^\infty \frac{v(s)\,ds}{V^3(s)}\int_0^s hV = \frac{1}{V^2(x)}\int_0^x hV + \int_x^\infty \frac{h}{V},$$

and (3.6) follows by Hardy's inequality.

Further, if $V(\infty) \in (0, \infty)$, then we write

$$\frac{1}{V^2(x)}\int_0^x hV = \left[\frac{1}{V^2(x)} - \frac{1}{V^2(\infty)}\right]\int_0^x hV + \frac{\int_0^x hV}{V^2(\infty)}$$
$$\leqslant 2\int_x^\infty \frac{v(s)\,ds}{V^3(s)}\int_0^s hV + \frac{\int_0^\infty hV}{V^2(\infty)} =: f_1(x) + \lambda\mathbf{1}.$$

We use the property (iii). The preceding arguments give us an estimate of the first term:

$$\left(\int_0^\infty (Tf_1)^q w\right)^{1/q} \ll C\left(\int_0^\infty h^p v^{1-p}\right)^{1/p}.$$

To estimate the second term, we employ the property (i), (3.5), and Hölder's inequality:

$$\Big(\int_0^\infty (T(\lambda\mathbf{1}))^q w\Big)^{1/q} \leqslant C\lambda\Big(\int_0^\infty v\Big)^{1/p} = C\frac{V^{1/p}(\infty)\int_0^\infty hV}{V^2(\infty)} \leqslant C\Big(\int_0^\infty h^p v^{1-p}\Big)^{1/p}.$$

The necessity of (3.6) follows from this.

The case $q=\infty$ is treated similarly. □

We consider the inequality

$$\Big(\int_0^\infty (Tf(t))^q w(t)\,dt\Big)^{1/q} \leqslant C\Big(\int_0^\infty (f(t))^p v(t)\,dt\Big)^{1/p}, \qquad f\in\mathfrak{M}^{\uparrow}, \tag{3.7}$$

and define $V_*(t) := \int_t^\infty v$.

Theorem 3.3. *Let $0<q\leqslant\infty$ and $1\leqslant p<\infty$, and let $T\colon \mathfrak{M}^+\to\mathfrak{M}^+$ be a positive operator. Then the condition (3.7) implies the inequality*

$$\Big(\int_0^\infty \Big(T\Big(\int_0^x h\Big)\Big)^q w\Big)^{1/q} \leqslant C\Big(\int_0^\infty h^p V_*^p v^{1-p}\Big)^{1/p}, \qquad h\in\mathfrak{M}^+. \tag{3.8}$$

If $V_*(0)=\infty$ and T is a monotone operator satisfying the conditions (i) and (ii), then (3.8) is sufficient for (3.7) to hold. If $0<V_*(0)<\infty$ and T is a monotone operator satisfying the conditions (i)–(iii), then (3.7) follows from (3.8) and (3.5).

Theorem 3.4. *Let $0<q<\infty$ and $1\leqslant p<\infty$, and let the operator $T\colon\mathfrak{M}^+\to\mathfrak{M}^+$ satisfy the conditions (i) and (ii). Then a sufficient condition for (3.7) is that*

$$\Big(\int_0^\infty \Big(T\Big(\frac{1}{V_*^2(x)}\int_x^\infty hV_*\Big)\Big)^q w\Big)^{1/q} \leqslant C\Big(\int_0^\infty h^p v^{1-p}\Big)^{1/p}, \qquad h\in\mathfrak{M}^+. \tag{3.9}$$

This condition is necessary if $V_*(0)=\infty$. If $0<V_*(0)<\infty$, then (3.7) is necessary for (3.9) to hold when the conditions (i)–(iii) and (3.5) are all satisfied.

3.2. Reduction theorems for quasi-linear operators when $0<p\leqslant q<\infty$.

Let $f\in\mathfrak{M}^{\downarrow}$. Then there exists a sequence $\{x_n\}\subset\mathbb{R}_+$ such that

$$\begin{aligned} f(x) &\approx \sum_n 2^{-n}\chi_{[0,x_n]}(x) = \sum_{n\colon x_n\geqslant x} 2^{-n}\chi_{[0,x_n]}(x) \\ &= \int_{[x,\infty)}\Big(\sum_n 2^{-n}\delta_{x_n}(s)\Big)\,ds =: \int_{[x,\infty)} h(s)\,ds, \end{aligned} \tag{3.10}$$

where $\delta_t(s)$ is the Dirac delta function at t. We note that

$$[f(x)]^r \approx \Big(\sum_n 2^{-n}\chi_{[0,x_n]}(x)\Big)^r \approx \sum_n 2^{-nr}\chi_{[0,x_n]}(x), \qquad r>0. \tag{3.11}$$

Theorem 3.5. *Let $0 < p \leqslant q < \infty$ and let $T\colon \mathfrak{M}^+ \to \mathfrak{M}^+$ be an operator satisfying both the conditions (i)–(iii) and the condition*

$$T\Big(\sum_n f_n\Big) \ll \Big(\sum_n [Tf_n]^p\Big)^{1/p} \tag{3.12}$$

for all $f_n \geqslant 0$. Then the inequality (3.1) on the cone $\mathfrak{M}^{\downarrow}$ is equivalent to any one of the following conditions:

$$\bigg(\int_0^\infty \bigg(\int_0^\infty [T\chi_{[0,s]}(x)]^p h(s)\,ds\bigg)^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_2^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.13}$$

$$\bigg(\int_0^\infty \Big[\sup_{s>0} T\chi_{[0,s]}(x) f(s)\Big]^q w(x)\,dx\bigg)^{1/q} \leqslant C_3 \bigg(\int_0^\infty f^p v\bigg)^{1/p}, \qquad f \in \mathfrak{M}^{\downarrow}, \tag{3.14}$$

$$\bigg(\int_0^\infty \bigg[\sup_{s>0} \big(T\chi_{[0,s]}(x)\big)^p \int_s^\infty h\bigg]^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_4^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.15}$$

or

$$\mathbf{D} := \sup_{t>0} \bigg(\int_0^\infty \big[T\chi_{[0,t]}(x)\big]^q w(x)\,dx\bigg)^{1/q} V^{-1/p}(t) < \infty. \tag{3.16}$$

Moreover,

$$C \approx C_2 = \mathbf{D} \approx C_3 = C_4. \tag{3.17}$$

Proof. It follows from Proposition 2.1 that (3.14) $\Leftrightarrow$ (3.15) with $C_3 = C_4$. The implication (3.14) $\Rightarrow$ (3.16) is obtained by applying (3.14) to the function $f_t(s) := \chi_{[0,t]}(s)$, $t > 0$. A similar argument proves the implication (3.1) $\Rightarrow$ (3.16). From the properties (i)–(iii) we see that for all $s > 0$

$$Tf(x) \geqslant T(\chi_{[0,s]}f)(x) \geqslant T\chi_{[0,s]}(x) f(s),$$

thereby proving the implication (3.1) $\Rightarrow$ (3.14). Let

$$k(x,s) := \big[T\chi_{[0,s]}(x)\big]^p \quad \text{and} \quad \mathbf{K}h(x) := \int_0^\infty k(x,s)h(s)\,ds.$$

Then the inequality (3.13) is equivalent to the boundedness of $\mathbf{K}\colon L^1_V \to L^{q/p}_w$, and hence by Theorem 1.1,

$$C_2^p = \|\mathbf{K}\|_{L^1_V \to L^{q/p}_w} = \mathbf{D}^p.$$

We assert that (3.13) $\Rightarrow$ (3.1). By (3.11) and (3.12),

$$\begin{aligned}(Tf^{1/p})(x) &\approx \bigg(T\Big(\sum_n 2^{-n}\chi_{[0,x_n]}\Big)^{1/p}\bigg)(x) \\ &\approx T\Big(\sum_n 2^{-n/p}\chi_{[0,x_n]}\Big)(x) \ll \Big(\sum_n 2^{-n}[T\chi_{[0,x_n]}(x)]^p\Big)^{1/p}.\end{aligned}$$

We observe that (3.1) is equivalent to the inequality

$$\left(\int_0^\infty (Tf^{1/p})^q w\right)^{p/q} \leqslant C^p \int_0^\infty fv, \qquad f \in \mathfrak{M}^\downarrow.$$

Using (3.11), we find that

$$\begin{aligned}\left(\int_0^\infty (Tf^{1/p})^q w\right)^{p/q} &\ll \left(\int_0^\infty \left(\sum_n 2^{-n}[T\chi_{[0,x_n]}(x)]^p\right)^{q/p} w(x)\,dx\right)^{p/q}\\ &= \left(\int_0^\infty \left(\int_0^\infty [T\chi_{[0,s]}(x)]^p h(s)\,ds\right)^{q/p} w(x)\,dx\right)^{p/q}\\ &\leqslant C_2^p \int_0^\infty hV = C_2^p \sum_n 2^{-n} V(x_n) \approx C_2^p \sum_n 2^{-n} \int_{(0,x_n]} v \approx C_2^p \int_0^\infty fv.\end{aligned}$$

Therefore, $C \ll C_2$, proving (3.17). □

A similar result holds also for the cone of non-decreasing functions.

Theorem 3.6. *Let $0 < p \leqslant q < \infty$ and let $T\colon \mathfrak{M}^+ \to \mathfrak{M}^+$ be an operator satisfying the conditions (i)–(iii) and the condition (3.12). Then the inequality (3.7) is equivalent to any one of the following conditions:*

$$\left(\int_0^\infty \left(\int_0^\infty [T\chi_{[s,\infty)}(x)]^p h(s)\,ds\right)^{q/p} w(x)\,dx\right)^{p/q} \leqslant C_2^p \int_0^\infty hV_*, \quad h \in \mathfrak{M}^+, \tag{3.18}$$

$$\left(\int_0^\infty \left[\sup_{s>0} T\chi_{[s,\infty)}(x) f(s)\right]^q w(x)\,dx\right)^{1/q} \leqslant C_3 \left(\int_0^\infty f^p v\right)^{1/p}, \quad f \in \mathfrak{M}^\uparrow, \tag{3.19}$$

$$\left(\int_0^\infty \left[\sup_{s>0} \big(T\chi_{[s,\infty)}(x)\big)^p \int_0^s h\right]^{q/p} w(x)\,dx\right)^{p/q} \leqslant C_4^p \int_0^\infty hV, \quad h \in \mathfrak{M}^+, \tag{3.20}$$

or

$$\mathbf{D}_* := \sup_{t>0} \left(\int_0^\infty [T\chi_{[t,\infty)}(x)]^q w(x)\,dx\right)^{1/q} V_*^{-1/p}(t) < \infty. \tag{3.21}$$

Moreover,

$$C \approx C_2 = \mathbf{D}_* \approx C_3 = C_4. \tag{3.22}$$

Next we consider the reverse inequality

$$\left(\int_0^\infty f^q w\right)^{1/q} \leqslant C \left(\int_0^\infty (Tf)^p v\right)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow. \tag{3.23}$$

Let $W_*(t) := \int_t^\infty w$.

Theorem 3.7. *Let $0 < p \leqslant q < \infty$ and let $T\colon \mathfrak{M}^+ \to \mathfrak{M}^+$ be an operator satisfying the conditions (i)–(iii) and the condition*

$$\left(\sum_n [Tf_n]^q\right)^{1/q} \ll T\left(\sum_n f_n\right) \tag{3.24}$$

for any $f_n \geqslant 0$. *Then the inequality* (3.23) *is equivalent to either of the conditions*

$$\int_0^\infty hW \leqslant C_2^q \biggl(\int_0^\infty \biggl(\int_0^\infty [T\chi_{[0,s]}(x)]^p h(s)\,ds\biggr)^{p/q} v(x)\,dx\biggr)^{q/p}, \qquad h \in \mathfrak{M}^+, \quad (3.25)$$

or

$$\mathfrak{D} := \sup_{t>0} W^{1/q}(t)\biggl(\int_0^\infty [T\chi_{[0,t]}(x)]^p v(x)\,dx\biggr)^{-1/p} < \infty. \qquad (3.26)$$

Moreover,

$$C \approx C_2 = \mathfrak{D}. \qquad (3.27)$$

Proof. The implication (3.23) ⇒ (3.26) is obvious. We assert that (3.26) ⇒ (3.25). In view of Minkowski's inequality,

$$\begin{aligned}\int_0^\infty hW &\leqslant \mathfrak{D}^q \int_0^\infty \biggl(\int_0^\infty [T\chi_{[0,t]}(x)]^p v(x)\,dx\biggr)^{q/p} h(t)\,dt \\ &\leqslant \mathfrak{D}^q \biggl(\int_0^\infty v(x)\biggl(\int_0^\infty [T\chi_{[0,t]}(x)]^q h(t)\,dt\biggr)^{p/q} dx\biggr)^{q/p}.\end{aligned}$$

Next, (3.23) is equivalent to the inequality

$$\int_0^\infty fw \leqslant C\biggl(\int_0^\infty (Tf^{1/q})^p v\biggr)^{q/p}, \qquad f \in \mathfrak{M}^{\downarrow}. \qquad (3.28)$$

Using (3.25), (3.10), (3.11), and (3.24), we have

$$\begin{aligned}\int_0^\infty fw \approx \int_0^\infty hW &\leqslant C_2^q \biggl(\int_0^\infty \biggl(\int_0^\infty [T\chi_{[0,s]}(x)]^q h(s)\,ds\biggr)^{p/q} v(x)\,dx\biggr)^{q/p} \\ &= C_2^q \biggl(\int_0^\infty \biggl(\sum_n 2^{-n}[T\chi_{[0,x_n]}(x)]^q\biggr)^{p/q} v(x)\,dx\biggr)^{q/p} \\ &= C_2^q \biggl(\int_0^\infty \biggl(\sum_n \bigl[T(2^{-n/q}\chi_{[0,x_n]})(x)\bigr]^q\biggr)^{p/q} v(x)\,dx\biggr)^{q/p} \\ &\leqslant C_2^q \biggl(\int_0^\infty \biggl[T\biggl(\sum_n 2^{-n/q}\chi_{[0,x_n]}\biggr)(x)\biggr]^p v(x)\,dx\biggr)^{q/p} \\ &\approx C_2^q \biggl(\int_0^\infty \biggl[T\biggl(\sum_n 2^{-n}\chi_{[0,x_n]}\biggr)^{1/q}(x)\biggr]^p v(x)\,dx\biggr)^{q/p} \\ &\approx C_2^q \biggl(\int_0^\infty \bigl(Tf^{1/q}\bigr)^p v\biggr)^{q/p}\end{aligned}$$

thereby establishing (3.27). □

Similar considerations allow one to characterize the inequality

$$\left(\int_0^\infty f^q w\right)^{1/q} \leqslant C\left(\int_0^\infty (Tf)^p v\right)^{1/p}, \qquad f \in \mathfrak{M}^\uparrow. \tag{3.29}$$

Theorem 3.8. *Let $0 < p \leqslant q < \infty$ and let $T\colon \mathfrak{M}^+ \to \mathfrak{M}^+$ be an operator satisfying the conditions (i)–(iii) and the condition (3.24). Then the inequality (3.29) is equivalent to either of the conditions*

$$\int_0^\infty hW_* \leqslant C_2^q\left(\int_0^\infty \left([T\chi_{[s,\infty)}(x)]^p h(s)\,ds\right)^{p/q} w(x)\,dx\right)^{q/p}, \qquad h \in \mathfrak{M}^+, \tag{3.30}$$

or

$$\mathfrak{D}_* := \sup_{t>0} W_*^{1/q}(t)\left(\int_0^\infty [T\chi_{[t,\infty)}(x)]^p v(x)\,dx\right)^{-1/p} < \infty. \tag{3.31}$$

Moreover,

$$C \approx C_2 = \mathfrak{D}_*. \tag{3.32}$$

Remark 3.1. Let T be a linear integral operator

$$Tf(x) := \int_0^\infty k(x,y)f(y)\,dy \tag{3.33}$$

with non-negative kernel. Then the condition (3.12) is satisfied for all $p \in (0,1]$, and so our Theorems 3.5 and 3.6 imply Theorem 4.1 in [97], Theorem 2.1, (a) in [66], and Theorem 1 in [6]. Similarly, the condition (3.24) holds for all $q \geqslant 1$. Therefore, we can extend Theorem 4.2 in [97] and Theorem 2.1, (b) in [66] to a wider range of variation of the integrability parameters by using Theorems 3.7 and 3.8.

