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Sturm–Liouville oscillation theory for impulsive problems

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Abstract. This paper extends the Sturm–Liouville oscillation theory on the distribution of zeros of eigenfunctions to the case of problems with strong singularities of the coefficients (of δ-function type). For instance, these are problems arising in the study of eigenoscillations of an elastic continuum with concentrated masses and localized interactions with the surrounding medium. The extension of the standard description of the problem is carried out by replacing the usual form of the ordinary differential equation

$$-(pu')' + qu = \lambda mu$$

by the substantially more general form

$$-(pu')(x) + (pu')(0) + \int_0^x u\,dQ = \lambda \int_0^x u\,dM$$

with absolutely continuous solutions whose derivatives, as well as the coefficients p, Q, M, belong to $\mathrm{BV}[0,l]$. The integral is understood in the Stieltjes sense.

Contents

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0. Foreword

0.1. The present paper is based on the replacement of the standard theory of the ordinary differential equation

$$-(pu')' + qu = f \tag{0.1}$$

by an analogous theory for a substantially more general equation

$$-(pu')(x) + (pu')(0) + \int_0^x u\,dQ = \int_0^x dF, \tag{A}$$

developed in the first part of the paper (see §§ 1 and 2). It is convenient to represent the equation (A) (in Stieltjes differentials) in the form

$$-d(pu') + u\,dQ = dF, \tag{dA}$$

which can be quite conditionally interpreted as

$$-\left(p\,\frac{du}{dx}\right)' + Q'u = F', \tag{A$'$}$$

where Q' and F' are generalized derivatives of the functions Q and F. In contrast to the last two equations, which are understood in the sense of distributions (or, more precisely, of functionals), the equation (A) has a pointwise meaning, which makes it basically similar to ordinary differential equations and enables one to study qualitative properties of solutions such as the distribution of zeros and extrema, the number of zeros and sign alternations, and so on, which cannot even be rigorously defined for distributions. If the coefficients p, Q, and F turn out to be smooth functions, then the equations (A), (dA), and (A$'$) are equivalent to the ordinary equation (0.1) with $q(x) = \dfrac{d}{dx}Q(x)$ and $f(x) = \dfrac{d}{dx}F(x)$.

Along with (A), we shall consider the equation with a parameter,

$$-(pu')(x) + (pu')(0) + \int_0^x u\,dQ = \lambda\int_0^x u\,dM, \tag{A$_\lambda$}$$

whose generalized form (in differentials) is

$$-d(pu') + u\,dQ = \lambda u\,dM. \tag{dA$_\lambda$}$$

0.2. Sturm oscillation theory: what does it mean? The Sturm–Liouville problem in the classical form

$$-(pu')' = \lambda u, \tag{0.2}$$

$$u(0) = u(l) = 0 \tag{0.3}$$

was posed and studied by Sturm in [1] almost two centuries ago when he was investigating heat propagation in an inhomogeneous rod (see the comments in [2]). The description of the oscillation properties of the eigenfunctions (the number of zeros, alternation of zeros, and so on) given by Sturm amazed scientists during the entire 19th century by the depth of mathematical penetration into physical problems. Even at the beginning of the 20th century these properties were referred to as *famous*, and Hilbert called them *astonishing* and *remarkable*.

In the mid-20th century Sturm's oscillation theorems were extended to equations of the more general form

$$-(pu')' + qu = \lambda m u \tag{0.4}$$

with positive continuous coefficients. However, by now the oscillation properties have been forgotten in general education courses, despite the fundamental role played by the Sturm–Liouville problem in science. One can read about these properties only in special and less accessible publications (like [2]–[4]), and the name of Sturm is mainly associated with the following comparison theorem in a standard university course.

Theorem (S-1). *For equations*

$$(pu')' + q_1 u = 0,$$
$$(pv')' + q_2 v = 0$$

with continuous coefficients p, q_1, q_2 $(p > 0)$ *on* $[0, l]$ *it follows from the inequality* $q_1(x) \leqslant q_2(x)$ *that for any non-trivial solution* $u(x)$ *of the first equation and between any two distinct zeros of this solution there is at least one zero of any solution* $v(x)$ *of the second equation provided that* $v(x)$ *is not collinear to* $u(x)$.

This theorem remains unused even in serious publications like [5]–[7], as if hanging in mid-air, together with the question of its relation to the problem (0.2)–(0.3), and it arises only in quite special problems in some special courses (for instance, as in [8]). Although an equation of the form (0.4) is one of the most widely used tools for describing problems in the most diverse areas of science and technology (it suffices just to recall the Schrödinger equation), the contemporary literature gives no understandable reason why Sturm's theorems amazed and fascinated scientists so much until the middle of the 20th century.

In fact, Theorem (S-1) was used by Sturm to prove the oscillation properties themselves, which we present now in somewhat modernized formulation and, for convenience, in two steps.

Theorem (S-2). *The spectrum of the problem* (0.2)–(0.3) *consists of an unbounded sequence of positive simple eigenvalues. If we enumerate them in ascending order,* $(0 <)\ \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, *then the corresponding eigenfunctions* $\varphi_0(x)$, $\varphi_1(x)$, $\varphi_2(x)$, ... *have the following properties*:

(i) $\varphi_0(x)$ *has no zeros inside* $(0, l)$;

(ii) $\varphi_k(x)$ *has exactly* k *zeros inside* $(0, l)$, *and each zero is simple* (*a node*);

(iii) *for any* k $(\in \{1, 2, \dots\})$ *there is exactly one node of the function* $\varphi_k(x)$ *between any neighbouring zeros of the function* $\varphi_{k+1}(x)$;

(iv) *for any* k *and any* $\alpha_0, \dots, \alpha_k$ *the generalized polynomial* $\alpha_0\varphi_0 + \cdots + \alpha_k\varphi_k$ *has at most* k *zeros in* $(0, l)$ (*that is,* $\{\varphi_i\}_0^k$ *is a Chebyshev system of order* k *on* $(0, l)$).

This oscillation theorem is complemented by the following theorem.

Theorem (S-3). *The eigenfunctions $\varphi_0, \varphi_1, \dots$ form an orthogonal basis in the space of sourcewise representable functions.*

0.3. In what follows, these theorems of Sturm are extended almost without modification to the case of the equation (A_λ) under the assumptions (0.3), that is, to the problem

$$\begin{cases} -(pu')(x) + (pu')(0) + \int_0^x u\,dQ = \lambda \int_0^x u\,dM, \\ u(0) = u(l) = 0 \end{cases}$$

in the class E of absolutely continuous functions whose derivatives are in $\mathrm{BV}[0,l]$.

0.4. Evolution of the oscillation theory. From the moment of its origin the Sturm–Liouville problem was constantly a topical subject in engineering and physical mathematics, even in its simplest form (0.2)–(0.3). Naturally, it became necessary to develop a theory for this problem. However, the scientific destiny of Theorems (S-2) and (S-3) turned out to be quite different.

Theorem (S-3), which arose from the problem of justifying the Fourier method, was assimilated in the context of integral operators with symmetric kernels already by the end of the 19th century and served as a basis for the subsequent development of spectral analysis (see, for instance, [9]), which is now an area of functional analysis. However, the role of this theorem seemed to be essential mainly for 'pure mathematics', and Theorem (S-3) was not a result which made the name of Sturm famous.

The researchers of the 19th century were impressed mainly by the oscillation properties, and especially by the physical nature of the terms (for example, the distribution of nodes). For elastic oscillations (with which the Sturm–Liouville problem (0.2)–(0.3) was associated) the properties of the eigenharmonics were observable, and they completely agreed with intuitive ideas about the shapes of eigenwaves (standing waves) of a vibrating string.

Attempts to extend Sturm's oscillation theorems to more complicated problems were initiated already in the 19th century. For example, Theorem (S-2) was extended by Stieltjes (see [2]) to the case of an elastic string with beads; in modern terms, this means that the distribution of masses $m(x)$ in the equation (0.4) (for $q \equiv 0$) coincides with a finite combination of δ-functions.

Almost half a century later, the Stieltjes result was extended (also for $q \equiv 0$) by Gantmakher and Krein [10] to the case of an arbitrary distribution of masses. This was done in terms of the problem

$$u(x) = \lambda \int_0^l K(x,s)u(s)\,dM(s),$$

where M is a non-decreasing function with infinitely many points of growth and $K(x,s)$ is the influence function of a homogeneous string (for which $p(x) \equiv \mathrm{const}$). This result used the very refined theory of Kellogg kernels based on tensor analysis.

A function $K(x,s)$ is a Kellogg kernel if the associated kernels

$$K\begin{pmatrix}\xi_1, & \dots, & \xi_m\\ \tau_1, & \dots, & \tau_m\end{pmatrix} = \begin{vmatrix} K(\xi_1,\tau_1) & \dots & K(\xi_1,\tau_m)\\ \dots & \dots & \dots \\ K(\xi_m,\tau_1) & \dots & K(\xi_m,\tau_m)\end{vmatrix}$$

are non-negative on the simplex $\xi_1 < \xi_2 < \dots < \xi_m$, $\tau_1 < \tau_2 < \dots < \tau_m$ and strictly positive on the 'diagonal' $\tau_i = \xi_i$ $(i = 1,\dots,m)$. These results, which involved a thorough analysis of the Green's function, became the basis for a broad circle of oscillation problems for higher-order equations (see the references in the papers [11]–[16]).

In the 1930s the need arose to investigate equations of the form (0.4) more general than (0.2) in which the coefficient q can be a combination of δ-functions (in the language of physics). Although publications about equations with distributional coefficients formed a rather broad front beginning in the 1960s (see, for instance, [17]–[29]), work on carrying spectral properties related to oscillations over to the case of impulsive singularities seemed to stop.

This can be understood. The apparatus of the theory of distributions existing at that time did not allow one to regard the equation (0.4) with δ-functions in the coefficients as a pointwise equality of functions of a scalar argument [30]. And neither the formulations nor the known methods for proving the properties (i)–(iv) in Theorem (S-2) can be carried over to the language of distributions. A maximal advance of the oscillation theory into the area of distributions was due to Myshkis [31], who obtained a weakened version of Theorem (S-1) for $q_1 = q_2$ for the equation $u'' + qu = 0$ with a distributional coefficient q.

A physically meaningful extension of Theorem (S-2) to more general problems began at the very end of the 20th century, involving the inclusion of δ-functions (as physicists understood them) in the coefficients with preservation of the usual pointwise understanding of the equation. This was the beginning of the extension of the Sturm–Liouville oscillation theory to the case of problems with singularities, summarized in part by the present paper (according to the results in [32]–[38]).

0.5. Concluding sketch of the present investigation. In outward appearance it looks quite simple: the assertions of Theorems (S-2) and (S-3) are literally preserved for a more general object than the ordinary differential equation of the form (0.4), namely, for the integro-differential equation (A_λ), where Q and M are non-decreasing functions, and the integrals are understood in the Stieltjes sense.

One can obtain the equation (A_λ) by formally integrating the canonical equation (0.4) and setting $\int_a^x q(x)\,dx = Q(x)$ and $\int_a^x m(x)\,dx = M(x)$. Corresponding to the Stieltjes problem of an elastic string with beads is the equation (A_λ) for $p(x) \equiv 1$ and $Q(x) \equiv \text{const}$ (that is, $q(x) \equiv Q'(x) \equiv 0$) if $M(x)$ is a piecewise constant (step) function. Similarly, a piecewise constant function $Q(x)$ corresponds to a linear combination of δ-functions in $q(x)$. Here *the equation* (A_λ) *is pointwise*, that is, it contains no Schwartz–Sobolev distributions.

Naturally, to rigorously prove the Sturm oscillation theorem, Theorem (S-2), we had to develop a new theory for the equation (A_λ) (and first of all, for the equation (A)) which is parallel to the theory of ordinary differential equations with the usual kind of coefficients.

Below we give a systematic exposition of the oscillation theory developed for the equation (A_λ) by the authors over the last decade. The use of the integro-differential form (A_λ) was suggested by the following Atkinson–Krein mathematical model of the Stieltjes string ([2], [39]):

$$u'_+(x) = u'_-(0) - \lambda \int_0^{x+0} u \, dM,$$

where $u'_+(x)$ is the right-hand derivative and $u'_-(0)$ is a certain 'extended value' of the derivative. The relation

$$-\frac{d}{dM} u'_+(x) = \lambda u(x)$$

suggested itself as a formal expression of the equation. Somewhat earlier, the last equation was obtained by Feller (see [2]) in a problem on diffusion [40]. Oscillation properties were not discussed in these papers.

0.6. Horizons of difficulties. In retrospect the investigation presented here appears as the following fairly transparent scheme going back to Sturm (and used, for instance, in [2] and [4]).

We denote by $u(x, \lambda)$ the solution of the equation $-(pu')' = \lambda u$ with the initial conditions $u(0) = 0$, $u'(0) = 1$. If the function $u(x, \lambda^*)$ vanishes at the right endpoint (that is, at $x = l$) for some $\lambda = \lambda^*$, then $\lambda = \lambda^*$ is an eigenvalue and $y(x) \equiv u(x, \lambda^*)$ is an eigenfunction. For $\lambda = 0$ the function $u(x, 0)$ obviously has no zeros on $(0, l]$.

One should next analyze how the number $N(\lambda) \stackrel{\text{def}}{=} N(u(x, \lambda))$ of zeros of the function $u(x, \lambda)$ on $(0, l]$ depends on λ. The comparison theorem, Theorem (S-1), helps here.

Suppose that the function $u(x, \lambda)$ has k zeros on $(0, l]$ for some λ. We denote them in increasing order by

$$(z_0 = 0) < z_1(\lambda) < z_2(\lambda) < \cdots < z_k(\lambda) \leqslant l.$$

It follows from Theorem (S-1) that, as λ increases, every zero point $z_i(\lambda)$ shifts to the left, that is, the number $N(u(x, \lambda))$ of zeros of the function $u(x, \lambda)$ on $(0, l]$ does not decrease. Moreover, each additional zero can appear as a result of shifting the rightmost zero (coinciding with the right endpoint $x = l$) to the left (as λ increases somewhat). Thus, the function $N(\lambda) = N(u(x, \lambda))$ turns out to be piecewise constant and increases by 1 every time the value λ passes the next point of the spectrum.

We refer to the above argument as the 'accumulation of zeros'. This reasoning is intelligible enough and was regarded until some time ago as sufficiently convincing, up to the point of being presented without additional comments as a proof of the oscillation properties by such authoritative authors as Levitan [4], Atkinson [2], and others. However, quite recently some complications were discovered, connected with almost obvious (or so it seemed earlier) circumstances in the study of oscillation properties (see [41], [38]) of the Sturm–Liouville problem on a graph (a spatial network [38]), and these circumstances have in fact a quite non-trivial nature.

Let us describe the circumstances, which require a sufficiently thorough investigation even in the case of smooth coefficients p, q, m of the equation (0.4).

The smooth dependence of the function $u_\lambda(x) \equiv u(x,\lambda)$ on λ is ensured by standard theorems. But how do the zeros of this function depend on λ?! A detailed analysis of a series of questions is necessary here.

