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On the Stefan problem

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Introduction

By the Stefan problem in the wide sense of the word one means at present the class of mathematical models describing heat, diffusion, or even thermo-diffusion processes accompanied by phase changes of the medium and absorption or emission of latent heat. Such processes occur, for example, in the formation of monocrystals, in several branches of contemporary metallurgical technology, in the formation and evolution of the interior structure of the earth and the polar ice as well as in a number of other fields of science and practice. The most characteristic peculiarity of these processes, which is the reason why their mathematical models are non-linear and difficult to analyse, is the "free" boundaries between different phases which are unknown beforehand (in the case of a homogeneous medium) or a "multiphase zone" (in a multicomponent medium), both of which are defined in terms of level surfaces of the functions describing the thermo-diffusion state of the system.

The first work devoted to this type of problem was the article [302] published by Lamé and Clapeyron in 1831, in which they studied the solidification of a homogeneous fluid filling a half-space and found at the temperature of the phase change at the initial moment under the influence of a constant temperature on the boundary. It was this paper that first established that the thickness of the solid phase (in a one-phase setting) is proportional to the square root of time. Almost thirty years passed before

Franz Neumann, according to Weber [339], considered in unpublished lectures a two-phase setting in a half-space, when at the initial moment a homogeneous medium is at a constant temperature greater than that of the phase change, and cooling is again effected by a constant temperature on the boundary. However, problems of this type were named after Joseph Stefan, who published in 1889 four papers ([327]-[330]) devoted to the study of heat and diffusion processes. In the first of these he considered a one-phase setting in a half-space, when crystallization of a fluid or melting of a solid takes place at the temperature of the phase change under the action of a constant temperature at the boundary. In the same paper he considered the two-phase "heat stroke" problem when at the initial moment two half-spaces come in contact with each other that are filled with different phases at constant temperatures other than that of the phase change. In these statements apart from the condition of isothermality, another relation (which is now called the "Stefan condition") is prescribed on the phase interfaces expressing the law of conservation of energy with the latent heat taken into account. All these statements are automodelled and were solved by means of probability integral. The third article is devoted to the one-phase problem on the freezing of water in a half-space at the temperature of the phase change under the action of variable temperature on the boundary. The author gives a formal solution of this problem in the form of a power series in the space variable, provides approximate solutions for the description of the evolution of polar ice, writes down a solution of "running wave" type and in its simplest form considers the statement which we could now call the "inverse Stefan problem".

There followed a quiet period of about four decades in the study of such problems. The next decades are characterized by an increasing scientific and practical interest in a description of phase change processes. This was furthered by Brillouin's survey [223] (1931) and also, without doubt, by the importance of technical applications. The main efforts were directed towards the study of one-dimensional problems and also of problems with cylindrical or spherical symmetry under conditions that ensure the existence of automodel solutions; the first attempts were also made (in 1930) to reduce the relevant problems in the one-dimensional case to integro-differential equations. (The main results and a bibliography until approximately the middle 50's can be found in [110], Ch. XI.) At the end of the 40's existence and uniqueness theorems were obtained (in a short time) for the one-dimensional Stefan problems by the method of reduction to integral equations of Volterra type; a beginning was made of the study of the problems of stability and of the asymptotic behaviour of a free boundary (as $t \to 0$ or $t \to \infty$); and methods of the approximate and numerical solution were studied intensively. (A historical survey, the main results, and a fairly complete bibliography until approximately the middle 60's can be found in [176].)

The beginning of the contemporary stage of the qualitative study of the Stefan problem, to which the present survey is mainly devoted, can be dated back to the end of the 50's and the beginning of the 60's, when new general methods of the study of linear and non-linear elliptic and parabolic equations and related problems were created and intensively developed. The work of Olga Oleinik ([163], [165], and [166] and others), devoted to the study of discontinuous problems of elliptic and parabolic type, has a direct connection to the Stefan problem. These new methods were applied to the many-dimensional quasilinear non-stationary Stefan problem: the concept of a generalized solution was used, existence and uniqueness theorems of generalized solutions were established, and the method of "smoothing" of coefficients in the theory of quasilinear parabolic equations was developed ([164], [108], etc.). And so the many-dimensional Stefan problem, in particular, the study of qualitative properties of a generalized solution of it, was made the order of the day, and also the main aims of the qualitative analysis of the problem were determined for the next decades. An approach conceptually similar to the discontinuous statement in the example of diffraction problems was worked out by Olga Ladyzhenskaya (see [128], Ch. III, §13 and Ch. V, §10 and the bibliography there).

The most important circumstance guaranteeing substantial progress in the analysis of generalized solutions of the many-dimensional Stefan problem at the beginning of the 70's was the introduction and further development of new methods created in mathematical physics at that time. We are speaking, first of all, of the method of variational inequalities, which at the beginning of the 60's emerged in potential theory and the mechanics of elastic-plastic media and in the subsequent decade was very intensively developed in its theoretical as well as practical aspects. The introduction of a new unknown function enabled Baiocchi [216] in 1971 to reduce a problem with a free boundary in the theory of filtration to an elliptic variational inequality, and two years later Duvaut [260] by using a similar substitution reduced the many-dimensional one-phase non-stationary Stefan problem to a parabolic variational inequality. Existence and uniqueness theorems for generalized ("weak") solutions were reobtained comparatively simply in this manner, however, the basic merit of the new approach consisted in that it potentially contained the tools of further research on the properties of a free boundary. The next outstanding step was made in 1975 in a paper by Friedman and Kinderlehrer [276], in which they found, in particular, conditions under which the free boundary in a cone-phase many-dimensional Stefan problem can be represented in a polar coordinate system by means of a continuous function that increases monotonically in the time variable and uniformly satisfies a Lipschitz condition in the angular variables. Approximately in these years (1976-1977) outstanding results were obtained mainly in papers of Caffarelli ([234], [235], [236]) in the study of properties of a free boundary in general problems "with an obstruction". In 1978 a synthesis of these achievements enabled Kinderlehrer and Nirenberg [295] to prove that the generalized solution of the many-dimensional one-phase non-stationary Stefan problem constructed by the method of variational inequalities is in fact the classical solution. A few years later the classical solubility was also established ([197]–[199]) for the "contact" one-phase many-dimensional Stefan problem, when the free boundary intersects the given one. The two-phase problem also admits a variational formulation [261], [263], and although in this way so far no substantial progress in the qualitative study of the properties of the unknown boundary has been achieved, the reduction to "pseudoparabolic" variational inequalities can serve as a basis for the numerical analysis of many-dimensional Stefan problems [286], [287].

The beginning of the 80's in the theory of the many-dimensional one-phase Stefan problem is marked by the appearance of a new general method based on modern forms of the abstract implicit function theorem. We are speaking of Nash's theorem [310], which was established at the end of the 50's and that form of it that was obtained in the research of J. Moser [307] (see also [161], Ch. 6). In the mid-70's in papers of Hörmander and Schaeffer the methods based on this theorem were applied to problems with an unknown boundary of hydrodynamic and geodesic origin and in 1981 in a paper of Hanzawa [284] to the many-dimensional Stefan problem. Under certain compatibility conditions (the order of which rises undoubtedly) this method makes it possible to establish the theorem on the classical local solubility in the time variable of the many-dimensional one-phase non-stationary Stefan problem. In similar terms, but by an entirely different method, the classical local solubility in the time variable was established recently [10], [101] also for many-dimensional Stefan problems with convection.

An entirely new point of view concerning the essence of the Stefan problem was expressed by Tikhonov and Samarkii [194] (first ed. 1951) at the beginning of the 50's. The main idea of this approach consists in the introduction of the notion of "effective" heat capacity, including also the latent heat of the phase change, concentrated on the phase interfaces. This makes it possible to write with the use of the Dirac delta-function a single quasilinear equation for the energy in the entire domain occupied by the heat-conducting medium, and Stefan's condition is a consequence of this equation. In this way an effective method of the numerical analysis of the non-stationary many-dimensional Stefan problem was worked out by means of the process of smoothing the coefficients of the resulting mathematical model [187] (see also [58]). An analogous approach to the Stefan problem was suggested by Albasiny [215] as the basis of a finite difference method for numerical analysis. The enthalpic form of the energy equation, based on the indicated idea also served as a starting point for the qualitative analysis of the many-dimensional Stefan problems in the papers [164] and [108] already cited.

At the beginning of the 70's the author of this survey and his collaborators started a study of the quasistationary formulations of the Stefan problem, which are important from the applied point of view. In 1973 for their approximate and numerical analysis they used [115] a variational method based on the theory of integral functionals with a variable domain of integration. The attempt made at that time to apply this method to the theory of permanent periodic waves [72] (see also the augmented English edition [256]) was applied to a modified statement of the established regimes of the Stefan problem in cylindrical domains. The questions arising in this approach of minimizing functionals of the indicated class, were the object of research in several papers by Bazalii and Shelepov [11]-[16]; the problem of an effective construction of a minimizing sequence, based on the Ritz method, was studied in [89], [111], [154], etc.; a computation of temperature fields and of crystallization fronts was carried out in [207] and [208]. The results thus obtained showed that the proposed variational approach was an effective method of the approximate and numerical analysis of the many-dimensional quasistationary Stefan problem in both the one-phase and multiphase settings. In recent years Borodin has shown [43], [44] that this approach combined with Rothe's method also makes it possible to obtain the classical solution of many-dimensional non-stationary two-phase problems globally in the time variable. A few years earlier another device of this kind, "the method of fibering into isothermals" was used by Meirmanov [141], [142] to establish the classical solubility (locally in the time variable) of the many-dimensional non-stationary Stefan problem.

Precise statements and ideas of proof of the majority of the basic results that exist at present in the theory of the Stefan problem, principally in its heat exchange interpretation, are contained in the corresponding chapters and sections of the survey. In conclusion of this brief introduction we wish to mention that there is an increasing interest in the Stefan problem, its generalizations and numerous applications, in the past decades. This stimulates, in particular, the discovery and application of approximate and numerical methods of the analysis of practically important statements. There is a long-standing tradition of this at the computing centers of the Moscow State University ([206], [174], [175], etc.) and the Latvian State University ([2], [168], etc). Abroad there have been several international conferences devoted to this range of problems ([308], [309], [267], etc.); the material of a similar conference held in the USSR is reflected in the collection of papers [195]. We also mention the books [62], [132], [120], etc. devoted to practical methods of solution of the simplest problems of phase changes and also the survey [119] devoted to the engineering method of the solution of crystallization problems and the extensive bibliography there. Modern methods of solving parabolic problems with a free boundary are contained in [312], where, together with [248], there is also an extensive bibliography of the Stefan problem.

Chapter I

GENERAL STATEMENT OF THE STEFAN PROBLEM AND SOME OF ITS VARIANTS

1.1. Equations of energy and momentum exchange.

We assume that the coefficient of heat conductivity has the structure

$$\lambda(T, \xi, \tau) = \lambda_0(T, \xi, \tau) + \sum_{j=1}^{m} \lambda_j(T, \xi, \tau)\,\eta(T - T_j); \tag{1.1}$$

here $\xi = (\xi_1, \xi_2, \xi_3)$ are the space variables, τ is the time variable, T the temperature, the λ_j ($j = 0, 1, \ldots, m$) are defined and have a certain smoothness on the closure of the set $(-\infty, \infty) \times D \times (0, \tau_0)$, where D is the domain filled with the medium, $\tau_0 \in (0, \infty)$ is given, the T_j ($j = 1, \ldots, m$) are the temperatures of the phase changes, and η is the characteristic function of the positive semi-axis. The heat capacity c and density ρ have similar representations.

We introduce the notation ($T_{m+1} = -\infty$, $T_0 = +\infty$):

$$D_{j,\tau} = \{\xi : \xi \in D,\ T_j < T(\xi, \tau) < T_{j-1}\},\quad \tau \in (0, \tau_0), \tag{1.2}$$

$$D_j = \bigcup_{0<\tau<\tau_0} (D_{j,\tau} \times \{\tau\}) \quad (j = 1, \ldots, m+1).$$

For a continuous $T(\xi, \tau)$ these sets are open, $D_{j,\tau}$ is the part of D occupied by the phase Φ_j at the moment of time τ. The relatively closed sets

$$\begin{cases} \Sigma_{j,\tau} = \{\xi : \xi \in D,\ T(\xi, \tau) = T_j\}, & \tau \in (0, \tau_0), \\ \Sigma_j = \bigcup_{0<\tau<\tau_0} (\Sigma_{j,\tau} \times \{\tau\}) & (j = 1, \ldots, m) \end{cases} \tag{1.3}$$

dividing them may have a quite complicated structure, in general.

In the simplest case the coefficients on the right-hand sides of the representations of the form (1.1) are constants, characteristic for the corresponding phase Φ_j. For $m = 1$ the quantities relating to the phase Φ_1 (or Φ_2) are equipped with a plus (or minus) sign.

Convection in the liquid (gas) phase are characterized by the velocity field $\vec{V}(\xi, \tau)$, which in the laminar case satisfies the Navier-Stokes equations (see, for example, [122], part II, Ch. II, A):

$$\rho \frac{d\vec{V}}{d\tau} = \vec{K} + \mathrm{div}_\xi \Pi, \tag{1.4}$$

where $\vec{K}$ is the vector of mass forces, Π is the viscous stress tensor, $d/d\tau = \partial/\partial\tau + \vec{V}\cdot\mathbf{grad}$ is the substantional derivative. To this we have to add the equation of continuity

$$\frac{d}{d\tau} \log \rho + \mathrm{div}_\xi \vec{V} = 0 \tag{1.5}$$

and the equation of state, connecting the pressure p, density ρ, and temperature T.

The general equation of energy exchange has the form (see ibidem, §10)

$$(1.6)\qquad c_v\rho\frac{dT}{d\tau}-Ap\frac{d}{d\tau}\log\rho=\operatorname{div}_\xi(\lambda\operatorname{grad}_\xi T)+f,$$

where c_v is the heat capacity per constant volume, A is the thermal equivalent of heat, and f are all possible heat sources.

With a view to simplification, in the planar (or axially symmetric) case, convection can be described by the equation

$$(1.7)\qquad \psi_{\xi_1\xi_1}+\psi_{\xi_2\xi_2}=-\tilde{\omega}\ \text{ in } D_+,\quad \psi=0\ \text{ on } \partial D_+,$$

where ψ is the flow function and $\tilde{\omega}$ is the given intensity of rotation. In the most general case the flow function ψ is determined from the Navier-Stokes equations in the Helmholtz form. For ρ = const, within the solid phase, (1.6) represents the usual equation of heat conductivity, which is sometimes also used for a fluid medium with "effective" characteristics.

1.2. **Boundary conditions.**
We assume that ∂D is the union of the closures of finitely many sufficiently smooth (compact or non-compact) manifolds Γ_j (j = 1, ..., n). The most general boundary condition describing the interaction of the process inside D with an outside medium of temperature $\overset{(0)}{T}$ can be represented in the form

$$(1.8)\qquad \chi_1\lambda\frac{\partial T}{\partial n}+\chi_2\alpha(T-\overset{(0)}{T})=g,\quad (\xi,\ \tau)\in\sum_{j=1}^{n}\Gamma_j\times(0,\ \tau_0),$$

where n in the outward normal; χ_1 and χ_2 are the "indicator" functions assuming the value 0 or 1 on each Γ_j, $\chi_1+\chi_2\geqslant 1$; $\alpha>0$ and g are given functions of (T, ξ, τ). For the corresponding choice of χ_1 and χ_2 we obtain from (1.8) the condition of Dirichlet, Neumann, or of the third boundary-value problem, as well as all possible combinations of them. It follows from experimental results quoted in [136], Ch. I, §11, for example, that $\alpha=a-b/T$, where a and b are positive constants, quite satisfactorily in the neighbourhood of the temperature of phase change.

Now we consider the conditions on the sets (1.3). In accordance with the physical meaning, these are the isothermal sets

$$(1.9)\qquad T(\xi,\ \tau)=T_j,\quad (\xi,\ \tau)\in\Sigma_j$$
$$(j=1,\ \ldots,\ m).$$

We denote by v_j the (signed) velocity of the motion of $\Sigma_{j,\tau}$ along the outward normal n_j to D_{j+1}. Assuming that $T(\xi,\ \tau)$ is continuously differentiable with respect to ξ and τ up to Σ_j and that $\Sigma_{j,\tau}$ is a smooth manifold, we obtain the formula $v_j=-T^{\pm}_{\tau}/|\operatorname{grad}T^{\pm}|$ as a consequence of (1.9). Let $\varkappa_j$ be the latent heat of the phase change per unit volume.

Balancing the energy in the neighbourhood of a point $\xi \in \Sigma_{j,\tau}$ and assuming that f in (1.6) is bounded, we obtain "the Stefan condition" (the plus sign indicates the phase with the higher temperature):

$$(1.10) \quad \lambda^+ \frac{\partial T^+}{\partial n_j} - \lambda^- \frac{\partial T^-}{\partial n_j} = -\varkappa_j v_j = \varkappa_j T_\tau^\pm / |\operatorname{grad}_\xi T^\pm| \text{ on } \Sigma_{j,\tau} \quad (j=1, \ldots, m),$$

on the right-hand side of which one has to take one of the two signs. In describing convection by means of effective parameters, the effective coefficient of heat conductivity λ^+ appears in (1.10). There is also a physical interpretation of (1.10) with a free term ([266], [267], [247], [338], etc.).

We assume that ρ undergoes a jump in the passage through $\Sigma_{j,\tau}$ and we denote by V_{n_j} the normal component of the velocity of the fluid on $\Sigma_{j,\tau}$. From the law of conservation of mass and from the condition of continuity of the medium we obtain

$$(1.11) \quad V_{n_j} = \left(1 - \frac{\rho^-}{\rho^+}\right) v_j = -\left(1 - \frac{\rho^-}{\rho^+}\right) T_\tau^\pm / |\operatorname{grad} T^\pm| \quad \text{on} \quad \Sigma_{j,\tau} \quad (j=1, \ldots, m).$$

If viscosity is taken into account, then the tangential component of V_{τ_j} is additionally equated to zero. Along with the Archimedean body force, (1.11) causes a natural convection in the liquid phase during crystallization.

Sometimes instead of (1.9) one considers the more general conditions:

$$(1.12) \qquad T(\xi, \tau) = \hat{T}_j(\xi, \tau), \qquad (\xi, \tau) \in \Sigma_j \quad (j=1, \ldots, m),$$

where the $\hat{T}_j$ are given functions. This is the way the requirement of "isothermality" looks in compressible media, in the core of the earth, etc.

1.3. The initial data.

We assume that $\hat{T}_0(\xi)$ is defined on the closure $D + \partial D$ and is piecewise continuous there, and we put

$$(1.13) \qquad T(\xi, 0) = \hat{T}_0(\xi), \qquad \xi \in D.$$

The presence of discontinuities on the right-hand side of (1.13) enables us to include the phenomenon of "heat stroke", when at the initial moment of time different phases with non-zero temperature overfall enter in the interaction. In the most important case $\hat{T}_0(\xi)$ undergoes discontinuities only along the initial position of the phase interfaces. By analogy with (1.2) and (1.3) we put

$$(1.14) \quad \begin{cases} D_{j,0} = \{\xi : \xi \in D, \quad T_j < \hat{T}_0(\xi) < T_{j-1}\} & (j=1, \ldots, m+1); \\ \Sigma_{j,0} = \{\xi : \xi \in D, \quad \hat{T}_0(\xi) = T_j\} & (j=1, \ldots, m). \end{cases}$$

In the general case these sets can have a very complicated structure, in particular, the $\Sigma_{j,0}$ can be of positive measure. If the $\Sigma_{j,0}$ are closed manifolds or manifolds with boundary and if on the complement to their union $\hat{T}_0(\xi)$ is continuous, then the $D_{j,0}$ are open. Although $T(\xi, \tau) \in C(\Omega)$ may cease to be continuous along Σ_j as $\tau \to +0$, we have

$$\text{(1.15)} \qquad \operatorname{dist}(\Sigma_{j,\tau}, \Sigma_{j,0}) \to 0 \quad \text{as} \quad \tau \to +0, \; \Sigma_{j,0} \subset \partial\Sigma_j$$

for non-empty $\Sigma_{j,0}$. Under the assumptions made, (1.13) must be satisfied at all points $\xi \in D \setminus \bigcup \Sigma_{j,0}$.

We assume that among the phases of the initial state there are liquid or gaseous phases with velocity distribution $\vec{V}_0(\xi)$. Extending $\vec{V}_0(\xi)$ to be identically zero on the remaining part of D, we put

$$\text{(1.16)} \qquad \vec{V}(\xi, 0) = \vec{V}_0(\xi), \qquad \xi \in D.$$

In the case of a viscous medium, in contrast to the ideal case, the right-hand side of this condition is continuous on $D + \partial D$.

1.4. Compatibility conditions.
The general Stefan problem reduces to the relations (1.4)–(1.6), (1.8)–(1.10), (1.13), and (1.16). To guarantee that its solution belongs to a preassigned function class, some a priori assumptions must be imposed on the initial data. The "natural" compatibility conditions ("conditions of the order minus 1") ensure the regularity of the surfaces (1.3), the continuity and boundedness of $T(\xi, \tau)$ on $D \times (0, \tau_0)$ and the local integrability of $\lambda \operatorname{grad}_\xi T$ on two-dimensional manifolds within the boundaries of D. Such solutions describe processes of a heat stroke when either the right-hand side of (1.13) becomes discontinuous or the right- and left-hand sides of (1.8) do not coincide at the initial moment of time. But if $T(\xi, \tau)$ on $(D + \partial D) \times [0, \tau_0]$ and (1.3) up to the moment $\tau = 0$ must have a certain degree of smoothness, then the compatibility conditions require, first of all, that the initial data have the corresponding smoothness, and secondly, that certain relations on ∂D and Σ_{j0} from (1.14) hold for $\tau = 0$. For the sake of simplicity we confine ourselves to important special cases.

If we assume that ρ is a known function of T, then (1.6) takes the form of the convective equation of heat conduction:

$$\text{(1.17)} \qquad c\rho \frac{dT}{d\tau} = \operatorname{div}_\xi (\lambda \operatorname{grad}_\xi T) + f, \qquad c = c_v - Ap\rho'\rho^{-2}.$$

In the absence of convection and heat sources and for constant c, ρ, and λ we obtain the usual homogeneous equation of heat conduction $T_\tau = a^2 \Delta_\xi T$, $a^2 = \lambda/\rho c$. If we also take into account that $\alpha = \text{const}$ in (1.8), then for (1.13) and (1.8) we can write the compatibility condition in the form

$$\text{(1.18)} \qquad \chi_1 \lambda \frac{\partial}{\partial n} \Delta_\xi^j \hat{T}_0(\xi) + \chi_2 \alpha \Delta_\xi^j [\hat{T}_0(\xi) - \overset{(0)}{T}] = \frac{\partial^j g}{\partial \tau^j},$$

$$j = 0, 1, \ldots, r; \quad \tau = 0, \quad \xi \in \partial D.$$

The number $r \geqslant 0$ is called the order of the compatibility conditions (1.13) and (1.8). For a general parabolic linear equation of order 2 the compatibility conditions are given in [128], Ch. IV, §5, and for the non-linear case (implicitly) in the arguments and results of Ch. V, §7 of the same book.

In passing to the compatibility conditions for (1.9), (1.10), and (1.13), we assume that $T(\xi, \tau)$ and $\hat{T}_0(\xi)$ are sufficiently smooth functions on the closure of their domains of definition. From (1.9) we obtain as $\tau \to +0$ that

$$\hat{T}_0(\xi) = T_j$$

on $\Sigma_{j,0}$; consequently, the normal to $\Sigma_{j,0}$ is defined in terms of $\hat{T}_0(\xi)$. Elinimating $T_\tau^\pm$ from the equation of heat conduction and then calculating in (1.10) the limit along Σ_j as $\tau \to +0$, we obtain a second compatibility condition of order zero on $\Sigma_{j,0}$. Similar arguments apply to the differentiated conditions (1.9) and (1.10) along Σ_j and to the heat conduction equation on both sides of Σ_j. In the end we arrive at the conclusion that the compatibility conditions of any order r_j have the form

$$\overset{(k)}{\psi}_j(\hat{T}_0^\pm, \partial_\xi \hat{T}_0^\pm, \ldots, \partial_\xi^{2r_j+2} \hat{T}_0^\pm) = 0 \quad \text{on} \quad \Sigma_{j,0}, \tag{1.19}$$

$$k = 1, \ldots, r_j; \; j = 1, \ldots, m,$$

where the $\overset{(k)}{\psi}_j$ are known elementary functions of the indicated variables. In the general case, r and r_j in (1.18) and (1.19) are distinct. In the one-phase quasilinear case the relations (1.19) are written out in detail in [142], §3.

1.5. Stationary and quasistationary problems.

In the general form above the Stefan problem has not yet been studied completely. In the search for simpler statements it can be assumed that the temperature fields do not depend on three but rather on two or one space variable ("two-" or "one-dimensional" problem) or that the state has already been established ("stationary" problem). It can also be assumed that $m = 1$ and only one of the phases has a non-constant temperature ("one-phase" problems).

More general than the "stationary" is the so-called "quasistationary" Stefan problem, which arises in cylindrical domains (of an arbitrary cross-section) if we assume that the phase interface moves along the generators at a constant velocity without changing its form, and that the entire phenomenon does not depend on the time variable in the corresponding moving coordinate system. The solution of the resulting elliptic problem with an unknown boundary generates a running wave in an absolute coordinate system ("crystallyzation soliton"). Such statements are important for some technological procedures (the growing of monocrystals, electrical slag melting, etc.).

1.6. Simplified variants.

In the quasistationary case $T(\xi_1, \xi_2, \xi_3\ \tau) = \hat{T}(\xi_1, \xi_2, \xi_3, -v\ \tau)$, and (1.9) and (1.10) are equivalent to the following [89]:

$$(1.20)\quad \begin{cases} (\lambda^-)^2|\nabla_\xi \hat{T}^-|^2 - (\lambda^+)^2|\nabla_\xi \hat{T}^+|^2 = \varkappa_j v\,(\lambda^+\hat{T}^+_{\xi_3} + \lambda^-\hat{T}^-_{\xi_3}) \overset{\text{def}}{\equiv} Q^2_{0j}, \\ \hat{T}(\xi_1, \xi_2, \xi_3) = T_j \quad (j = 1, \ldots, m) \text{ on } \hat{\Sigma}_j, \end{cases}$$

where $\hat{\Sigma}_j$ is a time-independent surface and the temperature of each phase satisfies a self-adjoint elliptic equation. If Q^2_{0j} from (1.20) is assumed to be known, then the entire problem becomes self-adjoint (see §4.1). So we obtain the "simplified" two-phase quasistationary Stefan problem. The "simplified" one-phase problem is obtained if we regard as known the sum of Q^2_j subtracted on the left in (1.20) and Q_{0j}. The construction of Q^2_{0j} and Q^2_j can be carried out by means of the "zero-th approximation", corresponding to the vanishing latent heat. The deviation of the solution of the simplified problems from the exact one can be estimated by means of two "discrepancies" in the one-phase and a single one in the two-phase version. The case $v = 0$ corresponds to the stationary Stefan problem in the exact statement.

Sometimes the transversal dimensions of the solid phase can be assumed to be small ([73] and [78]). The resulting versions of the Stefan problem can be studied fairly completely (see also [74], [79], [98], [179], [186], etc.). In multicomponent media, crystallization is accompanied by transport of the ingredient. The corresponding simplified versions [131], [36] can be studied qualitatively ([80], [81], and [85]).

1.7. On the inverse Stefan problem.

The heart of the Stefan problem for a medium with given characteristics is the determination of the operators $\Sigma_j(f, \hat{T}_0, \chi_1, \chi_2, g)$, which describe the unknown phase interfaces. The essence of the inverse Stefan problem consists in the fact that from the given surfaces Σ_j (or some parts of them) we have to determine the quantities generating them (or some part). The inverse Stefan problem is ill-posed. In the quasilinear case, for example, it reduces to a mixed elliptic problem in one of the phases and to an elliptic problem with Cauchy data on a free boundary in the other phase. So we arrive at an "optimization" setting [90] with heat (or temperature) flow as the "control". Under real-life conditions this flow is determined by a velocity v, which appears as a numerical parameter of the "control" [96] (see §3.3). A few years later in [178] the problem of optimal control was considered in the non-stationary one-phase problem when the deviation is measured by the $L_2(\Omega)$-norm and the role of control is played by the temperature on a given portion of the boundary.

