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# Analysis of linear differential equations by methods of the spectral theory of difference operators and linear relations

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**Abstract.** Many properties of solutions to linear differential equations with unbounded operator coefficients (their boundedness, almost periodicity, stability) are closely connected with the corresponding properties of the differential operator defining the equation and acting in an appropriate function space. The structure of the spectrum of this operator and whether it is invertible, correct, and Fredholm depend on the dimension of the kernel of the operator, the codimension of its range, and the existence of complemented subspaces. The notion of a state of a linear relation (multivalued linear operator) is introduced, and is associated with some properties of the kernel and range. A linear difference operator (difference relation) is assigned to the differential operator under consideration (or the corresponding equation), the sets of their states are proved to be the same, and necessary and sufficient conditions for them to have the Fredholm property are found. Criteria for the almost periodicity at infinity of solutions of differential equations are derived. In the proof of the main results, the property of exponential dichotomy of a family of evolution operators and the spectral theory of linear relations are heavily used.

Bibliography: 98 titles.

**Keywords:** linear differential operators, set of states of an operator, Fredholm operator, difference operators and difference relations, spectrum of an operator or linear relation, functions almost periodic at infinity.

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## 1. Main concepts and results

We consider a linear differential equation

$$-\frac{dx}{dt}+A(t)x=f(t), \qquad t\in\mathbb{J}, \tag{1.1}$$

where $\mathbb{J}$ is an infinite interval of the real line $\mathbb{R}$ and $A(t)\colon D(A(t))\subset X\to X$, $t\in\mathbb{J}$, is a family of closed linear operators in a complex Banach space $X$. Sometimes we shall call $X$ the *phase* space.

We assume that the Cauchy problem with $x(s)=x_0\in D(A(s))$, $t\geqslant s$, $t,s\in\mathbb{J}$, for the homogeneous differential equation

$$\frac{dx}{dt}=A(t)x, \qquad t\in\mathbb{J}, \tag{1.2}$$

is well posed. It gives rise to a family of evolution operators $\mathscr{U}\colon\Delta_{\mathbb{J}}\to\operatorname{End}X$, where $\Delta_{\mathbb{J}}=\{(t,s)\in\mathbb{J}\times\mathbb{J}\mid s\leqslant t\}$ and $\operatorname{End}X$ is the Banach algebra of bounded linear operators in $X$. Given the family $\mathscr{U}$, we can construct a differential operator

$$L=L_{\mathscr{U}}=-\frac{d}{dt}+A(t)$$

acting in a homogeneous space $\mathscr{F}(\mathbb{J},X)$ of functions on $\mathbb{J}$ taking values in $X$ (see § 2). One example here is the Banach space $C_b(\mathbb{J},X)$ of bounded continuous functions on $\mathbb{J}$.

This paper is concerned with existence and uniqueness questions for bounded solutions of the differential equation (1.1), where $f$ belongs to some homogeneous function space $\mathscr{F}(\mathbb{J},X)$ (for instance, $f$ can be a bounded function). These questions relate to the geometric (or qualitative) theory of differential equations, which goes back to Poincaré and Lyapunov.

The existence and uniqueness of bounded solutions are closely connected with the stability of solutions of differential equations. Starting from Lyapunov's ideas and results, Krein observed in [1] and in the later mimeographed notes [2] that many results of the stability theory for solutions can be obtained with the help of the theory of operators in Banach spaces. The departure from the very specific theory of operators in a finite-dimensional space (matrix analysis) simplifies many proofs and constructions and makes them more transparent. In this way the general approach was also good for the classical theory (see the monographs [3] by Demidovich, [4] by Coppel, and [5] by Hartman). Note that investigations of linear differential equations are not only important on their own, but also underlie the analysis of non-linear equations by means of linear approximations.

The monographs by Massera and Schäffer [6] and Dalecki and Krein [7] contain results in the case when the $A(t)$, $t \in \mathbb{J}$, are bounded operators in a Banach space. In connection with applications to parabolic partial differential equations the authors (of both monographs) pointed out the topical question of investigating the qualitative properties of solutions to equations with unbounded operator coefficients: "The authors clearly recognize that the arena of infinite-dimensional spaces requires the presence of unbounded operators, without which the present stability theory for systems with an infinite number of degrees of freedom would be impossible" (see [7]); "... we have entirely disregarded any possible extension of the theory to the case in which the values of $A$ are unbounded operators in $X$. Such an extension would certainly be of the greatest interest, especially in view of possible applications to partial differential equations" (see [6]).

Zhikov [8] (see also [9]) made a significant contribution to the theory of equations considered here by proving that the condition of the invertibility of the operator $L = -d/dt + A(t)$ in the space $C_b(\mathbb{R}, X)$ is equivalent to the condition of exponential dichotomy for solutions of (1.2). Note also Henry's monograph [10], where a geometric theory of semilinear parabolic equations was developed which made essential use of the invertibility property of the operator $L = -d/dt + A(t)$ in the Banach space $C_b(\mathbb{R}, X)$ under the assumption that the $A(t)$, $t \in \mathbb{R}$, are sectorial operators.

We point out that investigations of the well posedness of the Cauchy problem for an equation of the form (1.2) anticipated considerably the corresponding investigations in the qualitative theory of equations (1.1), (1.2). The case when $A(t) \equiv A$, $t \in \mathbb{R}_+ = [0, \infty)$, is a constant operator-valued function corresponds to the theory of operator semigroups (see [11] and [12]). The famous Hille–Phillips–Yosida–Feller–Miyadera theorem (see [11], Theorem 12.3.1 or [12], Theorem 3.3.8) gives a necessary and sufficient condition for the well posedness of the Cauchy problem for the differential equation (1.2) in terms of the resolvent of $A\colon D(A) \subset X \to X$. In [12] the reader can find many examples of partial differential equations of parabolic type when the corresponding operator is the generator of a strongly continuous operator semigroup (and hence gives rise to a well-posed Cauchy problem).

The current state of the problems discussed in this paper is closely connected with methods in the spectral theory of closed linear operators, linear relations (multivalued linear operators), difference operators and relations, and operator semigroups.

The theory of operator semigroups was applied to the geometric theory of differential equations of the form (1.1), (1.2) in [13]–[17]. The differential operator under consideration (generated by equation (1.1)) is the generator of a strongly continuous semigroup of difference operators. This opens the door to the use of the theory of operator semigroups in the qualitative theory of differential equations. The state of the corresponding theory up to 1999 was presented in detail by Chicone and Latushkin in their monograph [18]. In [19] linear difference relations and semigroups of such relations were used to investigate differential operators (equations) in function spaces on a half-axis.

The analysis of the operator $L$ (differential equation (1.1)) and more general operators generated by a family of evolution operators is carried out with the significant use of difference operators and linear difference relations on corresponding homogeneous spaces of vector sequences. We introduce the notion of the set of states of a linear relation (operator) for describing certain properties of the kernel and range of the relation (operator) and thereby revealing the extent to which it is continuously invertible. In particular, we obtain conditions ensuring the Fredholm property of difference operators and relations or differential operators. These results are closely connected with the asymptotic behaviour of solutions, the stability of solutions, and the problem of the existence of bounded solutions to differential equations (1.1) and (1.2).

Now we give the notions and definitions which we use the most (and which are sufficient for an understanding of this paper) and state some of the main results.

We make essential use of the concept of linear relation. Thus, in § 3 we give several notions in the theory of linear relations, notions needed in order to state many of the results below.

Let $\mathscr{X}$ and $\mathscr{Y}$ be Banach spaces, let $\operatorname{Hom}(\mathscr{X},\mathscr{Y})$ be the Banach space of bounded linear operators from $\mathscr{X}$ to $\mathscr{Y}$, and let $\operatorname{End}\mathscr{X}=\operatorname{Hom}(\mathscr{X},\mathscr{X})$.

An arbitrary linear subspace $\mathscr{A}$ of the Cartesian product $\mathscr{X}\times\mathscr{Y}$ of $\mathscr{X}$ and $\mathscr{Y}$ is called a *linear relation* between $\mathscr{X}$ and $\mathscr{Y}$ (a linear relation in $\mathscr{X}$ if $\mathscr{X}=\mathscr{Y}$). The set of linear relations between $\mathscr{X}$ and $\mathscr{Y}$ will be denoted by $LR(\mathscr{X},\mathscr{Y})$ (by $LR(\mathscr{X})$ if $\mathscr{Y}=\mathscr{X}$). A linear operator $A\colon D(A)\subset\mathscr{X}\to\mathscr{Y}$ is often identified with the (linear) relation $\mathscr{A}$ in $LR(\mathscr{X},\mathscr{Y})$ coinciding with the graph of $A$. The set of closed linear relations in $LR(\mathscr{X},\mathscr{Y})$ is denoted by $LRC(\mathscr{X},\mathscr{Y})$.

Any relation $\mathscr{A}\in LR(\mathscr{X},\mathscr{Y})$ has the inverse relation $\mathscr{A}^{-1}=\{(y,x)\in\mathscr{Y}\times\mathscr{X}\mid (x,y)\in\mathscr{A}\}\in LR(\mathscr{Y},\mathscr{X})$. A relation $LRC(\mathscr{X},\mathscr{Y})$ is said to be *continuously invertible* if $\mathscr{A}^{-1}\in\operatorname{Hom}(\mathscr{Y},\mathscr{X})$, which is equivalent to the simultaneous fulfilment of the following conditions: the kernel $\operatorname{Ker}\mathscr{A}=\{x\in\mathscr{X}\mid (x,0)\in\mathscr{A}\}$ of the relation $\mathscr{A}$ is zero and the range $\operatorname{Im}\mathscr{A}=\{y\in\mathscr{Y}\mid \text{there exists } x\in\mathscr{X} \text{ such that } (x,y)\in\mathscr{A}\}$ coincides with $\mathscr{Y}$.

Let $D(\mathscr{A})$ denote the domain of the relation $\mathscr{A}\in LR(\mathscr{X},\mathscr{Y})$, that is, $D(\mathscr{A})=\{x\in\mathscr{X}\mid (x,y)\in\mathscr{A} \text{ for some } y\in\mathscr{Y}\}$. For $x\in D(\mathscr{A})$, we let $\mathscr{A}x$ denote the set $\{y\in\mathscr{Y}\mid (x,y)\in\mathscr{A}\}$ and we let $\|\mathscr{A}x\|=\inf_{y\in\mathscr{A}x}\|y\|$. If $\mathscr{A}\in LRC(\mathscr{X},\mathscr{Y})$, then $D(\mathscr{A})$ is a Banach space with the graph norm $\|x\|_{\mathscr{A}}=\|x\|+\|\mathscr{A}x\|$.

The definitions below are essential for the statements of the main results.

**Definition 1.1.** Let $\mathscr{A}\in LRC(\mathscr{X},\mathscr{Y})$. Consider the following conditions:

1) $\operatorname{Ker}\mathscr{A}=\{0\}$ (that is, $\mathscr{A}$ is an injective relation);

2) $1\leqslant n=\dim\operatorname{Ker}\mathscr{A}\leqslant\infty$;

3) $\operatorname{Ker}\mathscr{A}$ is a complemented subspace of $D(\mathscr{A})$ (with the graph norm) or of $\mathscr{X}$;

4) $\overline{\operatorname{Im}\mathscr{A}} = \operatorname{Im}\mathscr{A}$, which is equivalent to the strict positivity of the quantity (*the minimum modulus of the relation* $\mathscr{A}$)

$$\gamma(\mathscr{A}) = \inf_{x\in D(\mathscr{A})\setminus\operatorname{Ker}\mathscr{A}} \frac{\|\mathscr{A}x\|}{\operatorname{dist}(x,\operatorname{Ker}\mathscr{A})} = \inf_{\substack{(x,y)\in\mathscr{A}\\ x\notin\operatorname{Ker}\mathscr{A}}} \frac{\|y\|}{\operatorname{dist}(x,\operatorname{Ker}\mathscr{A})}, \tag{1.3}$$

where $\operatorname{dist}(x,\operatorname{Ker}\mathscr{A}) = \inf_{x_0\in\operatorname{Ker}\mathscr{A}}\|x-x_0\|$;

5) the relation $\mathscr{A}$ is correct (uniformly injective), that is, $\operatorname{Ker}\mathscr{A} = \{0\}$ and $\gamma(\mathscr{A}) > 0$;

6) $\operatorname{Im}\mathscr{A}$ is a closed complemented subspace in $\mathscr{Y}$ (in particular, the quantity (1.3) is positive);

7) $\operatorname{Im}\mathscr{A}$ is a closed subspace of codimension $1 \leqslant m = \operatorname{codim}\operatorname{Im}\mathscr{A} \leqslant \infty$ in $\mathscr{Y}$;

8) $\operatorname{Im}\mathscr{A} = \mathscr{Y}$, that is, $\mathscr{A}$ is a surjective relation;

9) $\mathscr{A}$ is a continuously invertible relation.

If a relation $\mathscr{A}$ meets all the conditions in some set $S_0 = \{i_1,\dots,i_k\}$ of conditions, where $1 \leqslant i_1 < \dots < i_k \leqslant 9$, then we say that $\mathscr{A}$ is *in the state* $S_0$. We denote the set of states of $\mathscr{A}$ by $\operatorname{St}_{\mathrm{inv}}(\mathscr{A})$.

**Definition 1.2.** If a relation $\mathscr{A} \in LRC(\mathscr{X},\mathscr{Y})$ is in one of the states $\{1,7\}$, $\{2,7\}$, and $\{2,8\}$ with $n,m<\infty$, then it is called a *Fredholm* or $\Phi$*-relation*. If $\mathscr{A}$ is in one of the states $\{1,7\}$, $\{2,7\}$, and $\{2,8\}$ with only one of $m$ and $n$ finite, then it is called a *semi-Fredholm* relation ($\Phi_+$*-relation* if $n<\infty$ and $\Phi_-$*-relation* if $m<\infty$). The integer $\operatorname{ind}\mathscr{A} = \dim\operatorname{Ker}\mathscr{A} - \operatorname{codim}\operatorname{Im}\mathscr{A}$, where $\operatorname{codim}\operatorname{Im}\mathscr{A} = \dim(\mathscr{Y}/\operatorname{Im}\mathscr{A})$, is called the *index* of the (semi-)Fredholm relation $\mathscr{A}$.

**Definition 1.3.** A pair $E$, $F$ of closed subspaces of $\mathscr{X}$ is said to be *regular* if $\mathscr{X}$ is their direct sum: $\mathscr{X} = E \oplus F$.

**Definition 1.4.** Let $E$ and $F$ be closed subspaces of a Banach space $\mathscr{X}$. Consider the following conditions:

1) $E\cap F = \{0\}$;

2) $1 \leqslant n = \dim(E\cap F) \leqslant \infty$;

3) $E\cap F$ is a complemented subspace of $\mathscr{X}$;

4) $E+F$ is a closed subspace of $\mathscr{X}$, or equivalently, the quantity

$$\gamma(E,F) = \inf_{x\in E,\ x\notin F} \frac{\operatorname{dist}(x,F)}{\operatorname{dist}(x,E\cap F)} \quad (\leqslant 1) \tag{1.4}$$

is positive if $E$ does not lie in $F$;

5) $E\cap F = \{0\}$ and $\gamma(E,F) > 0$ (a correct pair of subspaces);

6) $E+F$ is a closed complemented subspace in $\mathscr{X}$;

7) $E+F$ is a closed subspace of codimension $1 \leqslant m \leqslant \infty$ in $\mathscr{X}$;

8) $E+F = \mathscr{X}$;

9) $\mathscr{X} = E\oplus F$.

If a pair $E$, $F$ of subspaces of $\mathscr{X}$ satisfies all the conditions in some set of conditions $S_0 = \{i_1,\dots,i_k\}$, where $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant 9$, then we say that the pair $E$, $F$ is in the *state* $S_0$. We denote the set of states of $E$, $F$ by $\operatorname{St}_{\mathrm{reg}}(E,F)$.

**Definition 1.5.** If a pair $E$, $F$ of closed subspaces of a Banach space $\mathscr{X}$ is in one of the states $\{1,7\}$, $\{2,7\}$, or $\{2,8\}$ with $n,m<\infty$ (in particular, the quantity in (1.4) is positive), then $E$, $F$ is said to be a *Fredholm* pair. If the pair $E$, $F$ is in one of the states $\{1,7\}$, $\{2,7\}$, or $\{2,8\}$ with only one of the integers $n$ and $m$ finite, then it is said to be *semi-Fredholm*. The number $\operatorname{ind}(E,F)=\dim(E\cap F)-\operatorname{codim}(E+F)$ is called the *index* of the (semi-)Fredholm pair $E$, $F$.

*Remark* 1.1. The equality $\mathrm{St}_{\mathrm{inv}}(\mathscr{A})=\mathrm{St}_{\mathrm{inv}}(\mathscr{B})$ for $\mathscr{A}\in LRC(\mathscr{X}_1,\mathscr{Y}_1)$ and $\mathscr{B}\in LRC(\mathscr{X}_2,\mathscr{Y}_2)$, where $\mathscr{X}_k$ and $\mathscr{Y}_k$, $k=1,2$, are Banach spaces, will indicate (in statements below) that each of the 9 properties in Definition 1.1 above holds (or does not hold) simultaneously for $\mathscr{A}$ and $\mathscr{B}$. The equality $\mathrm{St}_{\mathrm{inv}}(\mathscr{A})=\mathrm{St}_{\mathrm{reg}}(E,F)$ for $\mathscr{A}\in LR(\mathscr{X}_1,\mathscr{X}_2)$ and a pair $E$, $F$ of closed subspaces of a Banach space $\mathscr{X}$ indicates that if some property in Definition 1.1 holds for $\mathscr{A}$, then the property with the same number in Definition 1.4 holds for the pair of subspaces $E$, $F$, and conversely.

We shall define and investigate differential and difference operators with the use of a family of evolution operators, which do not necessarily correspond to a differential equation of the form (1.1).

Let $\mathbb{J}$ be a subset of $\mathbb{R}$ with the induced topology, and let $\Delta_{\mathbb{J}}=\{(t,s)\in\mathbb{J}^2: s\leqslant t\}\subset\mathbb{R}^2$. A map $\mathscr{U}\colon\Delta_{\mathbb{J}}\to\operatorname{End}X$ is called a (strongly continuous and exponentially bounded) *family of* (‘*forward*’) *evolution operators* on $\mathbb{J}$ if the following conditions are satisfied:

1) $\mathscr{U}(t,t)=I$ is the identity operator for each $t\in\mathbb{J}$;
2) $\mathscr{U}(t,s)\mathscr{U}(s,\tau)=\mathscr{U}(t,\tau)$ for $\tau\leqslant s\leqslant t$, $s,t,\tau\in\mathbb{J}$;
3) for each $x\in X$ the map $(t,s)\mapsto\mathscr{U}(t,s)x\colon\Delta_{\mathbb{J}}\to X$ is continuous;
4) there exist constants $M\geqslant 1$ and $\alpha\in\mathbb{R}$ such that

$$\|\mathscr{U}(t,s)\|\leqslant Me^{\alpha(t-s)},\qquad s\leqslant t,\quad s,t\in\mathbb{J}.$$

The family $\mathscr{U}\colon\Delta_{\mathbb{J}}\to\operatorname{End}X$ is called a *family of* (‘*backward*’) *evolution operators* if conditions 1), 3), and 4) in the above definition are fulfilled, while 2) is replaced by the condition

2′) $\mathscr{U}(s,\tau)\mathscr{U}(\tau,t)=\mathscr{U}(s,t)$ for all $s\leqslant\tau\leqslant t$ in $\mathbb{J}$.

If $\mathbb{J}\subset\mathbb{Z}$, then we say that the family $\mathscr{U}\colon\Delta_{\mathbb{J}}\to\operatorname{End}X$ is *discrete*. Unless otherwise stated, in what follows we deal with families of ‘forward’ evolution operators.

Let $\mathbb{R}_{-,a}=(-\infty,a]$, $\mathbb{R}_{a,+}=[a,\infty)$ and $\mathbb{R}_-=(-\infty,0]$, $\mathbb{R}_+=[0,\infty)$.

Families of evolution operators occur in a natural way in connection with the representation of solutions to the abstract Cauchy problem

$$x(s)=x_0\in D(A(s)),\qquad t\geqslant s,\quad t,s\in\mathbb{J},\tag{1.5}$$

for the linear differential equation (1.2), where $\mathbb{J}$ is one of the sets $[a,b]$, $\mathbb{R}_{-,a}$, $\mathbb{R}_{a,+}$, $\mathbb{R}$.

We say that a family of evolution operators $\mathscr{U}\colon\Delta_{\mathbb{J}}\to\operatorname{End}X$ *solves the abstract Cauchy problem* (1.2), (1.5) if for each $s\in\mathbb{J}$ there exists a subspace $X_s$ of $D(A(s))$ which is dense in $X$ and has the following properties: for each $x_0\in X_s$ the function $x(t)=\mathscr{U}(t,s)x_0$, $t\in\mathbb{J}$, is differentiable for $t\geqslant s$, $x(t)\in D(A(t))$ for every $t\in\mathbb{J}$, and equalities (1.5) and (1.2) hold. In this case we also say that the family $\mathscr{U}$ corresponds to the problem (1.5), (1.2).

If $f\colon \mathbb{J} \to X$ is a function in the linear space $L^1_{\mathrm{loc}}(\mathbb{J}, X)$ of locally integrable Bochner-measurable (classes of) functions defined on $\mathbb{J}$ and taking values in $X$, then by a (weak) *solution* of (1.1) (under the condition that the family $\mathscr{U}$ on $\mathbb{J}$ solves the Cauchy problem (1.5), (1.2)) we mean an arbitrary continuous function $x\colon \mathbb{J} \to X$ such that

$$x(t) = \mathscr{U}(t,s)x(s) - \int_s^t \mathscr{U}(t,\tau)f(\tau)\,d\tau \tag{1.6}$$

for all $s \leqslant t$ in $\mathbb{J}$.

We stress that the linear operators under consideration in this paper are constructed from an arbitrary family of evolution operators. Nevertheless, we call them 'differential operators'.

With any family of evolution operators $\mathscr{U}: \Delta_{\mathbb{J}} \to \operatorname{End} X$, where $\mathbb{J}$ is an interval of $\mathbb{R}$, we can associate the linear operator

$$\mathscr{L}_{\max} = \mathscr{L}_{\max,\mathbb{J}}\colon D(\mathscr{L}_{\max}) \subset L^1_{\mathrm{loc}}(\mathbb{J}, X) \to L^1_{\mathrm{loc}}(\mathbb{J}, X),$$

defined as follows. A continuous function $x\colon \mathbb{J} \to X$ is in $D(\mathscr{L}_{\max})$ if there exists a function $f \in L^1_{\mathrm{loc}}(\mathbb{J}, X)$ such that (1.6) holds for all $s \leqslant t$ in $\mathbb{J}$. The function $f$ is uniquely defined, so the definition of the operator $\mathscr{L}_{\max}$ is consistent (the consistency of the definition of $\mathscr{L}_{\max}$ follows from the fact that the points in $\mathbb{J}$ which are not Lebesgue points of an integrable function form a set of measure zero).

Let $\mathscr{F} = \mathscr{F}(\mathbb{J}, X)$ be a homogeneous function space (see § 2). Some instances of homogeneous spaces are given by the Banach spaces

$$L^p(\mathbb{J}, X) \ \text{ for } p \in [1,\infty], \quad S^p(\mathbb{J}, X) \ \text{ for } p \in [1,\infty], \quad C_b(\mathbb{J}, X)$$

defined in § 2.

In what follows we let $E$ be a closed linear subspace of $X$.