Suppose that $u(l,\lambda^*) = 0$ for some $\lambda = \lambda^*$, that is, λ^* is a point of the spectrum, and assume that the eigenfunction $u(x,\lambda^*)$ has m zeros on $(0,l)$, $(0<)\ z_1 < z_2 < \cdots < z_m < (z_{m+1} = l)$. As λ increases, these zero points are preserved. We represent their dependence on λ in the form of functions $z_1(\lambda)$, $z_2(\lambda)$, …, $z_m(\lambda)$, and $z_{m+1}(\lambda)$. As was already noted above, by Theorem (S-1) these functions are non-decreasing. However, to rigorously justify the scheme of 'accumulation of zeros', one must be sure that

(a) each of the functions $z_k(\lambda)$ is continuous;
(b) each of them is strictly monotone;
(c) neighbouring zeros $z_k(\lambda)$ and $z_{k+1}(\lambda)$ do not merge;
(d) zeros cannot be lost (do not disappear) as λ increases unboundedly;
(e) zeros do not accumulate near some interior point in $(0,l)$, that is, each zero $z_k(\lambda)$ inevitably approaches the point $x = 0\ (= z_0)$ without 'stalling' on the way.

Finally, one must understand what mechanism 'creates' the next additional zero of $u(x,\lambda)$ at the right endpoint and why no additional zeros appear inside $(0,l)$ by detaching from some interior zero.

What is known about the answers to these questions? Only that the functions $z_k(\lambda)$ are determined by the identities

$$u(z_k(\lambda),\lambda) = 0,$$

that is, these are implicit functions determined by the equation

$$u(z,\lambda) = 0. \tag{0.5}$$

It is known in advance about these implicit functions determined by the single equation (0.5) that there are many of them and that each of them is defined (however, this needs an explanation) on its own domain $\{\lambda_k < \lambda < \infty\}$, where λ_k is the corresponding point of the spectrum.

To characterize the behaviour of each of the branches of this multivalued implicit function, one must study the derivative $\dfrac{\partial u(x,\lambda)}{\partial \lambda}$ in a neighbourhood of each point λ^* of the spectrum of the original problem. For each of these values $\lambda = \lambda^*$ the corresponding function $h(x) \equiv \dfrac{\partial u(x,\lambda^*)}{\partial \lambda}$ turns out to be a solution of the equation

$$-(ph')' + qh = \lambda^* h + u(x,\lambda^*),$$

which is characteristic for adjoint functions. Investigations of the number of solutions of this equation in the context of oscillation theorems are nowhere to be found and have never been carried out, not even for the classical equation (0.4) on an interval.

In implementing the scheme of 'accumulation of zeros' for the equation (A_λ), we will have to find answers to all these questions for the problem with impulsive singularities. To this end, we must first construct a system of facts analogous to the theory of linear equations on an interval with ordinary coefficients.

1. Basic facts

In this section we study the equation

$$-(pu')(x) + (pu')(0) + \int_0^x u\,dQ = F(x) - F(0) \tag{A}$$

in the case of smooth coefficients p, Q, and F; this equation is equivalent to the ordinary equation

$$-(pu')' + qu = f \tag{1.1}$$

with $q = Q'$ and $f = F'$. The equation (A) inherits the universal property of the equation (1.1) for quite diverse problems in science and technology, from the Schrödinger equation in quantum mechanics to processes in electrical circuits, and for acoustic systems, nerve fibres, diverse waveguides, and so on.

The replacement of the usual ordinary differential equation (1.1) by the equation (A) broadens the class of problems admitting investigation and requires us to take into account the purely mathematical novelty of this object. What is really new here?

The coefficients p, Q, F in the equation (A) are discontinuous in general, and the integral is used in the Stieltjes sense. Preserving the explicit occurrence of the scalar argument, the equation (A) shows also its singular values; these are the points at which the derivative $u'(x)$ and the functions p, Q, F can be discontinuous and the values of the upper limit of the integral at which it becomes meaningless. The last condition is the most insidious circumstance (from the point of view of associations customary for elementary analysis) arising in the equation (A) due to the Stieltjes integral, namely, if in (A) a point x coincides with one of the discontinuity points ξ of the function $Q(x)$ and if $Q(\xi) \neq Q(\xi-0)$ and $Q(\xi) \neq Q(\xi+0)$ at the point, then the corresponding value $\int_0^\xi u(x)\,dQ(x)$ differs both from $\int_0^{\xi-0} u(x)\,dQ(x)$ and from $\int_0^{\xi+0} u(x)\,dQ(x)$, which makes the symbol $u'(\xi)$ in (A) meaningless, because $u'(\xi)$ stands for the common value of the left- and right-hand derivatives $u'_-(\xi)$ and $u'_+(\xi)$. Here each of the expressions $\int_0^{\xi-0} u\,dQ$ and $\int_0^{\xi+0} u\,dQ$, which arise as a rule in this very way in papers involving the use of the Stieltjes integral in differential equations (see, for instance, [2], [38], [37]), requires in fact additional explanation. For example, $\int_0^{\xi-0} u\,dQ$ stands for either the integral over the half-open interval $[0,\xi)$ or for the improper integral $\lim_{\delta\downarrow 0}\int_0^{\xi-\delta} u\,dQ$. Below we show that if $u(x)$ is absolutely continuous and its derivative $u'(x)$ has bounded variation, that at any point $\xi \in [0,l]$ there are right- and left-hand derivatives coinciding with the corresponding one-sided limits of $u'(x)$, that is, $u'_+(\xi) = u'(\xi+0)$ and $u'_-(\xi) = u'(\xi-0)$. This enables us, for instance, to identify the improper integral $\int_0^{\xi-0} f\,d\mu = \lim_{\varepsilon\downarrow 0}\int_0^{\xi-\varepsilon} f\,d\mu$ with the proper integral over the half-open interval $[0,\xi)$, that is, $\int_{[0,\xi)} f\,d\mu$. A deeper problem remains unsolved here, namely, the differentiability with respect to the upper limit. However, we shall discuss this in due time. In order to get rid of the indicated sources of possible misunderstandings, we replace every singular point ξ by the pair of symbols $\{\xi-0,\ \xi+0\}$.

Against the background of these 'nuances', we construct in this section a system of facts needed in what follows, a system analogous to the conventional theory of ordinary differential equations. We first briefly recall the notions and facts we need from the theory of the integral. For a more detailed exposition of the information presented below, see, for instance, [42]–[48].

The definition of the Stieltjes differential dG for any function $G(\cdot)$ in BV (this definition is given in 1.1.6) is absent in the literature known to the authors.

1.1. Preliminary facts.

1.1.1. *The Riemann–Stieltjes integral* $\int_0^l f(x)\,d\mu(x)$. This integral is defined for a pair of functions $f(x)$, $\mu(x)$ on $[0,l]$ by passing to the limit of the integral sums $\sum_{i=1}^{n} f(\xi_i)[\mu(x_{i+1})-\mu(x_i)]$, with the same stipulations as for the usual Riemann integral. It follows immediately from the definition that the integrals $\int_0^l f\,d\mu$ and $\int_0^l \mu\,df$ exist or do not exist simultaneously, and if they exist, then their sum is equal to $f(l)\mu(l)-f(0)\mu(0)$. The Stieltjes integral $\int_0^l f\,d\mu$ certainly exists if one of the functions $f(x)$, $\mu(x)$ is continuous and the other has bounded variation.

1.1.2. *The space* $\mathrm{BV}[0,l]$. This space is defined as the set of functions whose (total) variation $\mathrm{Var}_{[0,l]}\,u(x) = \sup_{0\leqslant x_0<x_1<\dots<x_k\leqslant l}\sum_{i=0}^{k-1}|u(x_{i+1})-u(x_i)|$ is bounded. For simplicity, we often write $V_0^l[u(x)]$ instead of $\mathrm{Var}_{[0,l]}\,u(x)$. Every function $u(x)$ in $\mathrm{BV}[0,l]$ admits a Jordan decomposition $u = u_1 - u_2$, where u_1 and u_2 are non-decreasing functions.

1.1.3. *Jumps of functions in* BV. For any $u(x)$ in $\mathrm{BV}[0,l]$ and at any point $\xi\in(0,l]$ (at any point $\xi\in[0,l)$) the left-hand (right-hand) limit exists, that is, the limit $u(\xi+0) = \lim_{x\to\xi,\,x>\xi} u(x)$ ($u(\xi-0) = \lim_{x\to\xi,\,x<\xi} u(x)$). By the simple jump of $u(x)$ at a point $x=\xi$ we mean the quantity $\Delta u(\xi) \stackrel{\text{def}}{=} u(\xi+0)-u(\xi-0)$ (we set $u(0-0)=u(0)$ and $u(l+0)=u(l)$). When speaking of a simple jump $\Delta u(\xi)$ for $0<\xi<l$, we ignore the true value $u(\xi)$ of the function at the point $x=\xi$ and take into account only the left-hand ($u(\xi-0)$) and right-hand ($u(\xi+0)$) limit values.

Everywhere below, we denote by S_u the set of discontinuity points of a function $u(x)$. For any $u(x)\in \mathrm{BV}[0,l]$ the set S_u is at most countable. We define the *jump function* $u_s(x)$ for $u(x)$ in $\mathrm{BV}[0,l]$ by

$$u_s(x)=\sum_{\xi\leqslant x}\Delta u(\xi). \tag{1.2}$$

For any $u(x)$ in $\mathrm{BV}[0,l]$ the difference $u_0(x)=u(x)-u_s(x)$ has equal left- and right-hand limits at any discontinuity point $\xi\in S_u$: $u_0(\xi-0)=u_0(\xi+0)$, that is, u_0 has a removable discontinuity there. Redefining the function $u_0(x)$ at these points by the common value of these limits, we denote the continuous function thus obtained by $\bar{u}_0(x)$.

It turns out that for the Stieltjes integral with $\mu(x)\in \mathrm{BV}[0,l]$ we have

$$\begin{aligned}\int_0^l f(x)\,d\mu(x) &= \int_0^l f(x)\,d\mu_0(x)+\int_0^l f(x)\,d\mu_s(x)\\ &= \int_0^l f(x)\,d\bar{\mu}_0(x)+\sum_{\xi\in S_\mu} f(\xi)\Delta\mu(\xi),\end{aligned} \tag{1.3}$$

and the expressions $d\mu_0$ and $d\bar{\mu}_0$ are equivalent under the sign of the (Stieltjes) integral.

In order to avoid confusion in formulations involving the (inessential for us) true values of functions in $\mathrm{BV}[0,l]$ at their discontinuity points, we assume that these functions are left continuous, that is, belong to $\mathrm{BV}_0[0,l]$.

1.1.4. *Riesz theorem.* The Stieltjes integral, which was seemingly not noticed by mathematicians for a long time, revealed its basic possibilities after Riesz proved the following fundamental fact.

For any continuous linear functional $l(u(\cdot))$ on the space $C[a,b]$ of continuous functions there is a function $g(\cdot)$ in $\mathrm{BV}[a,b]$ such that $l(u)=\int_a^b u(x)\,dg(x)$.

1.1.5. *Theorem on a transform of a measure. For any function $\sigma(x)$ in* $\mathrm{BV}[0,l]$ *and an arbitrary function $u(x)$ in $C[0,l]$*

$$\int_0^l \varphi\,d\mu=\int_0^l \varphi u\,d\sigma, \tag{1.4}$$

where $\mu(x)=\int_0^x u\,d\sigma$.

1.1.6. *The Stieltjes differential.* Turning to the basic meaning of the differential in the notion of integral, we define the Stieltjes differential of a function $g(x)$ in $\mathrm{BV}[0,l]$ to be the functional dg in $C^*[0,l]$ given by

$$(dg,\varphi)=\int_0^l \varphi\,dg \qquad (\varphi\in C[0,l]).$$

The reason for using the symbol dg to denote this functional will become clear from the properties discussed below.

The fact that the differential is homogeneous and additive is clear.

The norm of dg in $C^*[0,l]$ does not exceed the variation $V_0^l[g(x)]$. It follows immediately from the definition of the Stieltjes integral that $\int_\alpha^\beta dg = g(\beta) - g(\alpha)$, and $\int_{\xi-0}^{\xi+0} dg = g(\xi+0) - g(\xi-0)$.

If $g(\cdot) \in C^1[0,l]$, then we obviously have $dg = g'\,dx = \dfrac{dg}{dx}(x)\,dx$.

Proposition 1.1. *The Stieltjes differential* dg *is the zero functional in* $C^*[0,l]$ (*this is denoted by* $dg = 0$) *if and only if* $g(x) = \mathrm{const}$.

The theorem on a transform of a measure gives a meaning to formal multiplication of a continuous function $u(x)$ by a differential dg; namely, the following assertion holds.

Proposition 1.2. *If* $u(x) \in C[0,l]$ *and* $g(\cdot) \in \mathrm{BV}[0,l]$, *then there is a function* $h(x)$ *in* $\mathrm{BV}[0,l]$ *such that* $dh = u(x)\,dg$, *that is,* $u(x)dg$ *is the same object as* dg.

This implies the very useful formal relation

$$d\int_0^x \varphi\,dg = \varphi\,dg.$$

And this is a precise explanation for the fact that the original integro-differential equation (A_λ) is equivalent to the equality of general form

$$-d(pu') + u\,dQ = \lambda u\,dM, \qquad (\mathrm{dA}_\lambda)$$

while the inhomogeneous version (A) is equivalent to the equality

$$-d(pu') + u\,dQ = dF. \qquad (\mathrm{dA})$$

In what follows, we sometimes write (for brevity)

$$Du \equiv -d(pu') + u\,dQ.$$

It is interesting to note that the functional $l(u) = u(\xi)$, where $\xi \in (0,l)$, in the theory of distributions is identified with the Dirac delta function $\delta(x-\xi)$. Since

$$u(\xi) = \int_0^l u(x)\,d\Theta(x),$$

where $\Theta(x)$ is the Heaviside function, that is, $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x > 0$, it follows that this functional coincides with $d\Theta$, that is, $\delta(x) = d\Theta(x)$. Here we can also speak of a δ-function supported at one of the endpoints of the interval. For $\delta(x-l)$ the corresponding function generating the functional has the form $\Theta(x-l) = 0$ for $x < l$ and $\Theta(x-l) = 1$ for $x = l$.

1.1.7. *Differential inequalities.* We shall write $dg \geqslant 0$ and say that the differential dg is positive if the corresponding functional is non-negative, that is, $(dg, \varphi) = \int_0^l \varphi\, dg \geqslant 0$ for any non-negative continuous function $\varphi(x)$ on $[0, l]$.

Proposition 1.3. *For dg to be positive it is necessary and sufficient that the function $g(x)$ be non-decreasing on $[0, l]$.*

This assertion will be of importance for us, for example, in the study of solutions of a differential inequality of the form $Du \geqslant 0$, that is, $-d(pu') + u\, dQ \geqslant 0$, which for us is equivalent to the function $z(x) = -(pu')(x) + (pu')(0) + \int_0^x u\, dQ$ being non-decreasing on $[0, l]$. A similar pointwise description can also be given to the symbolic inequality $v_0(x)Du \geqslant 0$.

It will sometimes be convenient to use the notation

$$(Lu)(x) = -(pu')(x) + (pu')(0) + \int_0^x u\, dQ,$$

which is connected with Du by the formal equality $Du = d(Lu)$.