The statement of inverse problems goes back to Stefan's paper [330] in which also a solution in the form of a running wave for a half-space was considered. The next papers were, apparently, [134] and [135], also devoted to one-dimensional statements. In the recent paper [66] the one-dimensional quasilinear inverse problem is considered when not the free boundary, but rather the temperature on a given curve is assumed to be known. Problems of Stefan type are of theoretical and possibly practical interest in all possible combinations in the various phases of equations of elliptic, parabolic, and hyperbolic type.

CHAPTER II

THE ONE-DIMENSIONAL NON-STATIONARY PROBLEM

2.1. Quasilinear one-phase problem.

We consider it in the following statement $(T = T^+(\xi, \tau),\ \lambda = \lambda^+,\ f = \rho c)$:

$$(2.1)\quad \begin{cases} [\lambda(\xi, \tau, T)T_\xi]_\xi = f(\xi, \tau, T)T_\tau, & \tau > 0, \quad 0 < \xi < \delta(\tau); \\ \lambda_0(\tau)T_\xi(0, \tau) = g(\tau), & \tau > 0; \\ T[\delta(\tau), \tau] = 0, & \tau > 0, \quad \delta(0) = 0, \\ \varkappa \dfrac{d\delta(\tau)}{d\tau} = b - \lambda[\delta(\tau), \tau, 0]T_\xi[\delta(\tau), \tau], & \tau > 0, \quad b \geqslant 0. \end{cases}$$

We assume that the functions λ, f and g have continuous first and second derivatives and that

$$(2.2)\quad \begin{cases} 0 < \lambda^0 \leqslant \lambda(\xi, \tau, T) = \lambda_0(\tau) + \text{terms vanishing at } \xi = 0; \\ f(\xi, \tau, T) \geqslant f_0 > 0, \quad 0 < g_0 \leqslant -g(\tau)/\lambda_0(\tau) \leqslant g^0 < +\infty; \end{cases}$$
$$\lambda_\xi \geqslant 0, \quad \lambda_T \leqslant 0, \quad (\lambda_T/f)_T \leqslant 0,$$

where b, λ^0, f_0, g^0, and g_0 are constants. The Stefan condition (1.10) is obtained from the last equality of (2.1) for $b = 0$.

***Theorem* 2.1** [300]. *Let $\tau_0 > 0$ be given. Under the above hypotheses (2.1) has one and only one classical solution $\{T(\xi, \tau), \delta(\tau)\}$ defined for $0 \leqslant \xi \leqslant \delta(\tau)$, $0 \leqslant \tau \leqslant \tau_0$.*

The problem in question is equivalent to the integro-operator equation

$$(2.3)\quad \varkappa\delta(\tau) = \int_0^\tau [b - g(\sigma)]\,d\sigma -$$
$$- \int_0^{\delta(\tau)} d\eta \int_0^{T(\eta, \tau;\, \sigma)} f(\eta, \tau, \sigma_0)\,d\sigma_0 + \int_0^\tau d\sigma \int_0^{\delta(\sigma)} d\eta \int_0^{T(\eta, \sigma;\, \delta)} f_\tau(\eta, \sigma, \sigma_0)\,d\sigma_0,$$

in which $T(\xi, \tau;\ \delta)$ is a function satisfying the relations in the first three lines of (2.1) for the given $\delta(\tau)$. It is easy to obtain (2.3) from the last relation in (2.1) if for the elimination of $\lambda[\delta(\tau), \tau, 0]\,T_\xi[\delta(\tau), \tau]$ we integrate the first equality of (2.1) with respect to ξ and use the first line of (2.2). The solubility of (2.3) can be established by means of the Schauder fixed point theorem in the space $[0, \tau_0]$ (an account of [300] is also given in [176], Ch. VII, §1).

Even earlier (2.1) for $b = 0$ and constant λ, f, and g was studied by this method in [265] and the uniqueness in a less general statement was established in [258].

2.2. The quasilinear multiphase problem.
In the one-dimensional case $\xi = \xi_1$ the domain D is the strip $\{-\infty \leqslant \delta^0_{m+1} \leqslant \xi \leqslant \delta^0_0 \leqslant +\infty\}$. We assume that (1.2) is connected and we represent (1.3) in the form

$$\text{(2.4)} \qquad \Sigma_j\colon\ \xi = \delta_j(\tau). \quad (j = 1, \ldots, m);$$
$$\delta^0_{m+1} < \delta^0_m < \ldots < \delta^0_1 < \delta^0_0; \quad \delta^0_j = \delta_j(0).$$

We consider the quasilinear equation

$$\text{(2.5)} \qquad c\rho T_\tau = (\lambda T_\xi)_\xi + aT_\xi + a_0 T, \qquad (\xi, \tau) \in D_j$$
$$(j = 1, \ldots, m+1),$$

where a and a_0 like λ, c, and ρ have the structure (1.1). We assume that the coefficients in (2.5) and their first derivatives with respect to T satisfy a Hölder condition with respect to (ξ, τ) in $\bar{D}_j$, uniformly in $T \in [T_j, T_{j-1}]$ ($j = 1, \ldots, m+1$) and that

$$\text{(2.6)} \qquad 0 < \beta_0 \leqslant \lambda, \quad c, \rho \leqslant \beta^0 < +\infty, \quad (\xi, \tau, T) \in \bar{D} \times [0, \tau_0].$$

If D is finite, then conditions of the form (1.8)

$$\text{(2.7)} \qquad T(\delta^0_{m+1}, \tau) = g_{m+1}(\tau), \quad T(\delta^0_0, \tau) = g_0(\tau), \qquad \tau \in (0, \tau_0),$$

are given, in which g_{m+1} and g_0 are continuous on $[0, \tau_0]$ together with their first derivatives. The function $\hat{T}_0(\xi)$ in (1.13) is continuous on $\bar{D}$, and

$$\hat{T}_0(\delta^0_j) = T_j; \ T_j \leqslant \hat{T}_0(\xi) \leqslant T_{j-1}, \ \xi \in [\delta^0_j, \delta^0_{j-1}] \ (j = 1, \ldots, m);$$

it belongs to $C^{2+\nu}[\delta^0_j, \delta^0_{j-1}]$ $(j = 1, \ldots, m+1)$ with

$$\hat{T}'_0(\delta^0_j \pm 0) \geqslant 0 \ (j = 1, \ldots, m)$$

and in the presence of (2.7) it satisfies the compatibility conditions (of order zero) $g_{m+1}(0) = \hat{T}_0(\delta^0_{m+1})$, $g_0(0) = \hat{T}_0(\delta^0_0)$.

Besides (1.9), conditions of the form (1.10)

$$\text{(2.8)} \qquad \varkappa_j[\delta_j(\tau), \tau] \frac{d\delta_j(\tau)}{d\tau} = \lambda^- T^-_\xi - \lambda^+ T^+_\xi \text{ on } \Sigma_j \qquad (j = 1, \ldots, m),$$

must be satisfied, in which the $\varkappa_j(\xi, \tau)$ satisfy estimates of the form (2.6). We also assume that the first derivatives of λ, c, ρ, and $\varkappa_j$ with respect to ξ and τ are continuous and uniformly bounded.

***Theorem* 2.2** [139]. *Under the assumptions made above, the problem* (2.5), (2.7), (1.13), (1.9), *and* (2.8) *has a unique classical solution* $\{T, \delta_1, \ldots, \delta_m\}$ *and* $\delta_j(\tau) \in C^1[0, \tau_0]$, $T(\xi, \tau) \in W_4^{2,1}(D_j) \cap C^{2,1}(D_j) \cap C^{1,0}(\bar{D}_j)$ *if* D_j *is finite, and* $T(\xi, \tau) \in C^{2,1}(D_j) \cap C^{1,0}(\bar{D}_j)$ *if* D_j *is infinite* ($j = 1, \ldots, m$).

The proof is based on the inversion of the operator of differentiation in (2.8):

$$(2.9) \quad \delta_j(\tau) = \delta_j^0 + \int_0^\tau \{\lambda_{\Sigma_j}^- T_\xi^- [\delta_j(\sigma), \sigma; \vec{\delta}] - \lambda_{\Sigma_j}^+ T^+ [\delta_j(\sigma)\sigma; \vec{\delta}]\} \frac{d\sigma}{\varkappa_j[\delta_j(\sigma), \sigma]},$$

$$j = 1, \ldots, m,$$

where $\vec{\delta} = (\delta_1, \ldots, \delta_m)$ and $T(\xi, \tau; \vec{\delta})$ is the solution of the problem for given $\delta_j(\tau)$, on establishing the lower estimate $|\delta_j(\tau) - \delta_{j-1}(\tau)| \geqslant \delta_0 > 0$, and on the proof of the inclusions $T_\xi(\xi, \tau) \in H^{1/8, 1/16}(D_j)$ ($j = 1, \ldots, m+1$) by successive application of the Leray-Schauder principle to the operator on the right-hand side of (2.9).

In a more general setting the problem is studied in [196]. In this paper the equation of the energy is brought to the form (1.17) where f is assumed to depend on ξ, τ, T, T_ξ, δ_j, and δ_j', the requirement of isometry is stated in the form (1.12) and the condition (2.10) contains a free term of the form $\Phi_j(\xi, \tau, T, T_\xi^\pm)$ on Σ_j. Sufficient conditions for the existence of a classical solution are stated, the device of "linearization of interfaces" and results on the solubility of mixed problems for uniformly parabolic second-order equations are used, and the method of finite differences is justified for the relevant statement of the Stefan problem.

We also mention the papers [25]-[27] in which Rothe's method for a quasilinear two-phase Stefan problem is used to prove the existence of a classical solution and the case is studied when the compatibility conditions are not necessarily satisfied. In [259] the one-dimensional multiphase problem with intersecting free boundaries ("the vanishing of phases") is considered, the monotonicity and stabilization properties of these boundaries are studied, and an estimate is given for the time of the existence of the classical solution.

2.3. Piecewise continuous initial temperature ("heat stroke").
The following formulation is considered:

$$(2.10) \quad \begin{cases} c^\pm \rho^\pm T_\tau^\pm(\xi, \tau) = \lambda^\pm T_{\xi\xi}^\pm(\xi, \tau), \ (\xi, \tau) \in D^\pm; \\ T(\xi, 0) = \hat{T}_0(\xi), \quad \xi \in [0, b/2) \cup (b/2, \infty); \\ T_\xi^+(0, \tau) = 0, \quad 0 < \tau < \tau_1; \quad T_\xi^-(0, \tau) = 0, \quad \tau > \tau_1; \\ T[\delta(\tau), \tau] = T_1, \quad \varkappa_1 \dfrac{d\delta(\tau)}{d\tau} = \lambda^- T_\xi^- - \lambda^+ T_\xi^+ \text{ on } \Sigma_1, \quad 0 < \tau < \tau_1, \end{cases}$$

in which $b > 0$ is given and $\tau_1 \in (0, \infty)$ is not known beforehand. The function $\hat{T}_0(\xi)$ is continuous for $\xi \neq b/2$ and has the finite values $\hat{T}_0(b/2 \pm 0)$, which are, in general, distinct, and $\hat{T}_0(\xi) \geqslant T_1$ for $\xi \in [0, b/2]$ and $\hat{T}_0(\xi) \leqslant T_1$ for $\xi \geqslant b/2$; moreover, the derivative $\hat{T}'_0$ exists and is bounded for $\xi \leqslant b/2$.

Theorem 2.3 [84]. a) *Under the assumptions made above, the problem* (2.10) *is equivalent to a system of three non-linear integral equations for* $w^{\pm}(\tau)$, $\delta(\tau)$, *and the initial values* $w^{\pm}(0)$ *and with them also* α, *can be determined from a non-linear system of two transcendental equations.*

b) *The indicated system, and with it also* (2.10), *has a unique solution* $\{w^{\pm}(\tau), \delta(\tau)\}$ *continuous on* $[0, \tau]$ *at least for small* τ *and* $\hat{T}_0(b/2 - 0) - \hat{T}_0(b/2 + 0)$.

c) *There is a representation* ($a_{\pm}^2 = \lambda^{\pm}/\rho^{\pm}c^{\pm}$):

$$(2.11)\quad \begin{cases} \delta(\tau) = \frac{b}{2} + 2\varkappa_1^{-1}\alpha\sqrt{\tau}\,[1 + o(\sqrt{\tau})], \qquad \operatorname{sign}\alpha = \operatorname{sign} A, \\ A = \frac{\lambda^+}{a_+}\left[\hat{T}_0\left(\frac{b}{2} - 0\right) - T_1\right] - \frac{\lambda^-}{a_-}\left[T_1 - \hat{T}_0\left(\frac{b}{2} + 0\right)\right]. \end{cases}$$

d) *Let* $\hat{T}'_0(\xi) \leqslant 0$, $\xi \neq b/2$, $\hat{T}'_0(\xi) \in L(b/2, \infty)$, *and let* $a(0, \tau_1]$ *be the maximal segment on which* $w^{\pm}(\tau)$ *and* $\delta(\tau)$ *exist and are continuous. Then* τ_1 *is finite and* $\delta(\tau_1) = 0$.

The system equivalent to (2.10) is not a Volterra system, since the norm of the corresponding operator does not tend to zero as $\tau \to 0$. Its solubility follows from the implicit function theorem in the Banach space $C[0, \tau] \times C[0, \tau]$. The sign of A in (2.11) determines the direction of the process in the initial period. The interaction of the water and ice with discontinuous piecewise constant $\hat{T}_0(\xi)$ was first considered in Stefan's paper [327]. For subsequent research on such problems, see [110], Ch. XI.

2.4. Reduction of the Stefan problem to a Cauchy problem.
Suppose that in the space of the variables $\xi = \xi_1$, $\eta = \xi_2$ liquid phase at the initial moment of time occupies the half-plane $\xi \geqslant 0$, that $\hat{T}_0(0, \eta) = T_1$, and that on the axis $\xi = 0$ a homogeneous ($g = 0$) condition (1.8) is given for $\chi_1 = \chi_2 = 1$. We assume that

$$(2.12)\quad \Sigma_1\colon\ \xi = \delta(\eta, \tau), \quad -\infty < \eta < \infty, \quad \tau \geqslant 0; \quad \delta(\eta, 0) = 0,$$

for a sufficiently smooth $\delta(\eta, \tau)$. Applying Green's formula and the formulae for the jumps of the temperature potentials, we obtain the equation

$$(2.13)\quad \varkappa_1\delta_\tau(\eta, \tau) + \frac{\lambda^+}{2\pi a_+^2\tau}\int_{-\infty}^{\infty} d\zeta \int_0^{\infty} [\hat{T}_{0\xi} - \delta_\eta(\eta, \tau)\hat{T}_{0\zeta}] \times$$
$$\times e^{-\frac{[\delta(\eta, \tau) - \xi]^2 + (\eta - \zeta)^2}{4a_+^2\tau}}\, d\xi + \frac{\varkappa_1}{4\pi a_+^2}\int_0^{\tau} d\sigma \int_{-\infty}^{\infty} \delta_\sigma(\zeta, \sigma) \times$$
$$\times \frac{\delta(\eta, \tau) - \delta(\zeta, \sigma) - \delta_\eta(\eta, \tau)(\eta - \zeta)}{(\tau - \sigma)^2}\, e^{-\frac{[\delta(\eta, \tau) - \delta(\zeta, \sigma)]^2 + (\eta - \zeta)^2}{4a_+^2(\tau - \sigma)}}\, d\zeta = 0$$

if $\overset{(0)}{T} = T_1$. A similar equation can also be derived in the presence of heat sources at the initial period of the two-phase statement for $\overset{(0)}{T} < T_1$.

***Theorem* 2.4** [79]. *Under the assumptions made above, the non-stationary Stefan problem with two space variables is equivalent to the Cauchy problem* (2.12)-(2.13).

A theory of equations of the form (2.13) has not yet been worked out in general. A substantial simplification occurs when $\hat{T}_0$ does not depend on η; such an equation appeared in the mathematical model of hardened slag [73], [74]. These investigations were later continued in [98], [179], [180], [182], etc. We also mention [117], [118], [29], etc., which are devoted to the two-phase one-dimensional Stefan problem on a finite interval with Neumann boundary values at the ends and conditions of the form (1.12) on the unknown boundary, in the absence of one of the phases at the time $\tau = 0$.

CHAPTER III

THE QUASISTATIONARY MANY-DIMENSIONAL PROBLEM

3.1. The one-phase quasistationary problem.

For the sake of simplicity we consider the case of two space variables ξ_1, ξ_2. Let D be the lower semi-infinite strip $\xi_1 \in (-R, R)$, $R > 0$, $\xi_2 < 0$. Introducing the variables $x = \xi_1/R$, $y = \xi_2/R - v\tau$, $u = (T - T_1)/(T_1 - \overset{(0)}{T})^{-1}$ and denoting by $G_\gamma \subset G = \{(x, y): -1 < x < 1,\ y < 0\}$ the domain of the solid phase, where γ is the phase interface curve, which is symmetric with respect to $x = 0$ with the end-points $(\pm 1, 0)$, we arrive at the following dimensionless statement of the problem in question:

$$(3.1)\quad \begin{cases} u_{xx} + u_{yy} + \omega u_y = 0 \ \text{ in } \ G_\gamma, \quad \omega = R\rho c v/\lambda; \\ u_x \pm \omega_0 u = 0,\ x = \pm 1, \quad y < 0, \quad \omega_0 = R\alpha/\lambda; \\ u = 1,\ u_n = \varkappa \cos(n, y), \quad (x, y) \in \gamma, \qquad \varkappa = R\varkappa_1 v/\lambda^- (T_1 - \overset{(0)}{T}); \\ u = 0, \quad -1 < x < 1, \quad y = -\infty. \end{cases}$$

For fixed thermophysical and geometric characteristics (3.1) contains two independent dimensionless parameters, say, the Pekle number ω and the Nusselt number ω_0. The conditions on γ express the isothermicity of (1.9) and the Stefan relation (1.10) for $T^+ \equiv T_1$ in the quasistationary phase.

We put $u \equiv 1$ on $G \backslash \overline{G}_\gamma$ and consider the function $w(x, y)$, which is uniquely determined by the relations $w_y(x, y) = u(x, y)$, $w(x, 0) = 0$,

$x \in (-1, 1)$. We introduce the notation

$$(3.2)\quad \begin{cases} a_{\omega,\omega_0}(u, v) = \iint\limits_G e^{\omega y}[u_x v_x + u_y v_y]\,dx\,dy + \omega_0 \int\limits_\Gamma e^{\omega y} uv\,dy, \\ l(u) = -\int\limits_{-1}^{1} u(x, 0)\,dx + \omega \iint\limits_G e^{\omega y} u\,dx\,dy + \varkappa \iint\limits_G e^{\omega y}(u-y)_+\,dx\,dy, \end{cases}$$

where Γ is the vertical portion of the boundary ∂G, and $(u-y)_+$ is the non-negative part of $(u-y)$. Let $H^1_{\omega,\omega_0}(G)$ be the Hilbert space corresponding to the bilinear form in (3.2) for $\omega_0 > 0$. The functional $l(u)$ is convex, bounded, and lower semicontinuous.

Theorem 3.1 [87]. a) *If* (u, γ) *is the classical solution of* (3.1), *then* $w(x, y)$ *satisfies the variational inequality*

$$(3.3)\qquad a_{\omega,\omega_0}(w, v-w) + l(v) - l(w) \geqslant 0 \quad \text{for all } v \in H^1_{\omega,\omega_0}(G);$$

b) *for fixed* ω *and* ω_0 (3.3) *has a unique solution* $w(x, y;\ \omega, \omega_0) \in H^1_{\omega,\omega_0}(G)$;

c) (3.1) *has a classical solution if and only if* ω *and* ω_0 *satisfy the "solubility equation"* $w(1, 0;\ \omega, \omega_0) = 0$;

d) [100] *for every* $\omega_0 > 0$ *this equation has a unique strictly increasing solution,* $\omega = \omega(\omega_0)$ *and* $\omega(\omega_0) \to 0$ *as* $\omega_0 \to +0$.

Thus, the solution of (3.1) depends on a single real parameter ω_0. which determines ω uniquely, that is, the velocity v. Model problems of the form (3.1) for domains of finite height have been considered earlier in [38]-[41], [46], [47], and [197].

The one-dimensional two-phase quasistationary problem [17], [82], and [83] admits a complete study. In the last two papers the statement with a condition on the upper section is considered, which models heat and mass transfer simultaneously. Consequences of this are the phenomena of unsolubility or non-uniqueness of solutions. For a circular cylinder of radius $R \leqslant 5$cm there are the inequalities $v_{\min} \leqslant v \leqslant v_{\max}$ where for iron $v_{\min} = 6.8 \cdot 10^{-4}$ cm/sec, $v_{\max} = 6.5 \cdot 10^{-2}$ cm/sec.

3.2. Boundary properties of a free surface and of a heat flow.

In the quasistationary statements of the Stefan problem (see §1.5) the phase interfaces intersect a given part of the boundary of the relevant domain. The behaviour of the free boundary at the points of its intersection with the given boundary, in particular, the existence and magnitude of the angle under which they intersect, also determine the properties of the heat flows in closed domains. A complete study of these questions can be accomplished under the hypotheses of the preceding section.

Theorem 3.2 [88]. a) *The curve* γ *is symmetric with respect to* $x = 0$ *and strictly increasing for* $x \in (0, 1)$; *consequently, it admits the explicit representation* $y = y(x)$, $x \in (-1, 1)$; *the function* $y(x)$ *is analytic for* $x \in (-1, 1)$ *and admits the properties*

$$(3.4) \quad \begin{cases} y'(0) = 0, \quad 0 < y'(x) < +\infty, \quad x \in (0, 1); \\ \lim\limits_{x \to 1-0} y'(x) = y'(1-0) < +\infty; \; y''(x) > 0, \quad x \in [0, 1); \end{cases}$$

b) *the heat flow* (u_x, u_y) *is bounded in* G_γ, *and* u_y *is continuous on the closure* $\overline{G}_\gamma$;

c) *if* $(2\omega_0/\varkappa) \geqslant 1$, *then* u_x *has a discontinuity at* $(1, 0)$; *but if* $(2\omega_0/\varkappa) < 1$, *then for the continuity of* u_x *on* $\overline{G}_\gamma$ *it is necessary and sufficient that*

$$(3.5) \qquad \sin 2 \operatorname{arctg} y'(1-0) = \frac{2\omega_0}{\varkappa};$$

d) *the derivative* $y'(x)$ *satisfies a Hölder condition on the closed set* $[-1, 1]$.

The proof of the first three assertions relies on the fact that $y(x)$ is a root of the equation $w(x, y) - y = 0$, where w is the solution of (3.3). The last assertion of Theorem 3.2 (which is not included in [88]) requires an extension of $u(x, y)$ through Γ and an application of the theory of systems of singular integral equations. In the derivation of these equations, as in the proof of b), one uses the integral representation of the heat flow in terms of its boundary values, deduced without assuming the boundedness of the flow in G_γ.

3.3. The inverse two-phase quasistationary problem (problem of control). As an example we consider the statement (r, z are cylinder coordinates)

$$(3.6) \quad \begin{cases} u_{zz} + \dfrac{1}{r}(ru_r)_r + \omega u_z = 0, \quad 0 < r < 1, \quad z < 0; \\ u_r + \omega_0 u = 0, \quad r = 1, \quad z \leqslant 0; \\ u_z = f(r, v) \equiv a(r) v + b(r), \quad 0 \leqslant r \leqslant 1, \quad z = 0; \\ u = 1, \quad u_z^- - \lambda u_z^+ = \varkappa r_z^2 (1 + r_z^2)^{-1} \text{ on } \gamma; \\ u = 0, \quad 0 \leqslant r \leqslant 1, \quad z = -\infty, \end{cases}$$

in which $\lambda = \lambda^+/\lambda^-$, ω and ω_0 have the same meaning as in (3.1) and are piecewise constant; the functions $a(r)$ and $b(r)$ are known, $r = r(z)$ is the equation of the unknown boundary γ. Experimental and theoretical data tell us that in real-life statements v is an affine function of the force of the exterior stream; therefore, v can play the role of "control".

Let $l \geqslant 0$ be the height of the upper phase along $r = 1$, $h + l$ the total depth of this phase, and k the curvature of γ at $r = 1$. Suppose that γ

admits the representation

$$(3.7)\quad \begin{cases} z = z(r) = -(h+l) + a_1 r^2 + (h - a_1) r^4 + \ldots, \quad 0 \leqslant r \leqslant 1; \\ r^2(z) = 2(h+l+z)/[a_1 + \sqrt{a_1^2 + 4(h-a_1)(h+l+z)}], \\ -(h+l) \leqslant z \leqslant -l; \\ k = 2(6h - a_1)[1 + 4(2h - a_1)^2]^{-3/2}, \quad 0 < a_1 \leqslant 2h. \end{cases}$$

Theorem 3.3 [96]. *We assume that the curvatures of all isothermals differ little from k for r = 1. Then h and l for fixed values of the coefficients of the terms of the highest degree in* (3.7) *are single-valued functions of v and k with a finite domain of definition of the "boomerang" type in the first quadrant and the map* $(v, k) \to (h, l)$ *is one-sheeted.*

The most important consequence of this analysis is that for the solubility of (3.6) it is necessary that $v \in [v_{\min}, v_{\max}]$, $k \in [k_{\min}, k_{\max}]$, and $l \in [0, l_0]$, where the right-hand end-points of these intervals are finite, $v_{\min}$ and $k_{\min}$ are positive, and $l_0 \approx 1$. By the choice of the values of v the form of the liquid phase can be "controlled" by means of the criteria $l > 0$, $h + l \leqslant 2$.

3.4. Quasistationary problem with convection.
Let $\vec{v} = (0, v)$ be the vector of transfer velocity, $\vec{V}_+$ the velocity field in the liquid phase with respect to a moving coordinate system, $\vec{V}_- \equiv 0$ within the limits of the solid phase, and suppose that the relevant domain G be the semistrip $\{-1 < x < 1,\ y < 0\}$. We consider the statement

$$(3.8)\quad \begin{cases} b^{\pm}(\vec{V}_{\pm} - \vec{v})\vec{\nabla} u^{\pm} = \Delta u^{\pm} \quad \text{on} \quad G_+ \cup G_-; \\ [(\vec{V}_{\pm} - \vec{v})\vec{\nabla}]\vec{V}_+ + \vec{\nabla} p = \Delta \vec{V}_+/Re + f(u_+), \quad \vec{\nabla}\cdot\vec{V}_+ = 0 \ \text{in}\ G_+, \\ \vec{V}_+ = 0 \ \text{on}\ \partial G_+; \ u_y^+ = h(x),\ y = 0,\ -1 < x < 1,\ h(-x) = h(x); \\ u_x^{\pm} = 0, \quad -H < y < 0; \ u_x^{\pm} = \omega_0^{\pm} u^{\pm}, \ -\infty < y < -H, \ H > 0; \\ u^{\pm} = 1, \quad u_n^- - \lambda u_n^+ = v\varkappa/\sqrt{1 + y'^2(x)} \quad \text{on} \quad \gamma\colon\ y = y(x); \\ u^- = 0, \quad y = -\infty, \quad x \in (-1, 1), \end{cases}$$

in which $b^{\pm}$, ω_0, Re, v, $\varkappa$, and λ are given and positive, $f(u)$ is a Lipschitz function, $h(x)$ is a known positive function on $[-1, 1]$ belonging to $C^{2+\alpha}[-1, 1]$, $h'(-1) = h'(1) = 0$, and H is a given number. Let $w^{\pm}$ be the solution of the stationary problem without convection with the same conditions (3.8) for $v = 0$ and suppose that $-H < y_0(1) < 0$, $y_0(x) < 0$ for the free boundary $\gamma_0 : y = y_0(x)$ on $[-1, 1]$.

Theorem 3.4 [101]. *Under these hypotheses for sufficiently small v and Re* (3.8) *has a solution, and* $y(x) \in C^{2+\alpha}[-1, 1]$, $\vec{V} \in W_2^1(G_+) \cap C^{1+\alpha}(\bar{G}_+)$, $\nabla p \in L_r(G_+)$ $r > 0$, $u^{\pm} \in C^{2+\alpha}$ *on* $\bar{G}_{\pm} \setminus \{(\pm 1, -H)\}$.