3.3. Reduction theorems for Volterra operators when $0 < q < p \leqslant 1$.

Let u, v, and w be weight functions. We recall that $U(t) := \int_0^t u$, $V(t) := \int_0^t v$, and $W(t) := \int_0^t w$, and we define $U(y,x) := \int_x^y u$. For simplicity we assume that $0 < U(t) < \infty$, $0 < V(t) < \infty$, and $0 < W(t) < \infty$ for all $t > 0$ and $U(\infty) = V(\infty) = W(\infty) = \infty$. Let

$$K(x,y) := \int_0^y k(x,z)u(z)\,dz.$$

Theorem 3.9. *Let $0 < q < p \leqslant 1$ and let $k(x,y) \geqslant 0$ be a continuous Oinarov kernel. Then the following conditions are equivalent*:

$$\biggl(\int_0^\infty \biggl(\int_0^x k(x,y)f(y)u(y)\,dy\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant C_1\biggl(\int_0^\infty f^p v\biggr)^{1/p}, \quad f \in \mathfrak{M}^\downarrow, \tag{3.34}$$

$$\biggl(\int_0^\infty \biggl(\int_0^x K^p(x,y)h(y)\,dy + K^p(x,x)\int_x^\infty h(y)\,dy\biggr)^{q/p} w(x)\,dx\biggr)^{p/q} \leqslant C_2^p \int_0^\infty hV, \quad h \in \mathfrak{M}^+, \tag{3.35}$$

$$\biggl(\int_0^\infty \biggl(\sup_{0<y\leqslant x} K^p(x,y)\int_y^\infty h\biggr)^{q/p} w(x)\,dx\biggr)^{p/q} \leqslant C_3^p \int_0^\infty hV, \quad h \in \mathfrak{M}^+, \tag{3.36}$$

$$\biggl(\int_0^\infty \biggl(\sup_{0<y\leqslant x} K(x,y)f(y)\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant C_4\biggl(\int_0^\infty f^p v\biggr)^{1/p}, \quad f \in \mathfrak{M}^\downarrow, \tag{3.37}$$

$$\mathbb{B}^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \biggl(\int_{x_k}^{x_{k+1}} K^q(y,x_k)w(y)\,dy\biggr)^{r/q} V^{-r/p}(x_k) < \infty. \tag{3.38}$$

Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx \mathbb{B}. \tag{3.39}$$

Proof. The theorem is established by the implications (3.35) $\Rightarrow$ (3.34) $\Rightarrow$ (3.37) $\Leftrightarrow$ (3.36) $\Rightarrow$ (3.38) $\Rightarrow$ (3.35). The inequality (3.34) is equivalent to the inequality

$$\biggl(\int_0^\infty \biggl(\int_0^x k(x,y)f^{1/p}(y)u(y)\,dy\biggr)^q w(x)\,dx\biggr)^{p/q} \leqslant C_1^p \int_0^\infty fv, \qquad f \in \mathfrak{M}^\downarrow. \tag{3.40}$$

Let $f(x) = \int_x^\infty h$. Then by Minkowski's inequality,

$$\begin{aligned}\int_0^x k(x,z)f^{1/p}(z)u(z)\,dz &= \int_0^x \biggl(\int_z^\infty h\biggr)^{1/p} k(x,z)u(z)\,dz \\ &\leqslant \biggl(\int_0^\infty \biggl(\int_0^x \chi_{(z,\infty)}(y)k(x,z)u(z)\,dz\biggr)^p h(y)\,dy\biggr)^{1/p} \\ &\approx \biggl(\int_0^x K^p(x,y)h(y)\,dy + K^p(x,x)\int_x^\infty h(y)\,dy\biggr)^{1/p},\end{aligned}$$

and to prove that (3.35) $\Rightarrow$ (3.34) it remains to invoke Proposition 2.1.

Given any $f \in \mathfrak{M}^\downarrow$, we have

$$\int_0^x k(x,z)f(z)u(z)\,dz \geqslant \sup_{0<y\leqslant x}\int_0^y k(x,z)u(z)\,dz\, f(y) \geqslant \sup_{0<y\leqslant x} K(x,y)f(y).$$

Hence, (3.34) $\Rightarrow$ (3.37). It is clear that (3.37) $\Rightarrow$ (3.36), and the implication (3.36) $\Rightarrow$ (3.37) is proved by using Proposition 2.1 and Fatou's theorem.

Now assume that (3.36) holds, and let $\{x_n\} \subset \mathbb{R}_+$ be an arbitrary increasing sequence. For any $k \in \mathbb{Z}$, there exists a point $\varepsilon_k \in (x_k, x_{k+1})$ such that $V(\varepsilon_k) \leqslant 2V(x_k)$. Consider the function

$$h(x) := \sum_{k\in\mathbb{Z}} \frac{a_k}{x_k - \varepsilon_k} \chi_{(x_k,\varepsilon_k)}(x)$$

with an arbitrary sequence $\{a_k\} \subset \mathbb{R}_+$. Substituting this function in the inequality (3.36), we see that

$$\left(\sum_{k\in\mathbb{Z}} a_k^{q/p} \int_{x_k}^{x_{k+1}} K^q(x, x_k) w(x)\, dx\right)^{p/q} \leqslant 2C_3^q \sum_{k\in\mathbb{Z}} a_k V(x_k).$$

As a result, $\mathbb{B} \ll C_3$.

Thus, it remains only to show that (3.38) $\Rightarrow$ (3.35). By Oinarov's condition (4), this gives

$$K(x, y) \approx k(x, y) \int_0^y u(z)\, dz + \int_0^y k(y, z)u(z)\, dz = k(x, y)U(y) + K(y, y),$$

and it follows that (3.35) is equivalent to any one of the following three inequalities:

$$\left(\int_0^\infty \left(\int_0^x k(x, y)^p U(y)^p h(y)\, dy\right)^{q/p} w(x)\, dx\right)^{p/q} \leqslant C_2^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.41}$$

$$\left(\int_0^\infty \left(\int_0^x K(y, y)^p h(y)\, dy\right)^{q/p} w(x)\, dx\right)^{p/q} \leqslant C_2^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.42}$$

$$\left(\int_0^\infty \left(\int_x^\infty h(y)\, dy\right)^{q/p} K(x, x)^q w(x)\, dx\right)^{p/q} \leqslant C_2^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+. \tag{3.43}$$

By Lai's theorem (Theorem 5 in [54]), the inequality (3.41) holds if and only if

$$B_1^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} k(y, x_k)^q w(y)\, dy\right)^{r/q} \sup_{y\in(x_{k-1},x_k)} U(y)^r V^{-r/p}(y) < \infty,$$

$$B_2^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w(y)\, dy\right)^{r/q} \sup_{y\in(x_{k-1},x_k)} k^r(x_k, y)U(y)^r V^{-r/p}(y) < \infty,$$

and hence the inequality (3.42) is equivalent to the property

$$B_3^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w(y)\, dy\right)^{r/q} \sup_{y\in(x_{k-1},x_k)} K(y, y)^r V^{-r/p}(y) < \infty.$$

By the dual to Theorem 5 in [54], the inequality (3.43) is equivalent to

$$B_4^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_{k-1}}^{x_k} K(y, y)^q w(y)\, dy\right)^{r/q} V^{-r/p}(x_k) < \infty.$$

Let $y_k \in (x_{k-1}, x_k)$ be a point such that

$$\sup_{y\in(x_{k-1},x_k)} U(y)^r V^{-r/p}(y) = U(y_k)^r V^{-r/p}(y_k).$$

Further,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\biggl(\int_{x_k}^{x_{k+1}} k(y,x_k)^q w(y)\,dy\biggr)^{r/q} &\sup_{y\in(x_{k-1},x_k)} U(y)^r V^{-r/p}(y)\\ &\ll \sum_{k\in\mathbb{Z}}\biggl(\int_{y_k}^{y_{k+2}} K(y,y_k)^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_k)\\ &\ll \sum_{k\in\mathbb{Z}}\biggl(\int_{y_{2k}}^{y_{2k+2}} K(y,y_{2k})^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_{2k})\\ &\quad + \sum_{k\in\mathbb{Z}}\biggl(\int_{y_{2k+1}}^{y_{2k+3}} K(y,y_{2k+1})^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_{2k+1}) \ll \mathbb{B}^r.\end{aligned}$$

Consequently, $B_1 \ll \mathbb{B}$. If $y_k \in (x_{k-1}, x_k)$ is such that

$$\sup_{y\in(x_{k-1},x_k)} k^r(x_k,y) U(y)^r V^{-r/p}(y) = k^r(x_k,y_k) U(y_k)^r V^{-r/p}(y_k),$$

then

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\biggl(\int_{x_k}^{x_{k+1}} w(y)\,dy\biggr)^{r/q} &\sup_{y\in(x_{k-1},x_k)} k^r(x_k,y) U(y)^r V^{-r/p}(y)\\ &\ll \sum_{k\in\mathbb{Z}}\biggl(\int_{y_k}^{y_{k+2}} K(y,y_k)^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_k)\\ &\ll \sum_{k\in\mathbb{Z}}\biggl(\int_{y_{2k}}^{y_{2k+2}} K(y,y_{2k})^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_{2k})\\ &\quad + \sum_{k\in\mathbb{Z}}\biggl(\int_{y_{2k+1}}^{y_{2k+3}} K(y,y_{2k+1})^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_{2k+1}) \ll \mathbb{B}^r.\end{aligned}$$

Therefore, $B_2 \ll \mathbb{B}$. Given a point $y_k \in (x_{k-1}, x_k)$ for which

$$\sup_{y\in(x_{k-1},x_k)} K^r(y,y) V^{-r/p}(y) = K(y_k,y_k)^r V^{-r/p}(y_k),$$

we get that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\biggl(\int_{x_k}^{x_{k+1}} w(y)\,dy\biggr)^{r/q} &\sup_{y\in(x_{k-1},x_k)} K(y,y)^r V^{-r/p}(y)\\ &\ll \sum_{k\in\mathbb{Z}}\biggl(\int_{y_k}^{y_{k+2}} K(y,y_k)^q w(y)\,dy\biggr)^{r/q} V^{-r/p}(y_k)\end{aligned}$$

$$\ll \sum_{k\in\mathbb{Z}}\left(\int_{y_{2k}}^{y_{2k+2}} K(y,y_{2k})^q w(y)\,dy\right)^{r/q} V^{-r/p}(y_{2k})$$

$$+\sum_{k\in\mathbb{Z}}\left(\int_{y_{2k+1}}^{y_{2k+3}} K(y,y_{2k+1})^q w(y)\,dy\right)^{r/q} V^{-r/p}(y_{2k+1}) \ll \mathbb{B}^r,$$

so that $B_3 \ll \mathbb{B}$. Since $K(y,y) \ll K(y,x_{k-1})$ for $y \in (x_{k-1},x_k)$, we have $B_4 \ll \mathbb{B}$. Now $C_2 \ll \mathbb{B}$ by the above upper estimates. □

The next theorem gives a similar result for the cone of non-decreasing functions.

Theorem 3.10. *Let $0 < q < p \leqslant 1$. For a given continuous Oinarov kernel $k(x,y)$ let $K_*(y,x) = \int_y^\infty k(z,x)u(z)\,dz$. Then the following inequalities are equivalent:*

$$\left(\int_0^\infty\left(\int_x^\infty k(y,x)f(y)u(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C_1\left(\int_0^\infty f^p v\right)^{1/p}, \qquad f \in \mathfrak{M}^\uparrow,$$

$$\left(\int_0^\infty\left(\int_x^\infty K_*^p(y,x)h(y)\,dy + K_*^p(x,x)\int_0^x h(y)\,dy\right)^{q/p} w(x)\,dx\right)^{p/q}$$

$$\leqslant C_2^p \int_0^\infty hV_*, \qquad h \in \mathfrak{M}^+,$$

$$\left(\int_0^\infty\left(\sup_{y\geqslant x} K_*^p(y,x)\int_0^y h\right)^{q/p} w(x)\,dx\right)^{p/q} \leqslant C_3^p\int_0^\infty hV_*, \qquad h \in \mathfrak{M}^+,$$

$$\left(\int_0^\infty\left(\sup_{y\geqslant x} K_*(y,x)f(y)\right)^q w(x)\,dx\right)^{1/q} \leqslant C_4\left(\int_0^\infty f^p v\right)^{1/p}, \qquad f \in \mathfrak{M}^\uparrow,$$

$$\mathbb{B}_*^r := \sup_{\{x_k\}}\sum_{k\in\mathbb{Z}}\left(\int_{x_{k-1}}^{x_k} K_*^q(x_k,y)w(y)\,dy\right)^{r/q} V_*^{-r/p}(x_k) < \infty$$

(here, $V_*(t) := \int_t^\infty v$). Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx \mathbb{B}_*.$$

To prove the next theorem we need a fact of independent interest about Oinarov kernels.