Theorem 1.1 (on factorization). *Suppose that the homogeneous equation $Lu = 0$ has in E a solution without zeros on $[0, l]$. Then there are strictly positive functions $z_0(x)$ and $z_1(x)$ such that*

$$Du = -z_0\, d(z_1(z_0u)')$$

for any $u(x)$ in E, or $Lu(x) = -\int_0^x z_0\, dg$ for $g(x) = z_1(x)\frac{d}{dx}(z_0(x)u(x))$.

Proof. Let $\varphi(x)$ be a solution ($\varphi \in E$) of the equation $Du = 0$ and let $\varphi(x) > 0$ on $[0, l]$. We write $z_0(x) = 1/\varphi(x)$ and $z_1(x) = \varphi^2(x)p(x)$. In this case the desired factorization is obtained by a trivial manipulation based on the theorem on a transform of a measure.

In the classical non-oscillation theory a similar result plays a fundamental role and is connected with the names Pólya and Frobenius (see [14] and [49]).

The following theorem is a good demonstration of the fact that this factorization is effective.

Theorem 1.2. *Suppose that the equation $Lu = 0$ has in E a solution $\varphi(x)$ without zeros on $[0, l]$. Then every non-negative solution $u(x) \not\equiv 0$ of the inequality $Du \geqslant 0$ has no zeros in $(0, l)$, and moreover, $|u(0)| + |u'(0)| > 0$ and $|u(l)| + |u'(l)| > 0$.*

Proof. Let $u(x) \in E$ and $Du \geqslant 0$, where $u(x) \not\equiv 0$ and $u(x) \geqslant 0$. We use the previous theorem, setting $g(x) = z_1(x)\frac{d}{dx}(z_0(x)u(x))$. Since $z_0\, dg = -Du \leqslant 0$ and $z_0(x) > 0$, it follows that $dg \leqslant 0$, that is, the function $g(x)$ is non-increasing on $[0, l]$. If $g(x)$ changes sign, then this happens only once, and it is from plus to minus.

Let us consider the set $\Omega = \{x : u(x) > 0\}$. It is relatively open on $[0, l]$. We must prove that $\Omega \supset (0, l)$.

Suppose the contrary. Let (τ_0, τ_1) be one of the intervals forming Ω, and assume that $\tau_0 > 0$. Then $u(\tau_0) = 0$, and $u(x) > 0$ on (τ_0, τ_1). The same holds for $z_0(x)u(x)$. Therefore, the right-hand derivative $(z_0 u)'_+(\tau_0)$ is non-negative at the point $x = \tau_0$. Thus, $g(\tau_0 + 0) \geqslant 0$. Similarly (since $u(x) \geqslant 0$ to the left of $x = \tau_0$), we see that $g(\tau_0 - 0) \leqslant 0$. Since $g(x)$ is non-increasing (also at τ_0, that is, for the passage from $g(\tau_0 - 0)$ to $g(\tau_0 + 0)$), the last two inequalities must mean that $0 \geqslant g(\tau_0 - 0) \geqslant g(\tau_0 + 0) \geqslant 0$, that is, $g(\tau_0 - 0) = g(\tau_0 + 0) = 0$. This, together with the fact that the function $g(x)$ is non-increasing, implies that $g(x) \leqslant 0$ to the right of τ_0. Therefore, we also have $(z_0 u)' \leqslant 0$ to the right of τ_0. Then the function $(z_0 u)(x)$, which is non-increasing to the right of τ_0 and is zero at the point $x = \tau_0$, must satisfy the inequality $(z_0 u)(x) \leqslant 0$, which, together with the original assumption, means that $u(x) \equiv 0$ to the right of τ_0 and, in particular, on (τ_0, τ_1).

Thus, the case $\tau_0 \neq 0$ is impossible. Suppose now that $\tau_0 = 0$, that is, $u(0) = 0$. If $u'(0) = 0$, then $(z_0 u)'(0) = 0$ and $g(+0) = 0$. But then since $g(x)$ is non-increasing, we must have $g(x) \leqslant 0$ to the right of τ_0 (as above), that is, $(z_0 u)'(x) \leqslant 0$, and $(z_0 u)(x)$ is non-increasing and has a zero at $x = 0$. Therefore, $z_0 u \leqslant 0$, which, together with the original condition $u(x) \geqslant 0$, means that $u(x) \equiv 0$. The property $|u(l)| + |u'(l)| > 0$ can be proved in a similar way. This completes the proof of Theorem 1.2.

Remark 1.1. As will be proved below, the existence condition for a solution without zeros certainly holds if $dQ \geqslant 0$. However, we shall be able to discuss these conditions (*there is a solution such that*...) only if we have an existence theorem. We shall obtain such a theorem in the next section.

Remark 1.2. In the proof we in fact identify the right-hand derivative $(z_0 u)'_+(\tau_0)$ of the function $(z_0 u)$ with the right-hand limit

$$\lim_{\varepsilon \to +0} (z_0 u)'(\tau_0 + \varepsilon) = (z_0 u)'(\tau_0 + 0).$$

The same holds to the left of τ_0. This is one of the trickiest points here. In Theorem 1.4 we shall prove that this identification is admissible.

Remark 1.3. The importance of the last theorem is determined already by the fact that the influence function (Green's function) certainly satisfies the differential inequality of the form $Du \geqslant 0$.

1.2. Cauchy problem. Existence theorem. We begin a more thorough investigation of the main equation (A), that is,

$$-(pu')(x) + (pu')(0) + \int_0^x u\, dQ = \int_0^x dF.$$

Everywhere below, it is assumed that the functions $p(x)$, $Q(x)$, and $F(x)$ are continuous at the points $x = 0$ and $x = l$ and have bounded variation on $[0, l]$; moreover, $\inf_{[0,l]} p > 0$. By the theorem on a transform of a measure, the integral term on the left-hand side of (A) must belong to $\mathrm{BV}[0, l]$. But then the function $(pu')(x)$ must belong to the space $\mathrm{BV}[0, l]$ together with $u'(x)$.

1.2.1. *Space of admissible solutions.* Let us introduce the norm

$$\|u\| = \sup_{[0,l]} |u(x)| + V_0^l[u'(x)] \tag{1.5}$$

on the set E of absolutely continuous functions on $[0,l]$ whose derivatives belong to $\mathrm{BV}[0,l]$.

Theorem 1.3. *The space E is complete with respect to the norm* (1.5).

Proof. Let $\{u_n(x)\}$ be a Cauchy sequence with respect to the norm (1.5). Then it is a Cauchy sequence in $C[0,l]$ as well, and hence converges uniformly to some function $u_*(x) \in C[0,l]$. We assert that $u_*(\cdot) \in E$.

Consider the sequence $z_n(x) = u_n'(x)$. Since $\{u_n\}$ is bounded in E, there is a fixed finite number C such that

$$V_0^l[z_n] \leqslant C. \tag{1.6}$$

Since for any $z(\cdot)$ in $\mathrm{BV}[0,l]$ and any $\alpha, \beta \in [0,l]$ we have the inequality $|z(\beta) - z(\alpha)| \leqslant V_\alpha^\beta[z]$, it follows that

$$|u_n'(x)| \leqslant |u_n'(l)| + V_0^l[u_n'],$$

and by (1.6),

$$|u_n'(x)| \leqslant |u_n'(l)| + C \qquad (0 \leqslant x \leqslant l). \tag{1.7}$$

On the other hand,

$$u_n(l) = u_n(0) + \int_0^l u_n'(s)\,dx = u_n(0) + u_n'(l)l - \int_0^l x\,du_n',$$

and hence

$$lu_n'(l) = \int_0^l x\,du_n' + u_n(l) - u_n(0).$$

Therefore,

$$l|u_n'(l)| \leqslant lV_0^l[u_n'] + |u_n(l)| + |u_n(0)|.$$

The last two terms are uniformly bounded with respect to n by the Cauchy property of $\{u_n\}$ in the metric of $C[0,l]$, and the same holds for the first term by (1.6). This implies that $\sup_n |u_n'(l)| < \infty$, which, together with (1.7), means the existence of a constant c_2 such that $|u_n'| \leqslant c_2$. Thus, the sequence $z_n(x) = u_n'(x)$ is uniformly bounded, which, together with (1.6), implies that the conditions of Helly's second theorem are satisfied (see [45], Chap. VI, § 6.5), according to which the sequence $\{z_n\}$ is compact with respect to the weak topology (of pointwise convergence). Therefore, there is a subsequence $\{z_{n_k}\}$ which converges to some function $z^*(x)$. Then by Helly's first theorem ([45], Chap. VI, § 6.5), we have $z^*(x) \in \mathrm{BV}[0,l]$.

The convergence of the entire sequence $\{z_n\}$ to z^* follows now from its Cauchy property with respect to the semimetric $\rho(z, z^*) = V_0^l[z - z^*]$, which ensures the pointwise convergence of z_n to z^*. Therefore, $u_*' = z^*$. The final conclusion about convergence of $u_n(x)$ to $u_*(x)$ with respect to the norm (1.5) follows from the equality $u_n(x) = u_n(0) + \int_0^x u_n'(s)ds$. This completes the proof of Theorem 1.3.

Remark 1.4. The Banach space E introduced above is contained (in the set-theoretic sense) in the Sobolev space $W_1^1[0,l]$.

1.2.2. *Meaning of the equation* (A) *at singular points.* If ξ is one of the discontinuity points of one of the functions p, Q, or F, then jumps at this point of the derivative $u'(x)$ and of the integral term regarded as a function of the upper limit of integration are unavoidable.

Theorem 1.4. *For any function $g \in E$ the relations*

$$g'(\xi+0) = \lim_{x\to\xi+0} g'(x) = \lim_{\varepsilon\to 0+0} \frac{g(\xi+\varepsilon)-g(\xi)}{\varepsilon} = g'_+(\xi)$$

hold for $\xi < l$. A similar condition holds from the left at any point $\xi > 0$.

Proof. Let $\xi < l$. We assert first that the right-hand derivative $g'_+(\xi)$ exists at the point ξ. Since $g \in E$, it follows that the derivative $g'(x)$ belongs to $\mathrm{BV}[0,l]$, and thus there is a finite limit $\lim_{t\to\xi+0} g'(t) = g'(\xi+0)$. Let us show that $\lim_{\varepsilon\to+0} \dfrac{g(\xi+\varepsilon)-g(\xi)}{\varepsilon} = g'(\xi+0)$. Since $g(x)$ is absolutely continuous, we have $g(\xi+\varepsilon) = g(\xi) + \displaystyle\int_\xi^{\xi+\varepsilon} g'(t)\,dt$. Therefore,

$$\begin{aligned}\left|\frac{1}{\varepsilon}\bigl(g(\xi+\varepsilon)-g(\xi)\bigr) - g'(\xi+0)\right| &= \left|\frac{1}{\varepsilon}\int_\xi^{\xi+\varepsilon} g'(t)\,dt - g'(\xi+0)\right| \\ = \frac{1}{\varepsilon}\left|\int_\xi^{\xi+\varepsilon} \bigl(g'(t)-g'(\xi+0)\bigr)\,dt\right| &\leqslant \frac{1}{\varepsilon}\int_\xi^{\xi+\varepsilon} \bigl|g'(t)-g'(\xi+0)\bigr|\,dt,\end{aligned}$$

as was to be proved. The arguments for the left-hand derivatives are similar. This completes the proof of Theorem 1.4.

Thus, the equation (A) makes sense at $x = \xi - 0$ and at $x = \xi + 0$ for any problem point ξ at which one of the functions p, Q, and F can have a jump, and the following equality holds:

$$-\Delta(pu')(\xi) + u(\xi)\Delta Q(\xi) = \Delta F(\xi).$$

Assigning to x in (A) the true value $x = \xi$ of such a point does not make sense.

1.2.3. *Extended domain of the argument of the equation* (A). Denote by S_{A} the set of all points at which the functions $p(x)$, $Q(x)$, $F(x)$ have non-zero simple jumps, that is, distinct left- and right-hand limits. Let us remove the set S_{A} from $[0,l]$ and replace each point $\xi \in S_{\mathrm{A}}$ by the pair $\{\xi-0,\ \xi+0\}$. We assume that $\xi - 0 > x$ for any $x < \xi$ and $\xi + 0 < x$ for any $x > \xi$. Let $\overline{[0,l]}_{\mathrm{A}}$ be the set obtained from $[0,l]$ by replacing the points $\xi \in S_{\mathrm{A}}$ by the corresponding pairs $\{\xi-0,\ \xi+0\}$.

One can give the following correct definition of the set $\overline{[0,l]}_{\mathrm{A}}$ regarded as a one-dimensional metric space.

Taking a Jordan representation of the original coefficients p, Q, F in the form $p = p^+ - p^-$, $Q = Q^+ - Q^-$, and $F = F^+ - F^-$, we denote by $\sigma_{\mathrm{A}}(x)$ the following sum of non-decreasing functions:

$$\sigma_{\mathrm{A}}(x) = x + p^+(x) + p^-(x) + Q^+(x) + Q^-(x) + F^+(x) + F^-(x).$$

Without loss of generality we can assume that the function $\sigma(x)$ has discontinuities (full jumps) only at the points of the set S_{A}.

Let us equip the set $[0,l]\setminus S_{\mathrm{A}}$ with the metric $\rho(x,y)=|\sigma_{\mathrm{A}}(x)-\sigma_{\mathrm{A}}(y)|$. If $S_{\mathrm{A}}\neq\varnothing$, then this metric space is obviously not complete. Its standard metric completion coincides (up to isomorphism) with $\overline{[0,l]}_{\mathrm{A}}$ and induces a topology on this space.

It is clear that this space is disconnected and compact.

We consider the equation (A) on the set of values x in $\overline{[0,l]}_{\mathrm{A}}$ (thus, without permitting values of x in S_{A} for (A)). On $\overline{[0,l]}_{\mathrm{A}}$ the functions $p(\cdot)$, $Q(\cdot)$, and $F(\cdot)$ become continuous, because the values $p(\xi+0)$, $p(\xi-0)$, $Q(\xi+0)$, $Q(\xi-0)$, $F(\xi+0)$, and $F(\xi-0)$ of them which were limit values on $[0,l]$ now become true values at the corresponding points of $\overline{[0,l]}_{\mathrm{A}}$.

The continuity of the functions $u(\cdot)$ under consideration enables us to preserve the usual Riemann–Stieltjes meaning for the integral term in (A) at $x=\xi-0$ and $x=\xi+0$, regarding the previous limit values as true values.

Thus, it is as if we regard the equation (A) in two layers: the lower level is for the values $x\in[0,l]$ when speaking about the solutions $u(x)$ themselves (under the integral sign), and the second level is for the values x in the identity (A) with x taken from $\overline{[0,l]}_{\mathrm{A}}$. This now affects the definition of the Cauchy problem, namely, it is in the usual sense for $x\notin S_{\mathrm{A}}$, that is, the values of the solution $u(\xi)$ and its derivative $u'(\xi)$ are assumed to be given, but for $\xi\in S_{\mathrm{A}}$ one of the one-sided derivatives $u'(\xi-0)$ or $u'(\xi+0)$ can be prescribed in advance along with the value $u(\xi)$.