The weak solubility of stationary Stefan problems with convection was established in [225]-[227]. The classical solubility of the problem with convection described by an equation of the form (1.7) was established in [7], [8]. A similar many-dimensional stationary problem is given in [167].

The paper [105] is devoted to the numerical solution of the Stefan problem with convection within the framework of the boundary layer.

3.5. On model two-phase quasistationary problems.
In the exact setting, the many-dimensional two-phase quasistationary problem has been studied less completely than the one-phase one. As a supplement to the results of 3.4 we quote one result on the classical solubility of the following model problem:

$$
(3.9)\quad \begin{cases} \Delta w + \dfrac{\partial}{\partial y}\bar{\beta}(w) = 0 \text{ in } G\setminus\Sigma_1, \quad \bar{\beta}(w) = \beta_0 w + \mu\beta(w); \\ w = 0 \text{ for } y = 0, \quad w = c_2 > 0 \quad \text{for} \quad y = h; \\ w_n + \bar{\alpha}(w) = 0 \quad \text{for} \quad x = \pm 1, \ 0 < y < h; \\ \bar{\alpha}(w) \equiv \alpha_0 w + \mu\alpha(w) \geqslant 0; \\ w = c_1 \in (0, c_2), \ w_n^- - w_n^+ = \lambda(x)\cos(n, y) \text{ on } \Sigma_1, \end{cases}
$$

where $G = \{(x, y)\colon -1 < x < 1, \ 0 < y < h\}$, c_1, c_2, β_0, α_0 and μ are numerical parameters; $\beta(w)$, $\alpha(w)$ are twice continuously differentiable functions; $\lambda(x)$ is an even function defined on $[-1, 1]$ satisfying a Hölder condition, and $\lambda(1) = 0$. Let $w_0(x, y)$ be the solution of (3.9) for $\lambda = 0$, $\mu = 0$, and suppose that $w_{0y}(x, y) > 0$ in $\bar{G}$, as is true for $\beta_0 = 0$.

Theorem 3.5 [3]. *Under the hypotheses made above, for every pair* $(\lambda(x), \mu)$ *from some neighbourhood of zero in* $C^\alpha[0, 1] \times R$, $1/2 < \alpha < 1$, (3.9) *has a unique classical solution* (w, Σ_1) *and the function* $y(x;\ \lambda, \mu)$ *defining the free boundary* Σ_1 *belongs to* $C^{1+\alpha}[-1, 1]$ *and* $y'_x(0;\ \lambda, \mu) = 0$.

We also mention the article [173] devoted to the problem with a free boundary for two-dimensional second-order elliptic equations with variable coefficients. The Stefan condition is understood in a certain weak sense, and sufficient conditions are stated for a generalized solution from the class $W_2^1 \cap C$ with a Lipschitz phase interface curve. Yet another version of the model quasistationary two-phase Stefan problem is considered in [42], where the method of straight lines is used to prove the existence of a generalized solution from the class $W_2^1 \cap C$ with an increasing free boundary for $x \geqslant 0$.

Chapter IV

THE METHOD OF INTEGRAL FUNCTIONALS WITH A VARIABLE DOMAIN OF INTEGRATION

4.1. First variation; main lemma.
The variational approach to problems with free boundaries, based on the method of integral functions with a variable domain of integration, apparently goes back to [316] (see also [30] Chrs. IV and VII). Applied to stream and cavitational flows, it was developed in [277], [281], [282], etc., and

to the theory of steady-state waves in a number of aritcles summarized later in [72] (see also the enlarged English edition [256]). In this chapter this method is expounded as applied to thermophysical problems with a free boundary.

To begin with we consider the simplified one-phase quasistationary problem (see §1.6). Introducing the dimensionaless coordinates x, y, z and writing $S = \Sigma_1$, $G_s = G_-$, we arrive at the statement

$$(4.1)\quad \begin{cases} \mathcal{L}u \equiv (e^{\omega z}u_x)_x + (e^{\omega z}u_y)_y + (e^{\omega z}u_z)_z = 0 \quad \text{in} \quad G_s; \\ u_n + \omega_0 u = 0 \quad \text{on} \quad \partial G_s \setminus S; \quad u = 0 \quad \text{for} \quad z = -\infty; \\ u = 1, \quad |\text{grad}\, u|^2 = Q^2(x, y, z; v) \ \text{on}\ S, \\ Q^2 = \lambda^2 |\text{grad}\, u^+|^2 + \varkappa (1+\lambda) u_z^+, \quad \lambda = \lambda^+/\lambda^-, \end{cases}$$

where u is a dimensionless temperature, the parameters ω, ω_0, and $\varkappa$ have the same meaning as in (3.1), and u^+ is uniquely determined by the following conditions: 1) in the cylinder G it satisfies the equation in (4.1) for $\omega = \omega_+$; 2) on the upper section of S_0 ($z = 0$), it satisfies the dimensionless condition (1.8); 3) on $\partial G \setminus S_0$ it satisfies the condition in the second line of (4.1) for $\omega_0 = \omega_0^+$. The triple (u^+, u^-, S) is an approximate solution of the two-phase quasistationary Stefan problem; its deviation from the exact solution can be estimated by the discrepancies $\sup |u^+ - 1|$ and $\sup [(u_z^+ - \lambda u_z^-)/(u_z^- + \lambda u_z^+)]$ on S. In the two-phase statement only one discrepancy occurs.

(4.1) is of a variational nature (this property is alien to the original Stefan problem). We consider the functional

$$(4.2)\quad J(u, S) = \iint_{G_s} e^{\omega z} [u_x^2 + u_y^2 + u_z^2 + Q^2]\, dx\, dy\, dz + \omega_0 \int_{\partial G_s \setminus S} e^{\omega z} u^2\, d\sigma,$$

depending on the pair (u, S). A pair (u, S) is said to be "admissible" if it is sufficiently smooth and the second condition in the second line and the first condition in the third line of (4.1) are satisfied. The first variation of the functional (4.2) has the form

$$(4.3)\quad \delta J(u, S; \delta\bar{u}, \delta\vec{r}) = -2 \iint_{G_s} \delta\bar{u}\, \mathcal{L}u\, dx\, dy\, dz +$$
$$+ \int_S e^{\omega z} [Q^2 - |\text{grad}\, u|^2]\, \vec{n}\delta\vec{r}\, d\sigma + 2 \int_{\partial G_s \setminus S} e^{\omega z} [u_n + \omega_0 u]\, \delta\bar{u}\, d\sigma,$$

where $\delta\bar{u}$ is the variation of u for fixed S, $\delta\vec{r}$ is the variation of the independent variables, describing the passage from S to a "close" surface, and $\vec{n}$ is the outward normal to S. The following lemma is a consequence of (4.3).

Lemma **4.1** (the "main" lemma) [115]. *If* (u, S) *is the classical solution of* (4.1), *then it is also a critical point of the functional* (4.2) *on the indicated set of admissible pairs, that is, the first variation of* (4.3) *vanishes. The converse assertion is also true if* S *has no points of self-intersection.*

We emphasize that the second condition on S (together with the equality on $\partial D_s \setminus S$ and the equation in G_s) in (4.1) is "natural" for the variational problem with respect to (4.2) on the indicated set of admissible pairs. The two-phase version of the simplified quasistationary Stefan problem also is of a variational nature [111], [112].

4.2. Passage to a fixed domain (fibering into isothermals).
The map $(x, y, z) \to (x, y, u)$ takes G_s to a cylinder Δ in the strip $0 < u < 1$ with the same cross-section if $u_z > 0$. As the unknown, one can consider the solution $z(x, y, u)$ of the equation $u(x, y, z) - u = 0$. Then G_s is fibered into isothermals, and the value $u = 1$ corresponds to S. Putting $w(x, y, u) = \exp \omega z(x, y, u)$, $\omega > 0$, for convenience, instead of (4.2) we obtain the following functional with a fixed domain of integration:

$$(4.4)\quad J(u, S) = \tilde{J}(w) \overset{\text{def}}{=\!=} \frac{\omega_0}{\omega} \int\limits_{\partial\Delta \setminus \Gamma} u^2 w_u \, d\tilde{\sigma} + \frac{1}{\omega} \iint\limits_{\Delta} \left\{ w_x^2 + w_y^2 + \omega^2 w^2 + Q^2\left(x, y, \frac{1}{\omega}\log w;\ v\right) w_u^2 \right\} \frac{dx\, dy\, du}{w_u},$$

where Γ is the upper section $u = 1$, $\tilde{d}s = ds du$ and ds is the line element of the boundary of the cross-section of G. The variables (x, y, u) applied to (4.1) are similar to (x, y, ψ) (ψ is the stream function) used in their time by von Mises and Friedrichs in theoretical hydrodynamics. The passage from (4.2) to (4.4) clears the way to an approximate and numerical method for the construction of the critical pairs (u, S).

***Lemma* 4.2** [115]. *On the set of admissible pairs* (u, S), $u_z > 0$ *in* G_s, *the functional* (4.2) *admits the representation* (4.4). *A similar transformation is valid in the two-phase case* [111].

4.3. Second variation; uniqueness problem.
The conditions for the admissibility of $w(x, y, u)$ in (4.4) are as follows:
1) w is defined and positive on Δ, continuous on $\bar{\Delta}$, and vanishes for $u = 0$;
2) the first derivatives exist and are piecewise continuous on $\bar{\Delta}$;
3) $w_u(x, y, u) > 0$ in Δ. Let $\overset{(0)}{w}$ be the admissible function corresponding to the classical solution (u, S) of (4.1), $\overset{(1)}{w}$ any admissible function, and $\delta w = \overset{(1)}{w} - \overset{(0)}{w}$. Then for all $\varepsilon \in [0, 1]$ the function $\overset{(\varepsilon)}{w} = \overset{(0)}{w} + \varepsilon \delta w$ is admissible and

$$(4.5)\quad \frac{d^2}{d\varepsilon^2} \tilde{J}(\overset{(\varepsilon)}{w}) = \frac{2}{\omega} \iint\limits_{\Delta} [(\overset{(\varepsilon)}{w}_u \delta w_x - \overset{(\varepsilon)}{w}_x \delta w_u)^2 + (\overset{(\varepsilon)}{w}_u \delta w_y - \overset{(\varepsilon)}{w}_y \delta w_u)^2 + (\overset{(\varepsilon)}{w}_u \delta w - \overset{(\varepsilon)}{w} \delta w_u)^2] \frac{dx\, dy\, du}{\overset{(\varepsilon)}{w}{}_u^3} + \frac{2}{\omega^2} \int\limits_{\partial\Delta \cap \{u=1\}} Q Q_z (\delta w)^2 \frac{dx\, dy}{\overset{(\varepsilon)}{w}{}_u^3 (x, y, 1)}.$$

It is not hard to show that for $Q_z \geqslant 0$ the right-hand side of (4.5) vanishes only for $\delta w \equiv 0$. We also use the formula

$$\tilde{J}(\overset{(1)}{w}) = \tilde{J}(\overset{(0)}{w}) + \int_0^1 (1-\varepsilon) \frac{d^2 \tilde{J}(\overset{(\varepsilon)}{w})}{d\varepsilon^2}\, d\varepsilon. \tag{4.6}$$

***Theorem* 4.1** [115]. *Let* $QQ_z \geqslant 0$ *in* G. *Then* $J(u_1 S_1) > J(u, S)$, *for any admissible pair* (u_1, S_1) *and any function* w, *respectively. Consequently*, (4.1) *for fixed* ω *and* ω_0 *has at most one classical solution in the class of functions* $u_z > 0$.

In [277] this method of proving the uniqueness theorem was used as applied to stream flows (Q = const). A generalization of Theorem 4.3 to two-phase versions was given in [111] and [112].

4.4. The problem of minimization of the functional (4.4).
For simplicity we consider the case of two variables x, y and use an approach based on the Ritz method. Assuming that $Q(x, y)$ and the required temperature are even functions of x, we take as a basis the functions $u^k x^{2j}$. We arrive at the non-linear system of equations

$$\begin{cases} \dfrac{\partial}{\partial a_{pq}} \tilde{J}\left(\sum\limits_{k=1}^{m} \sum\limits_{j=0}^{m_k} a_{kj} x^{2j} u^k\right) + \tilde{\lambda} = 0, \quad \sum\limits_{k=1}^{m} \sum\limits_{j=0}^{m_k} a_{kj} - e^{\omega h} = 0 \\ \qquad (q = 0, 1, \ldots, m_p;\ p = 1, \ldots, m) \end{cases} \tag{4.7}$$

for the unknowns a_{kj} and the Lagrange multiplier $\tilde{\lambda}$; in a first approximation, the number $h \leqslant 0$ is equal to the depth of the isothermal $u^+ = 1$ for $x = 1$; better approximations are found by minimizing the discrepancies. We assume that $0 < \omega, \omega_0 < A < \pi^2/16$, $\lambda_0 \tan \lambda_0 = \omega_0$ and

$$\begin{cases} \hat{c}_0 e^{\mu_0 y} \leqslant Q(x, y) \leqslant \hat{c}_1 e^{\mu_0 y}, \quad Q_y(x, y) \geqslant 0 \text{ in } G,\ \hat{c}_0 > 0, \\ 0 < \hat{c}_1 \leqslant (1-A)\cos^2\sqrt{A}, \quad \mu_0 = -\omega/2 + ((\omega^2/4) + \lambda_0^2)^{1/2}. \end{cases} \tag{4.8}$$

The function $Q(x, y)$ constructed according to the method indicated in §4.1 satisfies (4.8).

***Theorem* 4.2** [91]. a) *Under the conditions* (4.8) *there is a non-empty set of pairs* (ω, ω_0) *in the neighbourhood of zero in the first quadrant for which the non-linear Ritz systems* (4.7) *have simultaneously unique solutions.*

b) *If* $Q_y[x, \frac{1}{\omega} \log w(x, 1)] \geqslant q(\omega)$ *for some* $q(\omega) > 0$, *then under the conditions in* a), *the sequence of successive Ritz approximations converges to* $\overset{(0)}{w}$ *in the norm of* $L_2(\Delta)$ *and* $W_2^1(\Delta \cap \{u > v\})$, $v \in (0, 1)$, *and the traces on* $u = 1$ *converge in* $L_2(-1, 1)$.

c) [153] *For* $u \geqslant v > 0$ *the convergence is uniform.*

The solubility of the system (4.7) for every approximation was established earlier in [89] and [113], and the numerical variational approach was realized in [207] and [208]. The single discrepancy in the two-phase version does not exceed a few percentage points, which has to be viewed as practically satisfactory, since the thermophysical parameters determined experimentally have a larger error. In [90] the variational approach is applied to the optimization of the Stefan problem. We also mention the paper [37] devoted to the quasistationary Stefan problem with a free boundary of small curvature.

4.5. The existence problem for simplified quasistationary Stefan problems. The method of integral functionals is also effective in the study of existence problems of classical solutions of the problems under discussion. The simplest of them, which is considered in [11] (see also [20]), corresponds to the planar version of (4.1) for $\omega = 0$, $Q = \text{const}$. A more general case was studied for $\omega > 0$ in [14]. We dwell in more detail on the following version of the two-phase simplified stationary problem:

$$(4.9)\quad \begin{cases} u_{xx}+u_{yy}=0 \text{ in } \Pi\setminus\gamma, \quad \Pi=\{(x,\ y):\ -1<x<1,\ 0<y<h\}; \\ u=0 \quad \text{for} \quad y=0; \quad u=c_1>0 \text{ on } \gamma, \quad u=c_2>c_1 \quad \text{for} \quad y=h; \\ u_n+\omega_0(u)=0 \quad \text{for} \quad x=\pm 1, \quad 0<y<h; \\ |\operatorname{grad} u^+|^2-|\operatorname{grad} u^-|^2=Q^2(x,\ y) \text{ on } \gamma. \end{cases}$$

To the case $Q^2 \equiv 0$ there corresponds the stationary Stefan problem [12], [13]. To (4.9) we assign the functional

$$(4.10)\quad J(u,\ \gamma)=\iint\limits_{G_\gamma}[|\operatorname{grad} u^-|^2+Q^2(x,\ y)]\,dx\,dy+ \iint\limits_{\Pi\setminus G_\gamma}|\operatorname{grad} u^+|^2\,dx\,dy+2\int_0^h dy\int_0^{v[u(1,\ y)]}\lambda(s)\,\omega_0(s)ds+ +2\int_0^h dy\int_0^{v[u(-1,\ y)]}\lambda(s)\,\omega_0(s)\,ds,$$

where G_γ is the domain of the solid phase of G^-, $v(u)$ is the function inverse to the solution $u(v)$ of the problem $u'(v) = \lambda(v)$, $u(0) = 0$, and λ is the coefficient of heat conductivity. The conditions of admissibility are written in the second line of (4.9). We also mention the article [159], which is devoted to the existence problem of the two-phase version of (4.1).

***Theorem* 4.3.** a) [15] *Suppose that $Q(x, y)$ is even in x, continuously differentiable in $\overline{\Pi}$, and satisfies the conditions $Q_x \leqslant 0$, $Q_y \geqslant 0$ for $x \geqslant 0$; suppose also that $\omega_0(u)$ is continuous, $\omega_0(0) = 0$ and $\omega_0(u) > 0$ for $u > 0$. Then (4.9) has a solution (u, γ) for which $u(x, y)$ is continuous on $\overline{\Pi}$, γ is symmetric with respect to the y-axis, is Jordan, and increases for $x \in (0, 1)$;*

suppose finally that the last relation in (4.9) *holds almost everywhere on* γ, *and the remaining ones hold in the classical sense. The functional* (4.10) *attains its smallest value at* (u, γ) *on the indicated class of admissible elements. If* $Q(x, y) \equiv 0$, *then* (4.9) *has a classical solution* (u, γ), $u_y > 0$ *in* Π, *which is unique in this class.*

b) [152] *Suppose that* $Q(x, y)$ *is analytic in* $G = \{-1 < x < 1, \ y < 0\}$ *and even in* x; *that* $Q_x \leqslant 0$, $Q_y \geqslant 0$ *for* $x \geqslant 0$, *and that* (4.8) *hold. Then on the set of pairs* (ω, ω_0) *as in Theorem* 4.2 a), *the planar one-phase problem* (4.1) *has the classical solution* (u, γ) $(S = \gamma)$ *for which* γ *is even, analytic at the interior points and increasing for* $x \in (0, 1)$; *moreover,* $u(x, y)$ *is even in* x *and* $u_x \leqslant 0$, $u_y \geqslant 0$ *for* $x \geqslant 0$ *in* G_γ. *The pair* (u, γ) *is a minimum place for the functional* (4.2) *on the set of pairs* (u, γ), $u = 1$ *on* γ.

In addition to what has been said we mention the paper [16]. Detailed proofs of b) were given in [154]. All papers cited generalize the methods worked out in [281] and [282] for axially symmetric cavitational streams. In [71], [72], and [256], this method is applied to the problem of permanent waves, and in [33] and [35] and other papers to the study of interior waves between liquids of different densities. In the above mentioned investigations of the author and his collaborators the assumptions of [71] were sharpened in that part which concerns the problem of minimization of functionals of the form (4.2) (see also [213]).

4.6. The non-stationary two-dimensional two-phase Stefan problem on a finite time interval.

We assume that the thermophysical parameters of the medium are piecewise constant and we introduce the dimensionless variables x_1, x_2, t and the temperature. Let $\Pi^+ = \{0 < x_1 < 1, \ 0 < x_2 < b\}$, $\Pi^+_{t_0} = \Pi^+ \times (0, \ t_0)$, $b > 0$, $t_0 > 0$; let S be the phase interface in the space (x_1, x_2, t) (corresponding to Σ_j for $j = 1$ in (1.3)) and let $\vec{N}$ be the unit normal vector to S, directed toward the side of higher temperature. The problem is considered in the following setting:

$$(4.11)\quad \begin{cases} u_t - a^2(u)\,\Delta u = 0 & \text{for } (x_1, x_2, t) \in \Pi^+_{t_0} \setminus S; \\ \lambda(u)\, u_n + \omega_0 u = 0 & \text{for } x_1 = 1, \ \ 0 < x_2 < b, \ \ 0 < t < t_0; \\ u_n = 0 & \text{for } x_1 = 0, \ \ 0 < x_2 < b, \ \ 0 < t < t_0; \\ u = 0 & \text{for } x_2 = 0, \ \ 0 < x_1 < 1, \ \ 0 < t < t_0; \\ u = q > 1 & \text{for } x_2 = b, \ \ 0 < x_1 < 1, \ \ 0 < t < t_0, \ \ q = \text{const}; \\ u = 1, \ \sum\limits_{i=1}^{2} (\lambda^- u^-_{x_i} - \lambda^+ u^+_{x_i}) \cos(N, x_i) + \varkappa \cos(N, t) = 0 \ \text{on } S; & \\ u(x_1, x_2, 0) = \psi(x_1, x_2) \ \text{in } \ \Pi^+ + S_0, & \\ \qquad S_0 = S \cap \{t = 0\}: \ x_2 = y_0(x_1), \ \ 0 < x_1 < 1. & \end{cases}$$

We assume that S_0 does not decrease on (0, 1) and that $\psi(x_1, x_2)$ satisfies the relations

$$(4.12)\quad \begin{cases} \psi \in W_2^1(\Pi^+) \cap C^\beta(\overline{\Pi}^+),\ 0<\beta<1, & \psi_{x_1} \leqslant 0,\quad \psi_{x_2} \geqslant 0 \text{ on } \Pi^+ \setminus S_0; \\ \psi = 0 \text{ on } \{0<x_1<1,\ x_2=0\}, & \psi = 1 \text{ on } S_0, \\ \psi = q \text{ on } \{0<x_1<1,\ x_2=b\}. \end{cases}$$

From (4.11) we easily obtain the identity

$$(4.13)\quad \iint_{\Pi_{t_0}^+} [a^2(u)\nabla u \nabla \eta + u_t \eta - \varkappa \chi_t \eta]\, dx\, dt + \omega_0 \int_{\{x_1=1,\ 0<x_2<b\}} u\eta\, dx_2\, dt = 0,$$

where $\eta \in W_2^{1,1}(\Pi_{t_0}^+)$ and η vanishes for $t = t_0$ and on $\{0 < x_1 < 1,\ x_2 = 0\} \cup \cup\ \{0 < x_1 < 1,\ x_2 = b\}$ and where $\chi(x, t)$ is the characteristic function of the solid phase of G^-. Applying Rothe's method to (4.13) and passing from the integral over t to the corresponding sum, we obtain finitely many elliptic problems with free boundaries, similar to the ones considered earlier, that is, of a variational nature. We write down the functional arising in the first step:

$$(4.14)\quad J(u,\ S;\ \psi,\ S_0) = \iint_{G_S^- \cup G_{S_0}^-} \left[a^2 |\nabla \bar{u}|^2 + \frac{1}{h}(u-\psi)^2\right] dx_1\, dx_2 +$$

$$+ \omega_0 \int_{\{x_1=1,\ 0<x_2<b\}} u^2\, dx_2 + \iint_{\Pi^+ \setminus (G_S \cup G_{S_0})} [a_+^2 |\nabla u^+|^2 + \frac{1}{h}(u-\psi)^2]\, dx_1\, dx_2 +$$

$$+ 2\varkappa \iint_{(G_S \cup G_{S_0}) \setminus G_{S_0}} \frac{1}{h}(1-u)\, dx_1\, dx_2.$$

Here G_S^- denotes the domain of the solid phase with the free boundary S, which is a Jordan curve with end-points on the vertical portions of the boundary $\partial\Pi^+$. The requirement of admissibility of the pair (u, S) includes also the relation

$$u \in W_2^1(\Pi^+) \cap C(\Pi^+ \cup \{0 < x_1 < 1,\ x_2 = 0\} \cup \{0 < x < 1,\ x_2 = b\},\ u = 0$$

on the lower base of Π^+, $u = 1$ on S, $u = q$ on the upper base of Π^+, $0 \leqslant u \leqslant 1$ on $G_S^- \cup G_{S_0}^-$ and $1 \leqslant u \leqslant q$ on the complement.

***Theorem* 4.4.** a) [43] *Under the conditions* (4.12) *there is a pair* (u, S) *at which the functional* (4.14) *attains its smallest value on the indicated set; moreover,* $u \in C^{\beta_1}(\overline{\Pi}^+)$, $0 < \beta_1 < 1$, *u is differentiable on* $\Pi^+ \setminus S$, $u_{x_1} \leqslant 0$, $u_{x_2} \geqslant 0$ *S is a non-decreasing curve, and* $G_{S_0}^- \subset G_S^-$.

b) [43] *Under the assumptions* (4.11) *has a unique solution* (u, S) *in the sense of the identity* (4.13), $u \in W^{1,1}(\Pi_{t_0}^+) \cap H^{\delta,\delta/2}(\overline{\Pi}_{t_0}^+)$, $0 < \delta < 1$, *the free* *boundary S satisfies a Lipschitz condition and the Stefan condition on it is satisfied almost everywhere.*

c) [44] *Suppose that* ψ *satisfies the conditions in* (4.11) *on the vertical portions of* $\partial\Pi^+$ *and* $a^2\Delta\psi = f$ *in* G_{S_0} *and* $a_+^2\Delta\psi = f$ *in* $\Pi^+\setminus\overline{G}_{S_0}$, *where* $a_- \geqslant a_+$, $f \in C^1(\overline{\Pi}^+)$, $f \leqslant 0$, $f_{x_1} \geqslant 0$, $f_{x_2} \leqslant 0$. *Then* (4.11) *has a unique solution for which the free boundary S is continuously differentiable at every interior point of it.*

This theorem shows that the method of integral functionals with a variable domain of integration combined with Rothe's method is effective in the solution of the existence problem of classical solutions even in the case of non-stationary many-dimensional two-phase Stefan problems on an arbitrary finite time interval.

Chapter V

THE MANY-DIMENSIONAL NON-STATIONARY PROBLEM

5.1. The enthalpic form of the energy equation; generalized solutions (the method of integral identities).

We introduce the notation

$$(5.1)\quad \begin{cases} H(T,\xi,\tau) = \int\limits_{T_{m+1}}^{T} (\rho c)(s,\xi,\tau)\,ds + \sum\limits_{j=1}^{m} \varkappa_j \eta(T-T_j), \quad T \neq T_i \\ \qquad\qquad (i=1,\ldots,m); \\ \Lambda(T,\xi,\tau) = \int\limits_{T_{m+1}}^{T} \lambda(s,\xi,\tau)\,ds,\ \Lambda_j(T,\xi,\tau) = \int\limits_{T_{m+1}}^{T} \lambda_{\xi_j}(s,\xi,\tau)\,ds \\ \qquad\qquad (j=1,2,3); \\ f_0(T,\xi,\tau) = f(T,\xi,\tau) + \int\limits_{T_{m+1}}^{T} (\rho c)_\tau(s,\xi,\tau)\,ds, \end{cases}$$

where the $\varkappa_j$ are from (1.10) and f is the constant term in (1.17). The first function in (5.1) is equal to the enthalpy (heat content) per unit volume and undergoes the jump $\varkappa_j$ in the passage through the phase interface. If $T(\xi, \tau)$ satisfies (1.17) without convection, then, as simple transformations show, within each phase D_j

$$(5.2)\qquad \frac{\partial H}{\partial \tau} = \sum_{j=1}^{3} \frac{\partial^2 \Lambda}{\partial \xi_j^2} - \sum_{j=1}^{3} \frac{\partial \Lambda_j}{\partial \xi_j} + f_0,$$

in which $T(\xi, \tau)$ is substituted in the functions in (5.1).

