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To the memory of Evgenii Frolovich Mishchenko

Smooth abnormal problems in extremum theory and analysis

A. V. Arutyunov

Abstract. A survey is given of results related to the inverse function theorem and to necessary and sufficient first- and second-order conditions for extrema in smooth extremal problems with constraints. The main difference between the results here and the classical ones is that the former are valid and meaningful without a priori normality assumptions.

Bibliography: 48 titles.

Keywords: abnormal point, Lagrange principle, necessary and sufficient second-order conditions for an extremum, 2-regularity, inverse function theorem, 2-normality, quadratic map.

Contents

1. Introduction

Analysis of non-linear extremal problems with constraints (problems on a conditional extremum) is based on the Lagrange principle, put forward by Lagrange at the end of the 18th century. A rigorous substantiation of the Lagrange principle for a broad class of problems required the serious efforts of many mathematicians and was mostly completed in the second half of the 20th century (see [1] and the bibliography there). In effect, this marked the end of the classical stage in the development of the theory. However, we shall explain below that the necessary first-order conditions in the form of the Lagrange principle are only meaningful when the

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constraints satisfy conditions of non-degeneracy (normality). In the abnormal case the Lagrange principle does not yield information.

The theory of inverse function theorems (more generally, implicit function theorems) is a closely related topic. Such theorems provide an important apparatus for the analysis of non-linear maps and are exceptionally important for theory and applications. The first inverse function theorems were proved in the second half of the 19th century, and by the end of the 20th century they had been extended to a maximally general situation which also includes non-smooth maps. However, all these results were obtained under the assumption that the map in question is non-degenerate (normal), which in the smooth case means that the derivative of the map at the point under consideration defines a surjective linear operator. Thus, by the end of the last century the classical stage in the development of the theory was basically complete (see, for instance, [2]). Using two of the problems described, we shall explain more thoroughly what abnormality means.

The theory of extremal problems. Let X be a vector space. We consider the problem of minimization with constraints

$$f_0(x) \to \min, \qquad F(x) = 0, \tag{1.1}$$

where $F\colon X \to Y = \mathbb{R}^k$ is a fixed map, and we seek the minimum of the given function $f_0\colon X \to \mathbb{R}$ on the admissible set $M = \{x \in X\colon F(x) = 0\}$. For simplicity we assume first that X is a Banach space (we can even consider $X = \mathbb{R}^n$) and that f_0 and F have continuous second-order derivatives in some neighbourhood of a point x_0 which is a local minimum point in the problem (1.1).[1] Then the Lagrange principle holds at x_0. To state this principle let us introduce the Lagrange function

$$L(x,\lambda) = \lambda^0 f_0(x) + \langle y^*, F(x)\rangle, \qquad \lambda = (\lambda^0, y^*), \quad \lambda^0 \in \mathbb{R}, \quad y^* \in Y^*, \tag{1.2}$$

where the $(k+1)$-dimensional vector $\lambda = (\lambda^0, y^*)$ and its components are called Lagrange multipliers and the angle brackets, as usual, denote the inner product.[2]

Theorem 1.1 (Lagrange principle). *Let x_0 be a local minimum point in the problem* (1.1). *Then there exists a Lagrange multiplier λ such that*

$$\frac{\partial L}{\partial x}(x_0, \lambda) = 0, \qquad \lambda^0 \geqslant 0, \quad \lambda \neq 0. \tag{1.3}$$

The Lagrange principle brings with it necessary first-order conditions for an extremum and is very well known (see [1], [3], and others).

We consider two cases. First, let x_0 be a normal point, which means that $\operatorname{im} F'(x_0) = Y$. (Russian authors often say that the Lyusternik condition holds at x_0, and besides *normal* the terms *non-degenerate* and *regular* point are also used.) Thus, if the minimum point x_0 is normal, then $\lambda^0 > 0$ by (1.3), and therefore, bearing in mind that the relations in (1.3) are positive-homogeneous with respect to λ, we can assume that $\lambda^0 = 1$. In this case a unique Lagrange multiplier

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[1] For simplicity we make excessive smoothness assumptions in the Introduction.

[2] The Lagrange multiplier y^* is taken in the topological dual space Y^* of Y. However, when Y is assumed to be a finite-dimensional arithmetic space, we shall identify Y and its dual and treat the Lagrange multiplier y^* as an element of Y.

exists, has the form $\lambda = (1, y^*)$, and satisfies the classical necessary second-order conditions

$$\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0 \qquad \forall\, x \in X \colon F'(x_0)x = 0 \tag{1.4}$$

(see [3], § 3.4, p. 287). Here the square brackets denote the action of the bilinear form.

Now we look at the second case: x_0 is an abnormal point, that is, $\operatorname{im} F'(x_0) \neq Y$. Then the Lagrange principle (1.3) holds at this point for $\lambda^0 = 0$ and an arbitrary $y^* \neq 0$ in the kernel $\ker F'(x_0)^*$ of the conjugate operator. This kernel is non-trivial, because $\operatorname{im} F'(x_0) \neq Y$. Thus, the Lagrange principle holds automatically at each abnormal point whatever the functional f_0 being minimized might be: it is just an direct consequence of the definition of abnormality. Hence, the Lagrange principle is of no use when we ask whether an abnormal point is extremal. As for the classical necessary second-order conditions (1.4), they can fail at an abnormal minimum point. Here is a simple two-dimensional example:

$$\begin{gathered} X = \mathbb{R}^2, \qquad f_0(x) = -|x|^2 \to \min, \\ F_1(x) = x_1^2 - x_2^2 = 0, \qquad F_2(x) = x_1 x_2 = 0, \end{gathered} \tag{1.5}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$. In this problem $x_0 = 0$ is the unique point satisfying the constraints, so of course it is a minimum point. However, conditions (1.4) do not hold at it for any Lagrange multiplier λ satisfying (1.3). We see that the Lagrange principle gives no information at an abnormal point, while the classical necessary second-order conditions can fail. Thus, we have the problem of finding meaningful necessary minimum conditions in the problem (1.1) when no a priori assumption is made that the point under consideration is normal.

The inverse function theorem. Let $F \colon X \to Y$ be a continuously differentiable map in a neighbourhood of a point $x_0 \in X$ and let $y_0 = F(x_0)$. The question is whether y_0 has a neighbourhood V such that for all $y \in V$ the equation

$$F(x) = y \tag{1.6}$$

has a solution $x(y)$ such that $x(y_0) = x_0$ and the map $x(\cdot)$ is continuous at y_0, or better, continuous in the whole neighbourhood V. If x_0 is a normal point, then the classical inverse function theorem answers this question in the affirmative, and the map $x(\cdot)$ can be taken to be continuously differentiable. However, this is no longer so when x_0 is an abnormal point. For example, in a neighbourhood of the origin the scalar equation $x_1^2 + x_2^2 = y$ has no solutions for $y < 0$, while the equation $x_1^2 - x_2^2 = y$ has infinitely many continuous solutions satisfying $x(0) = 0$, but all of them are non-differentiable at the origin, and moreover, they do not even satisfy a Lipschitz condition. Thus, we have the problem of finding conditions weaker than normality which ensure that equation (1.6) has a solution $x(\cdot)$ with the required properties.

The discussion of these two problems and others close to them is the subject of this survey. We shall focus on the latest progress and mostly leave aside the history of the problem and earlier results which were subsequently improved. In the next sections, §§ 2 and 3, we present necessary first- and second-order conditions for an extremum in problems with different types of constraints. Their distinctive feature is that they make sense and are derived without the a priori assumptions of normality, and in the normal case they become the classical conditions. The two sections are different in the approaches they present: in the first approach we obtain results in terms of the indices of the second derivatives of the classical Lagrange function L, while the investigations in § 3 are based on the generalized Lagrange function $\mathscr{L}_{\mathscr{A}}$. § 4 contains sufficient second-order conditions. Apart from the classical conditions, we give there sufficient conditions for problems with non-closed image and sufficient conditions in terms of the function $\mathscr{L}_{\mathscr{A}}$ for classes of abnormal problems in which the classical sufficient conditions are automatically degenerate. In § 5 we investigate quadratic problems, which present a typical example in the class of abnormal problems, and we prove several results in the theory of quadratic maps. § 6 contains various versions of inverse function theorems and implicit function theorems. Like the results mentioned above, they make sense and are obtained without a priori assumptions of normality, while in the normal case they become the classical results. In § 7, as applications of the abstract theory presented in §§ 2 and 3, we obtain necessary second-order optimality conditions in various optimal control problems. § 8 is devoted to applications of abstract results to bifurcation theory, sensitivity theory, controllability of dynamical control systems, and the theory of quadratic maps.

2. Necessary second-order conditions for an extremum. The index approach

An extremal problem with constraints of equality and inequality type. As before, let X be a vector space and $Y=\mathbb{R}^k$. Let $f_i\colon X\to\mathbb{R}$, $i=0,\dots,l$, be given functions and $f\colon X\to Y$ a fixed map. We consider an extremal problem with constraints of equality and inequality type:

$$f_0(x)\to\min,\qquad f_j(x)\leqslant 0,\quad j=1,\dots,l,\qquad f(x)=0. \tag{2.1}$$

Let $x_0\in X$ be a point satisfying the constraints in (2.1). For simplicity, in what follows we assume that all the indices corresponding to constraints of inequality type are active, that is, $f_j(x_0)=0$, $j=1,\dots,l$. This involves no loss of generality, because our considerations are local, so that constraints of the form $f_j(x_0)<0$ can be omitted.

The fixed functions f_j and the map $f\colon X\to Y$ are assumed to be smooth in the following sense. Let $\mathscr{M}$ be the set of finite-dimensional subspaces $M\subseteq X$, each endowed with the unique topology of a separable topological vector space. Let $\|\cdot\|_M$ be one of the (equivalent) norms generating this topology in M. In the vector space X we introduce the so-called finite topology: the open sets in X are precisely the sets having open intersection with each $M\in\mathscr{M}$. If X is infinite-dimensional, then in general the finite topology does not make it a topological vector space, since addition is generally discontinuous. On the other hand, the finite topology is stronger than any topology of a topological vector space on X. A local minimum in the finite topology is the weakest type of local minimum under consideration.

As for f, we assume that it is twice continuously differentiable in a neighbourhood of x_0 with respect to the finite topology τ. This means that for any finite-dimensional linear subspace M the restriction of f to M has continuous second derivatives in some neighbourhood of x_0 (depending on M). Thus, there exist a linear operator $A\colon X \to Y$, a bilinear map $Q\colon X \times X \to Y$, and a map $\alpha\colon X \to Y$ such that

$$f(x) = f(x_0) + A(x - x_0) + \frac{1}{2}Q[x - x_0, x - x_0] + \alpha(x - x_0) \qquad \forall\, x \in X,$$

and for each finite-dimensional subspace M

$$\frac{\|\alpha(x - x_0)\|}{\|x - x_0\|_M^2} \to 0, \qquad x \to x_0, \quad x \in M.$$

We denote the maps A and Q by $f'(x_0)$ $\Bigl($or $\frac{\partial f}{\partial x}(x_0)\Bigr)$ and $f''(x_0)$ $\Bigl($or $\frac{\partial^2 f}{\partial x^2}(x_0)\Bigr)$ and call them the first and second derivatives of f. The analogous assumptions hold for all the functions f_j. Smoothness with respect to the finite topology is the weakest assumption of all those that are usually considered.

It is important to note that the results below will be interesting also in the case when $X = \mathbb{R}^n$ and f and f_j have continuous second derivatives in the usual sense. So the reader who is not concerned with the infinite-dimensional context can assume that $X = \mathbb{R}^n$ with no loss of understanding of the general ideas.

Let us introduce the Lagrange function of the problem (2.1):

$$L(x, \lambda) = \sum_{j=0}^{l} \lambda^j f_j(x) + \bigl\langle y^*, f(x)\bigr\rangle, \qquad \lambda = (\lambda^0, \dots, \lambda^l, y^*), \quad \lambda^j \in \mathbb{R}, \quad y^* \in Y^*.$$

We consider the set of normalized Lagrange multipliers of the problem (2.1) that correspond to the point x_0 by the Lagrange principle:

$$\Lambda(x_0) = \biggl\{\lambda\colon \frac{\partial L}{\partial x}(x_0, \lambda) = 0,\ \lambda^j \geqslant 0 \ \ \forall\, j,\ \ \sum_{j=0}^{l} \lambda^j + |y^*| = 1\biggr\}. \tag{2.2}$$

(Note that here we drop the complementary slackness conditions $\lambda^j f_j(x_0) = 0$, $j = 1, \dots, l$, since by our assumption $f_j(x_0) = 0$ for $j = 1, \dots, l$.)

For non-negative integers s we introduce the sets $\Lambda_s(x_0)$ (some of them can be empty) of Lagrange multipliers $\lambda \in \Lambda(x_0)$ such that there exists a linear subspace $\Pi \subseteq X$ (depending on λ) for which

$$\operatorname{codim} \Pi \leqslant s, \qquad \Pi \subseteq \ker F'(x_0), \qquad \frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0 \quad \forall\, x \in \Pi. \tag{2.3}$$

Here and in what follows, $F = (f_1, \dots, f_l, f)$ acts from X to $\mathbb{R}^m$, and

$$m = k + l.$$

We consider the so-called critical cone

$$\mathscr{K}(x_0) = \bigl\{h \in X\colon \bigl\langle f_j'(x_0), h\bigr\rangle \leqslant 0,\ j = 0, \dots, l;\ f'(x_0)h = 0\bigr\}.$$

Obviously, it is non-empty and convex ($0 \in \mathscr{K}(x_0)$).

Theorem 2.1 (necessary second-order conditions). *In the problem* (2.1) *let* x_0 *be a local minimum point in the finite topology. Then the set* $\Lambda_m(x_0)$ *is non-empty, and moreover,*

$$\max_{\lambda\in\Lambda} \frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x] \geqslant 0 \quad \forall\, x \in \mathscr{K}(x_0), \qquad \Lambda = \Lambda_m(x_0). \tag{2.4}$$

The proof uses the following construction.

Lower estimate for the upper topological limit of a sequence of subspaces. If $\{\Pi_i\}$ is a sequence of subsets of a metric space, then let $\operatorname{Ls}\{\Pi_i\}$ denote its upper topological limit, which consists of all the limit points of sequences $\{x_i\}$ with $x_i \in \Pi_i$ for all i.

Let X be a Banach space and let $\{A_i\}_{i=1}^{\infty}$ be a sequence of continuous linear operators from X to $Y = \mathbb{R}^k$. We assume that $\{A_i\}$ converges in norm to a linear operator $A\colon X \to Y$. Let $M = \operatorname{Ls}\{\ker A_i\}$. Obviously, M is closed and non-empty ($0 \in M$), and $M \subseteq \ker A$. If A is surjective, then $M = \ker A$, but if not, then $M \neq \ker A$ in general and M is not even necessarily convex.

Theorem 2.2. *X has a closed subspace Π such that*

$$\operatorname{codim}\Pi \leqslant k, \qquad \Pi \subseteq \operatorname{Ls}\{\ker A_i\}, \qquad \Pi \subseteq \ker A.$$

This is an immediate consequence of the following result, which is also of independent interest.

Theorem 2.3. *Let $\{\Pi_i\}$ be a sequence of closed subspaces of X with* $\operatorname{codim}\Pi_i \leqslant k$ *for all i. Then there exists a closed subspace $\Pi \subseteq X$ such that*

$$\operatorname{codim}\Pi \leqslant k, \qquad \Pi \subseteq \operatorname{Ls}\{\Pi_i\}. \tag{2.5}$$

Theorem 2.3 shows that for each s the set $\Lambda_s(x_0)$ is closed and therefore compact when X is a Banach space and the maps f_i and f are smooth. We use this important property below. We defer the proof of Theorem 2.3 till the end of the section.

Proof of Theorem 2.1. Following [4]–[6], we divide the proof of Theorem 2.1 into three steps. First (Step I), assuming that X is finite-dimensional, we prove that $\Lambda_m(x_0)$ is non-empty. In Step II we prove the theorem for finite-dimensional X. Finally, in Step III we prove it in full generality (that is, we drop the assumption that $\dim X < \infty$).

Step I. Suppose that X is finite-dimensional. Then without loss of generality we can take $X = \mathbb{R}^n$. Let $\delta > 0$ be such that x_0 is a minimum point in the problem

$$f_0(x) \to \min, \qquad f_j(x) \leqslant 0, \quad j = 1,\dots,l, \qquad f(x) = 0, \qquad |x - x_0| \leqslant \delta.$$

We remove all the constraints in this problem (apart from the last one) using the penalty method. For positive integers i let

$$\varphi_i(x) = f_0(x) + i\biggl(\sum_{j=1}^{l}\bigl(f_j(x)^+\bigr)^4 + \bigl|f(x)\bigr|^4\biggr) + |x - x_0|^4,$$

where $a^+ = \max(a, 0)$, and consider the sequence of minimization problems

$$\varphi_i(x) \to \min, \qquad |x - x_0| \leqslant \delta,$$

which we call the i-problems. The i-problem is solvable, since φ_i is continuous and a closed ball in the finite-dimensional space X is compact. Let x_i be a solution of the i-problem.

We assert that $x_i \to x_0$. Indeed, since X is finite-dimensional, we can pass to a subsequence and assume the convergence $x_i \to \bar{x}$. Let us show that $\bar{x} = x_0$. We have

$$\begin{aligned} &\varphi_i(x_i) \leqslant \varphi_i(x_0) = f_0(x_0) \ \ \forall\, i \\ &\qquad\qquad \Rightarrow \lim_{i\to\infty} f_j(x_i) \leqslant 0,\ j = 1, \dots, l,\ f(x_i) \to 0 \\ &\qquad\qquad \Rightarrow f_j(\bar{x}) \leqslant 0,\ j = 1, \dots, l,\ f(\bar{x}) = 0 \\ &\qquad\qquad \Rightarrow f_0(\bar{x}) \geqslant f_0(x_0). \end{aligned}$$

Moreover, from the first inequality it follows that

$$\begin{aligned} &f_0(x_i) + |x_i - x_0|^4 \leqslant f_0(x_0) \ \ \forall\, i \\ &\qquad \Rightarrow f_0(\bar{x}) + |\bar{x} - x_0|^4 \leqslant f_0(x_0) \leqslant f_0(\bar{x}) \Rightarrow \bar{x} = x_0. \end{aligned}$$

Thus, we have shown that $x_i \to x_0$. Hence, for large i (which are the ones we consider in what follows) we have $|x_i - x_0| < \delta$, so the i-problem is locally equivalent to the problem of minimizing a smooth function without constraints, and hence the necessary first- and second-order conditions in this problem are as follows:

$$\varphi_i'(x_i) = 0, \quad \varphi_i''(x_i)[x, x] \geqslant 0 \quad \forall\, x.$$

Writing them out in detail, we have

$$\frac{\partial L}{\partial x}(x_i, \lambda_i) + \kappa_i \cdot 4(x_i - x_0)|x_i - x_0|^2 = 0, \tag{2.6}$$

$$\begin{aligned} \frac{\partial^2 L}{\partial x^2}(x_i, \lambda_i)[x, x] + 12\kappa_i \biggl(i \sum_{j=1}^{m} \bigl(f_j(x_i)^+\bigr)^2 \bigl|\langle f_j'(x_i), x\rangle\bigr|^2 \\ + i\bigl|f(x_i)\bigr|^2 \bigl|f'(x_i)x\bigr|^2 + 1(i)|x|^2 \biggr) \geqslant 0 \quad \forall\, x, \end{aligned} \tag{2.7}$$

where $1(i) \to 0$ as $i \to \infty$ and $\lambda_i = \kappa_i(1, \bar{\lambda}_i^1, \dots, \bar{\lambda}_i^l, \bar{y}_i^*)$, with

$$\bar{\lambda}_i^j = 4i(f_j(x_i)^+)^3, \quad j = 1, \dots, l, \qquad \bar{y}_i^* = 4i\bigl|f(x_i)\bigr|^2 f(x_i),$$

$$\kappa_i = \biggl(1 + \sum_{j=1}^{l} (\bar{\lambda}_i^j)^2 + |\bar{y}_i^*|^2\biggr)^{-1/2}.$$

By construction, $|\lambda_i| = 1$ and $\bar{\lambda}_i^j \geqslant 0$ for all i and all $j \leqslant l$. Hence, passing to a subsequence and then taking the limit in (2.6), we get that $\lambda_i \to \lambda \in \Lambda(x_0)$.

Now we show that a subspace Π satisfying (2.3) for $s = m$ exists. To do this we consider the linear operators $A_i = F'(x_i)$. By Theorem 2.2 there exists a subspace Π such that $\Pi \subseteq \ker F'(x_0)$, $\Pi \subseteq \operatorname{Ls}\{\ker A_i\}$, and $\operatorname{codim}\Pi \leqslant m$. Let $h \in \Pi$ be an arbitrary vector. By the definition of the upper topological limit there exist $h_i \in \ker F'(x_i)$ such that, after passing to a subsequence, we have $h_i \to h$. Substituting $x = h_i$ in (2.7) and passing to the limit as $i \to \infty$, we see that $\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[h, h] \geqslant 0$. Since $h \in \Pi$ is arbitrary, we have $\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0$ for all $x \in \Pi$. Thus, we have proved that $\lambda \in \Lambda_m(x_0)$.

Step II. As before, let $X = \mathbb{R}^n$. We shall prove (2.4). For convenience let $x_0 = 0$ and $f_0(x_0) = 0$. Let us introduce a function γ by

$$\gamma(\chi) = 0 \quad \forall\, \chi \leqslant 1, \qquad \gamma(\chi) = (\chi - 1)^4 \quad \forall\, \chi > 1.$$

We fix an arbitrary unit vector $h \in \mathscr{K}(x_0)$ such that $|h| = 1$ (if $\mathscr{K}(x_0) \neq \{0\}$, of course). For $\varepsilon = i^{-1}$, $i = 1, 2, \dots$, we consider the following minimization problem with respect to the variables $(x, \chi) \in X \times \mathbb{R}$:

$$f_\varepsilon(x, \chi) \to \min, \qquad f_j(x) - \chi f_j(\varepsilon h) \leqslant 0, \quad j = 1, \dots, l,$$
$$f(x) - \chi f(\varepsilon h) = 0, \qquad \chi \geqslant 0, \quad |x| \leqslant \delta;$$

we call it the ε-problem. Here δ is the same as above and

$$f_\varepsilon(x, \chi) = \tilde{f}_0(x) - \chi \tilde{f}_0(\varepsilon h) + \gamma(\chi), \qquad \tilde{f}_0(x) = f_0(x) + |x|^4.$$

For $\varepsilon < \delta$ (the only values we consider) the ε-problem is solvable, because the point $x = \varepsilon h$, $\chi = 1$ satisfies all the constraints in the problem, the ball $\{x\colon |x| \leqslant \delta\}$ is compact, and $\gamma(\chi)/\chi \to \infty$ as $\chi \to \infty$.

Among the solutions $(x_\varepsilon, \chi_\varepsilon)$ of the ε-problem there is one with $\chi_\varepsilon > 0$. Indeed, let $(x_\varepsilon, 0)$ be a solution. We assert that $x_\varepsilon = 0$. In fact, x_ε satisfies all the constraints in the problem (2.1), and if $x_\varepsilon \neq 0$, then

$$f_\varepsilon(x_\varepsilon, 0) = f_0(x_\varepsilon) + |x_\varepsilon|^4 \geqslant f_0(0) + |x_\varepsilon|^4 > 0 \quad \text{and} \quad f_\varepsilon(\varepsilon h, 1) = 0 \Rightarrow f_\varepsilon(x_\varepsilon, 0) \leqslant 0.$$

This contradiction shows that $x_\varepsilon = 0$, so that the minimum in the ε-problem is equal to zero. Hence, the point $(\varepsilon h, 1)$ also solves the problem, which proves the required result.

Let $\{x_\varepsilon, \chi_\varepsilon\}$ be a family of solutions of the ε-problems. Since $\gamma(\chi)/\chi \to \infty$ as $\chi \to \infty$, the sequence $\{\chi_\varepsilon\}$ is bounded. Hence, from the inequality $f_\varepsilon(x_\varepsilon, \chi_\varepsilon) \leqslant 0$ we have $\tilde{f}_0(x_\varepsilon) \leqslant \mathrm{const}\bigl|\tilde{f}_0(\varepsilon h)\bigr| \to 0$ as $\varepsilon \to 0$. Thus, $f_0(\tilde{x}) + |\tilde{x}|^4 \leqslant 0$ for any limit point $\tilde{x}$ of the sequence $\{x_\varepsilon\}$. At the same time, $f_0(\tilde{x}) \geqslant 0$, because $\tilde{x}$ satisfies all the constraints in (2.1). Hence $\tilde{x} = 0$, so that $x_\varepsilon \to 0$ as $\varepsilon \to 0$. Thus, for small $\varepsilon > 0$ (to which we confine ourselves) we have $|x_\varepsilon| < \delta$.