Lemma 3.1. *Let a measurable function $k(x,y) \geqslant 0$ on $\{(x,y)\colon x \geqslant y \geqslant 0\}$ satisfy the right-hand inequality in Oinarov's condition (4). Then there exists an $\alpha_D \in (0,1)$ such that, for all $\alpha \in (0,\alpha_D]$ and any sequence $x_1 \geqslant x_2 \geqslant \dots \geqslant x_n$,*

$$[k(x_1,x_n)]^\alpha \leqslant 2\sum_{i=1}^{n-1}[k(x_i,x_{i+1})]^\alpha. \tag{3.44}$$

Proof. We set

$$\alpha_D := \frac{\log 2}{\log[D(1+2D)]}.$$

Let us show that (3.44) holds for $\alpha = \alpha_D$. As a result, it will follow from Jensen's inequality that (3.44) holds for all $\alpha \in (0, \alpha_D]$. We proceed by induction. By (4) and Jensen's inequality,

$$k(x_1, x_3) \leqslant D(k(x_1, x_2) + k(x_1, x_3)) \leqslant D(k^\alpha(x_1, x_2) + k^\alpha(x_1, x_3))^{1/\alpha}.$$

Hence,

$$k^\alpha(x_1, x_3) \leqslant D^\alpha(k^\alpha(x_1, x_2) + k^\alpha(x_1, x_3)).$$

Since $D^\alpha \leqslant 2$ for $\alpha = \alpha_D$, it follows that (3.44) holds with $n = 3$. Assume that (3.44) holds for all $n = 3, \dots, k$ and define

$$a := \sum_{i=1}^{k} k^\alpha(x_i, x_{i+1}).$$

Then either $k^\alpha(x_1, x_2) \leqslant a/2$ or $k^\alpha(x_k, x_{k+1}) \leqslant a/2$. Without loss of generality we assume that $k^\alpha(x_1, x_2) \leqslant a/2$. Let j be the largest natural number such that

$$\sum_{i=1}^{j} k^\alpha(x_i, x_{i+1}) \leqslant \frac{a}{2}.$$

Then $j < k$ and $\sum_{i=1}^{j+1} k^\alpha(x_i, x_{i+1}) > a/2$, and thus $\sum_{i=j+2}^{k} k^\alpha(x_i, x_{i+1}) \leqslant a/2$. By the induction assumption,

$$k^\alpha(x_1, x_{j+1}) \leqslant 2\sum_{i=1}^{j} k^\alpha(x_i, x_{i+1}) \leqslant a,$$

$$k^\alpha(x_{j+2}, x_{k+1}) \leqslant 2\sum_{i=j+2}^{k} k^\alpha(x_i, x_{i+1}) \leqslant a,$$

and

$$k^\alpha(x_{j+1}, x_{j+2}) \leqslant a.$$

Consequently,

$$k(x_1, x_{j+1}) \leqslant a^{1/\alpha}, \quad k(x_{j+2}, x_{k+1}) \leqslant a^{1/\alpha}, \quad k(x_{j+1}, x_{j+2}) \leqslant a^{1/\alpha}.$$

Using (4), we see that

$$\begin{aligned} k(x_1, x_{k+1}) &\leqslant D(k(x_1, x_{j+1}) + k(x_{j+1}, x_{k+1})) \\ &\leqslant D(k(x_1, x_{j+1}) + D(k(x_{j+1}, x_{j+2}) + k(x_{j+2}, x_{k+1}))) \\ &\leqslant Da^{1/\alpha} + 2D^2 a^{1/\alpha} = D(1 + 2D)a^{1/\alpha}. \end{aligned}$$

As a result, if $\alpha = \alpha_D$, then

$$k^\alpha(x_1, x_{k+1}) \leqslant D^\alpha(1 + 2D)^\alpha a \leqslant 2a,$$

completing the induction step. □

In what follows we will need the well-known relation (see Proposition 2.1 in [35])

$$\sum_n 2^n \Big(\sum_{i=n}^{\infty} a_i\Big)^s \approx \sum_n 2^n a_n^s s > 0, \tag{3.45}$$

which holds for any sequence $\{a_k\} \subset \mathbb{R}_+$.

We put

$$U_k(y,x) := \int_x^y k(z,x)u(z)\,dz.$$

Theorem 3.11. *Let $0 < q < p \leqslant 1$ and let the kernel $k(x,y) \geqslant 0$ satisfy Oinarov's condition (4). Then the following assertions are equivalent:*

$$\bigg(\int_0^\infty \Big(\int_x^\infty k(y,x)f(y)u(y)\,dy\Big)^q w(x)\,dx\bigg)^{1/q} \leqslant C_1\Big(\int_0^\infty f^p v\Big)^{1/p}, \qquad f \in \mathfrak{M}^{\downarrow}, \tag{3.46}$$

$$\bigg(\int_0^\infty \Big(\int_x^\infty U_k^p(y,x)h(y)\,dy\Big)^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_2^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.47}$$

$$\bigg(\int_0^\infty \Big(\sup_{y\geqslant x} U_k(y,x)f(y)\Big)^q w(x)\,dx\bigg)^{1/q} \leqslant C_3\Big(\int_0^\infty f^p v\Big)^{1/p}, \qquad f \in \mathfrak{M}^{\downarrow}, \tag{3.48}$$

$$\bigg(\int_0^\infty \Big(\sup_{y\geqslant x} U_k^p(y,x)\int_y^\infty h\Big)^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_4^p \int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.49}$$

$$\mathbb{B}^r := \sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \Big(\int_{x_k}^{x_{k+1}} U_k^q(x_{k+1},y)w(y)\,dy\Big)^{r/q} V^{-r/p}(x_{k+1}) < \infty. \tag{3.50}$$

Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx \mathbb{B}. \tag{3.51}$$

Proof. The theorem will be established by the implications

$$(3.50) \Rightarrow (3.47) \Rightarrow (3.46) \Rightarrow (3.48) \Leftrightarrow (3.49) \Rightarrow (3.50).$$

We begin by proving that (3.50) $\Rightarrow$ (3.47). Let $\{x_n\} \subset \mathbb{R}_+$ be a sequence such that $2^n = \int_0^{x_n} w$. Setting $\Delta_n := [x_n, x_{n+1})$, we have

$$\begin{aligned} J := &\int_0^\infty \Big(\int_x^\infty U_k^p(y,x)h(y)\,dy\Big)^{q/p} w(x)\,dx \approx \sum_n 2^n \Big(\int_{x_n}^\infty U_k^p(y,x_n)h(y)\,dy\Big)^{q/p} \\ &= \sum_n 2^n \Big(\sum_{i=n}^\infty \int_{\Delta_i} U_k^p(y,x_n)h(y)\,dy\Big)^{q/p} \approx \sum_n 2^n \Big(\sum_{i=n}^\infty \int_{\Delta_i} U_k^p(y,x_i)h(y)\,dy\Big)^{q/p} \\ &\quad + \sum_n 2^n \Big(\sum_{i=n+1}^\infty k^p(x_i,x_n)\int_{\Delta_i}\Big(\int_{x_i}^y u\Big)^p h(y)\,dy\Big)^{q/p} \\ &\quad + \sum_n 2^n \Big(\sum_{i=n+1}^\infty U_k^p(x_i,x_n)\int_{\Delta_i} h(y)\,dy\Big)^{q/p}. \end{aligned}$$

By (3.45) and Lemma 3.1 with $\alpha = \min\{\alpha_D, p\}$, this gives

$$
\begin{aligned}
J \ll & \sum_n 2^n \Big(\int_{\Delta_n} U_k^p(y, x_n) h(y)\, dy \Big)^{q/p} \\
& + \sum_n 2^n \Big(\sum_{i=n+1}^{\infty} \Big(\sum_{j=n}^{i-1} k^{\alpha}(x_{j+1}, x_j) \Big)^{p/\alpha} \int_{\Delta_i} \Big(\int_{x_i}^{y} u \Big)^p h(y)\, dy \Big)^{q/p} \\
& + \sum_n 2^n \Big(\sum_{i=n+1}^{\infty} \Big(\sum_{j=n}^{i-1} U_k(x_{j+1}, x_j) \Big)^p \int_{\Delta_i} h(y)\, dy \Big)^{q/p} \\
& + \sum_n 2^n \Big(\sum_{i=n+2}^{\infty} \Big(\sum_{j=n+1}^{i-1} k(x_j, x_n) \int_{\Delta_j} u \Big)^p \int_{\Delta_i} h(y)\, dy \Big)^{q/p} \\
=: & \ \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}
$$

Estimation of the sum I. We have

$$
\begin{aligned}
\int_{\Delta_n} U_k^p(y, x_n) h(y)\, dy &= \int_{\Delta_n} U_k^p(y, x_n) V^{-1}(y) h(y) V(y)\, dy \\
&\leqslant \Big[\sup_{y \in \Delta_n} U_k^p(y, x_n) V^{-1}(y) \Big] \int_{\Delta_n} hV = U_k^p(y_n, x_n) V^{-1}(y_n) \int_{\Delta_n} hV
\end{aligned}
$$

for some $y_n \in \Delta_n$. By Hölder's inequality,

$$
\begin{aligned}
\mathrm{I} &\leqslant \sum_n 2^n U_k^q(y_n, x_n) V^{-q/p}(y_n) \Big(\int_{\Delta_n} hV \Big)^{q/p} \\
&\leqslant \Big(\sum_n 2^{nr/q} U_k^r(y_n, x_n) V^{-r/p}(y_n) \Big)^{q/r} \Big(\int_0^{\infty} hV \Big)^{q/p} =: B_1^q \Big(\int_0^{\infty} hV \Big)^{q/p}.
\end{aligned}
$$

Further,

$$
\begin{aligned}
B_1^r &\approx \sum_n \Big(\int_{\Delta_{n-1}} w \Big)^{r/q} U_k^r(y_n, x_n) V^{-r/p}(y_n) \\
&\leqslant \sum_n \Big(\int_{\Delta_{n-1}} w(x) U_k^q(y_n, x)\, dx \Big)^{r/q} V^{-r/p}(y_n) \\
&\leqslant \sum_n \Big(\int_{y_{n-2}}^{y_n} w(x) U_k^q(y_n, x)\, dx \Big)^{r/q} V^{-r/p}(y_n) \\
&= \sum_k \Big(\int_{y_{2k-2}}^{y_{2k}} w(x) U_k^q(y_{2k}, x)\, dx \Big)^{r/q} V^{-r/p}(y_{2k}) \\
&\qquad + \sum_k \Big(\int_{y_{2k-1}}^{y_{2k+1}} w(x) U_k^q(y_{2k+1}, x)\, dx \Big)^{r/q} V^{-r/p}(y_{2k+1}).
\end{aligned}
$$

If $z_k := y_{2k}$ and $z'_k := y_{2k+1}$, then by the definition of $\mathbb{B}^r$,

$$B_1^r \ll \sum_k \left(\int_{z_{k-1}}^{z_k} w(x) U_k^q(z_k, x)\,dx\right)^{r/q} V^{-r/p}(z_k) + \sum_k \left(\int_{z'_{k-1}}^{z'_k} w(x) U_k^q(z'_k, x)\,dx\right)^{r/q} V^{-r/p}(z'_k) \ll \mathbb{B}^r.$$

Thus,

$$I \ll \mathbb{B}^q \left(\int_0^\infty hV\right)^{q/p}. \tag{3.52}$$

Estimation of the sums II *and* IV. By Minkowski's inequality and (3.45),

$$\begin{aligned} \mathrm{II} &\leqslant \sum_n 2^n \left(\sum_{j=n}^\infty k^\alpha(x_{j+1}, x_j)\left(\sum_{i=j+1}^\infty \int_{\Delta_i}\left(\int_{x_i}^y u\right)^p h(y)\,dy\right)^{\alpha/p}\right)^{q/\alpha} \\ &\approx \sum_n 2^n k^q(x_{n+1}, x_n)\left(\sum_{i=n+1}^\infty \int_{\Delta_i}\left(\int_{x_i}^y u\right)^p h(y)\,dy\right)^{q/p} \\ &\leqslant \sum_n 2^n k^q(x_{n+1}, x_n)\left(\int_{x_{n+1}}^\infty \left(\int_{x_{n+1}}^y u\right)^p h(y)\,dy\right)^{q/p} =: J_0. \end{aligned}$$

Hence,

$$\mathrm{IV} = \sum_n 2^n \left(\sum_{i=n+2}^\infty \left(\sum_{j=n+1}^{i-1} k(x_j, x_n)\int_{\Delta_j} u\right)^p \int_{\Delta_i} h(y)\,dy\right)^{q/p}.$$

In view of Lemma 3.1 with $\alpha \in (0, p)$, this gives

$$k(x_j, x_n) \ll \left(\sum_{k=n}^{j-1} k^\alpha(x_{k+1}, x_k)\right)^{1/\alpha}.$$

By Minkowski's inequality,

$$\begin{aligned} \sum_{j=n+1}^{i-1} k(x_j, x_n)\int_{\Delta_j} u &\ll \sum_{j=n+1}^{i-1}\left(\sum_{k=n}^{j-1} k^\alpha(x_{k+1}, x_k)\right)^{1/\alpha}\int_{\Delta_j} u \\ &\leqslant \left(\sum_{k=n}^{i-2} k^\alpha(x_{k+1}, x_k)\left(\int_{x_{k+1}}^{x_i} u\right)^\alpha\right)^{1/\alpha}. \end{aligned}$$

Again by Minkowski's inequality,

$$\mathrm{IV} \ll \sum_n 2^n \biggl(\sum_{i=n+2}^{\infty} \biggl(\sum_{k=n}^{i-2} k^{\alpha}(x_{k+1}, x_k) \biggl(\int_{x_{k+1}}^{x_i} u \biggr)^{\alpha} \biggr)^{p/\alpha} \int_{\Delta_i} h \biggr)^{q/p}$$
$$\leqslant \sum_n 2^n \biggl(\sum_{k=n}^{\infty} k^{\alpha}(x_{k+1}, x_k) \biggl(\sum_{i=k+2}^{\infty} \biggl(\int_{x_{k+1}}^{x_i} u \biggr)^{p} \int_{\Delta_i} h \biggr)^{\alpha/p} \biggr)^{q/\alpha}.$$

Further,

$$\sum_{i=k+2}^{\infty} \biggl(\int_{x_{k+1}}^{x_i} u \biggr)^{p} \int_{\Delta_i} h \leqslant \sum_{i=k+2}^{\infty} \int_{x_i}^{x_{i+1}} h(y) \biggl(\int_{x_{k+1}}^{y} u \biggr)^{p} dy$$
$$\leqslant \int_{x_{k+1}}^{\infty} \biggl(\int_{x_{k+1}}^{y} u \biggr)^{p} h(y)\, dy.$$

Therefore, according to (3.45),

$$\mathrm{IV} \ll \sum_n 2^n \biggl(\sum_{k=n}^{\infty} k^{\alpha}(x_{k+1}, x_k) \biggl(\int_{x_{k+1}}^{\infty} \biggl(\int_{x_{k+1}}^{y} u \biggr)^{p} h(y)\, dy \biggr)^{\alpha/p} \biggr)^{q/\alpha}$$
$$\approx \sum_n 2^n k^q(x_{n+1}, x_n) \biggl(\int_{x_{n+1}}^{\infty} \biggl(\int_{x_{n+1}}^{y} u \biggr)^{p} h(y)\, dy \biggr)^{q/p} =: J_0.$$