1.2.4. *Existence and uniqueness theorem.*

Theorem 1.5. *For any $u_0,v_0\in\mathbb{R}$ and for any point $x_0\in\overline{[0,l]}_{\mathrm{A}}$ there is a unique solution $u(x)$ of the equation* (A) *such that*

$$u(x_0)=u_0,\qquad u'(x_0)=v_0. \tag{1.8}$$

Proof. Taking into account the initial conditions, the equation (A) becomes

$$(pu')(x)=\int_{x_0}^{x}u(s)\,dQ(s)-F(x)+p(x_0)v_0+F(x_0), \tag{1.9}$$

which can be rewritten in the form $u=Au+z$, where

$$(Au)(x)=\int_{x_0}^{x}\frac{1}{p(t)}\left(\int_{x_0}^{t}u(s)\,dQ(s)\right)dt, \tag{1.10}$$

$$z(x)=u_0+\int_{x_0}^{x}\big(p(x_0)v_0-F(t)+F(x_0)\big)\,\frac{dt}{p(t)}\,. \tag{1.11}$$

The operator A defined by the formula (1.10) acts from $C[0,l]$ to $C[0,l]$. Obviously, $z(x)\in C[0,l]$. We assert that the operator $I-A$ has an inverse. To prove this, it suffices to show that the spectral radius $\rho(A)$ is less than 1. In this case the resolvent operator $(I-A)^{-1}$ can be represented as the Neumann series $(I-A)^{-1}=\sum_{k=0}^{\infty}A^k$, which converges with respect to the operator norm due to the inequality $\rho(A) < 1$. Let us show that $\rho(A) = 0$ in our case. We apply the formula $\rho(A) = \lim_{n\to\infty} \sqrt[n]{\|A^n\|}$ and assert that

$$\sqrt[n]{\|A^n\|} \leqslant C^* \frac{1}{\sqrt[n]{n!}} \tag{1.12}$$

for some constant C^*. To show this, we prove that

$$|(A^n(\varphi))(x)| \leqslant C^n \|\varphi\| \frac{|x-x_0|^n}{n!} \tag{1.13}$$

for any positive integer n and arbitrary $\varphi(x)$ in $C[0,l]$, where $C = V_0^l[Q]/c_0$, $c_0 = \min_{\overline{[0,l]}_{\mathrm{A}}} p(x)$, and $\|\varphi\|$ is the norm of φ in $C[0,l]$.

We carry out the proof of the inequality (1.13) by induction. For $n = 1$ we have

$$\begin{aligned}
|(A(\varphi))(x)| &\leqslant \left|\int_{x_0}^{x} \frac{1}{p(t)} \int_{x_0}^{t} dQ(s)\,dt\right| \|\varphi\| \\
&\leqslant \frac{1}{c_0}\left|\int_{x_0}^{x} |Q(t)-Q(x_0)|\,dt\right| \|\varphi\| \leqslant C|x-x_0|\,\|\varphi\|,
\end{aligned}$$

as was to be proved. Assuming the validity of (1.13) for $n = k$, we obtain in succession

$$\begin{aligned}
|A(A^k\varphi)(x)| &\leqslant \left|\int_{x_0}^{x} \frac{1}{p(t)} \int_{x_0}^{t} (A^k\varphi)(s)\,dQ(s)\,dt\right| \\
&\leqslant \frac{\|\varphi\|}{c_0}\left|\int_{x_0}^{x}\left|\int_{x_0}^{t} \frac{|s-x_0|^k}{k!} C^k\,dQ(s)\right| dt\right| \\
&\leqslant \frac{\|\varphi\|}{c_0}\frac{C^k}{k!} V_0^l[Q]\left|\int_{x_0}^{x} |t-x_0|^k\,dt\right| = \frac{C^{k+1}}{(k+1)!}\,\|\varphi\|\,|x-x_0|^{k+1},
\end{aligned}$$

as was required for (1.13).

The fact that the function $u(x)$ belongs to the space E (which is connected with the fact that the derivative $u'(\cdot)$ belongs to the space $\mathrm{BV}[0,l]$) follows immediately from (1.9) by the theorem on a transform of a measure. This completes the proof of Theorem 1.5.

Remark 1.5. In traditional textbooks the corresponding proof (whose nature is similar in essence) does not explain the possibility of passing to the limit in an equality of the form $u_n = Au_n + z$. In fact, this is possible in the classical situation because $C^1[0,l]$ is complete and holds in our case because the space E is complete.

Remark 1.6. The following norms on the space E are obviously equivalent:

$$\begin{aligned}
\|u\|_E &= \max_{[0,l]} |u| + V_0^l[u'], \\
\|u\|_1 &= \max_{[0,l]} |u| + V_0^l[pu'], \\
\|u\|_2 &= \max\bigl\{\max_{[0,l]} |u|, V_0^l[u']\bigr\}.
\end{aligned}$$

1.2.5. *Continuous dependence of the solution on the parameters.*

Theorem 1.6. *Under the assumptions of Theorem 1.5 the solution of the problem (A), (1.8) depends continuously on the initial data u_0, v_0 and on the variations of the functions $p(\cdot)$, $Q(\cdot)$, and $F(\cdot)$ on $[0,l]$.*

The proof follows immediately from the continuity of $(I-A)^{-1}$ and from the explicit representation for $z(x)$.

We shall study the problem of continuous dependence on the parameter λ for the solution of the previous equation (A), that is,

$$-(pu')(x)+(pu')(0)+\int_0^x u\,dQ = F(x)-F(0),$$

with initial data depending on λ at some $x_0 \in \overline{[0,l]}_{\mathrm{A}}$:

$$u(x_0)=\psi_1(\lambda), \qquad u'(x_0)=\psi_2(\lambda). \tag{1.14}$$

Theorem 1.7. *If the functions $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are continuous with respect to λ, then the solution $u(x,\lambda)$ of (A) corresponding to the initial conditions (1.14) depends continuously on λ with respect to the norm (1.5).*

Proof. Let $v(x)$ be the solution of (A) with the conditions $v(x_0)=v'(x_0)=0$. Let $\varphi_1(x)$ and $\varphi_2(x)$ be the solutions of the homogeneous equation $Lu=0$, that is,

$$-(pu')(x)+(pu')(0)+\int_0^x u\,dQ=0,$$

with the conditions

$$u(x_0)=1, \quad u'(x_0)=0 \quad \text{and} \quad u(x_0)=0, \quad u'(x_0)=1,$$

respectively. Then the solution of the problem (A), (1.14) can be represented in the form

$$u(x,\lambda)=v(x)+\psi_1(\lambda)\varphi_1(x)+\psi_2(\lambda)\varphi_2(x).$$

Thus, the difference

$$u(x,\lambda)-u(x,\lambda_0)=\varphi_1(x)\big(\psi_1(\lambda)-\psi_1(\lambda_0)\big)+\varphi_2(x)\big(\psi_2(\lambda)-\psi_2(\lambda_0)\big)$$

can be estimated as follows:

$$\|u(x,\lambda)-u(x,\lambda_0)\| \leqslant |\psi_1(\lambda)-\psi_1(\lambda_0)|\,\|\varphi_1\|+|\psi_2(\lambda)-\psi_2(\lambda_0)\,\|\varphi_2\|,$$

where $\|\cdot\|$ is the norm in $C[0,l]$. Since

$$u'(x,\lambda)-u'(x,\lambda_0)=\varphi_1'(x)\big(\psi_1(\lambda)-\psi_1(\lambda_0)\big)+\varphi_2'(x)\big(\psi_2(\lambda)-\psi_2(\lambda_0)\big),$$

it follows that

$$V_0^l[u'(x,\lambda)-u'(x,\lambda_0)] \leqslant |\psi_1(\lambda)-\psi_1(\lambda_0)|V_0^l[\varphi_1']+|\psi_2(\lambda)-\psi_2(\lambda_0)|V_0^l[\varphi_2'].$$

Therefore,

$$\|u(x,\lambda)-u(x,\lambda_0)\|_E \leqslant |\psi_1(\lambda)-\psi_1(\lambda_0)|\,\|\varphi_1\|_E+|\psi_2(\lambda)-\psi_2(\lambda_0)|\,\|\varphi_2\|_E.$$

This completes the proof of Theorem 1.7.

We now consider the equation

$$- d(pu') + u\,dQ(\lambda, x) = dF(\lambda, x) \tag{1.15}$$

for $Q(\lambda, x) = Q_0(x) + \psi_1(\lambda)Q_1(x)$ and $F(\lambda, x) = F_0(x) + \psi_2(\lambda)F_1(x)$, where Q_0, Q_1, F_0, and F_1 are functions of bounded variation and $Q_1 \neq \text{const}$.

Theorem 1.8. *Let $u(x, \lambda)$ be the solution of (1.15) satisfying the conditions $u(x_0) = u_0$ and $u'(x_0) = v_0$ for some $x_0 \in \overline{[0, l]}_{\mathrm{A}}$. Then $u(x, \lambda)$ depends continuously on λ with respect to the norm (1.5) if $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are continuous, and $u(x, \lambda)$ is differentiable with respect to λ as many times as $\psi_1(\lambda)$ and $\psi_2(\lambda)$ are differentiable.*

The proof is quite similar to the previous arguments and is omitted because it is routine and cumbersome.

1.2.6. *Boundary-value problem.* Consider the homogeneous equation

$$-(pu')(x) + (pu')(0) + \int_0^x u\,dQ = 0. \tag{A_0}$$

It is clear that the set $\mathfrak{M}_0 \subset E$ of solutions of this equation is a linear subspace of E.

Lemma 1.1. $\dim \mathfrak{M}_0 = 2$.

Proof. This follows trivially from Theorem 1.5.

Following the traditions of the theory of ordinary differential equations, we refer to any basis in $\mathfrak{M}_0$ as a fundamental system of solutions of the homogeneous equation (A_0).

Lemma 1.2. *Let $u(x)$, $v(x)$ be a fundamental system of solutions of the homogeneous equation (A_0), and let $z(x)$ be a solution of the inhomogeneous equation (A). Then every solution $h(x)$ of the inhomogeneous equation is of the form $h(x) = \alpha_1 u(x) + \alpha_2 v(x) + z(x)$ for some α_1 and α_2.*

Proof. This is trivial, because $h - z$ belongs to $\mathfrak{M}_0$.

By the boundary-value problem we mean the equation (A) ($Du = dF$) under the conditions

$$l_1(u) = c_1, \qquad l_2(u) = c_2,$$

where l_1, l_2 are some linear functionals on E.

This problem is said to be *non-degenerate* if it is uniquely soluble for any c_1, c_2 and any $F(x) \in \mathrm{BV}[0, l]$.

Theorem 1.9. *For the boundary-value problem to be non-degenerate, it is necessary and sufficient that the corresponding homogeneous problem*

$$Du = 0, \qquad l_1(u) = 0, \quad l_2(u) = 0$$

have only the trivial solution $u(x) \equiv 0$.

Proof. By the previous lemma, if $u(x)$ is a solution of the inhomogeneous problem, then it has the form $u = \alpha_1 u_1 + \alpha_2 u_2 + z$, where z is some solution of the inhomogeneous equation (A) (which certainly exists by Theorem 1.5 under arbitrary but fixed initial conditions). The corresponding values of the coefficients α_1, α_2 can be found from the conditions $l_1(u) = c_1$, $l_2(u) = c_2$. This is a system of algebraic equations (with respect to α_1 and α_2), namely,

$$\alpha_1 l_1(\varphi_1) + \alpha_2 l_1(\varphi_2) = c_1 - l_1(z),$$
$$\alpha_1 l_2(\varphi_1) + \alpha_2 l_2(\varphi_2) = c_2 - l_2(z).$$

For the last system to be uniquely soluble, it is necessary and sufficient that its determinant

$$\begin{vmatrix} l_1(\varphi_1) & l_1(\varphi_2) \\ l_2(\varphi_1) & l_2(\varphi_2) \end{vmatrix}$$

be non-zero. But this condition is equivalent to the condition that the system arising in the same way from the homogeneous problem $Du = 0$, $l_1(u) = 0$, $l_2(u) = 0$ have only the trivial solution. This proves Theorem 1.9.

Below we are interested in the following special case of boundary conditions:

$$u(0) = 0, \qquad u(l) = 0.$$

For this problem to be non-degenerate, it is sufficient that every non-trivial solution of the homogeneous equation $Lu = 0$ have at most one zero on $[0, l]$. It is this condition that is the basis for the discussion below.

2. Linear theory

2.1. Homogeneous equation. Let us consider the homogeneous equation (A_0), that is,

$$Du = 0 \;\Leftrightarrow\; -pu'(x) + pu'(0) + \int_0^x u\, dQ = 0.$$

We note that if one of the functions p, Q is discontinuous at some point ξ, then at this point we have

$$-\, pu'(\xi + 0) + pu'(\xi - 0) + u(\xi)\Delta Q(\xi) = 0. \tag{2.1}$$

2.1.1. *Wronskian.* For a pair of functions φ_1, φ_2 in E we consider the determinant

$$W[\varphi_1, \varphi_2](x) = \begin{vmatrix} \varphi_1(x) & \varphi_2(x) \\ \varphi_1'(x) & \varphi_2'(x) \end{vmatrix}$$

on $\overline{[0, l]}_{A}$. If it is clear from the context what pair of functions φ_1, φ_2 is meant, then we shall write $W(x)$ instead of $W[\varphi_1, \varphi_2](x)$. At the singular points we distinguish between $W(\xi - 0)$ and $W(\xi + 0)$.

Lemma 2.1. *The following properties are equivalent for any two solutions φ_1, φ_2 of the homogeneous equation* (A_0):

(a) *The determinant $W[\varphi_1, \varphi_2](x)$ is non-zero at any point of $\overline{[0,l]}_A$;*

(b) *$W(x)$ differs from zero at least at one point of $\overline{[0,l]}_A$;*

(c) *the functions $\varphi_1(x)$ and $\varphi_2(x)$ are linearly independent.*

Proof. We prove the chain of implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

The implication (a) $\Rightarrow$ (b) is obvious. Let the property (b) hold and let x^* be a point of $\overline{[0,l]}_A$ such that $W(x^*) \neq 0$. If the functions $\varphi_1(x)$ and $\varphi_2(x)$ were linearly dependent, then the rows of the determinant would be proportional for any $x \in \overline{[0,l]}_A$, that is, we would have the equality $W(x) = 0$ for any x in $\overline{[0,l]}_A$, whereas $W(x^*) \neq 0$.

If the property (c) holds and $W(x^*) = 0$ for some x^* in $\overline{[0,l]}_A$, then the function $v(x) = \varphi_1(x)\varphi_2(x^*) - \varphi_2(x)\varphi_1(x^*)$ satisfies both the conditions $v(x^*) = v'(x^*) = 0$ and the equation $Lv = 0$, and therefore we must have $v(x) \equiv 0$ by Theorem 1.5, that is, the functions $\varphi_1(x)$ and $\varphi_2(x)$ are linearly dependent. This proves Lemma 2.1.

A function $G(x) \in \mathrm{BV}[0,l]$ is said to be continuous on $[0,l]$ if its left- and right-hand limits coincide (we redefine $G(x)$ at these points by the common value of these limits).

Lemma 2.2. *For any two solutions $\varphi_1(x)$, $\varphi_2(x)$ of the homogeneous equation* (A_0) *the function $p(x)W[\varphi_1, \varphi_2](x)$ is continuous on* $[0,l]$.