The solution $\{x_\varepsilon, \chi_\varepsilon\}$ of the ε-problem with $\chi_\varepsilon > 0$ satisfies the necessary conditions found in Step I: there exist a Lagrange multiplier $\lambda_\varepsilon = (\lambda_\varepsilon^0, \dots, \lambda_\varepsilon^l, y_\varepsilon^*)$ and a linear subspace Π_ε such that $|\lambda_\varepsilon| = 1$, $\lambda_\varepsilon^j \geqslant 0$ for all j,

$$L(\varepsilon h, \lambda_\varepsilon) + \lambda_\varepsilon^0 \varepsilon^4 = \lambda_\varepsilon^0 \gamma'(\chi_\varepsilon) \geqslant 0, \qquad \frac{\partial L}{\partial x}(x_\varepsilon, \lambda_\varepsilon) + 4\lambda_\varepsilon^0 x_\varepsilon |x_\varepsilon|^2 = 0,$$
$$\Pi_\varepsilon \subseteq \ker F'(x_\varepsilon), \qquad \operatorname{codim}\Pi_\varepsilon \leqslant m, \tag{2.8}$$
$$\frac{\partial^2 L}{\partial x^2}(x_\varepsilon, \lambda_\varepsilon)[x, x] + 12\lambda_\varepsilon^0 |x_\varepsilon|^2 |x|^2 \geqslant 0 \quad \forall\, x \in \Pi_\varepsilon.$$

Here we have benefited from being able to drop the constraint $\chi \geqslant 0$ in the Lagrange function for the ε-problem, which is possible because $\chi_\varepsilon > 0$ for the solution, so that the Lagrange multiplier corresponding to this constraint vanishes (by the complementary slackness conditions).

By passing to a subsequence we can assume that $\{\lambda_\varepsilon\}$ converges to a unit vector λ. Taking the limit in the relations obtained, we see that $\lambda \in \Lambda(x_0)$. By Theorem 2.3 there exists a subspace Π such that $\operatorname{codim}\Pi \leqslant m$ and $\Pi \subseteq \operatorname{Ls}\{\Pi_\varepsilon\}$. Passing to the limit as $\varepsilon \to 0$, we get as in Step I that

$$\Pi \subseteq \ker F'(x_0), \qquad \frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x] \geqslant 0 \quad \forall\, x \in \Pi.$$

Hence $\lambda \in \Lambda_m(x_0)$. Finally, expanding $L(\cdot,\lambda_\varepsilon)$ up to second-order terms and considering that $h \in \mathscr{K}(x_0)$, we get from the inequality in (2.8) that $\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[h,h] \geqslant 0$. Since h is arbitrary, this completes the argument in Step II.

Step III. We now prove the theorem in full generality. Let $h \in \mathscr{K}(x_0)$ be arbitrary and let $\widetilde{\mathscr{M}}$ be the set of subspaces $M \in \mathscr{M}$ such that $h \in M$ and $F'(x_0)(M) = \operatorname{im} F'(x_0)$. For an arbitrary $M \in \widetilde{\mathscr{M}}$ we consider the problem obtained from (2.1) by replacing X by M. As was shown in Step II, for this finite-dimensional problem we can find Lagrange multipliers λ_M such that

$$|\lambda_M| = 1, \qquad \lambda_M^j \geqslant 0 \quad \forall\, j, \qquad \frac{\partial L}{\partial x}(x_0,\lambda_M) \in M^\perp, \qquad \frac{\partial^2 L}{\partial x^2}(x_0,\lambda_M)[h,h] \geqslant 0,$$

and there exists a subspace $\Pi_M \subseteq \ker F'(x_0)$ of codimension at most m in M such that $\frac{\partial^2 L}{\partial x^2}(x_0,\lambda_M)[x,x] \geqslant 0$ for all $x \in \Pi_M$.[3] We denote the set of such λ_M by $\Lambda_m(M)$. Clearly, by Theorem 2.3 it is closed.

For any $M_1,\dots,M_s \in \widetilde{\mathscr{M}}$ we obviously have $\bigcap_{i=1}^s \Lambda_m(M_i) \supseteq \Lambda_m(M_1+\dots+M_s) \neq \varnothing$. Hence, the system of non-empty closed sets $\Lambda_m(M)$, $M \in \widetilde{\mathscr{M}}$, has the finite intersection property. Since the unit sphere in $\mathbb{R}^{m+1}$ is compact, the intersection $\bigcap_{M\in\widetilde{\mathscr{M}}}\Lambda_m(M)$ is non-empty. Obviously, for any λ in this intersection we have $\lambda \in \Lambda_m(x_0)$ and $\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[h,h] \geqslant 0$. The proof is complete.

Discussion of Theorem 2.1. If the problem (2.1) contains no constraints of inequality type (that is, $l = 0$), then we can replace the critical cone $\mathscr{K}(x_0)$ in the statement of the theorem by the subspace $\ker f'(x_0)$ (for if $h \in \ker f'(x_0)$ in this case, then either $h \in \mathscr{K}(x_0)$ or $(-h) \in \mathscr{K}(x_0)$). Furthermore, if x_0 is a normal point, then Theorem 2.1 gives the classical necessary second-order conditions (1.4), because for a normal point the set $\Lambda(x_0)$ is a singleton and therefore $\Lambda(x_0) = \Lambda_k(x_0)$ (in this case (2.4) is a consequence of the last equality, since $\Pi = \ker f'(x_0)$ at a normal point).

The strongest second-order conditions known previously for the problem (2.1) are due to Milyutin [7], [8]. We state them now. To do this we let $\Lambda_+(x_0)$ denote the set of $\lambda \in \Lambda(x_0)$ such that the quadratic form $\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)$ has finite index. Recall that the index $\operatorname{ind} q$ of a quadratic form q is the maximum dimension of a subspace on which the form is negative definite; equivalently, this is the minimum

[3] $M^\perp$ is the annihilator of the subspace M.

codimension of a subspace on which q is non-negative definite. Milyutin's theorem states that if x_0 is a local minimum point, then the condition obtained from (2.4) by replacing $\Lambda_m(x_0)$ by $\Lambda_+(x_0)$ is satisfied. However, if X is a finite-dimensional space, then obviously $\Lambda_+(x_0) = \Lambda(x_0)$, and hence if x_0 is an abnormal point for f, that is, $\operatorname{im} f'(x_0) \neq Y$, then Milyutin's conditions automatically hold (for then $\Lambda(x_0)$ contains two points, $\lambda_1 = (0, \dots, 0, y^*)$ and $\lambda_2 = -\lambda_1$) and provide no additional information. In any case, Milyutin's necessary conditions in [7] and [8] are weaker than the conditions in Theorem 2.1, since we always have $\Lambda_m(x_0) \subseteq \Lambda_+(x_0)$.

Necessary second-order conditions requiring that $\Lambda_m(x_0) \neq \varnothing$, without an a priori assumption of normality, were first obtained in [9] for the time-optimality problem, and then in [10] for the general optimal control problem and the mathematical programming problem. The conditions (2.4) were obtained in [4].

We point out an important feature which goes hand in hand with constraints of inequality type. Assume that the Mangasarian–Fromovitz regularity condition holds at x_0:

$$\operatorname{im} f'(x_0) = Y, \qquad \exists\, d \in \ker f'(x_0)\colon \quad \bigl\langle f_j'(x_0), d \bigr\rangle < 0, \quad j = 1, \dots, l,$$

which is a natural generalization of the Lyusternik condition to problems with inequalities. Then $\lambda^0 > 0$ for all $\lambda \in \Lambda(x_0)$, but there is not necessarily a 'universal' Lagrange multiplier λ for which $\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0$ for all $x \in \mathscr{K}(x_0)$. A corresponding example with $X = \mathbb{R}^3$ and $l = 4$ was presented in [11] (§ 2.4, p. 159).

Condition (2.4) in the theorem can be represented in the following equivalent form. For each $y \in \mathbb{R}^l \times \mathbb{R}^k$ let

$$\omega(y) = \inf\bigl\{f_0''(x_0)[x, x],\, x \in \Omega(y)\bigr\}$$
$$\forall\, y\colon \Omega(y) = \bigl\{x \in \mathscr{K}(x_0)\colon y = F''(x_0)[x, x]\bigr\} \neq \varnothing,$$

where this infimum can also be equal to $-\infty$. It is easy to see that (2.4) is equivalent to

$$\forall\, y\colon \Omega(y) \neq \varnothing \ \ \exists\, \lambda = (\lambda^0, \lambda^1, \dots, \lambda^l, y^*) \in \Lambda_m(x_0)\colon \quad \lambda^0 \omega(y) + \langle \bar\lambda, y\rangle \geqslant 0, \tag{2.9}$$

where $\bar\lambda = (\lambda^1, \dots, \lambda^l, y^*)$. Here if $\omega(y) = -\infty$, then the last inequality in (2.9) means that $\lambda^0 = 0$ and $\langle \bar\lambda, y\rangle \geqslant 0$.

Some generalizations. In the analysis of optimal control problems with geometric constraints (see § 7 below) it is convenient to use the following version of Theorem 2.1. Let X be a normed space and $C \subseteq X$ a closed convex cone. We consider the problem

$$f_0(x) \to \min, \qquad f_j(x) \leqslant 0, \quad j = 1, \dots, l, \qquad f(x) = 0, \quad x \in C, \tag{2.10}$$

which differs from (2.1) by the additional constraint $x \in C$.

We present necessary minimum conditions for the problem (2.10) at x_0. Let $\tilde\Lambda(x_0)$ be the set of normalized Lagrange multipliers λ corresponding to x_0 by the Lagrange principle for the problem (2.10):

$$\biggl\langle \frac{\partial L}{\partial x}(x_0, \lambda), x_0 \biggr\rangle = 0, \qquad \frac{\partial L}{\partial x}(x_0, \lambda) \in C^*, \qquad \lambda^j \geqslant 0 \quad \forall\, j, \qquad \sum_{j=0}^{l} \lambda^j + |y^*| = 1,$$

where $C^* = \{\xi \in X^* \colon \langle \xi, x\rangle \geqslant 0\ \forall\, x \in C\}$ is the dual cone of C. We set

$$\widetilde{\mathscr{K}}(x_0) = \bigl\{x \in C + \operatorname{span}\{x_0\} \colon \langle f_j'(x_0), x\rangle \leqslant 0\ \forall\, j,\ f'(x_0)x = 0\bigr\}.$$

Let $N = C \cap (-C)$ (so that N is the maximal linear subspace of C) and let $\tilde{\Lambda}_m(x_0)$ be the set of Lagrange multipliers $\lambda \in \tilde{\Lambda}(x_0)$ such that there exists a linear subspace $\Pi \subseteq N$ satisfying (2.3), where $s = m = k + l$ and codim is the codimension with respect to the subspace N.

Theorem 2.4. *In the problem* (2.10) *let* x_0 *be a local minimum point. Then*

$$\Lambda = \tilde{\Lambda}_m(x_0) \neq \varnothing, \qquad \max_{\lambda \in \Lambda} \frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0 \quad \forall\, x \in \widetilde{\mathscr{K}}(x_0). \tag{2.11}$$

The proof is similar to the proof of Theorem 2.1; it can be found in [6]. The structure of the cone $\widetilde{\mathscr{K}}(x_0)$ is important in (2.11): it is not necessarily closed. In fact, if the cone C is not polyhedral (cannot be represented as an intersection of finitely many half-spaces) and x_0 lies on the boundary of C but $x_0 \neq 0$, then the cone $C + \operatorname{span}\{x_0\}$ in the definition of $\widetilde{\mathscr{K}}(x_0)$ is not necessarily closed. On the other hand, since $\tilde{\Lambda}_m(x_0)$ is a compact set and the maximum function is continuous on this compact set, (2.11) holds for all $x \in \operatorname{cl} \widetilde{\mathscr{K}}(x_0)$. (Here and throughout, cl denotes the closure of a set.)

In this connection it seems natural to ask whether condition (2.11) survives the replacement of $\widetilde{\mathscr{K}}(x_0)$ by the cone

$$\bigl\{x \in \operatorname{cl}\bigl(C + \operatorname{span}\{x_0\}\bigr) \colon \langle f_j'(x_0), x\rangle \leqslant 0\ \forall\, j,\ f'(x_0)x = 0\bigr\}. \tag{2.12}$$

(Note that $\operatorname{cl}(C + \operatorname{span}\{x_0\})$ coincides with the tangent cone $T_C(x_0)$ to the convex cone C at x_0.) The example of the problem

$$f_0(x) = x_2 - x_1^2 \to \min, \qquad x \in C, \quad x_3 - 1 = 0, \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

where C is the closed conical hull of the set $\{x \colon x_2 \geqslant x_1^2,\ x_3 = 1\}$ and $x_0 = (0, 0, 1)$, answers this question in the negative (see [12] for details).

At the same time, in the second-order conditions (2.11) we can take the cone (2.12) in place of $\widetilde{\mathscr{K}}(x_0)$, but we must then add a certain term to the quadratic form $\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x]$ in (2.11). We can explain this by the example of the following more general problem (see [13]):

$$f_0(x) \to \min, \qquad F(x) \in C, \tag{2.13}$$

where $F \colon X \to Y = \mathbb{R}^k$, and C is an arbitrary closed (but not necessarily convex) subset of Y. We note that an even more general problem than (2.13) was considered in [13], a problem with the additional constraint $x \in \widetilde{C}$, where $\widetilde{C}$ is a subset of X closed in the finite topology (but not necessarily convex).

To state the theorem we introduce the requisite concepts. Let $y_0 = F(x_0) \in C$. Recall that a vector d is said to be tangent to C at a point y_0 if there exists a sequence $\{\varepsilon_n\} \downarrow 0$ such that

$$\operatorname{dist}(y_0 + \varepsilon_n d, C) = o(\varepsilon_n),$$

where $\operatorname{dist}(y, C) = \inf_{\xi \in C}\{\|\xi - y\|\}$ is the distance from y to C. The set of tangent vectors to C at y_0 forms a cone (the Bouligand cone), denoted by $T_C(y_0)$.

Let $N(C; y_0)$ denote the Mordukhovich normal cone to C at y_0 (see [13]). We define the second-order outer tangential set to C at $y_0 \in C$ in the direction $d \in T_C(y_0)$ by the formula

$$O_C^2(y_0, d) = \left\{ w \in Y : \exists \{\varepsilon_n\} \downarrow 0,\ \operatorname{dist}\left(y_0 + \varepsilon_n d + \frac{1}{2}\varepsilon_n^2 w, C \right) = o(\varepsilon_n^2) \right\}.$$

The linear subspace $P \subseteq Y$ is called a locally invariant subspace with respect to C at $y_0 \in C$ if there exists a $\delta > 0$ such that $(C \cap B(y_0, \delta)) + (P \cap B(0, \delta)) \subseteq C$, where $B(y, \delta)$ is the ball of radius δ about y. Obviously, the trivial subspace is locally invariant. Since Y is finite-dimensional, there exists a maximal locally invariant subspace with respect to inclusion; we denote it by $\mathscr{I}_C$. Let

$$\mathscr{K}^C(x_0) = \{x \colon F'(x_0)x \in T_C(y_0),\ \langle f_0'(x_0), x\rangle \leqslant 0\},$$
$$\lambda = (\lambda^0, y^*), \qquad \lambda^0 \in \mathbb{R}, \qquad y^* \in Y^*,$$
$$\Lambda(C; x_0) = \left\{ \lambda \colon \lambda^0 \geqslant 0,\ y^* \in N(C; y_0),\ \frac{\partial L}{\partial x}(x_0, \lambda) = 0,\ |\lambda| = 1 \right\}.$$

(Of course, here we define the Lagrange function L by (1.2).)

Consider the set of Lagrange multipliers $\lambda \in \Lambda(C; x_0)$ for which there exists a subset $\Pi \subseteq X$ (depending on λ) such that

$$\operatorname{codim} \Pi \leqslant k, \qquad F'(x_0)(\Pi) \subseteq \mathscr{I}_C, \qquad \frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0 \quad \forall\, x \in \Pi. \tag{2.14}$$

The set of these Lagrange multipliers will be denoted by $\Lambda_k(C; x_0)$.

Theorem 2.5. *In the problem (2.13) let x_0 be a local minimum point in the finite topology. Then the set $\Lambda_k(C; x_0)$ is non-empty, and moreover, for any $x \in \mathscr{K}^C(x_0)$ and $w \in O_C^2(F(x_0), F'(x_0)x)$*

$$\max_{\lambda \in \Lambda} \left(\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] - \langle y^*, w\rangle \right) \geqslant 0, \quad \textit{where } \lambda = (\lambda^0, y^*),\ \ \Lambda = \Lambda_k(C; x_0). \tag{2.15}$$

The proof is presented in [13]. By the minimax theorem (see [13], Appendix A), (2.15) implies the following assertion.

Under the hypotheses of Theorem 2.5 the following inequality holds for any convex set $\mathscr{T}(x) \subseteq O_C^2(F(x_0), F'(x_0)x)$:

$$\max_{\lambda \in \Lambda} \left(\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] - \sigma(y^*, \mathscr{T}(x)) \right) \geqslant 0, \quad \text{where } \Lambda = \operatorname{conv} \Lambda_k(C; x_0), \tag{2.16}$$

$\sigma(\cdot, T)$ is the support function of a set $T \subseteq X$, that is, $\sigma(y^*, T) = \sup_{y \in T}\langle y^*, y\rangle$ for $y^* \in Y$, and conv is the convex hull of a set.

Now we discuss Theorem 2.5. It generalizes in a natural way results in [14] and [15], where it was additionally assumed that either the Robinson regularity condition $0 \in \operatorname{int}\bigl(F(x_0) + \operatorname{im} F'(x_0) - C\bigr)$ holds or that $\operatorname{int} C \neq \varnothing$ (see [13] and [16] for details).

Let $F'(x_0)x \in \mathscr{I}_C$. Then $0 \in O^2_C\bigl(F(x_0), F'(x_0)x\bigr)$, and for $w = 0$ we get from (2.15) that

$$\max_{\lambda \in \Lambda} \frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] \geqslant 0.$$

If C is a convex set, then $\sigma\bigl(y^*, \mathscr{T}(x)\bigr) \leqslant 0$ for all y^* such that $(\lambda^0, y^*) \in \Lambda(C; x_0)$ for some $\lambda^0 \geqslant 0$ (see [14], p. 178 or [13], p. 4). On the other hand, if C is a convex polyhedral cone (for instance, in the case $C = \mathbb{R}^l_- \times \mathbb{R}^k$, where $\mathbb{R}^l_-$ is the non-positive orthant, we arrive at the problem (2.1)), then $\sigma(y^*, \mathscr{T}(x)) = 0$ for all such y^*.

At the same time, if C is not convex, then the term $\sigma(y^*, \mathscr{T}(h))$ can be strictly positive. For example, let $C = \bigl\{y = (y_1, y_2) \in \mathbb{R}^2 \colon y_1^l \geqslant y_2^m\bigr\}$, where l and m, $m < l \leqslant 2m$, are positive integers. We take $h = (1, 0) \in T_C(0)$. Then the set $O^2_C(0, h)$ contains a ball B about the origin, and hence $\sigma(\zeta, B) > 0$ for all $\zeta \neq 0$. Let $l = 2$, $m = 1$, $f_0(x) = x_1^2 - x_2$, $Y = \mathbb{R}^2$, and $F(x) \equiv x$. Then $x_0 = 0$ is a solution of the problem (2.13), $h \in \mathscr{K}^C(0)$, $y^* = -f_0'(0)$, but $\sigma(-f_0'(0), B) > 0$.

Abnormal problems. We return to the problem (2.1). If x_0 is a normal point for the map F, then the set $\Lambda(x_0)$ is a singleton and Theorem 2.1 gives the classical necessary second-order conditions (which become (1.4) for $l = 0$). Therefore, these conditions cannot be strengthened in the normal case. On the other hand, if x_0 is an abnormal point for F, that is, $F'(x_0)$ is not a surjective operator, then Theorem 2.1 can be improved by replacing $\Lambda_m(x_0)$ by a smaller set. We formulate the corresponding result.

Let $C \subseteq Y$ be a closed convex set, and define the inner second-order tangential set to C at $y \in C$ in the direction $d \in T_C(y)$ by

$$T^2_C(y, d) = \biggl\{w \in Y \colon \operatorname{dist}\biggl(y + \varepsilon d + \frac{1}{2}\varepsilon^2 w, C\biggr) = o(\varepsilon^2),\ \varepsilon \geqslant 0\biggr\}.$$

This is a closed convex set, and $T^2_C(y, d) \subseteq O^2_C(y, d)$, although the last inclusion may be strict (see [14]).

Theorem 2.6. *In the problem* (2.13) *let x_0 be a local minimum point in the finite topology and assume that* $\operatorname{im} F'(x_0) + \mathscr{I}_C \neq Y$ (*in this case x_0 is called an abnormal point*). *Then $\Lambda_{k-1}(C; x_0)$ is non-empty, and for each $x \in \mathscr{K}^C(x_0)$*

$$\max_{\lambda \in \Lambda} \biggl(\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] - \sigma\bigl(y^*, T^2_C\bigl(F(x_0), F'(x_0)x\bigr)\bigr)\biggr) \geqslant 0, \qquad \text{where} \quad \Lambda = \operatorname{conv} \Lambda_{k-1}(C; x_0). \tag{2.17}$$

Note that in Theorem 2.6 (just as in Theorem 2.5), we can replace the maximal (with respect to inclusion) local invariant subspace $\mathscr{I}_C$ by any other local invariant subspace, for instance, the trivial one.

A proof of Theorem 2.6 which uses results of real algebraic geometry is presented in [17], and also in [18] for $C = \{0\}$. For $C = \mathbb{R}^l_- \times \{0\} \subset \mathbb{R}^l \times \mathbb{R}^k$ we obtain the problem (2.1), and hence Theorem 2.6 implies the following theorem.

Theorem 2.7. *In the problem* (2.1) *let* x_0 *be a local minimum point in the finite topology, and let* x_0 *be an abnormal point for the map* F (*that is*, $\operatorname{im} F'(x_0) \neq Y$). *Then the set* $\Lambda_{m-1}(x_0)$ *is non-empty and*

$$\max_{\lambda\in\Lambda} \frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x] \geqslant 0 \quad \forall\, x \in \mathscr{K}(x_0), \quad \textit{where} \quad \Lambda = \Lambda_{m-1}(x_0). \tag{2.18}$$

Note that the example of the problem (1.5) shows that in Theorem 2.6 we cannot in general replace $(k-1)$ by $(k-2)$ even for $k \geqslant 2$.

The 'gap' between necessary and sufficient second-order conditions. A major characteristic of necessary second-order conditions is the size of the 'gap' which exists between them and sufficient second-order conditions. It is reasonable to treat such a gap as the smallest possible (in the class of necessary second-order conditions) if the necessary conditions become sufficient after implementing arbitrarily small perturbations, in the C^2-metric, of the function to be minimized and the map giving the constraints, without changing their values nor the values of their first derivatives at the point under consideration. In the case of the necessary second-order conditions obtained above we find out when such a gap in the problem (2.1) is smallest possible.

Let X be a Hilbert space. First we assume that x_0 is a normal point for F and that the necessary second-order conditions (2.4) hold at x_0. Then for an arbitrary $\varepsilon > 0$ the function $f_{0,\varepsilon}(x) = f_0(x)+\varepsilon|x-x_0|^2$ has a strict local minimum at x_0 under the constraints of the problem (2.1) in view of the sufficient second-order conditions (see [7], [5] or Theorem 4.1 in § 4 below), although the size of the neighbourhood in which x_0 is a minimum point can tend to zero as $\varepsilon \to 0$.