Let $a \in \mathbb{R}$ and let $\mathbb{J} \in \{\mathbb{R}_{-,a}, \mathbb{R}_{a,+}, \mathbb{R}\}$. In this paper we investigate the operators

$$\begin{aligned}
\mathscr{L} = \mathscr{L}_{\mathscr{U}}: D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X),\\
\mathscr{L}_E^{a,+}\colon D(\mathscr{L}_E^{a,+}) \subset \mathscr{F}(\mathbb{R}_{a,+}, X) \to \mathscr{F}(\mathbb{R}_{a,+}, X),\\
\mathscr{L}_E^{-,a}\colon D(\mathscr{L}_E^{-,a}) \subset \mathscr{F}(\mathbb{R}_{-,a}, X) \to \mathscr{F}(\mathbb{R}_{-,a}, X)
\end{aligned}$$

with respective domains

$$\begin{aligned}
D(\mathscr{L}) &= \{x \in D(\mathscr{L}_{\max,\mathbb{R}}) \cap \mathscr{F}(\mathbb{R}, X) \mid \mathscr{L}_{\max} x \in \mathscr{F}(\mathbb{R}, X)\},\\
D(\mathscr{L}_E^{a,+}) &= \{x \in D(\mathscr{L}_{\max,\mathbb{R}}) \cap \mathscr{F}(\mathbb{R}_{a,+}, X) \mid x(a) \in E,\\
&\qquad\qquad \mathscr{L}_{\max,\mathbb{R}_{a,+}} x \in \mathscr{F}(\mathbb{R}_{a,+}, X)\},\\
D(\mathscr{L}_E^{-,a}) &= \{x \in D(\mathscr{L}_{\max,\mathbb{R}_{-,a}}) \cap \mathscr{F}(\mathbb{R}_{-,a}, X) \mid x(a) \in E,\\
&\qquad\qquad \mathscr{L}_{\max,\mathbb{R}_{-,a}} x \in \mathscr{F}(\mathbb{R}_{-,a}, X)\}.
\end{aligned}$$

Here if $x$ lies in the domain of the corresponding operator, then the value of the latter on the function $x$ is taken to be $f = \mathscr{L}_{\max,\mathbb{J}}\, x$, so that the pair $(x, f)$ satisfies (1.6). If $a = 0$, then let

$$\mathscr{L}_E^{+} = \mathscr{L}_E^{a,+} \quad \text{and} \quad \mathscr{L}_E^{-} = \mathscr{L}_E^{-,a}.$$

We note that $\mathscr{L}$ and $\mathscr{L}_E^{-,a}$ are closed operators, whereas $\mathscr{L}_E^{a,+}$ is not necessarily closed (see [19], Example 5.1).

When a family $\mathscr{U}$ solves the Cauchy problem (1.2), (1.5), we shall occasionally use the notation $-d/dt + A(t)$ for the operators introduced above.

Let $\mathbb{J} \in \{\mathbb{R}_-, \mathbb{R}\}$. Then we have the well-defined semigroup $T\colon \mathbb{R}_+ \to \operatorname{End}\mathscr{F}$ of weighted shift operators of the form

$$(T(t)x)(s) = \mathscr{U}(s, s-t)x(s-t), \qquad s \in \mathbb{J}, \quad x \in \mathscr{F}, \quad t \geqslant 0, \tag{1.7}$$

in the Banach space $\mathscr{F} = \mathscr{F}(\mathbb{J}, X)$. For $\mathbb{J} = \mathbb{R}$ and a Hilbert space $X$ such a semigroup was introduced by Howland [20] in the Hilbert space $L^2(\mathbb{R}, X)$, in the case when the operators $\mathscr{U}(t, s)$, $t, s, \in \mathbb{R}$, are unitary. The following two theorems allow us to use the theory of operator semigroups and difference operators.

**Theorem 1.1** [13], [14]. *If $\mathscr{F}(\mathbb{R}, X) \in \{L^p(\mathbb{R}, X),\ p \in [1, \infty);\ C_0(\mathbb{R}, X)\}$, then the operator semigroup (1.7) is strongly continuous and has the generator $\mathscr{L} = \mathscr{L}_{\mathscr{U}}$.*

**Theorem 1.2.** *The spectra of the operators $T(t) \in \operatorname{End}\mathscr{F}(\mathbb{R}, X)$, $t \in \mathbb{R}_+$, are related by*

$$\sigma(T(t)) \setminus \{0\} = \exp\sigma(\mathscr{L})t = \{\exp(\lambda t); \lambda \in \sigma(\mathscr{L})\}.$$

For $\mathscr{F}(\mathbb{R}, X) \in \{L^p(\mathbb{R}, X),\ p \in [1, \infty);\ C_0(\mathbb{R}, X)\}$ Theorem 1.2 was proved in [13], [14], [16], and [17]. In fact, more general spaces were considered in [13] and [14]. Here and throughout, $C_0(\mathbb{J}, X)$ denotes the subspace of functions $x$ in $C_b(\mathbb{J}, X)$ such that $\lim_{|t|\to\infty} \|x(t)\| = 0$. There is no direct analogue of Theorem 1.1 for the operator $\mathscr{L}_E^{+}$, but in [19] we introduced a semigroup of difference relations on the Banach space $\mathscr{F}(\mathbb{R}_+, X)$ and proved an analogue of Theorem 1.2.

The application of difference operators in (1.7) and difference relations on the space $\mathscr{F}(\mathbb{J}, X)$ to the investigation of the operators $\mathscr{L}$ and $\mathscr{L}_E^{+}$ meets with difficulties because the use of the dual space of $\mathscr{F}(\mathbb{J}, X)$ is problematic. Moreover, it is difficult to apply the spectral theory of operators to $\mathscr{L}$ and $\mathscr{L}_E^{+}$ because of unbounded components of the spectra.

In this paper we study the differential operators under consideration by making essential use of difference operators and difference relations on homogeneous spaces of sequences of vectors. Usually, such operators and relations have bounded spectral components, and so for a given spectral component we can construct a Riesz projection. We can say that difference operators and difference relations play the same role in the investigation of differential equations on an infinite interval as the monodromy operator plays in the analysis of differential equations with periodic coefficients.

The notion of a homogeneous sequence space $\mathscr{F}(\mathbb{J}_d, X)$ is introduced in § 2.

Let $\mathbb{J}_d$ be one of the following subsets of the set of integers:

$$\mathbb{Z}, \quad \mathbb{Z}_{-,m} = (-\infty, m] \cap \mathbb{Z}, \quad \mathbb{Z}_{m,+} = [m, \infty) \cap \mathbb{Z}, \quad m \in \mathbb{Z},$$
$$\mathbb{Z}_- = \{n \in \mathbb{Z} \mid n \leqslant 0\}, \qquad \mathbb{Z}_+ = \{n \in \mathbb{Z} \mid n \geqslant 0\}.$$

One example of a homogeneous space is given by the Banach space $\ell^p(\mathbb{J}_d, X)$, $p \in [1,\infty]$, of sequences $x\colon \mathbb{J}_d \to X$ such that the following quantity (taken to be the norm) is finite:

$$\|x\|_p = \Big(\sum_{k\in\mathbb{J}_d} \|x(k)\|^p\Big)^{1/p}, \quad p \in [1,\infty); \qquad \|x\|_\infty = \sup_{k\in\mathbb{J}_d} \|x(k)\|, \quad p = \infty.$$

We shall occasionally denote a sequence $x\colon \mathbb{N} \to X$ by $(x_n)$.

Let $U\colon \mathbb{J}_d \to \operatorname{End} X$ be a bounded function and $E$ a closed subspace of the Banach space $X$.

We shall investigate the linear difference operators

$$\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X), \quad \mathscr{D}_E^{-,a}\colon D(\mathscr{D}_E^{-,a}) \subset \mathscr{F}(\mathbb{Z}_{-,a}, X) \to \mathscr{F}(\mathbb{Z}_{-,a}, X), \quad a \in \mathbb{Z},$$

and the difference relations $\mathscr{D}_E^{a,+} \in LRC(\mathscr{F}(\mathbb{Z}_{a,+}, X))$, $a \in \mathbb{Z}$, defined by

$$(\mathscr{D}x)(n) = x(n) - U(n)x(n-1), \qquad n \in \mathbb{Z}, \quad x \in \mathscr{F}(\mathbb{Z}, X), \tag{1.8}$$

$$(\mathscr{D}_E^{-,a}x)(n) = x(n) - U(n)x(n-1), \qquad n \leqslant a, \quad x \in D(\mathscr{D}_E^{-,a}), \tag{1.9}$$

where $D(\mathscr{D}_E^{-,a}) = \{x \in \mathscr{F}(\mathbb{Z}_{-,a}, X) \mid x(a) \in E\}$,

$$\begin{aligned}\mathscr{D}_E^{a,+} = \{(x,y) \in \mathscr{F}(\mathbb{Z}_{a,+}, X) \mid\ & y(n) = x(n) - U(n)x(n-1),\ n \geqslant a+1,\\ & y(a) = x(a) + x_0 \text{ for some } x_0 \in E\}.\end{aligned} \tag{1.10}$$

If $a = 0$, then let $\mathscr{D}_E^{-} = \mathscr{D}_E^{-,a}$, $\mathscr{D}_E^{+} = \mathscr{D}_E^{a,+}$. Note that in the definition of the difference relation $\mathscr{D}_E^{a,+}$ we can consider the function $U$ as being defined on $\mathbb{Z}_{a+1,+}$.

To investigate the operators $\mathscr{D}$ and $\mathscr{D}_E^{-,a}$ and the relations $\mathscr{D}_E^{a,+}$ we shall use the corresponding discrete families of evolution operators $\mathscr{U} = \mathscr{U}_d = \mathscr{U}_{\mathbb{J}_d}\colon \Delta_{\mathbb{J}_d} \to \operatorname{End} X$ defined by

$$\mathscr{U}_d(n,m) = \mathscr{U}_{\mathbb{J}_d}(n,m) = \begin{cases} U(n)U(n-1)\cdots U(m+1), & m < n,\\ I, & m = n,\end{cases} \tag{1.11}$$

where $m, n \in \mathbb{J}_d \in \{\mathbb{Z}, \mathbb{Z}_{-,a}, \mathbb{Z}_{a,+}\}$. We shall obtain results for difference operators and relations without assuming that the family $\mathscr{U}_d$ is generated by a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{J}} \to \operatorname{End} X$ with $\mathbb{J} \in \{\mathbb{R}_-, \mathbb{R}_+, \mathbb{R}\}$.

For the differential and difference operators (difference relations) under consideration we shall investigate the properties listed in Definition 1.1 using the notion of an exponential dichotomy for a family of evolution operators.

**Definition 1.6.** Let $\mathbb{J}$ be a subset of $\mathbb{R}$ with the topology inherited from $\mathbb{R}$. We say that a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{J}} \to \operatorname{End} X$ admits an exponential dichotomy on a subset $\Omega$ of $\mathbb{J}$, if there exist a bounded strongly continuous projection-valued function $P\colon \Omega \to \operatorname{End} X$ and constants $M_0 \geqslant 1$ and $\gamma > 0$ such that the following properties hold:

1) $\mathscr{U}(t,s)P(s) = P(t)\mathscr{U}(t,s)$ for $t \geqslant s$, $t, s \in \Omega$;

2) $\|\mathscr{U}(t,s)P(s)\| \leqslant M_0 \exp(-\gamma(t-s))$ for $s \leqslant t$, $s, t \in \Omega$;

3) for $s \leqslant t$, $s, t \in \Omega$, the restriction $\mathscr{U}_{t,s}\colon X'(s) \to X'(t)$ of the operator $\mathscr{U}(t,s)$ to the range $X'(s) = \operatorname{Im} Q(s)$ of the projection $Q(s) = I - P(s)$ complementary to $P(s)$ is an isomorphism between the spaces $X'(s)$ and $X'(t) = \operatorname{Im} Q(t)$ (we set $\mathscr{U}(s,t) \in \operatorname{End} X$ equal to $\mathscr{U}_{t,s}^{-1}$ on $X'(t)$ and to the zero operator on $X(t) = \operatorname{Im} P(t) \subset \mathscr{X}$);

4) $\|\mathscr{U}(s,t)\| \leqslant M_0 \exp \gamma(s-t)$ for all $t \geqslant s$ in $\Omega$.

We call the pair of projection-valued functions in this definition a *splitting pair* for the family $\mathscr{U}$. If $P = 0$ or $Q = 0$, then we say that the *trivial exponential dichotomy* holds for $\mathscr{U}$ on $\Omega$.

A similar definition of an exponential dichotomy is also given in the case when $\mathscr{U}$ is a family of 'backward' evolution operators.

**Theorem 1.3.** *Let $\mathscr{L}_E^+\colon D(\mathscr{L}_E^+) \subset \mathscr{F}(\mathbb{R}_+, X) \to \mathscr{F}(\mathbb{R}_+, X)$ be a closed linear operator in a homogeneous function space $\mathscr{F}(\mathbb{R}_+, X)$. Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{L}_E^+) = \operatorname{St}_{\mathrm{inv}}(\mathscr{D}_E^+), \tag{1.12}$$

*where the relation $\mathscr{D}_E^+$ is considered on the homogeneous sequence space $\mathscr{F}(\mathbb{Z}_+, X)$ associated with $\mathscr{F}(\mathbb{R}_+, X)$ (see Remark 2.1). In particular, $\mathscr{L}_E^+$ and $\mathscr{D}_E^+$ simultaneously have (or do not have) the (semi-)Fredholm property, and if they are (semi-)Fred-holm, then*

$$\begin{gathered}\dim \operatorname{Ker} \mathscr{L}_E^+ = \dim \operatorname{Ker} \mathscr{D}_E^+, \quad \operatorname{codim} \operatorname{Im} \mathscr{L}_E^+ = \operatorname{codim} \operatorname{Im} \mathscr{D}_E^+, \\ \textit{and} \quad \operatorname{ind} \mathscr{L}_E^+ = \operatorname{ind} \mathscr{D}_E^+. \end{gathered} \tag{1.13}$$

This theorem is proved at the end of § 4. Also there, in Remark 4.1, we clarify property 3) in Definition 1.1 for the operator $\mathscr{L}_E^+$. The next theorem is an immediate consequence of Theorems 1.3 and 1.7–1.9.

**Theorem 1.4.** *The Fredholm property of the operator $\mathscr{L}_E^+$ and its index are independent of the choice of the homogeneous space $\mathscr{F}(\mathbb{R}_+, X)$.*

The inclusion $\operatorname{St}_{\mathrm{inv}}(\mathscr{L}_E^+) \subset \operatorname{St}_{\mathrm{inv}}(\mathscr{D}_E^+)$ was proved in [19], Theorem 5.8. Furthermore, for each of the 9 properties in Definition 1.1, except for property 3) concerning the kernel being complemented, it was proved in [19] that if the property holds for $\mathscr{D}_E^+$, then it also holds for $\mathscr{L}_E^+$.

The analogues of Theorems 1.3 and 1.4 also hold for the operators $\mathscr{L}_E^-$ and $\mathscr{D}_E^-$.

**Theorem 1.5.** *The linear operator $\mathscr{L}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ in a homogeneous space $\mathscr{F}(\mathbb{R}, X)$ and the difference operator $\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X)$ in the homogeneous sequence space $\mathscr{F}(\mathbb{Z}, X)$ associated with $\mathscr{F}(\mathbb{R}, X)$ have the same sets of states*:

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{L}) = \operatorname{St}_{\mathrm{inv}}(\mathscr{D}). \tag{1.14}$$

*In particular, the operators $\mathscr{L}$ and $\mathscr{D}$ simultaneously have (or do not have) the (semi-)Fredholm property, and if they are (semi-)Fredholm, then*

$$\dim \operatorname{Ker} \mathscr{L} = \dim \operatorname{Ker} \mathscr{D}, \quad \operatorname{codim} \operatorname{Im} \mathscr{L} = \operatorname{codim} \operatorname{Im} \mathscr{D}, \quad \operatorname{ind} \mathscr{L} = \operatorname{ind} \mathscr{D}. \tag{1.15}$$

**Theorem 1.6.** *The Fredholm property of an operator $\mathscr{L}$ and its index are independent of the choice of the homogeneous space $\mathscr{F}(\mathbb{R}, X)$.*

Note that Theorem 1.6 is an immediate consequence of Theorems 1.5, 1.13, 1.15, and 1.16.

**Definition 1.7.** Let $\mathbb{J}$ be a subset of $\mathbb{R}$ containing a sequence $(t_n)$ such that $\lim_{n\to\infty} t_n = \infty$ (or $\lim_{n\to\infty} t_n = -\infty$). We say that a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{J}} \to \operatorname{End} X$ *is non-singular at* $+\infty$ (*at* $-\infty$) if there exists an $a \in \mathbb{J}$ such that the family $\mathscr{U}$ admits an exponential dichotomy on the set $\mathbb{J}\cap[a,\infty)$ (on $\mathbb{J}\cap(-\infty,a]$), respectively).

**Theorem 1.7.** *Let $\mathscr{U}\colon \Delta_{\mathbb{Z}_+} \to \operatorname{End} X$ be a family admitting an exponential dichotomy on the set $\mathbb{Z}_{a,+}$ (where $a\in\mathbb{Z}_+$) with splitting pair of projection-valued functions $P_+, Q_+\colon \mathbb{Z}_{a,+}\to \operatorname{End} X$. Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{D}_E^+) = \operatorname{St}_{\mathrm{inv}}(\mathscr{N}_a^+), \tag{1.16}$$

*where the operator $\mathscr{N}_a^+\colon E \to \operatorname{Im} Q_+(a)$ is defined by*

$$\mathscr{N}_a^+ x = Q_+(a)\mathscr{U}(a,0)x, \qquad x\in E. \tag{1.17}$$

**Theorem 1.8.** *Let $\mathscr{U}\colon \Delta_{\mathbb{R}_{a,+}} \to \operatorname{End} X$ be a family of evolution operators admitting an exponential dichotomy on $\mathbb{R}_{a,+}$ (where $a\in\mathbb{R}_+$) with splitting pair of projection-valued functions $P_+, Q_+\colon \mathbb{R}_{a,+}\to\operatorname{End} X$. Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{L}_E^+) = \operatorname{St}_{\mathrm{inv}}(\mathscr{N}_a^+), \tag{1.18}$$

*where the operator $\mathscr{N}_a^+\colon E\to \operatorname{Im} Q_+(a)$ is defined by* (1.17).

**Theorem 1.9.** *The following conditions are equivalent*:

1) *the operator $\mathscr{L}_E^+\colon D(\mathscr{L}_E^+)\subset \mathscr{F}(\mathbb{R}_+,X)\to\mathscr{F}(\mathbb{R}_+,X)$ is Fredholm*;

2) *there exists an $a\in\mathbb{R}_+$ such that the family $\mathscr{U}$ admits an exponential dichotomy on $[a,\infty)$ with a splitting pair of projection-valued functions $P_+,Q_+\colon[a,\infty)\to \operatorname{End} X$ such that the operator $\mathscr{N}_a^+\colon E\to\operatorname{Im} Q_+(a)$ defined by* (1.17) *is Fredholm.*

*If condition* 2) *holds, then*

$$\begin{gathered}\dim\operatorname{Ker}\mathscr{L}_E^+ = \dim\operatorname{Ker}\mathscr{N}_a^+, \qquad \operatorname{codim}\operatorname{Im}\mathscr{L}_E^+ = \operatorname{codim}\operatorname{Im}\mathscr{N}_a^+,\\ \textit{and}\quad \operatorname{ind}\mathscr{L}_E^+ = \operatorname{ind}\mathscr{N}_a^+.\end{gathered} \tag{1.19}$$

**Theorem 1.10.** *Assume that all the operators $\mathscr{U}(t,s)$, $s<t$, $s,t\in\mathbb{R}_+$, are compact and the family of evolution operators $\mathscr{U}\colon\Delta_{\mathbb{R}_+}\to\operatorname{End} X$ is non-singular at $+\infty$. Then the following conditions are equivalent*:

1) *the operator $\mathscr{L}_E^+\colon D(\mathscr{L}_E^+)\subset\mathscr{F}(\mathbb{R}_+,X)\to\mathscr{F}(\mathbb{R}_+,X)$ is Fredholm*;

2) *the relation $\mathscr{D}_E^+\in LRC(\mathscr{F}(\mathbb{Z}_+,X))$ is Fredholm*;

3) *$E$ is a finite-dimensional subspace of $X$.*

**Theorem 1.11.** *Let $\mathscr{U}\colon\Delta_{\mathbb{R}_+}\to X$ be a family of evolution operators admitting an exponential dichotomy on $\mathbb{R}_+$ with splitting pair of projection-valued functions $P,Q\colon\mathbb{R}_+\to\operatorname{End} X$. Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{L}_E^+) = \operatorname{St}_{\mathrm{inv}}(\mathscr{D}_E^+) = \operatorname{St}_{\mathrm{reg}}(E,\operatorname{Im} P(0)).$$

Importantly, if a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}_+} \to \operatorname{End} X$ is constructed for the differential equation (1.2) with function $A$ in $S^1(\mathbb{R}_+, \operatorname{End} X)$ and if $\mathscr{U}$ is non-singular at $+\infty$, then it admits an exponential dichotomy on $\mathbb{R}_+$. However, more general families of evolution operators do not have this property. If $\mathscr{L}_E^+$ is a Fredholm operator, then the family $\mathscr{U}$ is non-singular at $+\infty$, but $\mathscr{U}$ does not necessarily have the property of exponential dichotomy on $\mathbb{R}_+$ (on $\mathbb{Z}_+$; see Example 8.6).

Many of the results obtained in this paper for operators $\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X)$ and $\mathscr{L}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ hold under the following assumption.

**Assumption 1.1.** *Let $\mathbb{J} \in \{\mathbb{Z}, \mathbb{R}\}$. There exist $a, b \in \mathbb{J}$, $a \leqslant b$, such that the family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{J}} \to \operatorname{End} X$ admits an exponential dichotomy on the sets $\mathbb{J}_{-,a} = \{t \in \mathbb{J} \mid t \leqslant a\}$ and $\mathbb{J}_{b,+} = \{t \in \mathbb{J} \mid t \geqslant b\}$ with splitting pairs of projection-valued functions $P_-, Q_-\colon \mathbb{J}_{-,a} \to \operatorname{End} X$ and $P_+, Q_+\colon \mathbb{J}_{b,+} \to \operatorname{End} X$, respectively.*

Important results on the operator $\mathscr{L}$ were obtained by Zhikov [8], who proved that the operator $\mathscr{L}$ is invertible in the space $C_b(\mathbb{R}, X)$ if and only if the family $\mathscr{U}$ admits an exponential dichotomy. Subsequently, these results were presented in the monograph [9]. We note also that [8] and [9] put the use of operator theory in the analysis of qualitative properties of solutions of differential equations in a Banach space on a systematic basis.

It follows from the statements of the above results that we shall investigate the differential and difference operators (and equations) under consideration by making essential use of the notion of an exponential dichotomy for a family of evolution operators defining the operator in question. The property of exponential dichotomy of a family of evolution operators emerges in the most natural fashion when we look at the continuously invertible difference operator $\mathscr{D} = \mathscr{D}_{\mathscr{U}}$ defined by (1.8). Then the spectrum $\sigma(\mathscr{K})$ of the weighted shift operator $\mathscr{K} = I - \mathscr{D}$ contains no points in the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$. Hence $\sigma(\mathscr{K}) = \sigma_{\mathrm{int}} \cup \sigma_{\mathrm{out}}$, where $\sigma_{\mathrm{int}} = \{\lambda \in \sigma(\mathscr{K}) \mid |\lambda| < 1\}$.