We assume that (T, Σ_j) is a classical solution of the non-stationary many-dimensional Stefan problem:

$$
(5.3)\quad \begin{cases} \rho c \dfrac{\partial T}{\partial \tau} = \sum\limits_{j=1}^{3} \dfrac{\partial}{\partial \xi_j}\left(\lambda \dfrac{\partial T}{\partial \xi_j}\right) + f, \quad f = a(\xi, \tau)T + b(\xi, \tau); \\ \alpha(T - \overset{(0)}{T}) = g \text{ on } \partial D \times (0, \tau_0); \; T(\xi, 0) = \hat{T}_0(\xi), \quad \xi \in D; \\ T = T_j, \; \sum\limits_{i=1}^{3} \left[\lambda^{+}_{\Sigma_j} \dfrac{\partial T^+}{\partial \xi_i} - \lambda^{-}_{\Sigma_j} \dfrac{\partial T^-}{\partial \xi_i}\right] \cos(N, \xi_i) - \varkappa_j \cos(N, t) = 0 \\ \hfill \text{on } \Sigma_j, \end{cases}
$$

$$j = 1, \ldots, m.$$

Let $F(\xi, \tau)$ be a smooth function defined on $\overline{\Omega} = \overline{D} \times [0, \tau_0]$ and vanishing on the upper section $\tau = \tau_0$ and for $(\xi, \tau) \in \partial D \times [0, \tau_0]$. We multiply (5.2) by $F(\xi, \tau)$ and then integrate over each phase D_j in (1.2). Using the Gauss-Ostrogradskii formula, the boundary conditions on $T(\xi, \tau)$ and $F(\xi, \tau)$, the Stefan condition, and adding up the resulting equalities, we arrive at the identity

$$
(5.4)\quad \iiint\limits_{\Omega} \Big\{F_\tau H[T(\xi, \tau), \xi, \tau] + \Delta_\xi F \cdot \Lambda[T(\xi, \tau), \xi, \tau] +
$$

$$
+ F f_0[T(\xi, \tau), \xi, \tau] + \sum_{j=1}^{3} F_{\xi_j} \Lambda_j [T(\xi, \tau), \xi, \tau]\Big\} d\xi\, d\tau +
$$

$$
+ \iint\limits_{D} F(\xi, 0) H[\hat{T}_0(\xi), \xi, 0]\, d\xi + \int_0^{\tau_0} d\tau \int\limits_{\partial D} \frac{\partial F}{\partial n} \Lambda\left[\frac{g(\xi, \tau)}{\alpha} + T_0, \xi, \tau\right] d\sigma = 0
$$

for every function $F(\xi, \eta)$ in the indicated class. We assume that λ, ρ, and c satisfy the conditions of §1.1 and $f(T, \xi, \tau) = a_j(\xi, \tau)T + b_j(\xi, \tau)$, where $a_j(\xi, \tau)$ and $b_j(\xi, \tau)$ are bounded measurable functions. Under these assumptions the last three functions in (5.1) are defined and bounded for any bounded measurable function $T = T(\xi, \tau)$, while to the first function in (5.1), which is discontinuous as T passes through the T_j, for $T = T(\xi, \tau)$ at the points $T(\xi, \tau) = T_j$ some bounded values are assigned from the interval $[H(T_j - 0, \xi, \tau), H(T_j + 0, \xi, \tau)]$. With these conventions, a bounded measurable function $T(\xi, \tau)$ is called a generalized solution of the Stefan problem if (5.3) holds on the relevant class of functions $F(\xi, \tau)$.

***Theorem* 5.1** [164]. a) (5.3) *has at most one generalized solution if* $a_j(\xi, \tau)$ *and* $b_j(\xi, \tau)$ *do not depend on* j *and are smooth functions on* $\overline{D} \times [0, \tau_0]$;

b) *under the same assumptions* a) *holds also when the* $\varkappa_j$ *depend smoothly on* (ξ, τ), $\varkappa_j(\xi, \tau) \geqslant \varkappa > 0$, *in the class of generalized solutions whose free boundaries* Σ_j: $T(\xi, \tau) = T_j$ *are of measure zero*;

c) *we assume that the boundary ∂D is sufficiently smooth and λ, ρ, and c are piecewise constant functions of T and $\lambda \in C^2(\overline{\Omega})$; also,* ρ, c, $\varkappa_j \in C^1(\overline{\Omega})$, $\hat{T}_0(\xi) \in C^1(D)$, $g(\xi, \tau) \in C^2(\partial D \times [0, \tau_0])$.

Then (5.3) *has a generalized solution, which can be obtained as the limit of solutions of a quasilinear equation of the form* (5.2) *with "smoothed" functions* (5.1). *In the case of a single space variable there is uniform convergence, consequently, the generalized solution is continuous on* Ω.

A detailed account is contained in [166] (lecture 4) for a general parabolic operator with a single space variable. The method goes back to earlier work of the same author, which is devoted to the analysis of boundary-value problems for second-order equations of elliptic or parabolic type with discontinuous coefficients [163]. Somewhat earlier the many-dimensional multi-phase non-stationary Stefan problem with thermophysical parameters depending only on the temperature was discussed in [106]. To begin with, the uniqueness of a generalized solution in the sense of an identity of the form (5.4) was established by the method of [163], and then the method of finite differences, as proposed earlier in [215], was used to prove the existence of a generalized solution for any bounded functions $g(\xi, \tau)$ and $H[\hat{T}_0(\xi)]$ in (5.3) that are continuous almost everywhere. A complete exposition of these results was given in [107] and [108].

The papers just mentioned mark the beginning of the contemporary stage in the study of the Stefan problem, which is devoted mainly to the case of many space variables. We mention here the paper [269] in which the two-phase Stefan problem is considered for general parabolic second-order linear operators $\mathcal{L}$ with sufficiently smooth coefficients within each phase. As above, the generalized solution is sought in the class of bounded measurable functions satisfying an identity of the form (5.7) with the Laplace operator replaced by the adjoint operator to the elliptic part of $\mathcal{L}$; in particular, conditions ensuring the existence and uniqueness of a generalized solution are stated; under some additional restrictions this solution satisfies an energy estimate and, consequently, has a finite norm in $W_2^{1,1}(\Omega)$; a comparison theorem also holds, under the hypotheses of which the inequalities $g_1 \geqslant g$, $\hat{T}_{1,0}(\xi) \geqslant \hat{T}_0(\xi)$ imply that $T_1(\xi, \tau) \geqslant T(\xi, \tau)$ almost everywhere in Ω. In the same range of ideas and methods the non-stationary many-dimensional quasilinear two-phase Stefan problem is treated in the recent paper [311]: existence and uniqueness theorems are proved for the generalized solution, conditions are given that ensure the continuity of its dependence on the initial data, a comparison theorem is established, etc. In the one-dimensional case these results were established in [50].

5.2. Generalized solutions in Sobolev spaces.

We consider (5.3) under the assumption that $f \equiv 0$, $g \equiv -\alpha \overset{(0)}{T}$ and the thermophysical parameters of the medium under the preceding assumptions depend only on the temperature. Then (5.3) takes the form

$$(5.5)\quad \begin{cases} \dfrac{\partial b(\Lambda)}{\partial \tau} = B(\Lambda)\Lambda_\tau = \sum_{j=1}^{3} \dfrac{\partial^2 \Lambda}{\partial \xi_j^2}, \quad B(\Lambda) = (c\rho\lambda^{-1})(T); \\ \Lambda(\xi, \tau) = 0 \quad \text{on} \quad \partial D \times (0, \tau_0); \quad \Lambda(\xi, 0) = \hat{T}_0(\xi), \quad \xi \in D; \\ \Lambda = \Lambda_j = \int_{T_{m+1}}^{T_j} \lambda(s)\,ds, \ \sum_{i=1}^{3} [\Lambda^+_{\xi_i} - \Lambda^-_{\xi_i}] \cos(N, \xi_i) - \varkappa_j \cos(N, t) = 0 \\ \hfill \text{on } \Sigma_j, \\ j = 1, \ldots, m, \end{cases}$$

where $b(\Lambda)$ is an increasing function on $[0, \infty]$; $b'(\Lambda) = B(\Lambda)$ for

$$\Lambda \in (\Lambda_{j-1}, \Lambda_j) \quad (j = 1, \ldots, m), \ \Lambda_0 = 0;$$

$$b(\Lambda_j + 0) - b(\Lambda_j - 0) = -\varkappa_j \quad (j = 1, \ldots, m);$$

and $b(0) = 0$. If now $\eta(\xi, \tau)$ is an arbitrary element of $\overset{0}{W}{}_2^{1,1}(\Omega)$ (the closure of functions of compact support in Ω), then instead of (5.4) we obtain from (5.5) that

$$(5.6)\quad \iiint_\Omega \{-b(\Lambda)\eta_\tau + \operatorname{grad}_\xi \Lambda \cdot \operatorname{grad}_\xi \eta\}\, d\xi\, d\tau = 0 \ \text{ for all } \eta \in \overset{0}{W}{}_2^{1,1}(\Omega),$$

which must necessarily be satisfied by the classical solutions of the problem (5.5) (provided that it exists). We supplement the definition of $b(\Lambda)$ at the points Λ_j by arbitrary values in the interval $[b(\Lambda_j - 0), b(\Lambda_j + 0)]$ and we take the identity (5.6) and the initial condition in (5.5) as definition of the generalized solution of (5.5) in the class of bounded functions Λ from $\overset{0}{W}{}_2^{1,1}(\Omega)$ (that vanish on $\partial D \times (0, \tau_0)$). This definition in contrast to the one in §5.1, guarantees for the generalized solution of the Stefan problem an a priori degree of smoothness of elements in the Sobolev space $W_2^{1,1}(\Omega)$.

Theorem 5.2 ([128], Ch. V, §9). *Suppose that $\partial D \in H^{2+\beta}$ and that $B(\Lambda)$ is positive and piecewise smooth. Then* (5.5) *has one and only one generalized solution in the class of bounded functions from $W_2^{1,1}(\Omega)$ that vanish on $\partial D \times (0, \tau_0)$, provided that the initial distribution $\hat{T}_0(\xi)$, supplemented by zero on $\partial D \times (0, \tau_0)$, is the trace of a function belonging to $H^{2+\beta, 1+\beta/2}(\overline{Q})$, $\beta > 0$.*

The proof uses a method that was worked out earlier in connection with the general diffraction problem for parabolic equations (for results and a bibliography, see [128], Ch. III, §13); the uniqueness depends on an existence theorem for the generalized solution of a certain auxiliary mixed problem for a parabolic equation with bounded measurable coefficients; the existence uses the method of smoothing the coefficients.

By choosing (5.6) the $\eta(\xi, \tau)$ from the class $\overset{0}{C^\infty}(\Omega)$ we can verify that the energy equation in (5.5) can be treated also in the sense of distributions: we arrive at the following statement: for any element f from the dual space $D'(\Omega)$,

$$(5.7)\quad \begin{cases} \frac{\partial}{\partial\tau} b(\Lambda) - \sum_{i=1}^{3} \frac{\partial^2\Lambda}{\partial\xi_j^2} = f(\xi, \tau) \text{ in the sense of } D'(\Omega); \\ \alpha(T - \overset{(0)}{T}) = g \text{ on } \partial D \times (0, \tau_0);\ b(\Lambda)(\xi, 0) = b[\Lambda(\hat{T}_0)](\xi),\quad \xi \in D. \end{cases}$$

Theorem 5.3 [220]. *We assume that ∂D consists of two disjoint infinitely differentiable manifolds and that for $m = 1$ (the unique critical value of the temperature) $g/\alpha + \overset{(0)}{T} < T_1$ for the interior manifold and $g/\alpha + \overset{(0)}{T} > T_1$ for the exterior one. Suppose that $\hat{T}_0(\xi) \neq T_1$ almost everywhere and that there is a function $\widetilde{\Lambda}(\xi_0, \tau) \in W_2^1(Q)$ such that $\widetilde{\Lambda} = b[\Lambda(g/\alpha + \overset{(0)}{T})]$ on $\partial D \times (0, \tau_0)$ and $\widetilde{\Lambda}(\xi, 0) = b[\Lambda(\hat{T}_0(\xi))]$ on D. Then (5.7) has a unique solution $\Lambda(\xi, \tau)$ in the class $L_2(0, \tau_0; W_2^1(D))$ for any $f(\xi, \tau)$ from $L_2(\Omega)$.*

The proof is also given in the book [130] (Ch. 2, §3, 3.3). It is based on the general theory of monotone operators.

5.3. Numerical methods for the analysis of the many-dimensional problem. The enthalpic form of the energy equation and the smoothing of its coefficients are used in [187] as a basis for an effective numerical method of solving many-dimensional non-stationary quasilinear Stefan problems. Assuming that $T(\xi, \tau)$ is an infinitely differentiable function of (ξ, τ) and that $\text{grad}_{(\xi,\tau)} T(\xi, \tau) \neq 0$ along Σ_j: $T(\xi, \tau) = T_j$ $(j = 1, \dots, m)$ and differentiating the composite function $\eta[T(\xi, \tau) - T_j]$ with respect to τ in the sense of distributions, from the first formula in (5.1) we obtain

$$(5.8)\quad \frac{\partial}{\partial\tau} H[T(\xi, \tau), \xi, \tau] =$$

$$= \int_{T_{m+1}}^{T} (\rho c)_\tau (s, \xi, \tau)\, ds + \left\{(\rho c)[T(\xi, \tau), \xi, \tau] + \sum_{j=1}^{m} \varkappa_j \delta[T(\xi, \tau) - T_j]\right\} \frac{\partial T}{\partial\tau},$$

where $\delta(T - T_j)$ is the Dirac distribution on Σ_j. When we then take the energy equation in the form (5.2), we derive from this and from (5.1) the following equation for the temperature:

$$(5.9)\quad [(\rho c)(T, \xi, \tau) + \sum_{j=1}^{m} \varkappa_j \delta(T - T_j)]\, T_\tau = \text{div}_\xi (\lambda\, \text{grad}_\xi T) + f.$$

This leads to the equation of heat conduction in (5.3) inside each phase. As was shown in [187], under the assumption that the Σ_j are smooth and $T(\xi, \tau)$ is piecewise smooth, (5.9) also implies the Stefan condition on Σ_j, where also $T(\xi, \tau) = T_j$. Consequently, it is sufficient to adjoin to (5.9) the initial and boundary conditions on T (second line in (5.3)). (5.9) has the following intuitive physical meaning: the terms $\varkappa_j \delta(T = T_j)$ represent the condensed heat capacity on the phase interface, that occurs in the energy equation additvely along with the usual specific heat capacity ρc (per unit volume).

The smoothing method now consists in replacing the delta-function $\delta(T-T_j)$ by an approximately delta-shaped function $\delta(T-T_j, \Delta)$, where Δ is the length of the subinterval on which this function is different from zero. The same result is obtained if in the first formula (5.1) the discontinuous function $\eta(T-T_j)$ is replaced by a smooth $\eta(T-T_j, \Lambda)$ with preservation of the condition $\eta'(\xi, \Delta) = \delta(\xi, \Lambda)$. For simplicity we assume that the thermophysical parameters depend only on the temperature, and we introduce the "effective" heat capacity $(\widetilde{\rho c})(T) = (\rho c)(T) + \sum_{j=1}^{m} \varkappa_j \delta(T - T_j, \Delta)$ for sufficiently small $\Delta > 0$, assuming also that the integral of $\delta(T-T_j, \Delta)$ with respect to T on the set $(-\infty, \infty)$ is 1 (conservation of enthalpy). The functions $f(T)$ and $\lambda(T)$ are smoothed similarly. We then apply the finite difference method of the numerical solution to the resulting mixed problem for the quasilinear equation of heat conduction. A comparison with exact model solutions shows that the accuracy of this method is satisfactory. Another advantage of it is that the process of smoothing is independent of the number of space variables and the results depend weakly on the specific choice of $\delta(\xi, \Delta)$.

Simultaneously and independently, a similar approach was proposed in [58] to the numerical analysis of the many-dimensional multi-phase non-stationary quasilinear Stefan problem with a proof of convergence in $L_2(\Omega)$. For the Stefan problem in which the condition (1.9) of isothermality is replaced by more general conditions of the form (1.12) a numerical method of analysis was proposed in [189] (see also [53], [54], [60], [175], and the bibliographies in them).

A justification of the method of smoothing (in the space **C**) in the theory of mixed problems for quasilinear equations of heat conduction with piecewise smooth coefficients is contained in [80] (see also [81]; a detailed account is given in [86]). Here conditions are stated under which in the many-dimensional case, the solution of the problem with smoothed coefficients converges in the uniform metric to the unique solution of the original problem, and the latter solution has a piecewise continuous Hölder gradient with respect to the space variable ξ for almost all values of τ.

5.4. The method of parabolic variational inequalities.

We consider the one-phase non-stationary Stefan problem in a domain D whose boundary is the union of three parts $\Gamma_1', \Gamma_2', \Gamma_3'$, where $\Gamma_1' \cap \Gamma_3' = \emptyset$ and mes $\Gamma_1' > 0$. We assume that the thermophysical parameters of the medium are constant, that $T_1 = 0$ on Γ_1', that (1.8) holds for $\chi_1 = \chi_2 = 1$ that $g \equiv -\alpha \overset{(0)}{T}$ or that $T = \overset{(0)}{T}$ on Γ_2': the homogeneous Neumann condition ($\chi_1 = 1, \chi_2 = 0, g \equiv 0$) and the Dirichelt's zero condition ($T = 0$) on Γ_3'.

We also assume that the initial temperature distribution is identically zero ($\hat{T}_0(\xi) \equiv 0$). The temperature field $T(\xi, \tau)$ in the liquid phase $D_1 = \{(\xi, \tau): T(\xi, \tau) > 0\}$ is non-trivial if $\alpha > 0$. Now we introduce the notation:

$$(5.10)\quad \begin{cases} u(\xi, \tau) = \int\limits_0^\tau T(\xi, \sigma)\chi_{D_1}(\xi, \sigma)\, d\sigma, \quad (\xi, \tau) \in \Omega = D \times (0, \tau_0); \\ K = \{v \in W_2^1(D), \quad v \geqslant 0 \text{ in } D, \quad v = 0 \text{ on } \Gamma_3'\}; \\ K_1(\tau) = \{v(\xi, \tau) \in W_2^1(D), \quad v = \overset{(0)}{T}\tau \text{ on } \Gamma_1', \quad v \geqslant 0 \text{ on } D\} \end{cases}$$

$$\text{for all } \tau \in (0, \tau_0).$$

It is not hard to verify that $u(\xi, \tau)$ satisfies the relations

$$(5.11)\quad \begin{cases} u_\tau(\xi, \tau) - a_+^2 \Delta_\xi u(\xi, \tau) = -\varkappa_1/\lambda^+ \text{ in } D_1; \\ \lambda^+ u_n + \alpha(u - \overset{(0)}{T}\tau) = 0 \text{ or } u = \overset{(0)}{T}\tau \text{ on } \Gamma_1'; \; u_n = 0 \text{ on } \Gamma_2'; \\ u(\xi, 0) = 0 \text{ on } D; \quad u = 0, \quad \operatorname{grad}_\xi u = 0 \text{ on } \Sigma_1; \\ u(\xi, \tau) \in K \text{ or } K_1(\tau) \text{ for all } \tau \in (0, \tau_0); \; u(\xi, \tau) \equiv 0 \text{ outside } D_1, \end{cases}$$

which are equivalent to the Stefan problem in question.

***Theorem* 5.4** [260]. a) *If* (T, Σ_1) *is the classical solution of the one-phase non-stationary Stefan problem, then the function* $u(\xi, \tau)$ *in* (5.10) *satisfies the relations* (5.11);

b) *every such solution of* (5.11) *satisfies the variational inequality*

$$(5.12)\quad (u_\tau, v - u)_{L_2(D)} + a_+^2(\operatorname{grad}_\xi u, \operatorname{grad}_\xi(v - u))_{L_2(D)} + \frac{d}{\lambda^+}\int\limits_{\Gamma_1'} (u - \overset{(0)}{T}\tau)\, d\Gamma_1' \geqslant (-\varkappa_1(\lambda^+, v - u))_{L_2(D)} \quad \textit{for all } v \in K;$$

$$u(\xi, \tau) \in K \textit{ for all } \tau \in (0, \tau_0); \quad u(\xi, 0) = 0, \quad \xi \in D;$$

or, respectively, the inequality

$$(5.13)\quad (u_\tau, v - u)_{L_2(D)} + a_+^2(\operatorname{grad}_\xi u, \operatorname{grad}_\xi(v - u))_{L_2(D)} \geqslant (-\varkappa_1/\lambda^+, v - u)_{L_2(D)}$$

for all $v \in K_1(\tau)$; $u(\xi, \tau) \in K_1(\tau)$ *for all* $\tau \in (0, \tau_0)$; $u(\xi, 0) = 0$, $\xi \in D$;

c) *there is one and only one solution* u_α *of* (5.12) *in the class* $u_{\alpha\tau} \in L_2(0, \tau_0, \overset{0}{W}{}_2^1(D)) \cap L_\infty(0, \tau_0; W_2^1(D))$, *where* $W_2^1(D) = \{v \in W_2^1(D), \; v = 0 \textit{ on } \Gamma_3'\}$;

d)

$$(5.14)\quad \begin{cases} 0 \leqslant u_{\alpha_1}(\xi, \tau) \leqslant u_{\alpha_2}(\xi, \tau) \leqslant H\tau \textit{ for all } 0 \leqslant \alpha_1 \leqslant \alpha_2; \\ 0 \leqslant u_{\alpha\tau}(\xi, \tau) \leqslant H \textit{ for all } \tau \in (0, \tau_0), \end{cases}$$

where $H(\xi)$ is uniquely determined as the solution of the stationary problem $\Delta_\xi H = 0$ in D, $H = \overset{(0)}{T}$ on Γ'_1, $H_n = 0$ on Γ'_2, and $H = 0$ on Γ'_3;

e) *there is one and only one solution of* (5.13) *in the class* $u \in L_2(0, \tau_0; \overset{0}{W}{}_2^1(D)) \cap L_\infty(0, \tau_0; W_2^1(D))$, $u_\tau \in L_2(0, \tau_0; W_2^1(D))$; *if* $\alpha \to +\infty$, *then* $u_\alpha \to u$ *strongly in* $L_2(0, \tau_0; W_2^1(D))$ *and weakly in* $L_2(0, \tau_0; \overset{0}{W}{}_2^1(D))$, *and* $u_{\alpha\tau} \to u_{\tau}$* *weakly in* $L_\infty(D \times (0, \tau_0))$.

The introduction of a new unknown by the first formula in (5.10) goes back to [217], which is devoted to the problem with a free boundary in the theory of filtration. The proof of Theorem 5.4 is based on the penalty method ([130], Ch. 3, §§5, 6). The estimates (5.14) are proved by application of the maximum principle, and then d) is used in the proof. A detailed proof can be found in the book [103] (Appendix, §3). The solutions of the variational inequalities (5.12) and (5.13) obtained in Theorem 5.4 are taken as generalized ("weak") solutions of the original Stefan problem.

The two-phase non-stationary Stefan problems also admit a generalized statement in terms of certain variational inequalities [261], [263]. However, these inequalities belong to a class for which a theory has not yet been worked out. In the quasilinear case the reduction to a variational inequality in the one-phase Stefan problem was realized in [279]. An evolution problem of the form (5.3) for an elliptic energy equation ($c \equiv 0$) is also studied [247]. In conclusion we mention the recent paper [338], which is devoted to the degenerate quasilinear many-dimensional two-phase non-stationary Stefan problem; to begin with, under certain minimal assumptions on the enthalpy and the remaining initial data the existence of a generalized solution in the class T, $H(T) \in L_\infty(0, \tau_0; L_2(D))$ is established by the smoothing method (the thermophysical parameters of the medium depend only on the temperature) and then also in the class of weak solutions cases are studied when the latent heat of crystallization vanishes identically or the specific heat capacity can vanish ($c(T) \geqslant 0$).

5.5. Improving smoothness; the classical nature of generalized solutions. Under certain additional restrictions on the initial data the generalized solutions of the Stefan problem constructed above have an increased smoothness, in particular, properties of a classical solution. For a one-phase Stefan problem in the case of $n \geqslant 2$ space arguments the classical character of generalized solutions was established step-by-step by overcoming substantial mathematical difficulties.

In this direction we mention first of all the paper [276] in which the method of variational inequalities similar to the one developed in the preceding subsection was used to establish existence and uniqueness theorems for generalized solutions and to prescribe additional conditions that guarantee a certain original degree of smoothness of the free boundary.

It is assumed that the boundary ∂D of the domain D consists of two smooth components: a closed manifold Γ_1 and the boundary Γ_2 of a ball of sufficiently large radius R. At the initial moment of time the non-trivial (liquid) phase D_1 occupies the domain $D_{1,0} = \{\xi: \xi \in D, \ T_0(\xi) > 0\}$, $m = 1$, $T_1 = 0$, of positive measure, bounded by Γ_1 and a manifold $\Sigma_{1,0}$ that is smooth and disjoint from Γ_1 (see the notation (1.14)). We assume that (1.8) holds on $\Gamma_1 \times [0, \tau_0]$ for $\chi_1 = 0$, $\chi_2 = 1$ and $g(\xi, \tau) \in C^2(\Gamma_1 \times [0, \tau_0])$, while $\hat{T}_0(\xi) \in C^2(\bar{D}_{1,0})$ is positive on $\bar{D}_{1,0} \setminus \tilde{\Sigma}_{1,0}$. (In Theorem 5.4 we had $\hat{T}_0(\xi) \equiv 0$, $g(\xi, \tau) = 0$.) The same substitution as in (5.10) reduces the Stefan problem in question to the following:

$$
(5.15)\quad \begin{cases}
(u_\tau - a_+^2 \Delta_\xi u)(v-u) \geqslant f(v-u) \ \text{a.e. in } \Omega; \\
\text{for all } v \in W_2^1(D),\ v \geqslant 0 \text{ in } D,\ f = \hat{T}_0 \chi_{D_{1,0}} - \varkappa_1 (1 - \chi_{D_{1,0}})/\lambda^+; \\
u(\xi, \tau) = \psi \equiv \dfrac{1}{\alpha} \displaystyle\int_0^\tau g(\xi, \sigma)\, d\sigma + \overset{(0)}{T}\tau \ \text{ on } \Gamma_1 \times [0, \tau_0], \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(\xi, \tau) \equiv 0 \text{ on } \Gamma_2; \\
u(\xi, 0) \equiv 0, \quad \xi \in D.
\end{cases}
$$

In contrast to (5.12) and (5.13), here the inequality for $u(\xi, \tau)$ is written in pointwise form. A solution of (5.15) in one class or another of $L_p(0, \tau_0;\ W_0^2(D))$ is called a generalized solution of the original non-stationary one-phase Stefan problem. Such solutions were constructed ([276], §2) by the penalty method for an appropriate choice of the penalty function β_ε, namely: $\beta_\varepsilon(t) \in C^\infty(\mathbb{R}^1)$, $\beta_\varepsilon(t) \equiv 0$ for $t \geqslant \xi > 0$, $\beta_\varepsilon(0) = -1$, $\beta_\varepsilon'(t) > 0$, $\beta_\varepsilon''(t) \leqslant 0$ for $t \in (-\infty, \varepsilon)$. Then in §§3-4 of this paper some preliminary smoothness properties of the temperature and the free boundary are established. Before giving the exaxt statements we introduce some definitions and notation. Let $\Gamma_1 \in H^{2+\beta}$, $g(\xi, \tau) \in H^{2+\beta}(\Gamma_1 \times [0, \tau_0])$, $\hat{T}_0(\xi) \in H^{2+\beta}(D)$ and let $u(\xi, \tau)$ be a solution of (5.15) with a bounded non-negative derivative $u_\tau (0 \leqslant u_\tau \leqslant K)$, where the bound K depends only on the upper bounds of $|f|$ and $|\psi|$. Applying the Schauder estimates to $T(\xi, \tau) = u_\tau(\xi, \tau)$ near $\Gamma_1 \times [0, \tau_0]$ (see, for example, [128], Ch. IV, §5), we obtain the inequality $|A_{\Gamma_1} T| \leqslant Q$, where A_{Γ_1} is the tangential component of the Laplace operator on Γ_1 and the constant Q depends only on the norms of the initial data in the indicated spaces. Suppose that Γ_1 is star-like with respect to the origin and that $\rho = r(\theta)$ is its representation in polar coordinates.