Assume now that x_0 is an abnormal point for F. It turns out that everything then depends on whether or not the convex hull $\operatorname{conv}\Lambda_{m-1}(x_0)$ contains the origin. In fact, the maximum over $\Lambda = \Lambda_{m-1}(x_0)$ in (2.18) coincides with the maximum over the convex hull of this set. Hence if $0 \in \operatorname{conv}\Lambda$, then (2.18) necessarily holds. In this case condition (2.18) is satisfied for any function f_0 to be minimized, so it can give no additional information in the minimization problem (2.1). The situation here is the same as with the Lagrange multiplier rule at an abnormal point. Hence if $0 \in \operatorname{conv}\Lambda$, then in general we should not expect to make x_0 a local minimum in a perturbed problem by means of C^2-small perturbations of the functions f and f_j without changing their values nor the values of their first derivatives at x_0. This is confirmed by the example of the problem

$$f_0(x) = x_1 \to \min,$$
$$x_1x_2 = 0, \qquad x_1x_3 = 0, \qquad l = 0, \qquad m = k = 2, \qquad x = (x_1,x_2,x_3) \in \mathbb{R}^3.$$

Here $0 \in \operatorname{conv}\Lambda_1(0)$, and no perturbations of the type described above can make $x_0 = 0$ a local minimum point (this follows from Theorem 3.1 proved below). The case when $0 \notin \operatorname{conv}\Lambda$ is utterly different.

For integers $s \geqslant 0$ let $\mathscr{F}_s^2(x)$ be the set of $y^* \in Y^*$ with $|y^*| = 1$ such that $f'(x)^*y^* = 0$ and X has a subspace Π for which

$$\begin{gathered} \Pi \subseteq \ker f'(x), \qquad \operatorname{codim}\Pi \leqslant s, \\ \left\langle \frac{\partial^2 f}{\partial x^2}(x)[\xi,\xi], y^* \right\rangle \geqslant 0 \quad \forall\, \xi \in \Pi. \end{gathered} \tag{2.19}$$

Clearly, if x is a normal point for f, then $\mathscr{F}_s^2(x)=\varnothing$ for all s.

Definition. A map f is said to be 2-normal at a point x if $0\notin \operatorname{conv}\mathscr{F}_{k-1}^2(x)$ (in particular, if $\mathscr{F}_{k-1}^2(x)=\varnothing$). The map f is said to be 2-normal if it is 2-normal at each point.

Obviously, $y^*\in\mathscr{F}_{k-1}^2(x)\Leftrightarrow(0,y^*)\in\Lambda_{k-1}(x)$ (here $l=0$ and $m=k$). This is a geometric definition which is not convenient for testing 2-normality. In [5] (Chap. 1, § 1.9) the reader can find sufficient conditions for 2-normality in terms of f. It is also proved there that if $X=\mathbb{R}^n$ and $n\gg k$ (for instance, $n>2(k-2)$, $(n-k-1)(n-k)>2(k-1)$), then 2-normality is a generic property: in the space $C_s^3(\mathbb{R}^n,\mathbb{R}^k)$ with the Whitney topology (see [19], Chap. 2, § 1) the set of 2-normal maps is massive (that is, contains an intersection of countably many dense open subsets) and therefore dense.

Theorem 2.8. *Let X be a Hilbert space, and let f be a map that is 2-normal at a point x_0 at which the necessary second-order conditions* (2.18) *are satisfied. Then there exists a vector $\bar y\in Y$ such that for each $\varepsilon>0$ the point x_0 supplies a strict local minimum in the perturbed problem*

$$f_{0,\varepsilon}(x)=f_0(x)+\varepsilon|x-x_0|^2\to\min,$$
$$f_{j,\varepsilon}(x)=f_j(x)+\varepsilon|x-x_0|^2\leqslant 0,\ \ j=1,\dots,l,\qquad f_\varepsilon(x)=f(x)+\varepsilon|x-x_0|^2\bar y=0.$$

The proof is based on sufficient second-order conditions; it is presented in [5], Chap. 1, § 1.8.

Proof of Theorem 2.3. This is proved for an arbitrary Banach space in [5], but here we confine ourselves to a special case by assuming that X is a Hilbert space. This makes the argument considerably simpler. Theorem 2.3 first appeared in [9] just in the Hilbert case.

Without loss of generality we assume that $\operatorname{codim}\Pi_i=k$ for all i. In each orthogonal complement $\Pi_i^\perp$ we select some orthonormal basis $e_{i,1},\dots,e_{i,k}$. Passing to subsequences, we can assume that as $i\to\infty$ the sequence $\{e_{i,j}\}$ converges weakly to a vector e_j for each $j\in\{1,\dots,k\}$. We assert that $\Pi=\bigl\{x\in X\colon\langle e_j,x\rangle=0$ for all $j\in\{1,\dots,k\}\bigr\}$ is the required subspace. For the proof it suffices to find for an arbitrary point $x_0\in\Pi$ a sequence $\{x_i\}$ with $x_i\in\Pi_i$ for all i which converges to it. To do this, for each i we consider the minimization problem

$$|x-x_0|^2\to\min,\qquad \langle e_{i,j},x\rangle=0,\quad j=1,\dots,k.$$

We readily see that each of these problems is solvable; let x_i be the corresponding solution. By the Lagrange principle,

$$\exists\,\lambda_{i,j}\colon x_i-x_0=\sum_{j=1}^k\lambda_{i,j}e_{i,j}\Rightarrow|x_i-x_0|^2=\sum_{j=1}^k\lambda_{i,j}^2\Rightarrow|\lambda_{i,j}|\leqslant|x_i-x_0|\quad\forall\,i,j.$$

Hence, taking the inner product of the first equality with $x_i - x_0$, we have

$$|x_i - x_0|^2 = -\sum_{j=1}^{k} \lambda_{i,j}\langle e_{i,j}, x_0\rangle \leqslant \sum_{j=1}^{k} |\lambda_{i,j}|\,\bigl|\langle e_{i,j}, x_0\rangle\bigr| \leqslant |x_i - x_0|\sum_{j=1}^{k}\bigl|\langle e_{i,j}, x_0\rangle\bigr|$$
$$\Rightarrow |x_i - x_0| \leqslant \sum_{j=1}^{k}\bigl|\langle e_{i,j}, x_0\rangle\bigr| \to \sum_{j=1}^{k}\bigl|\langle e_j, x_0\rangle\bigr| = 0. \qquad \square$$

3. Necessary extremum conditions of the first and second order; 2-regularity

The generalized Lagrange function $\mathscr{L}_{\mathscr{A}}$. In this section we consider another approach to necessary conditions in extremal problems, based on the generalized Lagrange function $\mathscr{L}_{\mathscr{A}}$ and the concept of 2-regularity. For greater transparency we confine ourselves to problems with constraints of equality type,

$$f_0(x) \to \min, \qquad F(x) = 0, \tag{3.1}$$

where f_0 is a given function to be minimized, $F\colon X \to Y$ is a given map, and the spaces X and Y are as in the previous section. Let x_0 be a local minimum point in the problem under consideration (with respect to the finite topology). In what follows we assume that f_0 is twice continuously differentiable in a neighbourhood of x_0, F is thrice continuously differentiable in the finite topology (see § 2), and its third derivative satisfies a Lipschitz condition.

Now we introduce the requisite concepts. Let π denote the operator of orthogonal projection of Y onto $\bigl(\operatorname{im} F'(x_0)\bigr)^{\perp}$. We consider the non-empty cone $(0 \in H)$

$$H = H(x_0) = \bigl\{h \in X\colon F'(x_0)h = 0,\ F''(x_0)[h,h] \in \operatorname{im} F'(x_0)\bigr\}. \tag{3.2}$$

This cone is very important for investigations of the admissible set $M = \bigl\{x \in X\colon F(x) = 0\bigr\}$. In fact, it is easy to see that if h is a tangent vector to M at the point x_0, then $h \in H(x_0)$. However, it is much more important that the converse result also holds under certain additional but natural assumptions. To explain this we introduce an important notion.

For an arbitrary fixed $h \in H(x_0)$ let $G(x_0,h)\colon X \times \ker F'(x_0) \to Y$ be the linear operator defined by

$$G(x_0,h)(x_1,x_2) = F'(x_0)x_1 + F''(x_0)[h,x_2], \qquad x_1 \in X, \quad x_2 \in \ker F'(x_0).$$

Definition. The map F is said to be 2-regular at a point x_0 in a direction $h \in H(x_0)$ if the operator $G(x_0,h)$ is surjective.[4] A map F is said to be 2-regular at x_0 if it is 2-regular at x_0 in any non-zero direction $h \in H(x_0)$.

In another (although equivalent) form the notion of a 2-regular direction was introduced in [20]. Note that if a map is normal at some point, then it is obviously 2-regular at this point, but the converse fails in general.

[4] We use the word 'direction' as more intuitive, but treat it as a synonym for 'vector'.

Thus, as proved in [20], if F is a 2-regular map at a point x_0 in a direction $h \in H(x_0)$, then h is a tangent vector to M at x_0. (This also follows from Theorem 3.3 proved below.) In particular, if F is 2-regular at x_0, then $H(x_0)$ coincides with the tangent cone to M at x_0. So from a description of the tangent cone to an admissible set at a minimum point we can deduce necessary conditions for a minimum in an extremal problem.

Let us consider the function $\mathscr{L}_{\mathscr{A}}\colon X \times \mathbb{R} \times Y^* \times Y^* \times H(x_0) \to \mathbb{R}$ introduced in [20]:

$$\mathscr{L}_{\mathscr{A}}(x, \lambda_A, h) = \lambda^0 f_0(x) + \langle y_1^*, F(x)\rangle + \langle y_2^*, F'(x)h\rangle,$$
$$\lambda_A = (\lambda^0, y_1^*, y_2^*), \qquad \lambda^0 \in \mathbb{R}, \qquad y_1^*, y_2^* \in Y^*, \qquad h \in H(x_0).$$

The vector λ_A and its components are also called Lagrange multipliers. Note that λ_A, in comparison to the classical Lagrange multipliers λ considered above, has the additional component y_2^*, and the function $\mathscr{L}_{\mathscr{A}}$ differs from the Lagrange function L by the presence of the last term containing y_2^*. The vector h plays the role of a parameter in the function $\mathscr{L}_{\mathscr{A}}$.

Theorem 3.1. *Let x_0 be a local minimum point in the problem (3.1) with respect to the finite topology. Then for each $h \in H(x_0)$ there exists a λ_A such that*

$$\frac{\partial \mathscr{L}_{\mathscr{A}}}{\partial x}(x_0, \lambda_A, h) = 0, \tag{3.3}$$

$$y_1^* \in \operatorname{im} F'(x_0), \qquad y_2^* \in \bigl(\operatorname{im} F'(x_0)\bigr)^{\perp}, \qquad \lambda^0 \geqslant 0, \qquad \lambda^0 + |y_2^*| \neq 0. \tag{3.4}$$

Theorem 3.2. *Let x_0 be a local minimum point in the problem (3.1) with respect to the finite topology, and assume that the function f_0 is twice continuously differentiable in a neighbourhood of x_0 with respect to the finite topology.*

Then for each vector $h \in H(x_0)$ there exists a Lagrange multiplier λ_A satisfying (3.3), (3.4), and the relation

$$\frac{\partial^2 \mathscr{L}_{\mathscr{A}}}{\partial x^2}\Bigl(x_0, \lambda^0, y_1^*, \frac{1}{3} y_2^*, h\Bigr)[h, h] \geqslant 0. \tag{3.5}$$

Theorems 3.1 and 3.2 yield necessary conditions of orders 1 and 2. They were established in [20] in the case when X and Y are Banach spaces, the maps f_0 and F have sufficiently many derivatives, and the subspace $\operatorname{im} F'(x_0)$ is closed.

2-regularity. Before proving these results we discuss them. If x_0 is a normal point, then $y_2^* = 0$ by (3.4), $\mathscr{L}_{\mathscr{A}}$ becomes the ordinary Lagrange function L, and the Lagrange multiplier $\lambda_A = (\lambda^0, y_1^*, 0)$, without the last component equal to zero, becomes an ordinary Lagrange multiplier $\lambda = (\lambda^0, y^*)$, while Theorem 3.1 becomes the Lagrange principle (1.3). (To see the last assertion, let $h = 0$ in Theorem 3.1.) Incidentally, Theorem 3.2 becomes (1.4). Thus, when the normality condition is imposed, the above theorems are equivalent to the classical necessary second-order conditions.

If x_0 is an abnormal point and, moreover, F is not 2-regular in a direction $h \in H(x_0)$, then any Lagrange multiplier $\lambda_A = (0, y_1^*, y_2^*)$ with y_1^* a non-zero vector in $\bigl(\operatorname{im} G(x_0, h)\bigr)^{\perp}$ and $y_2^* = \pi y_1^*$ satisfies Theorem 3.1. Hence, Theorem 3.1 gives us no information in this case (in fact, $\lambda^0 = 0$ and the functional to be minimized is not involved in the conditions (3.3) and (3.4)): it simply expresses the lack of 2-regularity in the direction h. In addition, Theorem 3.2 contains no useful information in this case, because together with a Lagrange multiplier λ_A, $(-\lambda_A)$ also satisfies conditions (3.3) and (3.4), so that (3.5) holds automatically.

The most interesting case is the third, when x_0 is an abnormal point, but F is 2-regular in the direction h. Then it is easy to see that $\lambda^0 > 0$, and Theorem 3.1 provides useful information about the gradient of the function to be minimized, while Theorem 3.2 gives information about the second derivative of the same function. Furthermore, since the linear operator $G(x_0, h)$ is surjective, the Lagrange multipliers λ_A are determined by (3.3) and (3.4) uniquely up to normalization.

Proofs of Theorems 3.1 and 3.2. The proofs are based on the following result.

Theorem 3.3. *Let $h \in H(x_0)$ and let F be a 2-regular map at x_0 in the direction h. Then there exist an $h_2 \in X$ and a finite-dimensional subspace $\widetilde{X} \subseteq X$ such that*

$$F'(x_0)h_2 + \frac{1}{2}F''(x_0)[h,h] = 0, \quad F''(x_0)[h,h_2] + \frac{1}{6}F'''(x_0)[h,h,h] \in \operatorname{im} F'(x_0), \tag{3.6}$$

$$F\bigl(h(\varepsilon)\bigr) = 0 \ \ \forall\,\varepsilon, \quad h(\varepsilon) \in \widetilde{X} \ \ \forall\,\varepsilon, \quad \textit{where} \ \ h(\varepsilon) = x_0 + \varepsilon h + \varepsilon^2 h_2 + O(\varepsilon^3). \tag{3.7}$$

Proof. For convenience, here and in what follows we assume that $x_0 = 0$, and we omit x_0 in the notation for the derivatives at this point (for example, $f_0' = f_0'(x_0)$, $F'' = F''(x_0)$, and so on). Since Y has finite dimension, X contains a finite-dimensional subspace $\widetilde{X}$ such that the restriction of the linear operator $G(x_0,h)$ to $\widetilde{X} \times (\ker F' \cap \widetilde{X})$ is surjective, as before. Hence, replacing X by $\widetilde{X}$, we can assume that X is finite-dimensional.

In view of the 2-regularity in the direction h there exist $h_2, h_3 \in X$ such that

$$F'h_2 + \frac{1}{2}F''[h,h] = 0, \qquad F'h_3 + F''[h,h_2] + \frac{1}{6}F'''[h,h,h] = 0 \tag{3.8}$$

(see [5], § 5, pp. 72, 73 for greater detail). For a fixed ε we seek $h(\varepsilon)$ in the form

$$h(\varepsilon) = \varepsilon h + \varepsilon^2 h_2 + \varepsilon^3 h_3 + \varepsilon^2 r_2 + \varepsilon^3 r_1,$$

where the unknowns $r_1 = r_1(\varepsilon) \in X$ and $r_2 = r_2(\varepsilon) \in \ker F'$ are to be determined.

In a neighbourhood of the origin F has the representation

$$F(x) = F'x + \frac{1}{2}F''[x,x] + \frac{1}{6}F'''[x,x,x] + R(x)[x,x,x],$$

where $R(x)$ is a multilinear map for each x, $R(0) = 0$, and $R(\cdot)$ is locally Lipschitz. For a fixed $\varepsilon \neq 0$ we consider the equation $F\bigl(h(\varepsilon)\bigr) = 0$ with respect to $r_1 \in X$ and $r_2 \in \ker F'$. In view of the above representation and (3.8), after division by ε^3 this equation takes the form

$$G(x_0,h)(r_1,r_2) = \Delta(\varepsilon, r_1, r_2), \tag{3.9}$$

where Δ is a continuous map such that $\Delta(0, r_1, r_2) \equiv 0$, and on the unit ball the map $\Delta(\varepsilon,\cdot,\cdot)$ satisfies a Lipschitz condition with Lipschitz constant $k(\varepsilon) \to 0$ as

$\varepsilon \to 0$. Since the operator $G(x_0, h)$ is surjective (has the covering property), we can apply the theorem on the existence of coincidence points (Theorem 1 in [21]) to (3.9). This shows that for any ε with sufficiently small absolute value there exists a solution $r_1(\varepsilon)$, $r_2(\varepsilon)$ of equation (3.9) such that $|r_1(\varepsilon)| + |r_2(\varepsilon)| \leqslant \mathrm{const}\, |\varepsilon|$. $\square$

Proof of Theorem 3.1. By the above, it is sufficient to consider the case when F is 2-regular at $x_0 = 0$ in the direction h, so that is what we do. By Theorem 3.3 the vectors h and $(-h)$ are tangent to the set $M = \{x\colon F(x) = 0\}$ at $x_0 = 0$. Hence $\langle f_0', h\rangle = 0$.

Let $T_H(h)$ be the tangent cone to $H = H(x_0)$ at the point h. We consider the linear operator $\widetilde{G}(h)\colon X \to Y$ defined by $\widetilde{G}(h)x = F'x + \pi F''[h, x]$. Since $G(x_0, h)$ is surjective, so is $\widetilde{G}(h)$. We define the map $g\colon X \to Y$ by the formula $g(x) = \widetilde{G}(x)x$. Obviously, $H = \{x\colon g(x) = 0\}$ and h is a normal point for the map g. Hence, Lyusternik's theorem (on the tangent cone; see [1]) implies that $\ker \widetilde{G}(h) \subseteq T_H(h)$. Therefore, for any $x \in \ker \widetilde{G}(h)$ there exist a sequence $\{\alpha_i\}$ of positive numbers tending to zero and a sequence $\{a_i\}$ in X of vectors tending to zero such that $h_i = h + \alpha_i x + \alpha_i a_i \in H$ for all large i. However, by Theorem 3.3 all the vectors in H that are sufficiently close to h are also tangent to the set M. This applies to the vectors h_i for large i, so that $\langle f_0', h_i\rangle \leqslant 0 \Rightarrow \langle f_0', x\rangle \leqslant 0$ for all $x \in \ker \widetilde{G}(h)$. Hence $\langle f_0', x\rangle = 0$ for all $x \in \ker \widetilde{G}(h)$, since if x belongs to $\ker \widetilde{G}(h)$, then so does $(-x)$.

Thus, we have proved that

$$\langle f_0', x\rangle = 0 \quad \forall\, x \in X\colon\ F'x = 0, \quad \pi F''[h, x] = 0.$$

The annihilator lemma (see [1], p. 26) now shows that there exist y_1^* and y_2^* such that $\lambda_A = (1, y_1^*, y_2^*)$ satisfies (3.3) and (3.4). $\square$

Remark. It follows from the proof of Theorem 3.1 in [20] that if X is a Banach space, then it suffices in Theorem 3.1 to assume only that F is Fréchet-differentiable twice at x_0. We note that in [5] (§ 1.13) necessary extremum conditions of higher order are found.

Proof of Theorem 3.2. We take h_2 such that (3.6) and (3.7) hold. Then by (3.7)

$$\langle f_0', h_2\rangle + \frac{1}{2} f_0''[h, h] \geqslant 0. \tag{3.10}$$

Let $\lambda_A = (1, y_1^*, y_2^*)$ be a Lagrange multiplier satisfying (3.3) and (3.4). Then from (3.6) we get that $\langle f_0', h_2\rangle = \bigl(\langle y_1^*, F''[h, h]\rangle + \langle y_2^*, F'''[h, h, h]\rangle/3\bigr)/2$. Substitution of this in (3.10) yields (3.5). $\square$

The classical Lagrange principle in abnormal problems. The next result strengthens Theorem 3.1.

Theorem 3.4. *Let x_0 be a local minimum point in the problem* (3.1) *in the finite topology. If F is a 2-regular map in a direction $h \in H(x_0)$, then there exists a Lagrange multiplier λ_A such that conditions* (3.3) *and* (3.4) *hold, and also*

$$\bigl\langle y_2^*, F''(x_0)[x, x]\bigr\rangle = 0 \quad \forall\, x\colon\ F'(x_0)x = 0, \quad F''(x_0)[h, x] \in \operatorname{im} F'(x_0). \tag{3.11}$$

Proof. By Theorem 3.1 the vector h corresponds to a unique Lagrange multiplier with first component equal to 1, $\lambda_A = (1, y_1^*, y_2^*)$. As was shown in the proof of Theorem 3.1, for an arbitrary $x \in \ker \widetilde{G}(h)$ there exist a sequence $\{\alpha_i\}$ of positive numbers tending to zero and a sequence $\{a_i\} \subset X$ of vectors lying in some finite-dimensional subspace and tending to zero such that $h_i = h + \alpha_i x + \alpha_i a_i \in H(x_0)$ for all i, and the map F is 2-regular in each of the directions h_i. It then follows from Theorem 3.1 that for all i there exist Lagrange multipliers $\lambda_A^i = (1, y_{1,i}^*, y_{2,i}^*)$ such that (3.4) holds and

$$\frac{\partial \mathscr{L}_{\mathscr{A}}}{\partial x}(x_0, \lambda_A^i, h_i) = 0. \tag{3.12}$$

Since the operator $\widetilde{G}(h)$ is surjective, the sequences $\{y_{1,i}^*\}$ and $\{y_{2,i}^*\}$ are bounded, and thus, passing to subsequences, we can assume that $\{y_{s,i}^*\} \to \tilde{y}_s^*$ as $i \to \infty$, $s = 1, 2$. Obviously, $\tilde{\lambda}_A = (1, \tilde{y}_1^*, \tilde{y}_2^*)$ corresponds to the vector h by Theorem 3.1. Hence $\tilde{\lambda}_A = \lambda_A \Rightarrow \lambda_A^i \to \lambda_A$, and therefore, applying the left-hand side of (3.12) to the vector $x \in \ker \widetilde{G}(h)$, bearing in mind that $f_0' \in \bigl(\ker \widetilde{G}(h)\bigr)^\perp$ and $y_{2,i}^* \in \bigl(\operatorname{im} F'(x_0)\bigr)^\perp$, and passing to the limit as $i \to \infty$, we obtain (3.11). $\square$

The next simple example demonstrates that in Theorem 3.4 the assumption that F is 2-regular in the direction h is essential.

Example 3.1. For positive integers $1 \leqslant m < n$ and $a \in \mathbb{R}^n$ with $a \neq 0$ we consider the problem

$$f_0 = \langle a, x \rangle \to \min, \qquad F(x) = \sum_{i=1}^{m} x_i^2 + \sum_{i=m+1}^{n} x_i^4 = 0.$$

Its solution is the point $x_0 = 0$, and $H(0) = \{h \colon h_i = 0,\ i = 1, \dots, m\} \Rightarrow F''(0)h = 0$ for all $h \in H(0)$. Let $h \in H(0)$ with $h \neq 0$ be arbitrary. For any Lagrange multipliers λ_A corresponding to this h it follows from (3.3) and (3.4) that $\lambda^0 = 0 \Rightarrow y_2^* \neq 0$. At the same time, $F''(0) \neq 0$, so that for $y_2^* \neq 0$ the condition (3.11) fails. The point is that the map F is not 2-regular in any direction $h \in H(0)$.

In the problem (3.1) assume that the minimum is attained at an abnormal point x_0. Then by the Lagrange principle (1.3) this point corresponds to a Lagrange multiplier $\bar{\lambda} = (0, \bar{y}^*)$. At the same time, Theorem 3.4 provides sufficient conditions for the existence in this problem of, besides $\bar{\lambda}$, also the so-called normal Lagrange multiplier $\lambda = (1, y^*)$. This means the validity of the classical Lagrange principle, that is, the principle in the form stated by Lagrange himself:

$$\exists\, y^* \colon \quad f_0'(x_0) + F'(x_0)^* y^* = 0. \tag{3.13}$$

Lemma 3.1. *In the problem* (3.1) *let x_0 be a local minimum point in the finite topology and assume that the set of directions $h \in H(x_0)$ in which the map F is 2-regular at x_0 is non-empty. Suppose that at least one of the following assumptions holds*:

a) *for any non-zero* $y_2^* \in \bigl(\operatorname{im} F'(x_0)\bigr)^\perp$ *the restriction of the quadratic form* $\bigl\langle y_2^*, F''(x_0)[x,x]\bigr\rangle$ *to the subspace* $\ker F'(x_0)$ *has a zero cone containing no subspaces of codimension* $\operatorname{codim}\bigl(\operatorname{im} F'(x_0)\bigr)$;

b) F *is* 2*-normal at* x_0.