This representation for $\sigma(\mathscr{K})$ enables us to construct Riesz projections $\mathscr{P}_{\mathrm{int}}$ and $\mathscr{P}_{\mathrm{out}} = I - \mathscr{P}_{\mathrm{int}}$ for the spectral sets $\sigma_{\mathrm{int}}$ and $\sigma_{\mathrm{out}}$ so that $\sigma(\mathscr{K} \mid \mathscr{F}_{\mathrm{int}}) = \sigma_{\mathrm{int}}$ and $\sigma(\mathscr{K} \mid \mathscr{F}_{\mathrm{out}}) = \sigma_{\mathrm{out}}$, where $\mathscr{F}_{\mathrm{int}} = \operatorname{Im} \mathscr{P}_{\mathrm{int}}$ and $\mathscr{F}_{\mathrm{out}} = \operatorname{Im} \mathscr{P}_{\mathrm{out}}$. It can also be proved that $\mathscr{P}_{\mathrm{int}}$ and $\mathscr{P}_{\mathrm{out}}$ are the operators of multiplication by bounded operator-valued functions, and therefore $r(\mathscr{K}_{\mathrm{int}}) < 1$ (where $r(\,\cdot\,)$ is the spectral radius and $\mathscr{K}_{\mathrm{int}} = \mathscr{K} \mid \mathscr{F}_{\mathrm{int}}$ is the restriction of $\mathscr{K}$ to $\mathscr{F}_{\mathrm{int}}$), the operator $\mathscr{K}_{\mathrm{out}} = \mathscr{K} \mid \mathscr{F}_{\mathrm{out}}$ has a continuous inverse, and $r(\mathscr{K}_{\mathrm{out}}^{-1}) < 1$. Elaborating on the indicated properties (as done in [1], Theorem 6.1), we arrive at the properties 1)–4) in Definition 1.6 for an exponential dichotomy of the family of evolution operators $\mathscr{U}_d$ defined by (1.11) on the set $\mathbb{Z}$. If a family $\mathscr{U}_d$ is the restriction of a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ to $\Delta_{\mathbb{Z}}$, then $\mathscr{U}$ also admits an exponential dichotomy. We see that the following properties are equivalent (see [13] and [14]):

1) the operator $\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X)$ has a continuous inverse;

2) the operator $\mathscr{L} = \mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ has a continuous inverse;

3) the family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ admits an exponential dichotomy on $\mathbb{R}$.

If the continuous invertibility of the operator $\mathscr{D}$ (or $\mathscr{L}$) is replaced by the Fredholm property for it, then Assumption 1.1 holds for the family $\mathscr{U}_d$ (for $\mathscr{U}$) in view of Theorem 1.15 (Theorem 1.16, respectively). Example 8.6 gives a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ admitting an exponential dichotomy on $\mathbb{R}_-$ and $[1,\infty)$, but not on $\mathbb{R}$. By Theorems 1.15 and 1.16, if Assumption 1.1 holds, then the Fredholm property of the operators $\mathscr{D}_{\mathscr{U}}$ and $\mathscr{L}_{\mathscr{U}}$ depends on the linear operator

$$\mathscr{N}_{b,a}\colon \operatorname{Im} Q_-(a) \to \operatorname{Im} Q_+(b), \quad \mathscr{N}_{b,a}x = Q_+(b)\mathscr{U}(b,a)x, \quad x \in \operatorname{Im} Q_-(a). \tag{1.20}$$

This was defined in [20] and [21], where it was called the *node operator*. It is important to look at this operator, since it acts between subspaces of the phase space $X$ (rather than in $\mathscr{F}(\mathbb{J},X)$). By Theorems 1.12 and 1.13 stated below and also by their consequence Theorem 1.14, the properties of the operators $\mathscr{D}$ and $\mathscr{L}$ are most closely connected with the properties of $\mathscr{N}_{b,a}$ (their sets of states are the same).

Below we consider the difference operator $\mathscr{D} \in \operatorname{End}\mathscr{F}(\mathbb{Z},X)$ defined by (1.8) and the operator $\mathscr{L} = \mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R},X) \to \mathscr{F}(\mathbb{R},X)$. The operator $\mathscr{D}$ is constructed from a bounded function $U\colon \mathbb{Z} \to \operatorname{End} X$. Below we also use the family of evolution operators $\mathscr{U} = \mathscr{U}_d\colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$ defined by (1.11).

**Theorem 1.12.** *Suppose that Assumption 1.1 holds for the family* $\mathscr{U} = \mathscr{U}_d\colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$. *Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{D}) = \operatorname{St}_{\mathrm{inv}}(\mathscr{N}_{b,a}).$$

*In particular,* $\mathscr{D}$ *is (semi-)Fredholm if and only if the node operator* $\mathscr{N}_{b,a}$ *is. If* $\mathscr{D}$ *is Fredholm, then*

$$\dim\operatorname{Ker}\mathscr{D} = \dim\operatorname{Ker}\mathscr{N}_{b,a}, \quad \operatorname{codim}\operatorname{Im}\mathscr{D} = \operatorname{codim}\operatorname{Im}\mathscr{N}_{b,a}, \quad \operatorname{ind}\mathscr{D} = \operatorname{ind}\mathscr{N}_{b,a}.$$

The next result follows from Theorems 1.5 and 1.12.

**Theorem 1.13.** *Suppose that Assumption 1.1 holds for a family of evolution operators* $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$. *Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{L}) = \operatorname{St}_{\mathrm{inv}}(\mathscr{N}_{b,a}).$$

*In particular,* $\mathscr{L}$ *is (semi-)Fredholm if and only if the node operator* $\mathscr{N}_{b,a}$ *is. If* $\mathscr{L}$ *is (semi-)Fredholm, then*

$$\dim\operatorname{Ker}\mathscr{L} = \dim\operatorname{Ker}\mathscr{N}_{b,a}, \quad \operatorname{codim}\operatorname{Im}\mathscr{L} = \operatorname{codim}\operatorname{Im}\mathscr{N}_{b,a}, \quad \operatorname{ind}\mathscr{L} = \operatorname{ind}\mathscr{N}_{b,a}.$$

**Corollary 1.1.** *If the operator* $\mathscr{D}$ *(or* $\mathscr{L}$*) is (semi-)Fredholm in some homogeneous sequence (or function) space, then it is (semi-)Fredholm in any homogeneous space and its index (as well as the dimension of the kernel and the codimension of the range) is independent of the particular homogeneous space in which it acts.*

**Assumption 1.2.** *The family* $\mathscr{U}\colon \Delta_{\mathbb{J}} \to \operatorname{End} X$, *where* $\mathbb{J} \in \{\mathbb{Z},\mathbb{R}\}$, *admits an exponential dichotomy on the sets*

$$\mathbb{J}_- = \mathbb{J} \cap \mathbb{R}_- \quad \text{and} \quad \mathbb{J}_+ = \mathbb{J} \cap \mathbb{R}_+$$

*with splitting pairs of projection-valued functions* $P_-, Q_-\colon \mathbb{J}_- \to \operatorname{End} X$ *and* $P_+, Q_+\colon \mathbb{J}_+ \to \operatorname{End} X$, *respectively.*

**Theorem 1.14.** *Suppose that Assumption 1.2 holds for a family of evolution operators* $\mathscr{U}: \Delta_{\mathbb{R}} \to \operatorname{End} X$. *Then*

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{L}) = \operatorname{St}_{\mathrm{reg}}\bigl(\operatorname{Im} Q_-(0), \operatorname{Im} P_+(0)\bigr). \tag{1.21}$$

*In particular, the operator* $\mathscr{L}$ *is (semi-)Fredholm if and only if the subspaces* $\operatorname{Im} P_-(0)$ *and* $\operatorname{Im} Q_+(0)$ *form a (semi-)Fredholm pair. If* $\mathscr{L}$ *is a (semi-)Fredholm operator, then*

$$\begin{aligned} \dim \operatorname{Ker} \mathscr{L} &= \dim(\operatorname{Im} P_-(0) \cap \operatorname{Im} Q_+(0)), \\ \operatorname{codim} \operatorname{Im} \mathscr{L} &= \operatorname{codim}(\operatorname{Im} P_-(0) + \operatorname{Im} Q_+(0)), \\ \operatorname{ind} \mathscr{L} &= \operatorname{ind}\bigl(\operatorname{Im} P_-(0), \operatorname{Im} Q_+(0)\bigr). \end{aligned}$$

For the operator $\mathscr{D}$ we shall establish Theorem 5.4, whose result corresponds to Theorem 1.14. Equality (1.21) allows us to conclude that the set $\operatorname{St}_{\mathrm{inv}}(\mathscr{L})$ is independent of the homogeneous space in which $\mathscr{L}$ acts. Theorem 1.14 is a direct consequence of Theorems 1.5 and 5.4.

The first results on the Fredholm property of the ordinary differential operator

$$\mathscr{L} = -\frac{d}{dt} + A(t)\colon C_b^1(\mathbb{R}, X) \subset C_b(\mathbb{R}, X) \to C_b(\mathbb{R}, X),$$

where $A \in C_b(\mathbb{R}, \operatorname{End} X)$ and $\dim X < \infty$, were proved by Mukhamadiev in [22] and in his D.Sc. Thesis [23]. He obtained results on the Fredholm property of $\mathscr{L}$ in terms of the limiting operators and limiting solutions to the homogeneous equation (1.2). The first results on the Fredholm property of $\mathscr{L}$ (which went unnoticed) were obtained by Isaenko [24] using the property of exponential dichotomy of a family of evolution operators. He proved that the operator $\mathscr{L}$ is Fredholm if and only if the family of evolution operators admits an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$. For the operator $\mathscr{L} = -d/dt + A(t)$ in the Banach space $C_b(\mathbb{R}, \mathbb{C}^n)$ the same result was obtained by Palmer [25] and in [26], where $\mathscr{L}$ was considered to act in the Hilbert space $L^2(\mathbb{R}, \mathbb{C}^n)$.

In the following two theorems we give necessary and sufficient conditions for the Fredholm property of the difference operator $\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X)$ defined by (1.8) and of the operator $\mathscr{L} = \mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$.

**Theorem 1.15.** *The operator* $\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X)$ *is Fredholm if and only if the family* $\mathscr{U} = \mathscr{U}_d\colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$ *satisfies Assumption 1.1 and the node operator* $\mathscr{N}_{b,a}\colon \operatorname{Im} Q_-(a) \to \operatorname{Im} Q_+(b)$ *is Fredholm. If these operators are Fredholm, then their indices coincide, as do the dimensions of their kernels and the codimensions of their ranges.*

**Theorem 1.16.** *The operator* $\mathscr{L} = \mathscr{L}_{\mathscr{U}}$ *is Fredholm if and only if the family of evolution operators* $\mathscr{U}: \Delta_{\mathbb{R}} \to \operatorname{End} X$ *satisfies Assumption 1.1 and the node operator* $\mathscr{N}_{b,a}\colon \operatorname{Im} Q_-(a) \to \operatorname{Im} Q_+(b)$ *is Fredholm.*

The fact that the conditions in Theorems 1.15 and 1.16 are sufficient was proved in [43] and [21]. This also follows from Theorems 1.12 and 1.13.

The proof that these conditions are necessary was initially given under certain additional assumptions (see [27] for a reflexive space $X$ and [17], where the scheme of the proof was presented). It was observed in [29] that these additional assumptions were superfluous.

In [29] the necessity in Theorem 1.16 was proved for the spaces $L^p(\mathbb{R}, X)$ with $p \in [1, \infty)$ and $C_0(\mathbb{R}, X)$. For $C_b(\mathbb{R}, X)$ Theorem 1.16 is new. The history relating to the Fredholm property of differential operators was described in greater detail in [19] and [27]–[29], where examples of Fredholm operators were also discussed.

**Corollary 1.2.** *If $\mathscr{D}$ (or $\mathscr{L}$) is a Fredholm operator in some homogeneous sequence (function) space, then it is Fredholm in any homogeneous space and its index (as well as the dimension of its kernel and the codimension of the range) is independent of the homogeneous space in which it acts.*

**Theorem 1.17.** *If a family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ satisfies Assumption 1.1 with $Q_- = 0$ and $Q_+ = 0$ (a trivial dichotomy holds on $\mathbb{R}_{-,a}$ and $\mathbb{R}_{b,+}$), then the operator $\mathscr{L}$ has a continuous inverse.*

A similar result holds for the operator $\mathscr{D}$. It is stated in Theorem 5.2. Theorem 1.17 is a consequence of Theorem 5.2 and Theorem 1.5.

**Corollary 1.3.** *For a family of evolution operators $\mathscr{U}: \Delta_{\mathbb{R}} \to \operatorname{End} X$ assume that the limits*

$$\lim_{t\to\pm\infty} \mathscr{U}(t, t-1) = \mathscr{B}_\pm \in \operatorname{End} X$$

*exist in the uniform operator topology and their spectral radii satisfy $r(\mathscr{B}_\pm) < 1$. Then the operator $\mathscr{L}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ under consideration has a continuous inverse in any homogeneous space $\mathscr{F}(\mathbb{R}, X)$.*

**Corollary 1.4.** *Let $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ be a family satisfying the assumptions of Theorem 1.17 and let $B \in C_0(\mathbb{R}, \operatorname{End} X)$. Then the operator $\mathscr{L} + \mathscr{B}$, where $\mathscr{B}$ is the operator of multiplication by the function $B$ in $\mathscr{F}(\mathbb{R}, X)$, has a continuous inverse.*

Possible applications to partial differential equations were rather thoroughly described in [8], [9], [19], [14]–[17], [27]–[29]. Applications to the Wiener–Hopf equations were discussed in [19] and [28]. In § 8 we look at several examples of operators to which the above results can be applied.

## 2. Homogeneous function spaces

**2.1. Main function spaces.** We consider several spaces of vector-valued functions on an interval $\mathbb{J} \in \{[a, b], (-\infty, a], [a, \infty), \mathbb{R}\}$ which are most useful for us. All of them are Banach spaces lying in the linear space $L^1_{\mathrm{loc}}(\mathbb{J}, X)$ of locally integrable (Bochner measurable) functions on $\mathbb{J}$ taking values in a Banach space $X$. Here are some of them:

$L^p = L^p(\mathbb{J}, X)$, $p \in [1, \infty]$, the (Banach) space of functions $x \in L^1_{\mathrm{loc}}(\mathbb{J}, X)$ with finite quantity

$$\|x\|_p = \left(\int_{\mathbb{J}} \|x(\tau)\|^p \, d\tau\right)^{1/p}, \quad p \neq \infty, \qquad \|x\|_\infty = \operatorname*{ess\,sup}_{\tau\in\mathbb{J}} \|x(\tau)\|, \quad p = \infty$$

(taken to be the norm in the corresponding space);

$S^p = S^p(\mathbb{J}, X)$, $p \in [1, \infty)$, the Stepanov space of functions $x \in L^1_{\mathrm{loc}}(\mathbb{J}, X)$ with finite quantity

$$\|x\|_{S^p} = \sup_{t\in\mathbb{J}} \biggl(\int_0^1 \|x(s+t)\|^p\,ds\biggr)^{1/p}$$

if $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, while if $\mathbb{J} = \mathbb{R}_{-a}$, then the interval of integration is $[a-1, a]$, and if $\mathbb{J} = \mathbb{R}_{a,+}$, then it is $[a, a+1]$;

$C_0 = C_0(\mathbb{J}, X)$, the subspace of functions $x$ in $C_b(\mathbb{J}, X)$ such that $\lim_{t\in\mathbb{J},|t|\to\infty} \|x(t)\| = 0$.

We now suppose that $\mathbb{J}_d$ is one of the sets $\mathbb{Z}_-$, $\mathbb{Z}_+$, and $\mathbb{Z}$, with $\mathbb{J}_d = \mathbb{Z}_-$ if $\mathbb{J} = \mathbb{R}_{-,a}$, $\mathbb{J}_d = \mathbb{Z}_+$ if $\mathbb{J} = \mathbb{Z}_{a,+}$, and $\mathbb{J}_d = \mathbb{Z}$ if $\mathbb{J} = \mathbb{R}$.

We introduce yet another family of Banach spaces (Wiener amalgam spaces):

$$L^{p,q} = L^{p,q}(\mathbb{J}, X), \qquad p = 0 \text{ or } p \in [1, \infty], \quad q \in [1, \infty],$$

where $\mathbb{J} \in \{\mathbb{R}_{-,a}, \mathbb{R}_{a,+}, \mathbb{R}\}$. The Banach space $L^{p,q}(\mathbb{R}, X)$ consists of the functions $x \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ described as follows. We assign to a function $x$ the sequence $(x_n)$, $n \in \mathbb{Z}$, where $x_n(s) = x(s+n)$ for $s \in [0, 1]$ and $x_n \in L^1([0, 1], X)$. A function $x$ will be put in the space $L^{p,q}(\mathbb{R}, X)$ if $x_n \in L^q([0, 1], X)$ for $n \in \mathbb{Z}$ and the sequence $(x_n)$ belongs to $\ell^p(\mathbb{Z}, L^q([0, 1], X))$. For $p = 0$ the sequence $(x_n)$ is assumed to belong to the space $c_0(\mathbb{Z}, L^q([0, 1], X))$. The norm of a function $x \in L^{p,q}(\mathbb{R}, X)$ is taken to be

$$\|x\|_{p,q} = \begin{cases} \|(x_n)\|_p = \biggl(\displaystyle\sum_{n=\infty}^{\infty} \|x_n\|_q^p\biggr)^{1/p} & \text{for } p \in [1, \infty),\ \ q \in [1, \infty], \\ \displaystyle\sup_{n\in\mathbb{Z}} \|(x_n)\|_q & \text{for } p \in \{0, \infty\},\ \ q \in [1, \infty]. \end{cases}$$

The spaces $L^{p,q}(\mathbb{J}, X)$ for $\mathbb{J} \in \{\mathbb{R}_{-,a}, \mathbb{R}_{a,+}\}$ are similarly defined, but for $x \in L^{p,q}(\mathbb{J}, X)$ the sequence $(x_n)$ is one-sided. Clearly, $L^{p,p}(\mathbb{J}, X) = L^p(\mathbb{J}, X)$ for $p \in [1, \infty]$, and the Banach space $L^{\infty,q}(\mathbb{J}, X)$ coincides with the Stepanov space $S^q(\mathbb{J}, X)$ for $q \in [1, \infty)$. If $X \in \{\mathbb{R}, \mathbb{C}\}$, then we drop the symbol $X$ from the notation for the spaces. For example, $L^{p,q}(\mathbb{J}) = L^p(\mathbb{J}, X)$.

**2.2. Homogeneous spaces of functions and sequences.** We now axiomatize the class of function spaces in which the linear operators act that are constructed from a family of evolution operators $\mathscr{U} \colon \Delta_{\mathbb{J}} \to \operatorname{End} X$ with $\mathbb{J} \in \{\mathbb{R}_{-,a}, \mathbb{R}_{a,+}, \mathbb{J}\}$ and $a \in \mathbb{R}$. Let $\mathbb{J}(n)$ denote an interval of the form

$$\mathbb{J}(n) = \begin{cases} [n, n+1),\ n \in \mathbb{Z} & \text{if } \mathbb{J} = \mathbb{R}, \\ [a+n, a+n+1),\ n \in \mathbb{Z}_+ & \text{if } \mathbb{J} = \mathbb{R}_{a,+}, \\ (a+n-1, a+n],\ n \in \mathbb{Z}_- & \text{if } \mathbb{J} = \mathbb{R}_{-,a}. \end{cases}$$

**Definition 2.1.** We call the Banach space $\mathscr{F}(\mathbb{J})$ of measurable real functions on $\mathbb{J}$ a *homogeneous space of measurable functions* if the following conditions hold:

1) $\mathscr{F}(\mathbb{J})$ lies in $L^1_{\mathrm{loc}}(\mathbb{J}) = L^1_{\mathrm{loc}}(\mathbb{J}, \mathbb{R})$, and $\displaystyle\int_b^{b+1} |x(\tau)|\,d\tau \leqslant c\|x\|_{\mathscr{F}(\mathbb{R})}$ for $x \in \mathscr{F}(\mathbb{R})$, where $c > 0$ if $[b, b+1] \subset \mathbb{J}$;

2) $\mathscr{F}(\mathbb{J})$ is a Banach lattice, which means that if $\psi \in \mathscr{F}(\mathbb{J})$ and $\varphi\colon \mathbb{J} \to \mathbb{R}$ is a measurable function such that $|\varphi(t)| \leqslant |\psi(t)|$ for almost all $t \in \mathbb{R}_+$, then $\varphi \in \mathscr{F}(\mathbb{J})$ and $\|\varphi\| \leqslant \|\psi\|$;

3) if $\mathbb{J} = \mathbb{R}$, then the shift operators $S(t) \in \operatorname{End}\mathscr{F}(\mathbb{J})$ of the form

$$(S(t)x)(\tau) = x(\tau + t), \qquad \tau, t \in \mathbb{J}, \quad x \in \mathscr{F}(\mathbb{J}), \tag{2.1}$$

are defined and bounded, while if $\mathbb{J} = \mathbb{R}_{a,+}$, then the operators of the form (2.1) are defined and bounded, and for $t < 0$ in $\mathbb{J}$ this also holds for the operators

$$(S_+(t)x)(\tau) = \begin{cases} x(\tau + t) & \text{if } \tau + t \geqslant a, \\ 0 & \text{if } a > \tau + t, \end{cases}$$

and finally, if $\mathbb{J} = \mathbb{R}_{-,a}$ and $t \leqslant 0$, then the operators (2.1) are defined and bounded, and for $t > 0$ in $\mathbb{J}$ this also holds for the operators

$$(S_-(t)x)(\tau) = \begin{cases} 0 & \text{if } \tau + t > a, \\ x(\tau + t) & \text{if } \tau + t \leqslant a, \end{cases}$$

furthermore, $\sup_{t\in\mathbb{R}} \|S(t)\| < \infty$ and $\sup_{t>0} \|S_-(t)\| < \infty$;

4) the characteristic function $\chi_{[b,b+1]}$ of the interval $[b, b+1]$ ($b = 0$ if $\mathbb{J} = \mathbb{R}$, $b = a$ if $\mathbb{J} = \mathbb{R}_{a,+}$, and $b = -a$ if $\mathbb{J} = \mathbb{R}_{-,a}$) belongs to $\mathscr{F}(\mathbb{J}, X)$, and for each $x \in \mathscr{F}(\mathbb{J})$ the function $\widetilde{x}\colon \mathbb{R} \to \mathbb{R}$ defined by

$$\widetilde{x}(t) = \|\chi_{\mathbb{J}(n)} x\|_{S^1(\mathbb{J})}, \qquad t \in \mathbb{J}(n), \quad n \in \mathbb{J}_d,$$

belongs to $\mathscr{F}(\mathbb{J})$ and $\|\widetilde{x}\| \leqslant \|x\|$.

We consider the linear subspace $\mathscr{F}_s(\mathbb{J})$ of functions in the homogeneous space $\mathscr{F}(\mathbb{J})$ which are constant on each interval $\mathbb{J}(n)$, $n \in \mathbb{J}_d$. Starting from this closed subspace $\mathscr{F}_s(\mathbb{J})$, we define the Banach space $\mathscr{F}_s(\mathbb{J}_d)$ of real sequences $x\colon \mathbb{J}_d \to \mathbb{R}$ such that the function $\widetilde{x}\colon \mathbb{J} \to \mathbb{R}$ defined by the equality $\widetilde{x}(t) = x(n)$ for $t \in \mathbb{J}(n)$ and $n \in \mathbb{J}_d$ belongs to $\mathscr{F}_s(\mathbb{J})$, and we let $\|x\| = \|\widetilde{x}\|_{\mathscr{F}(\mathbb{J})}$. We call this Banach sequence space $\mathscr{F}(\mathbb{J}_d)$ the *space associated with* $\mathscr{F}(\mathbb{J})$. By definition it has properties analogous to those of the homogeneous space $\mathscr{F}(\mathbb{J})$.