***Theorem* 5.5** [276]. a) (5.15) *has a unique solution* $u(\xi, \tau)$ *satisfying the relations* $u \in L_\infty(0, \tau_0;\ W_p^2(D))$, $1 \leqslant p < \infty$, $u_\tau \in L_\infty(0, \tau_0;\ L_\infty(D))$, $0 \leqslant u_\tau(\xi, \tau) \leqslant K$ *a.e. in* $\Omega = D \times (0, \tau_0)$, *where* K *depends only on the upper bounds of* $|f|$ *and* $|\psi|$. *If* u_ε *denotes the classical solution of the problem with penalty, then* $u_\varepsilon \to u$ *weakly in* $W_p'(\Omega)$, $1 < p < \infty$, *weakly in* $W_p^2(D)$, $1 < p < \infty$, *for all* $\tau \in (0, \tau_0)$, *and consequently, uniformly in* Ω *and* *also* $u_{\varepsilon\xi_i} \to u_{\xi_i}$ *uniformly in D for all* $\tau \in (0, \tau_0)$. *If* $D_1(\tau) = \{\xi: u(\xi, \tau) > 0\}$, *then* $D_1(\tau) \subset D_1(\tau')$ *for* $\tau < \tau'$, *where* $\tau, \tau' \in [0, \tau_0]$, *and all the* $D_1(\tau)$ *are contained inside a ball of sufficiently large radius with boundary sphere* Γ_2; *finally,* $u(\xi, \tau) = 0$ *and* $\text{grad}_\xi u(\xi, \tau) = 0$ *on* $\Sigma_1 = \partial D_1 \cap \Omega$, *and in* D_1 *the equation in* (5.11) *is satisfied near* Σ_1.

b) *Let* $\hat{T}_0(\xi) = 0$ *on* $\tilde{\Sigma}_{1,0}$. *Then the function* $T(\xi, \tau) = u_\tau(\xi, \tau)$ *satisfies the energy inequality, that is,*

$$\sum_j \int_D u^2_{\tau\xi_j}(\xi, \tau)\, d\xi + \iint_\Omega u^2_{\tau\tau}(\xi, \tau)\, d\xi\, d\tau \leqslant C, \tag{5.16}$$

where C does not depend on $\tau \in (0, \tau_0)$.

c) *As before, suppose that* $\hat{T}_0(\xi) > 0$ *in* $D_{1,0}$ *and* $\hat{T}_0(\xi) = 0$ *on* $\tilde{\Sigma}_{1,0}$ *and*

$$\hat{T}_0(\xi) = \psi_\tau(\xi, 0) = \overset{(0)}{T} + g(\xi, 0)/\alpha, \ \xi \in \Gamma_1,$$

and

$$(\rho^2 \hat{T}_0(\xi))_\rho < 0 \ \ in \ \ D_{1,0}, \ \ \psi_\tau(\xi, \tau) - \psi_\tau(\xi, 0) - r^{-2}(\theta) Q\tau > 0 \ \ on \ \ \Gamma_1 \times (0, \tau_0), \tag{5.17}$$

where (ρ, θ) *are polar coordinates. Then the free boundary* $\Sigma_{1,\tau} = \Sigma_1 \times \{\tau\}$ *admits the representation* $\rho = \rho^*(\theta, \tau)$, $0 < \tau \leqslant \tau_0$, *in which* $\rho^*(\theta, \tau)$ *is a Lipschitz function in* θ *(with a Lipschitz constant independent of* τ*) and continuous in* τ; *moreover,* $u_\rho \leqslant 0$ *in* D_1.

d) *A function* $u(\xi, \tau) \in L_2(0, \tau_0; W^2_2(D))$ *satisfies* (5.15) *if and only if* $T(\xi, \tau) = u_\tau(\xi, \tau)$ *satisfies the corresponding integral identity of the form* (5.4).

When Γ_1 is the sphere $\rho = r > 0$, the second inequality in (5.17) is replaced by the more precise $\psi_\tau(\xi, \tau) - \psi_\tau(\xi, 0) - r^{-2} A\psi(\xi, \tau) > 0$ on $\Gamma_1 \times (0, \tau_0)$, where A is the spherical component of the Laplace operator. These conditions mean that the heat-generating surface Γ_1 must be "sufficiently planar" with respect to the growth of $\psi(\xi, \tau)$ in τ. Combined with the first condition in (5.17), they guarantee that $(\rho^2 u_\rho)_\rho \geqslant 0$ on $\Gamma_1 \times (0, \tau_0)$ (in the three-dimensional case) and consequently, $u \leqslant 0$ in D_1. See also [274], Ch. 1, §9.

The next step on the way to establishing the classical nature of a generalized solution was made in [235] under the additional assumption that

$$\sup_\Omega \text{vrai}\, |u_{\xi_i\xi_j}(\xi, \tau)| < +\infty \tag{5.18}$$

(the elliptic case is studied in [234] and complete proofs are given in [236]; see also [274], Ch. 2, §9).

***Theorem* 5.6** [235]. *Suppose that in some neighbourhood $W \subset D_1$ of $(\xi, \tau) \in \Sigma_1$ the function $u(\xi, \tau)$ in Theorem* 5.5, a) *also satisfies the local condition* (5.18). *If at a point $\xi_0 \in \Sigma_{1,\tau_0} \cap \partial W$, $\Sigma_{1,\tau} = \Sigma_1 \times \{\tau\}$, the complementary set of $(\Omega \setminus D_1) \times \{\tau\}$ has positive Lebesgue density, then in some neighbourhood of (ξ_0, τ_0) the free boundary Σ_1 is a smooth surface in Ω, and all second derivatives of $u(\xi, \tau)$ are continuous up to the indicated neighbourhood on Σ_1.*

Under the hypotheses of this theorem, the generalized solution in Theorem 5.5, a) is locally a classical solution of the one-phase many-dimensional non-stationary Stefan problem. But under the hypotheses of Theorem 5.5, c) every point of the surface $\Sigma_{1,\tau}$ has positive Lebesgue density with respect to the complement, since $\rho^*(\theta, \tau)$ satisfies a Lipschitz condition in θ, consequently, Theorem 5.6 is applicable globally to the entire free surface Σ_1.

The final step in the positive solution of the problem under consideration was made in [295]. Here to begin with (§1) a sketch of a proof of (5.18) is given by means of a method of the paper [221], next it is proved that a similar estimate holds for the second derivatives $u_{\tau\xi_i}(\xi, \tau)$ (§2) and then by using an elliptic analogue of Theorem 5.6 in [234] and some other results the assertion of Theorem 5.6 is proved for the function $u(\xi, \tau)$ in Theorem 5.5. Before stating rigorously this and other results of [295] we recall that a function $a(\xi, \tau) \in C^\infty$ is analytic in ξ and belongs to the second Jevret class in τ if

$$|\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}\partial_{\xi_3}^{\alpha_3}\partial_\tau^j a| \leqslant CM^{2j+|\alpha|}(2j + |\alpha|)!, \quad \alpha = \alpha_1 + \alpha_2 + \alpha_3,$$

where M and C are positive constants.

***Theorem* 5.7** [295]. a) *Let $u(\xi, \tau)$ be a function satisfying the first three assertions of Theorem 5.5 for $\varkappa_1(\xi, \tau) \in C^\infty(\overline{\Omega})$. (The estimate* (5.18) *is a consequence of them.) Then in the domain $\Omega = D \times (0, \tau_0)$ the free boundary Σ_1 is a C^1-manifold, $u_{\xi_i}(\xi, \tau) \in C^1(D_1 \cup \Sigma_1)$ $(i = 1, 2, 3)$, and the function g_1 representing $\Sigma_{1,\tau} = \Sigma_1 \times \{\tau\}$ locally has a positive derivative with respect to τ ($\Sigma_{1,\tau}$ is strictly increasing in τ); the pair (u, Σ_1) is a classical solution of the one-phase Stefan problem.*

b) *Under the same assumptions, $u(\xi, \tau) \in C^\infty(D_1 \cup \Sigma_1)$. If $\varkappa_1(\xi, \tau)$ is analytic in ξ and belongs to the second Jevret class, then so do $u(\xi, \tau)$ and $g_1(\xi, \tau)$ in some neighbourhood of every point of $D_1 \cup \Sigma_1$.*

c) *Let Γ_1 be a sphere of fixed radius and let $\psi(\xi, \tau) > 0$ be analytic on $\Gamma_1 \times (0, \tau_0)$. Then the free boundary Σ_1 is an analytic hypersurface in Ω.*

The infinite differentiability of the free boundary in the one-dimensional case was established earlier in [232] and [321], and its analyticity in the case of one space variable was proved in [272]. For the one-dimensional Stefan problem, as stated in [73], the analyticity of the free boundary was established in [181] by the method of extending integral equations equivalent to the original problem to the domain of complex values of the time variable. An account of the main results of this subsection in the one-dimensional case can be found in the book [298], Ch. VIII.

5.6. The one-phase quasilinear contact problem; the pseudoparabolic inequality.

Let D denote a cylindrical domain in the variables $\xi = (\xi_1, \xi_2, \dots, \xi_n)$, $n \geqslant 2$, situated in the layer $0 < \xi_n < +1$ and let Γ_1 and Γ_3 be the lower and upper bases of D, respectively; let $\Gamma_2 = \partial D \setminus \{\Gamma_1 \cup \Gamma_3\}$. We consider the following setting:

$$(5.19)\quad \begin{cases} c(T)\rho(T)\dfrac{\partial T}{\partial\tau} - \mathrm{div}_\xi[\lambda(T)\,\mathrm{grad}_\xi T] = 0 \text{ in } D_1; \\ T = T_0 + g(\xi, \tau)/\alpha \text{ on } \partial D_1 \cap \{\partial D \times (0, \tau_0)\}; \\ T(\xi, 0) = \hat{T}_0(\xi), \quad \xi \in D_{1,0}; \\ T = T_1, \ \lambda^- \sum\limits_{i=1}^n T_{\xi_i}\cos(N, \xi_i) + \varkappa_1 \cos(N, \tau) = 0 \text{ on } \Sigma_1, \end{cases}$$

in which N has the same meaning as in (4.11). Let

$$(5.20)\quad \begin{cases} \hat{T}_0 \in W_2^1(D), \ T_0 < \hat{T}_0(\xi) < T_1 \text{ in } D_{1,0}, \quad \hat{T}(\xi) = T_1 \text{ in } D \setminus D_{1,0}; \\ g \in L_2(0, \tau_0; W_2^{1/2}(\partial D)), \ g_\tau \in L_2(0, \tau_0; W_2^{-1/2}(\partial D)), \\ \qquad\qquad 0 \leqslant g \leqslant \alpha(T_1 - T_0); \\ \hat{T} = T_0 + g/\alpha \text{ for } \tau = 0, \quad \xi \in \partial D; \quad \lambda, c, \rho \in C(\mathbb{R}), \\ 0 < c_1 \leqslant \lambda, \rho, c \leqslant c_2 < +\infty, \ c_1, c_2 = \mathrm{const}. \end{cases}$$

Let $g_0(\xi, \tau)$ be a harmonic function in $\xi \in D$ taking values $T_0 + g(\xi, \tau)/\alpha$ on ∂D for $0 < \tau < \tau_0$, and let

$$(5.21)\quad \begin{cases} Q(z = \int\limits_0^z \lambda(s)\,ds \Big/ \int\limits_0^1 \lambda(s)\,ds, \quad B(z) = \int\limits_0^z \dfrac{(\rho c)(Q^{-1}(s))}{\lambda(Q^{-1}(s))}\,ds, \\ f = B[Q(\hat{T}_0)] + (\chi_{D_{1,0}} - 1)\,\varkappa_1\lambda^{-1}(T_1) \Big/ \int\limits_0^1 \lambda(s)\,ds, \\ w = F_1(T)(\xi, \tau) = \int\limits_0^\tau \{g_0(x, \eta) - Q[T(\xi, \eta)]\}\,d\eta, \\ \qquad\qquad T \equiv T_1 \text{ in } \Omega \setminus D_1, \\ K = \{v \in L_2(0, \tau_0; W_2^1(D)) : v \geqslant g_0 - 1 \text{ a.e. in } \Omega\}. \end{cases}$$

Here the last but one formula plays the role of a change of the unknown in (5.10). If (T, Σ_1) is a sufficiently smooth solution of (5.19), then w satisfies the variational pseudoparabolic inequality

$$\int_0^{\tau_0} (B(g-w_t)-\Delta_x w-f,\ v-w_t)_{L_2(D)}\,d\tau \geqslant 0, \tag{5.22}$$
$$\text{for all } v \in K;\quad w_t \in K,\quad w(x, 0)=0.$$

***Theorem* 5.8** [197]. a) *If* (T, Σ) *is a classical solution of* (5.19), *then w in* (5.21) *is a solution of the pseudoparabolic inequality* (5.22);

b) *under the assumptions* (5.20), *the variational inequality* (5.22) *has one and only one solution w; to it there corresponds a generalized solution* (T, Σ_1) *of* (5.19), *where* $T = \hat{F}_1^{-1}(w)$, $D_1 = \{(\xi, \tau) \in \Omega,\ T < T_1\}$, $T \in W_2^1(\Omega)$, $T = \hat{T}_0+g/\alpha$, *and* $T = \hat{T}_0$ *for* $\tau = 0$ *in the sense of the theory of traces, and* D_1 *is measurable (a brief account is also given in* [199]);

c) *we assume that in addition to the assumptions* (5.20) *the following are also satisfied:* $n = 2$, *the initial poistion* $\Sigma_{1,0}$ *of the free boundary* Σ_1 *in* $D^+ = D \cap \{x_1 > 0\}$ *is an increasing* C^1*-curve not intersecting the lower or upper base of D;* $\hat{T}_0 \in W_\infty^1(D^+)$, $T_{\xi_1} \leqslant 0$, $T_{\xi_2} \geqslant 0$ *a.e. in* D^+; $g \in L_\infty(0, \tau_0;\ W_2^2(\Gamma_2))$, $g_\tau \in L_2(\Gamma_2 \times (0, \tau_0))$, *where* Γ_2 *is the vertical portion of the boundary* ∂D; *the integral of* $(g_\tau - g_{\xi_2\xi_2})$ *on* $(0, \tau)$ *is non-positive a.e. on* $\Gamma_2 \times (0, \tau_0)$; $T = \hat{T}_0+g/\alpha$ *on* Γ_2 *and* $\hat{T}_0 = 0$ *on the lower base of* Γ_1. *Then* (5.19) *with coefficients* λ, ρ, *and c independent of T, has a unique classical solution, and* $T \in C^\infty(D_1 \cup \Sigma_1)$, *T and* Σ_1 *are analytic in* ξ *and belong to the second Jevret class in* τ.

Analogous results have been obtained for the case when instead of the Dirichlet conditions boundary-value conditions of the second and third kind are considered. The difference of c) from the corresponding assertions of the preceding sections consists in the fact that another geometry of D is considered and the free boundary intersects a fixed part of the boundary. b) can be proved by the method of regularization and is an improvement of the corresponding result in [280]. We also mention that similar results are contained in [197] also for the case when the coefficients of the differential equations depend on the point $\xi \in D$ (see §6.2). An analogous problem for the linear equation of heat conduction was considered earlier in [198].

5.7. The implicit function method.

In [284] this method was used to prove the classical solubility of the one-phase many-dimensional non-stationary Stefan problem in the following setting:

$$\left\{\begin{array}{l} T_\tau - a^2\Delta_\xi T = 0, \quad (\xi, \tau) \in D_1; \\ T(\xi, 0) = \hat{T}_0(\xi), \quad \xi \in D_{1,0}; \quad T(\xi, \tau) = T_0 + \frac{1}{\alpha} g(\xi, \tau) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \Gamma_1 \times [0, \tau_0]; \\ T = T_1, \quad \Phi_{\rho\tau} - (\lambda^+/\varkappa_1)\,\mathrm{grad}_\xi T \cdot \mathrm{grad}_\xi \Phi_\rho = 0 \text{ on } \Sigma_1. \end{array}\right. \tag{5.23}$$

As in §5.5, we assume that the boundary $\partial D_{1,0}$ of the initial domain of disposition of the solid phase $D_{1,0}$ consists of two connected disjoint manifolds Γ_1 and Γ_2; $\Gamma_1, \Gamma_2 \in C^\infty$. Under certain conditions on $\hat{T}_0$ and g the free boundary $\bar{\Sigma}_1$ is diffeomorphic to $\Gamma_1 \times [0, \tau_0]$ for sufficiently small $\tau_0 > 0$ and can therefore be described by a distance function ρ of Γ_2 in the space of the variables ξ. Correspondingly, the domain $D_1 \subset \Omega$ is diffeomorphic to $D_{1,0} \times (0, \tau_0)$. A sufficiently narrow strip N_0 around Γ_2 admits a local coordinate system of class C^∞ according to the rule $\xi \to (\omega(\xi), \lambda(\xi))$, where $\omega(\xi) \in \Gamma_2$ is the projection of ξ onto Γ_2 along the normal n_ω and $\lambda(\xi)$ is the distance from Γ_2 along n_ω. Now the function $\Phi_\rho(\xi, \tau)$ in the Stefan condition (5.23) can be defined as follows: $\Phi_0(\xi, \tau) = \lambda(\xi) - \rho[\omega(\xi), \tau]$, where $\rho \in C^\infty(\Gamma_2 \times [0, \tau_0])$. Consequently,

$$\bar{\Sigma}_1 = \{(\xi, \tau) \in N_0 \times [0, \tau_0]: \Phi_\rho(\xi, \tau) = 0\},$$

and the solution of (5.23) is a pair (ρ, T).

Let $C^{(r)}(\bar{D}) = H^r(\bar{D})$ be the ordinary Hölder space (see, for example, [128], Ch. I, §1). We also consider the weighted Hölder space $C^{(r)}(\bar{D} \times [0, \tau_0])$, $r \geqslant 0$, whose elements have derivatives $\partial_\xi^\alpha \partial_\tau^a f(\xi, \tau) \in C(\bar{D} \times [0, \tau_0])$ for $|\alpha| + 2a \leqslant [r]$ ($[r]$ is the integral part of r) and have finite norm $|f|_{(r)}$; to define it for any integer $i \geqslant 0$ we denote by $\langle f \rangle_{(i)}$ the sum of the norms of the derivatives of order $|\alpha| + 2a = i$ in $C(\bar{D} \times [0, \tau_0])$ and (for $i \geqslant 1$) the Hölder constants of order $1/2$ with respect to τ of the derivatives of order $|\alpha| + 2a = i - 1$; if r is not an integer, then $\langle f \rangle_r$ denotes the sum of the Hölder constants of order $r - [r]$ in ξ and of order $(r - [r])/2$ in τ of the derivatives of order $|\alpha| + 2a = [r]$ and the Hölder constants of order $(r - [r] + 1)/2$ in τ of the derivatives of order $|\alpha| + 2a = [r] - 1$. Now for an integer $r \geqslant 0$ we set $|f|_{(r)}$ equal to the sum of the $\langle f \rangle_{(i)}$ for $i = 0, 1, \ldots, r$ and in the case of a non-integral $r > 0$ we also add the number $\langle f \rangle_{(r)}$. The spaces $C^{(r)}$ on manifolds in (ξ, τ) are defined by means of partitioning the boundary and the local coordinates.

***Theorem* 5.9** [284]. *Let* $r_0 = n_0 + \varepsilon_0$, $n_0 \geqslant 7$ *an integer, and* $\varepsilon_0 \in (0, 1)$. *We assume that*: a) $\hat{T}_0 \in C^{(r_0+43)}(\bar{D}_{1,0})$, $g \in C^{(r_0+39)}(\Gamma_1 \times [0, \tau_0])$; b) *on* Γ_1 *the pair* $\{\hat{T}_0, T_0 + g/\alpha\}$ *satisfies the compatibility conditions* (1.18) *of the heat equation for* $\chi_1 = 0$, $\chi_2 = 1$ *up to the order* $[(r_0 + 39)/2]$; c) $\hat{T}_0(\xi) \geqslant T_1$ *on* $\bar{D}_{1,0}$; $T_0 + g/\alpha \geqslant T_1$ *on* $\Gamma_1 \times [0, \tau_0]$; $\lambda^+, \alpha_1 \in (0, \infty)$; d) $\hat{T}_0(\xi)$ *satisfies the compatibility conditions* (1.19) *up to the order* $[(r_0 + 39)/2] - 1$.

Then for sufficiently small $\tau_0 > 0$ *there is a pair* (ρ, T), $\rho \in C^{(r_0)}(\Gamma_2 \times [0, \tau_0])$, $\rho(\xi, 0) = 0$, $T \in C^{(r_0)}(\bar{D}_1)$, *that is the solution of* (5.23).

The idea of the technically quite complicated proof is simple. For a given ρ there is a unique solution of (5.23) without the last condition. This solution T_ρ can be obtained by standard methods, for example, those in [128], if as a preliminary we rewrite the problem in the cylindrical domain

$D_{1,\bullet} \times [0, \tau_0]$, using the function $\rho(\xi, \tau)$. The substitution of T_ρ and the formula for Φ_ρ indicated above in the left-hand side of the Stefan problem (5.23), generate an operator $F(\rho)$, and the problem itself reduces to the equation $F(\rho) = 0$. An analysis of the solubility of this equation is given in [310] on the basis the implicit function theorem, more precisely, on its version in [307]. Theorem 5.9 gives apparently excessive sufficient conditions for the existence of the (unique) classical solution of (5.23) locally in τ up to $\tau = 0$; that is, in particular, it answers the question left open in §5.5 on the behaviour of the free boundary as $\tau \to +0$.

5.8. The non-stationary many-dimensional Stefan problem with convection. Suppose that the boundary ∂D of the domain $D \subset \mathbf{R}^3$ consists of two connected manifolds Γ and Γ_0 of class $H^{4+\alpha}$, $0 < \alpha < 1$, and that Γ lies inside the domain with the boundary Γ_0. Let $\omega = (\omega_1, \omega_2)$ be any point of Γ_0, $\vec{\xi}(\omega)$ the corresponding point of $\mathbf{R}^3$, and $\vec{n}(\omega)$ the inward unit normal to Γ_0. Let γ_0 be a positive number such that the surfaces $\{\vec{\xi} = \vec{\xi}(\omega) \pm 2\vec{n}(\omega)\gamma,\ 0 < \gamma < \gamma_0\}$ do not have points of self-intersection and do not intersect Γ_0. We denote by $\eta(\omega, \tau)$ a function in $H^{2+\alpha, (2+\alpha)/2}\,\overline{(\Gamma_0 \times (0, \tau_0))}$, such that $\eta(\omega, 0) = 0$, $|\eta(\omega, \tau)| \leqslant \gamma_0$. Finally, let $\Omega_{\eta, \tau}$ be the domain bounded by the planes $\tau = 0$, $\tau = \tau_0$ and the surfaces $\Gamma \times [0, \tau_0]$, $\Gamma \times [0, \tau_0]$, $\Gamma_{\eta, \tau_0} = \{(\xi, \tau)\colon \vec{\xi} = \vec{\xi}(\omega) + \vec{n}(\omega)\eta(\omega, \tau)\}$.

For simplicity we consider the one-phase problem: we look for functions $\vec{V}(\xi, \tau)$, $p(\xi, \tau)$, $T(\xi, \tau)$ and $\eta(\omega, \tau)$ defined on $\overline{\Omega}_{\eta, \tau_0}$ and Γ_{η, τ_0}, respectively, satisfying the conditions

$$
(5.24)\quad \begin{cases}
T_\tau + \vec{V}\cdot\vec{\nabla} T - \sum\limits_{k=1}^{3} \dfrac{\partial}{\partial \xi_k}\left[\lambda(T)\dfrac{\partial T}{\partial \xi_k}\right] = 0 \text{ in } \Omega_{\eta, \tau_0}; \\
T(\xi, 0) = \hat{T}_0(\xi),\ \xi \in D;\ T = b(\xi, \tau) \text{ on } \Gamma \times (0, \tau_0);\ T = T_k \text{ on } \Gamma_{\eta, \tau_0}; \\
\vec{V}_\tau + \vec{V}\cdot\vec{\nabla}\vec{V} + \vec{\nabla} p = \Delta \vec{V}/\mathrm{Re} + f(T),\ \operatorname{div} \vec{V} = 0 \text{ in } \Omega_{\eta, \tau_0}; \\
\vec{V}(\xi, 0) = \vec{V}_0(\xi),\quad \xi \in D,\ \vec{V} = 0 \text{ on } [\Gamma \times (0, \tau_0)] \cup \Gamma_{\eta, \tau_0}; \\
\varkappa \cos(N, \tau) - \lambda \sum\limits_{k=1}^{3} T_{\xi_k} \cos(N, \xi_k) = 0 \text{ on } \Gamma_{\eta, \tau_0},
\end{cases}
$$

where $\vec{N} = \vec{N}(\omega, \tau)$ is the normal to Γ_{η, τ_0} directed towards the interior of Ω_{η, τ_0}, p is the exposure, and $\vec{V}$ is the velocity of the convective motion. We assume that $\lambda(T) \in H^3(\mathbf{R}^1)$, $f(T) \in H^1(\mathbf{R}^1)$, $b(\xi, \tau) \in H^{2+\alpha, 1+\alpha/2}$, $\vec{V}_0 \in H^{4+\alpha}(\overline{D})$, $\hat{T}_0 \in H^{4+\alpha}(\overline{D})$, also $b(\xi, \tau) \geqslant b_0 > 0$, $\lambda(T) \in [\nu, \nu^{-1}]$, $\nu = \text{const} > 0$, $|\hat{T}_{0n}| \geqslant \alpha_0 > 0$ on Γ_0, and that the compatibility conditions up to the first order are satisfied (see §1.4).

Theorem 5.10 [10]. *There is a sufficiently small number*

$$\tau_0 = \tau_0(\omega_0, \ \nu, \ |\hat{T}_0|^{(3+\alpha)}, \ |\vec{V}_0|^{(3+\alpha)}, \ |b|^{(2+\alpha)}, D) > 0,$$

such that (5.24) *has a solution with the properties*

$$\eta \in H^{2+\alpha, (2+\alpha)/2}(\Gamma_0 \times [0, \tau_0]), \quad \eta_\tau \in H^{1+\alpha, (1+\alpha)/2}(\Gamma_0 \times [0, \tau_0]),$$

$$T \in H^{2+\alpha, (2+\alpha)/2}(\bar{\Omega}_{\eta, \tau_0}), \ \vec{\nabla} p \in H^{\beta, \beta/2}(\bar{\Omega}_{\eta, \tau_0}), \ \vec{V} \in H^{2+\beta, (2+\beta)/2}(\bar{\Omega}_{\eta, \tau_0}),$$

$$0 < \beta < \alpha < 1.$$

The proof of this theorem relies on the Schauder fixed point theorem. As in [284], as a preliminary we reduce the problem to an equivalent one in a fixed domain $\bar{D} \times [0, \tau_0]$; then for a given $\vec{V}(\xi, \tau)$ we construct the temperature field $T(\xi, \tau)$ by the method of [6]; finally, the inverse of the hydrodynamical part of (5.24) is constructed by devices developed in the theory of Navier-Stokes systems. As a result, we obtain the classical solubility of the one-phase Stefan problem with convection locally in the time variable. The case of the two-phase problem is treated similarly.

The classical local solubility of the many-dimensional problem without convection in the time variable was established in [141] (proofs are given in [142]). The approach suggested here is based on the parabolic regularization of the Stefan condition and fibration into isothermal surfaces (see §4.2). We mention the paper [147], which is devoted to the one-phase many-dimensional Stefan problem with constant thermophysical parameters in the absence of the heat-carrying part of the boundary. The original problem is reformulated in terms of distinctive "Lagrange variables", the linearization of the problem is studied, etc.

We also mention the papers [251] and [317], which are devoted to non-stationary Stefan problems in a "finely grained" medium (the statement dates back to [305]). In these papers ε-approximations (for a fixed ε) of a periodic "grain" are considered and then problems of the passage to the corresponding limit situation as $\varepsilon \to 0$ are studied. A one-phase one-dimensional Stefan type problem has also been considered for a hyperbolic equation (see [319], 437–450). In conclusion we mention the papers [55] and [56] in which results concerning the classical solubility of a many-dimensional multi-phase Stefan problem are announced. The proofs in [57] are not convincing.