Then the classical Lagrange principle (3.13) *holds at* x_0.

Proof. Assume that a) holds, let $h \in H(x_0)$ be a vector such that F is 2-regular along h, and assume that a Lagrange multiplier $\lambda_A = (\lambda^0, y_1^*, y_2^*)$ corresponds to this vector by Theorem 3.4. We assert that $y_2^* = 0$. Indeed, by (3.11) the quadratic form $\bigl\langle y_2^*, F''(x_0)[x,x]\bigr\rangle$ vanishes on the subspace $\ker \widetilde{G}(h)$, which lies in the subspace $\ker F'(x_0)$ and has codimension $\operatorname{codim}\bigl(\operatorname{im} F'(x_0)\bigr)$ there. Hence $y_2^* = 0$, so $\lambda^0 > 0$ by (3.4). Then (3.13) follows from (3.3).

Assume that b) holds and (3.13) fails. Then $\lambda^0 = 0$ for all $\lambda \in \Lambda_{k-1}(x_0)$. Therefore by Theorem 2.7, for all $x \in \ker F'(x_0)$ there exists a $y^* \in \mathscr{F}_{k-1}^2(x_0)$ such that $\bigl\langle y^*, F''(x_0)[x,x]\bigr\rangle \geqslant 0$. Since F is 2-normal, by the separation theorem for convex sets there exists a $\bar{y} \in \bigl(\operatorname{im} F'(x_0)\bigr)^\perp$ such that $\langle y^*, \bar{y}\rangle > 0$ for all $y^* \in \operatorname{conv} \mathscr{F}_{k-1}^2(x_0)$. The map F is 2-regular in the direction h, therefore there exists a finite-dimensional subspace $\widetilde{X}$ such that $\widetilde{G}(h)\widetilde{X} = Y$. For ε small in absolute value we consider the system of equations

$$F'(x_0)x = 0, \qquad \pi F''(x_0)[x,x] + \varepsilon \bar{y}|x|^2 = 0 \tag{3.14}$$

with respect to $x \in \widetilde{X}$. For $\varepsilon = 0$ it has the solution $x = h$. We apply to this system the classical implicit function theorem at the normal point $x = h$ and deduce that if ε is small in absolute value, then equation (3.14) has a solution $h(\varepsilon)$ such that $h(\varepsilon) \to h$ as $\varepsilon \to 0$. In view of the foregoing, for each ε there exists a $y^*(\varepsilon) \in \mathscr{F}_{k-1}^2(x_0)$ such that $\bigl\langle y^*(\varepsilon), F''(x_0)\bigl[h(\varepsilon), h(\varepsilon)\bigr]\bigr\rangle \geqslant 0$. Taking the inner product of the second equation in (3.14) for $x = h(\varepsilon)$ ($\varepsilon > 0$) and $y^*(\varepsilon)$, and considering that $\bigl\langle y^*(\varepsilon), \bar{y}\bigr\rangle > 0$, we arrive at a contradiction to the previous inequality. □

Lemma 3.2. *Let* x_0 *be an abnormal point for* F *which is a local minimum point in the problem* (3.1) *with respect to the finite topology. Suppose that at least one of the following assumptions holds*:

c) *for each non-zero* $y^* \in \bigl(\operatorname{im} F'(x_0)\bigr)^\perp$ *the quadratic form* $\bigl\langle y^*, F''(x_0)[x,x]\bigr\rangle$ *has index greater than codim*$\bigl(\operatorname{im} F'(x_0)\bigr) - 1$ *on the subspace* $\ker F'(x_0)$, *or equivalently, the set* $\mathscr{F}_{k-1}^2(x_0)$ *is empty*;

d) *there exists a vector* $\bar{x} \in \ker F'(x_0)$ *such that* $\bigl\langle y^*, F''(x_0)[\bar{x},\bar{x}]\bigr\rangle < 0$ *for all* $y^* \in \mathscr{F}_{k-1}^2(x_0)$.

Then the classical Lagrange principle (3.13) *holds at* x_0.

Proof. Assume that (3.13) fails. Then $\lambda^0 = 0$ for all $\lambda \in \Lambda_{k-1}(x_0)$. Since x_0 is an abnormal point, Theorem 2.7 shows that $\Lambda_{k-1}(x_0)$ is non-empty. Hence, the set $\mathscr{F}_{k-1}^2(x_0)$ is non-empty too, and thus the assumption c) fails. Moreover, by Theorem 2.7 there exists a $y^* \in \mathscr{F}_{k-1}^2(x_0)$ such that $\bigl\langle y^*, F''(x_0)[\bar{x},\bar{x}]\bigr\rangle \geqslant 0$. Therefore, the assumption d) also fails. This contradiction completes the argument. □

Problems with non-closed image. The first proofs of Theorems 3.1 and 3.2 in [20] were in the case of Banach spaces X and Y, under the assumption that the subspace $\operatorname{im} F'(x_0)$ is closed. Subsequently, these theorems were generalized in [22] to the case when $\operatorname{im} F'(x_0)$ is not a priori closed (of course, the Banach space Y is infinite-dimensional). Here we present several of these results.

We assume that the maps f_0 and F are twice continuously Fréchet differentiable in a neighbourhood of x_0, and moreover, the second derivative of F satisfies a Lipschitz condition. As simple examples show (see [22], Examples 3 and 4), if the subspace $\operatorname{im} F'(x_0)$ is not closed at a local minimum point x_0 of the problem (3.1), then the Lagrange principle (1.3) may fail at this point. Moreover, even the necessary first-order condition $f_0'(x_0) \in \bigl(\ker F'(x_0)\bigr)^{\perp}$ may fail. Nevertheless, the following result holds.

Theorem 3.5. *Let x_0 be a local minimum in the problem* (3.1). *Then for any $h \in X$ with*

$$F'(x_0)h = 0, \qquad F''(x_0)[h,h] \in \operatorname{cl}\bigl(\operatorname{im} F'(x_0)\bigr) \tag{3.15}$$

such that the subspace $\operatorname{im} G(x_0,h)$ *is closed, there exist a $\lambda^0 \geqslant 0$ and a $y^* \in Y^*$ not simultaneously zero* (*and depending on h*) *satisfying the conditions*

$$\frac{\partial \mathscr{L}_h}{\partial x}(x_0,\lambda^0,y^*) \in \bigl(\ker F'(x_0)\bigr)^{\perp} \quad \textit{and} \quad F'(x_0)^* y^* = 0.$$

Furthermore, if $\operatorname{im} G(x_0,h) = Y$ (*that is, the map F is 2-regular at x_0 in the direction h*), *then $\lambda^0 > 0$.*

Here $\mathscr{L}_h$ is the family of functions depending on the parameter $h \in X$ and the variables $(x,\lambda^0,y^*) \in X \times \mathbb{R} \times Y^*$ and defined by the formula

$$\mathscr{L}_h(x,\lambda^0,y^*) = \lambda^0 f_0(x) + \bigl\langle y^*, F'(x)h \bigr\rangle.$$

Theorem 3.6. *Let x_0 be a local minimum in the problem* (3.1), *and assume that the range* $\operatorname{im} F'(x_0)$ *is dense in Y and that* $\operatorname{im} G(x_0,h) = Y$ *for some* $h \in \ker F'(x_0)$.

Then there exists a normal Lagrange multiplier $\lambda = (1, y^*)$, $y^* \in Y^*$, such that the classical Lagrange principle (3.13) *holds and, moreover,*

$$\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[h,h] \geqslant 0.$$

The proofs of both theorems are presented in [22]. In [23] these results were generalized to the problem (2.13), where C is an arbitrary closed convex subset of the Banach space Y. In this case the necessary second-order conditions for the problem (2.13) contain a 'sigma'-term, just as in (2.16).

As of now, the interconnections between the two approaches we have presented to the study of necessary extremum conditions in abnormal problems, that is, the index approach and the one based on the generalized Lagrange function $\mathscr{L}_{\mathscr{A}}$ and the concept of 2-regularity, are poorly understood. In fact, only the recent results in [24] show that if F does not have 2-regular directions at a point x_0, then for the problem (3.1) we have the conditions

$$\max_{\lambda\in\Lambda}\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x]\geqslant 0 \qquad \forall\, x\in\ker F'(x_0),$$
$$\text{where}\quad \Lambda=\bigl\{\lambda\in\Lambda_{k-1}(x_0)\colon \lambda^0=0\bigr\},$$

which are even stronger than the necessary conditions (2.4).

4. Sufficient second-order conditions for an extremum

Sufficient conditions in terms of the Lagrange function. Let us consider the problem (2.1). The previous two sections were devoted to necessary second-order conditions. We now consider sufficient second-order conditions, assuming that X and Y are normed spaces. Here it is important whether or not the spaces X and Y are finite-dimensional. We assume that all the functions f_j and the map f have two continuous Fréchet derivatives in a neighbourhood of the point x_0. If both X and Y are finite-dimensional, then sufficient second-order conditions at x_0 are well known for the problem (2.1):

$$\Lambda=\Lambda(x_0)\neq\varnothing,\qquad \max_{\lambda\in\Lambda}\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x]>0\quad \forall\, x\in\mathscr{K}(x_0)\colon x\neq 0. \tag{4.1}$$

Let X be infinite-dimensional and Y finite-dimensional. Then the conditions (4.1) are sufficient only for a local minimum in the finite topology but not for a local minimum with respect to the actual topology in X, even when there are no constraints (an example: $X=l_2$, $f_0(x)=\sum_{i=1}^{\infty}(i^{-1}x_i^2-x_i^4)$, $x_0=0$). Here we discuss only local minima with respect to the actual topology in X.

Theorem 4.1. *Let Y be a finite-dimensional space, and assume that the set $\Lambda=\Lambda(x_0)$ is non-empty and that there exists an $\varepsilon>0$ such that*

$$\sup_{\lambda\in\Lambda}\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x]\geqslant\varepsilon\|x\|^2\qquad \forall\, x\in\mathscr{K}(x_0). \tag{4.2}$$

Then x_0 is a strict local minimum in the problem (2.1).

This was proved in [5] (see § 1.7, Theorem 7.1).

Condition (4.2) leads us to the following problem: let X be a complete space and assume for simplicity that there are no constraints of inequality type and that x_0 is a normal point. Then (4.2) takes the form $\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x]\geqslant\varepsilon\|x\|^2$ for all $x\in\ker f'(x_0)$, from which it follows (see [1], § 7.2.2, pp. 307, 308) that the subspace $\ker f'(x_0)$ is homeomorphic to a Hilbert space. This is a significant deficiency of the sufficient condition (4.2), which can in fact be used only in Hilbert spaces, while in most cases of applications to infinite-dimensional problems neither X nor $\ker f'(x_0)$ is a Hilbert space. For example, in a non-linear optimal control problem we minimize over the Banach space $L_\infty[t_1,t_2]$ of measurable essentially bounded functions.

If Y is infinite-dimensional, then, as we show below, Theorem 4.1 holds under the assumptions that X and Y are complete spaces and the subspace $\operatorname{im} f'(x_0)$ is closed. The next example demonstrates that the assumption that $\operatorname{im} f'(x_0)$ is closed cannot be dropped here.

Example 4.1. Let $X = Y = l_2$ (Hilbert spaces). Consider the compact linear operator $A\colon l_2 \to l_2$ and the quadratic map $Q\colon l_2 \to l_2$ defined by the formulae $Ax = (x_1, x_2/2, \dots, x_j/j, \dots)$ and $Q(x) = (x_1^2, x_2^2, \dots, x_j^2, \dots)$. We look at the problem

$$f_0(x) = -|x|^2 \to \min, \qquad f(x) = Ax - Q(x) = 0.$$

In it the condition (4.2) is satisfied for $x_0 = 0$, since $(1,0) \in \Lambda(0)$ and $\ker f'(x_0) = \ker A = \{0\}$ by construction. However, $x_0 = 0$ is not a local minimum point.

Indeed, letting $x_i \in l_2$ be the sequence of elements with i^{-1} at the ith position and zeros elsewhere, we get that $f(x_i) = 0$, $f_0(x_i) < 0$ for all i, and $x_i \to 0$ as $i \to \infty$. In this example the point is that the subspace $\operatorname{im} f'(x_0)$ is not closed.

Problems with an infinite-dimensional image. For infinite-dimensional Y the assumption that $\operatorname{im} f'(x_0)$ is closed is quite restrictive. Our first goal is to present sufficient second-order conditions involving neither this assumption nor the assumption that $\ker f'(x_0)$ is homeomorphic to a Hilbert space.

We use a construction proposed in [25]. Fix a point $x_0 \in X$, and let $\{D_i\}$ be a sequence of subsets of X such that $x_0 \in D_i$ for all i and $\{D_i\}$ converges to x_0 in the sense that each sequence $\{x_i\}$ with $x_i \in D_i$ converges to x_0. For example, the D_i can be the balls $\bigl\{x\colon \|x - x_0\|_1 \leqslant \rho_i\bigr\}$ with $\rho_i \to 0+$, where $\|\cdot\|_1$ is a norm in X dominating the original norm.

We assume that f is a second-order Taylor map at x_0 with respect to $\{D_i\}$, that is, there exist a continuous linear operator $A\colon X \to Y$, a continuous symmetric bilinear map $B\colon X \times X \to Y$, and a numerical sequence $\{\alpha_i\} \to 0+$ such that for each i

$$f(x) = f(x_0) + A(x - x_0) + \frac{1}{2}B[x - x_0, x - x_0] + \Delta(x - x_0),$$
$$\bigl\|\Delta(x - x_0)\bigr\| \leqslant \alpha_i \|x - x_0\|^2 \quad \forall\, x \in D_i.$$

For convenience we use here the same notation $\|\cdot\|$ for the norms in X and Y. We denote the linear operator A, called the relative first differential, by $\frac{\partial f}{\partial x}(x_0)$, and the bilinear form B, called the relative second differential, by $\frac{\partial^2 f}{\partial x^2}(x_0)$. Note that these relative differentials A and B are not necessarily uniquely defined (everything depends on the choice of the sequence of sets $\{D_i\}$).

The functions f_j are also assumed to be second-order Taylor functions at x_0 with respect to $\{D_i\}$, and we use similar notation for their relative differentials and for partial derivatives of the Lagrange function with respect to the variable x. We note that even in the finite-dimensional case and with $D_i = \bigl\{x\colon \|x - x_0\| \leqslant i^{-1}\bigr\}$, a second-order Taylor function at some point is not even necessarily continuous in a neighbourhood of this point.

Again, for convenience we assume that $f_j(x_0) = 0$ for all $j \geqslant 1$. Let

$$\mathscr{K}_1(x_0) = \biggl\{x \in X\colon \biggl\langle \frac{\partial f_j}{\partial x}(x_0), x \biggr\rangle \leqslant 0,\ j = 0, \dots, l\biggr\}.$$

Theorem 4.2. *Let $\Lambda = \Lambda(x_0)$ be non-empty and assume that there exist a γ and an $\varepsilon > 0$ such that*

$$\sup_{\lambda \in \Lambda} \frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] + \gamma \biggl\|\frac{\partial f}{\partial x}(x_0) x\biggr\|^2 \geqslant \varepsilon \|x\|^2 \qquad \forall\, x \in \mathscr{K}_1(x_0). \tag{4.3}$$

Then x_0 is a strict minimum point on the set D_{i_0} in the problem (2.1) for some i_0, that is,

$$f_0(x) > f_0(x_0)\ \forall\, x \neq x_0\colon\ x \in D_{i_0},\ \ f_j(x) \leqslant 0,\ \ j = 1, \dots, l,\ \ f(x) = 0.$$

Proof. For convenience let $x_0 = 0$ and $f_0(x_0) = 0$. By Hoffman's lemma[5] [3],

$$\exists\, c_1 > 0\colon\ \ \operatorname{dist}\bigl(x, \mathscr{K}_1(x_0)\bigr) \leqslant c_1 \sum_{j=0}^{l} \max\biggl(\biggl\langle \frac{\partial f_j}{\partial x}(x_0), x \biggr\rangle, 0\biggr) \quad \forall\, x. \tag{4.4}$$

We give a proof by contradiction: assume that there exists a sequence $\{x_i\}$ such that

$$f(x_i) = 0, \qquad f_j(x_i) \leqslant 0, \qquad j = 0, \dots, l, \quad x_i \in D_i, \quad x_i \neq 0 \quad \forall\, i. \tag{4.5}$$

Since $\{x_i\}$ tends to zero, it follows from the representations for f and f_j that

$$\biggl\| \frac{\partial f}{\partial x}(x_0) x_i \biggr\| = o\bigl(\|x_i\|\bigr), \qquad \biggl\langle \frac{\partial f_j}{\partial x}(x_0), x_i \biggr\rangle \leqslant o\bigl(\|x_i\|\bigr), \qquad j = 0, \dots, l. \tag{4.6}$$

Hence by (4.4) there exists a sequence $\{\tilde{x}_i\}$ such that $x_i = \tilde{x}_i + o\bigl(\|\tilde{x}_i\|\bigr), \tilde{x}_i \neq 0$, and $\tilde{x}_i \in \mathscr{K}_1(x_0)$ for all i, where $o\bigl(\|\tilde{x}_i\|\bigr)/\|\tilde{x}_i\| \to 0$ as $i \to \infty$. By (4.3), for all i there exists a $\lambda_i = (\lambda_i^0, \dots, \lambda_i^l, y_i^*) \in \Lambda$ such that

$$\frac{\partial^2 L}{\partial x^2}(x_0, \lambda_i)[\tilde{x}_i, \tilde{x}_i] + \gamma \biggl\| \frac{\partial f}{\partial x}(x_0) \tilde{x}_i \biggr\|^2 \geqslant \frac{\varepsilon}{2} \|\tilde{x}_i\|^2. \tag{4.7}$$

We have

$$\begin{aligned} 0 &\overset{(1)}{\geqslant} \lambda_i^0 f_0(x_i) \overset{(2)}{\geqslant} L(x_i, \lambda_i) \overset{(3)}{=} L(x_0, \lambda_i) + \frac{\partial L}{\partial x}(x_0, \lambda_i) x_i \\ &\quad + \frac{1}{2} \frac{\partial^2 L}{\partial x^2}(x_0, \lambda_i)[x_i, x_i] + o\bigl(\|x_i\|^2\bigr) \overset{(4)}{=} \frac{1}{2} \frac{\partial^2 L}{\partial x^2}(x_0, \lambda_i)[\tilde{x}_i, \tilde{x}_i] + o\bigl(\|\tilde{x}_i\|^2\bigr) \\ &\overset{(5)}{=} \frac{1}{2} \biggl(\frac{\partial^2 L}{\partial x^2}(x_0, \lambda_i)[\tilde{x}_i, \tilde{x}_i] + \gamma \biggl\| \frac{\partial f}{\partial x}(x_0) \tilde{x}_i \biggr\|^2 \biggr) + o\bigl(\|\tilde{x}_i\|^2\bigr) \overset{(6)}{\geqslant} \frac{\varepsilon}{4} \|\tilde{x}_i\|^2 + o\bigl(\|\tilde{x}_i\|^2\bigr). \end{aligned}$$

Here the inequalities $\overset{(1)}{\geqslant}$ and $\overset{(2)}{\geqslant}$ follow from (4.5) and the non-negativity of the λ_i^j, the equality $\overset{(3)}{=}$ follows from the representations for f and f_j and the boundedness of the set Λ, $\overset{(4)}{=}$ holds because $\lambda_i \in \Lambda$ in view of the construction of the sequence $\{\tilde{x}_i\}$, $\overset{(5)}{=}$ holds because $\bigl\| \frac{\partial f}{\partial x}(x_0) x_i \bigr\|^2 = o\bigl(\|x_i\|^2\bigr)$ in view of (4.6), and the inequality $\overset{(6)}{\geqslant}$ follows from (4.7). Thus, we have $0 \geqslant \varepsilon \|x_i\|^2/4 + o\bigl(\|x_i\|^2\bigr)$. This contradiction proves the theorem. □

[5] Although in the statement of Hoffman's lemma it is usually assumed that X is complete, it follows immediately from its proof in [3] that for the polyhedral cone $\mathscr{K}_1(x_0)$ under consideration it also holds in an arbitrary normed space X.

Remark. In fact, we have proved more. Namely, $f_0(x) \geqslant f_0(x_0) + \varepsilon\|x - x_0\|^2/4$ for all x in D_{i_0} satisfying the constraints in the problem (2.1).

We discuss the theorem just proved. The case $D_i = \{x \colon \|x - x_0\| \leqslant i^{-1}\}$ corresponds to a local minimum in the problem (2.1). Let us compare the sufficient conditions (4.2) and (4.3). Obviously, (4.3) always implies (4.2). It turns out that if we impose the additional assumptions that X and Y are Banach spaces and $\operatorname{im} f'(x_0)$ is closed, then (4.2) implies (4.3). This is a consequence of the following generalization of a theorem of Finsler [26].

Let $A \colon X \to Y$ be a fixed continuous linear operator, and suppose that at least one of the following two assumptions holds: either Y is finite-dimensional, or X and Y are complete spaces and $\operatorname{im} A$ is closed. Let $B_\sigma(\cdot,\cdot) \colon X \times X \to \mathbb{R}$ be a family of symmetric bilinear maps depending on a parameter σ which ranges over a given set Σ. We assume that the B_σ are bounded uniformly with respect to σ, that is, there exists a $c_2 > 0$ such that $|B_\sigma(x_1, x_2)| \leqslant c_2\|x_1\|\,\|x_2\|$ for all $\sigma \in \Sigma$. Let $B_\sigma(x) = B_\sigma(x, x)$. We also consider the polyhedral cone $K = \{x \in X \colon \langle a_j, x\rangle \leqslant 0,\ j = 1, \dots, k\}$, where the $a_j \in X^*$ are fixed.

Lemma 4.1. *Assume that for some $\varepsilon > 0$*

$$\sup_{\sigma\in\Sigma} B_\sigma(x) \geqslant \varepsilon\|x\|^2 \qquad \forall\, x \in \ker A \cap K.$$

Then

$$\exists\,\gamma \geqslant 0\colon \quad \sup_{\sigma\in\Sigma} B_\sigma(x) + \gamma\|Ax\|^2 \geqslant \frac{\varepsilon}{4}\|x\|^2 \quad \forall\, x \in K.$$

Proof. Consider the cone $\widetilde{K} = \{x \in K \colon Ax = 0\}$. Estimating the distance from $\widetilde{K}$ to a point $x \in K$, we have from Hoffman's lemma that for some $c_3 > 0$ an arbitrary vector $x \in K$ can be represented as $x = a + b$, where $a \in K$, $Aa = 0$, and $c_3\|b\| \leqslant \|Ab\|$.

Let $x \in K$ be arbitrary. Using this representation, we get that for $\gamma > 0$

$$\begin{aligned}\sup_{\sigma\in\Sigma} B_\sigma(x) + \gamma\|Ax\|^2 &= \sup_{\sigma\in\Sigma} B_\sigma(a + b, a + b) + \gamma\|Ab\|^2\\ &\geqslant \sup_{\sigma\in\Sigma} B_\sigma(a) - 2c_2\|a\|\,\|b\| - c_2\|b\|^2 + \gamma c_3^2\|b\|^2\\ &\geqslant \varepsilon\|a\|^2 - 2c_2\|a\|\,\|b\| + (\gamma c_3^2 - c_2)\|b\|^2.\end{aligned}$$

We take $\gamma > 0$ such that the expression on the right-hand side of this inequality has the lower bound $\varepsilon(\|a\|^2 + \|b\|^2)/2$. Considering that $\|a\|^2 + \|b\|^2 \geqslant \|x\|^2/2$, we complete the proof. □

It follows from this lemma that if Y is a finite-dimensional space or if X and Y are Banach spaces and $\operatorname{im} f'(x_0)$ is closed, then the sufficient conditions (4.2) and (4.3) are equivalent. At the same time, we see from the above example that, without the assumption that $\operatorname{im} f'(x_0)$ is closed, the relation (4.2) is not sufficient for a local minimum. We give an example where the subspace $\operatorname{im} f'(x_0)$ is not closed, but the hypotheses of Theorem 4.2 are satisfied. Consider the problem

$$f_0(x) = |x|^2 - |\alpha Ax|^2 + r(x) \to \min, \qquad f(x) = Ax - Q(x) + R(x) = 0,$$

where $X = Y = l_2$, the linear operator A and the quadratic map Q are as in Example 4.1, α is an arbitrary constant, and r and R are arbitrary smooth maps whose first two derivatives vanish at zero. Obviously, all the assumptions of Theorem 4.2 hold at $x_0 = 0$, and therefore $x_0 = 0$ is a local minimum, although $\operatorname{im} f'(x_0) = \operatorname{im} A$ is not closed.