Thus,

$$\mathrm{II} + \mathrm{IV} \ll J_0.$$

Let $i \in \mathbb{Z}$. We set

$$A_i := \biggl\{ m \in \mathbb{Z} \colon 2^i \leqslant \sum_{n<m} 2^n k^q(x_{n+1}, x_n) < 2^{i+1} \biggr\} \tag{3.53}$$

and define $\mathscr{A} := \bigcup_k A_{i_k} \subset \mathbb{Z}$, where $A_{i_k} \neq \varnothing$ and $A_i = \varnothing$ for $i \notin \{i_k\}$. Next we define

$$n_{i_k} := \inf A_{i_k}, \quad n_{i_k}^{+} := \sup A_{i_k}; \qquad z_k := x_{n_{i_k}}, \quad z_k^{+} := x_{n_{i_k}^{+}}.$$

We have $n_{i_k} < n_{i_k}^{+} < n_{i_{k+1}}$. Assume for simplicity that $\operatorname{card} \bigcup_k \{i_k\} = \aleph_0$. We have

$$J_0 = \sum_k \sum_{n=n_{i_k}}^{n_{i_k}^{+}} 2^n k^q(x_{n+1}, x_n) \biggl(\int_{x_{n+1}}^{\infty} \biggl(\int_{x_{n+1}}^{y} u \biggr)^{p} h(y)\, dy \biggr)^{q/p}.$$

By Jensen's inequality and (3.45),

$$J_0 \ll \sum_k 2^{i_k}\bigg(\int_{z_k}^\infty \Big(\int_{z_k}^y u\Big)^p h(y)\,dy\bigg)^{q/p} = \sum_k 2^{i_k}\bigg(\sum_{j=k}^\infty \int_{z_j}^{z_{j+1}} \Big(\int_{z_k}^y u\Big)^p h(y)\,dy\bigg)^{q/p}$$

$$\approx \sum_k 2^{i_k}\bigg(\sum_{j=k}^\infty \int_{z_j}^{z_{j+1}} \Big(\int_{z_j}^y u\Big)^p h(y)\,dy\bigg)^{q/p}$$

$$+ \sum_k 2^{i_k}\bigg(\sum_{j=k}^\infty \Big(\sum_{l=k}^{j-1}\int_{z_l}^{z_{l+1}} u\Big)^p \int_{z_j}^{z_{j+1}} h\bigg)^{q/p}$$

$$\leqslant \sum_k 2^{i_k}\bigg(\sum_{j=k}^\infty \int_{z_j}^{z_{j+1}} \Big(\int_{z_j}^y u\Big)^p h(y)\,dy\bigg)^{q/p}$$

$$+ \sum_k 2^{i_k}\bigg(\sum_{j=k}^\infty \sum_{l=k}^{j-1}\Big(\int_{z_l}^{z_{l+1}} u\Big)^p \int_{z_j}^{z_{j+1}} h\bigg)^{q/p}$$

$$\approx \sum_k 2^{i_k}\bigg(\int_{z_k}^{z_{k+1}} \Big(\int_{z_k}^y u\Big)^p h(y)\,dy\bigg)^{q/p} + \sum_k 2^{i_k}\bigg(\sum_{l=k}^\infty \Big(\int_{z_l}^{z_{l+1}} u\Big)^p \int_{z_{l+1}}^\infty h\bigg)^{q/p}$$

$$\ll \sum_k 2^{i_k}\bigg[\sup_{z_k<y<z_{k+1}} \Big(\int_{z_k}^y u\Big)^p V^{-1}(y)\bigg]^{q/p}\bigg(\int_{z_k}^{z_{k+1}} hV\bigg)^{q/p}$$

$$+ \sum_k 2^{i_k}\bigg(\int_{z_k}^{z_{k+1}} u\bigg)^q \bigg(\int_{z_{k+1}}^\infty h\bigg)^{q/p} =: J_{0,1} + J_{0,2}.$$

Further, by Hölder's inequality,

$$J_{0,1} \leqslant \bigg(\sum_k 2^{i_k r/q} \sup_{z_k<y<z_{k+1}} \Big(\int_{z_k}^y u\Big)^r V^{-r/p}(y)\bigg)^{q/r}\bigg(\int_0^\infty hV\bigg)^{q/p}$$

$$=: B_2^q \bigg(\int_0^\infty hV\bigg)^{q/p}.$$

Let $z_k^0 \in [z_k, z_{k+1}]$ be such that

$$\sup_{z_k<y<z_{k+1}} \Big(\int_{z_k}^y u\Big)^r V^{-r/p}(y) = \Big(\int_{z_k}^{z_k^0} u\Big)^r V^{-r/p}(z_k^0).$$

Then

$$B_2^r = \sum_k 2^{i_k r/q}\Big(\int_{z_k}^{z_k^0} u\Big)^r V^{-r/p}(z_k^0).$$

According to (3.53),

$$2^{i_k} \leqslant \sum_{n<n_{i_k}} 2^n k^q(x_{n+1},x_n) = \sum_{n_{i_{k-2}}\leqslant n<n_{i_k}} 2^n k^q(x_{n+1},x_n) + \sum_{n<n_{i_{k-2}}} 2^n k^q(x_{n+1},x_n)$$

$$\leqslant \sum_{n_{i_{k-2}}\leqslant n<n_{i_k}} 2^n k^q(x_{n+1},x_n) + 2^{i_{k-2}+1}.$$

Thus,

$$2^{i_k} \leqslant 2 \sum_{n_{i_{k-2}} \leqslant n < n_{i_k}} 2^n k^q(x_{n+1}, x_n), \tag{3.54}$$

and hence

$$B_2^r \ll \sum_k \biggl(\sum_{n_{i_{k-2}} \leqslant n < n_{i_k}} 2^n k^q(x_{n+1}, x_n) \biggr)^{r/q} \biggl(\int_{z_k}^{z_k^0} u \biggr)^r V^{-r/p}(z_k^0).$$

We have

$$\begin{aligned} \sum_{n_{i_{k-2}} \leqslant n < n_{i_k}} 2^n k^q(x_{n+1}, x_n) &\approx \sum_{n_{i_{k-2}} \leqslant n < n_{i_k}} k^q(x_{n+1}, x_n) \int_{x_{n-1}}^{x_n} w(y)\, dy \\ &\ll \sum_{n_{i_{k-2}} \leqslant n < n_{i_k}} \int_{x_{n-1}}^{x_n} k^q(x_{n+1}, y) w(y)\, dy \ll \int_{z_{k-2}-1}^{z_k} k^q(z_k, y) w(y)\, dy \\ &\leqslant \int_{z_{k-3}}^{z_k} k^q(z_k, y) w(y)\, dy =: \varphi_k. \end{aligned}$$

Next,

$$\begin{aligned} \varphi_k^{1/q} \biggl(\int_{z_k}^{z_k^0} u \biggr) &= \biggl(\int_{z_{k-3}}^{z_k} w(y) \biggl(k(z_k, y) \int_{z_k}^{z_k^0} u(z)\, dz \biggr)^q dy \biggr)^{1/q} \\ &\ll \biggl(\int_{z_{k-3}}^{z_k} w(y) \biggl(\int_{z_k}^{z_k^0} k(z, y) u(z)\, dz \biggr)^q dy \biggr)^{1/q} \\ &\leqslant \biggl(\int_{z_{k-3}}^{z_k^0} w(y) \biggl(\int_{y}^{z_k^0} k(z, y) u(z)\, dz \biggr)^q dy \biggr)^{1/q}. \end{aligned} \tag{3.55}$$

Since $z_{k-3} \geqslant z_{k-4}^0$, it follows that

$$B_2^r \ll \sum_k \biggl(\int_{z_{k-4}^0}^{z_k^0} w(y) \biggl(\int_{y}^{z_k^0} k(z, y) u(z)\, dz \biggr)^q dy \biggr)^{r/q} V^{-r/p}(z_k^0).$$

We use the conventional approach and twice split the sum $\sum_k$ into the sum of sums $\sum_{k=2n} + \sum_{k=2n+1}$ to show that $B_2 \ll \mathbb{B}$.

Now let us prove the upper estimate for $J_{0,2}$. We have

$$J_{0,2} = \sum_k 2^{i_k} \biggl(\int_{z_k}^{z_{k+1}} u \biggr)^q \biggl(\int_{z_{k+1}}^{\infty} h \biggr)^{q/p}.$$

Given $j \in \mathbb{Z}$, we define

$$B_j := \biggl\{ m \in \mathbb{Z} \colon 2^j \leqslant \sum_{k<m} 2^{i_k} \biggl(\int_{z_k}^{z_{k+1}} u \biggr)^q < 2^{j+1} \biggr\}. \tag{3.56}$$

We set $\mathscr{B} := \bigcup_k B_{j_k} \subset \mathbb{Z}$, where $B_{j_k} \neq \varnothing$ and $B_j = \varnothing$ for $j \notin \{j_k\}$. Also, let

$$l_{j_k} := \inf B_{j_k}, \quad l_{j_k}^+ := \sup B_{j_k}; \qquad y_k := z_{l_{j_k}}, \quad y_k^+ := z_{l_{j_k}^+}.$$

In our setting, $l_{j_k} < l_{j_k}^+ < l_{j_{k+1}}$. We assume for simplicity that $\operatorname{card} \bigcup_k \{j_k\} = \aleph_0$. By Hölder's inequality,

$$\begin{aligned}
J_{0,2} &= \sum_j \sum_{k=l_j}^{l_j^+} 2^{i_k} \Big(\int_{z_k}^{z_{k+1}} u \Big)^q \Big(\int_{z_{k+1}}^{\infty} h \Big)^{q/p} \ll \sum_j 2^j \Big(\int_{y_j}^{\infty} h \Big)^{q/p} \\
&= \sum_j 2^j \Big(\sum_{k=j}^{\infty} \int_{y_k}^{y_k^+} h \Big)^{q/p} \approx \sum_k 2^k \Big(\int_{y_k}^{y_k^+} h \Big)^{q/p} \\
&\leqslant \sum_k 2^k V^{-q/p}(y_k) \Big(\int_{y_k}^{y_k^+} hV \Big)^{q/p} \leqslant \Big(\sum_k 2^{kr/q} V^{-r/p}(y_k) \Big)^{q/r} \Big(\int_0^{\infty} hV \Big)^{q/p} \\
&=: B_3^q \Big(\int_0^{\infty} hV \Big)^{q/p}.
\end{aligned}$$

According to (3.56), $2^j \leqslant 2 \sum_{l_{j-2} \leqslant k < l_j} 2^{i_k} \Big(\int_{z_k}^{z_{k+1}} u \Big)^q$. Hence,

$$B_3^r \ll \sum_j \Big(\sum_{l_{j-2} \leqslant k < l_j} 2^{i_k} \Big(\int_{z_k}^{z_{k+1}} u \Big)^q \Big)^{r/q} V^{-r/p}(y_j).$$

By (3.54) and (3.55), this gives

$$\begin{aligned}
B_3^r &\ll \sum_j \Big(\sum_{l_{j-2} \leqslant k < l_j} \int_{z_{k-3}}^{z_k} \Big(\int_y^{z_{k+1}} k(z,y) u(z)\, dz \Big)^q w(y)\, dy \Big)^{r/q} V^{-r/p}(y_j) \\
&\leqslant \sum_j \Big(\sum_{l_{j-2} \leqslant k < l_j} \int_{z_{k-3}}^{z_k} \Big(\int_y^{y_j} k(z,y) u(z)\, dz \Big)^q w(y)\, dy \Big)^{r/q} V^{-r/p}(y_j) \\
&\leqslant \sum_j \Big(\int_{y_{j-8}}^{y_j} \Big(\int_y^{y_j} k(z,y) u(z)\, dz \Big)^q w(y)\, dy \Big)^{r/q} V^{-r/p}(y_j),
\end{aligned}$$

whence the estimate $B_3 \ll \mathbb{B}$ follows.

Estimation of the sum III. We have $p \in (0,1]$, and thus by Jensen's inequality and (3.45),

$$\begin{aligned}
\mathrm{III} &\leqslant \sum_n 2^n \Big(\sum_{i=n+1}^{\infty} \sum_{j=n}^{i-1} U_k^p(x_{j+1}, x_j) \int_{\Delta_i} h \Big)^{q/p} \\
&= \sum_n 2^n \Big(\sum_{j=n}^{\infty} U_k^p(x_{j+1}, x_j) \int_{x_{j+1}}^{\infty} h \Big)^{q/p} \\
&\approx \sum_n 2^n U_k^q(x_{n+1}, x_n) \Big(\int_{x_{n+1}}^{\infty} h \Big)^{q/p} =: J_1.
\end{aligned}$$

Given $k \in \mathbb{Z}$, we define

$$G_k := \left\{ m \in \mathbb{Z}\colon 2^k \leqslant \sum_{n<m} 2^n \left(\int_{\Delta_n} k(z, x_n) u(z)\, dz \right)^q < 2^{k+1} \right\}.$$

As before, we assume for simplicity that the G_k are non-empty, and we define

$$\lambda_k := \inf G_k, \quad \lambda_k^+ := \sup G_k < \lambda_{k+1}.$$

One has

$$\begin{aligned} J_1 &= \sum_k \sum_{\lambda_k \leqslant n < \lambda_k^+} 2^n \left(\int_{\Delta_n} k(z, x_n) u(z)\, dz \right)^q \left(\int_{x_{n+1}}^\infty h \right)^{q/p} \ll \sum_k 2^k \left(\int_{x_{\lambda_k+1}}^\infty h \right)^{q/p} \\ &\leqslant \sum_k 2^k \left(\sum_{j=k}^\infty \int_{x_{\lambda_j}}^{x_{\lambda_{j+1}}} h \right)^{q/p} \approx \sum_k 2^k \left(\int_{x_{\lambda_k}}^{x_{\lambda_{k+1}}} h \right)^{q/p} \\ &\leqslant \sum_k 2^k V^{-q/p}(x_{\lambda_k}) \left(\int_{x_{\lambda_k}}^{x_{\lambda_{k+1}}} hV \right)^{q/p} \leqslant \left(\sum_k 2^{kr/q} V^{-r/p}(x_{\lambda_k}) \right)^{q/r} \left(\int_0^\infty hV \right)^{q/p} \\ &=: B_4^q \left(\int_0^\infty hV \right)^{q/p}. \end{aligned}$$

Since

$$\begin{aligned} 2^k &\ll \sum_{\lambda_{k-2} \leqslant n < \lambda_k} 2^n \left(\int_{\Delta_n} k(z, x_n) u(z)\, dz \right)^q \\ &\approx \sum_{\lambda_{k-2} \leqslant n < \lambda_k} \int_{\Delta_{n-1}} w(y)\, dy \left(\int_{\Delta_n} k(z, x_n) u(z)\, dz \right)^q \\ &\ll \sum_{\lambda_{k-2} \leqslant n < \lambda_k} \int_{\Delta_{n-1}} \left(\int_y^{x_{\lambda_k}} k(z, y) u(z)\, dz \right)^q w(y)\, dy \\ &\leqslant \int_{x_{\lambda_{k-3}}}^{x_{\lambda_k}} \left(\int_y^{x_{\lambda_k}} k(z, y) u(z)\, dz \right)^q w(y)\, dy, \end{aligned}$$

we see that

$$B_4^r \ll \sum_k \left(\int_{x_{\lambda_{k-3}}}^{x_{\lambda_k}} \left(\int_y^{x_{\lambda_k}} k(z, y) u(z)\, dz \right)^q w(y)\, dy \right) V_{x_{\lambda_k}}^{-r/p} \ll \mathbb{B}^r.$$

This completes the proof of the implication (3.50) $\Rightarrow$ (3.47).