CHAPTER VI

STABILITY AND STABILIZATION. OPEN QUESTIONS

6.1. L_2-stability and L_p-stabilization of temperature.

Let D be a domain in the space of the variables $\xi = (\xi_1, \ldots, \xi_n)$, $n \geqslant 2$, bounded by disjoint manifolds Γ_1 and Γ_2 of class $C^{2+\alpha}$, $\alpha > 0$. We consider the following statement of the Stefan problem:

$$(6.1)\quad \begin{cases} \alpha^j \dfrac{\partial T}{\partial \tau} = \displaystyle\sum_{i,k=1}^{n} a_{ik}(\xi) \frac{\partial^2 T}{\partial \xi_i \partial \xi_k} + \sum_{i=1}^{n} b_i(\xi) \frac{\partial T}{\partial \xi_i} + c^j(\xi) T, \\ \qquad\qquad (\xi, \tau) \in D_j,\ j = 1, 2; \\ T = T_0 + g_i(\xi, \tau)/\alpha \text{ on } \Gamma_i, \quad i = 1 \text{ or } 2;\ T(\xi, 0) = \hat{T}(\xi),\ \xi \in D; \\ T = T_1, \displaystyle\sum_{i,k=1}^{n} (a_{ik} T^-_{\xi_i} \Phi_{\xi_k} - a_{ik} T^+_{\xi_i} \Phi_{\xi_k}) = \varkappa_1 \Phi_\tau \text{ on } \Sigma_1, \end{cases}$$

where $\Phi(\xi, \tau) > 0$ in D_2, $\Phi(\xi, \tau) < 0$ in D_1, and $\Phi(\xi, \tau) = 0$ on Σ_1. We assume that: a) the functions a_{ik}, $\nabla_\xi a_{ik}$, $\nabla^2 a_{ik}$, b_i, $\nabla_\xi b$, b_i, and c^j are continuous in D^-, $c^j(\xi) \leqslant 0$, and that the matrix a_{ik} satisfies the condition of uniform ellipticity with constants μ_1 and μ_2; b) $T_0 + g/\alpha \geqslant \gamma_1 > T_1$ on $\Gamma_1 \times [0, \tau_0]$, $T_0 + g/\alpha \leqslant \gamma_2 < T_1$ on $\Gamma_2 \times [0, \tau_0]$, $\gamma_1, \gamma_2 = \text{const}$;
c) $\hat{T}_0(\xi) \geqslant T_1$ in $\bar{D}_{1,0}$, $\hat{T}_0(\xi) \leqslant T_1$ in $\bar{D}_{2,0}$, and $\hat{T}_0(\xi) = T_1$ only on $\tilde{\Sigma}_{1,0}$; thus, $(-1)^i \hat{T}_0(\xi) \geqslant \gamma_3 > T_1$, $\gamma_3 = \text{const}$ in some δ-neighbourhood of Γ_i ($i = 1$ or 2);
d) $\hat{T}_0(\xi) \in C(\bar{D}) \cap W_2^1(D)$, and there is a function $\psi(\xi, \tau)$ that is Hölder continuous together with $\nabla_\xi \psi$, $\nabla_\xi^2 \psi$ and ψ_τ in the closed domain $\bar{\Omega}$, $\Omega = D \times (0, \tau_0)$, and $\psi = T_0 + g/\alpha$ on $\partial D \times [0, \tau_0]$ and $\psi = \hat{T}_0(\xi)$ in the indicated δ-neighbourhood of the boundary ∂D. In the relevant case $H(T) = \alpha_2(T - T_{m+1})$ for $T_{m+1} < T < T_1$ and $H(T) = \alpha_1(T - T_{m+1}) + \varkappa_1$ for $T \geqslant T_1$. The generalized solution of (6.1) is defined by menas of an identity of the form (5.4) with Δ_ξ replaced by the operator adjoint to the right-hand side of the first equation in (6.1). We denote by M the set of pairs $(g, \hat{T})$ satisfying the conditions listed above with fixed constants $\gamma_1, \gamma_2, \gamma_3, \delta$ and $A \geqslant \max |\hat{T}_0| + \max |g|$. Next, we denote by $g_\infty(\xi)$ a function from $C^{2+\alpha}(\partial D)$ that is positive on Γ_1 and negative on Γ_2, and we consider the function $w(\xi)$ defined by the conditions

$$(6.2)\quad \sum_{i,k=1}^{n} a_{ik}(\xi) \frac{\partial^2 w}{\partial \xi_i \partial \xi_k} + \sum_{i=1}^{n} b_i(\xi) \frac{\partial w}{\partial \xi_i} + c(\xi, w) w = 0 \text{ in } D,$$

$$w = g_\infty(\xi) \quad \text{for} \quad \xi \in \partial D,$$

where $c(\xi, w) = C^1(\xi)$ for $w < T_1$ and $c(\xi, w) = C^2(\xi)$ for $w > T_1$.

***Theorem* 6.1** [269]. a) *Under the assumptions made above there is a constant B depending only on the class M, the constants* α^1, α^2, $\varkappa_1$, μ_1, μ_2, τ_0 *and the least upper bounds of the norms of the coefficients of* (6.1) *such that*

$$(6.3)\quad \begin{cases} \|T-\tilde{T}\|^2_{L_2(\Omega)} \leqslant B\{\|\hat{T}_0-\hat{T}_0\|^2_{L_2(D)} + \|\psi-\tilde{\psi}\|^2_{L_2(\Omega)} + \\ \qquad + \|\nabla_\xi\psi-\nabla_\xi\tilde{\psi}\|^2_{L_2(\Omega)}\}, \\ \|T-\tilde{T}\|^2_{L_2(\Omega)} \leqslant B\Big\{\|\hat{T}_0-\tilde{\hat{T}}_0\|^2_{L_2(D)} + \int_0^{2\varepsilon}\int_D [(\psi-\tilde{\psi})^2 + \\ \quad + |\nabla_\xi\psi-\nabla_\xi\tilde{\psi}|^2]\,d\xi\,d\tau + \int_\varepsilon^{\tau_0}\int_{\partial D}[(g-\tilde{g})^2+|\nabla_\xi g-\nabla_\xi\tilde{g}|^2]\,d\sigma_\xi\,d\tau\Big\}, \\ \qquad 0<2\varepsilon<\tau_0, \end{cases}$$

where $(g, \hat{T}_0)$ *and* $(\tilde{g}, \tilde{\hat{T}}_0)$ *are arbitrary elements of M and T and* $\tilde{T}$ *are the corresponding generalized solutions of* (6.1), ε *is fixed*;

b) *we assume in addition that the assumptions made earlier are satisfied up to* $\tau_0=\infty$ *and that* $c^i(x)\equiv 0$ $(i=1$ *or* $2)$ *and* $g(\xi,\tau)$ *are bounded on* $\partial D\times[0,\infty)$ *together with their first derivatives and the Hölder constant of* $\nabla_\xi g(\xi,\tau)$ *with respect to the argument* $\xi\in\partial D$;

$$(6.4)\qquad N=\int_\varepsilon^\infty\int_{\partial D}[(g-g_\infty)^2+|\nabla_\xi g-\nabla_\xi g_\infty|^2]\,d\sigma_\xi\,d\tau<+\infty,$$

$$\int_0^\infty\int_{\partial D}|g_\tau|\,d\sigma_\xi\,d\tau<\infty;$$

then

$$(6.5)\quad \int_0^\infty\int_D|T(\xi,\tau)-w(\xi)|^2\,d\xi\,d\tau\leqslant$$

$$\leqslant B\Big\{N+\|w(\xi)-\hat{T}_0(\xi)\|^2_{L_2(D)}+\int_0^{2\varepsilon}\int_D[(w-\psi)^2+|\nabla_\xi w-\nabla_\xi\psi|^2]\,d\xi\,d\tau\Big\},$$

where B has the same properties as in a);

c) *for* $n\geqslant 2$ *and* $p<2n/(n-2)$ L_p*-stabilization holds*:

$$(6.6)\qquad \int_D|T(\xi,\tau)-w(\xi)|^p\,d\xi\to 0 \quad as \quad \tau\to\infty,$$

and for $n=1$ *the convergence* $T(\xi,\tau)\to w(\xi)$ *as* $\tau\to\infty$ *is uniform in* $\xi\in\bar{D}$.

In particular, T can be a classical solution of (6.1) (under certain conditions from the preceding chapter) and then (6.3) gives estimates for the deviation from its generalized solution $\tilde{T}$ corresponding to the approximate initial data. These estimates also imply the uniqueness of generalized solutions and if g does not depend on $\tau (g\equiv g_\infty)$, then, by (6.5), $T(\xi,\tau)\equiv w(\xi)$

identically, $(\xi, \tau) \in D \times (0, \tau_0)$. Theorem 6.1 admits generalizations to the case of a quasilinear energy equation for a multi-phase problem in domains with an aribtrary number of boundary manifolds Γ_i, and to more general boundary conditions of the form (1.8), etc. ([269], §6, see also [274], Ch. 5, §10). When the results of §3.5 are taken into account, it seems plausible that Theorem 6.1 also holds when there is a separate energy equation in each phase. We also mention that in [311] the L_2-continuous dependence of the generalized solution on the initial data is also established for a general quasilinear two-phase Stefan problem with a non-linear boundary condition within the framework of being L_2-well-posed in the sense of Hadamard.

6.2. W_2^1-stabilization of temperature and mes-stabilization of the phase in a one-phase contact problem.

Let D be the same domain as in §5.6. We consider the setting

$$(6.7)\quad \begin{cases} \dfrac{\partial T}{\partial \tau} - \operatorname{div}_\xi [A(\xi) \operatorname{grad}_\xi T] = 0 \ \text{ in } D_1; \\ T = T_0 \ \text{ on } \Gamma_1 \times [0, \tau_0], \ T = T_0 + g(\xi, \tau)/\alpha \ \text{ on } \Gamma_2 \times [0, \tau_0], \ \text{ or} \\ A\nabla_\xi T \cdot n + \alpha(\xi)[T - T_0] = 0 \ \text{ on } \Gamma_{1,2} \times [0, \tau_0], \ A\nabla_\xi T \cdot n = 0 \\ \qquad\qquad\qquad\qquad \text{on } \partial D_1 \cap \Gamma_3 \times [0, \tau_0]; \\ T = T_1, \ \sum\limits_{i,j=1}^{n} A_{ij} T_{\xi_i} \cos(N, \xi_j) + \varkappa_1 \cos(N, \tau) = 0 \ \text{ on } \Sigma_1, \\ T(\xi, 0) = \hat{T}_0(\xi), \quad \xi \in D_{1,0}. \end{cases}$$

Here A is a given uniformly positive definite symmetric matrix. We assume that

$$(6.8)\quad \begin{cases} \hat{T}_0(\xi) \in W_2^1(D), \ T_0 < \hat{T}(\xi) < T_1 \ \text{ in } D_{1,0}, \ \hat{T}_0(\xi) \equiv T_1 \ \text{ in } D \setminus D_{1,0}; \\ \alpha(\xi) \geqslant \alpha_0 > 0 \ \text{ on } \Gamma_1 \cup \Gamma_2; \ g(\xi, \tau) \in L_2(0, \tau_0; \ W_2^{3/2}(\partial D)), \\ 0 \leqslant g(\xi, \tau) \leqslant \alpha(T_1 - T_0), \quad g(\xi, 0) = 0 \ \text{ on } \overline{\Gamma}_1 \cap \overline{\Gamma}_2; \\ \hat{T}_0(\xi) = T_0 + g(\xi, \tau)/\alpha \ \text{ for } \ \tau = 0, \ \xi \in \Gamma_2, \quad \hat{T}_0(\xi) = T_0 \ \text{ for } \xi \in \Gamma_1, \end{cases}$$

and we introduce the notation

$$(6.9)\quad \begin{cases} w(\xi, \tau) = -\displaystyle\int_0^\tau T(\xi; \sigma)\, d\sigma, \ v(\xi, \tau) = w(\xi, \tau)/\tau; \\ (Az, v) = \displaystyle\int_D A\nabla_\xi z \cdot \nabla_\xi v \, d\xi + \int_{\partial D} \tilde{\alpha}(\xi) zv \, ds \ \text{ for all } z, v \in W_2^1(D); \\ \psi(z) = \dfrac{\varkappa_1}{2} \displaystyle\int_D \{|z + \tau| - (z + \tau)\}\, d\xi, \ f = -\hat{T}_0(\xi) - \varkappa_1 [1 - \chi_{D_{1,0}}(\xi)]; \end{cases}$$

$K = W_2^1(D)$ in the case of Dirichlet conditions or

$$K=\Big\{z:\ z\in W_2^1(D),\quad z=-T_0\tau \text{ on } \Gamma_1,\quad z=-T_1\tau \text{ on } \Gamma_3,$$

$$z=-T_0\tau-\frac{1}{\alpha}\int_0^\tau g(\xi,\sigma)\,d\sigma \text{ on } \Gamma_2\Big\},$$

where $\tilde{\alpha}(\xi)\equiv 0$ in the first case and $\tilde{\alpha}(\xi)=\alpha(\xi)$ on $\Gamma_1\cup\Gamma_2$, $\tilde{\alpha}(\xi)\equiv 0$ on Γ_3 in the second case.

Theorem 6.2 [197]. a) *If* (T,Σ_1) *is a classical solution of* (6.7), *then* $w(0)=0$ *and*

$$(6.10)\quad (w_\tau, z-w)_{L_2(D)}+(Aw, z-w)+\psi(z)-\psi(w)\geqslant(f, z-w)_{L_2(D)}\quad \text{for all } z\in K$$

almost everywhere on $(0,\tau_0)$; *under conditions* (6.8) *this inequality has one and only one solution* $w(\tau)\in K$, $w(0)=0$, *which is, by definition, a generalized solution of* (6.7);

b) *the generalized solution of* (6.7) *has the following properties*: $T(\xi,\tau)\in L_\infty(D\times(0,\tau_0))$, $T_0\leqslant T(\xi,\tau)\leqslant T_1$ *almost everywhere, and the domain* $D_{1,\tau}$ *occupied by the solid phase is monotone expanding (generally speaking, in the wide sense of the word) as* τ *increases*;

c) *in the case of a boundary condition of the third kind the generalized solution* (T,Σ_1) *of* (6.7) *has the properties*

$$(6.11)\quad \operatorname{mes}(D\setminus D_{1,\tau})=O(\tau^{-2}),\quad \left\|\frac{1}{\tau}\int_0^\tau T(\xi,\sigma)\,d\sigma-T_0\right\|_{W_2^1(D)}=O(\tau^{-1});$$

d) *suppose that* $T_\infty(\xi)$ *is a solution of the Dirichlet problem*

$$(6.12)\qquad \operatorname{div}_\xi[A(\xi)\operatorname{grad}_\xi T_\infty]=0 \text{ in } D,\quad T_\infty=T_0+g_\infty(\xi) \text{ on } \partial D,$$

and that

$$(6.13)\quad \nu(\tau)=\left\|\frac{1}{\alpha\tau}\int_0^\tau g(\xi,\sigma)\,d\sigma-g_\infty\right\|_{W_2^{1/2}(\partial D)}\to 0 \quad \text{as}\quad \tau\to\infty.$$

Then

$$(6.14)\qquad \left\|\frac{1}{\tau}\int_0^\tau T(\xi,\sigma)\,d\sigma-T_\infty(\xi)\right\|_{W_2^1(D)}=O(\tau^{-1}+\nu(\tau));$$

if $g_\infty(\xi)\geqslant -T_1+d$ *almost everywhere on* ∂D *for some constant* $d>0$, *then*

$$(6.15)\qquad \operatorname{mes}(D\setminus D_{1,\tau})=O(\tau^{-2}+\nu^2(\tau)).$$

We also mention that $w(\xi,\tau)$ satisfies the energy equation in (6.7) almost everywhere in D_1, and in the case of heat exchange also the conditions in the third line of (6.7) almost everywhere on $\partial D\times(0,\tau_0)$. Moreover, a sufficient condition for (6.13) is that $g(\xi,\tau)\to\alpha g_\infty(\xi)$ in $W_2^{1/2}(\partial D)$ as $\tau\to\infty$.

6.3. Stabilization of temperature and of a free boundary in the uniform metric.

Suppose again that the geometry of D is the same as in §5.6. We consider the setting (in the dimensionless case):

$$(6.16)\quad \begin{cases} \gamma(u)\,u_t - \Delta_x u + \alpha(u) = 0 \text{ in } D_i \subset D \times (0,\infty)) \quad (i=1 \text{ or } 2); \\ u(x,0) = u_0(x), \quad x \in \overline{D}; \quad u_n(x,t) = 0 \text{ on } \Gamma_2 \times [0,\infty); \\ u(x',0,t) = f_1(x',t), \quad (x',t) \in \Gamma_1 \times [0,\infty); \; u(x',1,t) = f_2(x',t), \\ \qquad (x',t) \in \Gamma_3 \times [0,\infty); \\ 0 \leqslant u(x,t) < 1 \text{ on } \overline{D}_1 \setminus \Sigma_1, \quad u(x,t) > 1 \text{ on } \overline{D}_2 \setminus \Sigma_1; \\ u = 1, \; u_x^- - u_x^+ = \lambda\varphi_t / \sqrt{1 + |\nabla_x \varphi|^2} \text{ on } \Sigma_1\colon x_n = \varphi(x',t). \end{cases}$$

We assume that $\alpha(u)$ and $\gamma(u)$ are continuously differentiable functions of $u \in [0, \infty)$ and that $u_0(x)$ and $f_i(x', t)$ $(i = 1$ or $2)$ are continuous and satisfy the compatibility conditions guaranteeing the existence of a classical solution of (6.16) for $t \in [0, \infty)$, and that

$$(6.17)\quad \begin{cases} 0 \leqslant u_0(x) \leqslant 1 \text{ in } \overline{D}_{1,0}, \quad u_0(x) > 1 \text{ in } \overline{D}_{2,0} \setminus \Sigma_{1,0}; \\ 0 \leqslant f_1(x',t) \leqslant 1, \; (x',t) \in \Gamma_1 \times [0,\infty); \quad f_2(x',t) > 1, \\ \qquad (x',t) \in \Gamma_3 \times [0,\infty); \\ \gamma(u) \geqslant \gamma_0 > 0, \; \alpha'(u) \geqslant 0, \quad \alpha(0) = 0, \quad \lambda > 0; \\ f_1(x',t) \to 0, \; f_2(x',t) \to p > 1 \text{ as } t \to \infty, \text{ uniformly in } x'. \end{cases}$$

We also consider the corresponding one-dimensional stationary problem:

$$(6.18)\quad \begin{cases} v_{xx} - \alpha(v) = 0, \quad x \in (0,1), \quad x \neq h_p \in (0,1); \\ v(0) = 0, \quad v(1) = p, \quad 0 \leqslant v < 1 \text{ on } [0, h_p), \\ v > 1 \text{ на } (h_p, 1]; \\ v^{\pm}(h_p) = 1, \quad v_x^-(h_p) - v_x^+(h_p) = 0 \end{cases}$$

and we write $u_\infty(x) = v(x_n)$, $\Sigma_{1,\infty}\colon x_n = h_p$.

***Theorem* 6.3** [24]. a) *The stationary problem* (6.18) *has one and only one solution* (v, h_p); *if* (u, Σ_1) *is a classical solution of* (6.16), *then it tends to* $(u_\infty, \Sigma_{1,\infty})$, *uniformly in x and x', respectively, as* $t \to \infty$;

b) *suppose that for some constants* $A > 0$, $\mu > 0$

$$|f_1(x',t)| \leqslant Ae^{-\mu t}, \quad |f_2(x',t) - p| \leqslant Ae^{-\mu t}, \quad t \geqslant 0.$$

Then there are positive constants B, B', ν, ν' *such that*

$$(6.19)\quad \begin{cases} \Sigma_1 \subset \{(x,t)\colon |x_n - h_p| \leqslant Be^{-\nu t}, \quad t \geqslant 0\}, \\ |u(x,t) - u_\infty(x)| \leqslant B'e^{-\nu' t}, \quad x \in \overline{D}, \quad t \geqslant 0; \end{cases}$$

c) *a similar statement holds if* $e^{-\mu t}$ *is replaced by* $(1 + t)^{-\mu}$.

Earlier studies dealt with the process of exit of a non-stationary solution to a quasistationary regime [17] and also with the case of a general one-dimensional quasilinear problem [18], [22]. Closely related problems were treated in [19], [21], and [23]. We also mention the papers [28] and [29], in which the problem of the stabilization in the uniform metric of the solution of a one-dimensional Stefan problem was studied on a finite interval $0 \leqslant x \leqslant l$ when Neumann conditions are prescribed at the end-points and the conditions of isothermality have the form (1.12) with a right-hand side independent of τ.

In the case of infinite domains D the free boundary may move away to infinity. Such a one-dimensional problem concerning the freezing of soil was already considered in Stefan's paper [329]. For general one-phase problems in the half-space $x \geqslant 0$ for the equation of heat conduction with a Neumann condition at the end-point $x = 0$, the asymptotic behaviour of the free boundary was described in [200], Ch. VIII, §3 (see also [269], §7). In [69] conditions are investigated for the free boundary to move to infinity for a general linear one-dimensional parabolic equation on the entire axis. The situation can change if not the temperature, but rather the heat flow has positive values at infinity ([73], [98], [180], etc.).

The stability and stabilization in the neighbourhood of a stationary solution in Hölder spaces are considered in [5], [144].

6.4. Stability of the Stefan problem with convection.
We consider a two-phase problem with spherical symmetry, when D is the layer $R_0 < r < R$, Σ_1 can be represented by spherical coordinates (r, φ, θ) in the form $r = \psi(\varphi, \theta, \tau) \in (R_0, R)$, and (1.8) reduces to the spedification of constant temperatures $T_b < T_1$, $T_H > T_1$ for $r = R_0$ and $r = R$, respectively. We take the energy equation in the liquid phase in the form (1.17) with constants ρ, c, and λ and $f \equiv 0$, and in the solid phase in the form of the homogeneous equation of heat conduction. Convection is described by (1.5) and the system (1.4) with the right-hand side $\{-\nabla p + \rho[\nabla w + \nu\Delta \vec{V}]\}$, where $w = -gr$ is the potential created by the relevant heat-conducting mass. On the unknown boundary Σ_1 the conditions (1.9) of isothermality and Stefan's condition (1.10) must be satisfied and also the condition of adhesion for $r = R$ and the vanishing of the tangential components of Σ_1 on $\vec{V}$. For the unknown $T^{\pm}$, $\vec{V}$, p, and $\Sigma_{1,\tau}$, the corresponding initial data are also prescribed. Here only on the right-hand side of the equations of motion ∇w we put $\rho = \rho(T^+) \equiv \rho^+[1-\alpha(T^+ - T_1)]$, where $\alpha = \text{const}$ is the coefficient of expansion, and in all remaining cases $\rho = \rho^+$.

The problem in question has a spherically symmetric stationary solution in the absence of mixing in the liquid phase. The original problem can be linearized in the neighbourhood of this solution; from the resulting linear problem the time τ can be eliminated by means of the exponential function.

The spectral problem thus obtained can then be studied by the method of separation of variables. As a result of this analysis, the following assertion is obtained.

***Theorem* 6.4** [99]. *Let $\rho^+ = \rho^-$. Then the spectrum of the corresponding spectral problem is located inside the left half-plane; in other words, the original non-stationary Stefan problem with convection is stable in the linear approximation as $\tau \to \infty$.*

Stability in the planar case and in the case of cylindrical symmetry was considered in [126] and [127].

6.5. Some unsolved problems.

1. To develop a general theory of an integro-differential equation of the form (2.13) and, by the same token, to give an analysis of the corresponding Cauchy problem equivalent to the non-stationary Stefan problem.

2. To study many-dimensional "piecewise continuous" statements where the initial data are not continuous (see §2.3) and the boundary of the relevant domain is not smooth. So far in this direction only simplified formulations (the model of "fine hell") have been studied ([78], [79], etc.).

3. To give a complete analysis of many-dimensional statements of the inverse Stefan problem (see §§1.7 and 3.3) and to work out methods for approximate and numerical solutions.

4. To study the boundary properties of the phase interfaces in the many-dimensional non-stationary contact Stefan problems (compare with §3.2).

5. In the statement of the Stefan problem, to take account of both forced and natural convection caused by the Archemedean forces, the dependence of the density of the medium on the phase ("contraction"), etc.

6. To lay the foundations of a calculus of variations "in the large" for many-dimensional integral functionals of the form (4.2) with a variable domain of integration (on the two-dimensional case, see [72], [256]) and to work out methods of search for their critical points.

7. To study problems of Stefan type with all possible combinations of equations of elliptic, parabolic, and hyperbolic types in the various phases (see §1.5 and [319], 437).

8. To study stability and stabilization problems of solutions of the Stefan problem in many-dimensional statements in Hölder function spaces, among them cases when a solution does not exist or is not unique in the limit case (§3.1, [82]).

9. To find methods of numerical and approximate solutions of many-dimensional Stefan problems in cases when the existence of a classical solution has been established.

10. The mathematical modelling of processes of phase change during simultaneous heat and mass exchanges goes beyond the scope of this survey. However, we mention it here as a problem of great theoretical and practical significance.

References

[1] A.A. Abbaev and M.E. Sapozhnikov, On the existence of solutions of an optimal problem of Stefan type for quasi-linear equations, in: *Issledovaniya po nekotorym voprosam konstruktivnoi teorii funktsii i differentsial'nykh urvanenii* (Investigations of some questions of constructive function theory and differential equations), Aznefkhim, Baku 1981, 131-135.

[2] N.A. Abbaev, *Matematicheskoe opisanie protsessov kristallizatsii* (Mathematical description of crystallization processes), Zinatne, Riga 1980, 1-178. MR **82b**:80001.

[3] B.V. Bazalii, On a quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1976**, no. 1, 3-5. MR **54** # 8026.

[4] ———, Investigation of a two-phase Stefan problem in the neighbourhood of a stationary solution, Uspekhi Mat. Nauk **37**:4 (1982), 117.

[5] ———, Stability of smooth solutions of the two-phase Stefan problem, Dokl. Akad. Nauk SSSR, **262** (1982), 265-269. MR **83f**:35057.

[6] ———, Investigation of a two-phase Stefan problem in the neighbourhood of a stationary solution, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1983**, no. 12, 3-7. MR **85k**:35234.

[7] ———, Convective heat transfer in the Stefan problem, Uspekhi Mat. Nauk **38**:5 (1983), 152.

[8] ———, The stationary two-phase Stefan problem with convective heat transfer in a liquid phase, in: *Mat. fizika* (Math. physics), Naukova Dumka, Kiev.

[9] ———, On the instability and branching of solutions of the quasi-stationary thermodiffusion Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1985**, no. 4.

[10] ——— and S.P. Degtryarëv, Classical solubility of the non-stationary Stefan problem with convection, Dokl. Akad. Nauk SSSR **286**:6 (1986).

[11] ——— and V.Yu. Shelepov, On a mixed problem with free boundary for the Laplace equation, Dokl. Akad. Nauk SSSR **209** (1973), 320-323. MR **47** # 9041. = Soviet Phys. Dokl. **18** (1973), 170-172.

[12] ——— and ———, On a stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSSR Ser. A, **1974**, no. 1, 5-8. MR **49** # 7602.

[13] ——— and ———, On the two-phase stationary Stefan problem, Mat. Fiz. **1974**, no. 16, 52-61. MR **49** # 5559.

[14] ——— and ———, On a thermophysical quasi-stationary problem with free boundary, in: *Kraevye zadachi teploprovodnosti* (Boundary-value problems of heat conduction), Izdanie Inst. Mat. Akad. Nauk Ukrain. SSR, Kiev 1975, 45-56. MR **54** # 4361.

[15] ——— and ———, A generalization of the stationary Stefan problem, in: *Mat. Fizika* (Math. physics), Naukova Dumka, Kiev 1975, no. 27, 65-81. MR **57** # 10232.

[16] ——— and ———, On a class of non-linear problems with free boundary, in: Proc. All-Union Conf. on Partial Differential Equations, Moscow State Univ., Moscow 1978, 264-265.

[17] ——— and ———, The asymptotic behaviour of the solution of a Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1978**, no. 12, 1059-1061. MR **80b**:35015.

[18] ——— and ———, On the stabilization of the solution of the Stefan problem for a quasi-linear equation, in: *Kraevye zadachi matematicheskoi fiziki* (Boundary-value problems of mathematical physics), Naukova Dumka, Kiev 1979, 24-39.