Non-linear optimal control problems can be formulated in a natural way as minimization problems in the space $X = L_\infty = L_\infty[t_1, t_2]$. In this space (4.2) fails, and if we endow L_∞ with the L_2-norm, then the space is no longer complete and we cannot use Theorem 4.1 either. On the other hand, Theorem 4.2 does not require X to be complete, so it is applicable.

We explain the above by looking at the Lagrange problem

$$\dot{x} = f(x, u, t), \quad t \in [t_1, t_2], \qquad \int_{t_1}^{t_2} f_0(x, u, t)\, dt \to \min \tag{4.8}$$

with fixed left-hand endpoint $x(t_1)$ and with constraints of equality and inequality type for the right-hand endpoint $x(t_2)$. As X we take the Euclidean space of bounded measurable functions $u(\cdot)$ on the given interval $[t_1, t_2]$ and import to X the inner product from $L_2[t_1, t_2]$, which generates the norm $\|\cdot\|$. As Y we take a finite-dimensional space whose dimension is determined by the number of constraints at the endpoints. As the D_i we take the balls $D_i = \bigl\{u \colon \|u - u_*\|_{L_\infty} \leqslant i^{-1}\bigr\}$, where u_* is the admissible control being investigated in the minimization problem. It is straightforward to verify that if the vector-valued function f and the function f_0 are twice continuously differentiable with respect to the finite-dimensional variables (x, u), then Theorem 4.2 can be applied and yields sufficient conditions for a weak local minimum in the problem (4.8).

Sufficient conditions for abnormal problems. The sufficient conditions in Theorems 4.1 and 4.2 are stated in terms of the classical Lagrange function L and hold without a priori regularity assumptions. However, when x_0 is an abnormal point, these conditions cannot hold for some large classes of problems. We shall explain this; for simplicity we confine ourselves to the problem (3.1) from the previous section and assume that X is a Banach space, Y is finite-dimensional, f_0 has two and F has three continuous derivatives in a neighbourhood of a point x_0, and the third derivative of F satisfies a Lipschitz condition in this neighbourhood.

Let x_0 be an abnormal point solving the problem (3.1), and assume that the classical Lagrange multiplier rule (3.13) fails at this point, that is, there does not exist a $y^* \in Y^*$ such that $f_0'(x_0) + F'(x_0)^* y^* = 0$, and also the cone $H(x_0)$ is distinct from zero. (If X is finite-dimensional and x_0 is not an isolated point of the set $\bigl\{x \colon F(x) = 0\bigr\}$, then this cone is distinct from zero.) Then the assumptions of Theorems 4.1 and 4.2 cannot hold, because in this case

$$\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[x, x] = 0 \quad \forall\, \lambda \in \Lambda(x_0), \quad \forall\, x \in H(x_0) \subseteq \operatorname{Ker} F'(x_0), \quad H(x_0) \neq \{0\}.$$

In terms of the generalized Lagrange function $\mathscr{L}_{\mathscr{A}}$ we state sufficient second-order conditions for an extremum which remain useful in the situation just described, but which hold only under the following assumptions of strong 2-regularity.

Definition. A map F is said to be strongly 2-regular at a point x_0 if there exists an $\varepsilon > 0$ such that

$$\widetilde{G}(h)(B_X) \supseteq \varepsilon B_Y \qquad \forall\, h \in X\colon \|h\| = 1,\ \big\|\widetilde{G}(h)h\big\| \leqslant \varepsilon,$$

with B_X and B_Y the unit balls about 0 in X and Y, respectively, $\widetilde{G}(h)x = F'(x_0)x + \pi F''(x_0)[h, x]$, and π the operator of orthogonal projection of Y onto $\big(\operatorname{im} F'(x_0)\big)^{\perp}$.

Note that if X is finite-dimensional, then strong 2-regularity is equivalent to 2-regularity. This is not so in infinite-dimensional spaces. There are tests for strong 2-regularity in a Hilbert space in [27] and [28].

Theorem 4.3. *Let F be a strongly 2-regular map at a point x_0. Assume that there exists a $\delta > 0$ such that for each $h \in H(x_0) \setminus \{0\}$ there are Lagrange multipliers $\lambda_A = (\lambda^0, y_1^*, y_2^*)$ with $|\lambda_A| = 1$ satisfying (3.3) and (3.4) such that*

$$\frac{\partial^2 \mathscr{L}_{\mathscr{A}}}{\partial x^2}\Big(x_0, \lambda^0, y_1^*, \frac{1}{3}y_2^*, h\Big)[h, h] > \delta \|h\|^2. \tag{4.9}$$

Then x_0 is a strict local minimum point in the problem (3.1).

This theorem was proved in [5], Chap. 1, § 1.14.

It follows from Theorem 4.3 that if F is a strongly 2-regular map at a point x_0 and $H(x_0) = \{0\}$, then x_0 is a strict local minimum point in the problem (3.1). Furthermore, if X is finite-dimensional, then it is sufficient to take $\delta = 0$ in (4.9). Now we give an example where the assumptions of Theorem 4.3 are satisfied, but those of Theorems 4.1 and 4.2 are not.

Example 4.2. Let $k = 2$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n = X$, and consider the problem

$$f_0(x) = x_1 + \sum_{i=2}^{n} x_i^2 \to \min,$$

$$F_1(x) = x_1(x_1 - 2x_n) = 0, \qquad F_2(x) = \frac{1}{2}x_1^2 + \sum_{i=2}^{n-1} x_i^2 - x_n^2 = 0.$$

Here $x_0 = 0$ satisfies the Lagrange multiplier rule, but each Lagrange multiplier $\lambda \in \Lambda(0)$ has the form $\lambda = (0, y^*)$. Hence $\frac{\partial^2 L}{\partial x^2}(x_0, \lambda)[h, h] = 0$ for all $h \in H(x_0) = \big\{x\colon F(x) = 0\big\}$ and all $\lambda \in \Lambda(0)$, so that (4.2) fails.

On the other hand, $h_1 = 0$ for each $h \in H(0)$, and the map F is 2-regular at zero. Therefore, for each λ_A satisfying (3.3) and (3.4) we have

$$\lambda^0 > 0,\quad y_1^* = 0 \quad \Rightarrow \quad \forall\, h \in H(0) \setminus \{0\}\ \ \exists\, \lambda_A = (\lambda^0, 0, y_2^*)\colon$$
$$\frac{\partial^2 \mathscr{L}_{\mathscr{A}}}{\partial x^2}\Big(0, \lambda^0, 0, \frac{1}{3}y_2^*, h\Big)[h, h] = \lambda^0 \sum_{i=2}^{n} h_i^2 > 0.$$

In contrast to Theorems 4.1 and 4.2, we assume in Theorem 4.3 that F is strongly 2-regular. The next example shows that this assumption is essential.

Example 4.3. Let $k = 1$ and $x = (x_1, x_2) \in \mathbb{R}^2 = X$. Obviously, $x_0 = 0$ is not a minimum point in the problem

$$f_0(x) = -|x|^2 \to \min, \qquad F(x) = x_1^2 + x_2^3 = 0.$$

At the same time, $H(x_0) = \bigl\{h = (h_1, h_2) \colon h_1 = 0\bigr\}$, and thus (4.9) holds for $\lambda_A = (0, 0, 1)$ and $\delta = 1$. The point here is that F is not 2-regular at the origin.

Sufficient higher-order conditions in the problem (2.1) were obtained in Chap. 1 of [5] (§ 1.14), while sufficient second-order conditions in the problem (2.13) with C a closed convex set were obtained in [15].

5. Quadratic maps and quadratic problems

Sign-definiteness of a quadratic form on an intersection of quadrics. A typical example of an extremal problem in which the extremum is always attained at an abnormal point is given by the following classical algebraic problem. Let Q_i, $i = 0, \dots, k$, be $k + 1$ symmetric $n \times n$ matrices which define the quadratic forms $q_i(x) = \langle Q_i x, x\rangle$ on $X = \mathbb{R}^n$. It is required to find conditions ensuring that

$$q_0(x) \geqslant 0 \qquad \forall\, x\colon\ q_1(x) = q_2(x) = \cdots = q_k(x) = 0, \tag{5.1}$$

where n and k are fixed positive integers. This problem is of interest both in itself and also for applied problems (see [29]–[31], among others). Obviously, (5.1) is equivalent to the following: $x_0 = 0$ is a minimum point in the problem

$$q_0(x) \to \min, \qquad q_i(x) = 0, \quad i = 1, \dots, k. \tag{5.2}$$

However, this point is necessarily abnormal.

Applying the necessary second-order conditions (2.18) in Theorem 2.7 to (5.2), we see that if (5.1) holds, then

$$\Lambda_{k-1} \neq \{\varnothing\}, \qquad \max_{\lambda \in \Lambda_{k-1}} \biggl(\lambda^0 q_0(x) + \sum_{j=1}^{k} y_j^* q_j(x)\biggr) \geqslant 0 \quad \forall\, x \in X, \tag{5.3}$$

where

$$\Lambda_s = \biggl\{\lambda = (\lambda^0, y^*)\colon \lambda^0 \geqslant 0,\ \operatorname{ind}\biggl(\lambda^0 q_0 + \sum_{j=1}^{k} y_j^* q_j\biggr) \leqslant s,\ |\lambda| = 1\biggr\}.$$

However, for some of the vectors $\lambda \in \Lambda_{k-1}$ we can have $\lambda^0 = 0$. In this case the conditions (5.3) will no longer depend on the form q_0, and may not yield information as a result. This occurs when $0 \in \operatorname{conv} \Lambda_{k-1}$, because (5.3) then holds automatically. Here is a corresponding example: $k = 2$, $q_1(x) = x_1 x_2$, and the forms q_0 and q_2 are arbitrary. Obviously, $\operatorname{conv} \Lambda_1$ contains the segment $(0, \lambda_1, 0)$ with $|\lambda_1| \leqslant 1$, and thus (5.3) holds for any forms q_0 and q_2, including ones for which (5.1) fails (for example, for $q_0(x) = -|x|^2$ and $q_2 = 0$).

Our immediate goal is to find conditions ensuring that if (5.1) holds, then in (5.3) we can confine ourselves to λ with $\lambda^0 > 0$. The next (obvious) example shows that without additional assumptions this is impossible. Moreover, it can happen that $\lambda^0 = 0$ for all $\lambda \in \Lambda_{k-1}$. In fact, let $k = 1$, $n = 2$, $q_0(x) = x_1 x_2$, and $q_1(x) = x_1^2$. Then obviously, $q_0(x) = 0$ if $q_1(x) = 0$, but $\lambda^0 = 0$ for all $\lambda \in \Lambda_0$.

Let us introduce some notation required in what follows. We consider the bilinear map $\widetilde{Q}\colon X \times X \to Y = \mathbb{R}^k$ which associates with arbitrary $x_1, x_2 \in X$ the vector $y = \widetilde{Q}[x_1, x_2]$ with ith component $y_i = \langle Q_i x_1, x_2 \rangle$. Since the matrices Q_i are symmetric, this bilinear map $\widetilde{Q}$ is also symmetric. We set $Q(x) = \widetilde{Q}[x, x]$. The map $Q\colon X \to Y$ is said to be quadratic. For fixed $h \in X$ let Qh be the linear operator from X to Y acting by the formula $Qh(x) = \widetilde{Q}[h, x]$. For $y^* \in Y^*$ let

$$y^* Q = \sum_{i=1}^{k} y_i^* Q_i, \qquad y^* Q(x) = \sum_{i=1}^{k} y_i^* q_i(x).$$

For a closed convex cone C in Y we consider the cone

$$K = \bigl\{x \in X\colon Q(x) \in C,\ x \neq 0\bigr\}.$$

Definition. The quadratic map Q is said to be regular at a point $h \in K$ with respect to the cone C if

$$\operatorname{im} Qh - C = Y.$$

The quadratic map Q is said to be regular with respect to the cone C if it is regular at each point $h \in K$. For $C = \{0\}$ the points in K are called non-trivial zeros of the quadratic map Q. The regularity of Q at a point h where $Q(h) = 0$ means that the vectors $Q_1 h, \dots, Q_k h$ are linearly independent, so this point h is called a regular zero. The regularity of Q means that all its non-trivial zeros (if there are any) are regular.

The quadratic map Q is said to be surjective if $Q(X) = Y$. If Q has a regular zero h, then Q is surjective. Indeed, $Q(h) = 0$ and the vectors $Q_1 h, \dots, Q_k h$ are linearly independent. Hence, applying the classical inverse function theorem to the equation $Q(x) = y$ at $x = h$, we see that the Q-image of the unit-ball neighbourhood of h contains a neighbourhood of zero, and since Q is positive-homogeneous, it is surjective.

For $y \in (Q(X) - C)$ let $\omega(y)$ be the infimum in the problem

$$q_0(x) \to \inf, \qquad Q(x) - y \in C, \tag{5.4}$$

and let $\omega(y) = +\infty$ for all $y \notin \bigl(Q(X) - C\bigr)$. The function ω thus defined is positive-homogeneous, and if

$$q_0(x) \geqslant 0 \qquad \forall\, x\colon Q(x) \in C, \tag{5.5}$$

then $\omega(0) = 0$. Let $S = \bigl\{y \in Y\colon |y| = 1\bigr\}$ be the unit sphere.

Lemma 5.1. *Assume that (5.5) holds and, furthermore, if $h \in K$ and $q_0(h) = 0$, then Q is regular at h with respect to the cone C. Then the function ω is bounded below on S, that is, there exists a $d > 0$ such that $\omega(y) \geqslant -d$ for all $y \in S$.*

Proof. Assume the converse. Then there exist sequences $\{x_i\} \subset X$ and $\{\tilde{\eta}_i\} \subset C$ such that $Q(x_i) - \tilde{\eta}_i = y_i \in S$ for all i and $q_0(x_i) \to -\infty$, so that $|x_i| \to \infty$. Let $h_i = x_i|x_i|^{-1}$ and $\eta_i = \tilde{\eta}_i|x_i|^{-2}$. Since $|h_i| = 1$, by passing to a subsequence we can assume that $h_i \to h$, and therefore $\eta_i \to \eta \in C$, because $\bigl|Q(h_i) - \eta_i\bigr| = |x_i|^{-2} \to 0$. Obviously, $q_0(h) \leqslant 0$, and also $Q(h) = \eta \Rightarrow h \in K$. Hence $q_0(h) = 0$, and thus the quadratic map Q is regular at the point h with respect to the cone C, so that it satisfies the Robinson condition. Therefore, by Robinson's theorem [16], in a neighbourhood of h the distance to the set K has the estimate $\operatorname{dist}(h_i, K) \leqslant \operatorname{const} \times \operatorname{dist}(Q(h_i), C) \leqslant \operatorname{const}|y_i|\,|x_i|^{-2} = \operatorname{const}|x_i|^{-2}$, because $Q(h_i) - y_i|x_i|^{-2} = \eta_i \in C$. Recall that dist denotes the distance from a point to a set, and also, const will be our notation for a constant whose concrete value is not important for us. Hence, for large indices i there exist an $\tilde{h}_i \in K$ and a ξ_i such that $h_i = \tilde{h}_i + \xi_i$ and $|\xi_i| \leqslant \operatorname{const}|x_i|^{-2}$. Then

$$q_0(h_i) = q_0(\tilde{h}_i) + q_0(\xi_i) + 2\langle Q_0\tilde{h}_i, \xi_i\rangle \geqslant -\operatorname{const}|\xi_i| \geqslant -\operatorname{const}|x_i|^{-2}$$

(because $q_0(\tilde{h}_i) \geqslant 0$ by (5.5)), and thus $q_0(x_i) = |x_i|^2 q_0(h_i) \geqslant -\operatorname{const}$ for all i. This is a contradiction. □

Corollary. *Under the assumptions of Lemma 5.1*

$$q_0(x) + d\bigl|Q(x) - \eta\bigr| \geqslant 0 \qquad \forall\, x \in X, \quad \eta \in C, \tag{5.6}$$

where $d > 0$ is the constant in the lemma.

In fact, since the function ω is positive-homogeneous, $\omega(y) + d|y| \geqslant 0$ for all y. Let $\eta \in C$ and $x \in X$. Substituting $y = Q(x) - \eta$ in the above inequality and bearing in mind that $q_0(x) \geqslant \omega(y)$, we obtain (5.6).

For $\varepsilon \geqslant 0$ and integers $s \geqslant 0$ let

$$\Lambda_s(C) = \bigl\{\lambda = (\lambda^0, y^*) \in \Lambda_s \colon y^* \in C^0\bigr\}, \qquad \Lambda_s^\varepsilon(C) = \bigl\{\lambda \in \Lambda_s(C) \colon \lambda^0 \geqslant \varepsilon\bigr\},$$

where $C^0 = \bigl\{y^* : \langle y^*, y\rangle \leqslant 0 \text{ for all } y \in C\bigr\}$ is the polar cone of C.

In what follows we assume that C is a polyhedral cone, that is, it can be described by finitely many homogeneous linear inequalities.

Theorem 5.1. *Assume that the conditions* (5.5) *hold, and also that if $h \in K$ and $q_0(h) = 0$, then Q is regular at h with respect to the cone C. Then there exists an $\varepsilon > 0$ such that*

$$\Lambda = \Lambda_{k-1}^\varepsilon(C) \neq \varnothing, \qquad \max_{\lambda \in \Lambda}\bigl(\lambda^0\omega(y) + \langle y^*, y\rangle\bigr) \geqslant 0 \quad \forall\, y. \tag{5.7}$$

Note that we do not exclude the case $K = \varnothing$. In this case the inequality in (5.7) holds automatically for $y \notin \bigl(Q(X) - C\bigr)$, because $\omega(y) = +\infty$ for such y.

For the proof of the theorem we require the following lemma.

Lemma 5.2. *Let $\varphi\colon X \to \mathbb{R}$ and $\Phi\colon X \to Y$ be fixed smooth maps and let d be a positive constant. Assume that the function $f(x, \eta) = \varphi(x) + d|\Phi(x) + \eta|$ attains its minimum with respect to $x \in X$ and $\eta \in C$ at a point (x_0, η_0) such that*

$$\Phi(x_0) + \eta_0 = 0, \qquad \operatorname{im}\Phi'(x_0) + \operatorname{span}\Gamma = Y, \quad \eta_0 \in \operatorname{ri}\Gamma,$$

where Γ *is a face of the cone* C, *and* ri *denotes the relative interior of a convex set. Then there exists a* $y^* \in C^0$ *such that* $|y^*| \leqslant d$,

$$\varphi'(x_0) + \Phi'(x_0)^* y^* = 0, \tag{5.8}$$

and for $q_0 = \varphi''(x_0)$ *and* $Q = \Phi''(x_0)$

$$q_0(x) + y^* Q(x) \geqslant 0 \qquad \forall\, x \colon \Phi'(x_0)x \in \operatorname{span} \Gamma. \tag{5.9}$$

The proof of the lemma is similar to the proof of Proposition 2 in [32], where the case $\Gamma = \{0\}$ was treated.

We now prove Theorem 5.1. Since ω is a positive-homogeneous function, it is sufficient to show that there exists an $\varepsilon > 0$ such that the set $\Lambda = \Lambda^{\varepsilon}_{k-1}(C)$ is non-empty and the inequality in (5.7) holds for each $y \in S$. Let us show this.

By assumption the cone C has finitely many faces. For each face we consider the orthogonal projection of Y onto the orthogonal complement of the linear hull of this face. Omitting the zero operator (if it is present) and renumbering the remaining operators, we let them be $P_1, \dots, P_{r_0}$.

We fix an arbitrary point $y \in S$. By Sard's theorem, the set of critical values of each smooth map $P_r Q \colon X \to \operatorname{im} P_r$ has Lebesgue measure zero. Hence, the set of points which are regular values of each of the maps $P_r Q$, $r = 1, \dots, r_0$, is dense in Y. Moreover, each of these maps is positive-homogeneous. Then there exists a sequence $\{y_s\}$ converging to y such that $y_s \in S$ for all s and each $P_r y_s$ is a regular value of all the maps $P_r Q$, $r = 1, \dots, r_0$.

By Lemma 5.1, (5.6) holds for some $d > 0$. For a positive integer i we set $q_{0,i}(x) = q_0(x) + i^{-1}|x|^2$ and $Q_{0,i} = Q_0 + i^{-1} I$, where I is the identity matrix. We fix an arbitrary index s and consider the family of problems

$$f_i(x, \eta) = q_{0,i}(x) + d\bigl|Q(x) - \eta - y_s\bigr| \to \inf, \qquad x \in X, \quad \eta \in C,$$

which depend on the positive integer parameter i. They are called the i-problems. By Lemma 5.1 the infimum in each i-problem is finite. We assert that it is attained. Indeed, let $\{x_j, \eta_j\}$ be a minimizing sequence in the i-problem. Then $f_i(x_j, \eta_j) \geqslant i^{-1}|x_j|^2 - d|y_s|$ by (5.6), so if $|x_j| \to \infty$, then $f_i(x_j, \eta_j) \to \infty$ as $j \to \infty$, which is impossible. Hence, the sequence $\{x_j\}$ is bounded and therefore $\{\eta_j\}$ is also bounded. Passing to subsequences, we obtain two convergent sequences, so that their limits give a solution of the i-problem.

Let $x_{i,s}, \eta_{i,s}$ be some solution of the i-problem. For fixed i we consider two cases.

The first case: $Q(x_{i,s}) - \eta_{i,s} \neq y_s$. Then the function f_i being minimized in the i-problem is smooth in a neighbourhood of the solution $(x_{i,s}, \eta_{i,s})$. Therefore, by the classical necessary conditions of orders 1 and 2,

$$\frac{\partial f_i}{\partial x}(x_{i,s}, \eta_{i,s}) = 2(Q_{0,i} + y^*_{i,s} Q)x_{i,s} = 0, \quad \frac{\partial f_i}{\partial \eta}(x_{i,s}, \eta_{i,s}) = -y^*_{i,s} \in -C^0, \tag{5.10}$$

$$q_{0,i}(x) + y^*_{i,s} Q(x) \geqslant 0 \qquad \forall\, x \in \ker Q x_{i,s}, \tag{5.11}$$

where $y^*_{i,s} = d\bigl(Q(x_{i,s}) - \eta_{i,s} - y_s\bigr)/\bigl|Q(x_{i,s}) - \eta_{i,s} - y_s\bigr|$. Let us analyse these relations.

First let $Q(x_{i,s}) \neq 0$. Then we set $\Pi_i = \ker Q x_{i,s} + \operatorname{span}\{x_{i,s}\}$. By (5.10) and (5.11) we have $q_{0,i}(x) + y^*_{i,s} Q(x) \geqslant 0$ for all $x \in \Pi_i$. Moreover, $\operatorname{codim} \Pi_i \leqslant k - 1$

because $x_{i,s} \notin \ker Qx_{i,s}$, since $(Qx_{i,s})x_{i,s} = Q(x_{i,s}) \neq 0$. Hence, in view of (5.10) we have

$$y^*_{i,s} \in C^0, \qquad |y^*_{i,s}| \leqslant d,$$
$$\lambda^0_{i,s}(1, y^*_{i,s}) \in \Lambda_{k-1} \quad \text{for} \quad \lambda^0_{i,s} = \bigl(1 + |y^*_{i,s}|^2\bigr)^{-1/2}. \tag{5.12}$$

Taking the inner product of the first formula in (5.10) and $x_{i,s}$, and using the definition of $y^*_{i,s}$, we get that

$$q_{0,i}(x_{i,s}) + d\bigl|Q(x_{i,s}) - \eta_{i,s} - y_s\bigr| + \langle y^*_{i,s}, y_s\rangle = -\langle y^*_{i,s}, \eta_{i,s}\rangle \geqslant 0.$$

On the other hand,

$$q_{0,i}(x) + d\bigl|Q(x) - \eta - y_s\bigr| \\ \geqslant q_{0,i}(x_{i,s}) + d\bigl|Q(x_{i,s}) - \eta_{i,s} - y_s\bigr| \quad \forall\, x \in X, \ \ \forall\, \eta \in C, \tag{5.13}$$

because $(x_{i,s}, \eta_{i,s})$ solves the i-problem. Thus, by the previous inequality

$$\min\bigl\{q_{0,i}(x) + d\bigl|Q(x) - \eta - y_s\bigr|, \ x \in X, \ \eta \in C\bigr\} + \langle y^*_{i,s}, y_s\rangle \geqslant 0. \tag{5.14}$$

Let $Q(x_{i,s}) = 0$. Taking the inner product of (5.10) and $x_{i,s}$, we get that $q_{0,i}(x_{i,s}) = 0$. Hence, $(0, \eta_{i,s})$ is also a solution of the i-problem. Substituting the point $x = 0$ for $x_{i,s}$ in (5.11), we have $q_{0,i}(x) + y^*_{i,s}Q(x) \geqslant 0$ for all x, which yields (5.12). The proof of (5.14) is similar.