The implication (3.47) $\Rightarrow$ (3.46) follows from Minkowski's inequality.

The implications (3.46) $\Rightarrow$ (3.48) $\Rightarrow$ (3.49) are easy.

To prove the last implication (3.49) $\Rightarrow$ (3.50), we take an arbitrary increasing sequence $\{x_k\} \subset \mathbb{R}_+$ and let $\varepsilon_k \in (x_k, x_{k+1})$ be such that $V(\varepsilon_k) \leqslant 2V(x_k)$. Let

$$h(x) := \sum_k \frac{a_k}{\varepsilon_k - x_k} \frac{\chi_{(x_k, \varepsilon_k)}(x)}{V(x_k)},$$

where $\{a_k\} \subset [0, \infty)$ and $\sum_k a_k < \infty$. It follows that

$$\int_0^\infty hV = \sum_k \frac{a_k}{(\varepsilon_k - x_k)V(x_k)} \int_{x_k}^{\varepsilon_k} V(x)\,dx \leqslant \sum_k \frac{a_k V(\varepsilon_k)}{V(x_k)} \leqslant 2\sum_k a_k.$$

Writing

$$\begin{aligned} H &:= \int_0^\infty \left[\sup_{y\geqslant x}\left(\int_x^y k(z,x)u(z)\,dz\right)^p \int_y^\infty h\right]^{q/p} w(x)\,dx \\ &= \sum_k \int_{x_k}^{x_{k+1}} \left[\sup_{y\geqslant x}\left(\int_x^y k(z,x)u(z)\,dz\right)^p \int_y^\infty h\right]^{q/p} w(x)\,dx \\ &\geqslant \sum_k \int_{x_k}^{x_{k+1}} \left(\int_x^{x_{k+1}} k(z,x)u(z)\,dz\right)^q w(x)\,dx \int_{x_{k+1}}^\infty h, \end{aligned}$$

we have

$$\int_{x_{k+1}}^\infty h = \sum_{n\geqslant k+1} \frac{a_n}{V(x_n)} \geqslant \frac{a_{k+1}}{V(x_{k+1})}.$$

Hence,

$$\begin{aligned} H &\geqslant \sum_k \left(\int_{x_k}^{x_{k+1}} \left(\int_x^{x_{k+1}} k(z,x)u(z)\,dz\right)^q w(x)\,dx\right) \frac{a_{k+1}^{q/p}}{V^{q/p}(x_{k+1})} \\ &=: \sum_k b_k a_{k+1}^{q/p}. \end{aligned}$$

Then by (3.49)

$$\left(\sum_k b_k a_{k+1}^{q/p}\right)^{p/q} \ll C_4^p \sum_k a_{k+1}$$

for all $\{a_k\} \subset \mathbb{R}_+$. Consequently,

$$\left(\sum_k b_k^{r/q}\right)^{p/r} \ll C_4^p,$$

which is equivalent to $\mathbb{B} \ll C_4$. This completes the proof of Theorem 3.11. □

The next theorem is concerned with the important case of Theorem 3.11 when $k(x, y) \equiv 1$ and a criterion can be given in an integral form. This setting was studied by Goldman [33], [34]. We will obtain a criterion in a different form and give a simpler proof. Let $\varphi(x) := W^{-1}(4W(x))$, where $W^{-1}(t) := \inf\{s \geqslant 0 \colon W(s) = t\}$ is the inverse function of $W(t) := \int_0^t w$.

Theorem 3.12. *Let $0 < q < p \leqslant 1$. Then the following conditions are equivalent:*

$$\bigg(\int_0^\infty \bigg(\int_x^\infty fu\bigg)^q w(x)\,dx\bigg)^{1/q} \leqslant C_1\bigg(\int_0^\infty f^p v\bigg)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow, \tag{3.57}$$

$$\bigg(\int_0^\infty \bigg(\int_x^\infty U^p(y,x)h(y)\,dy\bigg)^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_2^p\int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.58}$$

$$\bigg(\int_0^\infty \bigg(\sup_{y\geqslant x} U^p(y,x)\int_y^\infty h\bigg)^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_3^p\int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.59}$$

$$\bigg(\int_0^\infty \Big(\sup_{y\geqslant x} U(y,x)f(y)\Big)^q w(x)\,dx\bigg)^{1/q} \leqslant C_4\bigg(\int_0^\infty f^p v\bigg)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow, \tag{3.60}$$

$$\bigg(\int_0^\infty \bigg(\sup_{x\leqslant y\leqslant\varphi(x)} U^p(y,x)\int_y^\infty h\bigg)^{q/p} w(x)\,dx\bigg)^{p/q} \leqslant C_5^p\int_0^\infty hV, \qquad h \in \mathfrak{M}^+, \tag{3.61}$$

$$\bigg(\int_0^\infty \Big(\sup_{x\leqslant y\leqslant\varphi(x)} U(y,x)f(y)\Big)^q w(x)\,dx\bigg)^{1/q} \leqslant C_6\bigg(\int_0^\infty f^p v\bigg)^{1/p}, \qquad f \in \mathfrak{M}^\downarrow, \tag{3.62}$$

$$\bigg(\int_0^\infty \bigg[\bigg(\int_x^{\varphi(x)} U^p(y,x)h(y)\,dy\bigg)^{q/p} + U^q(\varphi(x),x)\bigg(\int_{\varphi(x)}^\infty h\bigg)^{q/p}\bigg] w(x)\,dx\bigg)^{p/q} \leqslant C_7^p\int_0^\infty hV \tag{3.63}$$

for all $h \in \mathfrak{M}^+$,

$$\mathbb{B}^r := \sup_{\{x_k\}}\sum_{k\in\mathbb{Z}}\bigg(\int_{x_k}^{x_{k+1}} U^q(x_{k+1},y)w(y)\,dy\bigg)^{r/q} V^{-r/p}(x_{k+1}) < \infty, \tag{3.64}$$

$$\mathbf{B}_1 + \mathbf{B}_2 < \infty, \tag{3.65}$$

where

$$\mathbf{B}_1^r := \int_0^\infty \sup_{x\leqslant y\leqslant\varphi(x)} U^r(y,x)V^{-r/p}(y)W^{r/p}(x)w(x)\,dx, \tag{3.66}$$

$$\mathbf{B}_2^r := \int_0^\infty \bigg(\int_0^x w(y)U^q(\varphi(y),y)\,dy\bigg)^{r/p} V^{-r/p}(\varphi(x))w(x)U^q(\varphi(x),x)\,dx. \tag{3.67}$$

Moreover,

$$C_1 \approx C_2 \approx C_3 = C_4 \approx C_5 \approx C_6 \approx C_7 \approx \mathbb{B} \approx \mathbf{B}_1 + \mathbf{B}_2. \tag{3.68}$$

Proof. We note first that (3.60) $\Leftrightarrow$ (3.64) comes from Theorem 4.4 in [110]. The implication (3.60) $\Rightarrow$ (3.59) is obvious, and the implication (3.59) $\Rightarrow$ (3.60) follows from Proposition 2.1 and Fatou's theorem. For any function $f \in \mathfrak{M}^\downarrow$,

$$\int_x^\infty fu \geqslant \sup_{y\geqslant x}\int_x^y fu \geqslant \sup_{y\geqslant x} U(y,x)f(y).$$

Therefore, (3.57) ⇒ (3.60). The inequality (3.57) is equivalent to the following:

$$\left(\int_0^\infty\left(\int_x^\infty f^{1/p}u\right)^q w\right)^{p/q} \leqslant C_1^p\left(\int_0^\infty fv\right)^{1/p}, \qquad f\in\mathfrak{M}^{\downarrow}. \tag{3.69}$$

If $f(x)=\int_x^\infty h$, then by Minkowski's inequality,

$$\int_x^\infty f^{1/p}u = \int_x^\infty\left(\int_z^\infty h\right)^{1/p} u(z)\,dz \leqslant \left(\int_x^\infty U^p(y,x)h(y)\,dy\right)^{1/p},$$

and now (3.58) ⇒ (3.57) follows by Proposition 2.1.

We assert that (3.59) ⇒ (3.58). To this end, we let $A^{p/q}$ and $B^{p/q}$ denote the left-hand sides of (3.58) and (3.59), respectively. Assuming that (3.59) holds, we have

$$\begin{aligned} A &:= \sum_n\int_{\Delta_n}\left(\int_x^\infty U^p(y,x)h(y)\,dy\right)^{q/p} w(x)\,dx \\ &\approx \sum_n 2^n\left(\int_{\Delta_n} U^p(y,x_n)h(y)\,dy\right)^{q/p} \\ &\quad + \sum_n 2^n\left(\sum_{i=n+1}^\infty U^p(x_i,x_n)\int_{\Delta_i} h(y)\,dy\right)^{q/p} =: A_1 + A_2. \end{aligned} \tag{3.70}$$

Using Jensen's inequality and (3.45), we conclude that

$$\begin{aligned} A_2 &:= \sum_n 2^n\left(\sum_{i=n+1}^\infty\left(\sum_{j=n}^{i-1} U(x_{j+1},x_j)\right)^p\int_{\Delta_i} h(y)\,dy\right)^{q/p} \\ &\ll \sum_n 2^n\left(U^p(x_{n+1},x_n)\int_{x_{n+1}}^\infty h(y)\,dy\right)^{q/p}. \end{aligned} \tag{3.71}$$

Similarly, for the constant B,

$$\begin{aligned} B &:= \int_0^\infty\left(\sup_{y\geqslant x} U^p(y,x)\int_y^\infty h\right)^{q/p} w(x)\,dx \\ &= \sum_n\int_{\Delta_n}\left(\sup_{y\geqslant x} U^p(y,x)\int_y^\infty h\right)^{q/p} w(x)\,dx \approx \sum_n 2^n\left(\sup_{y\geqslant x_n} U^p(y,x_n)\int_y^\infty h\right)^{q/p} \\ &= \sum_n 2^n\left(\sup_{i\geqslant n}\sup_{y\in\Delta_i} U^p(y,x_n)\int_y^\infty h\right)^{q/p} \approx \sum_n 2^n\left(\sup_{i\geqslant n}\sup_{y\in\Delta_i} U^p(y,x_i)\int_y^\infty h\right)^{q/p} \\ &\quad + \sum_n 2^n\left(\sup_{i\geqslant n+1} U^p(x_i,x_n)\int_{x_i}^\infty h\right)^{q/p} \approx \sum_n 2^n\left(\sup_{y\in\Delta_n} U^p(y,x_n)\int_y^\infty h\right)^{q/p} \\ &\quad + \sum_n 2^n\left(\sup_{i\geqslant n+1} U^p(x_i,x_n)\int_{x_i}^\infty h\right)^{q/p} =: B_1 + B_2. \end{aligned}$$

If (3.59) holds, then

$$B_i \ll C_3^q \bigg(\int_0^\infty hV \bigg)^{q/p}, \qquad i = 1, 2. \tag{3.72}$$

By (3.71),

$$A_2 \ll B_2 \ll C_3^q \bigg(\int_0^\infty hV \bigg)^{q/p}. \tag{3.73}$$

By Hölder's inequality,

$$\begin{aligned}
A_1 &\approx \sum_n 2^n \bigg(\int_{\Delta_n} U^p(y, x_n) V^{-1}(y) h(y) V(y)\, dy \bigg)^{q/p} \\
&\leqslant \sum_n 2^n \Big(\sup_{y \in \Delta_n} U^p(y, x_n) V^{-1}(y) \Big)^{q/p} \bigg(\int_{\Delta_n} hV \bigg)^{q/p} \\
&\leqslant \bigg(\sum_n 2^{nr/q} \Big(\sup_{y \in \Delta_n} U^p(y, x_n) V^{-1}(y) \Big)^{r/p} \bigg)^{q/r} \bigg(\sum_n \int_{\Delta_n} hV \bigg)^{q/p} \\
&=: \mathbb{D}^q \bigg(\int_0^\infty hV \bigg)^{q/p}.
\end{aligned}$$

From (3.72),

$$\sum_n 2^n \bigg(\sup_{y \in \Delta_n} U^p(y, x_n) \int_y^{x_{n+1}} h \bigg)^{q/p} \ll C_3^q \bigg(\int_0^\infty hV \bigg)^{q/p}. \tag{3.74}$$

Let $H_n \colon L_V^1[\Delta_n] \to L^\infty[\Delta_n]$ be an operator of the form

$$H_n h(y) := U^p(y, x_n) \int_y^{x_{n+1}} h.$$

Then by Theorem 1.1,

$$d_n := \|H_n\|_{L_V^1[\Delta_n] \to L^\infty[\Delta_n]} = \sup_{y \in \Delta_n} U^p(y, x_n) V^{-1}(y).$$

Let $h_n \in L_V^1[\Delta_n]$ be a function such that

$$\sup_{y \in \Delta_n} U^p(y, x_n) \int_y^{x_{n+1}} h_n \geqslant \frac{d_n}{2} \int_{\Delta_n} h_n V.$$

It now follows from (3.74) that

$$C_3^q \gg \sup_{h\geqslant 0} \frac{\sum_n 2^n \Big(\sup_{y\in\Delta_n} U^p(y,x_n)\int_y^{x_{n+1}} h\Big)^{q/p}}{\Big(\int_0^\infty hV\Big)^{q/p}}$$

$$\geqslant \sup_{h=\sum_n a_n h_n} \frac{\sum_n 2^n a_n^{q/p}\Big(\sup_{y\in\Delta_n} U^p(y,x_n)\int_y^{x_{n+1}} h\Big)^{q/p}}{\Big(\sum_n a_n \int_{\Delta_n} hV\Big)^{q/p}}$$

$$\gg \sup_{h=\sum_n a_n h_n} \frac{\sum_n 2^n d_n^{q/p}\Big(a_n\int_{\Delta_n} hV\Big)^{q/p}}{\Big(\sum_n a_n\int_{\Delta_n} hV\Big)^{q/p}} = \mathbb{D}^q.$$

Hence, $\mathbb{D} \ll C_3$ and

$$A_1 \ll \mathbb{D}^q\bigg(\int_0^\infty hV\bigg)^{q/p} \ll C_3^q\bigg(\int_0^\infty hV\bigg)^{q/p}. \tag{3.75}$$

This together with (3.73) shows that (3.59) $\Rightarrow$ (3.58).