**Definition 2.2.** The *homogeneous space of measurable vector-valued functions corresponding to the homogeneous space* $\mathscr{F}(\mathbb{J})$ is defined to be the Banach space $\mathscr{F}(\mathbb{J}, X)$ of (Bochner) measurable functions $x\colon \mathbb{J} \to X$ such that the function $\overline{x}\colon \mathbb{J} \to \mathbb{R}$ with $\overline{x}(t) = \|x(t)\|$ for $t \in \mathbb{J}$ belongs to $\mathscr{F}(\mathbb{J})$, with the norm $\|x\| = \|\overline{x}\|_{\mathscr{F}(\mathbb{J})}$.

*Remark* 2.1. If $\mathscr{F}(\mathbb{J}, X)$ is the homogeneous space consisting of the functions corresponding to the homogeneous space $\mathscr{F}(\mathbb{J})$, then it has the corresponding properties 1)–4) in Definition 2.1. Let $\mathscr{F}(\mathbb{J}_d, X)$ be the Banach space of sequences $x\colon \mathbb{J}_d \to X$ with the following properties: the sequence $\widetilde{x}(n) = \|x(n)\|$, $n \in \mathbb{J}_d$, belongs to $\mathscr{F}(\mathbb{J}_d)$ and $\|x\|_{\mathscr{F}(\mathbb{J}_d,X)} = \|\overline{x}\|_{\mathscr{F}(\mathbb{J}_d)}$. We can define the Banach space $\mathscr{F}(\mathbb{J}_d, X)$ as the set of sequences of functions $x\colon \mathbb{J}_d \to X$ such that the function $y\colon \mathbb{J} \to X$ with $y(t) = x(n)$ for $t \in \mathbb{J}(n)$ and $n \in \mathbb{J}_d$ belongs to $\mathscr{F}(\mathbb{J}, X)$. Let $\|x\|_{\mathscr{F}(\mathbb{J}_d,X)} = \|y\|_{\mathscr{F}(\mathbb{J})}$. We call the resulting sequence space the *homogeneous sequence space* associated with $\mathscr{F}(\mathbb{J}, X)$. The pair of spaces $\mathscr{F}(\mathbb{J}, X)$, $\mathscr{F}(\mathbb{J}_d, X)$ is called an *associated pair*.

*Remark* 2.2. The Banach spaces

$$S^p(\mathbb{J},X),\quad p\in[1,\infty),\quad \text{and}\quad L^{p,q}(\mathbb{J},X),\quad p=0 \text{ or } p\in[1,\infty],\quad q\in[1,\infty],$$

are homogeneous spaces of measurable functions, and the corresponding associated sequence spaces are $\ell^\infty(\mathbb{J}_d,X)$ and $\ell^p(\mathbb{J}_d,X)$ for $p\neq 0$, respectively. For $p=0$ the space $L^{0,q}(\mathbb{J},X)$ is associated with $c_0(\mathbb{J}_d,X)$.

*Remark* 2.3. It follows immediately from Definition 2.2 that

$$L^{1,\infty}(\mathbb{J},X)\subset\mathscr{F}(\mathbb{J},X)\subset S^1(\mathbb{J},X)$$

for each homogeneous space $\mathscr{F}(\mathbb{J},X)$, and

$$\ell^1(\mathbb{J}_d,X)\subset\mathscr{F}(\mathbb{J}_d,X)\subset\ell^\infty(\mathbb{J}_d,X)$$

for the homogeneous sequence space $\mathscr{F}(\mathbb{J}_d,X)$ (associated with $\mathscr{F}(\mathbb{J},X)$).

*Remark* 2.4. If $\mathscr{F}(\mathbb{R},X)$ is a homogeneous function space, then for any $\alpha\neq 0$ and $\beta$ in $\mathbb{R}$ the function space $\mathscr{F}_{\alpha,\beta}(\mathbb{R},X)=\{y(t)=x(\alpha t+\beta), t\in\mathbb{R}\mid x\in\mathscr{F}(\mathbb{R},X)\}$ with the norm $\|y\|=\|x\|$ is also homogeneous.

In addition to homogeneous spaces of measurable functions we shall consider the Banach space of bounded continuous functions $C_b(\mathbb{J},X)$ and its subspace $C_0(\mathbb{J},X)=\{x\in C_b(\mathbb{J},\mathbb{X})\mid \lim_{|t|\to\infty}\|x(t)\|=0\}$.

We use the notation $\mathscr{F}(\mathbb{J},X)$ both for homogeneous spaces of measurable functions and for $C_b(\mathbb{J},X)$ or $C_0(\mathbb{J},X)$. The spaces associated with $C_b(\mathbb{J},X)$ and $C_0(\mathbb{J},X)$ are the sequence spaces $\ell^\infty(\mathbb{J}_d,X)$ and $c_0(\mathbb{J}_d,X)$, respectively. In [1] Krein introduced the notions of a homogeneous space of measurable functions $\mathscr{F}(\mathbb{R}_+,X)$ and a homogeneous space of vector sequences. Also in [1] he distinguished subspaces important for the investigation of the operators and relations under consideration, and obtained estimates for the norms of operators in homogeneous spaces of functions and sequences. By analogy, all these notions and results can be defined and proved in a natural way for homogeneous spaces of functions and sequences defined on $\mathbb{J}\in\{\mathbb{R}_{-,a},\mathbb{R}\}$ and $\mathbb{J}_d\in\{\mathbb{Z}_-,\mathbb{Z}\}$, respectively. We can use the norm estimates given in [1] in the proofs.

Let $\mathscr{F}(\mathbb{J},X)$ be a homogeneous space of measurable functions or one of the spaces $C_b(\mathbb{J},X), C_0(\mathbb{J},X)$. Let $\mathscr{F}_c(\mathbb{J},X)$ denote the minimal closed subspace of $\mathscr{F}(\mathbb{J},X)$ containing the functions with compact support in $\mathscr{F}(\mathbb{J},X)$. If $\mathscr{F}(\mathbb{J}_d,X)$ is a homogeneous sequence space, then $\mathscr{F}_c(\mathbb{J}_d,X)$ denotes the minimal subspace of $\mathscr{F}(\mathbb{J}_d,X)$ containing the finite sequences in $\mathscr{F}(\mathbb{J}_d,X)$. If $\mathscr{F}(\mathbb{J},X)=C_b(\mathbb{J},X)$, then let $\mathscr{F}_c(\mathbb{J},X)=C_0(\mathbb{J},X)$.

The homogeneous spaces $\mathscr{F}=L^{p,q}(\mathbb{J},X)$, $p=0$, $p\in[1,\infty)$, $q\in[1,\infty]$, and $\mathscr{F}=L^p(\mathbb{J},X)$, $p\in[1,\infty)$, have the property that $\mathscr{F}=\mathscr{F}_c$. The space $(S^p(\mathbb{J},X))_c$, $p\in[1,\infty)$, coincides with $L^{0,p}(\mathbb{J},X)$, and $(L^\infty(\mathbb{J},X))_c=L^{0,\infty}(\mathbb{J},X)$. In addition, note that $(\ell^p(\mathbb{J}_d,X))_c=(\ell^p(\mathbb{J}_d,X))$ for $p\in[1,\infty)$, and $(\ell^\infty(\mathbb{J}_d,X))_c=c_0(\mathbb{J}_d,X)$.

## 3. States of the operator $\mathscr{N}_a^+$ and the relation $\mathscr{D}_E^+$; proofs of Theorems 1.7–1.11

Most of the notions and results of the theory of linear relations used here can be found in [19] and [30] (see also the monographs [31] and [32]). For a linear relation

$\mathscr{A} \in LR(\mathscr{X},\mathscr{Y})$ the conjugate relation $\mathscr{A}^* \in LR(\mathscr{Y}^*,\mathscr{X}^*)$, is defined by

$$\mathscr{A}^* = \{(\eta,\xi) \in \mathscr{Y}^* \times \mathscr{X}^* \mid \langle y,\eta\rangle = \langle x,\xi\rangle \ \forall\,(x,y) \in \mathscr{A}\}.$$

The properties of the conjugate relation used here can be found in [19] and [30]–[38].

**3.1. Kernels and ranges of the operator $\mathscr{N}_a^+$ and the relation $\mathscr{D}_E^+$.** In the first part of this section we consider a discrete family of evolution operators $\mathscr{U} = \mathscr{U}_d \colon \Delta_{\mathbb{Z}_+} \to \operatorname{End} X$ satisfying the following assumption.

**Assumption 3.1.** *There exists an $a \in \mathbb{Z}_+$ such that the family $\mathscr{U}$ admits an exponential dichotomy on the set $\mathbb{Z}_{a,+}$ with splitting pair of projection-valued functions $P_+, Q_+ \colon \mathbb{Z}_{a,+} \to \operatorname{End} X$.*

In what follows we assume without loss of generality that $a = 0$, that is, we will look at the relation $\mathscr{D}_E^+$. Its kernel has the representation

$$\operatorname{Ker} \mathscr{D}_E^+ = \{x \in \mathscr{F}(\mathbb{Z}_+, X) \mid x(n) = \mathscr{U}(n,0)x_0,\ n \geqslant 0,\ x_0 \in E\}.$$

We also consider the bounded operator $B_d \colon X \to \mathscr{F}(\mathbb{Z}_+, X)$ given by

$$(B_d x)(n) = \begin{cases} \mathscr{U}(n,0)x, & 0 \leqslant n \leqslant a-1, \\ \mathscr{U}(n,0)P_+(a)x, & n \geqslant a, \end{cases} \qquad n \in \mathbb{Z}_+, \quad x \in X.$$

*Remark* 3.1. It follows immediately from the formula for $B_d$ that the sequence $B_d x$ belongs to the closed subspace $\operatorname{Ker} \mathscr{D}_E^+$ of $\mathscr{F}(\mathbb{Z}_+, X)$ if and only if the vector $x \in X$ lies in the subspace $\operatorname{Ker} \mathscr{N}_a^+$ of $X$, where the operator $\mathscr{N}_a^+$ is defined by (1.17).

Remark 3.1 has the following implication.

**Lemma 3.1.** *The operator $B_d \colon X \to \mathscr{F}(\mathbb{Z}_+, X)$ realizes an isomorphism of the subspaces $\operatorname{Ker} \mathscr{N}_a^+$ and $\operatorname{Ker} \mathscr{D}_E^+$.*

Lemma 3.1 and Remark 3.1 have the following consequence.

**Lemma 3.2.** *If $\operatorname{Ker} \mathscr{D}_E^+$ is a complemented subspace of $\mathscr{F}(\mathbb{Z}_+, X)$ and $\mathscr{F}_0$ is a closed complement of it, that is,*

$$\mathscr{F}(\mathbb{Z}_+, X) = \operatorname{Ker} \mathscr{D}_E^+ \oplus \mathscr{F}_0,$$

*then $\operatorname{Ker} \mathscr{N}_a^+$ is a complemented subspace of $X$ and*

$$X = \operatorname{Ker} \mathscr{N}_a^+ \oplus B_d^{-1}(\mathscr{F}_0).$$

In the description of the subspace $\operatorname{Im} \mathscr{D}_E^+$ we shall use the bounded linear operators $\Phi_a^0, \Phi_a \colon \mathscr{F}(\mathbb{Z}_+, X) \to X$ defined by the formulae

$$\Phi_a^0 x = Q_+(a) \sum_{k=0}^{a-1} \mathscr{U}(a,k)x(k) + \sum_{k \geqslant a+1} \mathscr{U}(a,k)Q_+(k)x(k),$$
$$\Phi_a x = \Phi_a^0 x + Q_+(a)x(a), \qquad x \in \mathscr{F}(\mathbb{Z}_+, X).$$

**Lemma 3.3.** *The following equalities hold for the range* $\operatorname{Im}\mathscr{D}_E^+$ *of the relation* $\mathscr{D}_E^+$:

$$\operatorname{Im}\mathscr{D}_E^+ = \{z \in \mathscr{F}(\mathbb{Z}_+, X) \mid z(n) = y(n),\ n \neq a,\ \text{and } z(a) = y_a - \Phi_a^0 y + \mathscr{N}_a^+ x_0 \text{ for some } y \in \mathscr{F}(\mathbb{Z}_+, X),\ y_a \in \operatorname{Im} P_+(a),\ \text{and } x_0 \in E\},$$
$$\operatorname{Im}\mathscr{D}_E^+ = \Phi_a^{-1}(\operatorname{Im}\mathscr{N}_a^+).$$

**3.2. Proofs of Theorems 1.7 and 1.8.** The fact that the relation $\mathscr{D}_E^+$ and the operator $\mathscr{N}_a^+$ have the same states (the equalities (1.16)) follows from Lemmas 3.1–3.3. Lemma 3.1 shows that the kernels of the relation $\mathscr{D}_E^+$ and the operator $\mathscr{N}_a^+$ have the same dimension. By Lemma 3.3 the subspaces $\operatorname{Im}\mathscr{D}_E^+$ and $\operatorname{Im}\mathscr{N}_a^+$ have the same codimension in $\mathscr{F}(\mathbb{Z}_+, X)$ and $\operatorname{Im} Q_+(a)$, respectively.

For the proof of Theorem 1.8 it is sufficient to cite Theorem 1.7, which we have already proved, and Theorem 1.3, which we will prove (independently) in § 4.

**3.3. Proofs of Theorems 1.9–1.11.** It follows from Theorem 5.7 in [19] that the operator $\mathscr{L}_E^+$ and the relation $\mathscr{D}_E^+$ have (or do not have) the Fredholm property simultaneously and their indices coincide. It is easy to see that if $\mathscr{D}_E^+$ is Fredholm, then the family $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{Z}_{a,+}$ for some $a \in \mathbb{Z}_+$. By [19], Lemma 6.2, $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{R}_{a,+}$. Equalities (1.19) follow from equality (1.18) in Theorem 1.8. The proof of Theorem 1.9 is complete.

Theorems 1.10 and 1.11 are immediate consequences of Theorem 1.9.

## 4. Proofs of Theorems 1.2, 1.3, 1.5

We consider an associated pair of homogeneous spaces $\mathscr{F}(\mathbb{R}, X)$, $\mathscr{F}(\mathbb{Z}, X)$ and the operators $\mathscr{L}\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$, $\mathscr{D} \in \operatorname{End}\mathscr{F}(\mathbb{Z}, X)$ defined in § 1 (the difference operator $D$ is defined by (1.8)) for a family of evolution operators $\mathscr{U}: \Delta_{\mathbb{R}} \to \operatorname{End} X$. We shall establish several lemmas and theorems concerning their kernels and ranges, and these will yield Theorem 1.5. The corresponding results for the operator $\mathscr{L}_E^+$ and relation $\mathscr{D}_E^+$ were proved in [19], § 5. Since the schemes of the proofs will be the same and some of the results for $\mathscr{L}$ and $\mathscr{D}$ (and a broad class of homogeneous spaces) were actually established in [14], some of the proofs here will be omitted or considerably abridged, and we shall refer to corresponding results in [14] and [19].

**4.1. Proofs of Theorems 1.2 and 1.5.** The proof of Theorem 1.2 is similar to the proof of Theorem 2 in [14]. The boundedness of the operators on $\mathscr{F}(\mathbb{R}, X)$ was actually shown in [19], § 2.

The proof of Theorem 1.5 uses the operator $B\colon \mathscr{F}(\mathbb{Z}, X) \to \mathscr{F}(\mathbb{R}, X)$ with $(Bx)(s) = -\varphi(s)\mathscr{U}(s, n-1)x(n-1)$ for $n \in \mathbb{Z}$ and $s \in [n-1, n]$ (where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is a periodic function with period 1 such that $\varphi(s) = 6s(1-s)$ for $s \in [0, 1]$), which realizes an isomorphism of the kernels $\operatorname{Ker}\mathscr{L}$ and $\operatorname{Ker}\mathscr{D}$ of the operators $\mathscr{L}$ and $\mathscr{D}$. For the description of the ranges of $\mathscr{L}$ and $\mathscr{D}$ we can use the bounded operators

$$\mathscr{B}\colon \mathscr{F}(\mathbb{Z}, X) \to \mathscr{F}(\mathbb{R}, X), \qquad \mathscr{B}x = -\varphi Bx, \quad x \in \mathscr{F}(\mathbb{Z}, X),$$
$$C\colon \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{Z}, X),$$
$$(Cy)(n) = -\int_{n-1}^{n} \mathscr{U}(n, \tau)y(\tau)\, d\tau, \qquad n \in \mathbb{Z}, \quad y \in \mathscr{F}(\mathbb{R}, X).$$

They are used to describe the ranges $\operatorname{Im}\mathscr{L}$ and $\operatorname{Im}\mathscr{D}$ by the same scheme as in Lemmas 5.7–5.12 and Theorems 5.1–5.6 of [19], where the operator $\mathscr{L}_E^+$ and the relation $\mathscr{D}_E^+$ were considered. From the analogues of these results we can immediately deduce Theorem 1.5, that is, the equalities (1.14) and (1.15).

### 4.2. Proof of Theorem 1.3.

*Remark* 4.1. The operator $\mathscr{L}_E^+\colon D(\mathscr{L}_E^+)\subset \mathscr{F}(\mathbb{R}_+,X)\to \mathscr{F}(\mathbb{R}_+,X)$ in Theorem 1.3 is not necessarily closed (a corresponding example can be found in [19]): it is closed if and only if its kernel $\operatorname{Ker}\mathscr{L}_E^+$ is closed. In this connection there arises the problem of interpreting property 3) in Definition 1.1 in the context of Theorem 1.3. There, in relation to the operator $\mathscr{L}_E^+$, this property is understood as follows: $\operatorname{Ker}\mathscr{L}_E^+$ is a complemented subspace either of $\mathscr{F}(\mathbb{R}_+,X)$ or of $D(\mathscr{L}_E^+)$ with the norm

$$\|x\|=\|x(0)\|_X+\|x\|_{\mathscr{F}}+\|\mathscr{L}_E^+x\|_{\mathscr{F}},\qquad x\in D(\mathscr{L}_E^+) \tag{4.1}$$

(see [19]).

The space $D(\mathscr{L}_E^+)$ with this norm is a Banach space. If $\mathscr{F}(\mathbb{R}_+,X)=C_b(\mathbb{R}_+,X)$, then $\mathscr{L}_E^+$ is a closed operator and the norm (4.1) in $D(\mathscr{L}_E)$ is equivalent to the graph norm in $D(\mathscr{L}_E^+)$. In the results below we consider the space $D(\mathscr{L}_E^+)$ with the norm (4.1). The subspace $\operatorname{Ker}\mathscr{L}_E^+$ is closed in $D(\mathscr{L}_E^+)$ with the norm (4.1).

In the proof of the next result we use the representation for functions in $D(\mathscr{L}_E^+)$ obtained in [19] (formula (5.4)). Let $\lambda_0\in\mathbb{C}$ be a complex number with $\operatorname{Re}\lambda_0>\alpha$, where $\alpha\in\mathbb{R}$ is taken from the definition of the family of evolution operators (see § 1). Then the operator $\mathscr{L}_{\{0\}}^+-\lambda_0 I$ has a continuous inverse, and each function $x\in D(\mathscr{L}_E^+)$ has a representation

$$\begin{aligned}x(t)&=\bigl((\mathscr{L}_{\{0\}}^+-\lambda_0 I)^{-1}(\mathscr{L}_E^+-\lambda_0 I)x\bigr)(t)+e^{-\lambda_0 t}\mathscr{U}(t,0)x(0)\\&=-\int_0^t e^{-\lambda_0(t-\tau)}\mathscr{U}(t,\tau)((\mathscr{L}_E^+-\lambda_0 I)x)(\tau)\,d\tau+e^{-\lambda_0 t}\mathscr{U}(t,0)x(0),\quad t\geqslant 0,\end{aligned} \tag{4.2}$$

where $x(0)\in E$. It follows from (4.2) that this is a consistent definition and the operator

$$J_+\colon D(\mathscr{L}_E^+)\to\mathscr{F}(\mathbb{Z}_+,X),\quad (J_+x)(n)=x(n),\quad n\in\mathbb{Z}_+,\quad x\in D(\mathscr{L}_E^+),$$

is bounded.

**Theorem 4.1.** *If* $\operatorname{Ker}\mathscr{D}_E^+$ *is a complemented subspace of* $\mathscr{F}(\mathbb{Z}_+,X)$ *and*

$$\mathscr{F}(\mathbb{Z}_+,X)=\operatorname{Ker}\mathscr{D}_E^+\oplus\mathscr{F}_0, \tag{4.3}$$

*where* $\mathscr{F}_0$ *is a closed subspace of* $\mathscr{F}(\mathbb{Z}_+,X)$, *then* $\operatorname{Ker}\mathscr{L}_E^+$ *is a complemented subspace of* $\mathscr{D}(\mathscr{L}_E^+)$ *(with the norm* (4.1)*) and*

$$D(\mathscr{L}_E^+)=\operatorname{Ker}\mathscr{L}_E^+\oplus J_+^{-1}(\mathscr{F}_0). \tag{4.4}$$

*Proof.* It follows from the definition of $J_+$ that it realizes an isomorphism between $\operatorname{Ker}\mathscr{L}_E^+$ and $\operatorname{Ker}\mathscr{D}_E^+$. If $y\in\operatorname{Ker}\mathscr{L}_E^+\cap J_+^{-1}(\mathscr{F}_0)$, then $J_+y\in\mathscr{F}_0$ and $J_+y\in\operatorname{Ker}\mathscr{D}_E^+$. Hence $y=0$. All this shows that the decomposition (4.4) follows from (4.3). □

In the next theorem we use the linear operator $\Phi_E\colon \mathscr{D}_E^+ \to D(\mathscr{L}_E^+)$ constructed in [19], § 5. As shown in Lemma 5.9 in [19],

$$\mathscr{L}_E^+\Phi_E(x_0,f)=\mathscr{B}^+f, \qquad (x_0,f)\in\mathscr{D}_E^+, \tag{4.5}$$

where $\mathscr{B}^+\colon \mathscr{F}(\mathbb{Z}_+,X)\to\mathscr{F}(\mathbb{R}_+,X)$ is the operator defined in the same way as $\mathscr{B}\colon \mathscr{F}(\mathbb{Z},X)\to\mathscr{F}(\mathbb{R},X)$.