The second case: $Q(x_{i,s}) - \eta_{i,s} = y_s$. Each point in a polyhedral cone lies in the relative interior of one of its faces. Then C has a face Γ such that $\eta_{i,s} \in \operatorname{ri}\Gamma$. First let $\operatorname{span}\Gamma \neq Y$. Taking the corresponding projection operator $P_r \neq 0$ onto $(\operatorname{span}\Gamma)^{\perp}$, we have $P_r\eta_{i,s} = 0 \Rightarrow P_rQ(x_{i,s}) = P_ry_s$. Hence, $\operatorname{im} Qx_{i,s} + \operatorname{span}\Gamma = Y$ by construction, and thus the assumptions of Lemma 5.2 hold for the i-problem. By the lemma there exist $y^*_{i,s}$ satisfying (5.10) and (5.12). Here we have used (5.9) and the fact that the subspace $\Pi_i = \bigl\{x\colon Q[x_{i,s}, x] \in \operatorname{span}\Gamma\bigr\}$ has codimension at most $(k-1)$. On the other hand, if $\operatorname{span}\Gamma = Y$, then $\eta_{i,s} \in \operatorname{int} C$, which easily shows that conditions (5.10) and (5.12) hold for $y^*_{i,s} = 0$.

Taking the inner product of the first equality in (5.10) and $x_{i,s}$, we get that $q_{0,i}(x_{i,s}) + \bigl\langle y^*_{i,s}, Q(x_{i,s})\bigr\rangle = 0$. Then using (5.13) and taking account of the relations $Q(x_{i,s}) - \eta_{i,s} - y_s = 0$ and $\langle y^*_{i,s}, \eta_{i,s}\rangle \leqslant 0$, we arrive at (5.14).

Thus, by considering the above two cases we have proved that there exist $y^*_{i,s}$ such that (5.12) and (5.14) hold. For each fixed s we select from the bounded sequence $\{y^*_{i,s}\}$ a convergent subsequence, so that $y^*_{i,s} \to y^*_s$ as $i \to \infty$ for some y^*_s with $|y^*_s| \leqslant d$. Similarly, passing from $\{y^*_s\}$ to a subsequence, we can assume that $y^*_s \to \hat{y}^*$ as $s \to \infty$ for some $\hat{y}^*$. We fix $x \in X$ and $\eta \in C$ and pass to the limit in (5.12) and (5.14): first, for fixed s as $i \to \infty$, and then as $s \to \infty$. Since the set Λ_{k-1} is closed, we have $|\hat{y}^*| \leqslant d$, $\hat{y}^* \in C^0$,

$$\lambda = (\lambda^0, y^*) \in \Lambda_{k-1}(C) \qquad \text{for} \quad \lambda^0 = \bigl(1 + |\hat{y}^*|^2\bigr)^{-1/2}, \quad y^* = \lambda^0\hat{y}^*,$$
$$\lambda^0 \inf\bigl\{q_0(x) + d\bigl|Q(x) - \eta - y\bigr|, \ x \in X, \ \eta \in C\bigr\} + \langle y^*, y\rangle \geqslant 0.$$

From the last inequality we get that $\lambda^0 q_0(x) + \langle y^*, y\rangle \geqslant 0$ for all x such that $Q(x) - y \in C \Rightarrow \lambda^0\omega(y) + \langle y^*, y\rangle \geqslant 0$. Let $\varepsilon = (1 + d^2)^{-1/2}$. Then $\lambda^0 \geqslant \varepsilon$, because $|\hat{y}^*| \leqslant d$. The proof is complete.

The above proof of Theorem 5.1 is based on an analysis of the problem

$$f(x,\eta)=q_0(x)+d\bigl|Q(x)-\eta-y\bigr|\to\inf,\qquad x\in X,\quad \eta\in C, \tag{5.15}$$

in which by Lemma 5.1 the infimum is finite for each y. If the infimum in (5.15) could be attained for each y, then in the proof of the theorem we could do without the family of i-problems, simply by analysing the minimum points of f. If the form q_0 is positive on K, then by increasing d if necessary we easily deduce from Lemma 5.1 that each minimizing sequence in the problem (5.15) is bounded, and therefore the minimum in the problem is attained. However, even for $k=1$ this does not hold without the additional assumption that q_0 is positive on K. Indeed, let $C=\{0\}$, $q_0(x)=x_1^2$, and $Q(x)=q_1(x)=x_1x_2$. Then for any $d>0$ and $y\in Y\setminus\{0\}$ the infimum in (5.15) is zero, but it is not attained.

Examples in [32] show that even for $C=\{0\}$ the following cases are possible: all the assumptions of Theorem 5.1 hold, but the infimum in (5.4) is not attained whatever the choice of $y\neq 0$; for each $\alpha\in\mathbb{R}$ there exists a $y=y(\alpha)$ such that the infimum in (5.4) is equal to α, but is not attained; the quadratic map Q is surjective and each $y\neq 0$ is a regular value of it, but all the non-trivial zeros of Q are irregular and q_0 vanishes on K, while $\omega(y)=-\infty$ for all $y\neq 0$.

Theorem 5.1 has the following consequence.

Theorem 5.2. *Assume that $q_0(x)>0$ for all $x\in K$. Then there exists an $\varepsilon>0$ such that*

$$\Lambda=\Lambda_{k-1}^{\varepsilon\varepsilon}(C)\neq\varnothing\quad\text{and}\quad \max_{\lambda\in\Lambda}\bigl(\lambda^0q_0(x)+y^*Q(x)\bigr)\geqslant\varepsilon|x|^2\quad\forall\,x\in X,$$

where the set $\Lambda_s^{\varepsilon\varepsilon}(C)$ consists of those $\lambda=(\lambda^0,y^*)$ such that $|\lambda|=1$, $\lambda^0\geqslant\varepsilon$, $y^*\in C^0$, and there exists a linear subspace $\Pi\subset X$ such that

$$\operatorname{codim}\Pi\leqslant s,\qquad \lambda^0q_0(x)+y^*Q(x)\geqslant\varepsilon|x|^2\quad\forall\,x\in\Pi.$$

Theorems 5.1 and 5.2 describe necessary conditions for q_0 to be non-negative definite and positive definite on the cone K, respectively. Obviously, they are also sufficient.

Unfortunately, Theorems 5.1 and 5.2 are essentially finite-dimensional, and their statements can fail if $\dim X=\infty$. Here is an example of this.

Example 5.1. Let X be an infinite-dimensional Hilbert space, let $Y=\mathbb{R}$, let $C=\{0\}$, let $A\colon X\to X$ be a symmetric compact positive-definite linear operator, let $q_0(x)=-|x|^2$, and let $q_1(x)=\langle Ax,x\rangle$. Then $K=\varnothing$, but at the same time $\Lambda_0^{\varepsilon}=\varnothing$ for all $\varepsilon>0$, because the Legendrian quadratic form $(-q_0/2+\lambda_1q_1)$ has finite index for each λ_1 (see [1], § 6.2), and thus $\operatorname{ind}(q_0+\lambda_1q_1)=\infty$ for all λ_1.

Example 5.2 (see [32], Example 3). Let $n=4$, $k=2$, $C=\{0\}$, $q_0(x)=x_2x_4$, $q_1(x)=x_1x_2$, and $q_2(x)=-x_1^2+x_2^2+x_3^2$. It can be immediately verified that the irregular zeros have the form $x=(0,0,0,x_4)$, while all the other non-trivial zeros are regular. Furthermore, if $Q(x)=0$, then $x_2=0\Rightarrow q_0(x)=0$. At the same time, $\omega\bigl((0,1)\bigr)=-\infty$, so that neither the assertion of Lemma 5.1 nor the inequality (5.7) in Theorem 5.1 holds.

This example shows that if (5.1) holds, but the quadratic map Q has at least one irregular non-trivial zero h such that $q_0(h)=0$, then Theorem 5.1 can fail even for $C=\{0\}$. However, Theorem 5.2 has the following consequence.

For $\varepsilon>0$ let $\tilde{\Lambda}(\varepsilon)$ be the set of $\lambda=(\lambda^0,y^*)$ such that $|\lambda|=1$, $\lambda^0>0$, $y^*\in C^0$, and there exists a linear subspace $\Pi\subset X$ such that

$$\operatorname{codim}\Pi\leqslant k-1,\quad \lambda^0q_0(x)+y^*Q(x)+\varepsilon\lambda^0|x|^2>0\quad \forall\,x\in\Pi\setminus\{0\}.$$

Corollary. *Assume that* (5.5) *holds. Then for each* $x\in X\setminus\{0\}$

$$\forall\,\varepsilon>0\quad \exists\,\lambda=\lambda(\varepsilon)=(\lambda^0,y^*)\in\tilde{\Lambda}(\varepsilon)\colon\ \lambda^0q_0(x)+y^*Q(x)+\varepsilon\lambda^0|x|^2>0. \tag{5.16}$$

This follows by applying Theorem 5.2 to the quadratic form $q_{0,\varepsilon}=q_0(x)+\varepsilon|x|^2$.

Let $\tilde{\Lambda}_{k-1}(C)=\operatorname{Ls}\bigl\{\tilde{\Lambda}(1/i)\bigr\}$ be the upper topological limit of the sequence of sets $\bigl\{\tilde{\Lambda}(1/i)\bigr\}$. If (5.5) holds, then

$$\tilde{\Lambda}=\tilde{\Lambda}_{k-1}(C)\neq\varnothing,\qquad \max_{\lambda\in\tilde{\Lambda}}\bigl(\lambda^0\omega(y)+\langle y^*,y\rangle\bigr)\geqslant 0\quad\forall\,y. \tag{5.17}$$

This follows from the family of conditions (5.16), because each limit point of a sequence $\{\lambda_i\}$ with $\lambda_i\in\tilde{\Lambda}(1/i)$ for all i belongs to $\tilde{\Lambda}_{k-1}(C)$.

In turn, (5.17) implies condition (5.3) (in which the set Λ_{k-1} must be replaced by $\Lambda_{k-1}(C)$ if the cone C is distinct from zero), because obviously $\tilde{\Lambda}_{k-1}(C)\subseteq\Lambda_{k-1}(C)$. In general the converse is not true, as the following example shows: $n=2$, $k=1$, $C=\{0\}$, $q_0(x)=-|x|^2$, and $q_1(x)=x_1^2$. Here (5.3) holds but (5.17) fails, since obviously $\tilde{\Lambda}(\varepsilon)=\varnothing$ for all $\varepsilon>0\Rightarrow\tilde{\Lambda}_{k-1}(C)=\varnothing$. Thus, in the general case condition (5.3), which is necessary for the quadratic form to be non-negative on the cone K, is weaker than (5.17) and therefore is weaker than the family of conditions (5.16).

Application to the theory of extremal problems. Let us return to the problem (3.1), in which we take $X=\mathbb{R}^n$. We shall present conditions ensuring that in (2.18) we can manage with only the Lagrange multipliers for which $\lambda^0>0$.

Lemma 5.3. *Let* x_0 *be an abnormal point, let* F *be a* 2*-regular map at this point, and assume both the classical Lagrange principle* (3.13) *and the quadratic growth condition*

$$\exists\,\delta_1,\delta_2>0\colon\ f_0(x)-f_0(x_0)\geqslant\delta_1|x-x_0|^2\quad\forall\,x\colon\ F(x)=0,\quad |x-x_0|\leqslant\delta_2.$$

Then there exists an $\varepsilon>0$ *such that*

$$\Lambda=\bigl\{\lambda\in\Lambda_{k-1}(x_0)\colon\lambda^0\geqslant\varepsilon\bigr\}\neq\varnothing$$

$$\text{and}\quad \max_{\lambda\in\Lambda}\frac{\partial^2L}{\partial x^2}(x_0,\lambda)[x,x]\geqslant 0\quad\forall\,x\in\ker F'(x_0).$$

Proof. In view of (3.13), we can assume without loss of generality that $f_0'(x_0)=0$. Since F is 2-regular at x_0, by Theorem 1 in [28] the set $\bigl\{x\colon F(x)=0\bigr\}$ is locally diffeomorphic in a neighbourhood of this point to the cone $H(x_0)$ in a neighbourhood of zero. From the last assumption of the lemma we get that $f_0''(x_0)[x,x]>0$ for all $x\in H(x_0)$ with $x\neq0$. Using Theorem 5.2, we finish the proof. □

Let us return to the sufficient extremum conditions (4.1). Comparing them to the necessary conditions (2.18), we see that they contain different sets $\Lambda(x_0)$ and $\Lambda_{k-1}(x_0)$ of Lagrange multipliers. This disparity seems unnatural. However, replacing the set $\Lambda_{k-1}(x_0)$ in the necessary conditions by the larger set $\Lambda(x_0)$ is unreasonable, because weaker necessary conditions will simply hold automatically in abnormal problems.

In this connection we can ask the following: if we replace $\Lambda(x_0)$ in (4.1) by the smaller set $\Lambda_{k-1}(x_0)$, will this make the sufficient conditions (4.1) weaker? Formally yes, but the following lemma demonstrates that the sufficient conditions (4.1) do not actually become weaker after such a replacement.

Lemma 5.4. *Let x_0 be an abnormal point and assume that* (4.1) *holds. Then*

$$\max_{\lambda\in\Lambda}\frac{\partial^2 L}{\partial x^2}(x_0,\lambda)[x,x]>0 \quad \forall\, x\in\ker F'(x_0), \quad x\neq 0, \qquad \Lambda=\Lambda_{k-1}(x_0).$$

This follows from Theorem 5.2 (see the details in [33], pp. 35, 36).

Surjectiveness and non-trivial zeros of quadratic maps. We ask about a possible link between non-trivial zeros of a quadratic map and its surjectiveness. As already pointed out, if the quadratic map Q has a regular zero, then Q is surjective. At the same time (see [30]), all generic quadratic maps are regular. Hence, if a generic quadratic map has a non-trivial zero, then it is surjective. The converse is not true: the simplest example of a surjective quadratic map with no non-trivial zeros is given by

$$Q\colon \mathbb{R}^2\to\mathbb{R}^2, \qquad Q(x)=(x_1x_2,x_1^2-x_2^2). \tag{5.18}$$

However, there are no such examples for $n\gg k$. Namely, the following result holds.

Lemma 5.5. *Let Q be a surjective quadratic map and let*

$$n>k^2-1. \tag{5.19}$$

Then the set of its non-trivial zeros is non-empty ($K\neq\varnothing$).

The proof of this lemma is based on Theorem 5.1; it is presented in [32]. The lemma refines Proposition 1 in [30], which states that if a quadratic map Q is essentially surjective (this means that each quadratic map close to Q is surjective too) and (5.19) holds, then the set of non-trivial zeros of Q is non-empty. Incidentally, it is at present unclear whether there exists a surjective quadratic map which is not essentially surjective.

The estimate (5.19) in Lemma 5.5 can be improved. For instance, if $k=2$, then the lemma holds already for $n\geqslant 3$. We consider the question of when the image $Q(X)$ of the quadratic map is convex or almost convex (a set M is said to be almost convex [31] if there exists a convex set C such that $C\subset M\subset \operatorname{cl} C$). This question is interesting both in itself and for applications (see [31] and the bibliography there for details). We note only that for $k=2$ the set $Q(X)$ is convex [29], whereas for $k\geqslant 3$ it may not even be almost convex. The following map is an example of this:

$$Q\colon \mathbb{R}^3\to\mathbb{R}^3, \qquad Q(x)=(x_1x_2,x_1x_3,x_2x_3).$$

In it $y_1y_2y_3 \geqslant 0$ for all $y = (y_1, y_2, y_3) \in Q(\mathbb{R}^3)$, but at the same time $Q(\mathbb{R}^3)$ contains three straight lines: $y_1 = y_2 = 0$, $y_2 = y_3 = 0$, and $y_1 = y_3 = 0$.

For the most common case $Q(X)$ is convex thanks to the presence of a regular zero, because then $Q(X) = Y$. Here we present weaker conditions ensuring that $Q(X)$ is almost convex. For integers $s \geqslant 0$ we set

$$\mathscr{Y}_s = \{y^* : y^* \neq 0,\ \mathrm{ind}(y^*Q) \leqslant s\}. \tag{5.20}$$

Lemma 5.6. *Assume that*

$$\exists\, h \in X\colon \quad \bigl\langle y^*, Q(h)\bigr\rangle < 0 \quad \forall\, y^* \in \mathscr{Y}_{k-1} \setminus \mathscr{Y}_0. \tag{5.21}$$

Then

$$\mathrm{int}(\mathscr{Y}_0^*) \subseteq Q(X) \subseteq \mathscr{Y}_0^*.$$

As above, here $\mathscr{Y}_0^*$ is the conjugate cone of $\mathscr{Y}_0$.

Corollary. *If* (5.21) *holds, then the set* $Q(X)$ *is almost convex, that is, there exists a convex set* D *such that* $D \subset Q(X) \subset \mathrm{cl}\, D$.

The proofs of these results are presented in [32].

Quadratic forms vanishing on an intersection of quadrics. Let a_i, $i = 1, \dots, k$, be fixed linear functionals. If a linear functional a_0 vanishes on the subspace $\bigl\{x\colon \langle a_i, x\rangle = 0,\ i = 1, \dots, k\bigr\}$, then by the annihilator lemma (see [1]) there exist α_i such that $a_0 = \sum_{i=1}^{k} \alpha_i a_i$. We shall investigate the analogous question for quadratic maps.

We say that a form q_0 annihilates a cone $\bigl\{x\colon Q(x) = 0\bigr\}$ if $q_0(x) = 0$ for all x such that $Q(x) = 0$. The question is whether in this case

$$\exists\, y^* \in Y^*\colon \quad Q_0 = y^*Q? \tag{5.22}$$

Already for $k = 1$ the example when Q_1 is the identity matrix shows that without additional assumptions about Q the answer is negative (since then any form q_0 vanishes on the cone $\bigl\{x\colon Q(x) = 0\bigr\} = \{0\}$).

We present conditions on Q ensuring that (5.22) holds for each quadratic form q_0 vanishing on $\bigl\{x\colon Q(x) = 0\bigr\}$. (A cone $\bigl\{x\colon Q(x) \in C\bigr\}$ of more general form was considered in [34].)

Theorem 5.3. *Let* $\mathrm{conv}\, \mathscr{Y}_{2(k-1)}$ *be an acute cone* (*that is, a cone containing no non-trivial subspaces*) *and assume that*

$$\exists\, h \in X\colon \quad \bigl\langle Q(h), y^*\bigr\rangle < 0 \quad \forall\, y^* \in \mathscr{Y}_{k-1}. \tag{5.23}$$

Then (5.22) *holds for each quadratic form* q_0 *vanishing on* $\bigl\{x\colon Q(x) = 0\bigr\}$.

Proof. By Lemma 6.1 established below (see also Lemma 2 in [35]) and (5.23) the map Q is surjective. Since the cone $\mathrm{conv}\, \mathscr{Y}_{2(k-1)}$ is acute, its polar cone has a non-empty interior. Hence

$$\exists\, \hat{h} \in X\colon \quad \bigl\langle y^*, Q(\hat{h})\bigr\rangle < 0 \quad \forall\, y^* \in \mathscr{Y}_{2(k-1)}. \tag{5.24}$$

In the minimization problem (5.2) the minimum is attained at the abnormal point $x_0 = 0$, so by (2.18) (see Theorem 2.7) there exists a $\lambda = (\lambda^0, y^*) \in \Lambda_{k-1}$ such that $\lambda^0 q_0(\hat{h}) + \big\langle y^*, Q(\hat{h})\big\rangle \geqslant 0$. Then $\lambda^0 > 0$ (for if $\lambda^0 = 0$, then $y^* \neq 0$, $\operatorname{ind}(y^*Q) \leqslant (k-1)$, and $\langle y^*, Q(\hat{h})\rangle \geqslant 0$, in contradiction to (5.24)). Hence

$$\exists\, y_1^*\colon\ \operatorname{ind}(Q_0 + y_1^*Q) \leqslant k-1, \quad q_0(\hat{h}) + \big\langle y_1^*, Q(\hat{h})\big\rangle \geqslant 0.$$

Similarly, if in the problem (5.2) considered above we replace q_0 by $(-q_0)$, then we have

$$\exists\, y_2^*\colon\ \operatorname{ind}(-Q_0 + y_2^*Q) \leqslant k-1, \quad -q_0(\hat{h}) + \big\langle y_2^*, Q(\hat{h})\big\rangle \geqslant 0.$$

We shall prove that $y_2^* = -y_1^*$. Assume the contrary: $y_1^* + y_2^* \neq 0$. Then bearing in mind that the index of a sum of quadratic forms is not larger than the sum of their indices, we get that $(y_1^* + y_2^*) \in \mathscr{Y}_{2(k-1)}$, $\big\langle (y_1^* + y_2^*), Q(\hat{h})\big\rangle \geqslant 0$. This is a contradiction to (5.24). Thus, $y_2^* = -y_1^*$. Hence setting $y^* = y_1^*$, $\widetilde{Q}_0 = Q_0 + y^*Q$, and $\tilde{q}_0(x) = \langle \widetilde{Q}_0 x, x\rangle$, we see that

$$\operatorname{ind}\widetilde{Q}_0 \leqslant k-1, \qquad \operatorname{ind}(-\widetilde{Q}_0) \leqslant k-1, \qquad \tilde{q}_0(\hat{h}) = 0. \tag{5.25}$$

We assert that $\widetilde{Q}_0 = 0$. Assume the contrary. Then $\tilde{q}_0 \neq 0$. Hence, by (5.24) there exists an $\tilde{h}$ sufficiently close to $\hat{h}$ such that

$$\big\langle \mu^*, Q(\tilde{h})\big\rangle < 0 \quad \forall\, \mu^* \in \mathscr{Y}_{2(k-1)}, \qquad \tilde{q}_0(\tilde{h}) \neq 0. \tag{5.26}$$

The quadratic form $\tilde{q}_0$ constructed also vanishes on $\big\{x\colon Q(x) = 0\big\}$. Hence, using arguments analogous to those above, we get for it and the vector $\tilde{h}$ that

$$\begin{aligned} \exists\, \tilde{y}^*\colon\ &\operatorname{ind}(\widetilde{Q}_0 + \tilde{y}^*Q) \leqslant k-1, \quad \operatorname{ind}\big(-(\widetilde{Q}_0 + \tilde{y}^*Q)\big) \leqslant k-1, \\ &\tilde{q}_0(\tilde{h}) + \big\langle \tilde{y}^*, Q(\tilde{h})\big\rangle = 0. \end{aligned} \tag{5.27}$$

From the first inequality in (5.25) and the second inequality in (5.27),

$$\begin{aligned} 2(k-1) \geqslant \operatorname{ind}\big(\widetilde{Q}_0 - (\widetilde{Q}_0 + \tilde{y}^*Q)\big) &= \operatorname{ind}(-\tilde{y}^*Q) \\ \Rightarrow \operatorname{ind}(-\tilde{y}^*Q) &\leqslant 2(k-1). \end{aligned}$$

Similarly, from the second inequality in (5.25) and the first inequality in (5.27) we have $\operatorname{ind}(\tilde{y}^*Q) \leqslant 2(k-1)$. Thus, $\pm\tilde{y}^* \in \mathscr{Y}_{2(k-1)} \Rightarrow \tilde{y}^* = 0$, because $\operatorname{conv}\mathscr{Y}_{2(k-1)}$ is an acute cone. However, since $\tilde{y}^* = 0$, the equality in (5.27) yields $\tilde{q}_0(\tilde{h}) = 0$, contradicting (5.26) and showing that $\widetilde{Q}_0 = 0 \Rightarrow Q_0 + y^*Q = 0$. □

The theory of quadratic maps is now at its initial stage of development, so we present some problems in this theory which have not yet been solved.