To prove the remaining equivalences we observe that

$$C_5 \leqslant C_3, \quad C_7 \leqslant C_2, \quad C_6 \leqslant C_4, \quad C_6 = C_5 \leqslant C_7.$$

Therefore, if we show that $C_2 \ll C_5$, then it is clear that $C_2 \approx C_5 \approx C_6 \approx C_7$. According to what was shown above,

$$A := \int_0^\infty \bigg(\int_x^\infty U^p(y,x)h(y)\,dy\bigg)^{q/p} w(x)\,dx \approx A_1 + A_2,$$

and

$$A_2 \ll \sum_n 2^n U^q(x_{n+1},x_n)\bigg(\int_{x_{n+1}}^\infty h\bigg)^{q/p}.$$

We have $\varphi(x_{n-1}) = x_{n+1}$, and hence

$$U^q(x_{n+1},x_n)\bigg(\int_{x_{n+1}}^\infty h\bigg)^{q/p} = \sup_{x_n\leqslant y\leqslant x_{n+1}} U^q(y,x_n)\bigg(\int_{x_{n+1}}^\infty h\bigg)^{q/p}$$

$$\leqslant \sup_{x_n\leqslant y\leqslant x_{n+1}} U^q(y,x_n)\bigg(\int_y^\infty h\bigg)^{q/p} = \sup_{x_n\leqslant y\leqslant \varphi(x_{n-1})} U^q(y,x_n)\bigg(\int_y^\infty h\bigg)^{q/p}$$

$$\leqslant \sup_{x\leqslant y\leqslant\varphi(x)} U^q(y,x)\bigg(\int_y^\infty h\bigg)^{q/p}, \qquad x\in(x_{n-1},x_n).$$

Consequently,

$$A_2 \ll \sum_n \biggl(\int_{x_{n-1}}^{x_n} w\biggr) U^q(x_{n+1}, x_n) \biggl(\int_{x_{n+1}}^{\infty} h\biggr)^{q/p}$$
$$\leqslant \sum_n \int_{x_{n-1}}^{x_n} \biggl[\sup_{x\leqslant y\leqslant \varphi(x)} U^p(y,x) \biggl(\int_y^\infty h\biggr)\biggr]^{q/p} w(x)\,dx$$
$$\leqslant \int_0^\infty \biggl[\sup_{x\leqslant y\leqslant \varphi(x)} U^p(y,x) \biggl(\int_y^\infty h\biggr)\biggr]^{q/p} w(x)\,dx \leqslant C_5^q \biggl(\int_0^\infty hV\biggr)^{q/p}.$$

Since $x_{n-1} \leqslant x \leqslant x_n$, we have $x_{n+1} = \varphi(x_{n-1}) \leqslant \varphi(x)$, and therefore

$$\sup_{x\leqslant y\leqslant \varphi(x)} U^p(y,x) \biggl(\int_y^\infty h\biggr) \geqslant \sup_{x_n\leqslant y\leqslant \varphi(x_{n-1})} U^p(y,x) \biggl(\int_y^\infty h\biggr),$$
$$\sup_{x_n\leqslant y\leqslant x_{n+1}} U^p(y,x) \biggl(\int_y^\infty h\biggr) \geqslant \sup_{x_n\leqslant y\leqslant x_{n+1}} U^p(y,x_n) \biggl(\int_y^{x_{n+1}} h\biggr).$$

This together with the preceding estimate shows that

$$\sum_n 2^n \sup_{x_n\leqslant y\leqslant x_{n+1}} U^q(y,x_n) \biggl(\int_y^{x_{n+1}} h\biggr)^{q/p} \ll C_5^q \biggl(\int_y^\infty hV\biggr)^{q/p}. \tag{3.76}$$

By the same argument as after the inequality (3.74),

$$A_1 \ll C_5^q \biggl(\int_y^\infty hV\biggr)^{q/p}.$$

Therefore, $C_2 \ll C_5$.

Our next assertion is that

$$C_2 \ll \mathbf{B}_1 + \mathbf{B}_2. \tag{3.77}$$

From (3.75) we know that

$$A_1 \ll \mathbb{D}^q \biggl(\int_y^\infty hV\biggr)^{q/p},$$

where

$$\mathbb{D}^r := \sum_n 2^{nr/q} \Bigl(\sup_{y\in\Delta_n} U^p(y,x_n) V^{-1}(y)\Bigr)^{r/p}.$$

We have

$$\mathbb{D}^r \approx \sum_n \int_{\Delta_{n-1}} W^{r/p} w \Bigl(\sup_{y\in\Delta_n} U^p(y,x_n) V^{-1}(y)\Bigr)^{r/p}$$
$$\leqslant \sum_n \int_{\Delta_{n-1}} W^{r/p}(x) w(x) \Bigl(\sup_{x\leqslant y\leqslant x_{n+1}} U^p(y,x) V^{-1}(y)\Bigr)^{r/p} dx$$
$$\leqslant \sum_n \int_{\Delta_{n-1}} W^{r/p}(x) w(x) \Bigl(\sup_{x\leqslant y\leqslant \varphi(x)} U^p(y,x) V^{-1}(y)\Bigr)^{r/p} dx \leqslant \mathbf{B}_1^r, \tag{3.78}$$

and hence

$$A_1 \ll \mathbf{B}_1^q \Big(\int_y^\infty hV \Big)^{q/p}.$$

Further,

$$\begin{aligned}
A_2 &\ll \sum_n \Big(\int_{\Delta_{n-1}} w \Big) U^q(x_{n+1}, x_n) \Big(\int_{x_{n+2}}^\infty h \Big)^{q/p} \\
&\quad + \sum_n 2^n U^q(x_{n+1}, x_n) \Big(\int_{\Delta_{n+1}}^\infty h \Big)^{q/p} \\
&\leqslant \sum_n \int_{\Delta_{n-1}} w(x) U^q(\varphi(x), x) \Big(\int_{\varphi(x)}^\infty h \Big)^{q/p} dx \\
&\quad + \sum_n 2^n U^q(x_{n+1}, x_n) V^{-q/p}(x_{n+1}) \Big(\int_{\Delta_{n+1}}^\infty hV \Big)^{q/p} \\
&\leqslant \int_0^\infty U^q(\varphi(x), x) \Big(\int_{\varphi(x)}^\infty h \Big)^{q/p} w(x)\, dx \\
&\quad + \Big(\sum_n 2^{nr/q} U^r(x_{n+1}, x_n) V^{-r/p}(x_{n+1}) \Big)^{q/r} \Big(\int_0^\infty hV \Big)^{q/p}.
\end{aligned}$$

Using Remark 1.2 and (3.78), we find that

$$A_2 \ll \mathbf{B}_2^q \Big(\int_0^\infty hV \Big)^{q/p} + \mathbf{B}_1^q \Big(\int_0^\infty hV \Big)^{q/p},$$

thereby proving (3.77).

By (3.63),

$$\Big(\int_0^\infty U^q(\varphi(x), x) \Big(\int_{\varphi(x)}^\infty h \Big)^{q/p} w(x)\, dx \Big)^{p/q} \leqslant C_6^p \int_0^\infty hV.$$

This together with Remark 1.2 gives $\mathbf{B}_2 \ll C_6$. Since $C_5 \approx C_6$, our assertion will follow if we can show that $\mathbf{B}_1 \ll C_5$. We have

$$\begin{aligned}
\mathbf{B}_1^r &\ll \sum_n 2^{nr/q} \sup_{x_n \leqslant y \leqslant \varphi(x_{n+1})} U^r(y, x_n) V^{-r/p}(y) \\
&= \sum_n 2^{nr/q} \sup_{x_n \leqslant y \leqslant x_{n+3}} U^r(y, x_n) V^{-r/p}(y) \\
&= \sum_{i=0}^2 \sum_k 2^{(3k+i)r/q} \sup_{x_{3k+i} \leqslant y \leqslant x_{3(k+1)+i}} U^r(y, x_{3k+i}) V^{-r/p}(y).
\end{aligned}$$

Further, by the estimate (3.76) and the analogous expansion, this gives

$$\sum_k 2^{3k}\biggl(\sup_{x_{3k+i}\leqslant y\leqslant x_{3(k+1)+i}} U^p(y,x_{3k+i})\int_y^{x_{3(k+1)+i}} h\biggr)^{q/p} \ll C_5^q\biggl(\int_0^\infty hV\biggr)^{q/p}, \qquad i=0,1,2,$$

and hence, as in the proof of the estimate $\mathbb{D}\ll C_3$,

$$\sum_k 2^{3kr/q}\sup_{x_{3k+i}\leqslant y\leqslant x_{3(k+1)+i}} U^r(y,x_{3k+i})V^{-r/p}(y)\ll C_5^r.$$

This proves the estimate $\mathbf{B}_1\ll C_5$ and completes the proof of Theorem 3.12. □

The following result is a symmetric variant of Theorem 3.12. Let $\psi(x) := W_*^{-1}(4W_*(x))$, where $W_*^{-1}(t):=\inf\{s\geqslant 0\colon W_*(s)=t\}$.

Theorem 3.13. *Let $0<q<p\leqslant 1$. Then the following conditions are equivalent:*

$$\biggl(\int_0^\infty\biggl(\int_0^x fu\biggr)^q w(x)\,dx\biggr)^{1/q}\leqslant \overline{C}_1\biggl(\int_0^\infty f^pv\biggr)^{1/p}, \qquad f\in\mathfrak{M}^\uparrow, \tag{3.79}$$

$$\biggl(\int_0^\infty\biggl(\int_0^x U^p(x,y)h(y)\,dy\biggr)^{q/p} w(x)\,dx\biggr)^{p/q}\leqslant \overline{C}_2^p\int_0^\infty hV_*, \qquad h\in\mathfrak{M}^+,$$

$$\biggl(\int_0^\infty\biggl(\sup_{0<y\leqslant x} U^p(x,y)\int_0^y h\biggr)^{q/p} w(x)\,dx\biggr)^{p/q}\leqslant \overline{C}_3^p\int_0^\infty hV_*, \qquad h\in\mathfrak{M}^+,$$

$$\biggl(\int_0^\infty\biggl(\sup_{0<y\leqslant x} U(x,y)f(y)\biggr)^q w(x)\,dx\biggr)^{1/q}\leqslant \overline{C}_4\biggl(\int_0^\infty f^pv\biggr)^{1/p}, \qquad f\in\mathfrak{M}^\uparrow,$$

$$\biggl(\int_0^\infty\biggl(\sup_{\psi(x)\leqslant y\leqslant x} U^p(x,y)\int_0^y h\biggr)^{q/p} w(x)\,dx\biggr)^{p/q}\leqslant \overline{C}_5^p\int_0^\infty hV_*, \qquad h\in\mathfrak{M}^+,$$

$$\biggl(\int_0^\infty\biggl(\sup_{\psi(x)\leqslant y\leqslant x} U(y,x)f(y)\biggr)^q w(x)\,dx\biggr)^{1/q}\leqslant \overline{C}_6\biggl(\int_0^\infty f^pv\biggr)^{1/p}, \qquad f\in\mathfrak{M}^\uparrow,$$

$$\biggl(\int_0^\infty\biggl[\biggl(\int_{\psi(x)}^x U^p(x,y)h(y)\,dy\biggr)^{q/p}+U^q(x,\psi(x))\biggl(\int_0^{\psi(x)} h\biggr)^{q/p}\biggr]w(x)\,dx\biggr)^{p/q} \leqslant \overline{C}_7^p\int_0^\infty hV_*$$

for all $h\in\mathfrak{M}^+$,

$$\overline{\mathbb{B}}^r := \sup_{\{x_k\}}\sum_{k\in\mathbb{Z}}\biggl(\int_{x_k}^{x_{k+1}} U^q(y,x_k)w(y)\,dy\biggr)^{r/q} V_*^{-r/p}(x_k)<\infty,$$

$$\overline{\mathbf{B}}_1+\overline{\mathbf{B}}_2<\infty,$$

where

$$\overline{\mathbf{B}}_1^r := \int_0^\infty \sup_{\psi(x)\leqslant y\leqslant x} U^r(x,y)V_*^{-r/p}(y)W_*^{r/p}(x)w(x)\,dx,$$

$$\overline{\mathbf{B}}_2^r := \int_0^\infty \left(\int_x^\infty w(y)U^q(y,\psi(y))\,dy\right)^{r/p} V_*^{-r/p}(\varphi(x))w(x)U^q(x,\psi(x))\,dx.$$

Moreover,

$$\overline{C}_1 \approx \overline{C}_2 \approx \overline{C}_3 = \overline{C}_4 \approx \overline{C}_5 \approx \overline{C}_6 \approx \overline{C}_7 \approx \overline{\mathbb{B}}.$$

Remark 3.2. Theorem 3.13 supplements [44].