[19] B.V. Bazali and V.Yu. Shelepov, On the rate of stabilization of the solution of the Stefan problem, Uspekhi Mat. Nauk **35**:4 (1980), 178-179.
[20] ——— and ———, Variational methods in the mixed problem of thermal equilibrium with free boundary, Mat. Fiz. **1980**, no. 27, 62-74. MR **82b**:35046.
[21] ——— and ———, On the stabilization of the solution of the Stefan problem, in: *Granichnye zadachi matematicheskoi fiziki* (Limit problems of mathematical physics), Naukova Dumka, Kiev 1981, 5-7.
[22] ——— and ———, Estimates for the rate of stabilization of the solution of the Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1981**, no. 4, 3-6. MR **84d**:35017.
[23] ——— and ———, On an estimate of the rate of stabilization of the solution of the Stefan problem, Mat. Fiz. no. 30 (1981), 35-41. MR **83m**:35064.
[24] ——— and ———, On the problem of stabilization of the solution of the Stefan problem, in: *Uravneniya v chastnykh proizvodnykh i zadachi so svobodnoi granichei* (Partial differential equations and free boundary problems), Naukova Dumka, Kiev 1983, 20-24.
[25] R.D. Bachelis and V.G. Melamed, On the solution of the quasi-linear two-phase Stefan problem by the method of lines under weak restrictions on the input data of the problem, Zh. Vychisl. Mat. i Mat. Fiz. **12** (1972), 828-829.
= USSR Comp. Math. and Math. Phys. **12**:3 (1972), 342-343.
[26] ——— and ———, Solution of a Stefan type quasi-linear problem by the method of lines, VINITI, no. 3942-72.
[27] ——— and ———, Solution of a Stefan type quasi-linear problem by the method of lines in the case of an initial function not differentiable and not compatible with the boundary conditions, VINITI, no. 3942-72.
[28] G.I. Bizhanova, Stabilization of the solution of the second boundary-value problem of Stefan, Izv. Akad. Nauk Kazakh. SSR Ser. Fiz.-Mat. **1980**, no. 5, 12-17. MR **82b**:35151.
[29] ———, Investigation of the second boundary-value problem of Stefan with non-constant temperature on the free boundary, Thesis, Alma-Ata 1982.
[30] G. Birkhoff and E.H. Zarantonello, Jets, wakes, and cavities, Academic Press, New York 1957. MR **19**-486.
Translation: *Strui, sledy i kaverny,* Mir, Moscow 1964.
[31] V.A. Bogatyrëv, On the uniqueness of the solution of a non-linear problem, Ukrain. Mat. Zh. **25** (1973), 347-352. MR **51** # 1094.
= Ukrainian Math. J. **25** (1973), 277-281.
[32] ———, On a non-linear problem with free boundary, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1973**, no. 10, 867-869.
[33] ———, Investigation of a two-component non-linear boundary-value problem by the method of integral functionals with a variable domain of integration, Thesis, Donetsk 1973.
[34] ———, Uniqueness theorem of the weak solution of a non-linear problem, Mat. Fiz. no. 17 (1975), 81-85. MR **56** # 3447.
[35] ———, Solubility of a non-linear problem with free boundary, in: *Kraevye zadachi dlya uravnenii v chastnykh proizvodnykh* (Boundary-value problems for partial differential equations), Naukova Dumka, Kiev 1978, 5-13. MR **80j**:35042.
[36] V.T. Borixov, V.V. Vinogradov, A.I. Dukhin, et al. The applicability of the theory of a quasi-equilibrium two-phase zone to the description of the crystallization of an ingot, Izv. Akad. Nauk SSSR Metally, **1971**, no. 6, 104-109.
[37] M.A. Borodin, Some non-linear problems of heat conduction, Mat. Fiz. no. 14 (1973), 8-14.

[38] M.A. Borodin, An existence theorem for the solution of a single-phase quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1976**, no. 7, 582-585. MR **55** # 847.
[39] ———, A one-phase quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1977**, no. 9, 775-777. MR **58** # 11851.
[40] ———, A one-phase quasi-stationary Stefan problem, in: *Kraevye zadachi dlya uravnenii v chasnykh proizvodnykh* (Boundary-value problems for partial differential equations), Naukova Dumka, Kiev 1978, 13-21. MR **80j**:35095.
[41] ———, The spatial one-phase quasi-stationary Stefan problem, Uspekhi Mat. Nauk **35**:4 (1980), 177.
[42] ———, Solubility of a two-phase quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1982**, no. 2, 3-5. MR **84a**:35106.
[43] ———, Solubility of a two-phase non-stationary Stefan problem, Dokl. Akad. Nauk SSSR **263** (1982), 1040-1042. MR **84a**:35107.
= Soviet Math. Dokl. **25** (1982), 469-471.
[44] ———, The classical solubility of a two-phase non-stationary Stefan problem, Uspekhi Mat. Nauk **38**:5 (1983), 152.
[45] ———, A two-phase non-stationary Stefan problem, Mat. Sb. 1986.
[46] ——— and U. Felgenhauer, A one-phase quasi-linear Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1978**, no. 2, 99-102. MR **57** # 16978.
[47] ——— and ———, An axiosymmetric one-phase Stefan problem, Mat. Fiz. no. 24 (1978), 74-76. MR **80h**:35069.
[48] B.M. Budak and V.N. Vasil'eva, The solution of the inverse Stefan problem, in: *Reshenie zadachi Stefana* (Solution of the Stefan problem), Moscow State Univ., Moscow 1970, 65-86. MR **57** # 3679.
[49] ———, F.P. Vasil'ev, and A.B. Uspenskii, Difference methods for solving certain boundary-value problems of Stefan type, in: *Chislennye metody v gazovoi dinamiki* (Numerical methods in gas dynamics), Moscow State Univ., Moscow 1965, no. IV, 139-183. MR **34** # 7126.
[50] ——— and Yu.L. Gaponenko, The solution of the Stefan problem containing a quasi-linear parabolic equation and a quasi-linear boundary condition, Moscow State Univ., Moscow 1974, 235-313. MR **57** # 6844.
[51] ——— and N.L. Gol'dman, On the stability of a quasi-equilibrium inverse Stefan problem for a quasi-lienar parabolic equation, in: *Priblizhennye metody resheniya zadach* (Approximate methods of solving problems), Moscow State Univ., Moscow 1972, 172-202.
[52] ——— and ———, The inverse Stefan problem with an unknown flow on the boundary for a quasi-linear parabolic equation, in: *Priblizhennye metody resheniya zadach optimal'nogo upravleniya i nekotorykh nekorrektnykh zadach* (Approximate methods for solving problems of optimal control and of some ill-posed problems, Moskow State Univ., Moscow 1972, 147-171.
[53] ———, ———, A.T. Egorova, and A.B. Uspenskii, The method of straightening of fronts for the solution of Stefan type problems in the multidimensional case, in: *Vychislitel'nye metody i programmirovanie* (Computing methods and programming) VIII, 103-120, Moscow State Univ., Moscow 1967. MR **44** # 3513.
[54] ——— and V.G. Melamed, Numerical solution of a Stefan type problem for a quasi-linear parabolic system, in: *Vychislitel'nye metody i programmirovanie* (Computing methods and programming) VIII, 121-138, Moscow State Univ., Moscow 1967. MR **44** # 3512.

[55] B.M. Budak and M.Z. Moskal, The classical solution of the multidimensional Stefan problem, Dokl. Akad. Nauk SSSR **184** (1969), 1263-1266. MR **41** # 8845.
= Soviet Math. Dokl. **10** (1969), 219-223.
[56] ——— and ———, The classical solution of a multidimensional many-phase Stefan type problem in a domain with a piecewise smooth boundary, Dokl. Akad. Nauk SSSR **191** (1970), 751-754. MR **41** # 7299.
= Soviet Math. Dokl. **11** (1970), 427-431.
[57] ——— and ———, The classical solution of a first boundary-value problem of Stefan for the multidimensional heat equation in a coordinate parallelepiped, in: *Resheniya zadachi Stefana* (Solutions of the Stefan problem), Moscow State Univ., Moscow 1971, 87-133. MR **58** # 1634.
[58] ———, E.N. Solov'ëva, and A.B. Uspenskii, A difference method with smoothing of coefficients for the solution of the Stefan problem, Zh. Vychisl. Mat. i Mat. Fiz. **5** (1965), 828-840. MR **33** # 8109.
= USSR Comp. Math. and Math. Phys. **5**:5 (1965), 59-76.
[59] Yu.A. Bychkov and S.V. Iordanskii, Instability of the boundary of the phase inteface in the process of phase change, Zh. Prikl. Mekh. i Tekhn. Fiz. **1980**, no. 5, 45-51.
= J. Appl. Mech. Tech. Phys. **21** (1980), 612-617.
[60] F.P. Vasil'ev, A variant of the method of straight lines for the solution of Stefan type problems, in: *Vychislitel'nye metody i programmirovanie* (Computing methods and programming) VIII, 139-164, Moscow State Univ., Moscow 1967. MR **44** # 3516.
[61] ———, A difference method of solving Stefan type problems for quasi-linear parabolic equations with discontinuous coefficients, Dokl. Akad. Nauk SSSR **157** (1964), 1280-1283. MR **29** # 5393.
= Soviet Math. Dokl. **5** (1964), 1109-1113.
[62] A.I. Veinik, *Teoriya zatverdevaniya otlivki* (The thoery of solidification of ingots), Mashgiz, Moscow 1960.
[63] T.D. Venttsel', A free boundary problem for the heat equation, Dokl. Akad. Nauk SSSR **131** (1960), 1000-1003. MR **22** # 8225.
= Soviet Math. Dokl. **1** (1960), 358-361.
[64] N.L. Gol'dman, On the solution of some inverse Stefan type problems for a quasi-linear parabolic equation, Proc. Conf. Young Scientists Fac. Comp. Math. and Cybernetics, Moscow State Univ., Moscow 1973, 12-21.
[65] ———, On the solution of a class of inverse Stefan type problems for a quasi-linear parabolic equation, in: *Resheniya zadach tipa Stefana* (Solutions of Stefan type problems), Moscow State Univ., Moscow 1974, 151-164.
[66] ———, On a class of non-linear inverse problems with an unknown boundary, Differentsial'nye Uravneniya, **19** (1983), 608-617. MR **84m**:35120.
= Differential Equations **19** (1983), 443-451.
[67] R. Glowinski, J.-L. Lions, and R. Trémolières, Analyse numerique des inéquations variationelles (2 vols), Dunod, Paris 1976. MR **58** # 31697a, b.
Translation: *Chislennoe issledovanie variatsionnykh neravenstv,* Mir, Moscow 1979.
[68] A.A. Golubeva, A standard program for the numerical solution of the Stefan problem for one-dimensional parabolic equations by a difference method with "fractional steps in x and t", in: *Metody resheniya kraevykh i obratnykh zadach teploprovodnosti* (Methods for the solution of boundary-value and inverse problems of heat conduction), Moscow State Univ., Moscow 1975, 38-56. MR **55** #4716.

[69] Lien-Kun Ku, The behaviour of the solution of the Stefan problem as time increases unboundedly, Dokl. Akad. Nauk SSSR **138** (1961), 263-266.
= Soviet Math. Dokl. 2 (1961), 570-573.
[70] V.N. Gusakov, I.I. Danilyuk, and A.V. Kostikov, The numerical analysis of the Stefan problem with heat shock, in: *Mat. Fizika i nelin. mekhanika* (Mathematical physics and non-linear mechanics), Naukova Dumka, Kiev 1986.
[71] I.I. Danilyuk, An existence theorem in a non-linear problem with free boundary, Ukrain. Mat. Zh. **20** (1968), 25-53. MR **37** # 456.
= Ukrainian Math. J. **20** (1968), 25-29.
[72] ———, Integral functionals with a variable domain of integration, Trudy Mat. Inst. Steklov **118** (1972), 1-112. MR **48** # 7055.
= Proc. Steklov Inst. Math. **118** (1972), 1-144.
[73] ———, On a variant of the two-phase Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1973**, no. 9, 783-787. MR **50** # 9187.
[74] ———, The crystallization process in the formation of hardened slag, Mat. Fiz. no. 17 (1975), 99-111. MR **58** # 25422.
[75] ———, On a crystallization problem of Stefan type, in: *Kraevye zadachi matematicheskoi fiziki i smeshannye voprosy teorii funktsii* (Boundary-value problems of mathematical physics and related questions of the theory of functions), II, Nauka, Leningrad 1979, 26-34. MR **81a**:80003.
[76] ———, The variational approach to the quasi-stationary Stefan problem, in: *Differentsial'nye i integral'nye uravneniya. Kraevye zadachi* (Differential and integral equations. Boundary value problems), Tbilisi Univ., Tbilisi 1979, 75-89.
[77] ———, A thermophysical problem with free boundary, Proc. 7th Conf. on Problems of Mathematical Physics, Karl-Marx-Stadt, vol. 1, 1979, 1-7.
[78] ———, A multidimensional non-stationary Stefan problem in domains with non-smooth boundary, Dokl. Akad. Nauk Ukrain. SSR Ser, A, **1980**, no. 11, 11-15. MR 82i:35082.
[79] ———, A variant of the multidimensional non-stationary Stefan problem in domains with a non-regular boundary, Mat. Fiz. no. 30 (1981), 48-59. MR **83k**:80005.
[80] ———, On the mixed problem for the quasilinear equation of heat conduction with piecewise continuous coefficients, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1981**, no. 7, 3-7.
[81] ———, On the mixed problem for the quasi-linear equation of heat conduction with piecewise continuous coefficients, Uspekhi Mat. Nauk **36**:4 (1981), 219-220.
[82] ———, On the two-phase quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1982**, no. 1, 6-10. MR **84a**:35108.
[83] ———, On a variant of the two-phase Stefan problem, in: *Obshchaya teoriya granichnykh zadach* (General theory of boundary value problems), Naukova Dumka, Kiev 1983, 73-81. MR **85m**:80003.
[84] ———, On the Stefan problem with heat shock, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1983**, no. 3, 9-14. MR **85d**:35051.
[85] ———, On the initial-boundary-value problem for the quasi-linear equation of heat conduction with discontinuous coefficients, in: *Uravneniya v chastnykh proizvodnykh i zadachi so svobodnoi granitsei* (Partial differential equations and problems with a free boundary), Naukova Dumka, Kiev 1983, 42-48. MR **85m**:35031.
[86] ———, On the initial-boundary-value problem for the quasi-linear equation of heat conduction with piecewise continuous coefficients, in: *Problemy matematiki i mekhaniki* (Problems of mathematics and mechanics), Nauka (Siberian Branch) Novosibirsk 1983, 81-94.

[87] I.I. Danilyuk, On the multidimensional one-phase quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1984**, no. 1, 13-17.
[88] ———, Boundary properties of the free boundary in the quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1985**, no. 10.
[89] ——— and V.E. Kashkakha, A non-linear Ritz system, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1973**, no. 10, 870-874. MR **48** # 3270.
[90] ——— and A.S. Minenko, An optimization problem with free boundary, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1976**, no. 5, 389-392. MR **56** # 9354.
[91] ——— and ———, The Ritz method in a non-linear problem with free boundary, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1978**, no. 4, 291-294. MR **58** # 13779.
[92] ——— and ———, A variational thermophysical problem with free boundary, Proc. Conf. on Mixed Boundary-Value Problems and Problems with a Free Boundary, Stuttgart 1978.
[93] ——— and ———, An approach to the analysis of the stationary Stefan problem in the presence of convection in the liquid phase, in: *Mat. fizika i nelin. mekhanika* (Mathematical physics and non-linear mechanics), Naukova Dumka, Kiev 1985.
[94] ——— and M.V. Oleinik, On the uniqueness of the solution of a non-linear problem with free boundary, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1972**, no. 3, 202-205. MR **46** # 2023.
[95] ——— and ———, The uniqueness theorem of a non-linear problem with free boundary, in: *Kraevye zadachi dlya uravnenii v chastnykh proizvodnykh* (Boundary-value problems for partial differential equations), Naukova Dumka, Kiev 1979, 21-33.
[96] ——— and ———, The control of the unknown boundary in the two-phase quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1983**, no. 4, 8-13. MR **85a**:80003.
[97] ——— and ———, On the inverse Stefan problem, Uspekhi Mat. Nauk **38**:5 (1983), 153-154.
[98] ——— and S.V. Salei, A variant of the two-phase Stefan problem in the presence of heat sources, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1975**, no. 11, 972-976. MR **53** # 7239.
[99] ——— and ———, The stability of the Stefan problem with convection and spherical symmetry, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1986**.
[100] ——— and S.I. Timchenko, Numerical analysis of the equation of solubility in the one-phase two-dimensional quasi-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1984**, no. 9, 6-9.
[101] S.P. Degtryarev, The classical solubility of a quasi-stationary Stefan problem with convection, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1985**, no. 9.
[102] ———, The classical solubility of the general stationary Stefan problem with convection, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1986**, no. 7.
[103] G. Duvaut and J.-L. Lions, Les inequations en mécanique et en physique, Dunod, Paris 1972. MR **57** # 4778.
Translation: Inequalities in mechanics and physics, Springer, Berlin-New York 1976. MR **58** # 25191.
Translation: *Neravenstva v mekhanike i fizike*, Nauka, Moscow 1980.
[104] V.P. Evlev, A method of solving Stefan type problems, in: A collection of papers in mathematics, Yakutsk 1975, 8-9.

[105] I.F. Zherebyat'ev and M.A. Tulegenov, Numerical solution of the problem of solidification of a moving melted substance, in: *Analiticheskie, chislennye i analogovye metody v zadachakh teploprovodnosti* (Analytic, numerical and analogue methods in problems of heat conduction), Naukova Dumka, Kiev 1977, 114-119.

[106] S.L. Kamenomostskaya, On the Stefan problem, Scientific lectures to high schools **1**:1 (1958), 60-62.

[107] ———, Some problems for equations of parabolic type with an unknown boundary, Thesis, Moscow State University, Moscow 1958.

[108] ———, On the Stefan problem, Mat. Sb. **53** (1961), 488-514. MR **25** # 5292.

[109] M.A. Kanibolotskii, The Stefan problem with boundary conditions of the fourth kind, Zh. Prikl. Mekh. i Tekhn. Fiz. **1973**, no. 4, 189-192.
= J. Appl. Mech. Tech. Phys. **14** (1973), 601-603.

[110] H.S. Carslaw and J.C. Jaeger, Conduction of heat in solids, Clarendon Press, Oxford 1947. MR **9**-188.
Translation: *Teploprovodnost' tverdykh tel,* Nauka, Moscow 1964.

[111] V.E. Kashkakha, A variational method in the study of some non-linear problems with free boundary, Thesis, Donetsk 1974.

[112] ———, On the approximate computation of the crystallization front of bars of rectangular cross-section, Mat. Fiz. no. 16 (1974), 93-96.

[113] ———, The Ritz method in the study of two-phase quasi-stationary Stefan type problems, Mat. Fiz. no. 17 (1975), 128-137. MR **56** # 7257.

[114] ———, The unconditional solubility of a non-linear Ritz system, in: *Kraevye zadachi matematicheskoi fiziki* (Boundary-value problems of mathematical physics), Naukova Dumka, Kiev 1979, 93-96.

[115] ——— and I.I. Danilyuk, A non-linear spatial problem with free boundary, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1973**, no. 2, 119-123. MR **47** # 3833.

[116] E.I. Kim and G.I. Bizhanova, Study of the second boundary-value problem of Stefan for small values of time, Vestnik Akad. Nauk Kazakh. SSR **1981**, no. 6, 76-86.

[117] ——— and ———, On a class of integro-differential equations, Vestnik Akad. Nauk Kazakh. SSR **1982**, no. 5.

[118] ———, V.T. Omel'chenko, and S.N. Kharin, *Matematicheskie medeli teplovykh protsessov v elektricheskikh kontaktkh* (Mathematical models of heat processes in electrical contact), Nauka, Alma-Ata 1977. MR **58** # 14426.

[119] L.A. Kozdoba, Methods of solving problems of solidification (a survey), Fizika i khimiya obrabotki materialov, **1973**, no. 2, 41-59.

[120] ———, *Metody resheniya nelineinykh zadach teploprovodnosti* (Methods of solving non-linear heat conduction problems), Nauka, Moscow 1975.

[121] V.S. Kolesov and A.V. Uglanov, The maximum principle for Stefan problems, in: *Issledovaniya po ustoichivosti i teorii kolebanii* (Stability and perturbation theory), Yaroslavl' 1981, 54-63.

[122] N.E. Kochin, I.A. Kibel', and N.V. Roze, *Teoreticheskaya gidrodinamika,* part II, Fizmatgiz, Moscow 1963, 1-727. MR **30** # 5598.
Translation: Theoretical hydrodynamics, Interscience, New York-London-Sydney 1964.

[123] S.N. Krozhkov, On some problems with unknown boundaries for the heat conduction equation, Zh. Prikl. Mat. Mekh. **31** (1967), 1009-1020. MR **37** # 6618.
= J. Appl. Math. Mech. **31** (1967), 1014-1024.

[124] S.N. Krozhkov and S.L. Yakubov, The solubility of a certain class of problems with unknown boundary for the heat conduction equation; asymptotic behaviour of solutions as $t \to \infty$, Izv. Akad. Nauk UzSSR Ser. Fiz.-Mat. Nauk, **1978**, no. 6, 26-28. MR **80c**:35104.

[125] M.M. Lavrent'ev, *O nekotorykh nekorrektnykh zadachakh matematicheskoi fiziki* (Some ill-posed problems of mathematical physics), Siberian Branch of the USSR Acad. of Sci., Novosibirsk 1962. MR **36** # 5487, MR **36** # 7363.

[126] Yu.P. Ladikov, V.I. Vasil'ev, and I.V. Kazachkov, The stability of hardened slag in steel aggregates, Avtomatika, **1980**, no. 4, 80-85.

[127] ——— and V.F. Tkachenko, The influence of free convection on the stability of a planar crystallization front, Zh. Prikl. Mekh. Tekh. Fiz., **1971**, no. 6, 99-106. = J. Appl. Mech. Tech. Phys. **12** (1971), 870-876.

[128] O.A. Ladyzhenskaya, V.A. Solonnikov, and N.N. Ural'tseva, *Lineinye i kvazilineinye uravneniya parabolicheskogo tipa,* Nauka, Moscow 1967. MR **39** # 3159a.
Translation: Linear and quasilinear equations of parabolic type, AMS, Providence, RI, 1968. MR **39** # 3159b.

[129] J.-L. Lions, On partial differential inequalities, Uspekhi Mat. Nauk **53**:2 (1971), 205-263. MR **45** # 5539.

[130] ———, Qeulques méthodes de résolution des problèmes aux limites non linéaires, Dunod; Gauthier Villars, Paris 1969. MR **41** # 4326.
Translation: *Nekotorye metody resheniya nelineinykh kraevykh zadach,* Mir, Moscow 1972.

[131] M.K. Likht and S.B. Kuz'ninskaya, On the solidification of binary alloys in a temperature interval, Fizika Metallov i Metallovedeniya **11** (1961), 878-882.

[132] B.Ya. Lyubov, *Teoriya kristallizatsiya v bol'shikh ob'emakh* (The theory of crystallization in large volumes), Nauka, Moscow 1975.

[133] V.V. Mansurov and N.G. Fomina, Instability of the automodel front of the crystallization of a one-component fusion, Inzh.-Fiz. Zh. **45** (1983), 636-639.

[134] G.A. Martynov, On propagation of heat in a two-phase medium for a given law of motion of the phase boundary, Zh. Tekh. Fiz. **25** (1955), 1754-1767. MR **19**-710.

[135] ———, On the solution of the inverse problem of Stefan for a half-space when the phase boundary moves according to a linear law, Dokl. Akad. Nauk SSSR **109** (1956), 279. MR **18**-314.

[136] B.I. Medovar, V.L. Shevtsov, G.S. Marinskii, et al, *Teplovye protsessy pri elektroshlakovom pereplave* (Heat processes in an electrical melting), Naukova Dumka, Kiev 1978.

[137] A.M. Meirmanov, The Stefan problem that arises in the theory of combustion, Dinamika Sploshn. Sredy no. 11 (1972), 74-82. MR **56** # 10462.

[138] ———, A generalized solution of the Stefan problem in a medium with lumped capacity, Dinamika Sploshn. Sredy no. 10 (1972), 85-101. MR **56** # 17529.

[139] ———, Multiphase Stefan problem for quasi-linear parabolic equations, Dinamika Sploshn. Sredy no. 13 (1973), 74-86. MR **56** # 12602.

[140] ———, The unique solubility and asymptotic behaviour as $t \to \infty$ of the solution of a Stefan type problem arising in hydraulics, Dinamika Sploshn. Sredy no. 30 (1977), 98-111. MR **58** # 3927.

[141] ———, On the classical solubility of the multidimensional Stefan problem, Dokl. Akad. Nauk SSSR **249** (1979), 1309-1312. MR **81a**:35042. = Soviet Math. Dokl. **20** (1979), 1426-1429.

[142] A.M. Meirmanov, On the classical solution of the multi-dimensional Stefan problem for quasi-linear parabolic equations, Mat. Sb. **112** (1980), 170-192. MR **81j**:35055. = Math. USSR-Sb. **40** (1981), 157-178.
[143] ———, The Stefan problem: approximate modelling, Uspekhi Mat. Nauk **36**:4 (1981), 201.
[144] ———, On those solutions of the two-dimensional two-phase Stefan problem which are close to one-dimensional ones, Dinamika Sploshn. Sredy no. 50 (1981), 135-149. MR **83a**:35046.
[145] ———, Approximate models in the two-phase Stefan problem ("lumped capacity"), Dinamika Sploshn. Sredy no. 52 (1981), 56-77.
[146] ———, An example of the non-existence of a classical solution of the Stefan problem, Dokl. Akad. Nauk SSSR **258** (1981), 547-549. MR **84g**:35071. = Soviet Math. Dokl. **23** (1981), 564-566.
[147] ——— and V.V. Pukhnachëv, Lagrange coordinates in the Stefan problem, Dinamika Sploshn. Sredy no. 47 (1980), 90-111.
[148] A.S. Melikulov, The method of variation of the boundary in the one-phase quasi-stationary problem of crystallization, Dokl. Akad. Nauk UzSSR, **1982**, no. 8, 10-12.
[149] Methods of solving boundary-value and inverse problems of heat conduction, Proc. Comp. Centre, Moscow State Univ., Moscow 1975.
[150] A.T. Mikulets, The classical solution of the Stefan problem for systems of equations of parabolic type with exterior fronts, Dokl. Akad. Nauk MSSR **94** (1976), 14-24. MR **54** # 8029.
[151] A.S. Minenko, On an optimization problem, Mat. Fiz. no. 23 (1978), 74-77.
[152] ———, On a thermophysical problem with free boundary, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1979**, no. 6, 413-416. MR **58** # 23935.
[153] ———, The convergence of the Ritz method in a problem with free boundary, in: *Kraevye zadachi matematicheskoi fiziki* (Boundary-value problems of mathematical physics), Naukova Dumka, Kiev 1979, 139-145.
[154] ———, A thermophysical problem with free boundary, Thesis, Donetsk 1979.
[155] ———, The solubility of a thermophysical problem with free boundary, in: *Granichnye zadachi dlya differentsial'nykh uravnenii* (Boundary-value problems for differential equations), Naukova Dumka, Kiev 1980, 104-115. MR **84a**:35294.
[156] ———, On a non-linear problem with free boundary, in: *Granichnye zadachi matematicheskoi fiziki* (Boundary-value problems of mathematical physics), Naukova Dumka, Kiev 1981, 85-86.
[157] ———, The solubility of a non-linear thermophysical problem, Mat. Fiz. no. 30 (1981), 77-83. MR **83m**:80007.
[158] ———, Certain estimates of the solution of the quasi-stationary one-phase Stefan problem, Mat. Fiz. no. 31 (1982), 102-105. MR **84a**:35295.
[159] ———, On the variational method of studying the quasi-stationary Stefan problem, in: *Uravneniya v chastnykh proizvodnykh i zadachi so svobodnoi granitsei* (Partial differential equations and problems with a free boundary), Naukova Dumka, Kiev 1983, 82-85.
[160] Din Chi Nguen, A problem with free boundary for a parabolic equation, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1966**, no. 2, 40-54; **1966**, no. 5, 51-62. MR **33** # 2967; MR **33** # 7706.
[161] L. Nirenberg, Topics in non-linear functional analysis, Courant Inst. of Math. Sci., New York Univ., New York 1974. MR **58** # 7672.
Translation: Lektsii po nelineinomu funktsional'nomu analizu, Mir, Moscow 1977.
[162] M.V. Oleinik, On the existence and uniqueness of the solution of certain non-linear problems with free boundary, Thesis, Donetsk 1980.