Problem 1. Let $Q\colon \mathbb{R}^n \to \mathbb{R}^k$ be a surjective quadratic map. Is it true that any sufficiently close quadratic map is also surjective? (We understand the closeness of quadratic maps in an obvious sense: the matrices defining theme are pairwise close.) A positive answer was given in [35] under the additional assumption that the cone $\operatorname{conv}\mathscr{Y}_k$ is acute. The answer in the general case is not yet known.

Problem 2. Let $Q\colon \mathbb{R}^n \to \mathbb{R}^k$ be a surjective quadratic map. Is it true that $0 \in \operatorname{int} Q(B)$? Here B is the unit ball in $\mathbb{R}^n$.

Problem 3. Let $Q\colon \mathbb{R}^n \to \mathbb{R}^k$ be a quadratic map with no non-trivial zeros. Then the set $\mathscr{Y}_{k-1}$ is non-empty. This follows from application of Theorem 2.7 to the problem (5.2) for $q_0(x) \equiv -|x|^2$.

For $k = 2, 3, 5, 9$ [36] contains examples of quadratic maps with no non-trivial zeros, but with $\mathscr{Y}_{k-2} = \varnothing$. The question is whether there are examples of such quadratic maps for $k \neq 2, 3, 5, 9$?

6. Inverse function theorems

The index approach. Let $F\colon X \to Y$, where X is a linear space and $Y = \mathbb{R}^k$. Fix $x_0 \in X$ and let $y_0 = F(x_0)$. We present conditions ensuring that there exists a neighbourhood V of y_0 such that for each $y \in V$ equation (1.6) has a solution $x(y)$ for which $x(y_0) = x_0$ and the map $x(\cdot)$ is continuous at y_0 in the finite topology and satisfies some a priori bounds.

Theorem 6.1. *Suppose that the map F is twice continuously differentiable in a neighbourhood of x_0 with respect to the finite topology τ. Assume that*

$$\exists\, h \in X\colon \quad F'(x_0)h = 0, \quad \bigl\langle y^*, F''(x_0)[h,h]\bigr\rangle < 0 \quad \forall\, y^* \in \mathscr{F}_k^2(x_0). \tag{6.1}$$

Then there exist a constant const, *a neighbourhood V of the point y_0, and a finite-dimensional linear subspace $\widetilde{\Pi} \subseteq X$ such that*

$$\begin{gathered} \forall\, y \in V \quad \exists\, x(y) \in \widetilde{\Pi}\colon\ F\bigl(x(y)\bigr) = y, \\ \bigl\|x(y) - x_0\bigr\|_{\widetilde{\Pi}} \leqslant \operatorname{const}\bigl(|y - y_0| + \bigl|\pi(y - y_0)\bigr|^{1/2}\bigr). \end{gathered} \tag{6.2}$$

As above, here π is the orthogonal projection of Y onto $\bigl(\operatorname{im} F'(x_0)\bigr)^{\perp}$ and $\mathscr{F}_s^2(x)$ is the set of $y^* \in Y^*$ such that $|y^*| = 1$, $F'(x)^*y^* = 0$, and X has a subspace Π such that (2.19) holds.

Moreover, if $0 \notin \operatorname{conv} \mathscr{F}_k^2(x_0)$ and (6.2) is satisfied, then (6.1) holds. However, it was observed in § 2 (see also [5], Chap. 1, § 1.9) that for $n \gg k$ generic maps have the property that $0 \notin \operatorname{conv} \mathscr{F}_k^2 x)$ for all x, and therefore for them Theorem 6.1 gives not only sufficient but also necessary conditions for an inverse function $x(\cdot)$ to exist.

If x_0 is a normal point, then $\mathscr{F}_k^2(x_0) = \varnothing$, $\pi = 0$, and Theorem 6.1 becomes in fact a classical inverse function theorem (here is easy to prove separately that the map $x(\cdot)$ can be taken to be smooth). On the other hand, if x_0 is abnormal, then $\pi \neq 0$ and for the solution we have only the estimate (6.2) with orders 1 and 1/2 in the arguments, in contrast to the linear estimate in the classical case.

A proof of the above results in the general case based on necessary second-order conditions in § 2 is presented in [35]. To describe the central idea of the proof we establish the following result, which yields Theorem 6.1 in the special case when $x_0 = 0$ and F is a quadratic map.

Lemma 6.1. *Let $Q\colon \mathbb{R}^n \to Y = \mathbb{R}^k$, $k \geqslant 2$, be a quadratic map and assume that*

$$\exists\, h \in \mathbb{R}^n\colon \quad \bigl\langle Q(h), y^*\bigr\rangle < 0 \quad \forall\, y^* \in \mathscr{Y}_{k-2} \tag{6.3}$$

(*see* (5.20)). *Then there exists a constant* const *such that*

$$\forall\, y \in Y \quad \exists\, x(y) \in \mathbb{R}^n\colon \quad Q\bigl(x(y)\bigr) = y, \quad \bigl|x(y)\bigr| \leqslant \mathrm{const}\,|y|^{1/2}.$$

Proof. Assume the converse. Then it is easy to see that for each i there exists a $y_i \in Y$ such that $|y_i| = 1$, and the point $x = 0$, $\chi = 0$ is a solution in the minimization problem

$$|x|^2 - i\chi \to \min, \qquad Q(x) - \chi y_i = 0, \qquad (x,\chi) \in \mathbb{R}^n \times \mathbb{R}.$$

By the necessary second-order conditions (2.18) for this problem, there exist $\lambda_i^0 \geqslant 0$, $y_i^* \in Y$ with $\lambda_i^0 + |y_i^*| = 1$, and a linear subspace $\widetilde{\Pi}_i \subseteq \mathbb{R}^n \times \mathbb{R}$ such that

$$\operatorname{codim} \widetilde{\Pi}_i \leqslant k-1, \qquad \chi = 0 \quad \forall\,(x,\chi) \in \widetilde{\Pi}_i, \qquad i\lambda_i^0 + \langle y_i^*, y_i\rangle = 0,$$
$$\bigl\langle y_i^*, Q(x)\bigr\rangle + \lambda_i^0 |x|^2 \geqslant 0 \quad \forall\, x \in \Pi_i = \bigl\{x \in \mathbb{R}^n\colon (x,\chi) \in \widetilde{\Pi}_i\bigr\},$$
$$\bigl\langle y_i^*, Q(h)\bigr\rangle + \lambda_i^0 |h|^2 \geqslant 0.$$

Then $\operatorname{codim} \Pi_i \leqslant k-2 \Rightarrow \operatorname{ind}(y_i^* Q + \lambda_i^0 I) \leqslant k-2$ (I is the identity matrix). Furthermore, $\lambda_i^0 \to 0 \Rightarrow |y_i^*| \to 1$. By going to a subsequence we can assume that $y_i^* \to y^*$, $|y^*| = 1$. Passing to the limit as $i \to \infty$, we get that $\bigl\langle y^*, Q(h)\bigr\rangle \geqslant 0$, and then $y^* \in \mathscr{Y}_{k-2}$ by Theorem 2.3. This is a contradiction to (6.3). □

For $k = 1$ we must replace (6.3) by (5.23) or by $\mathscr{Y}_0 = \varnothing$: these relations are equivalent for $k = 1$.

Lemma 6.1 gives sufficient conditions for quadratic maps to be surjective. It is important to point out that in view of Example 4 in [37] the example of the map (5.18) shows that there is not necessarily a continuous map $x(\cdot)$ in Lemma 6.1.

2-regularity. A mere solution of (1.6) is often insufficient for applications: one must find solutions $x \in K$, where K is a fixed subset of Y. Of course, along with abnormality, the constraint $x \in K$ brings additional complications to the problem.

Let us consider the following problem, which is more general than (1.6): find a solution to the equation

$$F(x,\sigma) = 0, \qquad x \in K, \tag{6.4}$$

where $F\colon X \times \Sigma \to Y$ is a given map, X and Y are Banach spaces, Σ is a topological space, and $K \subseteq X$ is a closed convex cone. Suppose that there exist an $x_0 \in K$ and a $\sigma_0 \in \Sigma$ such that $F(x_0, \sigma_0) = 0$.

Under the assumption that F is smooth, it follows from Robinson's stability theorem [16], [14] that if the Robinson regularity condition

$$\frac{\partial F}{\partial x}(x_0,\sigma_0)\bigl(K + \operatorname{span}\{x_0\}\bigr) = Y \tag{6.5}$$

holds (for $K = X$ the Robinson condition is the condition of normality), then the point σ_0 has a neighbourhood O such that for each $\sigma \in O$ there exists an $x(\sigma) \in K$ for which

$$F\bigl(x(\sigma),\sigma\bigr) \equiv 0, \quad \bigl\|x(\sigma) - x_0\bigr\| \leqslant \mathrm{const}\bigl\|F(x_0,\sigma)\bigr\| \qquad \forall\,\sigma \in O. \tag{6.6}$$

Though the question of whether the map $x(\cdot)$ can be taken to be continuous was not discussed in the papers cited, the following result holds.

Theorem 6.2 (classical implicit function theorem). *Let $F\colon X\times\Sigma\to Y$ be continuous in a neighbourhood of a point (x_0,σ_0), assume that F is strictly x-differentiable at this point uniformly with respect to σ, and assume that the Robinson condition (6.5) is satisfied. Then the point σ_0 has a neighbourhood O in which there exists a continuous map $x(\cdot)\colon O\to K$ such that (6.6) holds.*

Let us now consider implicit function theorems without the a priori imposition of the Robinson condition. As for F, we assume that it is twice continuously x-differentiable uniformly with respect to σ in a neighbourhood of (x_0,σ_0) and that for each fixed σ its second partial derivative $\frac{\partial^2 F}{\partial x^2}(\cdot,\sigma)$, regarded as a symmetric bilinear map, satisfies a Lipschitz condition with respect to x with Lipschitz constant independent of σ (see [38] for details). We assume that the maps $F(x_0,\cdot)$, $\frac{\partial F}{\partial x}(x_0,\cdot)$, and $\frac{\partial^2 F}{\partial x^2}(x_0,\cdot)$ are continuous in a neighbourhood of σ_0.

Let

$$\mathscr{K}=K+\operatorname{span}\{x_0\},\qquad C=\frac{\partial F}{\partial x}(x_0,\sigma_0)(\mathscr{K}).$$

We assume that the linear span $\operatorname{span}C$ of the cone C is closed and is a topologically complemented subspace. Let π denote the continuous linear operator which projects Y onto a complementary subspace to $\operatorname{span}C$.

Definition. Let

$$h\in\mathscr{K},\qquad \frac{\partial F}{\partial x}(x_0,\sigma_0)h=0,\qquad -\frac{\partial^2 F}{\partial x^2}(x_0,\sigma_0)[h,h]\in C. \tag{6.7}$$

Then the map F is said to be 2-regular at a point (x_0,σ_0) with respect to K in the direction h if

$$\frac{\partial F}{\partial x}(x_0,\sigma_0)(\mathscr{K})+\frac{\partial^2 F}{\partial x^2}(x_0,\sigma_0)\biggl[h,\mathscr{K}\cap\ker\frac{\partial F}{\partial x}(x_0,\sigma_0)\biggr]=Y. \tag{6.8}$$

In the case when $\operatorname{ri}C\neq\varnothing$ another definition of 2-regularity, which is equivalent to the above, was presented in [39]. For $K=X$ both definitions become the 2-regularity condition discussed in the previous sections.

Theorem 6.3. *Let $\operatorname{ri}C\neq\varnothing$ and assume that there exists an $h\in X$ such that the map F is 2-regular at a point (x_0,σ_0) with respect to K in the direction h (that is, (6.7) and (6.8) hold). Then for each $l\in\operatorname{ri}C$ there exist a neighbourhood O of σ_0, a $\delta>0$, and a continuous map $x(\cdot)\colon O\to K$ such that*

$$F\bigl(x(\sigma),\sigma\bigr)=0\qquad\forall\,\sigma\in O, \tag{6.9}$$

$$\begin{aligned}\bigl\|x(\sigma)-x_0\bigr\|\leqslant\operatorname{const}\bigl(&\Delta_1(\sigma)+\Delta_2(\sigma)+\bigl\|F(x_0,\sigma)\bigr\|\\&+\rho\bigl(-F(x_0,\sigma),C_\delta\bigr)^{1/2}\bigr)\qquad\forall\,\sigma\in O,\end{aligned} \tag{6.10}$$

where

$$\Delta_1(\sigma)=\sup\biggl\{\biggl\|\pi\frac{\partial F}{\partial x}(x_0,\sigma)x\biggr\|,\ x\in\operatorname{span}K,\ \|x\|\leqslant 1\biggr\},$$

$$\Delta_2(\sigma)=\sup\biggl\{\biggl\|\frac{\partial F}{\partial x}(x_0,\sigma)x\biggr\|,\ x\in\ker\frac{\partial F}{\partial x}(x_0,\sigma_0)\cap\mathscr{K},\ \|x\|\leqslant 1\biggr\},$$

and $C_\delta=\operatorname{cone}\bigl(B_\delta(l)\bigr)\cap\operatorname{span}C$ (cone denotes the conical hull of a set).

Making the additional assumption

$$\text{the cone}\quad \frac{\partial F}{\partial x}(x_0,\sigma_0)(K)\quad \text{is a subspace,} \tag{6.11}$$

we can improve the estimate (6.10) in Theorem 6.3 by dropping the term $\Delta_2(\sigma)$. Namely, the following result holds.

Theorem 6.4. *Under the assumptions of Theorem 6.3 suppose that condition (6.11) also holds. Then the point σ_0 has a neighbourhood in which there exists a continuous map $x(\cdot)\colon O\to K$ such that both (6.9) and the estimate*

$$\|x(\sigma)-x_0\| \leqslant \mathrm{const}\bigl(\Delta_1(\sigma)+\|F(x_0,\sigma)\|+\|\pi F(x_0,\sigma)\|^{1/2}\bigr) \qquad \forall\,\sigma\in O \tag{6.12}$$

hold.

The classical implicit function theorem follows from Theorem 6.4 (for if the Robinson condition holds, then $\pi=0$). Theorem 6.4 also yields a generalized implicit function theorem concerned with solving the equation $F(x,\sigma)=y$ with respect to $x\in K$ (see Theorem 5 in [38] for details).

From Theorem 6.4 and the results of [24] we can deduce Theorem 6.1, even improving it in the abnormal case. Namely, let x_0 be an abnormal point and assume that

$$\exists\, h\in X\colon\quad F'(x_0)h=0,\quad \bigl\langle y^*,F''(x_0)[h,h]\bigr\rangle<0 \quad \forall\, y^*\in \mathscr{F}^2_{k-1}(x_0). \tag{6.13}$$

Then there exist a neighbourhood V of $y_0=F(x_0)$, a finite-dimensional linear subspace $\widetilde{\Pi}\subseteq X$, and a continuous map $x(\cdot)\colon V\to\widetilde{\Pi}$ such that (6.2) holds.

Indeed, by (6.13) and Theorem 2 in [23] there is a direction $h\in X$ such that F is 2-regular at x_0 along h. Hence the required result follows from Theorem 6.4 with $K=X$, because F is linear in σ under the assumptions of Theorem 6.1, and then the term Δ_1 in (6.12) vanishes for $K=X$.

Theorems 6.3 and 6.4 hold only provided that the cone $\operatorname{ri} C$ is non-empty. However, if Y is infinite-dimensional, then a convex cone C in Y can have empty relative interior. Therefore, we present an implicit function theorem which holds without the a priori assumption that $\operatorname{ri} C$ is non-empty.

Theorem 6.5. *Let F be a 2-regular map at (x_0,σ_0) with respect to K in some direction $h\in X$ (that is, (6.7) and (6.8) hold). Then the point σ_0 has a neighbourhood O in which there exists a continuous map $x(\cdot)\colon O\to K$ such that (6.9) holds and*

$$\|x(\sigma)-x_0\| \leqslant \mathrm{const}\bigl(\Delta_1(\sigma)+\Delta_2(\sigma)+\|F(x_0,\sigma)\|^{1/2}\bigr) \qquad \forall\,\sigma\in O.$$

These results were obtained in [38] and [40], where there are also some examples given as illustrations. We note that the estimates for $\|x(\sigma)-x_0\|$ in these theorems have orders 1 and 1/2 with respect to the arguments, in contrast to the linear estimate in the classical implicit function theorem.

For $K=X$ an implicit function theorem at an abnormal point was obtained in [41], where under the assumption that F is 2-regular in some direction it was proved that there exists a map $x(\cdot)$ satisfying (6.9) and the estimate (6.12) with $\Delta_1=0$. The question of the continuity of $x(\cdot)$ was not treated in [41].

Equations with non-closed image. We return to equation (1.6) in the case when X and Y are Banach spaces. Theorems 6.3–6.5 can also be applied to (1.6), but under the assumption that the range $\operatorname{im} F'(x_0)$ is closed. However, if x_0 is an abnormal point and the space Y is infinite-dimensional, then this assumption can be a burden. Here we present an inverse function theorem in which we do not assume that $\operatorname{im} F'(x_0)$ is closed. Let F be a map with two continuous derivatives such that the second derivative is Lipschitz-continuous in a neighbourhood of x_0.

Theorem 6.6. *Let F be a 2-regular map in some direction h satisfying conditions (3.15). Then there exist $r, c, c_1 > 0$ such that for each $l \in \operatorname{im} F'(x_0)$ and each $y \in B_Y\bigl(y_0, r(l)\bigr)$ there is an $x(y) \in B_X(x_0, r)$ for which*

$$F\bigl(x(y)\bigr) = y, \quad \|x(y) - x_0\| \leqslant c_1\bigl(b(l)\|y - y_0\| + \|y - y_0\|^{1/2}\|\theta(y - y_0) - \theta(l)\|^{1/2}\bigr),$$

where $y_0 = F(x_0)$, $B_X(x, \rho)$ is the ball in X of radius ρ about x, $\theta(y) = y/\|y\|$ for $y \neq 0$, $\theta(0) = 0$, $b(y) = \inf\bigl\{\|x\| \colon F'(x_0)x = \theta(y)\bigr\}$ for $y \in \operatorname{im} F'(x_0)$, and $r(l) = \min\bigl\{cr/b(l), cr^2\bigr\}$.

For $l = 0$ this theorem has an important consequence.

Corollary. *Under the assumptions of Theorem 6.6 there exist $r, c > 0$ such that*

$$\forall\, y \in B_Y(y_0, r) \quad \exists\, x(y)\colon \quad F(x(y)) = y, \quad \bigl\|x(y) - x_0\bigr\| \leqslant c\|y - y_0\|^{1/2}.$$

Theorem 6.7. *Let F be a 2-regular map in some direction $h \in H(x_0)$ (see (3.2)). Then there exist constants $r, c, c_1, c_2 > 0$ such that for each $l \in \operatorname{im} F'(x_0)$ and each $y \in B_Y\bigl(y_0, r(l)\bigr)$*

$$\exists\, x(y) \in B_X(x_0, r)\colon \ F\bigl(x(y)\bigr) = y, \quad x(y) = x_0 + t(y)(h + \xi + \chi_1) + O\bigl(t^2(y)\bigr),$$

where $r(l)$ is as defined above,

$$t(y) = \max\Bigl\{c_1\|y - y_0\|b(l)r^{-1},\ c_2\|y - y_0\|^{1/2}\bigl\|\theta(y - y_0) - \theta(l)\bigr\|^{1/2}r^{-1/2}\Bigr\},$$

and ξ and χ_1 are arbitrary solutions of a certain linear-quadratic system of equations (explicitly written out in Theorem 2 in [23]).

This theorem includes the assertion that the required ξ and χ_1 exist. The proofs of both theorems and examples illustrating them can be found in [22].

7. Necessary second-order conditions in optimal control problems

The Lagrange problem. As one application of the results in § 2 we obtain necessary conditions for a weak minimum in an optimal control problem. We start with the Lagrange problem, that is, the problem without constraints on the controls:

$$\dot{x} = f(x, u, t), \qquad t \in [t_1, t_2], \tag{7.1}$$

$$K_1(p) \leqslant 0, \quad K_2(p) = 0, \quad \text{where} \quad p = (x_1, x_2), \quad x_1 = x(t_1), \quad x_2 = x(t_2), \tag{7.2}$$

$$J = J(p, u) = K_0(p) + \int_{t_1}^{t_2} f^0(x, u, t)\, dt \to \min. \tag{7.3}$$

Here $t \in [t_1, t_2]$ is the time, and the moments of time $t_1 < t_2$ are fixed; $x \in \mathbb{R}^n$ is the phase variable, and $u \in \mathbb{R}^m$ is the control parameter; f is a given n-dimensional vector-valued function, and K_0 and f^0 are scalar functions. The functions K_1 and K_2 take values in the arithmetic spaces of dimension $d(K_1)$ and $d(K_2)$, respectively. (Throughout, $d(z)$ is the dimension of a vector z.)

As for the functions K_0, K_1, and K_2, they are assumed to be twice continuously differentiable. The functions f^0 and f are twice continuously differentiable with respect to (x, u) for almost all t. Moreover, together with all their partial derivatives of orders 1 and 2 with respect to (x, u), they are measurable in t for any fixed (x, u) and are bounded and continuous in (x, u) on each bounded set uniformly with respect to x, u, and t. We take the space of functions $u = u(\cdot) \in L^m_\infty[t_1, t_2]$ for the class of admissible controls.

As for the constraints (7.2), called the endpoint constraints, the following condition of regularity is assumed: for each p satisfying (7.2),

$$\operatorname{rank} \frac{\partial K_2}{\partial p}(p) = d(K_2),$$
$$\exists\, \tilde{p} = (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^{2n}\colon\ \tilde{p}\frac{\partial K_2}{\partial p}(p) = 0, \quad \left\langle \frac{\partial K_{1,j}}{\partial p}(p), \tilde{p} \right\rangle < 0$$
$$\forall\, j\colon\ K_{1,j}(p) = 0, \quad j = 1, \dots, d(K_1).$$

A pair of vector-valued functions $(x(t), u(t))$, $t \in [t_1, t_2]$, is called an admissible process if $u(\cdot)$ is an admissible control and $x(\cdot)$ is the corresponding solution of (7.1) which satisfies the endpoint constraints (7.2). The original problem consists in finding the minimum of the functional J on the set of admissible processes.

We set $K = (K_1, K_2)$. Let $H\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \times (\mathbb{R}^n)^* \times \mathbb{R} \to \mathbb{R}$ and $l\colon \mathbb{R}^{2n} \times \mathbb{R}^{1+d(K)} \to \mathbb{R}$ be the Pontryagin function and the endpoint Lagrange function defined by the formulae

$$H(x, u, t, \psi, \lambda^0) = \langle f(x, u, t), \psi \rangle - \lambda^0 f^0(x, u, t),$$
$$l(p, \lambda) = \lambda^0 K_0(p) + \langle \lambda^1, K_1(p) \rangle + \langle \lambda^2, K_2(p) \rangle, \qquad \lambda = (\lambda^0, \lambda^1, \lambda^2) \in \mathbb{R}^{1+d(K)},$$

where $\lambda^0 \in \mathbb{R}$, $\lambda^s \in \mathbb{R}^{d(K_s)}$, $s = 1, 2$, and ψ is an n-dimensional column vector.

We say that an admissible process $(x_0(\cdot), u_0(\cdot))$ satisfies the Euler–Lagrange equation if there exists a vector λ with

$$|\lambda| = 1, \qquad \lambda^0 \geqslant 0, \qquad \lambda^1 \geqslant 0, \qquad \langle \lambda^1, K_1(p_0) \rangle = 0 \tag{7.4}$$

such that the vector-valued function ψ solving the Cauchy problem

$$\dot{\psi} = -\frac{\partial H}{\partial x}\big(x_0(t), u_0(t), t, \psi, \lambda^0\big), \qquad \psi(t_1) = \frac{\partial l}{\partial x_1}(p_0, \lambda) \tag{7.5}$$

satisfies

$$\psi(t_2) = -\frac{\partial l}{\partial x_2}(p_0, \lambda), \quad \text{where} \quad p_0 = \big(x_0(t_1), x_0(t_2)\big), \tag{7.6}$$

and

$$\frac{\partial H}{\partial u}\big(x_0(t), u_0(t), t, \psi(t), \lambda^0\big) = 0 \qquad \dot{\forall}\, t, \tag{7.7}$$

where $\dot{\forall}\, t$ means 'for almost all t'. Note that since the endpoint constraints are regular, we have $\lambda^0 + \big|\psi(t)\big| > 0$ for all t.