3.4. Further results and comments. First we present a complete characterization of the inequality (3.34).

Theorem 3.14. *Let $0 < q, p < \infty$ and let $k(x,y) \geqslant 0$ be a measurable kernel. Then the inequality (3.34) with the best constant C_1 holds for all $f \in \mathfrak{M}^{\downarrow}$ if and only if*:

(i) $0 < p \leqslant 1$ *and* $p \leqslant q < \infty$, *and in this case*

$$C_1 = C_2 := \sup_{x\in(0,\infty)} \left(\int_0^\infty K^q(x,\min(x,y))w(y)\,dy\right)^{1/q} V^{-1/p}(x) < \infty;$$

(ii) $q = 1 < p < \infty$, *and in this case* $C_1 \approx C_3$, *where*

$$C_3 := \left(\int_0^\infty \left(\int_x^\infty \left(\int_y^\infty k(z,y)w(z)\,dz\right) V^{-1}(y)u(y)\,dy\right)^{p'} v(x)\,dx\right)^{1/p'} < \infty.$$

If $k(x,y)$ is an Oinarov kernel, then:

(iii) $0 < q < p \leqslant 1$ *and $k(x,y)$ is continuous, and in this case*

$$C_1 \approx C_4 := \left(\sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} K^q(y,x_k)w(y)\,dy\right)^{r/q} V^{-r/p}(x_k)\right)^{1/r} < \infty;$$

(iv) $1 < p \leqslant q < \infty$, *and in this case* $C_1 \approx C_5 + C_6 + C_7$, *where*

$$C_5 := \sup_{x\in(0,\infty)} \left(\int_0^x K^q(y,y))w(y)\,dy\right)^{1/q} V^{-1/p}(x) < \infty,$$

$$C_6 := \sup_{x\in(0,\infty)} W_*^{1/q}(x)\left(\int_0^x K^{p'}(x,y)V^{-p'}(y)v(y)\,dy\right)^{1/p'} < \infty,$$

$$C_7 := \sup_{x\in(0,\infty)} \left(\int_x^\infty k^q(y,x)w(y)\,dy\right)^{1/q} \left(\int_0^x U^{p'}(y)V^{-p'}(y)v(y)\,dy\right)^{1/p'} < \infty;$$

(v) $1 < q < p < \infty$, *and in this case* $C_1 \approx C_8 + C_9 + C_{10}$, *where*

$$C_8 := \left(\int_0^\infty \left(\int_0^x K^q(y,y)w(y)\,dy\right)^{r/p} K^q(x,x)w(x)V^{-r/p}(x)\,dx\right)^{1/r} < \infty,$$

$$C_9 := \left(\int_0^\infty \left(\int_x^\infty k^q(y,x)w(y)\,dy\right)^{r/q} \times \left(\int_0^x U^{p'}(y)V^{-p'}(y)v(y)\,dy\right)^{r/p'} U^{p'}(x)V^{-p'}(x)v(x)\,dx\right)^{1/r} < \infty,$$

$$C_{10} := \left(\int_0^\infty W_*^{r/p}(x)w(x)\left(\int_0^x K^{p'}(x,y)V^{-p'}(y)v(y)\,dy\right)^{r/p'} dx\right)^{1/r} < \infty;$$

(vi) $0 < q < 1 < p < \infty$, *and in this case* $C_1 \approx C_8 + C_{11} + C_{12} < \infty$, *where*

$$C_{11} := \left(\sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} w\right)^{r/q} \left(\int_{x_{k-1}}^{x_k} K^{p'}(x_k,y)V^{-p'}(y)v(y)\,dy\right)^{r/p'}\right)^{1/r} < \infty,$$

$$C_{12} := \left(\sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \left(\int_{x_k}^{x_{k+1}} k^q(y,x_k)^q w(y)\,dy\right)^{r/q} \times \left(\int_{x_{k-1}}^{x_k} U^{p'}(y)V^{-p'}(y)v(y)\,dy\right)^{r/p'}\right)^{1/r} < \infty.$$

Proof. The assertion (i) follows from [66] and (ii) from Theorem 2.5. The assertion (iii) is Theorem 3.9. The assertions (iv) and (v) are obtained from Theorem 7 in [78], and (vi) can be derived using Theorems 3.1 and 1.3. In view of Theorem 1.4 we can replace the constants C_{11} and C_{12} by the integral ones. □

For the extreme cases $p = \infty$ and $q = \infty$ we have the following results.

Theorem 3.15. *Let* $0 < p < \infty$ *and let* $k(x,y) \geqslant 0$ *be a measurable kernel. Then the inequality*

$$\operatorname*{ess\,sup}_{x\in(0,\infty)} \left(\int_0^x k(x,y)f(y)u(y)\,dy\right)w(x) \leqslant C\left(\int_0^\infty f^p v\right)^{1/p} \tag{3.80}$$

holds for all $f \in \mathfrak{M}^{\downarrow}$ *if and only if:*
(i) $0 < p \leqslant 1$, *and in this case*

$$C = \sup_{x\in(0,\infty)} \left(\operatorname*{ess\,sup}_{y\in(0,\infty)} K(x,\min(x,y))w(y)\right)V^{-1/p}(x) < \infty;$$

(ii) $1 < p < \infty$, *and in this case*

$$C = \operatorname*{ess\,sup}_{s\in(0,\infty)} w(s)\left(\int_0^s \left(\int_t^s k(s,y)u(y)V^{-1}(y)\,dy\right)^{p'} v(t)\,dt\right)^{1/p'} < \infty.$$

Theorem 3.16. *Let $\|\cdot\|_X$ be an arbitrary quasi-norm on $\mathfrak{M}^+$ and let $k(x,y)\geqslant 0$ be a measurable kernel on $\{(x,y)\colon x\geqslant y\geqslant 0\}$. Then the inequality*

$$\left\|\int_0^x k(x,y)f(y)\,dy\right\|_X \leqslant C\|fv\|_\infty$$

holds for all $f\in\mathfrak{M}^{\downarrow}$ if and only if

$$C=\left\|\int_0^x \frac{k(x,y)\,dy}{\operatorname{ess\,sup}_{z\in(0,y)} v(z)}\right\|_X<\infty.$$

To state the next theorem we need the following notation:

$$V_k(x):=\int_x^\infty k(y,x)\frac{u(y)}{V^2(y)}\,dy, \tag{3.81}$$

$$V_1(x):=\int_x^\infty \frac{u(y)}{V^2(y)}\,dy. \tag{3.82}$$

Theorem 3.17. *Let $0<q,p<\infty$ and let $k(x,y)\geqslant 0$ be an Oinarov kernel. Then the inequality (3.46) holds for all $f\in\mathfrak{M}^{\downarrow}$ with the best constant C_1 if and only if:*

(i) $0<p\leqslant 1$ *and* $p\leqslant q<\infty$, *and in this case*

$$C_1=C_2:=\sup_{x\in(0,\infty)}\left(\int_0^x U_k^q(x,y)w(y)\,dy\right)^{1/q}V^{-1/p}(x)<\infty;$$

(ii) $0<q<p\leqslant 1$, *and in this case*

$$C_1\approx C_3:=\left(\sup_{\{x_k\}}\sum_{k\in\mathbb{Z}}\left(\int_{x_k}^{x_{k+1}} U_k^q(x_{k+1},y)w(y)\,dy\right)^{r/q}V^{-r/p}(x_{k+1})\right)^{1/r}<\infty;$$

(iii) $1<p\leqslant q<\infty$, *and in this case* $C_1\approx A_1+A_2+A_3$, *where*

$$A_1:=\sup_{x\in(0,\infty)}\left(\int_x^\infty V_k^q w\right)^{1/q}[V(x)]^{1+1/p'},$$

$$A_2:=\sup_{x\in(0,\infty)}W^{1/q}(x)\left(\int_x^\infty (VV_k)^{p'}v\right)^{1/p'},$$

$$A_3:=\sup_{x\in(0,\infty)}\left(\int_0^x k^q(x,s)w(s)\,ds\right)^{1/q}\left(\int_x^\infty (VV_1)^{p'}v\right)^{1/p'}$$
$$+\sup_{x\in(0,\infty)}W^{1/q}(x)\left(\int_x^\infty k^{p'}(s,x)(VV_1)^{p'}(s)v(s)\,ds\right)^{1/p'};$$

(iv) $1 < q < p < \infty$, *and in this case* $C_1 \approx B_1 + B_2 + B_3$, *where*

$$B_1 := \left(\int_0^\infty \left(\int_x^\infty V_k^q w\right)^{r/p} \left(\int_0^x V^{p'} v\right)^{r/p'} V_k^q(x) w(x)\, dx\right)^{1/r},$$

$$B_2 := \left(\int_0^\infty W^{r/p}(x) \left(\int_x^\infty (VV_k)^{p'} v\right)^{r/p'} w(x)\, dx\right)^{1/r},$$

$$B_3 := \left(\int_0^\infty \left(\int_0^x k^q(x,s) w(s)\, ds\right)^{r/q} \left(\int_x^\infty (VV_1)^{p'} v\right)^{r/q'} (V(x)V_1(x))^{p'} v(x)\, dx\right)^{1/r} + \left(\int_0^\infty W^{r/p}(x) \left(\int_x^\infty k^{p'}(s,x)(V(s)V_1(s))^{p'} v(s)\, ds\right)^{r/q'} w(x)\, dx\right)^{1/r};$$

(v) $q = 1 \leqslant p < \infty$, *and in this case* $C_1 = C_7$, *where*

$$C_7 := \left(\int_0^\infty V^{p'}(x) v(x)\, dx \left(\int_x^\infty \frac{u(y)\, dy}{V^2(y)} \int_0^y k(y,s) w(s)\, ds\right)^{p'}\right)^{1/p'}$$

for $p > 1$ *and*

$$C_7 := \sup_{x \in (0,\infty)} V(x) \left(\int_x^\infty \frac{u(y)\, dy}{V^2(y)} \int_0^y k(y,s) w(s)\, ds\right)$$

for $p = 1$;

(vi) $0 < q < 1 < p < \infty$, *and in this case* $C_1 \approx B_1 + B_2 + B_4$, *where*

$$B_4 := \sup_{x_k} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} w\right)^{r/q} \left(\int_{x_{k-1}}^{x_k} k^{p'}(x_k,s)(V(s)V_1(s))^{p'} v(s)\, ds\right)^{r/p'}\right)^{1/r} + \sup_{x_k} \left(\sum_k \left(\int_{x_k}^{x_{k+1}} k^{p'}(s,x_k) w(s)\, ds\right)^{r/q} \times \left(\int_{x_{k-1}}^{x_k} (V(s)V_1(s))^{p'} v(s)\, ds\right)^{r/p'}\right)^{1/r}.$$

Proof. The assertion (i) comes from [66], and the assertion (ii) is Theorem 3.11. Using Theorem 3.1 with $q = 1 \leqslant p < \infty$, we reduce (3.46) to the inequality

$$\int_0^\infty h(s) V(s) \left(\int_s^\infty \frac{u(y)}{V^2(y)} \left(\int_0^y k(y,x) w(x)\, dx\right) dy\right) \leqslant C_1 \left(\int_0^\infty h^p v^{1-p}\right)^{1/p}, \tag{3.83}$$

$h \in \mathfrak{M}^+$. The assertion (v) now follows from Theorem 1.1. From the decomposition

$$\begin{aligned}
&\int_x^\infty k(y,x)\frac{u(y)}{V^2(y)}\int_0^y h(z)V(z)\,dz\,dy\\
&\quad= \int_0^x h(z)V(z)\,dz \int_x^\infty k(y,x)\frac{u(y)}{V^2(y)}\,dy + \int_x^\infty k(y,x)\frac{u(y)}{V^2(y)}\int_x^y h(z)V(z)\,dz\,dy\\
&\quad= \int_0^x h(z)V(z)\,dz \int_x^\infty k(y,x)\frac{u(y)}{V^2(y)}\,dy + \int_x^\infty h(z)V(z)\int_z^\infty k(y,x)\frac{u(y)}{V^2(y)}\,dy\,dz\\
&\quad\approx \int_0^x h(z)V(z)\,dz \int_x^\infty k(y,x)\frac{u(y)}{V^2(y)}\,dy + \int_x^\infty h(z)V(z)\int_z^\infty k(y,z)\frac{u(y)}{V^2(y)}\,dy\,dz\\
&\qquad+ \int_x^\infty k(z,x)h(z)V(z)\int_z^\infty \frac{u(y)}{V^2(y)}\,dy\,dz
\end{aligned}$$

it follows in the remaining cases that the inequality (3.46) reduces by Theorem 3.1 to a characterization of the following three inequalities on the cone $\mathfrak{M}^+$:

$$\left(\int_0^\infty \left(V_k(x)\int_0^x hV\right)^q w(x)\,dx\right)^{1/q} \leqslant C_{4,1}\left(\int_0^\infty h^p v^{1-p}\right)^{1/p}, \tag{3.84}$$

$$\left(\int_0^\infty \left(\int_x^\infty hVV_k\right)^q w(x)\,dx\right)^{1/q} \leqslant C_{4,2}\left(\int_0^\infty h^p v^{1-p}\right)^{1/p}, \tag{3.85}$$

$$\left(\int_0^\infty \left(\int_x^\infty k(s,x)V(s)V_1(s)h(s)\,ds\right)^q w(x)\,dx\right)^{1/q} \leqslant C_{4,3}\left(\int_0^\infty h^p v^{1-p}\right)^{1/p}, \tag{3.86}$$

where V_k and V_1 are defined by (3.81) and (3.82), respectively. Using Theorem 1.2 and results in [92], we deduce that $C_{4,1} \approx A_1$ for $1 < p \leqslant q < \infty$ and $C_{4,1} \approx B_1$ for $0 < q < p < \infty$ and $p > 1$. Similarly, $C_{4,2} \approx A_2$ for $1 < p \leqslant q < \infty$ and $C_{4,2} \approx B_2$ for $0 < q < p < \infty$ and $p > 1$. By Theorem 1.2 again, $C_{4,3} \approx A_3$ for $1 < p \leqslant q < \infty$ and $C_{4,3} \approx B_3$ for $1 < q < p < \infty$. Theorem 1.3 implies that $C_{4,3} \approx B_4$ for $0 < q < 1 < p < \infty$. □

In the special case $k(y,x) = 1$ we have, in addition to Theorem 3.12, the following entirely integral characterization of the inequality (3.57).

Corollary 3.1. *Let $0 < q, p < \infty$. Then the inequality (3.57) with the best constant C_1 holds for all $f \in \mathfrak{M}^\downarrow$ if and only if:*

(i) $0 < q < p = 1$, *and in this case* $C_1 \approx B_{1,0} + B_{2,0}$, *where*

$$B_{1,0} := \left(\int_0^\infty V^{q/(1-q)}(x)\left(\int_x^\infty V_1^q w\right)^{q/(1-q)} V_1^q(x)w(x)\,dx\right)^{(1-q)/q},$$

$$B_{2,0} := \left(\int_0^\infty \left[\sup_{t\in(x,\infty)} V(t)V_1(t)\right]^{q/(1-q)} W^{q/(1-q)}(x)w(x)\,dx\right)^{(1-q)/q};$$

(ii) $1 < p \leqslant q < \infty$, *and in this case* $C_1 \approx A_{1,0} + A_{2,0}$, *where*

$$A_{1,0} := \sup_{x\in(0,\infty)} \biggl(\int_x^\infty V_1^q w\biggr)^{1/q} [V(x)]^{1+1/p'},$$
$$A_{2,0} := \sup_{x\in(0,\infty)} W^{1/q}(x)\biggl(\int_x^\infty (VV_1)^{p'} v\biggr)^{1/p'};$$

(iii) $1 < q < p < \infty$, *and in this case* $C_1 \approx \mathbf{B}_{1,0} + \mathbf{B}_{2,0}$, *where*

$$\mathbf{B}_{1,0} := \biggl(\int_0^\infty \biggl(\int_x^\infty V_1^q w\biggr)^{r/p} [V(x)]^{r+r/p'} V_1^q(x) w(x)\,dx\biggr)^{1/r},$$
$$\mathbf{B}_{2,0} := \biggl(\int_0^\infty W^{r/p}(x)\biggl(\int_x^\infty (VV_1)^{p'} v\biggr)^{r/p'} w(x)\,dx\biggr)^{1/r};$$

(iv) $q = 1 \leqslant p < \infty$, *and in this case* $C_1 = C_{7,0}$, *where*

$$C_{7,0} := \biggl(\int_0^\infty V^{p'}(x) v(x)\,dx \biggl(\int_x^\infty \frac{W(y)u(y)\,dy}{V^2(y)}\biggr)^{p'}\biggr)^{1/p'}$$

when $p > 1$, *and with the usual modification for* $p = 1$;

(v) $0 < q < 1 < p < \infty$, *and in this case* $C_1 \approx \mathbf{B}_{1,0} + \mathbf{B}_{2,0}$.