**Theorem 4.2.** *Let* $\operatorname{Ker}\mathscr{L}_E^+$ *be a closed complemented subspace of* $D(\mathscr{L}_E^+)$ *with the norm* (4.1) *and let*

$$D(\mathscr{L}_E^+)=\operatorname{Ker}\mathscr{L}_E^+\oplus\widetilde{\mathscr{F}},$$

*where* $\widetilde{\mathscr{F}}$ *is a closed subspace of* $D(\mathscr{D}_E^0)$. *Then the Banach space* $\mathscr{F}(\mathbb{Z}_+,X)$ *has the direct sum decomposition*

$$\mathscr{F}(\mathbb{Z}_+,X)=\operatorname{Ker}\mathscr{D}_E^+\oplus\widetilde{\mathscr{F}}_d, \tag{4.6}$$

*where*

$$\widetilde{\mathscr{F}}_d=\mathscr{P}_1\mathscr{D}_E^0, \qquad \mathscr{D}_E^0=\Phi_E^{-1}(\widetilde{\mathscr{F}}),$$
$$\mathscr{P}_1\colon \mathscr{F}(\mathbb{Z}_+,X)\times\mathscr{F}(\mathbb{Z}_+,X)\to\mathscr{F}(\mathbb{Z}_+,X), \qquad \mathscr{P}_1(x_1,x_2)=x_1,$$
$$(x_1,x_2)\in\mathscr{F}(\mathbb{Z}_+,X)\times\mathscr{F}(\mathbb{Z}_+,X).$$

*Proof.* Equality (4.5) shows that the operator $\Phi_E\colon\mathscr{D}_E^+\to D(\mathscr{L}_E^+)$ is continuous. Since $\mathscr{D}_E^0$ is a closed subspace of $\mathscr{F}(\mathbb{Z}_+,X)\times\mathscr{F}(\mathbb{Z}_+,X)$, it follows from the definition of $\mathscr{P}_1$ that the subspace $\widetilde{\mathscr{F}}_d$ is closed. To prove (4.6) it is sufficient to observe that the domain of the relation $\mathscr{D}_E^+$ coincides with the whole of $\mathscr{F}(\mathbb{Z}_+,X)$, and for each pair $x_0$, $f\in\mathscr{F}(\mathbb{Z}_+,X)$ we have $\Phi_E(x_0,f)\in\operatorname{Ker}\mathscr{L}_E^+$ if and only if $x_0\in\operatorname{Ker}\mathscr{D}_E^+$. This equivalence follows from the construction of $\Phi_E$. □

*Proof of Theorem* 1.3. The fact that the operator $\mathscr{L}_E^+$ and the relation $\mathscr{D}_E^+$ simultaneously satisfy the same conditions in Definition 1.1 and, in particular, equalities (1.12) and (1.13) follow from [19], § 5 and from Theorems 4.1 and 4.2 proved in this paper. □

## 5. States of the operator $\mathscr{D}$ and the node operator $\mathscr{N}_{b,a}$; proof of Theorems 1.15 and 1.12

We consider the difference operator $\mathscr{D}\in\operatorname{End}\mathscr{F}(\mathbb{Z},X)$ defined by (1.8) and constructed from a bounded function $U\colon\mathbb{Z}\to\operatorname{End}X$. The corresponding family of evolution operators $\mathscr{U}_d\colon\Delta_{\mathbb{Z}}\to\operatorname{End}X$ is defined by (1.11).

Let $m\in\mathbb{N}$, $m\geqslant 2$. From $\mathscr{D}$ we construct an operator $\mathscr{D}_m\in\operatorname{End}\mathscr{F}(\mathbb{Z},X)$. It is constructed for the function $U_m\colon\mathbb{Z}\to\operatorname{End}X$ given by $U_m(n)=U(nm)U(nm-1)\cdots U(nm-m+1)=\mathscr{U}(nm,(n-1)m+1)$, $n\in\mathbb{Z}$, and $\mathscr{U}\colon\Delta_{\mathbb{Z}}\to\operatorname{End}X$ is the family of evolution operators of the form (1.11).

The operator $\mathscr{D}_m$ is defined by the formula

$$(\mathscr{D}_m x)(x)=x(n)-U_m(n)x(n-1), \qquad n\in\mathbb{Z}, \quad x\in\mathscr{F}(\mathbb{Z},X).$$

The family $\mathscr{U}_m$ of evolution operators corresponding to $U_m$ has the form $\mathscr{U}_m(n,k)=\mathscr{U}(mn,mk)$, $k\leqslant n$.

The next theorem is significant for the investigation of the operator $\mathscr{D}$.

**Theorem 5.1.** *The following equality holds*:

$$\mathrm{St}_{\mathrm{inv}}(\mathscr{D}_m) = \mathrm{St}_{\mathrm{inv}}(\mathscr{D}).$$

*Remark* 5.1. Let $\mathscr{F}(\mathbb{Z}_-, X)$ be a homogeneous sequence space. Then

$$\mathscr{F}(\mathbb{Z}_+, X) = \{y\colon \mathbb{Z}_+ \to X \mid y(k) = x(-k) \quad \text{for some } x \in \mathscr{F}(\mathbb{Z}_-, X), \\ k \in \mathbb{Z}_+, \text{ with } \|y\| = \|x\|_{\mathscr{F}(\mathbb{Z}_-, X)}\}$$

is a homogeneous sequence space.

In a similar way (using a reflection), for a given homogeneous function space $\mathscr{F}(\mathbb{R}_-, X)$ we define a homogeneous function space $\mathscr{F}(\mathbb{R}_+, X)$. We denote the isometries (reflections) constructed in the definition of $\mathscr{F}(\mathbb{Z}_+, X)$ and $\mathscr{F}(\mathbb{R}_+, X)$ by

$$\mathscr{V}_d\colon \mathscr{F}(\mathbb{Z}_-, X) \to \mathscr{F}(\mathbb{Z}_+, X) \quad \text{and} \quad \mathscr{V}\colon \mathscr{F}(\mathbb{R}_-, X) \to \mathscr{F}(\mathbb{R}_+, X).$$

Now we consider the difference operator $\mathscr{D}_E^-\colon D(\mathscr{D}_E^-) \subset \mathscr{F}(\mathbb{Z}_-, X) \to \mathscr{F}(\mathbb{Z}_-, X)$ (see formula (1.8)). A direct calculation shows that

$$\widetilde{\mathscr{D}}_E^+ = \mathscr{V}_d \mathscr{D}_E^- \mathscr{V}_d^{-1},$$

where $\widetilde{\mathscr{D}}_E^+\colon D(\widetilde{\mathscr{D}}_E^+) \subset \mathscr{F}(\mathbb{Z}_+, X) \to \mathscr{F}(\mathbb{Z}_+, X)$ is the operator

$$(\widetilde{\mathscr{D}}_E^+ x)(n) = x(n) - U(n+1)x(n+1), \qquad n \in \mathbb{Z}_+, \quad x \in D(\widetilde{\mathscr{D}}_E^+), \\ D(\widetilde{\mathscr{D}}_E^+) = \{x \in \mathscr{F}(\mathbb{Z}_+, X) \mid x(0) \in E\}.$$

Thus, the study of $\mathscr{D}_E^-$ can be reduced to the study of $\widetilde{\mathscr{D}}_E^+$ (these operators are similar, so that $\mathrm{St}_{\mathrm{inv}}(\mathscr{D}_E^-) = \mathrm{St}_{\mathrm{inv}}(\widetilde{\mathscr{D}}_E^+)$). The converse also holds.

The properties of the relation $\mathscr{D}_E^+$ (see formula (1.10)) were discussed in [39]–[42]. If $\mathscr{F} = \mathscr{F}(\mathbb{Z}, X) = \mathscr{F}_c(\mathbb{Z}, X)$, then the dual space $\mathscr{F}^*$ of $\mathscr{F}$ is identified with the dual′ space $\mathscr{F}' = \mathscr{F}'(\mathbb{Z}_-, X^*)$ (here we use Theorem 3.1 in [19]).[1]

We assume that $\mathscr{F} = \mathscr{F}(\mathbb{Z}_-, X) = \mathscr{F}_c(\mathbb{Z}_-, X)$, so that we can identify the dual space $\mathscr{F}(\mathbb{Z}_-, X)^*$ and the dual′ space $\mathscr{F}' = \mathscr{F}'(\mathbb{Z}_-, X^*)$ (by Theorem 3.1 in [19]). It follows from the definition of the conjugate relation that $(\mathscr{D}_E^-)^* \in LRC(\mathscr{F}')$ consists of the pairs $(\eta, \xi) \in \mathscr{F}' \times \mathscr{F}'$ such that

$$\langle y, \eta \rangle = \sum_{n \leqslant 0} \langle y(n), \eta(n) \rangle = \sum_{n \leqslant 0} \langle x(n), \xi(n) \rangle = \langle x, \xi \rangle$$

for all $x \in \mathscr{F}$ with $x(0) \in E$ such that $\mathscr{D}_E^- x = y$. Hence $(\mathscr{D}_E^-)^*$ can be defined by

$$(\mathscr{D}_E^-)^* = \{(\eta, \xi) \in \mathscr{F}' \times \mathscr{F}' \mid \xi(n) = \eta(n) - U(n+1)^* \eta(n+1), \\ n \leqslant -1, \ \xi(0) + \eta_0, \ \eta_0 \in E^\perp = F\}.$$

[1] *Editor's note.* Readers of the English version of [19] may be confused as a result of a questionable translation there. In the original Russian version of [19], as in the original Russian version of this paper, the author uses distinct terms to distinguish between the standard dual space and a certain subspace of that space defined in the context of sequence spaces, which we refer to here as the dual′ space. Unfortunately, the dual space and this subspace of it are not distinguished in the terminology of the English version of [19].

Clearly, $D((\mathscr{D}_E^-)^*) = \mathscr{F}'$ and $(\mathscr{D}_E^-)^*0 = \{\xi \in \mathscr{F}' \mid \xi(0) \in E^\perp,\ \xi(k) = 0,\ k \leqslant -1\}$. The operator $(\mathscr{D}_X^-)^*$ belongs to $\operatorname{End} \mathscr{F}'$. In the same way we prove that the conjugate relation to the operator $\widetilde{\mathscr{D}}_E^+$ (which is similar to $\mathscr{D}_E^-$) in Remark 5.3 below has the form

$$(\widetilde{\mathscr{D}}_E^+)^* = \{(\eta, \xi) \in \mathscr{F}'(\mathbb{Z}_+, X^*) \times \mathscr{F}'(\mathbb{Z}_+, X^*) \mid \xi(n) = \eta(n) - U(-n+1)^*\eta(n-1),\\ n \geqslant 1,\ \xi(0) = \eta(0) + \eta_0,\ \eta_0 \in E^\perp\}.$$

*Remark* 5.2. The relations $(\mathscr{D}_E^-)^*$ and $(\widetilde{\mathscr{D}}_E^+)^*$ are similar. They are related by the equality

$$(\widetilde{\mathscr{D}}_E^+)^* = \widetilde{\mathscr{V}}_d (\mathscr{D}_E^-)^* \widetilde{\mathscr{V}}_d^{-1},$$

where $\widetilde{\mathscr{V}}_d \colon \mathscr{F}'(\mathbb{Z}_-, X^*) \to \mathscr{F}'(\mathbb{Z}_+, X^*)$ is the operator $(\widetilde{\mathscr{V}}_d\xi)(n) = \xi(-n)$, $n \leqslant 0$, $\xi \in \mathscr{F}'(\mathbb{Z}_-, X^*)$. It is important to note that the relation $(\widetilde{\mathscr{D}}_E^+)^* \in LRC(\mathscr{F}'(\mathbb{Z}_+, X^*))$ coincides with the relation $\mathscr{D}_{E^\perp}^+$ constructed using the function $\widetilde{U}(n) = U(-n+1)^*$, $n \in \mathbb{Z}_+$, that is, it belongs to the class of difference relations investigated in § 4 and [19]. Hence we can use the results obtained there.

Thus, for the relation $\mathscr{D}_E^-$ we have the analogues of all the results obtained in [19] and in § 3 of the present paper.

**5.1. Proof of Theorem 1.15.** In [43] (see also [27]) our study of the operator $\mathscr{D}$ used the two difference operators $\mathscr{D}^- = D_X^{-,-1} \in \operatorname{End} \mathscr{F}(\mathbb{Z}_- \setminus \{0\}, X)$ and $\mathscr{D}^+ = \mathscr{D}_{\{0\}}^+ \in \operatorname{End} \mathscr{F}(\mathbb{Z}_+, X)$ defined by the equalities

$$(\mathscr{D}^- x)(n) = x(n) - U(n)x(n-1), \qquad n \leqslant -1, \quad x \in \mathscr{F}_- = \mathscr{F}(\mathbb{Z}_- \setminus \{0\}, X),$$

$$(\mathscr{D}^+ x)(n) = (\mathscr{D}_{\{0\}}^+ x)(n) = \begin{cases} x(0), & n = 0, \\ x(n) - U(n)x(n-1), & n \geqslant 1,\ x \in \mathscr{F}_+ = \mathscr{F}(\mathbb{Z}_+, X), \end{cases}$$

and also the operator

$$\mathscr{D}_0 \colon \mathscr{F}_- \to \mathscr{F}_+, \quad (\mathscr{D}_0 x)(n) = \begin{cases} U(0)x(-1), & n = 0, \\ 0, & n \geqslant 1, \end{cases} \qquad x \in \mathscr{F}_-.$$

The homogeneous spaces $\mathscr{F}_-$ and $\mathscr{F}_+$ are obtained from $\mathscr{F}(\mathbb{Z}, X)$ by restricting the sequences in $\mathscr{F}(\mathbb{Z}, X)$ to $\mathbb{Z}_- \setminus \{0\}$ and $\mathbb{Z}_+$, respectively. Thus, $\mathscr{F} = \mathscr{F}(\mathbb{Z}, X) = \mathscr{F}_- \times \mathscr{F}_+$. The spaces $\mathscr{F}_-$ and $\mathscr{F}_+$ can also conveniently be regarded as subspaces of $\mathscr{F}$, so that $\mathscr{F} = \mathscr{F}_- \oplus \mathscr{F}_+$. In either representation for $\mathscr{F}$ the operator $\mathscr{D}$ is given by the matrix

$$\begin{pmatrix} \mathscr{D}^- & 0 \\ \mathscr{D}_0 & \mathscr{D}^+ \end{pmatrix}. \tag{5.1}$$

We now use the following lemma from [44] (p. 23) relating to operators with matrix representation (5.2).

**Lemma 5.1.** *Let $\mathscr{X}$ be a Banach space that is a direct sum $\mathscr{X}_1 \oplus \mathscr{X}_2$ of two closed subspaces $\mathscr{X}_1$ and $\mathscr{X}_2$, and let $A \in \operatorname{End} \mathscr{X}$ be an operator with matrix representation*

$$\begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}, \qquad A_{11} \in \operatorname{End} \mathscr{X}_1, \quad A_{21} \in \operatorname{Hom}(\mathscr{X}_1, \mathscr{X}_2), \quad A_{22} \in \operatorname{End} \mathscr{X}_2. \tag{5.2}$$

*Then $A$ is Fredholm if and only if the following conditions hold*:

1) $\operatorname{Im} A_{11}$ *is a closed subspace of* $\mathscr{X}_1$ *and* $\operatorname{codim}\operatorname{Im} A_{11} < \infty$;

2) $\operatorname{Im} A_{22}$ *is a closed subspace of* $\mathscr{X}_2$ *and* $\dim\operatorname{Ker} A_{22} < \infty$;

3) $\mathscr{X}_1^0 = \{x \in \mathscr{X}_1 \mid x \in \operatorname{Ker} A_{11}$ *and* $A_{21}x \in \operatorname{Im} A_{22}\}$ *is a finite-dimensional subspace of* $\mathscr{X}_1$;

4) $\mathscr{X}_2^0 = \operatorname{Im} A_{22} + A_{21}(\operatorname{Ker} A_{11})$ *has finite codimension in* $\mathscr{X}_2$.

*If properties* 1)–4) *hold, then*

$$\dim\operatorname{Ker} A = \dim\operatorname{Ker} A_{22} + \dim \mathscr{X}_1^0, \qquad \operatorname{codim}\operatorname{Im} A = \operatorname{codim}\operatorname{Im} A_{11} + \operatorname{codim} \mathscr{X}_2^0.$$

*Proof of Theorem* 1.15. Let $\mathscr{D}$ be a Fredholm operator. We look at the space $E = U(0)X_0(-1)$, where $X_0(-1) = \{x(-1) \mid x \in \operatorname{Ker}\mathscr{D}_-\}$. Note that the subspace $\mathscr{D}(\operatorname{Ker}\mathscr{D}_-) \cap \mathscr{F}_+(\mathbb{Z}, X)$ is closed. It consists of the sequences $y \in \mathscr{F}_+(\mathbb{Z}, X)$ that have the form $y(k) = 0$, $k \geqslant 1$, $y(0) = -U(0)x(-1)$, $x(-1) \in X_0(-1)$. Therefore, $E$ is a closed subspace of $X$. It follows immediately from the definition of $E$ that

$$\operatorname{Im}\mathscr{D}_+ + \mathscr{D}_0(\operatorname{Ker}\mathscr{D}_-) = \operatorname{Im}\mathscr{D}_E^+.$$

We now see from property 4) in Lemma 5.1 that the range $\operatorname{Im}\mathscr{D}_E^+$ of the relation $\mathscr{D}_E^+ \in LRC(\mathscr{F}_+)$ has finite codimension in $\mathscr{F}_+$.

Next we prove that the kernel $\operatorname{Ker}\mathscr{D}_E^+$ of $\mathscr{D}_E^+$ is finite dimensional. Let $\widetilde{x} \in \operatorname{Ker}\mathscr{D}_E^+$. Then $\widetilde{x}(0) \in E$ and $\widetilde{x}(n) = \mathscr{U}(n,0)x_0$, $n \geqslant 0$. By the definition of $E$ the vector $\widetilde{x}(0)$ has a representation $\widetilde{x}(0) = U(0)x_0$ with $x_0 \in X_0(-1)$. Hence there exists a sequence $x \in \mathscr{F}_-$ such that $x \in \operatorname{Ker}\mathscr{D}_-$ and $x(-1) = x_0$.

We extend $\widetilde{x}$ onto $\mathbb{Z}_-$ as a sequence $y \in \mathscr{F}(\mathbb{Z}, X)$ by setting $y(n) = \widetilde{x}(n)$ for $n \geqslant 0$, and $y(n) = x(n)$ for $n \leqslant -1$. It follows from this definition that $y \in \operatorname{Ker}\mathscr{D}$. Therefore, $\dim\operatorname{Ker}\mathscr{D}_E^+ \leqslant \dim\operatorname{Ker}\mathscr{D} < \infty$.

Thus we have shown that $\mathscr{D}_E^+$ is a Fredholm relation. Hence there exists $m \in \mathbb{Z}_+$ such that the family $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{Z}_{m,+}$ with some splitting pair $P_+, Q_+ \colon \mathbb{Z}_{m,+} \to \operatorname{End} X$.

Passing if necessary to a similar operator $S(-m)\mathscr{D}S(m)$ defined by means of the operator-valued function $U_m(n) = U(n-m)$ for $n \in \mathbb{Z}$ (see Remark 5.3 in § 5.2), we now assume that $m = 0$.

Let us prove the Fredholm property of the operator $\mathscr{D}_F^{-,(-1)} = \mathscr{D}_{F,-}$ defined in $\mathscr{F}_-$ by formula (1.9), where $E = F = U(0)^{-1}\operatorname{Im} P_+(0)$, with domain

$$D(\mathscr{D}_{F,-}) = \{x \in \mathscr{F}_- \mid x(-1) \in U(0)^{-1}(\operatorname{Im} P_+(0)) = F\}.$$

We consider the subspace $\mathscr{F}_-^0$ of $\mathscr{F}(\mathbb{Z}, X)$ defined by

$$\mathscr{F}_-^0 = \{x \in \mathscr{F}(\mathbb{Z}, X) \mid x(k) = 0,\ k \geqslant 1,\ x(0) \in \operatorname{Ker} Q_+(0) = \operatorname{Im} P_+(0)\}.$$

We claim that $\operatorname{Im}\mathscr{D}_{F,-} = \{\widetilde{y} \in \mathscr{F} \mid \widetilde{y}$ is the restriction of some sequence $y$ in $\operatorname{Im}\mathscr{D} \cap \mathscr{F}_-^0\}$.

Let $y \in \operatorname{Im}\mathscr{D} \cap \mathscr{F}_-^0$. Then there exists a sequence $x \in \mathscr{F}(\mathbb{Z}, X)$ such that

$$y(n) = x(n) - U(n)x(n-1), \quad n \leqslant 0; \qquad y(n) = 0, \quad n \geqslant 1; \qquad y(0) \in \operatorname{Im} P_+(0).$$

Then $P_+(0)y(0) = y(0) = P_+(0)x(0) - P_+(0)U(0)x(-1)$. It follows from the equalities $x(n) = \mathscr{U}(n,0)x(0)$, $n \geqslant 0$, and the fact that $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{Z}_+$ that $x(0) \in \operatorname{Im} P_+(0)$. Since $y(0) \in \operatorname{Im} P_+(0)$, from the equality $x(0) - y(0) = U(0)x(-1)$ we get that $U(0)x(-1) \in \operatorname{Im} P_+(0)$, that is, $x(-1) \in F$.

Now let $y_- \in \operatorname{Im} \mathscr{D}_{F,-}$. Then there exists a sequence $x_- \in \mathscr{F}_-$ such that $x_-(-1) \in F$ and $y_-(n) = x_-(n) - U(n)x_-(n-1)$, $n \leqslant -1$. Let $x(n) = x_-(n)$ for $n \leqslant -1$ and $x(n) = U(n)x(n-1)$ for $n \geqslant 0$. Clearly, $\mathscr{D}x = y \in \mathscr{F}(\mathbb{Z}, X)$, where the restriction of the sequence to $\mathbb{Z}_- \setminus \{0\}$ is equal to $y_-$.

It follows from the above equality and Remark 5.1 that $\operatorname{Im} \mathscr{D}_{F,-}$ has finite codimension in $\mathscr{F}_-$ not exceeding $\operatorname{codim} \operatorname{Im} \mathscr{D}$.

We now prove that the kernel $\operatorname{Ker} \mathscr{D}_{F,-}$ of the operator $\mathscr{D}_{F,-}$ in question is finite dimensional. Let $x_0 \in \operatorname{Ker} \mathscr{D}_{F,-}$. Then $x_0(-1) = \mathscr{U}(0,n)x_0(n)$, $n \leqslant -1$, where $U(0)x_0(-1) \in \operatorname{Im} P_+(0)$. We consider the sequence $\widetilde{x}_0 \in \mathscr{F}(\mathbb{Z}, X)$ such that $\widetilde{x}_0(n) = x_0(n)$ for $n \leqslant -1$ and $\widetilde{x}_0(n) = \mathscr{U}(n,0)U(0)x(-1)$ for $n \geqslant 0$. Since $U(0)x(-1) \in \operatorname{Im} P_+(0)$, it follows that $\widetilde{x}_0$ decays exponentially as $n \to \infty$. Hence $\widetilde{x}_0 \in \mathscr{F}(\mathbb{Z}, X)$. We see from the construction of the sequence that $\widetilde{x}_0 \in \operatorname{Ker} \mathscr{D}$, so $\dim \operatorname{Ker} \mathscr{D}_{F,-} \leqslant \dim \operatorname{Ker} \mathscr{D} < \infty$ by the above.

Thus we have shown that $\mathscr{D}_{F,-}$ is a Fredholm operator. In view of Remark 5.1, $\mathscr{U}$ is a non-singular family on $-\infty$, that is, there exist $a, b \in \mathbb{Z}$, $a \leqslant b$, such that $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{Z}_{-,a}$ and $\mathbb{Z}_{b,+}$ with splitting pairs of projection-valued functions $P_-, Q_- \colon \mathbb{Z}_{-,a} \to \operatorname{End} X$ and $P_+, Q_+ \colon \mathbb{Z}_{b,+} \to \operatorname{End} X$. In this case it was proved in [43] for the operator $\mathscr{D}$ acting in any of the spaces $l^p(\mathbb{Z}, X)$ with $p \in [1, \infty]$ and $c_0(\mathbb{Z}, X)$ that the node operator $\mathscr{N}_{b,a} \colon \operatorname{Im} Q_-(a) \to \operatorname{Im} Q_+(a)$ defined by (1.20) is Fredholm. The proof in [43] also goes through without modification for the restriction of $\mathscr{D}$ to the subspace $\mathscr{F}_c(\mathbb{Z}, X)$, which is $\mathscr{D}$-invariant. The Fredholm property of $\mathscr{N}_{b,a}$ also follows from Theorem 1.12. From [43] and Theorem 1.12 we see that the conditions of this theorem are sufficient. (This also follows from Theorem 5.3 below.) $\square$

**5.2. Proof of Theorem 1.12.** Now (and throughout the rest of this section) we assume that the family $\mathscr{U}_d \colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$ is non-singular at $\pm\infty$, that is, the following assumption holds.