Proof. Let $x = s - 0$, where $s \in S_A$. Since $p(s+0)\varphi_i'(s+0) - p(s-0)\varphi_i'(s-0) = \varphi_i(s)\Delta Q(s)$ $(i = 1, 2)$, we have

$$p(s+0)W(s+0) - p(s-0)W(s-0)$$
$$= -\varphi_1(s+0)\varphi_2(s+0)\Delta Q(s) + \varphi_1(s+0)\varphi_2(s+0)\Delta Q(s) = 0.$$

The case $x = s + 0$ can be treated in a similar way.

If the functions p and Q are continuous at a point x, then so are $\varphi_1'(x)$ and $\varphi_2'(x)$ (this follows from the equation (A_0) and the fact that $p > 0$), and thus the function pW is continuous at x. This proves Lemma 2.2.

2.1.2. *Main property of the Wronskian.*

Theorem 2.1. *$p(x)W(x) \equiv \mathrm{const}$ on $\overline{[0,l]}_A$ for any pair of solutions $\varphi_1(x)$, $\varphi_2(x)$ of the equation* (A_0).

Proof. This theorem is based on the following lemma.

Lemma 2.3. *Let $h(x) \in \mathrm{BV}[0,l]$ and let $\varphi(x) = V_0^x[h]$. Then the following relations hold for any $\xi < l$:*

$$\lim_{\varepsilon\to+0} V_{\xi+0}^{\xi+\varepsilon}[h] = \lim_{\varepsilon\to+0}\big(\varphi(\xi+\varepsilon) - \varphi(\xi+0)\big) = 0, \tag{2.2}$$

and $\lim_{\varepsilon\to+0}\big(\varphi(\xi-0) - \varphi(\xi-\varepsilon)\big) = 0$ for any $\xi > 0$. In other words, the function $\varphi(x) = V_0^x[h]$ is right continuous at any 'point' $\xi + 0$ and left continuous at any 'point' $\xi - 0$.

Proof. If $h(\xi-0) = h(\xi+0)$, then the assertion is obvious. If a discontinuity point ξ of the function $h(x)$ lies to the left of l, then the proof of (2.2) follows easily from the readily verifiable equality $V_0^x[h] = V_0^{\xi+0}[h] + V_{\xi+0}^x[h]$ for $\xi < x$. The proof for a point $\xi > 0$ is similar.

We proceed to the proof of the theorem. We extend the definition of the function $pW(x)$ by continuity to the entire interval $[0, l]$. Let us prove first that $(pW)'(x+0) \equiv 0$. For an arbitrary function $h(x)$ we write $\Delta_\varepsilon h(x) = h(x+\varepsilon) - h(x)$ for $\varepsilon > 0$. Since φ_1 and φ_2 belong to $\mathfrak{M}_0$,

$$\begin{aligned}\frac{\Delta_\varepsilon (pW)(x)}{\varepsilon} = \frac{\Delta_\varepsilon \varphi_1(x)}{\varepsilon} \int_0^{x+\varepsilon} \varphi_2\, dQ + \frac{p(0){\varphi_2}'(0)\Delta_\varepsilon \varphi_1(x)}{\varepsilon} \\ - \frac{\Delta_\varepsilon \varphi_2(x)}{\varepsilon} \int_0^{x+\varepsilon} \varphi_1\, dQ - \frac{p(0){\varphi_1}'(0)\Delta_\varepsilon \varphi_2(x)}{\varepsilon} \\ + \frac{1}{\varepsilon}\Big(\varphi_1(x) \int_{x+0}^{x+\varepsilon} \varphi_2\, dQ - \varphi_2(x) \int_{x+0}^{x+\varepsilon} \varphi_1\, dQ \Big).\end{aligned}$$

Let us show that the limit of the last term as $\varepsilon \to +0$ vanishes. Denoting the variable of integration (with respect to dQ) by s, we obtain

$$\begin{aligned}\Big| \frac{1}{\varepsilon} \int_{x+0}^{x+\varepsilon} \big(\varphi_1(x)\varphi_2(s) - \varphi_2(x)\varphi_1(s)\big)\, dQ(s) \Big| \\ \leqslant \frac{1}{\varepsilon}\Big(\max_{x\leqslant s\leqslant x+\varepsilon} |\varphi_1(x)\varphi_2(s) - \varphi_2(x)\varphi_1(s)| \Big) V_{x+0}^{x+\varepsilon}[Q].\end{aligned}$$

Note that

$$\begin{aligned}|\varphi_1(x)\varphi_2(s) - \varphi_2(x)\varphi_1(s)| \leqslant \|\varphi_1\| \cdot |\varphi_2(s) - \varphi_2(x)| + \|\varphi_2\| \cdot |\varphi_1(x) - \varphi_1(s)| \\ \leqslant \|\varphi_1\| \cdot \Big| \int_s^x |\varphi_2'(\tau)|\, d\tau \Big| + \|\varphi_2\| \cdot \Big| \int_s^x |\varphi_1'(\tau)|\, d\tau \Big|.\end{aligned}$$

Since $\varphi_1, \varphi_2 \in E$, it follows that the variations of the functions φ_1' and φ_2' are finite, and thus there is a constant c_0 for which $|\varphi_2'(\tau)| \leqslant c_0$ and $|\varphi_1'(\tau)| \leqslant c_0$. Then

$$|\varphi_1(x)\varphi_2(s) - \varphi_2(x)\varphi_1(s)| \leqslant (\|\varphi_1\| + \|\varphi_2\|)c_0\varepsilon,$$

and hence

$$\frac{1}{\varepsilon} \max_{x\leqslant s\leqslant x+\varepsilon} |\varphi_1(x)\varphi_2(s) - \varphi_2(x)\varphi_1(s)| \leqslant (\|\varphi_1\| + \|\varphi_2\|)c_0.$$

Since $V_{x+0}^{x+\varepsilon}[Q] \to 0$ as $\varepsilon \to +0$, we obtain the desired relation. This yields

$$\begin{aligned}(pW)'(x+0) = \varphi_1'(x+0) \int_0^{x+0} \varphi_2\, dQ + p(0)\varphi_2'(0)\varphi_1'(x+0) \\ - \varphi_2'(x+0) \int_0^{x+0} \varphi_1\, dQ - p(0)\varphi_1'(0)\varphi_2'(x+0). \qquad (2.3)\end{aligned}$$

Each of the functions $\varphi_1(x)$ and $\varphi_2(x)$ must satisfy the equality

$$p(x+0)\varphi'(x+0) - p(0)\varphi'(0) = \int_0^{x+0} \varphi\, dQ,$$

therefore, $(pW)'(x+0) \equiv 0$ by (2.3).

The arguments for the left-hand derivative $(pW)'(x-0)$ are similar. This completes the proof of Theorem 2.1.

2.2. Non-degenerate boundary-value problem. We denote by E_0 the set of functions in E with $u(0) = u(l) = 0$ and consider the problem

$$Du = dF, \qquad u(0) = u(l) = 0. \tag{2.4}$$

In the standard theory of boundary-value problems the possibility of integral representations of solutions is realized by constructing the Green's function defined by a system of axioms, as in [6] and [50]–[52]. However, as was noted recently [53], this approach is incorrect. And it is even impossible to apply this approach to our problem. For this reason, we follow a more physical path.

2.2.1. *The influence function. A rigorous definition.* We give an exact description of the influence function, starting from the physical or, more precisely, the variational motivation for the equation (A). At the physical level, the influence function $K(x,s)$ is defined as the deformation of the original system under the action of a unit force applied at the point $x = s$.

We recall that for an elastic continuum the potential energy corresponding to a virtual (fictitious) shape $u(x)$ arising under the influence of an external force $f(x)\,dx = dF(x)$ is expressed by the functional

$$\Phi(u) = \int_0^l p\,\frac{(u')^2}{2}\,dx + \int_0^l \frac{u^2}{2}\,dQ - \int_0^l u\,dF.$$

If the external load has unit value and is applied only at a point $x = s$, then the work of this force is equal to $u(s)$, and the last term in the representation of Φ becomes $u(s) = \int_0^l u\,d\theta(x-s)$, where $\theta(x)$ is the classical Heaviside function.

According to variational principles, the function $K(x,s)$ minimizes the functional

$$\varphi(u) = \int_0^l p\,\frac{u'^2}{2}\,dx + \int_0^l \frac{u^2}{2}\,dQ - u(s).$$

One can readily see that a function $u_0(x)$ provides the minimum of this functional if and only if

$$\int_0^l pu_0'h'\,dx + \int_0^l u_0 h\,dQ - h(s) = 0$$

for any admissible function $h(x)$ (that is, for $h(x)$ belonging to E_0). Setting $z(x) = \int_0^x u\,dQ$ and $h(s) = \int_0^l h(x)\,d\theta(x-s)$ here, we see after integrating the first two terms by parts that

$$\int_0^l h\,d(-(pu')(x) + z(x) - \theta(x-s)) = 0,$$

and hence the corresponding Stieltjes differential, which is zero, must be generated by a constant, that is, $-(pu')(x) + z(x) - \theta(x-s) \equiv \text{const}$, and this means (by the definition of $z(x)$) that

$$-(pu')(x) + \int_0^x u\,dQ = \theta(x-s) - (pu')(0). \tag{2.5}$$

Definition 1. By the influence function $K(x,s)$ of the original problem (2.4) we mean the solution at any $s \in (0,l)$ of the equation (2.5) under the conditions $u(0) = u(l) = 0$.

2.2.2. *Main properties of the influence function.* The properties listed below can be obtained immediately from (2.5).

1. For any $s_0 \in (0,l)$ the function $g(x) = K(x,s_0)$ satisfies the homogeneous equation $Lu = 0$ on the intervals $(0,s_0)$ and (s_0,l).

2. For any $s_0 \in (0,l)$ the function $g(x) = K(x,s_0)$ satisfies the following equalities on the diagonal $x = s_0$:

(α) if ξ is not a discontinuity point of Q, then $-\Delta(pg')(s_0) = 1$,

(β) if ξ is a discontinuity point of Q, then $-\Delta(pg')(s_0) + g(s_0)\Delta Q(s_0) = 1$.

Theorem 2.2. *Suppose that the equation* $Lu = 0$ *has a solution without zeros on* $[0,l]$. *Then* $K(x,s) > 0$ *for any* x *and* s *distinct from* 0 *and* l. *Moreover, if the function* $Q(x)$ *is non-decreasing, then* $\max_{x\in[0,l]} K(x,s) = K(s,s)$.

Proof. We denote by $\varphi(x)$ a positive solution of the equation $Lu = 0$ which has no zeros on the interval $[0,l]$. Let us show that the function $g(x) = K(x,s_0)$ is non-negative for any $s_0 \in (0,l)$. By the factorization theorem (Theorem 1.1), we have

$$d\left(\varphi^2 p\,\frac{d}{dx}\left(\frac{g}{\varphi}\right)\right) = -\varphi\,d\theta(x-s) \leqslant 0,$$

that is, the function $\varphi^2 p\,\dfrac{d}{dx}\left(\dfrac{g}{\varphi}\right)$ is non-increasing on $[0,l]$. Thus, the function $\dfrac{d}{dx}\left(\dfrac{g}{\varphi}\right)$ can change sign at most once, and only from plus to minus. Since $\varphi(x)$ is positive, it follows from the conditions $g(0) = g(l) = 0$ that $g(x) \geqslant 0$ on $[0,l]$.

Since the function $g(x)$ satisfies the inequality $Du \geqslant 0$, it follows from Theorem 1.2 that $g(x) > 0$ on $(0,l)$. For $x < s_0$ we have

$$(pg')(x) = (pg')(0) + \int_0^x g(s)\,dQ(s),$$

and it is clear that $(pg')(0) \geqslant 0$, which implies that $(pg')(x) > 0$ on $(0,s_0)$, and hence $g'(x) > 0$ at the same points x. This means that $g(x)$ increases on $(0,s_0)$. Similarly, $g(x)$ is strictly decreasing on (s_0,l). Thus, $g(x)$ has a unique maximum at $x = s_0$. This completes the proof of Theorem 2.2.

2.2.3. *One-pair representation of the influence function.* Suppose that the problem (2.4) is non-degenerate. Let $\varphi_1(x)$, $\varphi_2(x)$ be the solutions of the homogeneous equation $Lu = 0$ that satisfy the conditions

$$\varphi_1(0) = 0, \qquad \varphi_1'(0) = 1, \tag{2.6}$$

$$\varphi_2(l) = 0, \qquad \varphi_2'(l) = -1. \tag{2.7}$$

Theorem 2.3. *The influence function $K(x,s)$ of the problem (2.4) exists and can be represented as*

$$K(x,s) = \frac{1}{p(0)\varphi_2(0)} \begin{cases} \varphi_1(s)\varphi_2(x) & \text{for } 0 \leqslant s \leqslant x \leqslant l, \\ \varphi_2(s)\varphi_1(x) & \text{for } 0 \leqslant x \leqslant s \leqslant l. \end{cases} \tag{2.8}$$

Proof. Since the problem is non-degenerate, we have $\varphi_2(0) \neq 0$ and $\varphi_1(l) \neq 0$. One can readily see that the system $\{\varphi_1, \varphi_2\}$ is a basis in $\mathfrak{M}_0$.

The function $K(x,s)$ (as a solution) satisfies the equation $Lu = 0$ with respect to x (for $x \neq s$). Therefore,

$$K(x,s) = \begin{cases} c_1(s)\varphi_1(x), & 0 \leqslant x < s, \\ c_2(s)\varphi_2(x), & s < x \leqslant l, \end{cases} \tag{2.9}$$

for some functions $c_1(s)$ and $c_2(s)$. Since the function $g(x) = K(x, s_0)$ must be continuous at $x = s_0$, it follows that

$$c_1(s_0)\varphi_1(s_0) = c_2(s_0)\varphi_2(s_0). \tag{2.10}$$

The function $g(x) = K(x, s_0)$ must satisfy the equation $Du = dF$ for $F(x) = \Theta(x - s_0)$, which implies that $\Delta(pg')(s_0) = g(s_0)\Delta Q(s_0) - 1$. In turn, this must mean that

$$c_2(s_0)(p\varphi_2')(s_0 + 0) - c_1(s_0)(p\varphi_1')(s_0 - 0) = K(s_0, s_0)\Delta Q(s_0) - 1.$$

Replacing $c_2(s_0)$ by $c_1(s_0)\varphi_1(s_0)/\varphi_2(s_0)$ here according to (2.10), we obtain

$$c_1(s_0)\frac{\varphi_1(s_0)(p\varphi_2)'(s_0+0)}{\varphi_2(s_0)} - c_1(s_0)(p\varphi_1')(s_0 - 0) + 1 = K(s_0, s_0)\Delta Q(s_0). \tag{2.11}$$

Since $\varphi_1(x)$ satisfies the homogeneous equation, we have

$$(p\varphi_1')(s_0 - 0) = (p\varphi_1)'(s_0 + 0) - \varphi_1(s_0)\Delta Q(s_0).$$

Substituting the expression thus obtained into (2.11) and taking into account the equality

$$c_1(s_0)\varphi_1(s_0) = K(s_0, s_0),$$

we obtain

$$c_1(s_0)\varphi_1(s_0)\frac{(p\varphi_2)'(s_0+0)}{\varphi_2(s_0)} - c_1(s_0)(p\varphi_1')(s_0 + 0) + 1 = 0,$$

that is, we have lost the term with the jump $\Delta Q(s_0)$, and (the main point) only right-hand limit values appear.