To state the results dual to the two previous results, we introduce the following notation:

$$V_*(t) := \int_t^\infty v, \qquad W_*(t) := \int_t^\infty w, \qquad U_k^*(x,y) := \int_y^x k(x,y)u(z)\,dz,$$
$$V_k^*(x) := \int_0^x k(x,y)\frac{u(y)}{V_*^2(y)}\,dy, \qquad V_1^*(x) := \int_0^x \frac{u(y)}{V_*^2(y)}\,dy.$$

Theorem 3.18. *Let* $0 < q, p < \infty$ *and let* $k(x,y) \geqslant 0$ *be an Oinarov kernel. Then the inequality*

$$\biggl(\int_0^\infty \biggl(\int_0^x k(x,y)f(y)u(y)\,dy\biggr)^q w(x)\,dx\biggr)^{1/q} \leqslant \mathscr{C}_1 \biggl(\int_0^\infty f^p v\biggr)^{1/p}$$

with the best constant $\mathscr{C}_1$ *holds for all* $f \in \mathfrak{M}^\uparrow$ *if and only if*:

(i) $0 < p \leqslant 1$ *and* $p \leqslant q < \infty$, *and in this case*

$$\mathscr{C}_1 = \mathscr{C}_2 := \sup_{x\in(0,\infty)} \biggl(\int_x^\infty [U_k^*(y,x)]^q w(y)\,dy\biggr)^{1/q} V_*^{-1/p}(x) < \infty;$$

(ii) $0 < q < p \leqslant 1$, *and in this case*

$$\mathscr{C}_1 \approx \mathscr{C}_3 := \biggl(\sup_{\{x_k\}} \sum_{k\in\mathbb{Z}} \biggl(\int_{x_k}^{x_{k+1}} [U_k^*((y,x_k)]^q w(y)\,dy\biggr)^{r/q} V_*^{-r/p}(x_k)\biggr)^{1/r} < \infty;$$

(iii) $1 < p \leqslant q < \infty$, *and in this case* $\mathscr{C}_1 \approx \mathscr{A}_1 + \mathscr{A}_2 + \mathscr{A}_3$, *where*

$$\mathscr{A}_1 := \sup_{x\in(0,\infty)} \left(\int_0^x [V_k^*]^q w\right)^{1/q} [V_*(x)]^{1+1/p'},$$

$$\mathscr{A}_2 := \sup_{x\in(0,\infty)} W_*^{1/q}(x)\left(\int_0^x (V_* V_k^*)^{p'} v\right)^{1/p'},$$

$$\mathscr{A}_3 := \sup_{x\in(0,\infty)} \left(\int_x^\infty k^q(s,x)w(s)\,ds\right)^{1/q} \left(\int_0^x (V_* V_1^*)^{p'} v\right)^{1/p'}$$
$$+ \sup_{x\in(0,\infty)} W_*^{1/q}(x)\left(\int_0^x k^{p'}(x,s)(V_* V_1^*)^{p'}(s)v(s)\,ds\right)^{1/p'};$$

(iv) $1 < q < p < \infty$, *and in this case* $\mathscr{C}_1 \approx \mathscr{B}_1 + \mathscr{B}_2 + \mathscr{B}_3$, *where*

$$\mathscr{B}_1 := \left(\int_0^\infty \left(\int_0^x [V_k^*]^q w\right)^{r/p} [V_*(x)]^{r+r/p'} [V_k^*]^q(x)w(x)\,dx\right)^{1/r},$$

$$\mathscr{B}_2 := \left(\int_0^\infty W_*^{r/p}(x)\left(\int_0^x (V_* V_k^*)^{p'} v\right)^{r/p'} w(x)\,dx\right)^{1/r},$$

$$\mathscr{B}_3 := \left(\int_0^\infty\left(\int_x^\infty k^q(s,x)w(s)\,ds\right)^{r/q}\left(\int_0^x (V_* V_1^*)^{p'} v\right)^{r/q'} (V_*(x)V_1^*(x))^{p'} v(x)\,dx\right)^{1/r}$$
$$+ \left(\int_0^\infty W_*^{r/p}(x)\left(\int_0^x k^{p'}(x,s)(V_*(s)V_1^*(s))^{p'} v(s)\,ds\right)^{r/q'} w(x)\,dx\right)^{1/r};$$

(v) $q = 1 \leqslant p < \infty$, *and in this case* $\mathscr{C}_1 = \mathscr{C}_7$, *where*

$$\mathscr{C}_7 := \left(\int_0^\infty V_*^{p'}(x)v(x)\,dx\left(\int_0^x \frac{u(y)\,dy}{V_*^2(y)}\int_y^\infty k(s,y)w(s)\,ds\right)^{p'}\right)^{1/p'}$$

for $p > 1$ *and*

$$\mathscr{C}_7 := \sup_{x\in(0,\infty)} V_*(x)\left(\int_0^x \frac{u(y)\,dy}{V_*^2(y)}\int_y^\infty k(s,y)w(s)\,ds\right)$$

for $p = 1$;

(vi) $0 < q < 1 < p < \infty$, *and in this case* $\mathscr{C}_1 \approx \mathscr{B}_1 + \mathscr{B}_2 + \mathscr{B}_4$, *where*

$$\mathscr{B}_4 := \sup_{x_k}\left(\sum_k\left(\int_{x_{k-1}}^{x_k} w\right)^{r/q}\left(\int_{x_k}^{x_{k+1}} k^{p'}(s,x_k)(V_*(s)V_1^*(s))^{p'} v(s)\,ds\right)^{r/p'}\right)^{1/r}$$
$$+ \sup_{x_k}\left(\sum_k\left(\int_{x_{k-1}}^{x_k} k^{p'}(x_k,s)w(s)\,ds\right)^{r/q}\left(\int_{x_k}^{x_{k+1}} (V_*(s)V_1^*(s))^{p'} v(s)\,ds\right)^{r/p'}\right)^{1/r}.$$

In the case $k(y,x) = 1$ we have, in addition to Theorem 3.13, the following entirely integral characterization of the inequality (3.79).

Corollary 3.2. *Let $0 < q, p < \infty$. Then the inequality*

$$\left(\int_0^\infty \left(\int_0^x f(y)u(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C_{1,0}^*\left(\int_0^\infty f^p v\right)^{1/p}$$

with the best constant $C_{1,0}^*$ holds for all $f \in \mathfrak{M}^\uparrow$ if and only if:

(i) $0 < q < p = 1$, *and in this case $C_{1,0}^* \approx B_{1,0}^* + B_{2,0}^*$, where*

$$B_{1,0}^* := \left(\int_0^\infty V_*^{q/(1-q)}(x)\left(\int_0^x [V_1^*]^q w\right)^{q/(1-q)} [V_1^*]^q(x)w(x)\,dx\right)^{(1-q)/q},$$

$$B_{2,0}^* := \left(\int_0^\infty \left[\sup_{t\in(0,x)} V_*(t)V_1^*(t)\right]^{q/(1-q)} W_*^{q/(1-q)}(x)w(x)\,dx\right)^{(1-q)/q};$$

(ii) $1 < p \leqslant q < \infty$, *and in this case $C_{1,0}^* \approx A_{1,0}^* + A_{2,0}^*$, where*

$$A_{1,0}^* := \sup_{x\in(0,\infty)} \left(\int_0^x [V_1^*]^q w\right)^{1/q} [V_*(x)]^{1+1/p'},$$

$$A_{2,0}^* := \sup_{x\in(0,\infty)} W_*^{1/q}(x)\left(\int_0^x (V_* V_1^*)^{p'} v\right)^{1/p'};$$

(iii) $1 < q < p < \infty$, *and in this case $C_{1,0}^* \approx \mathbf{B}_{1,0}^* + \mathbf{B}_{2,0}^*$, where*

$$\mathbf{B}_{1,0}^* := \left(\int_0^\infty \left(\int_0^x [V_1^*]^q w\right)^{r/p} [V_*(x)]^{r+r/p'} [V_1^*]^q(x)w(x)\,dx\right)^{1/r},$$

$$\mathbf{B}_{2,0}^* := \left(\int_0^\infty W_*^{r/p}(x)\left(\int_0^x (V_* V_1^*)^{p'} v\right)^{r/p'} w(x)\,dx\right)^{1/r};$$

(iv) $q = 1 \leqslant p < \infty$, *and in this case $C_{1,0}^* = C_{7,0}^*$, where*

$$C_{7,0}^* := \left(\int_0^\infty V_*^{p'}(x)v(x)\,dx\left(\int_0^x \frac{W_*(y)u(y)\,dy}{V_*^2(y)}\right)^{p'}\right)^{1/p'}$$

for $p > 1$, and with the usual modification for $p = 1$;

(v) $0 < q < 1 < p < \infty$, *and in this case $C_{1,0}^* \approx \mathbf{B}_{1,0}^* + \mathbf{B}_{2,0}^*$.*

Let $k(x,y) \geqslant 0$ be an Oinarov kernel. One of the curious applications of reduction theorems (in the backward direction!) consists in finding an integral criterion for the inequalities

$$\left(\int_0^\infty \left(\int_0^x k(x,y)h(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty h^p \sigma\right)^{1/p}, \qquad h \in \mathfrak{M}^+, \quad (3.87)$$

and

$$\left(\int_0^\infty \left(\int_x^\infty k(y,x)h(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty h^p \sigma\right)^{1/p}, \qquad h \in \mathfrak{M}^+, \quad (3.88)$$

for $0 < q < 1 \leqslant p < \infty$ when

$$k(x,y) = \int_y^x u(s)\,ds =: U(x,y).$$

Note that the answer here is much simpler than that provided by Theorems 1.3 and 1.4. We will give the solution for (3.88); similar arguments apply for (3.87). We set

$$S(x) := \int_x^\infty \sigma^{1-p'}, \quad 1 < p < \infty, \qquad v_\sigma(x) := [S(x)]^{-p}[\sigma(x)]^{1-p'},$$
$$V_\sigma(x) := \int_0^x v_\sigma, \quad V_{1,\sigma}(x) := \int_x^\infty S^{2(p-1)}u.$$

Corollary 3.3. *Let $0 < q < 1 \leqslant p < \infty$. Then the inequality*

$$\left(\int_0^\infty \left(\int_x^\infty U(y,x)h(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C\left(\int_0^\infty h^p\sigma\right)^{1/p}, \qquad h \in \mathfrak{M}^+, \tag{3.89}$$

holds if and only if:

(i) $0 < q < 1 < p < \infty$, *and in this case* $C \approx B_{1,1} + B_{2,1}$, *where*

$$B_{1,1} := \left(\int_0^\infty \left(\int_x^\infty [V_{1,\sigma}]^q w\right)^{r/p} [S(x)]^{(1-2p)r/p'}[V_{1,\sigma}]^q(x)w(x)\,dx\right)^{1/r},$$
$$B_{2,1} := \left(\int_0^\infty W^{r/p}(x)\left(\int_0^x (V_\sigma V_{1,\sigma})^{p'}v_\sigma\right)^{r/p'} w(x)\,dx\right)^{1/r};$$

(ii) $0 < q < 1 = p$, *and in this case* $C \approx B_{1,2} + B_{2,2}$, *where*

$$B_{1,2} := \left(\int_0^\infty V_\sigma^{q/(1-q)}(x)\left(\int_0^x [V_{1,\sigma}]^q w\right)^{q/(1-q)} [V_{1,\sigma}]^q(x)w(x)\,dx\right)^{(1-q)/q},$$
$$B_{2,2} := \left(\int_0^\infty \left[\sup_{t\in(x,\infty)} V_\sigma(t)V_{1,\sigma}(t)\right]^{q/(1-q)} W^{q/(1-q)}(x)w(x)\,dx\right)^{(1-q)/q}.$$

Proof. We begin with (i). Note that

$$V_\sigma(x) = \int_0^x [S(y)]^{-p}[\sigma(y)]^{1-p'}\,dy \approx ([S(x)]^{1-p} - [S(0)]^{1-p})$$

and

$$[V_\sigma(x)]^p[v_\sigma(x)]^{1-p} = ([S(x)]^{1-p} - [S(0)]^{1-p})^p[S(x)]^{p(p-1)}\sigma(x) \leqslant \sigma(x).$$

By Theorem 3.1, the inequality

$$\left(\int_0^\infty \left(\int_x^\infty U(y,x)h(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C_1\left(\int_0^\infty h^pV_\sigma^pv_\sigma^{1-p}\right)^{1/p} \tag{3.90}$$

is equivalent to the inequality

$$\left(\int_0^\infty \left(\int_x^\infty u(y)f(y)\,dy\right)^q w(x)\,dx\right)^{1/q} \leqslant C_2\left(\int_0^\infty f^p v_\sigma\right)^{1/p}, \quad f \in \mathfrak{M}^\downarrow, \tag{3.91}$$

which is characterized by Corollary 3.1, so that

$$\begin{aligned} C_2 &\approx B_{1,1} + B_{2,1}, & 0 < q < 1 < p < \infty; \\ C_2 &\approx B_{1,2} + B_{2,2}, & 0 < q < 1 = p. \end{aligned}$$

As a result,

$$\begin{aligned} C &\ll B_{1,1} + B_{2,1}, & 0 < q < 1 < p < \infty; \\ C &\ll B_{1,2} + B_{2,2}, & 0 < q < 1 = p. \end{aligned}$$

To prove the reverse inequalities we assume that (3.89) holds with a constant $C \in (0,\infty)$. Then (3.89) holds with the same constant C also for $\sigma_\varepsilon := \sigma + \varepsilon$ and $w_\delta := w\chi_{(\delta,1/\delta)}$ instead of σ and w, respectively, where $\varepsilon > 0$ and $\delta \in (0,1)$. For these weight functions $S(0) = \infty$, and thus $B^{\varepsilon,\delta}_{i,j} \ll C$, where $B^{\varepsilon,\delta}_{i,j} < \infty$ is the constant B with the modified weights. By the Lebesgue dominated convergence theorem, $\lim_{\delta\to 0}\bigl(\lim_{\varepsilon\to 0} B^{\varepsilon,\delta}_{i,j}\bigr) = B_{i,j}$, from which the reverse inequality follows.

Case (ii) is treated similarly. $\square$

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