**Assumption 5.1.** *For some integers $a$ and $b$ with $a \leqslant b$ the family $\mathscr{U}_d$ admits an exponential dichotomy on $\mathbb{Z}_{-,a}$ and $\mathbb{Z}_{b,+}$ with splitting pairs of projection-valued functions $P_-, Q_- \colon \mathbb{Z}_{-,a} \to \operatorname{End} X$ and $P_+, Q_+ \colon \mathbb{Z}_{b,+} \to \operatorname{End} X$.*

We consider the node operator

$$\mathscr{N}_{b,a} \colon \operatorname{Im} Q_-(a) \to \operatorname{Im} Q_+(b), \qquad \mathscr{N}_{a,b}x = Q_+(b)\mathscr{U}(b,a)x, \quad x \in \operatorname{Im} Q_-(a).$$

The following remark shows that we can limit ourselves to the case $a = -1$, $b = 0$.

*Remark* 5.3. The operator $\bigl(S(b)\mathscr{D}S(-b)x\bigr)(n) = x(n) - U(n+b)x(n-1)$, $n \in \mathbb{Z}$, $x \in \mathscr{F}(\mathbb{Z}, X)$, is similar to $\mathscr{D}$, so

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{D}) = \operatorname{St}_{\mathrm{inv}}(S(b)\mathscr{D}S(-b)).$$

The family of evolution operators $\mathscr{U}_b$ constructed for the functions $U_b(n) = U(n+b)$, $n \in \mathbb{Z}$, $b \in \mathbb{Z}$, admits an exponential dichotomy on $\mathbb{Z}_-$ and $\mathbb{Z}_{m,+}$, where $m = b - a$, with splitting pairs $P_\pm(n+b)$, $Q_\pm(n+b)$, $n \in \mathbb{Z}$. We now apply Theorem 5.1 to the difference operator $S(b)\mathscr{D}S(-b)$, and conclude that it (and therefore also the operator $\mathscr{D}$) has the same set of invertibility states as the operator $\widetilde{\mathscr{D}} \in \operatorname{End}\mathscr{F}(\mathbb{Z}, X)$ constructed using the functions $\widetilde{U}_m(n) = U(mn+b)\cdots U(mn-m+b+1) = \mathscr{U}(mn+b, m(n-1)+b)$. The corresponding family of evolution operators $\widetilde{\mathscr{U}_d}$ has the form

$$\widetilde{\mathscr{U}_d}(n,k) = \mathscr{U}_d((b-a)n+b, (b-a)k+b), \qquad k \leqslant n, \quad k,n \in \mathbb{Z}. \tag{5.3}$$

The family (5.3) admits an exponential dichotomy on $\mathbb{Z}_{-,-1} = \mathbb{Z}_- \setminus \{0\}$ and $\mathbb{Z}_+$ with splitting pairs of projection-valued functions

$$\widetilde{P}_-, \widetilde{Q}_- \colon \mathbb{Z}_- \setminus \{0\} \to \operatorname{End} X, \qquad \widetilde{P}_+, \widetilde{Q}_+ \colon \mathbb{Z}_+ \to \operatorname{End} X,$$
$$\widetilde{P}_-(n) = P_-((b-a)n+b), \quad n \leqslant -1, \qquad \widetilde{P}_+(n) = P_+((b-a)n+b), \quad n \geqslant 1,$$
$$\widetilde{Q}_- = I - \widetilde{P}_-, \qquad \widetilde{Q}_+ = I - \widetilde{P}_+.$$

The node operator defined by the family $\widetilde{\mathscr{U}} \colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$, has the form

$$\widetilde{\mathscr{N}} \colon \operatorname{Im}\widetilde{Q}_-(-1) \to \operatorname{Im}\widetilde{Q}_+(0),$$
$$\widetilde{\mathscr{N}}x = \widetilde{Q}_+(0)\widetilde{\mathscr{U}_d}(0,-1)x = Q_+(b)\mathscr{U}_d(b,a)x = \mathscr{N}_{b,a}, \quad x \in \operatorname{Im} Q_-(a) = \operatorname{Im}\widetilde{Q}_-(0),$$

that is, it coincides with the node operator $\mathscr{N}_{b,a}$. Thus, $\mathrm{St}_{\mathrm{inv}}(\mathscr{D}) = \mathrm{St}_{\mathrm{inv}}(\widetilde{\mathscr{D}})$ and $\mathscr{N}_{b,a} = \widetilde{\mathscr{N}}$.

As follows from this observation, we can assume without loss of generality that this family $\mathscr{U}$ of evolution operators admits an exponential dichotomy on the sets $\mathbb{Z}_-\setminus\{0\}$ and $\mathbb{Z}_+$, with splitting pairs $P_-, Q_- \colon \mathbb{Z}_-\setminus\{0\} \to \operatorname{End} X$ and $P_+, Q_+ \colon \mathbb{Z}_+ \to \operatorname{End} X$ and with the node operator

$$\mathscr{N} \colon \operatorname{Im} Q_-(-1) \to \operatorname{Im} Q_+(0), \qquad \mathscr{N}x = Q_+(0)U(0)x, \quad x \in Q_-(-1),$$

where we take into account that $\mathscr{U}(0,-1) = U(0)$.

Precisely these conditions on $\mathscr{U}$, established with the help of the node operator, were assumed to hold in [43], in the investigation of a difference operator $\mathscr{D} \in \operatorname{End} l^p(\mathbb{Z}, X)$, $p \in [1,\infty]$. Since the proofs of most of the results in [43] also hold for the operator $\mathscr{D}$ in a homogeneous space, we use some of these results in what follows.

**Theorem 5.2.** *Suppose that Assumption 1.1 with $Q^{\pm} = 0$ holds for a family $\mathscr{U}_d \colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$. Then the operator $\mathscr{D}$ has a continuous inverse.*

*Proof.* By Remark 5.1 above we can limit ourselves to the case $a=-1$, $b=0$. We represent the matrix (5.1) of the operator $\mathscr{D} \in \mathscr{F}(\mathbb{Z}, X)$ as

$$\begin{pmatrix} \mathscr{D}^- & 0 \\ \mathscr{D}_0 & \mathscr{D}^+ \end{pmatrix} = \begin{pmatrix} \mathscr{D}^- & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ \mathscr{D}_0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & \mathscr{D}^+ \end{pmatrix}.$$

In this factorization of $\mathscr{D}$ each factor has a continuous inverse operator (by the conditions $Q_- = 0$ and $Q_+ = 0$). Hence so does $\mathscr{D}$. □

We consider the linear operator $K\colon \operatorname{Ker}\mathscr{N} \to \operatorname{Ker}\mathscr{D}$ defined by

$$(Kx_0)(n) = \begin{cases} \mathscr{U}_d(n,-1)x_0 = \mathscr{U}_d(n,-1)Q_-(-1)x_0, & n \leqslant -1, \\ \mathscr{U}_d(n,0)U(0)x_0, & n \geqslant 0, \end{cases} \qquad x_0 \in \operatorname{Ker}\mathscr{N}. \tag{5.4}$$

**Theorem 5.3.** *The operator $K$ is well defined and bounded, and it realizes an isomorphism between the subspaces $\operatorname{Ker}\mathscr{N}$ and $\operatorname{Ker}\mathscr{D}$.*

This is a consequence of Theorem 4 in [43]. Note that if $x_0 \in \operatorname{Ker}\mathscr{D}$, then because there is an exponential dichotomy for $\mathscr{U}$ on $\mathbb{Z}_- \setminus \{0\}$ and $\mathbb{Z}_+$, there exist constants $M_0 > 0$ and $q_0 \in (0,1)$ such that $\|x_0(n)\| \leqslant M_0 q_0^{|n|}$, $n \in \mathbb{Z}$. Therefore, the kernel $\operatorname{Ker}\mathscr{D}$ of the operator is independent of the choice of the homogeneous space $\mathscr{F}(\mathbb{Z}, X)$.

**Lemma 5.2.** *Let $\operatorname{Ker}\mathscr{N} \subset \operatorname{Im} Q_-(-1)$ be a complemented subspace of $\operatorname{Im} Q_-(-1)$, and let*

$$\operatorname{Im} Q_-(-1) = \operatorname{Ker}\mathscr{N} \oplus X_1, \tag{5.5}$$

*where $X_1$ is a closed subspace of $\operatorname{Im} Q_-(-1)$. Then*

$$\mathscr{F}(\mathbb{Z}, X) = \operatorname{Ker}\mathscr{D} \oplus \mathscr{F}_1, \tag{5.6}$$

*where $\mathscr{F}_1 = \{x \in \mathscr{F} \mid x(-1) \in \widetilde{X}_1\}$ and $\widetilde{X}_1 = X_1 \oplus \operatorname{Im} P_-(-1)$.*

*Proof.* It suffices to verify that the representation (5.6) follows from (5.5) and the representation (5.4) for sequences in $\operatorname{Ker}\mathscr{D}$.

We introduce the operator $\mathscr{A}\colon \mathscr{F}(\mathbb{Z}, X) \to \mathscr{F}(\mathbb{Z}, X)$ by

$$(\mathscr{A}x)(n) = \begin{cases} \mathscr{U}_d(n,-1)Q_-(-1)x(-1), & n \leqslant -1, \\ \mathscr{U}_d(n,0)P_+(0)U(0)Q_-(-1)x(-1), & n \geqslant 0, \end{cases} \qquad x \in \mathscr{F}(\mathbb{Z}, X).$$

It follows from the definition that $\mathscr{A} \in \operatorname{End}\mathscr{F}(\mathbb{Z}, X)$ and $\mathscr{A}^2 = \mathscr{A}$, that is, $\mathscr{A}$ is a projection. In particular, $\operatorname{Im}\mathscr{A}$ is a closed subspace of $\mathscr{F}(\mathbb{Z}, X)$. $\square$

**Lemma 5.3.** *The equality $\mathscr{A}x = x$ holds for each $x \in \operatorname{Ker}\mathscr{D}$.*

*Proof.* Let $x \in \operatorname{Ker}\mathscr{D}$. Then (5.4) shows that

$$x(n) = \begin{cases} \mathscr{U}_d(n,-1)Q_-(-1)x_0, & n \leqslant -1, \\ \mathscr{U}_d(n,0)U(0)x_0, & n \geqslant 0, \end{cases}$$

where $x_0 \in \operatorname{Ker}\mathscr{N}$ and

$$\begin{aligned} &(\mathscr{A}x)(n) = \mathscr{U}_d(n,-1)Q_-(-1)x_0, && n \leqslant -1, \\ &(\mathscr{A}x)(n) = \mathscr{U}_d(n,0)P_+(0)U(0)Q_-(-1)x_0 = \mathscr{U}_d(n,0)U(0)Q_-(-1)x_0, && n \geqslant 0. \end{aligned}$$

Hence $\mathscr{A}x = x$. $\square$

Assume that $\mathscr{F}(\mathbb{Z}, X)$ has a decomposition

$$\mathscr{F}(\mathbb{Z}, X) = \operatorname{Ker} \mathscr{D} \oplus \mathscr{F}_1, \tag{5.7}$$

where $\mathscr{F}_1$ is a closed subspace of $\mathscr{F}(\mathbb{Z}, X)$. Then we see from this decomposition and Lemma 5.2 that

$$\operatorname{Im} \mathscr{A} = \operatorname{Ker} \mathscr{D} \oplus \mathscr{F}_0. \tag{5.8}$$

Since $\mathscr{A} \operatorname{Ker} \mathscr{D} = \operatorname{Ker} \mathscr{D}$, it follows that $\mathscr{F}_0$ is a closed subspace of $\operatorname{Im} \mathscr{A}$. Let $A_0\colon \operatorname{Im} \mathscr{A} \to \operatorname{Im} Q_-(-1)$ be the operator defined by $A_0 x = x(-1) \in \operatorname{Im} Q_-(-1)$, $x \in \operatorname{Im} \mathscr{A}$. It follows from the representation for $\operatorname{Im} \mathscr{A}$ that $A_0$ is an isomorphism. Applying $A_0^{-1}$ to both sides of (5.8), we obtain

$$\operatorname{Im} Q_-(-1) = \operatorname{Ker} \mathscr{N} \oplus A_0^{-1}(\mathscr{F}_0). \tag{5.9}$$

Thus we have proved the following result.

**Lemma 5.4.** *Assume that the kernel of $\mathscr{D}$ is complemented in $\mathscr{F}(\mathbb{Z}, X)$ (the decomposition (5.7) holds). Then the kernel of the node operator $\mathscr{N}$ is complemented in* $\operatorname{Im} Q_-(-1)$ *and* (5.9) *holds.*

Let us return to the matrix representation (5.1) for $\mathscr{D}$ with respect to the decomposition $\mathscr{F}(\mathbb{Z}, X) = \mathscr{F}_- \oplus \mathscr{F}_+$. It shows that we have the following result (see the proof of Theorem 1.15 and Theorem 5.1).

**Lemma 5.5.** *The equalities*

$$\operatorname{Im} \mathscr{D}_- = \operatorname{Im} \mathscr{D} \cap \mathscr{F}_-,$$
$$\operatorname{Im} \mathscr{D}_+ + \mathscr{D}_0(\operatorname{Ker} \mathscr{D}_-) = \operatorname{Im} \mathscr{D}_E^+, \quad \mathscr{D}_E^+ \in LRC(\mathscr{F}_+)$$

*hold with $E = U(0)(\operatorname{Im} Q_-(-1))$. The subspace $\operatorname{Im} \mathscr{D}$ is closed if and only if the range $\operatorname{Im} \mathscr{D}_E^+$ of the relation $\mathscr{D}_E^+$ is closed.*

We note that under the assumptions of Lemma 5.5 the relation $\mathscr{D}_E^+$ is defined by (1.10), where $a = 0$ and the subspace $E$ of $X$ is not necessarily closed.

**Lemma 5.6.** *If $\operatorname{Im} \mathscr{D}_E^+$ is a closed subspace of $\mathscr{F}_+$, then the subspace $E \subset X$ is closed.*

*Proof.* Let $(x_n)$ be a sequence of vectors in $E$ converging to $x_0 \in X$. Then the sequence $\widetilde{x}_n \in \mathscr{F}_+$ with $\widetilde{x}_n(0) = x_n$ and $\widetilde{x}_n(k) = 0$ for $k \geqslant 1$ belongs to $\operatorname{Im} \mathscr{D}_E^+$ and converges to the vector $\widetilde{x}_0 \in \mathscr{F}_+$ with $\widetilde{x}_0(0) = x_0$ and $\widetilde{x}_0(k) = 0$ for $k \geqslant 1$. Hence $\widetilde{x}_0 \in \operatorname{Im} \mathscr{D}_E^+$, so that $x_0 \in E$. □

**Lemma 5.7.** *The following equivalence holds:*

$$\overline{\operatorname{Im} \mathscr{D}} = \operatorname{Im} \mathscr{D} \quad \Leftrightarrow \quad \overline{\operatorname{Im} \mathscr{N}} = \operatorname{Im} \mathscr{N}.$$

*Proof.* Let $\overline{\operatorname{Im} \mathscr{D}} = \operatorname{Im} \mathscr{D}$. Then it follows from Lemma 5.5 that $\operatorname{Im} \mathscr{D}_E^+$ is a closed subspace of $\mathscr{F}_+$, and $E$ is closed in $X$ by Lemma 5.6. Since the family $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{Z}_+$, it follows from Theorem 1.11 that $\mathrm{St}_{\mathrm{inv}}(\mathscr{D}_E^+) = \mathrm{St}_{\mathrm{reg}}(\operatorname{Im} P_+(0), E)$. Therefore, the subspace $\operatorname{Im} P_+(0) + E$ is closed (see property 4) in Definition 1.4). The fact that $\operatorname{Im}\mathscr{N}$ is closed follows from the representation of the subspace $\operatorname{Im} P_+(0) + E$ in the form

$$\begin{aligned}\operatorname{Im} P_+(0) + E &= \operatorname{Im} P_+(0) + \operatorname{Im}(U(0)Q_-(-1)) \\ &= \operatorname{Im} P_+(0) + \operatorname{Im}(P_+(0)U(0)Q_-(-1)) + \operatorname{Im}\mathscr{N} = \operatorname{Im} P_+(0) \oplus \operatorname{Im}\mathscr{N}.\end{aligned} \tag{5.10}$$

Conversely, let $\overline{\operatorname{Im}\mathscr{N}} = \operatorname{Im}\mathscr{N}$. Then $\operatorname{Im} P_+(0) + E$ is a closed subspace of $X$ by the same Theorem 1.11. Hence $\overline{\operatorname{Im}\mathscr{D}_E} = \operatorname{Im}\mathscr{D}_E = \operatorname{Im}\mathscr{D} \cap \mathscr{F}_+$. Since $\operatorname{Im}\mathscr{D} \cap \mathscr{F}_- = \mathscr{F}_-$, the subspace $\operatorname{Im}\mathscr{D}$ is closed in $\mathscr{F}(\mathbb{Z}, X)$. □

**Lemma 5.8.** *If* $\operatorname{Im}\mathscr{D}$ *is a closed subspace of* $\mathscr{F}(\mathbb{Z}, X)$ *and*

$$\mathscr{F}(\mathbb{Z}, X) = \operatorname{Im}\mathscr{D} \oplus \widetilde{\mathscr{F}}, \tag{5.11}$$

*where* $\widetilde{\mathscr{F}}$ *is a closed subspace, then* $\operatorname{Im}\mathscr{N}$ *is a closed complemented subspace of* $\operatorname{Im} Q_+(0)$.

*Proof.* That $\operatorname{Im}\mathscr{N}$ is closed follows from Lemma 5.7. Let $\mathscr{P}_-$, $\mathscr{P}_+$ be a pair of projections realizing a decomposition $\mathscr{F}(\mathbb{Z}, X) = \mathscr{F}_- \oplus \mathscr{F}_+$, which means that $\operatorname{Im}\mathscr{P}_\mp = \mathscr{F}_\mp$. Then

$$\mathscr{F}_+ = \operatorname{Im}\mathscr{D}_E^+ \oplus \mathscr{P}_+(\widetilde{\mathscr{F}}) \tag{5.12}$$

by Lemma 5.5 and (5.11), where $\mathscr{P}_+(\widetilde{\mathscr{F}})$ and $\operatorname{Im}\mathscr{D}_E^+$ are closed subspaces. Since the family $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{Z}_+$, it follows from Theorem 1.11 that $\mathrm{St}_{\mathrm{inv}}(\mathscr{D}_E^+) = \mathrm{St}_{\mathrm{reg}}(E, \operatorname{Im} P_+(0))$. Then (5.12) and property 6) in Definition 1.4 show that $E + \operatorname{Im} P_+(0)$ is a closed complemented subspace of $X$ (it is closed by Lemma 5.6). The representation (5.10) for $E + \operatorname{Im} P_+(0)$ shows that $\operatorname{Im}\mathscr{N}$ is complemented in $\operatorname{Im} Q_+(0)$. □

**Lemma 5.9.** *If* $\operatorname{Im}\mathscr{N}$ *is a closed complemented subspace of* $\operatorname{Im} Q_+(0)$, *then* $\operatorname{Im}\mathscr{D}$ *is closed and complemented in* $\mathscr{F}(\mathbb{Z}, X)$.

*Proof.* By Lemma 5.7 the subspace $\operatorname{Im}\mathscr{D}_E^+$ is closed, and therefore $E$ is a closed subspace of $X$. Since $\operatorname{Im}\mathscr{N}$ is complemented in $\operatorname{Im} Q_+(0)$, the subspace $E + \operatorname{Im} P_+(0)$ is closed and complemented in $X$ by equalities (5.10). Hence Theorem 1.11 gives us that $\operatorname{Im}\mathscr{D}_E^+$ is complemented in $\mathscr{F}_+$, and therefore $\operatorname{Im}\mathscr{D}$ is complemented in $\mathscr{F}(\mathbb{Z}, X)$ by Lemma 5.5. □

*Proof of Theorem* 1.12. For the kernels of $\mathscr{D}$ and $\mathscr{N}_{b,a}$ properties 1)–3) in Definition 1.1 hold (or do not hold) simultaneously by Theorem 5.1 and Lemmas 5.1 and 5.4. Lemmas 5.7, 5.8, and 5.9 show that the other properties in Definition 1.1 also hold simultaneously for our operators $\mathscr{D}$ and $\mathscr{N}_{b,a}$. □

**Theorem 5.4.** *Suppose that a family of evolution operators* $\mathscr{U}_d\colon \Delta_{\mathbb{Z}} \to \operatorname{End} X$ *satisfies Assumption* 1.2. *Then*

$$\mathrm{St}_{\mathrm{inv}}(\mathscr{D}) = \mathrm{St}_{\mathrm{reg}}(\operatorname{Im} Q_-(0), \operatorname{Im} P_+(0)).$$

*The operator* $\mathscr{D}$ *is (semi-)Fredholm if and only if the subspaces* $\operatorname{Im} Q_-(0), \operatorname{Im} P_+(0)$ *form a (semi-)Fredholm pair. If* $\mathscr{D}$ *is (semi-)Fredholm, then*

$$\begin{aligned}\dim \operatorname{Ker} \mathscr{D} &= \dim(\operatorname{Im} Q_-(0) \cap \operatorname{Im} P_+(0)),\\ \operatorname{codim} \operatorname{Im} \mathscr{L} &= \operatorname{codim}(\operatorname{Im} Q_-(0) + \operatorname{Im} P_+(0)),\\ \operatorname{ind} \mathscr{L} &= \operatorname{ind}(\operatorname{Im} Q_-(0), \operatorname{Im} P_+(0)).\end{aligned}$$

*Proof.* We have $a = b = 0$, so the node operator $\mathscr{N}$ has the form

$$\mathscr{N}\colon \operatorname{Im} Q_-(0) \to \operatorname{Im} Q_+(0), \quad \mathscr{N}x = Q_+(0)x, \quad x \in \operatorname{Im} Q_-(0).$$

All the results in this section which relate to the kernels and ranges of $\mathscr{D}$ and $\mathscr{N}$ are valid, therefore

$$\operatorname{St}_{\mathrm{inv}}(\mathscr{D}) = \operatorname{St}_{\mathrm{inv}}(\mathscr{N}).$$

It follows directly from the form of $\mathscr{N}$ and Definitions 1.1 and 1.4 that $\operatorname{St}_{\mathrm{inv}}(\mathscr{N}) = \operatorname{St}_{\mathrm{reg}}(\operatorname{Im} Q_-(0), \operatorname{Im} P_+(0))$. □

## 6. Almost periodicity criteria for solutions of differential equations

When we investigate the differential equation (1.1) with constant operator coefficient $A(t) \equiv A\colon D(A) \subset X \to X$ which is the generator of a $C_0$-semigroup $U\colon \mathbb{R}_+ \to \operatorname{End} X$ ([11], [12], [45], [46]), methods of harmonic analysis are especially important.

We give several definitions used in what follows.

Let $T\colon \mathbb{R} \to \operatorname{End} \mathscr{X}$ (where $\mathscr{X}$ is a complex Banach space) be a strongly continuous isometric representation. We treat the Banach space $L^1(\mathbb{R}) = L^1(\mathbb{R}, \mathbb{C})$ as a Banach algebra with convolution of functions as multiplication. The formula

$$fx = \int_{\mathbb{R}} f(\tau)T(-\tau)x\,d\tau, \qquad f \in L^1(\mathbb{R}), \quad x \in X,$$

endows the Banach space $\mathscr{X}$ with the structure of a Banach $L^1(\mathbb{R})$-module (see [46]–[48]). Let $\widehat{f}\colon \mathbb{R} \to \mathbb{C}$ denote the Fourier transform of a function $f \in L^1(\mathbb{R})$.