This implies easily that

$$c_1(s_0)p(s_0 + 0)W[\varphi_1, \varphi_2](s_0 + 0) + \varphi_2(s_0) = 0,$$

and hence

$$c_1(s_0) = -\frac{1}{(pW)(s_0 + 0)}\,\varphi_2(s_0).$$

The denominator in the last expression, that is, the function $(pW)(x)$, is a constant (by Theorem 2.1) equal to

$$(pW)(s_0+0) = (pW)(l) = (pW)(0) = -p(l)\varphi_1(l) = -p(0)\varphi_2(0).$$

Therefore,

$$c_1(s_0) = \frac{\varphi_2(s_0)}{p(0)\varphi_2(0)} = \frac{\varphi_2(s_0)}{p(l)\varphi_1(l)}\,.$$

In a similar way one can show that $c_2(s) = \dfrac{\varphi_1(s)}{p(0)\varphi_2(0)}$ for any s. This completes the proof of Theorem 2.3.

Corollary 2.1. *The influence function is symmetric, that is,* $K(x,s) = K(s,x)$.

Corollary 2.2. *The influence function is jointly continuous on the square* $[0,l]\times[0,l]$.

2.2.4. *Integral invertibility of a non-degenerate problem.*

Theorem 2.4. *Suppose that the problem* (2.4) *is non-degenerate and let* $K(x,s)$ *be its influence function. Then for any* $F(x)$ *in* $\mathrm{BV}[0,l]$ *the corresponding solution* $u(x)$ *of the problem* (2.4) *can be represented in the form*

$$u(x) = \int_0^l K(x,s)\,dF(s). \tag{2.12}$$

Proof. We denote the right-hand side of (2.12) by $v(x)$, that is,

$$v(x) = \frac{\varphi_2(x)}{p(0)\varphi_2(0)}\int_0^x \varphi_1\,dF + \frac{\varphi_1(x)}{p(0)\varphi_2(0)}\int_x^1 \varphi_2\,dF.$$

The equalities $v(0) = v(l) = 0$ are obvious. To prove the equality $Dv = dF$, we show first that $v(\cdot)\in E$. For arbitrary $\alpha \leqslant \beta$

$$\begin{aligned} v(\beta)-v(\alpha) &= \frac{1}{p(0)\varphi_2(0)}\left(\big(\varphi_2(\beta)-\varphi_2(\alpha)\big)\int_0^\beta \varphi_1\,dF + \big(\varphi_1(\beta)-\varphi_1(\alpha)\big)\int_\beta^1 \varphi_2\,dF\right) \\ &\quad + \frac{1}{p(0)\varphi_2(0)}\int_\alpha^\beta \Big(\big(\varphi_2(\alpha)-\varphi_2(s)\big)\varphi_1(s) + \big(\varphi_1(s)-\varphi_1(\alpha)\big)\varphi_2(s)\Big)\,dF(s), \end{aligned}$$

which implies the absolute continuity of the function $v(x)$.

Let us show that the derivative $v'(x)$ of $v(x)$ is defined by the equality

$$v'(x) = \frac{\varphi_2'(x)}{p(0)\varphi_2(0)}\int_0^x \varphi_1\,dF + \frac{\varphi_1'(x)}{p(0)\varphi_2(0)}\int_x^1 \varphi_2\,dF. \tag{2.13}$$

Let $\Delta_\varepsilon h(x) = h(x+\varepsilon) - h(x)$, where $\varepsilon > 0$. We carry out the proof for the right-hand derivative (the arguments for the left-hand derivative are similar). We have

$$\begin{aligned} \frac{\Delta_\varepsilon v}{\varepsilon} &= \frac{1}{p(0)\varphi_2(0)}\,\frac{\Delta_\varepsilon \varphi_2}{\varepsilon}\int_0^{x+\varepsilon} \varphi_1\,dF + \frac{1}{p(0)\varphi_2(0)}\,\frac{\Delta_\varepsilon \varphi_1}{\varepsilon}\int_{x+\varepsilon}^1 \varphi_2\,dF \\ &\quad + \frac{1}{p(0)\varphi_2(0)}\int_{x+0}^{x+\varepsilon} \frac{\varphi_2(x+0)\varphi_1(s)-\varphi_1(x+0)\varphi_2(s)}{\varepsilon}\,dF(s). \end{aligned}$$

As in the case of Theorem 2.1, one can show that

$$\lim_{\varepsilon\to+0}\frac{1}{\varepsilon}\bigg(\int_{x+0}^{x+\varepsilon}\big(\varphi_2(x+0)\varphi_1(s)-\varphi_1(x+0)\varphi_2(s)\big)\,dF(s)\bigg)=0,$$

which implies (2.13). It follows from (2.13) that $v'\in \mathrm{BV}[0,l]$, and thus $v\in E$.

Let us show now that $Dv=dF$. First, we have

$$\int_0^x v\,dQ=\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_2(s)\int_0^s\varphi_1\,dF\,dQ \\ +\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_1(s)\int_s^1\varphi_2\,dF\,dQ.$$

Both the iterated integrals are applied here to the continuous function $\varphi_1\varphi_2$, and therefore by Fubini's theorem (see, for instance, [45]) we can interchange the limits of integration. Using the equalities $D\varphi_1=0$ and $D\varphi_2=0$ here, we obtain

$$\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_2(s)\int_0^s\varphi_1\,dF\,dQ \\ =\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_1(t)\big((p\varphi_2')(x)-(p\varphi_2')(t)\big)\,dF(t).$$

Similarly, by the identity

$$p(0)\varphi_1'(0)+\int_0^x\varphi_1\,dQ=p(x)\varphi_1'(x)$$

and the conditions (2.6), we have

$$\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_1(s)\int_s^l\varphi_2\,dF\,dQ \\ =\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_2(t)\big(p(t)\varphi_1'(t)-p(0)\big)\,dF(t) \\ +\frac{1}{p(0)\varphi_2(0)}\int_x^l\varphi_2(t)\big(p(x)\varphi_1'(x)-p(0)\big)\,dF(t) \\ =\frac{1}{p(0)\varphi_2(0)}\int_0^x\varphi_2 p\varphi_1'\,dF-\frac{1}{\varphi_2(0)}\int_0^x\varphi_2\,dF \\ +\frac{p(x)\varphi_1'(x)}{p(0)\varphi_2(0)}\int_x^l\varphi_2\,dF-\frac{1}{\varphi_2(0)}\int_x^l\varphi_2\,dF.$$

Substituting the expression obtained for $\displaystyle\int_0^x v\,dQ$ into (2.4) and using the equality

$$p(t)\big(\varphi_2(t)\varphi_1'(t)-\varphi_1(t)\varphi_2'(t)\big)=-p(0)\varphi_2(0)$$

(which follows from Theorem 2.1), that is, $(pW)(t)=-p(0)\varphi_2(0)$, we obtain a true equality. Thus, the function $v(x)$ is a solution of the problem (2.4).

2.3. Distribution of zeros. We consider the homogeneous equation $Lu = 0$, that is,

$$Du = 0 \Leftrightarrow -(pu')(x) + (pu')(0) + \int_0^x u\,dQ = 0.$$

Proposition 2.1. *Every non-trivial solution in E of the equation $Lu = 0$ can have only finitely many zeros on the interval $[0, l]$.*

Proof. If $u(\xi_n) = 0$ and if $u(x)$ is a non-trivial solution of the homogeneous equation, then at any accumulation point ξ^* of the sequence $\{\xi_n\}$ we must have the equalities $u(\xi^*) = u'(\xi^*) = 0$, which is impossible.

2.3.1. *Alternation of zeros.* Consider two equations

$$d(pu') = u\,dQ_1, \tag{2.14}$$

$$d(pv') = v\,dQ_2. \tag{2.15}$$

Theorem 2.5 (a comparison theorem). *Let $dQ_1 \geqslant dQ_2$, that is, let the function $Q_1 - Q_2$ be non-decreasing on $[0, l]$. Let $\xi_1 < \xi_2$ be neighbouring zeros of a non-trivial solution $u(x)$ of the equation (2.14). Then any solution $v(x)$ of (2.15) not collinear to $u(x)$ has a zero between ξ_1 and ξ_2.*

Proof. Let $u(x) > 0$ for any $x \in (\xi_1, \xi_2)$. Suppose that $v(x) > 0$ for any $x \in [\xi_1, \xi_2]$. We consider the case in which the functions p, Q_1, Q_2 are all continuous at ξ_1 and one of them is discontinuous at ξ_2 (the other cases are treated similarly). Since

$$pu'(x) - pu'(0) = \int_0^x u\,dQ_1,$$

it follows from the theorem on a transform of a measure that for any continuous function $\varphi(x)$ we have

$$\int_{\xi_1}^{\xi_2 - 0} \varphi\,d(pu') = \int_{\xi_1}^{\xi_2 - 0} \varphi u\,dQ_1,$$

which gives

$$\int_{\xi_1}^{\xi_2 - 0} v\,d(pu') = \int_{\xi_1}^{\xi_2 - 0} uv\,dQ_1$$

for $\varphi = v$. Similarly,

$$\int_{\xi_1}^{\xi_2 - 0} u\,d(pv') = \int_{\xi_1}^{\xi_2 - 0} uv\,dQ_2.$$

Therefore, after transforming the integrals $\int_{\xi_1}^{\xi_2 - 0} u\,d(pv')$ and $\int_{\xi_1}^{\xi_2 - 0} v\,d(pu')$ by integrating by parts, we have for their difference

$$v(\xi_2)p(\xi_2 - 0)u'(\xi_2 - 0) = v(\xi_1)p(\xi_1)u'(\xi_1) + \int_{\xi_1}^{\xi_2 - 0} uv\,d(Q_1 - Q_2). \tag{2.16}$$

Since $u'(\xi_1) > 0$, the right-hand side of (2.16) is strictly positive. On the other hand, $u'(\xi_2 - 0) < 0$, an obvious contradiction. Thus, $v(x)$ has a zero in the interval (ξ_1, ξ_2).

For $dQ_1 = dQ_2$ we obtain a corollary.

Theorem 2.6 (alternation theorem). *Let $\varphi_1(x)$ and $\varphi_2(x)$ be linearly independent solutions of the equation* (A_0). *Then there is at least one zero of the function $\varphi_2(x)$ between any two neighbouring zeros of $\varphi_1(x)$, and vice versa.*

2.3.2. *Non-oscillation of the homogeneous equation.* As we have seen, a non-trivial solution in E of the equation (A_0) can have at most finitely many zeros. It turns out that the most important case occurs if there is at most one zero.

Definition 2. An equation $Du = 0$ ($Lu = 0$) is said to be *n*on-oscillating on the interval $[0, l]$ if any non-trivial solution of the equation has at most one zero on $[0, l]$.

The simplest example of this equation is $-d(pu') = 0$, that is, (A_0) with $dQ = 0$. For any solution $\varphi(x)$ of this equation we must have $(p\varphi') = \text{const}$, and the function $\varphi(x)$ is clearly strictly monotone. By the comparison theorem (Theorem 2.5), this gives the following result.

Proposition 2.2. *For an equation $-d(pu')+u\,dQ = 0$ to be non-oscillating on $[0, l]$ it is sufficient that $dQ \geqslant 0$, that is, that the function $Q(x)$ be non-decreasing.*

In the calculus of variations the so-called Jacobi condition is common. It is defined using the notion of conjugate point. We recall that a point ξ is said to be conjugate to a point η for an equation $-(pu') + qu = 0$ if there is a non-trivial solution with zeros at the points $x = \eta$ and $x = \xi$. The Jacobi condition is that there are no points on $[a, b]$ (other than a) that are conjugate to a.

Theorem 2.7. *The following conditions are equivalent*:

(a) *the homogeneous equation $Du = 0$ is non-oscillating on $[0, l]$*;
(b) *there are no points on $[0, l]$ conjugate to $x = 0$ and distinct from $x = 0$, and similarly, there are no points conjugate to $x = l$ and distinct from $x = l$*;
(c) *there is a non-negative solution of the equation $Du = 0$ on $[0, l]$ such that $u(0) > 0$ (or $u(l) > 0$)*;
(d) *there is a strictly positive solution of the equation $Du = 0$ on $[0, l]$.*

Proof. We prove the chain of implications (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (a). First, we need the following lemma.

Lemma 2.4. *For the homogeneous equation* (A_0) *every non-trivial solution on $[0, l]$ with constant sign has no zeros in the interval $(0, l)$.*

Proof. Let $u(x)$ be a solution of the equation (A_0) and let $u(x) \geqslant 0$ on $[0, l]$. Let $0 < \xi < l$ be a point at which $u(\xi) = 0$. This means that the function $u(x)$ has a minimum at the point $x = \xi$, and therefore the derivative $u'(x)$ must change sign at this point together with $(pu')(x)$. If $\xi \notin S_{\mathrm{A}}$, then pu' is continuous at the point, and therefore $(pu')(\xi) = 0$, that is, $u'(\xi) = 0$, which together with the equality $u(\xi) = 0$ implies the identity $u(x) \equiv 0$ by Theorem 1.5, which is impossible. Suppose now that $\xi \in S_{\mathrm{A}}$, that is, $u'(\xi - 0) \neq u'(\xi + 0)$. In this case $u'(\xi-0) \leqslant 0$ and $u'(\xi+0) \geqslant 0$. On the other hand, it follows from the equality (2.1) that $(pu')(\xi + 0) - (pu')(\xi - 0) = 0$ (by the condition $u(\xi) = 0$), and therefore $(pu')(\xi-0) = (pu')(\xi+0)$. As was noted above, $u'(\xi-0) = u'(\xi+0) = 0$, and thus, together with the condition $u(\xi) = 0$, we have the zero Cauchy problem, and hence $u(x) \equiv 0$. Lemma 2.4 is proved.

Let us proceed to the proof of the theorem. The implication (a) $\Rightarrow$ (b) is trivial. Let (b) hold. Taking a solution $u(x)$ with the conditions $u(l) = 0$ and $u'(l) = -1$, we see that it cannot have zeros on $[0, l]$ distinct from $x = l$ by (b), and therefore $u(x) > 0$ on $[0, l)$. Thus, (b) $\Rightarrow$ (c).

Let (c) hold, and let $u(x)$ be a non-negative solution with $u(0) > 0$. It follows from the inequality $u(x) \geqslant 0$ (by the above lemma) that $u(x)$ cannot have zeros interior to $[0, l]$, that is, $u(x) > 0$ on $[0, l)$. Let $v(x)$ be a similar solution which is non-zero at $x = l$. The sum $u(x) + v(x)$ is a solution which is strictly positive on $[0, l]$, that is, (c) $\Rightarrow$ (d).

We prove the implication (d) $\Rightarrow$ (a). Let $u(x)$ be a strictly positive solution on $[0, l]$. If some other solution $v(x)$ had two zeros on $[0, l]$, then by the theorem on alternation of zeros, the function $u(x)$ would have at least one zero between these two zeros, which is impossible. This completes the proof of Theorem 2.7.