Let $\Lambda(x_0(\cdot), u_0(\cdot))$ denote the set of vectors λ corresponding to an admissible process $\big(x_0(\cdot), u_0(\cdot)\big)$ by virtue of the Euler–Lagrange equation; this set gives us necessary first-order conditions for an extremum which are weaker in general than the Pontryagin maximum principle and which are equivalent to it in the case of problems linear in the control. A process satisfying the Euler–Lagrange equation is called an extremal.

We consider an extremal $\big(x_0(\cdot), u_0(\cdot)\big)$. From the endpoint constraints (7.2) we shall again remove for convenience all the constraints of inequality type corresponding to the inactive indices and assume that $K_1(p_0) = 0$. The system of equations in variations corresponding to this extremal has the form

$$\begin{gathered} \frac{d}{dt}\delta x = \delta x \frac{\partial f}{\partial x}\big(x_0(t), u_0(t), t\big) + \delta u(t) \frac{\partial f}{\partial u}\big(x_0(t), u_0(t), t\big), \\ \delta u \in L_\infty^m[t_1, t_2]. \end{gathered} \tag{7.8}$$

We look at the space $X = \mathbb{R}^n \times L_\infty^m[t_1, t_2]$ and in it the cone $\mathscr{K}\big(x_0(\cdot), u_0(\cdot)\big)$ of critical directions consisting of the pairs $(\xi, \delta u)$ such that

$$\big(\delta x(t_1), \delta x(t_2)\big) \frac{\partial K_1}{\partial p}(p_0) \leqslant 0, \qquad \big(\delta x(t_1), \delta x(t_2)\big) \frac{\partial K_2}{\partial p}(p_0) = 0. \tag{7.9}$$

Here and below, δx is the solution of the system of equations in variations (7.8) corresponding to δu and with the initial condition $\delta x(t_1) = \xi$. Let

$$\mathscr{N}_K = \bigg\{(\xi, \delta u) \in X \colon \big(\delta x(t_1), \delta x(t_2)\big) \frac{\partial K}{\partial p}(p_0) = 0\bigg\}.$$

Obviously, $\mathscr{N}_K$ is the maximal linear subspace of $\mathscr{K}\big(x_0(\cdot), u_0(\cdot)\big)$.

For $\lambda \in \Lambda\big(x_0(\cdot), u_0(\cdot)\big)$ we define a quadratic form $\Gamma(\lambda)$ on X:

$$\begin{aligned} \Gamma(\xi, \delta u; \lambda) = {} & \frac{\partial^2 l}{\partial p^2}(p_0, \lambda)\big[\big(\delta x(t_1), \delta x(t_2)\big), \big(\delta x(t_1), \delta x(t_2)\big)\big] \\ & - \int_{t_1}^{t_2} \frac{\partial^2 H}{\partial (x,u)^2}\big(x_0(t), u_0(t), t, \psi(t), \lambda^0\big)\big[\big(\delta x(t), \delta u(t)\big), \big(\delta x(t), \delta u(t)\big)\big]\, dt. \end{aligned}$$

Let Φ be the fundamental matrix of the system of equations in variations (7.8), that is, the solution of the homogeneous linear system $\frac{d}{dt}\Phi = \Phi \frac{\partial f}{\partial x}\big(x_0(t), u_0(t), t\big)$ with $\Phi(t_1) = I$, where I is the identity matrix. We set

$$A = \frac{\partial K}{\partial x_1}(p_0) + \Phi(t_2)\frac{\partial K}{\partial x_2}(p_0), \quad B(t) = \frac{\partial f}{\partial u}\big(x_0(t), u_0(t), t\big)\Phi^{-1}(t)\Phi(t_2)\frac{\partial K}{\partial x_2}(p_0)$$

and consider the *extended controllability matrix*

$$D = A^*A + \int_{t_1}^{t_2} B^*(t)B(t)\,dt \tag{7.10}$$

(of size $d(K) \times d(K)$). Let d be the dimension of the kernel of D. Finally, let $\Lambda_\theta = \Lambda_\theta\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ denote the set of $\lambda \in \Lambda\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ such that the restriction of the form $\Gamma(\lambda)$ to the subspace $\mathscr{N}_K$ has index at most $\theta = \min(d, 2n)$.

Theorem 7.1. *Let $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ be an admissible process which supplies a weak minimum in the problem (7.1)–(7.3). Then $\Lambda_\theta \neq \varnothing$ and for arbitrary $(\xi, \delta u) \in \mathscr{K}$ satisfying*

$$\biggl\langle \bigl(\delta x(t_1), \delta x(t_2)\bigr), \frac{\partial K_0}{\partial p}(p_0) \biggr\rangle + \int_{t_1}^{t_2} \biggl\langle \bigl(\delta x(t), \delta u(t)\bigr), \frac{\partial f^0}{\partial(x,u)}\bigl(x_0(t), u_0(t), t\bigr) \biggr\rangle\,dt \leqslant 0 \tag{7.11}$$

the following inequality holds:

$$\max_{\lambda \in \Lambda_\theta} \Gamma(\xi, \delta u; \lambda) \geqslant 0. \tag{7.12}$$

The proof consists in reformulating the optimal control problem (7.1)–(7.3) in the abstract form (2.1) and applying to it Theorem 2.1, after which the result must be deciphered. Namely, by the standard existence and uniqueness theorems, for any $\bigl(x_1, u(\cdot)\bigr) \in X$ sufficiently close to $\bigl(x_0(t_1), u_0(\cdot)\bigr)$ the Cauchy problem

$$\dot{x} = f\bigl(x, u(t), t\bigr), \qquad x(t_1) = x_1, \qquad t \in [t_1, t_2],$$

has a unique solution, which we denote by $x(\cdot)$. For $\bigl(x_1, u(\cdot)\bigr)$ as above we set

$$F_s\bigl(x_1, u(\cdot)\bigr) = K_s\bigl(x_1, x(t_2)\bigr), \qquad s = 1, 2,$$
$$f_0\bigl(x_1, u(\cdot)\bigr) = K_0\bigl(x_1, x(t_2)\bigr) + \int_{t_1}^{t_2} f^0\bigl(x(t), u(t), t\bigr)\,dt.$$

Let us consider the problem

$$f_0\bigl(x_1, u(\cdot)\bigr) \to \min, \qquad F_1\bigl(x_1, u(\cdot)\bigr) \leqslant 0, \qquad F_2\bigl(x_1, u(\cdot)\bigr) = 0. \tag{7.13}$$

The point $\bigl(x_0(t_1), u_0(\cdot)\bigr)$ supplies a local minimum in this problem with respect to the finite topology. Hence, we can apply the necessary conditions of orders 1 and 2 from Theorem 2.1. After standard transformations (see [3], § 4.1 for details) the Lagrange principle (2.2) gives rise to the Euler–Lagrange conditions, while (2.4), after similar transformations, gives rise to (7.12). Here it is used that the subspace $\mathscr{N}_K \subseteq X$ has codimension $d(K) - d$ (see Proposition 1 in [42]) and $\mathscr{N}_K$ coincides with the kernel of the derivative of the map $F = (F_1, F_2)\colon X \to \mathbb{R}^{d(K)}$ at the point $\bigl(x_0(t_1), u_0(\cdot)\bigr)$.

We discuss the theorem. It is known that for an extremal $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ the condition $d = 0$ is tantamount to the controllability of the system of equations in variations (7.8) (the normality condition), which ensures the local controllability of the system (7.1) in a neighbourhood of $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$. If one or both of the endpoints

x_1 and x_2 of the trajectory are free (so that the map K does not depend on the corresponding variable x_1 or x_2), then $d = 0$. On the other hand, if at least one endpoint, x_1 or x_2, is fixed in advance, then we can take $\theta = \min(d, n)$.

Assume that there are no endpoint constraints of inequality type (so $d(K_1) = 0$). Then we can omit the assumption (7.11) in the statement of the theorem. In addition if $d = 0$, then $\Lambda\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ consists of a unique point λ, and the theorem guarantees that the form $\Gamma(\lambda)$ is non-negative on $\mathscr{K}$, which is the classical necessary second-order condition. On the other hand, if an extremal supplies a weak minimum but is abnormal (that is, $d \neq 0$), then a $\lambda \in \Lambda\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ such that $\Gamma(\lambda)$ is non-negative on $\mathscr{K}$ does not necessarily exist: a corresponding example is due to E. J. McShane and goes back to 1941 (see [11]).

Problems with mixed constraints. We consider the optimal control problem with mixed constraints

$$R_1(x, u, t) \leqslant 0, \qquad R_2(x, u, t) = 0, \tag{7.14}$$

where the $R_s\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^{d(R_s)}$, $s = 1, 2$, are given maps satisfying the same smoothness assumptions as f. An admissible process in this problem is a pair of vector-valued functions $\bigl(x(t), u(t)\bigr)$, $t \in [t_1, t_2]$, in which $u(\cdot)$ is an admissible control and the corresponding solution $x(\cdot)$ of equation (7.1) must satisfy the endpoint constraints (7.2) and the mixed constraints $R_1\bigl(x(t), u(t), t\bigr) \leqslant 0$ and $R_2\bigl(x(t), u(t), t\bigr) = 0$ for almost all t.

Consider an admissible process $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$. We assume that the mixed constraints are regular on this process, that is, there exists an $\varepsilon_0 > 0$ such that for almost all $t \in [t_1, t_2]$ the matrix $\frac{\partial R}{\partial u}(t)$ contains a minor of order $\bigl|I_{\varepsilon_0}(t)\bigr|$ with modulus at least ε_0 located in the last $d(R_2)$ columns and the columns with indices $i \in I_{\varepsilon_0}(t)$. Here $R = (R_1, R_2)$, $|I|$ is the cardinal number of a set I, and

$$I_\varepsilon(t) := \bigl\{i\colon 1 \leqslant i \leqslant d(R_1),\ R_{1,i}(t) \geqslant -\varepsilon\bigr\}$$

for $\varepsilon \geqslant 0$; we also use the notation

$$R_1(t) := R_1\bigl(x_0(t), u_0(t), t\bigr), \qquad \frac{\partial R}{\partial u}(t) := \frac{\partial R}{\partial u}\bigl(x_0(t), u_0(t), t\bigr), \qquad \text{and so on.}$$

We say that an admissible process $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ satisfies the Euler–Lagrange equation in the problem with mixed constraints if there exist a vector λ satisfying (7.4) and a vector-valued function

$$\begin{gathered} \eta = (\eta_1, \eta_2)\colon\ \eta_s \in L_\infty^{d(R_s)}[t_1, t_2], \qquad s = 1, 2, \\ \eta_1(t) \geqslant 0, \qquad \bigl\langle \eta_1(t), R_1(t)\bigr\rangle = 0 \qquad \dot\forall\, t, \end{gathered}$$

such that the vector-valued function ψ solving the Cauchy problem

$$\dot\psi = -\frac{\partial H}{\partial x}\bigl(x_0(t), u_0(t), t, \psi, \lambda^0\bigr) + \eta(t)\frac{\partial R}{\partial x}(t)^*, \qquad \psi(t_1) = \frac{\partial l}{\partial x_1}(p_0, \lambda),$$

satisfies conditions (7.6) and

$$\frac{\partial H}{\partial u}\bigl(x_0(t), u_0(t), t, \psi(t), \lambda^0\bigr) - \eta(t)\frac{\partial R}{\partial u}(t)^* = 0 \qquad \dot\forall\, t.$$

Using the regularity of the mixed constraints and Gronwall's inequality, we get that each vector λ corresponds by virtue of the Euler–Lagrange conditions to a unique pair $\bigl(\psi(\cdot),\eta(\cdot)\bigr)$. Let $\Lambda_R\bigl(x_0(\cdot),u_0(\cdot)\bigr)$ be the set of λ such that for some vector-valued function η the pair (λ,η) corresponds to the process $\bigl(x_0(\cdot),u_0(\cdot)\bigr)$ by virtue of the Euler–Lagrange equation in the problem with mixed constraints.

Let $\mathscr{K}_R$ denote the set of pairs $(\xi,\delta u)\in X$ satisfying (7.9) and the conditions

$$\biggl\langle \delta x(t),\frac{\partial R_{1,i}}{\partial x}(t)\biggr\rangle+\biggl\langle \delta u(t),\frac{\partial R_{1,i}}{\partial u}(t)\biggr\rangle\leqslant 0 \qquad \forall\, i\in I_0(t),$$
$$\delta x(t)\frac{\partial R_2}{\partial x}(t)+\delta u(t)\frac{\partial R_2}{\partial u}(t)=0 \qquad \dot{\forall}\, t.$$

In X we consider the subspace $\mathscr{N}_R$ of all $(\xi,\delta u)$ such that

$$\biggl\langle \delta x(t),\frac{\partial R_{1,i}}{\partial x}(t)\biggr\rangle+\biggl\langle \delta u(t),\frac{\partial R_{1,i}}{\partial u}(t)\biggr\rangle= 0 \qquad \forall\, i\in I_0(t),$$
$$\delta x(t)\frac{\partial R_2}{\partial x}(t)+\delta u(t)\frac{\partial R_2}{\partial u}(t)=0 \qquad \dot{\forall}\, t.$$

For $\lambda\in\Lambda_R\bigl(x_0(\cdot),u_0(\cdot)\bigr)$ we define a quadratic form $\Gamma_R(\lambda)$ on X by

$$\begin{aligned}\Gamma_R(\xi,\delta u;\lambda)&=\Gamma(\xi,\delta u;\lambda)\\&\quad+\int_{t_1}^{t_2}\biggl\langle \eta(t),\frac{\partial^2 R}{\partial(x,u)^2}\bigl(x_0(t),u_0(t),t\bigr)\bigl[\bigl(\delta x(t),\delta u(t)\bigr),\bigl(\delta x(t),\delta u(t)\bigr)\bigr]\biggr\rangle\,dt.\end{aligned}$$

Let $\Lambda_{\theta_R}=\Lambda_{\theta_R}\bigl(x_0(\cdot),u_0(\cdot)\bigr)$ denote the set of $\lambda\in\Lambda_R\bigl(x_0(\cdot),u_0(\cdot)\bigr)$ such that the restriction of the form $\Gamma_R(\lambda)$ to the subspace $\mathscr{N}_K\cap\mathscr{N}_R$ has index at most $\theta_R=\min(d_R,2n)$. Here d_R is the dimension of the kernel of the extended controllability matrix D_R, which is explicitly written in [42]. In the particular case when the map R is independent of x, D_R can be calculated by the formula (7.10) with $B(t)$ replaced by the matrix $B_R(t)=P(t)B(t)$, where $P(t)$ is the operator of orthogonal projection of $\mathbb{R}^m$ onto the subspace

$$\biggl\{u\in\mathbb{R}^m\colon \biggl\langle\frac{\partial R_{1,i}}{\partial u}(t),u\biggr\rangle=0\ \forall\, i\in I_0(t),\ u\frac{\partial R_2}{\partial u}(t)=0\biggr\}.$$

Theorem 7.2. *Let $\bigl(x_0(\cdot),u_0(\cdot)\bigr)$ be an admissible process which supplies a weak minimum in the problem with mixed constraints* (7.1)–(7.3), (7.14). *Then $\Lambda_{\theta_R}\neq\varnothing$ and for arbitrary $(\xi,\delta u)\in\mathscr{K}_R$ satisfying* (7.11),

$$\max_{\lambda\in\Lambda_{\theta_R}}\Gamma_R(\xi,\delta u;\lambda)\geqslant 0. \tag{7.15}$$

The proof of Theorem 7.2 is essentially as follows. First we introduce an additional control $v\in\mathbb{R}^{d(R_1)}$ and consider the problem (7.1)–(7.3) with the mixed constraints

$$R_1(x,u,t)-v=0, \qquad R_2(x,u,t)=0, \qquad v\leqslant 0;$$

obviously, it is equivalent to the original problem (7.1)–(7.3), (7.14). Since the mixed constraints (7.14) are regular, we can solve the system $R_1(x,u,t)-v=0$,

$R_2(x,u,t)=0$ locally with respect to some of the variables and reduce the original problem to one which involves only geometric constraints on the controls $v\leqslant 0$ instead of mixed constraints (see [42] for details).

Restating the problem under consideration as a minimization problem in the Banach space $\mathbb{R}^n\times L_\infty^m[t_1,t_2]$, we obtain the problem (7.13) with an additional constraint on the control: $v(\cdot)\in C$, where $C=\bigl\{v(\cdot)\in L_\infty^{d(R_1)}\colon v(t)\leqslant 0$ for almost all $t\bigr\}$. Applying Theorem 2.4 to this problem (see [43] for details), we see that $\Lambda_{\theta_R}\neq\varnothing$ and (7.15) holds for arbitrary $(\xi,\delta u)\in\widetilde{\mathscr{K}}$. Here $\widetilde{\mathscr{K}}$ consists of those $(\xi,\delta u)\in X$ such that (7.9) and (7.11) hold and the condition $\delta v\in C+\operatorname{span}\{v_0\}$ is satisfied, where δv and v_0 are the first $d(R_1)$ components of the vector-valued functions δu and u_0, respectively. However, by Theorem 2 in [12] the closure of the set $\widetilde{\mathscr{K}}$ coincides with the set of $(\xi,\delta u)\in\mathscr{K}_R$ such that (7.11) holds. Furthermore, as noted in § 2 (see also Lemma 1 in [12]), the set Λ_{θ_R} is compact, and therefore the maximum function $\gamma(\xi,\delta u)=\max_{\lambda\in\Lambda_{\theta_R}}\Gamma(\xi,\delta u;\lambda)$ is continuous. The required result holds because the function γ is non-negative on $\widetilde{\mathscr{K}}$.

In optimal control problems with phase constraints and non-fixed time the necessary second-order conditions which require that the set Λ_s of Lagrange multipliers be non-empty for an appropriate choice of s were deduced in [9]–[11]. In these papers necessary conditions of the first order were taken in the form of the Pontryagin maximum principle rather than in the form of the Euler–Lagrange equation.

8. Some applications

Bifurcation theory. Let X, Σ, and Y be Banach spaces and let $F\colon X\times\Sigma\to Y$ be a fixed map which is twice continuously differentiable in a neighbourhood of a point (x_0,σ_0). We assume that the subspace $\ker\frac{\partial F}{\partial x}(x_0,\sigma_0)$ is topologically complemented and the subspace $\operatorname{im}\frac{\partial F}{\partial x}(x_0,\sigma_0)$ has finite codimension (and hence is closed). Let $F(x_0,\sigma)=0$ for all σ in some neighbourhood of σ_0.

Recall that (x_0,σ_0) is a bifurcation point if there exists a sequence $\bigl\{(x_i,\sigma_i)\bigr\}$ with $x_i\neq 0$ for all i that converges to (x_0,σ_0) and is such that $F(x_i,\sigma_i)=0$ for all i. If x_0 is a normal point, that is, $\operatorname{im}\frac{\partial F}{\partial x}(x_0,\sigma_0)=Y$, then bifurcations are described in terms of the kernel $\ker\frac{\partial F}{\partial x}(x_0,\sigma_0)$. Bifurcation points (x_0,σ_0) where x_0 is abnormal are of particular interest.

Using the results of § 2, we can find sufficient conditions for a bifurcation also in the abnormal case. Consider the point (x_0,σ_0) mentioned above. First, using the Lyapunov–Schmidt construction, we reduce the problem to the case when $Y=\mathbb{R}^k$ and $\frac{\partial F}{\partial x}(x_0,\sigma_0)=0$. Then we consider the minimization problem

$$f_0(x,\sigma):=\langle x-x_0,x^*\rangle\to\min,\qquad F(x,\sigma)=0,$$

where x^* is an arbitrary non-zero continuous linear functional on X. If (x_0,σ_0) is not a local minimum point in this problem, then obviously it is a bifurcation point. Therefore, using Theorem 2.7 (more precisely, writing out the negation of condition (2.18)), we obtain sufficient conditions for a bifurcation (see [44]).

Sensitivity theory. In applications it is often necessary to investigate a whole family of extremal problems instead of a single problem, for example,

$$f_0(x,\sigma) \to \min, \qquad F(x,\sigma) = 0,$$

where $\sigma \in \Sigma$ is a parameter. We are interested in the dependence on σ of the minimum points in these problems or at least the minimum values $\omega(\sigma)$, or we can even ask whether the set $\{x\colon F(x,\sigma) = 0\}$ of admissible points is non-empty for all σ close to the distinguished value σ_0 of the parameter for which we known the solution x_0. For instance, the continuity of the minimum function ω means that the problem is well posed. If x_0 is a normal point, then these questions have been well studied with the use of the classical implicit function theorem (see [14] and the bibliography there). However, if x_0 is an abnormal point, then we can no longer use this machinery. Nevertheless, also in this case the upper and lower semicontinuity of ω has been investigated using the implicit function theorems in § 6 and closely related results in [45], and upper and lower bounds for this function have been obtained, as well as asymptotic expansions of it. In [46] these results were extended to a problem with constraints $F(x,\sigma) \in C$, C being a closed convex cone.

Controllability theory. We consider a control system (7.1) with the initial condition $x(t_1) = x_1$, where x_1 is fixed. Let $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ be a given admissible process. We ask whether it is locally controllable, that is, whether we can use admissible controls sufficiently close to $u_0(\cdot)$ to attain an arbitrary point in a neighbourhood of the point $x_0(t_2)$ at the time t_2 by moving along the trajectory of the control system (7.1). If the linear system of equations in variations (7.8) is controllable for the process under consideration, that is, the extended controllability matrix (7.10) has a null kernel, then the classical inverse function theorem gives us an affirmative answer to this question. But if the kernel is distinct from zero, so that the process $\bigl(x_0(\cdot), u_0(\cdot)\bigr)$ is abnormal, then the classical inverse function theorem cannot be used for studying local controllability. However, also in this case the implicit and inverse function theorems in § 6 yield sufficient conditions for local controllability (see [47]).

The theory of quadratic maps. In the recent paper [48] two open problems in the theory of quadratic maps were stated among others. The first is as follows: for a fixed quadratic map $Q = (q_1, \dots, q_k)\colon X = \mathbb{R}^n \to \mathbb{R}^k$ find sufficient conditions in terms of the matrices Q_i which ensure that the set $K = \{x\colon Q(x) = 0,\ x \neq 0\}$ of non-trivial zeros is empty. The second problem consists in finding sufficient conditions ensuring that

$$\begin{gathered} \max\{q_1(x), \dots, q_k(x)\} > 0 \quad \forall\, x \neq 0 \\ \bigl(\text{or } \max\{q_1(x), \dots, q_k(x)\} \geqslant 0 \quad \forall\, x\bigr). \end{gathered} \tag{8.1}$$

The solution of these problems is based on the results in § 5. For integers $s \geqslant 0$ we set

$$\mathscr{Y}_s^+ = \{y^* \neq 0\colon \operatorname{ind}^+ y^*Q \leqslant s\}, \qquad \widetilde{\mathscr{Y}_s} = \{y^* \in \mathscr{Y}_s\colon y^* \geqslant 0\},$$
$$\widetilde{\mathscr{Y}_s^+} = \{y^* \in \mathscr{Y}_s^+\colon y^* \geqslant 0\},$$

where $\operatorname{ind}^+ q$ is the non-negative index of the quadratic form q, that is, the number of non-positive eigenvalues of its matrix. Note that in contrast to $\mathscr{Y}_s$, the cone $\mathscr{Y}_s^+$ can have a non-closed intersection with the unit sphere.

Theorem 8.1. *The condition $K=\varnothing$ is equivalent to the condition*

$$\forall\, x\in X\setminus\{0\}\quad \exists\, y^*\in\mathscr{Y}_{k-1}^+\colon\ \bigl\langle y^*,Q(x)\bigr\rangle>0.$$

If (5.19) holds, then the condition $K=\varnothing$ is equivalent to the condition

$$\forall\, x\in X\setminus\{0\}\quad \exists\, y^*\in\mathscr{Y}_{k-2}^+\colon\ \bigl\langle y^*,Q(x)\bigr\rangle>0.$$

Theorem 8.2. *The first condition in (8.1) is equivalent to the condition*

$$\forall\, x\in X\setminus\{0\}\quad \exists\, y^*\in\widetilde{\mathscr{Y}}_{k-2}^+\colon\ \bigl\langle y^*,Q(x)\bigr\rangle>0.$$

The second condition in (8.1) is equivalent to the condition

$$\forall\, x\in X\quad \exists\, y^*\in\widetilde{\mathscr{Y}}_{k-2}\colon\ \bigl\langle y^*,Q(x)\bigr\rangle\geqslant 0.$$

We see that Theorem 8.1 solves the first problem and Theorem 8.2 solves the second. The proof of Theorem 8.1 is based on Theorem 5.2, while Theorem 8.2 is a consequence of Theorem 8.1.

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