**Definition 6.1.** The *Beurling spectrum of a vector* $x$ in $\mathscr{X}$ is the set $\Lambda(x) = \{\lambda_0 \in \mathbb{R} \mid fx \neq 0$ for any $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\lambda_0) \neq 0\}$ of real numbers.

Below we use some properties of the Beurling spectrum of vectors in $\mathscr{X}$ (see [46], [47] for details).

**Definition 6.2.** Let $\lambda_0 \in \mathbb{R}$. A bounded net of functions $(f_\alpha)$ in $L^1(\mathbb{R})$ (where $\alpha$ ranges over a directed set $\Omega$) is called a $\lambda_0$-*net* if

1) $\widehat{f_\alpha}(0) = 1$ for each $\alpha \in \Omega$;
2) $\lim f_\alpha * f = 0$ for any $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\lambda_0) = 0$.

Here are examples of $\lambda_0$-nets in the algebra $L^1(\mathbb{R})$: $g_\alpha(t) = f_\alpha(t)\exp(i\lambda_0 t)$ and $\psi_\alpha(t) = \varphi_\alpha(t)\exp(i\lambda_0 t)$, $\alpha > 0$, where the 0-nets $(f_\alpha)$ and $(\varphi_\alpha)$ are defined by

$$f_\alpha(t) = \begin{cases} \alpha\exp(-\alpha t), & t \geqslant 0, \\ 0, & t < 0, \end{cases} \qquad \alpha > 0, \tag{6.1}$$

$$\varphi_\alpha(t) = \begin{cases} (2\alpha)^{-1}, & t \in [-\alpha, \alpha], \\ 0, & t \notin [-\alpha, \alpha], \end{cases} \qquad \alpha > 0. \tag{6.2}$$

Here for $(f_\alpha)$ the set $\Omega = (0, \infty)$ is oriented in the descending direction and for $(\varphi_\alpha)$ in the ascending direction.

**Definition 6.3.** A number $\lambda_0$ in the Beurling spectrum $\Lambda(x_0)$ of a vector $x_0 \in \mathscr{X}$ is called an *ergodic point in the spectrum of* $x_0$ if the limit $\lim f_\alpha x$ exists for some $\lambda_0$-net $(f_\alpha)$ in $L^1(\mathbb{R})$ (and therefore for any $\lambda_0$-net).

The set of ergodic points of $x_0 \in \mathscr{X}$ is often denoted by $\Lambda_{\mathrm{erg}}(x_0)$. We call the set

$$\Lambda_B(x) = \{\lambda_0 \in \Lambda_{\mathrm{erg}}(x) \mid \lim f_\alpha x = x_0 \neq 0 \\ \text{for some } \lambda_0\text{-net } (f_\alpha)\}$$

the *Bohr spectrum* of $x \in \mathscr{X}$. Since this limit is independent of the choice of the net $(f_\alpha)$ in $L^1(\mathbb{R})$ (see [47], Chap. II), it follows that

$$\begin{aligned}\lim_{\alpha\to\infty} 2\alpha^{-1}\int_{-\alpha}^{\alpha} T(s)e^{-i\lambda s}x\,ds &= \lim_{\varepsilon\to 0}\varepsilon\int_0^{\varepsilon^{-1}} e^{-\varepsilon\tau}T(\tau)e^{-i\lambda\tau}x\,d\tau \\ &= \lim_{0<\varepsilon\to 0}\varepsilon R(\varepsilon + i\lambda, iA)x, \end{aligned}\tag{6.3}$$

where $iA\colon D(A) \subset X \to X$ is the generator of the group $T\colon \mathbb{R} \to \operatorname{End}\mathscr{X}$ of operators under consideration. Note that $\sigma(A) \subset \mathbb{R}$ and that the nets used in (6.3) are defined by (6.1) and (6.2).

**Definition 6.4.** A vector $x \in \mathscr{X}$ is called an *almost-periodic vector* if its orbit $\{T(\tau)x,\ \tau \in \mathbb{R}\}$ is precompact in $\mathscr{X}$.

The set of almost-periodic vectors in $\mathscr{X}$ is a closed subspace of $\mathscr{X}$, which we denote below by $AP(\mathscr{X}) = AP(\mathscr{X}, T)$. Notions and results on almost-periodic vectors can be found in [47] and [48].

If $x \in AP(\mathscr{X})$, then let $x \sim \sum_{\lambda\in\Lambda_B(x)} x_\lambda$ denote the *Fourier series of the vector* $x$. The vector $x_\lambda$, $\lambda \in \Lambda_B(x)$, is defined by $x_\lambda = \lim f_\alpha x$ for some $\lambda$-net $(f_\alpha)$. We remark that the set $\Lambda_B(x)$ is finite or countable and

$$T(\tau)x_\lambda = \exp(i\lambda\tau)x_\lambda, \qquad \tau \in \mathbb{R}, \quad \lambda \in \Lambda_B(x). \tag{6.4}$$

If $\mathscr{X}_0$ is a closed submodule of $\mathscr{X}$ which is invariant under the operators $T(t)$, $t \in \mathbb{R}$, then the quotient space $\mathscr{X}/\mathscr{X}_0$ can also be endowed with a Banach module structure by means of the representation $\widetilde{T}(t)\widetilde{x} = \widetilde{T(t)}x = T(t)x + \mathscr{X}_0$, that is, $f\widetilde{x} = \widetilde{fx}$ for each $x \in \mathscr{X}$. The Beurling spectrum $\Lambda(\widetilde{x})$ of the equivalence class $\widetilde{x} = x + \mathscr{X}_0$ containing $x$ will be denoted by $\Lambda(x, \mathscr{X}_0)$. If $\mathscr{X}_0 = AP(\mathscr{X})$, then we call $\Lambda(x, \mathscr{X}_0)$ the *non-almost-periodicity set of the vector* $x \in \mathscr{X}$.

**Theorem 6.1** [48]. *For a vector $x \in \mathscr{X}$ assume that $\Lambda(x, AP(\mathscr{X}))$ is finite or countable. Then $x$ is almost periodic if and only if each limit point of $\Lambda(x, AP(\mathscr{X}))$ is ergodic for $x$. In particular, if $\Lambda(x, AP(\mathscr{X}))$ has no finite limit points, then $x \in AP(\mathscr{X})$.*

Now let $\mathscr{X} = C_{b,u}(\mathbb{R}, X)/C_0(\mathbb{R}, X)$. Acting on $\mathscr{X}$ is the strongly continuous group of isometric operators

$$T(t)\widetilde{x} = \widetilde{S(t)x} = S(t)x + C_0(\mathbb{R}, X), \qquad t \in \mathbb{R}.$$

Therefore, using the representation $T$, we can endow $\mathscr{X}$ with the structure of a Banach $L^1(\mathbb{R})$-module.

This module structure is defined by the formula

$$\varphi\widetilde{x} = \widetilde{\varphi * x}, \qquad (\varphi * x)(t) = \int_{\mathbb{R}} \varphi(t-s)x(s)\,ds, \quad t \in \mathbb{R}, \tag{6.5}$$

where $\varphi \in L^1(\mathbb{R})$, $x \in C_{b,u}$, and $\widetilde{x} = x + C_0(\mathbb{R}, \mathbb{X})$.

*Remark* 6.1. We can extend each $x_+ \in C_{b,u}(\mathbb{R}_+, X)$ to a function $x \in C_{b,u}(\mathbb{R}, X)$ so that $\lim_{t\to-\infty} x(t) = 0$. Then the (equivalence) class of $\widetilde{x} \in \mathscr{X}$ is independent of such an extension to $\mathbb{R}$, and thus the Banach space $C_{b,u}(\mathbb{R}_+, X)/C_0(\mathbb{R}_+, X)$ is isometrically embedded in $\mathscr{X} = C_{b,u}(\mathbb{R}, X)/C_0(\mathbb{R}, X)$ as a closed submodule, which we denote by $\mathscr{X}_+$. The group of operators $T(t)$, $t \in \mathbb{R}$, is well defined on $\mathscr{X}_+$.

Let $\mathbb{J}$ be one of the intervals $\mathbb{R}_+$ and $\mathbb{R}$.

**Definition 6.5.** A function $x \in C_{b,u}(\mathbb{J}, X)$ is said to be *almost periodic* at infinity (at $\infty$) if $\widetilde{x} \in AP(\mathscr{X})$ ($\widetilde{x} \in AP(\mathscr{X}_+)$ if $x \in C_{b,u}(\mathbb{R}_+, X)$). The set of functions in $C_{b,u}(\mathbb{J}, X)$ which are almost periodic at infinity will be denoted by $AP_\infty(\mathbb{J}, X)$.

**Definition 6.6.** A function $x \in C_{b,u}(\mathbb{J}, X)$ is *slowly varying* at infinity (at $\infty$) if $S(t)x - x \in C_0(\mathbb{J}, X)$ for each $t \in \mathbb{J}$.

The set of functions in $C_{b,u}(\mathbb{J}, X)$ slowly varying at $\infty$, which we denote by $C_{s\ell,\infty}(\mathbb{J}, X)$, is a closed subspace. One example of a function in $C_{s\ell,\infty}(\mathbb{R}, \mathbb{C})$ is $x(t) = \sin\ln(1+t^2)$, $t \in \mathbb{R}$.

From the definition of the Beurling spectrum of a vector and Definition 6.6 we obtain the following result.

**Lemma 6.1.** *A function $x \in C_{b,u}(\mathbb{J}, X)$ is slowly varying at infinity if and only if one of the following two equivalent conditions is fulfilled*: 1) $\Lambda(\widetilde{x}) = \{0\}$; 2) $f\widetilde{x} = \widehat{f}(0)\widetilde{x}$ *for each* $f \in L^1(\mathbb{R})$, *where $\widetilde{x}$ is the class in $\mathscr{X}$ containing $x$.*

We note that the subspace $C_{s\ell,\infty}(\mathbb{R}_+, X)$ coincides with the set of functions in $C_{b,u}(\mathbb{R}_+, X)$ which are stationary at infinity (see the definition in [7], Chap. III, § 6).

**Definition 6.7.** Let $x \in AP_\infty(\mathbb{J}, X)$ and let

$$\widetilde{x} \sim \sum_{n\geqslant 1} y_n, \quad \Lambda_B(\widetilde{x}) = \{\lambda_1, \lambda_2, \dots\}, \quad \Lambda(y_n) = \{\lambda_n\},$$

be a Fourier series of a class $\widetilde{x} \in AP(\mathscr{X})$. Then the series

$$x \sim \sum_{n \geqslant 1} x_n, \tag{6.6}$$

where $x_n$ is a representative of the class $y_n \in AP(\mathscr{X})$, is called a *Fourier series of* $x$.

We note that for a function $x \in AP_\infty(\mathbb{J}, X)$ its Fourier series (6.6) is not unique. It follows from (6.5) and Lemma 6.1 that each function $x_n$, $n \geqslant 1$, has a representation $x_n(t) = x_n^0(t)\exp(i\lambda_n t)$, $t \in \mathbb{R}$, where $x_n^0 \in C_{s\ell,\infty}(\mathbb{J}, X)$.

Hence the following result holds (here we use Theorem 3.67 in [47]).

**Theorem 6.2.** *A function $x \in C_{b,u}(\mathbb{J}, X)$ is almost periodic at $\infty$ if and only if for each $\epsilon > 0$ there exist functions $x_1, \dots, x_n \in C_{b,u}(\mathbb{J}, X)$ representable as $x_k(t) = x_k^0(t)\exp(i\lambda_k t)$, $t \in \mathbb{R}$, with $\lambda_k \in \mathbb{R}$ and $x_k^0 \in C_{s\ell,\infty}(\mathbb{R}, X)$ such that*

$$\sup_{t \in \mathbb{J}} \left\| x(t) - \sum_{k=1}^{n} x_k^0(t)\exp(i\lambda_k t) \right\| < \epsilon.$$

We pass to the differential equation (1.1) with function $f \in AP_\infty(\mathbb{J}, X)$ and constant operator coefficient $A(t) \equiv A$, $t \in \mathbb{J}$, which is the generator of a $C_0$-semigroup $U \colon \mathbb{R}_+ \to \operatorname{End} X$. By a bounded (weak) solution of this equation we mean a function $x \in C_b(\mathbb{J}, X)$ satisfying

$$x(t) = U(t-s)x(s) - \int_s^t U(t-\tau)f(\tau)\,d\tau, \qquad t, s \in \mathbb{J}, \quad s \leqslant t, \tag{6.7}$$

so that $x \in D(\mathscr{L})$ with $\mathscr{L} = -d/dt + A$ if $\mathbb{J} = \mathbb{R}$, and $x \in D(\mathscr{L}_X^+)$ if $\mathbb{J} = \mathbb{R}_+$. It follows from (6.7) that $x \in C_{b,u}(\mathbb{J}, X)$. Consequently, the subspace $C_{b,u}(\mathbb{J}, X)$ is invariant under the operator $\mathscr{L}$ (if $\mathbb{J} = \mathbb{R}$) or $\mathscr{L}_X^+$ (if $\mathbb{J} = \mathbb{R}_+$). In what follows we take the restrictions of these operators to $C_{b,u}(\mathbb{J}, X)$, denoting them by the same symbols $\mathscr{L}$ and $\mathscr{L}_X$.

**Theorem 6.3.** *For each solution $x_0 \in C_{b,u}(\mathbb{J}, X)$ of the differential equation (1.1) with $f \in AP_\infty(\mathbb{J}, X)$,*

$$\Lambda(\widetilde{x}_0, AP_\infty(\mathbb{J}, X)) \subset \sigma(A) \cap i\mathbb{R}. \tag{6.8}$$

*The function $x_0$ is almost periodic at infinity if $\sigma(A) \cap i\mathbb{R}$ is finite or countable and each limit point of the set $\Lambda(\widetilde{x}_0, AP_\infty(\mathbb{J}, X))$ is ergodic for the class $x_0 + C_0(\mathbb{J}, X)$. In particular, $x_0 \in AP_\infty(\mathbb{J}, X)$ if the set $\sigma(A) \cap i\mathbb{R}$ has no finite limit points.*

*Proof.* First let $\mathbb{J} = \mathbb{R}$. We choose $\lambda_0 \in \mathbb{R}$ such that $\lambda_0 > \alpha$, where $\alpha$ is the number in the definition of the family of evolution operators (see § 1). Then the operator $\mathscr{L} - \lambda_0 I$ has a continuous inverse, and this inverse $B = (\mathscr{L} - \lambda_0 I)^{-1} \in \operatorname{End} C_{b,u}(\mathbb{R}, X)$ has a representation $Bx = G_0 * x$, $x \in C_{b,u}(\mathbb{R}, X)$, where $G_0(\tau) = U(\tau)\exp(-\lambda_0\tau)$ for $\tau \geqslant 0$ and $G_0(\tau) = 0$ for $\tau < 0$. Thus, $B$ commutes with the convolution operators:

$$B(\varphi * x) = \varphi * Bx, \qquad \varphi \in L^1(\mathbb{R}), \quad x \in C_{b,u}(\mathbb{R}, X).$$

Therefore, $\mathscr{L}$ has the same property. In particular, $\varphi * x_0 \in D(\mathscr{L})$ and

$$\mathscr{L}(\varphi * x_0) = \varphi * f \in AP_\infty(\mathbb{R}, X) \tag{6.9}$$

for each $\varphi \in L^1(\mathbb{R})$. Let $i\mu \notin \sigma(A) \cap i\mathbb{R}$, where $\mu_0 \in \mathbb{R}$. Then the resolvent of $A$ is defined in a neighbourhood $iV_0 \subset i\mathbb{R}$ of the point $i\mu_0$. Let $[\mu_0 - \delta, \mu_0 + \delta_0] \subset V_0$, where $\delta_0 > 0$. We take an infinitely differentiable function $\widehat{\varphi}_0 \colon \mathbb{R} \to \mathbb{C}$ such that $\widehat{\varphi}_0(\mu_0) \neq 0$ and $\operatorname{supp} \widehat{\varphi}_0 \subset [\mu_0 - \delta, \mu_0 + \delta]$. It is the Fourier transform of some $\varphi_0 \in L^1(\mathbb{R})$, and the function

$$\widehat{F}(\lambda) = \begin{cases} \widehat{\varphi}_0(\lambda) R(i\lambda, A), & \lambda \in [\mu_0 - \delta, \mu_0 + \delta], \\ 0, & \lambda \notin [\mu_0 - \delta, \mu_0 + \delta], \end{cases}$$

is the transform of some integrable function $F \colon \mathbb{R} \to \operatorname{End} X$. Then from (6.9) we obtain

$$F * \mathscr{L}(\varphi * x_0) = \varphi_0 * x_0 = F * \varphi * f \in AP_\infty(\mathbb{R}, X).$$

Therefore, it follows from the definition of the non-almost-periodicity set that $\mu_0 \notin \Lambda(\widetilde{x}_0, AP_\infty(\mathbb{R}, X))$.

Thus, we have proved (6.8) for functions on $\mathbb{R}$.

Now let $\mathbb{J} = \mathbb{R}_+$, so that $\mathscr{L}_X^+ x_0 = f$. We take a continuously differentiable function $\varphi \colon \mathbb{R} \to \mathbb{R}$ such that $\operatorname{supp} \varphi \subset [1, \infty)$ and $\varphi \equiv 1$ on $[2, \infty)$. In what follows, $\varphi x_0$ will be a function vanishing on $\mathbb{R}_-$ and equal to the product of $\varphi$ and $x_0$ on $\mathbb{R}_+$. Then $\varphi x_0 \in D(\mathscr{L})$ and $\mathscr{L}(\varphi x_0) = \dot{\varphi} x_0 + \varphi f$, where $\varphi f$ is set equal to zero on $\mathbb{R}_-$. We have thus reduced the case $\mathbb{J} = \mathbb{R}_+$ to the previous case. □

**Theorem 6.4.** *Let $U \colon \mathbb{R}_+ \to \operatorname{End} X$ be a uniformly bounded semigroup, let $\sigma(A) \cap i\mathbb{R}$ be a finite or countable set, and let $x_0 \in X$. The function $x \colon \mathbb{R}_+ \to X$, $x(t) = U(t)x_0$, is almost periodic at infinity if the limit $\lim_{0<\epsilon\to 0} \epsilon R(\epsilon + i\lambda_0, A)x_0$ exists at each limit point $i\lambda_0$ of the set $\sigma(A) \cap i\mathbb{R}$.*

*Proof.* This follows from Theorems 6.3 and 6.2, with $g_\alpha(t) = f_\alpha(t)\exp(i\lambda_0 t)$, $\alpha > 0$, as the $\lambda_0$-net, where $(f_\alpha)$ is the net in (6.1). □

**Theorem 6.5.** *Let $U$ be a uniformly bounded semigroup and let $\sigma(A) \cap i\mathbb{R}$ be a finite or countable set. Then $U$ is strongly stable if and only if one of the following conditions is fulfilled:*

1) $\lim_{0<\epsilon\to 0} \epsilon R(i\lambda_0 + \epsilon, A)x_0 = 0$ *for each vector $x_0 \in X$ and any $i\lambda_0 \in \sigma(A) \cap i\mathbb{R}$;*
2) *the conjugate operator $A^*$ has no eigenvalues in $i\mathbb{R}$.*

*Proof.* This result, obtained in [49] and [50], is a consequence of the ergodic theorem stating that conditions 1) and 2) are equivalent (see, for instance, Theorem 2.2.6 in [47]). □

## 7. Comments on the main notions and some of the results in §§ 2–6

First we point out several additional results. In [51]–[58] the authors investigated difference and differential inclusions and gave a description of the spectra of differential and difference operators in weighted spaces (see [54]–[58]). Estimates for the norm of the inverse operator of $\mathscr{L} = \mathscr{L}_u$ and applications to the proof of the Gearhart–Prüss theorem were presented in [59]–[72]. The method of similar operators was applied to the splitting of differential operators in [73]–[80].

**7.1. On the choice of spaces and terminology.** The spaces $L^{p,q}=L^{p,q}(\mathbb{R},\mathbb{C})$, $1\leqslant p,\ q\leqslant\infty$, were first considered in [81], where they were called 'amalgams' of $L^q$ and $\ell^p$ (and denoted by $(L^q,\ell^p)$). The idea of such spaces was proposed by Wiener in the following cases: the spaces $L^{2,1}$ and $L^{\infty,2}$ were defined in [82] and the spaces $L^{1,\infty}$ and $L^{\infty,1}$ were defined in [83]. For this reason authors often call them Wiener amalgam spaces. The space $S^p(\mathbb{R},\mathbb{C})$, $p\in[1,\infty)$, which is isomorphic to $L^{\infty,p}(\mathbb{R},\mathbb{C})$, was used by Stepanov [84] in his definition of the space of almost-periodic functions $AP(S^p)$. The paper [85] contains several interesting results on harmonic analysis in the spaces $L^{p,q}$ and investigates their dual spaces.

If $X$ is a Hilbert space, then the Hilbert space $L^2(\mathbb{R},X)=L^2$ can be quite useful. For instance, if $\mathscr{L}=-d/dt+A\colon D(\mathscr{L})\subset L^2\to L^2$, then Theorem 10.2 in [59] gives the precise value of the norm of $\mathscr{L}^{-1}$, which enabled us in [59], § 10 to give explicit estimates for the quantities $\|\mathscr{L}^{-1}\|_p$ in all the spaces $L^p(\mathbb{R},X)$ and $S^p(\mathbb{R},X)$ with $p\in[1,\infty]$.

The first result on the equivalence of the property of continuous invertibility for an ordinary differential operator $\mathscr{L}=-d/dt+A(t)\colon D(\mathscr{L})\subset L^2(\mathbb{R},\mathbb{C}^n)\to L^2(\mathbb{R},\mathbb{C}^n)$ with $A\in C_b(\mathbb{R},\operatorname{End}\mathbb{C}^n)$ to the property of exponential dichotomy for the family of evolution operators $\mathscr{U}\colon\Delta_{\mathbb{R}}\to\operatorname{End}\mathbb{C}^n$ constructed for $A$ (for the differential equation (1.2)) was obtained in [26] (see also [86]).

In [87] the differential equation (1.1) was considered in general function spaces like the homogeneous space $\mathscr{F}(\mathbb{R}_+,X)$. If $X$ is a non-reflexive Banach space, then $L^p(\mathbb{R},X)^*$, $p\in[1,\infty)$, is not isomorphic to the space $L^q(\mathbb{R},X^*)$ with $q^{-1}+p^{-1}=1$ (see [88]). Hence there is a problem regarding the use of conjugate operators for the analysis of the differential operators in question.

The use of difference operators and relations made it possible to apply the machinery of conjugate operators and relations. In [10] Henry used a family of difference operators to study a differential operator. In [89] a (single) difference operator $\mathscr{D}\in\operatorname{End}\mathscr{F}(\mathbb{Z},X)$ was used to investigate the correctness of the operator $\mathscr{L}=\mathscr{L}_u\colon D(\mathscr{L})\subset\mathscr{F}(\mathbb{R},X)\to\mathscr{F}(\mathbb{R},X)\in\{C_b(\mathbb{R},X),S^p(\mathbb{R},X)\}$.

In almost all the author's papers cited, starting with [90], [13], and [14], the analysis of differential operators was carried out with the help of corresponding difference operators and difference relations (see also [91], [27], [29], [92], [54]–[58]). The spectral theory of linear relations was used in the process. Difference relations were first used in [19], where we introduced the relation $\mathscr{D}_E^+\in LRC(\mathscr{F}(\mathbb{Z}_+,X))$ to investigate the operator $\mathscr{L}_E^+\in LRC(\mathscr{F}(\mathbb{R}_+,X))$.