The next result for smooth functions p and Q is equivalent to the classical de la Vallée–Poussin theorem.

Theorem 2.8. *For an equation $Du = 0$ to be non-oscillating it is necessary and sufficient that the equation $Du = u\,dH$ have a strictly positive solution for some non-decreasing function $H(x)$.*

Proof. The necessity follows from the previous theorem for $dH = 0$. Let $\varphi \in E$ and let $D\varphi = \varphi\,dH$ with $dH \geqslant 0$. Then the equation $Du - u\,dH = 0$, that is, $-d(pu') + u\,dQ_1 = 0$ with $Q_1 \equiv Q - H$, has a strictly positive solution on $[0, l]$ and does not oscillate on $[0, l]$. Since $dQ \leqslant dQ_1$, the non-oscillation of the original equation follows from an analogue of the Sturm comparison theorem, Theorem 2.5.

2.3.3. *Non-oscillation of differential inequalities.* A homogeneous equation $Du = 0$ is said to be *critically non-oscillating* on the interval $[0, l]$ if it does not oscillate on any interval $[a, b] \subset [0, l]$ distinct from $[0, l]$ but oscillates on $[0, l]$.

This means that the point $x = l$ is conjugate to the point $x = 0$ but there are no conjugate points in $(0, l)$. In other words, $[0, l]$ is an interval of critical non-oscillation if the homogeneous equation $Lu = 0$ has a non-trivial solution strictly positive on $(0, l)$ and with zeros at the endpoints.

Proposition 2.3. *Let the homogeneous equation $Du = 0$ be critically non-oscillating on $[0, l]$. Then every solution $u(x) \not\equiv 0$ of the differential inequality $Du \geqslant 0$ such that u is non-negative on $[0, l]$ has no zeros on the interval $(0, l)$. Moreover, $u'(0) \neq 0$ ($u'(l) \neq 0$) if $u(0) = 0$ ($u(l) = 0$).*

Proof. Since the equation $Du = 0$ is non-oscillating on $[\varepsilon, l]$ and on $[0, l - \varepsilon]$ for $\varepsilon > 0$, our assertion follows from Theorem 1.2.

Theorem 2.9. *Let the equation $Du = 0$ be critically non-oscillating on $[0, l]$. Then every non-trivial solution $z(x)$ of the inequality $Du \geqslant 0$ with the conditions*

$$u(0) \geqslant 0, \qquad u(l) \geqslant 0 \tag{2.17}$$

transforms all these inequalities into equalities, that is, is a solution of the problem

$$Dz = 0, \qquad z(0) = 0, \quad z(l) = 0. \tag{2.18}$$

The following example shows to what extent the circle of problems under consideration deepens the usual qualitative properties discussed in the traditional theory of ordinary differential equations. According to the last theorem, a function $u(x)$ satisfying the inequalities

$$-u'' \geqslant u, \quad u(0) \geqslant 0, \quad u(\pi) \geqslant 0,$$

cannot be other than $u(x) \equiv C \sin x$ for some $C = \text{const}$.

Proof. Let $v(x)$ be a positive solution of the problem (2.18) on the interval $(0, l)$. Let $u(x)$ be a non-trivial solution of the inequality $Du \geqslant 0$ and let u satisfy (2.17). We consider the function $\varphi = u/v$, which is continuous on $(0, l)$. Assume that $\varphi \neq \text{const}$. If the greatest lower bound $\lambda_0 = \inf_{(0,l)} \varphi$ is achieved at one of the interior points $x_0 \in (0, l)$, then the function $h = u - \lambda_0 v$ is a non-negative solution of the inequality $Du \geqslant 0$ and vanishes at the point $x_0 \in (0, l)$, which thus contradicts the previous theorem. Let $\inf_{(0,l)} \varphi$ be attained at one of the boundary points of $(0, l)$, for instance, at the point $x = 0$. If $\lambda_0 > -\infty$, then it follows from the equality $v(0) = 0$ that $u(0) = 0$. But then $\lambda_0 = \lim_{x\to 0}(u(x)/v(x)) = u'(0)/v'(0)$, and the function $h = u - \lambda_0 v$, which is non-negative on $[0, l]$ and satisfies the differential inequality $Du \geqslant 0$, would have both zero value and zero derivative at the point $x = 0$, which would contradict the previous theorem.

Suppose now that $\lambda_0 = \inf_{(0,l)} \varphi = -\infty$. By the inequality $u(0) \geqslant 0$, this is possible only if $u(0) = 0$. Since $v'(0) > 0$ and the limit of $\varphi = u/v$ as $x \to 0$ is equal to $u'(0)/v'(0)$, it follows that the equality $\lambda_0 = -\infty$ is impossible. Thus, the function $\varphi = u/v$ is constant. Theorem 2.9 is proved.

Theorem 2.10. *Let $v_0(x)$ be a non-trivial solution of the problem (2.18), that is,*

$$Du = 0, \qquad u(0) = u(l) = 0,$$

let $u(x)$ be a solution of the inequality

$$v_0(x) Du \geqslant 0 \qquad (x \in (0, l)), \tag{2.19}$$

and let the equality $p(\xi - 0)u'(\xi - 0) = p(\xi + 0)u'(\xi + 0)$ hold at any zero point ξ of the function $v_0(x)$. Suppose that $u(0) = 0$ and $v_0'(l - 0)u(l) \leqslant 0$. Then the functions $v_0(x)$ and $u(x)$ are collinear, that is, the identity $u(x) = Cv_0(x)$ holds for some constant C.

Proof. We carry out the proof by induction on the number of zeros of the function $v_0(x)$ on $(0, l)$. For $k = 0$ (in which case the function $v_0(x)$ is of constant sign on $(0, l)$) the assertion follows from the previous theorem.

Suppose that the theorem holds for any function $v_0(x)$ with k zeros on $(0, l)$. If $\{\xi_i\}_1^{k+1}$ are the zeros of some solution $z_0(x)$ of the problem (2.18), then all the conditions of the previous theorem are satisfied on the interval $(0, \xi_1)$, which implies that $u(x) \equiv C_0 z_0(x)$ on $(0, \xi_1)$ for some C_0. All the conditions of the theorem we are proving now hold on the interval (ξ_1, l), and $z_0(x)$ has k zeros there. By the induction hypothesis, there is a C_1 such that $u(x) \equiv C_1 z_0(x)$ on (ξ_1, l). Let us now show that $C_0 = C_1$. To this end, we note first that the function pz_0' is continuous at the point ξ_1. If ξ_1 belongs to S_{A}, then we have the equality $(pz_0')(\xi_1 + 0) - (pz_0')(\xi_1 - 0) = 0$, which means precisely that $pz_0'(x)$ is continuous at ξ_1. If $\xi_1 \notin S_{\mathrm{A}}$, then the continuity of $pz_0'(x)$ is obvious. The equality $C_0 = C_1$ now follows easily from the continuity of the functions pu' and pz_0' at ξ_1. Thus, $u(x) \equiv C_0 z_0(x)$ on the entire interval $(0, l)$. This completes the proof of Theorem 2.10.

3. Sturm–Liouville spectral problem

Let the function $Q(x)$ be non-decreasing and the function $M(x)$ be strictly increasing on $[0, l]$ and let the functions $p(x)$, $Q(x)$, and $M(x)$ be continuous at the points $x = 0$ and $x = l$. We consider the problem

$$\begin{cases} -d(pu') + u\,dQ = \lambda u\,dM, \\ u(0) = u(l) = 0 \end{cases} \tag{3.1}$$

in the class E of absolutely continuous functions on $[0, l]$ with derivatives in $\mathrm{BV}[0, l]$. All functions are assumed to be real.

A number $\lambda = \lambda^*$ (which can be complex) is called a point of the spectrum (or a spectral point) of the problem (3.1) if the problem becomes degenerate at this value of λ, that is, has a non-trivial solution $u^*(x)$. Such a solution is called an eigenfunction, and the corresponding spectral point λ^* is called an eigenvalue.

3.1. Structure of the spectrum. We assert that the spectrum of the problem (3.1) is non-empty and consists of positive eigenvalues which are simple (in the sense of both algebraic and geometric multiplicity).

3.1.1. *Discreteness and simplicity of the spectrum.* Suppose that the function $Q(x)$ is non-decreasing on the interval $[0, l]$, that is, $dQ \geqslant 0$. We consider the auxiliary problem

$$\begin{cases} -d(pu') + u\,dQ = dF, \\ u(0) = u(l) = 0. \end{cases} \tag{3.2}$$

Since this problem is non-degenerate (by $dQ \geqslant 0$), it admits an influence function $K(x, s)$.

If λ^* is a spectral point and $u^*(x)$ is the corresponding eigenfunction, then, setting $dF = \lambda^* u^*\,dM$, that is,

$$F(x) = \lambda^* \int_0^x u^*\,dM,$$

we can invert the corresponding boundary-value problem $Du = dF$, $u(0) = u(l) = 0$ by using the influence function and thereby obtain the equality

$$u^*(x) = \lambda^* \int_0^l K(x, s) u^*(s)\,dM(s). \tag{3.3}$$

Theorem 3.1. *The operator*

$$Au(x) = \int_0^l K(x,s)u(s)\,dM(s)$$

acts on the space $C[0,l]$ *and is compact (completely continuous).*

Proof. This theorem follows in a rather standard way (see [45]) from the joint continuity of the kernel $K(x,s)$ of the operator.

Thus, the equation (3.3) turns out to be in the domain of applicability of the general Riesz–Schauder theory, and this enables us to immediately establish the following property.

Corollary 3.1. *The spectrum of the operator* A *consists of eigenvalues, it is at most countable, and the only possible accumulation point of the eigenvalues of* A *is the point* $\lambda = 0$.

We note that the term *spectrum* can have two meanings here. Properly speaking, the points of the spectrum of the boundary-value problem are characteristic values of the operator A rather than its eigenvalues. More precisely, if λ is an eigenvalue of the boundary-value problem, then $\mu = 1/\lambda$ is an eigenvalue of the operator A, and conversely. Therefore, for instance, the spectrum of the integral operator can accumulate only near zero, whereas that of the boundary-value problem can accumulate only in a 'neighbourhood of infinity'.

Thus, the spectrum of the problem (3.1) consists of eigenvalues, it is at most countable, and the only possible accumulation point of the eigenvalues is the point at infinity.

Theorem 3.2. *The algebraic and geometric multiplicities of any eigenvalue are equal to one.*

Proof. Let $\varphi_1(x)$ and $\varphi_2(x)$ be two eigenfunctions corresponding to an eigenvalue λ_0. Then $\varphi_1(0) = 0$ and $\varphi_2(0) = 0$, and hence the Wronskian $W[\varphi_1, \varphi_2](0)$ vanishes, that is, the system of functions φ_1, φ_2 is linearly dependent. Thus, $\varphi_1(x) = c\varphi_2(x)$.

Let us prove the absence of adjoint functions. Suppose that $u(x)$ is an adjoint function. Then $u(x)$ is a solution of the equation

$$Du = \lambda_0 u\,dM + \varphi\,dM$$

with the conditions $u(0) = u(l) = 0$, where $\varphi(x)$ is an eigenfunction of the original problem corresponding to the eigenvalue λ_0. Thus, the function $u(x)$ satisfies the equality $D_1 u = \varphi\,dM$ for $D_1 u = Du - \lambda_0 u\,dM$, which implies that u satisfies the equation $\varphi D_1 u = \varphi^2\,dM \geqslant 0$, and the assumptions of Theorem 2.10 turn out to be satisfied. By this theorem, $\varphi \equiv 0$.

3.1.2. *Real-valuedness and positivity of the spectrum.*

Theorem 3.3. *Every eigenvalue of the problem (3.1) is positive. The corresponding eigenfunctions can be chosen to be real-valued.*

Proof. The spectrum is real because the influence function is symmetric. We assert that the eigenvalues are positive. Let $u(x)$ be a real-valued eigenfunction corresponding to an eigenvalue λ. Then

$$-d(pu') + u\,dQ = \lambda u\,dM,$$

and by the theorem on a transform of a measure,

$$-\int_0^l u\,d(pu') + \int_0^l u^2\,dQ = \lambda \int_0^l u^2\,dM.$$

Consequently,

$$\int_0^l p(u')^2\,dx + \int_0^l u^2\,dQ = \lambda \int_0^l u^2\,dM,$$

which shows that $\lambda > 0$.

3.1.3. *Non-emptiness of the spectrum.* Let us prove now that the spectrum of the problem (3.1) is non-empty.

Lemma 3.1. *Let $\xi \in (0, l)$ be a zero of a non-trivial solution $u_0(x)$ of the equation $Du = \lambda u\,dM$. Then upon passage through the point ξ the function $u_0(x)$ changes sign, that is, the point ξ is a node.*

Proof. Let $u_0(\xi) = 0$. Suppose that the function $u_0(x)$ is of constant sign in some neighbourhood of the point ξ. Then the derivatives $u_0'(\xi - 0)$ and $u_0'(\xi + 0)$ vanish (the proof is just like that in Lemma 2.4), and this implies the equality $u_0(x) \equiv 0$, which is impossible.

Theorem 3.4. *Let the function $Q(x)$ be non-decreasing on the interval $[0, l]$ and let the function $M(x)$ be strictly increasing on $[0, l]$. Then there is a finite positive number λ^* such that the problem (3.1) has a non-trivial solution for $\lambda = \lambda^*$.*

Proof. The integral operator Au generated by the influence function has a kernel continuous on the square $0 \leqslant x,\ s \leqslant l$ and strictly positive interior to it. Therefore, by a theorem of M. G. Krein [54], this operator has an eigenfunction which is strictly positive in $(0, l)$ and corresponds to an eigenvalue λ_0 such that all the spectral points of the operator satisfy the inequality $|\lambda| < \lambda_0$.

3.2. Oscillation properties of the eigenfunctions. The goal of the present subsection is to complete the proof of the main oscillation theorem for the problem

$$Du = \lambda u\,dM, \qquad u(0) = u(l) = 0. \tag{3.4}$$

To this end, we introduce a function $u(x, \lambda)$ satisfying the equation

$$-d(pu') + u\,d(Q - \lambda M) = 0$$

and the conditions

$$u(0) = 0, \qquad u'(0) = 1.$$

It is clear that if $u(x,\lambda)$ vanishes at the right endpoint $x = l$ for some $\lambda = \lambda^*$, then λ^* is an eigenvalue, and the function $z(x) = u(x,\lambda^*)$ is an eigenfunction corresponding to this eigenvalue. Thus, the set of solutions of the equation

$$u(l,\lambda) = 0$$

with respect to λ contains the spectrum of the original problem. We shall study the solutions of this equation, analyzing the dependence on λ for the solutions $x(\lambda)$ of the more general equation

$$u(x,\lambda) = 0$$

and fixing the values of λ such that $x(\lambda) = l$.

3.2.1. *Method of accumulation of zeros.* Let us extend the coefficients p, Q, M of the original equation to the right of the point $x = l$, that is, to the set $[l,\infty)$, in such a way that they become continuous at $x = l$ and the functions p and Q are constant to the right of l, whereas M is a linear increasing function on $[l,\infty)$ ($M(x) = m_0x+c$ with $m_0 > 0$). The solutions of this extended equation are defined on $[0,\infty)$, and they coincide with solutions of the original equation on $[0,l]$. We keep the original notation for the extended coefficients. On $[l,\infty)$ this equation becomes

$$-d(p_0u') = \lambda m_0 u\,dx,$$

that is, $-p_0u'' = \lambda m_0 u$ (here $p_0 = p(l)$). Extending the corresponding solution $u(x,\lambda)$ of the problem with $u(0) = 0$, $u'(0) = 1$ to $[l,\infty)$, we note that this function has infinitely many zeros on $[l,\infty)$ for any $\lambda > 0$, and thus infinitely many zeros on $[0,\infty)$.

Denote by $z_0(\lambda), z_1(\lambda), \dots, z_k(\lambda), \dots$ the zeros of $u(x,\lambda)$ on $(0,\infty)$ in increasing order. All of them are simple zeros of $u(x,\lambda)$ continuously dependent on λ. By Theorem 2.5, each of the functions $z_k(\lambda)$ is strictly decreasing as a function of λ if its value belongs to the ray $(0,\infty)$.

For the value of λ coinciding with the leading eigenvalue λ_0 we obviously have the equality $z_0(\lambda_0) = l$. For $\lambda = 0$ the function $u(x,0)$ has no zeros on $(0,l]$, because the equation $-d(pu') + u\,dQ = 0$ is non-oscillating on $[0,l]$ (since $dQ \geqslant 0$). Therefore, $\lambda_0 > l$. As λ increases continuously, all the zero points $z_i(\lambda)$ move continuously to the left without stopping anywhere (this is to be proved). When one of these points, say $z_k(\lambda)$, coincides with l, the corresponding solution $u(x,\lambda)$, which vanishes at the point $x = l$, turns out to be an eigenfunction of the problem (3.4), and the value λ for which $z_k(\lambda) = l$ turns out to be an eigenvalue. Since the occurrence of $z_k(\lambda)$ at the point l must be preceded by the passage of the preceding zeros $z_0(\lambda), z_1(\lambda), \dots, z_{k-1}(\lambda)$ through the same point, it follows that the equality $z_k(\lambda) = l$ determines λ_k, that is, the kth eigenvalue.

One can readily predict the character of the forthcoming difficulties by using the same function $u(x,\lambda)$. The connection between the zeros of this function with parameter λ and their evolution as λ varies is determined by the equation $u(x,\lambda) = 0$ in the form of an implicit function $x(\lambda)$. This function is certainly multivalued (for any λ the function $u(x,\lambda)$ can have many zeros with respect to x, and the number of zeros on $[0,l]$ increases as λ increases). It is convenient to treat this multivaluedness by distinguishing continuous branches.

3.2.2. *Main theorem.*

Theorem 3.5. *Let the function $Q(x)$ be non-decreasing and let $M(x)$ be strictly increasing on $[0, l]$. Then the spectrum Λ of the problem (3.4) consists of an unbounded sequence of real strictly positive simple eigenvalues $\lambda_0 < \lambda_1 < \cdots$. Moreover, the eigenfunction $\varphi_k(x)$ corresponding to λ_k has exactly k zeros in $(0, l)$ and changes sign at each of them; the zeros of $\varphi_k(x)$ and $\varphi_{k+1}(x)$ alternate.*

As was proved above, any point of the spectrum Λ is real, strictly positive, and simple.

We denote by $Z(\lambda)$ the set of zeros of $u_\lambda(x)$ on $[0, l]$, where $u_\lambda(x)$ denotes the function $u(x, \lambda)$ satisfying the conditions

$$Du = \lambda u\, dM, \qquad u(0) = 0, \quad u'(0) = 1.$$

Lemma 3.2. *Under the assumptions of the theorem, there is a countable family $\{\zeta_k(\lambda)\}_{k=1}^{\infty}$ of continuous and strictly decreasing functions with respective domains $(\lambda_{k-1}, +\infty)$ and with values in $(0, l)$ such that $\zeta_k(\lambda_{k-1} + 0) = l$ and such that the set $Z(\lambda)$ of zeros of the function u_λ is $\{\zeta_1(\lambda), \dots, \zeta_k(\lambda)\}$ for any $\lambda \in \mathbb{R}$.*

This lemma implies the assertion of the theorem about the number of zeros of the eigenfunctions of the problem under consideration.

Proof. This central lemma follows from a series of auxiliary considerations.

We denote by $U_\delta(x_0, \lambda_0)$ the set of points $(x; \lambda)$ in $(0, +\infty) \times \mathbb{R}$ with $|x - x_0| < \delta$ and $|\lambda - \lambda_0| < \delta$.

Lemma 3.3. *Suppose that $u(x_0, \lambda_0) = 0$, $u(x, \lambda)$ is continuous in some neighbourhood $U_\delta(x_0, \lambda_0)$ of the point (x_0, λ_0) and has continuous partial derivative u'_λ with finite variation for fixed $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, and the derivative u'_x (which can have discontinuities in general) has finite variation on the interval $[x_0 - \delta, x_0 + \delta]$ for any fixed $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. If the derivative $u'_x(\tau, \lambda)$ is non-zero and of constant sign for any $\tau \in \overline{[x_0 - \delta, x_0 + \delta]}_{\mathrm{A}}$ and $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$, then there is a rectangle $\{x_0 - \delta_1 < x < x_0 + \delta_1,\ \lambda_0 - \delta_2 < \lambda < \lambda_0 + \delta_2\}$ inside which the equation $u(x, \lambda) = 0$ determines x as a single-valued function of λ for $\lambda_0 - \delta_2 < \lambda < \lambda_0 + \delta_2$ taking the value x_0 at $\lambda = \lambda_0$ and having a derivative that is defined on $(\lambda_0 - \delta_2, \lambda_0 + \delta_2)$ and has finite variation on this set.*

The proof is quite natural but routine, so we omit it because it is cumbersome.

Lemma 3.4. *For any number $\lambda_* > 0$ there are numbers $\varepsilon > 0$ and $\delta > 0$ such that for any zero z of the function $u(x, \lambda_*)$ (that is, for any $z \in Z(\lambda_*) = \{x \in (0, l) \mid u_{\lambda_*}(x) = 0\}$) there is a unique function $\zeta\colon U_\delta(\lambda_*) \to U_\varepsilon(z)$ satisfying the conditions*

1) $\zeta(\lambda_*) = z$,
2) $u_\lambda(\zeta(\lambda)) \equiv 0$,

and ζ is decreasing and continuous on $U_\delta(\lambda_*)$. If $\lambda_* \in \Lambda$ here, then the numbers $\varepsilon > 0$ and $\delta > 0$ can be chosen in such a way that a function $\zeta_1\colon (\lambda_*, \lambda_* + \delta) \to U_\varepsilon(l)$ satisfying the conditions

1) $\zeta_1(\lambda_* + 0) = l$,
2) $u_\lambda(\zeta_1(\lambda)) \equiv 0$,

exists and is unique, and ζ_1 is decreasing and continuous on $(\lambda_*, \lambda_* + \delta)$.

As above, $U_\delta(\xi)$ denotes the δ-neighbourhood $(\xi-\delta,\xi+\delta)$ of the point ξ.

Proof. We write $D_\lambda u = Du - \lambda u\,dM$. Let $z \in Z(\lambda_*)$. Substituting the function $u(x,\lambda)$ in the equation $Du = \lambda u\,dM$ and differentiating the identity thus obtained with respect to λ (this differentiation is admissible by Theorem 1.8), we get that

$$D(u'_\lambda) = \lambda u'_\lambda\,dM + u\,dM.$$

After substituting λ_*, we obtain

$$D_{\lambda_*}h \equiv Dh - \lambda_* h\,dM = u\,dM, \tag{3.5}$$

where $h(x) = \dfrac{\partial}{\partial\lambda}u(x,\lambda_*)$. Multiplying the equality (3.5) by $u_{\lambda_*}(x)$, we obtain

$$u_{\lambda_*}D_{\lambda_*}h = u^2_{\lambda_*}\,dM \geqslant 0,$$

where $h(0)=0$. If the inequality $h(z)u'(z-0,\lambda_*) \leqslant 0$ holds, then one can apply Theorem 2.10 on $(0,z)$, deducing from the inequality that the identity $D_{\lambda_*}h \equiv 0$ holds on $(0,z)$. This, together with the equality $D_{\lambda_*}h = u_{\lambda_*}\,dM$, means that u_{λ_*} is trivial on $(0,z)$, which contradicts the condition that the set of zeros of u_{λ_*} is finite. Thus, $h(z)u'(z-0,\lambda_*) > 0$, which implies the first part of the lemma after using the implicit function theorem (since $Z(\lambda_*)$ is finite).

The second part of the lemma (concerning the case $\lambda_* \in \Lambda$) is established by the same arguments, the only difference being that the implicit function theorem is 'one-sided'. Lemma 3.4 is proved.

Lemma 3.5. *The number of zeros is constant on any interval* (ν_1,ν_2) *containing no spectral points, that is,* $|Z(\lambda)| \equiv \mathrm{const}$ *on* (ν_1,ν_2).

Proof. Since u_λ is uniformly continuous with respect to λ, it follows that the set $G_Z = \{(\lambda;x) \in \mathbb{R}\times[0,l] \mid u_\lambda(x)=0\}$ is closed. Indeed, if $(\lambda_k;x_k) \in G_Z$ and $(\lambda_k;x_k) \to (\lambda_0;x_0)$, then $x_0 \in [0,l]$ and

$$|u_{\lambda_0}(x_0)| = |u_{\lambda_k}(x_k) - u_{\lambda_0}(x_0)| \leqslant |u_{\lambda_k}(x_k) - u_{\lambda_0}(x_k)| + |u_{\lambda_0}(x_k) - u_{\lambda_0}(x_0)|.$$

It remains to use the uniform convergence of u_{λ_k} to u_{λ_0} and the continuity of u_{λ_0}.

Let $\sigma \in (\nu_1,\nu_2)$. By Lemma 3.4 we have $|Z(\lambda)| \geqslant |Z(\sigma)|$ in some neighbourhood of σ; therefore, if $|Z(\lambda)| \not\equiv \mathrm{const}$ in any sufficiently small neighbourhood of σ, then there is a sequence $\sigma_k \to \sigma$ such that $|Z(\sigma_k)| > |Z(\sigma)|$. Thus, there are at least $|Z(\sigma)|+1$ termwise distinct sequences $\{z^i_k\}_{k=1}^\infty \subset (0,1)$ $(i=1,\dots,|Z(\sigma)|+1$; $z^i_k \neq z^j_k$ for $i \neq j)$ such that $u_{\sigma_k}(z^i_k)=0$ for all i and k, and no two of these sequences can converge to the same point of $Z(\sigma)\cup\{l\}$ by Lemma 3.4 (the sequences $\{z^i_k\}_{k=1}^\infty$ can be assumed to converge, because $[0,l]$ is compact). As a consequence, there is an i_0 such that $z^{i_0}_k$ converges to 0. But since $Z(\sigma_k)$ is finite, this contradicts the condition that $u_{\sigma_k}(x)$ cannot have zeros, for example, on an interval of non-oscillation of the equation $L_{\sigma+1}u=0$ (and such an interval abutting on $x=0$ certainly exists).

This proves that $|Z(\lambda)| \equiv |Z(\sigma)|$ in some neighbourhood of the point σ. Since σ is arbitrary, this implies that there is a covering of the interval (ν_1,ν_2) by intervals on which $|Z(\lambda)|$ is constant. By the Heine–Borel lemma, any closed interval contained in (ν_1, ν_2) admits a finite subcovering of this covering, which implies that $|Z(\lambda)|$ is constant on any closed interval in (ν_1, ν_2). This proves Lemma 3.5.

Remark 3.1. One can prove similarly that if $\nu_2 \in \Lambda$ under the assumptions of the previous lemma, then $|Z(\lambda)| = |Z(\nu_2)|$ for any $\lambda \in (\nu_1, \nu_2]$. The only difference is that in the case $\sigma = \nu_2$ one must consider left-sided neighbourhoods of the point ν.

Lemma 3.6. *If* $\lambda_* \in \Lambda$, *then* $|Z(\lambda_* - 0)| = |Z(\lambda_*)| = |Z(\lambda_* + 0)| - 1$, *that is, when* λ *passes through a spectral point of the problem* (3.1), *the number of zeros of* $u_\lambda(x)$ *increases by exactly* 1.

Proof. The equality $|Z(\lambda_* - 0)| = |Z(\lambda_*)|$ follows from the last remark. By the second part of Lemma 3.4 we have $|Z(\lambda_* + 0)| > |Z(\lambda_*)|$, and therefore if $|Z(\lambda_*)| \neq |Z(\lambda_* + 0)| - 1$, then $|Z(\lambda_* + 0)| \geqslant |Z(\lambda_*)| + 2$. This, together with the inequality $|Z(\lambda_*) \cup \{l\}| < |Z(\lambda_* + 0)|$, contradicts Lemma 3.4.

4. Bibliographical comments

As was noted in the Foreword, the present paper is devoted to an exposition of the theory extending Sturm's oscillation theorems to the case of equations with impulsive coefficients.

Sturm's oscillation theory served well the needs of the scientific and technological advances of the 19th century and sufficed for problems of oscillation theory, the theory of critical stresses of columns and beams, and other important problems in the rapid development of science and technology. Under conditions for which the structure of the spectrum was obviously simple (discrete, real-valued), the qualitative properties of the eigenfunctions (amplitude functions) reflecting the manifest properties of harmonic oscillations, standing waves, and shapes in loss of stability (like the number of nodes, the number of extremum points, their mutual arrangement, and so on) were most significant, and the Sturm oscillation theory was devoted to these very problems.

These results on the form of eigenoscillations began to be carried over to the case of non-smooth systems (irregularly loaded elastic continua) at the beginning of the 20th century. Papers of Stieltjes on a string with beads and of Krein and Gantmakher on arbitrarily loaded rods and papers of Kellogg showed a direction of investigations in the interests of physical oscillation theory. This is the direction to which the present paper has been devoted.

In the mid-20th century, under the influence of questions in theoretical physics (quantum mechanics), interest arose in the structure of the spectrum, its asymptotic behaviour, completeness of the spectrum, diverse properties of the continuous spectrum, the structure of singular components of the spectrum (spectral gaps, instability zones), trace problems, and so on. The spectral theory of differential operators developed rapidly and became a backbone of functional analysis. This direction was stirred to activity by problems in theoretical physics (quantum mechanics) and attracted the attention of a rather broad circle of researchers, which led to the writing of fundamental monographs like [55]–[62] and subsequently of many hundreds of papers (see the references in [63]–[78]). One can obtain a definite impression of the investigations in this direction by scanning the references in [72]–[78].

It should be noted that cases involving diverse impulsive perturbations (singular potentials) of Sturm–Liouville (Schrödinger) differential operators have been quite actively studied during the past decade. The deepest recent results are connected mainly with the names of A. A. Shkalikov and his students [80]–[85], B. S. Mityagin [86]–[88], V. A. Mikhailets [79], R. O. Hryniv [Griniv] and Ya. V. Mykytyuk [Mikityuk] [89], and others.

The problems treated in the present paper do not require modern methods of spectral analysis because of the extremely simple structure of the spectrum. We have developed approaches related to fundamental ideas from the beginning of the 20th century and involving a pointwise interpretation of differential equations.

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