It should be pointed out that in the proof of Theorem 1.15, which is based on the representation (8.1), we make essential use of the relation $\mathscr{D}_E^+$ for a suitable subspace $E$ and of the difference operator $\mathscr{D}_{F,-}$, whose domain is not dense if $F\neq X$.

Definitions 1.1–1.5 were significant for the statements of the central results in this paper. The first of these definitions, in a very similar form, was given in [19] and the monographs [31] (Definition III.6.1) and [34].

We remark further that the spectral theory of linear relations is also used in solvability problems and the construction of solutions to the Cauchy problem with $x(0)=x_0$ for the differential equation

$$F\dot{x}(t)=Gx(t),\qquad t\geqslant 0,$$

where $F, G \in \operatorname{Hom}(\mathscr{X}, \mathscr{Y})$ ($\mathscr{X}$ and $\mathscr{Y}$ are Banach spaces) with $\operatorname{Ker} F \neq \{0\}$. It was shown in [30] that the Cauchy problem for this equation is solvable if and only if the Cauchy problem with $x(0) = x_0$ is solvable for the differential inclusion

$$\dot{x}(t) \in \mathscr{A} x(t), \qquad t \in \mathbb{R}_+,$$

where $\mathscr{A} = F^{-1}G$ is a linear relation on $\mathscr{X}$.

**7.2. Comments on the central results.** Most properties of the operator $\mathscr{L}_E^+$ and the relation $\mathscr{D}_E^+$ in Theorem 1.3 (see equality (1.12)) were established in [19]. In Theorems 4.1 and 4.2 we establish properties not proved in [19].

Theorems 1.7–1.11 contain some of our central results in this paper. Results very close to these were presented in [19], but they were obtained under an additional assumption. For a reflexive space $X$ and for invertible operators $\mathscr{U}(t,\tau)$, $\tau < t$, $\tau$, $t \in \mathbb{R}_+$, the statement of Theorem 1.9 was obtained in [27] for the operators $\mathscr{L}_{\{0\}}^+$ and $\mathscr{L}_X^+$. Theorems 1.7–1.11 are particularly important when the family of evolution operators (the 'coefficients' of the differential operator) is stationary at $+\infty$. Theorems 1.7–1.10 can be used for substantiation of the finite section method [93]–[95].

Theorems 1.12–1.16 are among our most important results in this paper. In the proofs of Theorems 1.12 and 1.13 we make essential use of the results obtained for the operator $\mathscr{D}_E^-$ and the relation $\mathscr{D}_E^+$.

In [96] Didenko considered the spectral theory of differential operators in function spaces on a finite interval.

We remark finally that the state of the theory discussed here up to the year 1999 was described in [18]. That monograph gave details of the history of this theory, and for the theory of differential operators presented the first results obtained with the use of difference operators (the results in the present author's paper [14] were expounded). The notion of a function slowly varying at infinity defined on a locally compact Abelian group was introduced in the paper [98].

## 8. Examples

**Example 8.1.** Let $A\colon \mathbb{J} \to \operatorname{End} X$, where $\mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$, be in the Stepanov space $S^1(\mathbb{J}, \operatorname{End} X)$. Then (see [7]) there exists a family of evolution operators $\mathscr{U} : \mathbb{J} \times \mathbb{J} \to \operatorname{End} X$ solving the Cauchy problem (1.2), (1.5) and representable as $\mathscr{U}(t,s) = U(t)U(s)^{-1}$, $s, t \in \mathbb{J}$, where the operator-valued function $U\colon \mathbb{J} \to \operatorname{End} X$ (the Cauchy function) solves (for almost all $t \in \mathbb{J}$) the operator differential equation

$$\dot{X} + A(t)X = 0, \quad t \in \mathbb{J}, \qquad X(0) = I.$$

The results in this paper include the corresponding results from [6] and [7], but some of them are new, for example, results of the analysis of the operator $\mathscr{L}_E\colon D(\mathscr{L}_E) \subset \mathscr{F}(\mathbb{R}_+, X) \to \mathscr{F}(\mathbb{R}_+, X)$ (the operators $\mathscr{L}_{\{0\}}$ and $\mathscr{L}_X$ were treated in [6] and [7]). Since the operators $U(t)$, $t \in \mathbb{J}$, have continuous inverses, an exponential dichotomy for the family $\mathscr{U}$ on some infinite interval $(-\infty, a]$ or $[b, \infty)$ implies an exponential dichotomy for $\mathscr{U}$ on any interval of the corresponding form $(-\infty, a']$ with $a' > a$ or $[b', \infty)$ with $b' < b$, respectively. Thus, in studying the operator $\mathscr{L} = -d/dt + A(t)\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}_+, X) \to \mathscr{F}(\mathbb{R}_+, X)$ under the assumption of an exponential dichotomy for the family $\mathscr{U}: \mathbb{R}\times\mathbb{R} \to \operatorname{End} X$ on the intervals $(-\infty, 0]$ and $[0,\infty)$ it is natural to turn to Theorem 1.14. In particular, this theorem yields the following result of Pliss [97], which is often used in the theory of dynamical systems (we state it using the terminology of our paper).

**Theorem 8.1.** *Let* $\dim X < \infty$ *and let* $\mathscr{U}: \mathbb{R}\times\mathbb{R} \to \operatorname{End} X$ *be a family admitting an exponential dichotomy on* $\mathbb{R}_-$ *and* $\mathbb{R}_+$ *with splitting pairs* $P_\pm, Q_\pm : \mathbb{R}_\pm \to \operatorname{End} X$ *of projection-valued functions such that* $\operatorname{Im} Q_-(0) + \operatorname{Im} P_+(0) = X$. *Then the operator* $\mathscr{L} = -d/dt + A(t)\colon D(\mathscr{L}) \subset C_b(\mathbb{R}, X) \to C_b(\mathbb{R}, X)$ *is surjective.*

**Example 8.2.** Consider equation (1.1) with $X = L^2(\Omega, \mathbb{C})$ and $\Omega$ a bounded domain with smooth boundary in $\mathbb{R}^n$. The family of linear differential operators

$$A(t)\colon H_0^m(\Omega) \cap H^{2m}(\Omega) \subset L_2(\Omega) \to L_2(\Omega), \qquad t \in \mathbb{J} \in \{\mathbb{R}_-, \mathbb{R}_+\}$$

(where $H_0^m(\Omega)$ and $H^{2m}(\Omega)$ are Sobolev spaces, $m \geqslant 1$; see [5]) is defined in terms of the family of differential expressions

$$(\ell_t y)(n) = \sum_{|\alpha| \leqslant 2m} a_\alpha(t,u)(D^\alpha y)(u), \qquad t \geqslant 0,$$

and the Dirichlet problem on the boundary $\partial\Omega$ of $\Omega$. The functions $a_\alpha : \mathbb{R}_+ \times \Omega \to \mathbb{C}$, where $|\alpha| \leqslant 2m$, belong to the space $C_b(\mathbb{R}_+, C^k(\Omega))$ with $k \in \mathbb{N}$ sufficiently large and are Lipschitz continuous with respect to the first variable. Moreover, assume that the family of differential expressions $\ell_t$, $t \geqslant 0$, is uniformly elliptic.

It follows from our assumptions that the elliptic operators $A(t)$, $t \in \mathbb{J}$, are the generators of analytic operator semigroups. Furthermore, the hypotheses of the Sobolevski–Tanabe theorem hold, and therefore the Cauchy problem on $\mathbb{J}$ is well posed and there exists a family of evolution operators $\mathscr{U}: \Delta_{\mathbb{J}} \to \operatorname{End} L_2$ enabling the Cauchy problem to be solved. Thus the operator $\mathscr{L}_E^+ = -d/dt + A(t)$ is defined in any homogeneous function space $\mathscr{F}(\mathbb{R}_+, X)$, and we can apply the results obtained here to this operator.

We note also that all the operators $\mathscr{U}(t,s)$, $s < t$, $s,t \in \mathbb{J}$, are compact. Suppose that Assumption 1.1 holds for the family $\mathscr{U}$. Then $\operatorname{Im} Q_-(a)$ and $\operatorname{Im} Q_+(b)$ are finite-dimensional subspaces of $L_2(\Omega)$, and therefore $\mathscr{L} = -d/dt + A(t)\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ is a Fredholm operator if $\mathbb{J} = \mathbb{R}$.

**Example 8.3.** Let $A\colon D(A) \subset X \to X$ be the generator of a strongly continuous operator semigroup $U: \mathbb{R}_+ \to \operatorname{End} X$. We consider the differential equation (1.2) for $t \in \mathbb{J} \in \{\mathbb{R}_+, \mathbb{R}\}$. The corresponding family of evolution operators $\mathscr{U}: \Delta_{\mathbb{J}} \to \operatorname{End} X$ has the form

$$\mathscr{U}(t,s) = T(t-s), \qquad s \leqslant t, \quad s,t \in \mathbb{J}.$$

By Theorem 1.5 the differential operator

$$\mathscr{L} = -\frac{d}{dt} - A\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$$

has a continuous inverse if and only if the difference operator $\mathscr{D} \in \operatorname{End} \mathscr{F}(\mathbb{Z}, X)$ with $(\mathscr{D}x)(n) = x(n) - T(1)x(n-1)$ for $n \in \mathbb{Z}$ and $x \in \mathscr{F}(\mathbb{Z}, X)$ has one. It follows from the results in [30] that $\mathscr{D}$ has a continuous inverse if and only if the following condition holds (this is called the *hyperbolicity* condition for the semigroup $U$):

$$\sigma(U(1)) \cap \mathbb{T} = \varnothing, \tag{8.1}$$

where $\mathbb{T} = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ is the unit circle. This condition shows that

$$\sigma(A) \cap i\mathbb{R} = \varnothing, \qquad \sup_{\lambda\in\mathbb{R}} \|R(i\lambda, A)\| < \infty. \tag{8.2}$$

The conditions in (8.2) do not necessarily mean that (8.1) holds. However, if $X$ is a Hilbert space, then it follows from the Gearhart–Prüss theorem (see [60], [61], and [59]) that (8.1) must hold. From (8.1) we deduce the representation

$$\sigma(U(1)) = \sigma_{\mathrm{int}} \cup \sigma_{\mathrm{out}},$$

where $\sigma_{\mathrm{int}} = \{\lambda \in \sigma(U(1)) \mid |\lambda| < 1\}$. Let $P_{\mathrm{int}}$ and $P_{\mathrm{out}}$ be the Riesz projections constructed for the spectral sets $\sigma_{\mathrm{int}}$ and $\sigma_{\mathrm{out}}$, respectively. Then $X = X_{\mathrm{int}} \oplus X_{\mathrm{out}}$, where $X_{\mathrm{int}} = \operatorname{Im} P_{\mathrm{int}}$ and $X_{\mathrm{out}} = \operatorname{Im} P_{\mathrm{out}}$. The subspace $X_{\mathrm{int}}$ is said to be *stable*, while $X_{\mathrm{out}}$ is said to be *unstable*.

If $\mathscr{L} = -d/dt + A\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ is a Fredholm operator, then the family $\mathscr{U}$ admits an exponential dichotomy on $\mathbb{R}_+$ by Theorem 1.16. Then it follows from [19], Theorem 10.1 that the operator semigroup $U$ is hyperbolic, that is, (8.1) holds, and therefore $\mathscr{L}$ has a continuous inverse.

**Example 8.4** (Petrovskii-correct systems of differential equations). Let $p(\xi) = (p_{kj})_{k,j=1}^m$, $\xi \in \mathbb{R}^n$, $n \in \mathbb{N}$, be a matrix with polynomial entries $p_{kj}\colon \mathbb{R}^n \to \mathbb{C}$, $p_{kj}(\xi) = \sum_{|\alpha|\leqslant N_{kj}} a_\alpha \xi^\alpha$. Here $\alpha \in \mathbb{N}_+^n$ and $a_\alpha \in \mathbb{C}$ are quantities depending on $k$ and $j$. In the Hilbert space the operator $A = p(i\partial)$, $\partial = (\partial_1, \dots, \partial_n)$, $i^2 = -1$, is defined in terms of the Fourier transformation $\mathscr{F}$ as $A = \mathscr{F}^{-1} p(\,\cdot\,)\mathscr{F}$. It is an operator with matrix coefficients and has symbol $p$. We say that $A$ is *Petrovskii-correct* if for some $\omega \in \mathbb{R}$ the spectrum $\sigma(p(\xi))$ of the matrix $p(\xi)$ satisfies $\sigma(p(\xi)) \subset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda \leqslant \omega\}$ for all $\xi \in \mathbb{R}^n$. In this case $A$ generates a strongly continuous operator semigroup $T\colon \mathbb{R}_+ \to \operatorname{End} L_2(\mathbb{R}^n)$ and has domain $D(A)$ equal to the Sobolev space $W_2^N(\mathbb{R}^n)$ of order $N$. This is a hyperbolic semigroup (the operator $T(1)$ has the property $\sigma(T(1)) \cap \mathbb{T} = \varnothing$) when the sets $\sigma(p(\xi))$, $\xi \in \mathbb{R}^n$, are uniformly bounded away from $i\mathbb{R}$. Then the family $\mathscr{U}(t,s) = T(t-s)$, $s \leqslant t$, $s,t \in \mathbb{R}_+$, admits an exponential dichotomy on $\mathbb{R}_+$. In this case the stable and unstable subspaces are infinite dimensional if they are non-zero.

**Example 8.5.** Let $S\colon L^p(\mathbb{R}_+, X) \to L^p(\mathbb{R}_+, X) = L^p$ be the Wiener–Hopf operator

$$(Sx)(t) = x(t) - \int_0^\infty \mathscr{K}(t-s)x(s)\,ds, \qquad t \in \mathbb{R}_+, \quad x \in L^p,$$

with integrable kernel $\mathscr{K}: \mathbb{R} \to \operatorname{End} X$ having the symbol

$$W(\lambda) = I - \int_{\mathbb{R}} e^{i\lambda t}\mathscr{K}(t)\,dt = I - NR(\lambda, A)M, \qquad \lambda \in \mathbb{R}, \tag{8.3}$$

where $iA\colon D(A)\subset Y\to Y$ (here $Y$ is an auxiliary Banach space) is the generator of a $C_0$ operator semigroup $T\colon \mathbb{R}_+\to \operatorname{End} Y$ satisfying $\sigma(T(1))\cap\mathbb{T}=\varnothing$ (so that $\sigma(A)\cap\mathbb{R}=\varnothing$). The linear operators $M\colon X\to Y$ and $N\colon Y\to X$ are bounded. In particular, for $X=\mathbb{C}^m$ it was proved in [34], § XIII.4 that if the symbol $W$ is a rational function, then such a representation holds for some operators $A\in \operatorname{End}\mathbb{C}^n$, $M\colon\mathbb{C}^m\to\mathbb{C}^n$, and $N\colon\mathbb{C}^n\to\mathbb{C}^m$, where $n\in\mathbb{N}$ is some integer. Taking an appropriate Banach space $X$, we can represent any function $W=I-\widehat{\mathscr{K}}$ which is holomorphic in some domain $\mathbb{C}_\alpha=\{\lambda\in\mathbb{C}\mid |\operatorname{Im}\lambda|<\alpha\}$ in the form (8.3) with $Y=\mathbb{C}$. One can easily show (for $X=\mathbb{C}^n$ see [34], Lemma 18.5.1) that $S$ can be represented in the form $S=I+i\widetilde{N}\mathscr{L}_E^{-1}\widetilde{M}$, where $\widetilde{M}\colon L^p(\mathbb{R}_+,X)\to L^p(\mathbb{R}_+,Y)$ and $\widetilde{N}\colon L^p(\mathbb{R}_+,Y)\to L^p(\mathbb{R}_+,X)$ are the operators of multiplication by the operators $M$ and $N$, respectively, and

$$\mathscr{L}_E^+=-\frac{d}{dt}+iA\colon D(\mathscr{L}_E^+)\subset L^p(\mathbb{R}_+,Y)\to L^p(\mathbb{R}_+,Y),\qquad E=\operatorname{Im}P_{\mathrm{out}},$$

where $P_{\mathrm{out}}$ is the Riesz projection constructed for the spectral set $\sigma_{\mathrm{out}}=\{\lambda\in\sigma(T(1))\mid |\lambda|>1\}$ of the operator $T(1)$. This representation enables one to establish the following theorem (for $X=\mathbb{C}^n$ see [34], Theorem 18.5.3).

**Theorem 8.2.** *Let $\widetilde{\mathscr{L}_E^+}=\mathscr{L}_E^+-MN=-d/dt+A-MN\colon D(\mathscr{L}_E^+)\subset L^p(\mathbb{R}_+,X)\to L^p(\mathbb{R}_+,X)$, $p\in[1,\infty]$. Then*

$$\operatorname{Ker} S=N(\operatorname{Ker}\widetilde{\mathscr{L}_E^+}),\qquad \operatorname{Im} S=M^{-1}(\operatorname{Im}\widetilde{\mathscr{L}_E^+})$$

*and $S$ is a Fredholm operator if and only if $\widetilde{\mathscr{L}_E^+}$ is Fredholm. Moreover, these operators have the same index. In particular, $S$ is invertible if and only if $\widetilde{\mathscr{L}_E^+}$ is, and in this case $S^{-1}=(I-i\widetilde{N}\widetilde{\mathscr{L}_E^{-1}})\widetilde{M}$.*

We can now ask a natural question: does the Fredholm property of the operator $\mathscr{L}=\mathscr{L}_{\mathscr{U}}$ (or $\mathscr{D}$) in some space $\mathscr{F}=\mathscr{F}_c$ imply an exponential dichotomy for the family $\mathscr{U}$? The following example answers this question in the negative.

**Example 8.6.** Let $A_0\colon\mathscr{D}(A_0)\subset X\to X$ be the generator of a $C_0$ operator semigroup $T_0\colon\mathbb{R}_+\to\operatorname{End}X$ such that $T_0(1)$ is a Fredholm operator and $\operatorname{Ker}T_0(1)^*\neq\{0\}$. We look at the function $A(t)=A_0$ for $t\in[0,1)$, $A(t)=(\ln 2)I$ for $t\in[1,\infty)$ and the differential operator $\mathscr{L}_X^+=-d/dt+A(t)$ in some homogeneous function space $\mathscr{F}=\mathscr{F}(\mathbb{R}_+,X)$ such that $\mathscr{F}=\mathscr{F}_c$. Then $\mathscr{U}(1,0)=T_0(1)$ and $\mathscr{U}(n,n-1)=2I$ for $n\geqslant 2$, and the corresponding difference relation $\mathscr{D}_X^+\in LRC(\mathscr{F}(\mathbb{Z}_+,X))$ is defined by

$$\begin{aligned}\mathscr{D}_X^+=\{(x,y)\in\mathscr{F}(\mathbb{Z}_+,X)\mid y(n)=x(n)-2x(n-1),\ n\geqslant 2,\\ y(1)=x(1)-T_0(1)x(0)\}.\end{aligned}$$

The family $\mathscr{U}$ admits the trivial exponential dichotomy on the set $\mathbb{N}$ with $P = 0$ and $Q = I$, so we have the representations

$$\begin{aligned}
\operatorname{Ker} \mathscr{D}_X^+ &= \{x \in \mathscr{F}(\mathbb{Z}_+, X) \mid x(0) \in \operatorname{Ker} T_0(1),\ x(k) = 0,\ k \geqslant 1\},\\
\operatorname{Im} \mathscr{D}_X^+ &= \{f \in \mathscr{F}(\mathbb{Z}_+, X) \mid f(1) \in \operatorname{Im} T_0(1)\} = \overline{\operatorname{Im} \mathscr{D}_X^+},\\
\operatorname{Ker} \mathscr{D}_X^* &= \{\xi \in \mathscr{F}'(\mathbb{Z}_+, X) \mid \xi(0) = 0,\ \xi(1) \in \operatorname{Ker} T_0(1)^*,\\
&\qquad\qquad \xi(n) = 2^{-n+1}\xi(1),\ n \geqslant 2\} \neq \{0\}.
\end{aligned}$$

Thus, $\mathscr{D}_X^+$ is a Fredholm relation, and it follows from [19], Lemma 8.7 that the family $\mathscr{U}$ does not admit an exponential dichotomy on $\mathbb{Z}_+$. It follows from [19], Lemma 6.2 that the family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ of the form

$$\mathscr{U}(t,s) = \begin{cases} e^{2(t-s)} I, & 1 \leqslant s \leqslant t < \infty,\\ T_0(t-s), & 0 \leqslant s \leqslant t \leqslant 1,\\ I, & s \leqslant t \leqslant 0, \end{cases}$$

extended to $\Delta_{\mathbb{R}}$ in the natural way (we keep the notation $\mathscr{U}$ for the extensions), admits an exponential dichotomy on $\mathbb{R}_-$ and $[1,\infty)$. However, it does not have this property on $[0,\infty)$. In this case the corresponding splitting pairs $P_-, Q_-\colon \mathbb{R}_- \to \operatorname{End} X$ and $P_+, Q_+\colon \mathbb{R}_+ \to \operatorname{End} X$ are $P_- = I$, $Q_- \equiv 0$ and $P_+ \equiv 0$, $Q_+ \equiv I$. Hence the node operator

$$\mathscr{N}_{1,0}\colon \operatorname{Im} Q_-(0) = \{0\} \to \operatorname{Im} Q_+(1) = X$$

is zero. Thus, the family of evolution operators $\mathscr{U}\colon \Delta_{\mathbb{R}} \to \operatorname{End} X$ we have constructed corresponds to the differential operator $\mathscr{L} = -d/dt + A(t)\colon D(\mathscr{L}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ with $A(t) = 0$ for $t > 1$. Since $\mathscr{N}_{1,0}$ is semi-Fredholm, so is $\mathscr{L}_{\mathscr{U}}$ (by Theorem 1.13). The operator $\mathscr{N}_{1,0}$ is injective, and hence $\mathscr{L}_{\mathscr{U}}\colon D(\mathscr{L}_{\mathscr{U}}) \subset \mathscr{F}(\mathbb{R}, X) \to \mathscr{F}(\mathbb{R}, X)$ is an injective operator.

**Example 8.7.** Let $T\colon \mathbb{R}_+ \to \operatorname{End} X$ be a bounded $C_0$-semigroup with generator $A$ such that $\sigma(A) \cap i\mathbb{R} = \{i\lambda_1, \dots, i\lambda_n\}$ is a finite set. From the results in § 6 we obtain the following theorem.

**Theorem 8.3.** *The semigroup $T$ has the representation*

$$T(t) = \sum_{k=1}^{n} B_k(t) e^{i\lambda_k t} + B_0(t), \qquad t \geqslant 0, \tag{8.4}$$

*where the $B_k \in C_{b,u}(\mathbb{R}_+, \operatorname{End} X)$ are operator-valued functions slowly varying at infinity, and $B_0\colon \mathbb{R}_+ \to \operatorname{End} X$ is a strongly continuous function such that $\lim_{t\to\infty} B_0(t)x = 0$ for each $x \in X$.*

We note that the functions $B_k$, $k = 1, \dots, n$, in (8.4) can be taken to extend to $\mathbb{C}$ as entire functions of an (arbitrarily small) exponential type such that $\lim_{t\to\infty} \|B_k'(t)\| = 0$, $k = 1, \dots, n$. This possibility is ensured by Lemma 6.1.

An analogue of Theorem 8.3 holds when $\sigma(A) \cap i\mathbb{R}$ is a countable set without limit points in $i\mathbb{R}$. It should be stressed that the quantities $i\lambda_1, \dots, i\lambda_n$ in the hypothesis of the theorem do not necessarily belong to different connected components of $\sigma(A)$.

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