[163] O.A. Oleinik, Discontinuous solutions of non-linear differential equations, Uspekhi Mat. Nauk **12**:3 (1957), 3-73. MR **20** # 1055.

[164] ———, On a method of solving the general Stefan problem, Dokl. Akad. Nauk SSSR **135** (1960), 1054-1057. MR **23** # A2644.
= Soviet Math. Dokl. **1** (1960), 1350-1353.

[165] ———, Boundary-value problems for linear equations of elliptic and parabolic type with discontinuous coefficients, Izv. Akad. Nauk SSSR Ser. Mat. **25** (1961), 3-20. MR **23** # A1136.

[166] ———, *Some non-linear problems of the theory of partial differential equations*, Proc. 1st Math. Summer School, Part II, Naukova Dumka, Kiev 1964, 117-255. MR **33** # 6123.

[167] ——— and S.N. Kurzhkov, Certain non-linear problems for equations of elliptic type, Uspekhi Mat. Nauk **15**:5 (1960), 203-206.

[168] Optimization of regimes of solidification of a continuous ingot, Zinatne, Riga 1977, 1-147.

[169] A.G. Petrova, The local solubility of the thermodiffusion Stefan problem, Dinamika Sploshn. Sredy no. 58 (1982), 156-163. MR **85e**:80004.

[170] A.A. Pomerantsev, On the theory of melting and combustion of a body, (the Stefan problem), Proc. 1st Conf. on Heat and Mass Exchange, 1961, vol. 3 (1963).

[171] M.A. Pudovkin, Solution of the two-phase Stefan problem for a cylindrical domain, in: *Issledovanie po prikladnoi matematike* (Investigations in applied mathematics), Kazan' State Univ., Kazan' 1945, 97-107.

[172] E.V. Radkevich, An approach to the solution of problems of filtration theory, Dokl. Akad. Nauk SSSR **249** (1979), 1321-1324. MR **81e**:35035.
= Soviet Math. Dokl. **20** (1979), 1440-1443.

[173] ——— and A.S. Melikulov, On the solubility of a two-phase quasi-stationary problem of crystallization, Dokl. Akad. Nauk SSSR **265** (1982), 58-62. MR **84f**:35063.
= Soviet Phys. Dokl. **27** (1982), 540-542.

[174] Solutions of Stefan problems, Proc. Comp. Centre, Moscow State Univ., Moscow 1971.

[175] Solutions of Stefan type problems, Proc. Comp. Centre, Moscow State Univ., Moscow 1974.

[176] L.I. Rubinshtein, *Problema Stefana* (The Stefan problem), Izdat. Zvaizgne, Riga 1967. MR **36** # 5488.

[177] Yu.S. Ryazantsev, Calculation of density variation which occurs during phase transformation in Stefan's problem, Prikl. Mat. Mekh. **25** (1961), 1143-1144.
= J. Appl. Math. Mech. **25** (1961), 1706-1708.

[178] K. Sage, Optimal control by systems with free boundaries by means of variational inequalities, in: *Vychislitel'nye metody v prikladnoi matematike* (Numerical methods in applied mathematics), Nauka, Novosibirsk 1982, 72-86.

[179] S.V. Salei, The global solubility of a Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1979**, no. 6, 424-428.

[180] ———, A problem with free boundary in the presence of heat sources, in: *Kraevye zadachi matematicheskoi fiziki* (Boundary-value problems of mathematical physics), Naukova Dumka, Kiev 1979, 172-192.

[181] ———, The analyticity of the free boundary in a Stefan problem, in: *Granichnye zadachi dlya differentsial'nykh uravnenii* (Boundary-value problems for differential equations), Naukova Dumka, Kiev 1980, 157-166. MR **84a**:35297.

[182] ———, The global solubility of a problem with free boundary, in: *Granichnye zadachi matematicheskoi fiziki* (Boundary-value problems of mathematical physics), Naukova Dumka, Kiev 1981, 117-119.

[183] S.V. Salei, On the rate of convergence of the free boundary to its limit in a Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1981**, no. 6, 19-23. MR **82m**:35067.
[184] ———, A Stefan problem in a cylindrical domain with axial symmetry, Mat. Fiz. no. 30 (1981), 90-96. MR **84j**:35085.
[185] ———, The asymptotic behaviour of the solution of a Stefan problem, in: *Uravneniya v chastnykh proizvodnykh i zadachi so svobodnoi granitsei* (Partial differential equations and problems with a free boundary), Naukova Dumka, Kiev 1982, 97-101.
[186] ———, A Stefan problem in the presence of heat sources, Thesis, Donetsk 1982.
[187] A.A. Samarskii and B.D. Moiseenko, An efficient scheme for through computation in a many-dimensional Stefan problem, Zh. Vychisl. Mat. i Mat. Fiz. **5** (1965), 816-827. MR **34** # 3807.
= USSR Comp. Math. and Math. Phys. **5**:5 (1965), 43-58.
[188] Yu.A. Samoilovich, *Sistemnyi analiz kristallizatsii slitka* (Systematic analysis of the crystallization of an ingot), Naukova Dumka, Kiev 1983.
[189] E.N. Solov'eva, On the solution of the one-dimensional Stefan problem in a gradient medium, in: *Vychislitel'nye metody i programmirovanie* (Numerical methods and programming), Proc. Comp. Centre, Moscow State Univ. no. 26 (1982), 44-51.
[190] ——— and A.B. Uspenskii, Schemes of through computation of the numerical solution of the Stefan problem for the two-dimensional equation of heat conduction, in: *Metody resheniya kraevykh i obratnykh zadach teploprovodnosti* (Methods of solving boundary-value and inverse problems of heat conduction), Moscow State Univ., Moscow 1975, 24-37. MR **55** # 6881.
[191] *Teplovye protsessy pri nepreryvnom let'e stali* (Heat processes in the continuous casing of steel), Metallurgiya, Moscow 1982.
[192] A.N. Tikhonov, On equations of heat conduction for several variables, Bull. Moscow State Univ. Ser. A1, **1938**, no. 9.
[193] ———, Ill-posed problems, in: *Vychislitel'nye metody i programmirovanie* (Computing methods and programming), Proc. Comp. Centre, Moscow State Univ. no. 8 (1967), 3-33.
[194] ——— and A.A. Samarskii, *Uravneniya matematicheskoi fiziki,* 3rd Ed. Nauka, Moscow 1966. MR **35** # 1982.
Translation of 2nd ed.: Equations of mathematical physics, Macmillan, New York 1963. MR **29** # 2498.
[195] *Uravneniya v chastnykh proizvodnykh i zadachi so svobodnoi granitsei* (Partial differential equations and problems with free boundary), Naukova Dumka, Kiev 1983.
[196] A.B. Uspenskii, A method of rectifying the fronts for many-front one-dimensional Stefan type problems, Dokl. Akad. Nauk SSSR **172** (1967), 61-64. MR **35** # 4550.
= Soviet Phys. Dokl. **12** (1967), 26-29.
[197] U. Felgenhauer, Certain multidimensional Stefan problems, Thesis, Donetsk 1981.
[198] ———, A one-phase non-stationary Stefan problem, Dokl. Akad. Nauk Ukrain. SSR Ser. A, **1981**, no. 1, 30-32. MR **84g**:35077.
[199] ———, On the solubility of the one-dimensional quasi-linear Stefan problem, Dokl. Akad. Nauk Ukrain. SSR, **1982**, no. 5, 24-27.
[200] A. Friedman, Partial differential equations of parabolic type, Prentice-Hall, Englewood Cliffs, NJ, 1964. MR **31** # 6062.
Translation: *Uravneniya s chastnymi proizvodnymi parabolicheskogo tipa,* Mir, Moscow 1968.

[201] N.V. Khusnutdinova, On the behaviour of solutions of the Stefan problem as time increases unboundedly, Dinamika Sploshn. Sredy no. 2 (1969), 168-178.
[202] A.B. Tsibulis, Elliptic equations with discontinuous non-linearities and certain problems of optimal control, Thesis, Riga 1982.
[203] ———, The theorem of solubility of a non-linear elliptic boundary-value problem and its application in problems of optimal control, Latv. Mat. Ezhegodnik, no. 27 (1983), Riga, 131-140.
[204] O.M. Chekmarëva, On the motion of the phase interface surface for large values of time in an axisymmetrical Stefan problem, Zh. Tekhn. Fiz. **45** (1975), 203-213.
[205] F.L. Chernous'ko, On the solution of non-linear problems of heat conduction in a medium with phase changes, Zh. Prikl. Mekh. Tekhn. **1969**, no. 2, 6-14. = J. Appl. Mech. Tech. Phys. **10** (1969), 167-175.
[206] *Chislennye metody v gazovoi dinamike* (Numerical methods in gas dynamics), Moscow State Univ. **1965**, no. 4.
[207] V.V. Shapovalenko, Computation of the temperature field and crystallization front of a hollow cylindrical rod, Mat. Fiz. no. 22 (1977), 98-102.
[208] ———, Minimization of a functional with a variable domain of integration in the three-dimensional case, in: *Teoriya optimal'nykh protsessov* (The theory of optimal processes), Akad. Naul Ukrain. SSR, Inst. Kibernet., 1977, Preprint No. 11, 25-30. MR **58** # 30672.
[209] G.I. Shishkin, A heat problem with free boundary, Dokl. Akad. Nauk SSSR **197** (1971), 1276-1279. MR **43** # 3649. = Soviet Math. Dokl. **12** (1971), 682-685.
[210] ———, A problem with free boundary for a system of parabolic equations, in: *Chislennye metody mekhaniki sploshnoi sredy* (Numerical methods of the mechanics of a continuous medium), Comp. Centre, Akad. Nauk SSSR **3**:2 (1972). MR **51** # 13460.
[211] ———, A Stefan type problem with a discontinuous moving boundary, Dokl. Akad. Nauk SSSR **224** (1975), 1276-1278. MR **54** # 8030. = Soviet Math. Dokl. **16** (1975), 1409-1411.
[212] S. Yakubov, The uniqueness of the solution of a certain class of problems with unknown boundary for a parabolic equation of second order, Izv. Akad. Nauk SSSR Ser. Fiz. Mat. **4** (1975), 35 - 39. MR **53** # 1026.
[213] A. Acker, Concerning Daniljuk's existence theorem for boundary value problems with the Bernoulli condition, Proc. Amer. Math. Soc. **80** (1980), 451-454. MR **82b**:49028.
[214] ———, Interior free boundary problems for the Laplace equation, Arch. Rat. Mech. Anal. **75** (1981), 157-168. MR **82f**:35185.
[215] E.L. Albasiny, Proc. Inst. Electr. Engin. Part B **103** (1956), Supp. 1, 158-164.
[216] C. Baiocchi, Sur une problème à frontiere libre traduisant le filtrage de liquides à traverse des milieux poreux, C.R. Acad. Sci. Paris sér. A **273** (1971), 1215-1217. MR **45** # 6265.
[217] ———, Problème à frontière libre en hydraulique, C.R. Acad. Sci. Paris, **278** (1974), 1201-1204. MR **50** # 2673.
[218] ———, Problèmes à frontière libre et inéquations variationelles, C.R. Acad. Sci. Paris sér A **283** (1976), 29-32. MR **55** # 846.
[219] G. Borgiolli and E. Benedetto, Stefan problems with non-linear boundary conditions; the polygonal method, ZAMM, **58** (1978), 539-546. MR **80a**:65208.
[220] H. Brézis, On some degenerate non-linear parabolic equations, Proc. Symposium on non-linear Functional Analysis, Chicago, April 1968. MR **42** # 8346.

[221] H. Brézis and D. Kinderlehrer, The smoothness of solutions to non-linear variational inequalities, Indiana Univ. Math. J. **23** (1974), 831-844. MR **50** # 13881.
[222] ——— and A. Friedman, Estimates on the support of solutions of parabolic variational inequalities, Illinois J. Math. **20**:1 (1976), 82-97. MR **52** # 11327.
[223] M. Brillouin, Sur quelques problèrnes nonrésoloues de la physique mathématique classique, Propagation de la fusion. Ann. de l'Inst. H. Poincaré, **1** (1931), 285-308.
[224] J.R. Cannon, Two phase moving boundary problems, Conf. on moving boundary problems (Tennessee, September 1977), Academic Press, New York 1978, 3-24.
[225] ——— and E. Di Benedetto, The steady state Stefan problem with convection, with mixed temperature and non-linear heat flux boundary conditions, in: Free boundary problems, Rome 1980, 231-265. MR **83c**:35114.
[226] ——— and ———, On the existence of weak solutions to an n-dimensional Stefan problem with non-linear boundary conditions, SIAM J. Math. Anal. **11** (1980), 632-642. MR **81j**:35058.
[227] ———, ———, and G.H. Knightly, The steady state Stefan problem with convection, Arch. Rat. Mech. Anal. **73** (1980), 79-97. MR **81a**:35085.
[228] ——— and J. Douglas Jr, The stability of the boundary in a Stefan problem, Ann. Scuola Norm. Sup. Piza, **21** (1967), 89-91. MR **42** # 4891.
[229] ———, D.B. Henry, and D.H. Kotlow, Continuous differentiability of the free boundary for weak solutions of the Stefan problem, Bull. Amer. Math. Soc. **80** (1974), 45-48. MR **48** # 11768.
[230] ——— and C.D. Hill, Remarks on a Stefan problem, J. Math. Mech. **17** (1967), 433-442. MR **36** # 1854.
[231] ——— and ———, Existence, uniqueness, stability and monotone dependence in a Stefan problem for the heat equation, J. Math. Mech. **17** (1967), 1-19. MR **42** # 4893.
[232] ——— and ———, On the infinite differentiability of the free boundary in a Stefan problem, J. Math. Anal. Appl. 22 (1968), 385-397. MR **37** # 610.
[233] ——— and M. Primicerio, Remarks on the one-phase Stefan problem for the heat equation with the flux prescribed on the fixed boundary, J. Math. Anal. Appl. **35** (1971), 361-373. MR **43** # 4357.
[234] L.A. Caffarelli, The smoothness of the free surface in a filtration problem, Arch. Rat. Mech. Anal. **63** (1976), 77-86. MR **54** # 13696.
[235] ———, The regularity of elliptic and parabolic free boundaries, Bull. Amer. Math. Soc. **82** (1976), 616-618. MR **53** # 8662.
[236] ———, The regularity of free boundaries in higher dimensions, Acta. Math. **139** (1977), 155-184. MR **56** # 12601.
[237] ———, Some aspects of the one-phase Stefan problem, Indiana Univ. Math. J. **27** (1978), 73-77. MR **57** # 6368.
[238] ———, The Stefan problem in n dimensions, Proc. Internat. Meeting "Recent methods in Non-linear Analysis" (Rome 1978), Pitagora, Bologna 1979, 13-24.
[239] ———, A remark on the Hausdorff measure of the free boundary and the convergence of coincidence sets, Boll. Unione Mat. Ital. **18** (1981), 1297-1299. MR **82i**:35078.
[240] ——— and L.C. Evans, Continuity of the temperature for the two phase Stefan problem, Arch. Rat. Mech. Anal. **81** (1983), 199-220. MR **84g**:35070.
[241] ——— and A. Friedmann, The one-phase Stefan problem and the porous medium diffusion equation. Continuity of the solution in n space dimensions, Proc. Nat. Acad. Sci. USA **75** (1978), 2084. MR **81h**:35021.

[242] L.A. Caffarelli and A. Friedman, Continuity of the temperature in the Stefan problem, Indiana Univ. Math. J. **28** (1979), 53-70. MR **80i**:35104.
[243] P.L. Chambré, On the dynamics of phase growth, Quart. J. Mech. Appl. Math. **9** (1956), 224-233. MR **18**-358.
[244] J. Chadam and P. Ortoleva, The stabilizing effect of surface tension on the development of the free boundary in a planar one-dimensional Cauchy-Stefan problem, IMA J. Appl. Math. **30** (1983), 57-66. MR **84j**:35157.
[245] M. Chipot, Sur la regularité de la solution d'inequations variationelles elliptiques, C.R. Acad. Sci. Paris **288** (1979), 543-546. MR **80d**:35042.
[246] J. Crank, The mathematics of diffusion, Clarendon Press, Oxford 1956. MR **18**-616.
[247] A.B. Crowley, On the weak solution of moving boundary problems, J. Inst. Math. Appl. **24** (1979), 43-57. MR **80d**:65129.
[248] C.W. Cryer, A bibliography of free boundary problems, Math. Research Center, Technical Summary, University of Wisconsin, Madison, report no. 1793, 1977.
[249] A. Damlamian, Some results on the multi-phase Stefan problem, Commun. Partial Diff. Equ. 2 (1977), 1017-1044. MR **58** # 6694.
[250] ———, Homogenization of the Stefan problem, C.R. Acad. Sci. Paris, ser. A-B, **289** (1979), 9-12. MR **80f**:35067.
[251] ———, The homogenization of the Stefan problem and related topics, in: Free boundary problems, part II, Rome 1980, 267-275.
[252] ———, How to homogenize a non-linear diffusion equation: Stefan's problem, SIAM J. Math. Ann. **12** (1981), 306-314. MR **82f**:35187.
[253] ——— and N. Kenmochi, The Stefan problem with non-stationary boundary conditions, Hiroshima Math. J. **10** (1980), 271-293.
[254] P.V. Danckwerts, Unsteady-state diffusion or heat conduction with moving boundary, Trans. Faraday Soc. **46** (1950), 701-712.
[255] I.I. Danilyuk, Über eine Klasse von Problemen mit freiem (unbekanntem) Rand für elliptische Gleichungen, in: Lecture Notes in Math. **561** (1976), 487-501. MR **58** # 1579.
[256] ———, On integral functionals with a variable domain of integration, Amer. Math. Soc., Providence, RI, 1976, 1-144. MR **53** # 8992.
[257] ———, On the direct and inverse two-phase quasilinear Stefan problem, ICM-82, Short Comm., XI, 47.
[258] J. Douglas Jr, Uniqueness theorem for the solution of a Stefan problem, Proc. Amer. Math. Soc. 8 (1957), 402-408. MR **19**-1060.
[259] ———, J.R. Cannon, and C.D. Hill, A multiboundary Stefan problem and disappearence of phases, J. Math. and Mech. **17** (1967), 21-34. MR **42** # 4892.
[260] M.G. Duvaut, Résolution d'un problème de Stefan (Fusion d'un bloc de glace à zero degré), C.R. Acad. Sci. Paris, ser. A, **276** (1973), 1461-1463. MR **48** # 6688.
[261] ———, Solution of two phase Stefan problem by variational inequality, in: Proc. Symp. on moving boundary problems, Oxford, March 1974, 25-27.
[262] ———, Etude de problèmes unilateraux en mécanique par des méthodes variationnelles, in: New variational techniques in mathematical physics, Rome 1974, 43-102.
[263] ———, Two phase Stefan problem with varying specific heat coefficients, An. Acad. Brasil. Cience **47** (1975), 377-380. MR **58** # 19876.
[264] C.M. Elliott and J.R. Ockendon, Weak and variational methods for moving boundary problems, Pitman, London 1982. MR **83i**:35157.

[265] G.H. Evans II, A note on the eixstence of a solution to a problem of Stefan, Quart. Appl. Math. **9** (1951), 185-193. MR **13**-243.
[266] A. Fasano and M. Primicerio, General free boundary problems for the heat equation. II, J. Math. Ann. Appl. **57** (1977), 694-723; II, **58** (1977), 202-231; III, **59** (1977), 1-14. MR **58** # 6695a, b, c.
[267] Free boundary problems, Proc. Sem. (Pavia), 1st Naz. Alta Mat. Roma 1980.
[268] Free boundary problems, Proc. Sympos. (Montecatini), Pitman, London 1982.
[269] A. Friedman, The Stefan problem in several space variables, Trans. Amer. Math. Soc. **132** (1968), 51-87. MR **37** # 3209.
[270] ———, One domensional Stefan problem with non-monotone free boundary, Trans. Amer. Math. Soc. **133** (1968), 89-114. MR **37** # 3210.
[271] ———, The shape and smoothness of the free boundary for some elliptic variatonal inequalities, Indiana Univ. Math. J. **25** (1976), 103-118. MR **52** # 14614.
[272] ———, Analyticity of the free boundary for the Stefan problem, Arch. Rat. Mech. Anal. **61** (1976), 97-125. MR **53** # 11227.
[273] ———, Time dependent free boundary problems, SIAM Review **21** (1979), 213-221. MR **81g**:76097.
[274] ———, Variational principles and free boundary problems, Wiley, New York 1982. MR **84e**:35153.
[275] ——— and R. Jensen, Convexity of the free boundary in the Stefan problem and in the dam problem, Arch. Rat. Mech. Anal. **67** (1978), 1-24. MR **82i**:35100.
[276] ——— and D. Kinderlehrer, A one phase Stefan problem, Indiana Univ. Math. J. **25** (1975), 1005-1035. MR **52** # 6190.
[277] K. Friedriehs, Über ein Minimum Problem für Potentialströmungen mit freiem Rnade, Math. Ann. **109** (1933), 60-82.
[278] U. Felgenhauer, Monotonie und Glätte des freien Randes bei einem Stefanschen Problem, Wiss. Z. Tech. Hochsch. Magdeburg **24**:5 (1980), 29-31. MR **83c**:35121.
[279] F. Gastaldi, About the possibility of setting Stefan like problems in variational form, Boll. Un. Mat. Ital. A (5) **16** (1979), 148-156. MR **80c**:35103.
[280] H. Gajewski, Stabilitätsaussagen für einige Aufgaben mit freier Grenze, Z. Angew. Math. Mech. **57** (1977), 439-447. MR **57** # 8506.
[281] P.R. Garabedian, H. Lewy, and M. Schiffer, Axially symmetric cavitational flow, Ann. Math. **56** (1952), 560-602. MR **14**-810.
[282] ——— and D.C. Spencer, Extremal method in cavitational flow, J. Rat. Mech. Anal. **1** (1952), 359-409. MR **14**-102.
[283] C. Gerhardt, Regularity of solutions of non-linear variational inequalities, Arch. Rat. Mech. Anal. **52** (1973), 389-393. MR **51** # 13435.
[284] Ei-Ichi Hanzawa, Classical solutions of the Stefan problems, Tôhoku Math. J. **33** (1981), 297-335. MR **82k**:35065.
[285] C.D. Hill and D.B. Kotlow, Classical solutions in the large of a two-phase free boundary problem, J. Arch. Rat. Mech. Anal. **45** (1972), 63-78; **47** (1972), 369-379. MR **51** # 6149; **48** # 11771.
[286] K. Ichikawa and N. Kikuchi, A one-phase multidimensional Stefan problem by the method of variational inequalities, Internat. J. Numer. Meth. Eng. **14** (1979), 1197-1220. MR **80e**:65113.
[287] ———, A two-phase multidimensional Stefan problem by the method of variational inequalities, Internat. J. Numer. Meth. Eng. **14** (1979), 1221-1240.
[288] R. Jensen, The smoothness of the free boundary in the Stefan problem with supercooled water, Ill. J. Math. **22** (1978), 623-629. MR **81g**:35057.

[289] J.W. Jerome, Existence and approximation of weak solutions of the Stefan problem with non-monotone non-linearities, in: Lecture notes in Math. **506** (1976), 108-156. MR **57** # 8077.
[290] ———, Non-linear equations of evolution and a generalized Stefan problem, J. Differential Equations **26** (1977), 240-261. MR **58** # 1659a.
[291] D. Kinderlehrer, The smoothness of the solution of the boundary solution obstacle problem, J. Math. Pure Appl. **60** (1981), 193-212. MR **84j**:49011.
[292] ———, Variational inequalities and free boundary problems, Bull. Amer. Math. Soc. **84** (1978), 7-26. MR **57** # 6759.
[293] ——— and L. Nirenberg, The smoothness of the free boundary in the one phase Stefan problem, Séminare Brézis, 1975.
[294] ——— and ———, Regularity in free boundary problems, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), 373-391. MR **55** # 13066.
[295] ——— and ———, The smoothness of the free boundary in the one phase Stefan problem, Comm. Pure Appl. Math. **31** (1978), 257-282. MR **82b**:35152.
[296] ——— and ———, Analyticity on the boundary of solutions of non-linear second-order parabolic equations, Comm. Pure Appl. Math. **31** (1978), 283-338. MR **57** # 888.
[297] ———, ———, and J. Spruck, Regularity in elliptic free boundary problems. I, J. d'Analyse Math. **34** (1978), 86-119; II, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **6** (1979), 637-683. MR **83d**:35060; **83d**:35061.
[298] ——— and G. Stampacchia, An introduction to variational inequalities and their applications, Academic Press, New York 1980. MR **81g**:49013.
[299] I.I. Kolodner, Free boundary problem for the heat equation with applications to problems of change of phase, Comm. Pure. Appl. Math. **9** (1956), 1-31. MR **19**-285.
[300] W.T. Kyner, An existence and uniqueness theorem for a non-linear Stefan problem, J. Math. Mech. **8** (1959), 483-498. MR **26** # 1630.
[301] A. Lacey and M. Shillor, The existence and stability of regions with superheating in the classical two-phase one-dimensional Stefan problems with heat sources, IMA J. Appl. Math. **30** (1983), 215-230. MR **84i**:80004.
[302] G. Lamé and B.P. Clapeyron, Mémoire sur la solidification par refroidissement d'un globe solid, Ann. Chem. Phys. **47** (1831), 250-256.
[303] H. Lewy and G. Stampacchia, On the regularity of the solution of a variational inequality, Comm. Pure Appl. Math. 22 (1969), 153-188. MR **40** # 816.
[304] J.-L. Lions, On free surface problems: methods of variational and quasivariational inequalities, Lect. Notes in Math. **461** (1975), 129-148. MR **53** # 3821.
[305] ———, Asymptotic behaviour of solutions of variational inequalities with highly oscillating coefficients, Lect. Notes in Math. **503** (1976), 30-55. MR **58** # 29088.
[306] G.H. Meyer, Multidimensional Stefan problem, SIAM J. Numer. Anal. **10** (1973), 522-538. MR **48** # 10139.
[307] J. Moser, A new technique for the construction of solutions of non-linear differential equations, Proc. Nat. Acad. Sci. USA **47** (1961), 1824-1831. MR **24** # A2695.
[308] J.R. Ockendon and W.R. Hodgkins (eds), Moving boundary problems in heat flow and diffusion, Clarendon Press, Oxford 1975.
[309] D.C. Wilson, A.D. Solomon, and P.T. Boggs (eds), Moving boundary problems, Academic Press, New York-London 1978.
[310] J. Nash, The imbedding problem for Riemannian manifolds, Ann. of Math. (2) **63** (1956), 20-63. MR **17**-782.

[311] M. Niezgódka and I. Pawlow, A generalized Stefan problem in several space variables, Appl. Math. Optim. **9** (1983), 193-224. MR **84g**:35166.

[312] J.A. Nitsche, A finite element method for parabolic free boundary problems, in: Free boundary problems, Rome 1980, 277-318.

[313] O.A. Oleinik, On Stefan-type free boundary problems for parabolic equations, Seminari 1962-1963 di analisi, algebra, geometria e topologia, vol. 1, Rome 1965, 388-403.

[314] M. Primicerio, Problemi a contorno libero per equazione della diffusione, Rend. Sem. Mat. Torino **32** (1973/1974), 183-206. MR **57** # 16993.

[315] ———, Qualitative properties of some one-dimensional parabolic free boundary problems, in: Free boundary problems, Proc. Sem. (Pavia), 1st Naz. Alta Mat. Rome 1980, 451-460.

[316] D. Riabuschinsky, Sur un problème de variation, C.R. Acad. Sci. Paris **185** (1927), 840-841.

[317] J.F. Rodrigues, Free boundary convergence in the homogenisation of the one-phase Stefan problem, Trans. Amer. Math. Soc. **274** (1982), 297-305. MR **83k**:35084.

[318] L.I. Rubinstein, The Stefan problem: comments on its present state, J. Inst. Math. Appl. **24** (1979), 259-277. MR **80j**:35096.

[319] ———, Application of the integral equations technique to the solution of several Stefan problems, in: Free boundary problems, part II, Rome 1980, 383-450.

[320] ———, Global stability of the Neuman solution of the two-phase Stefan problem, IMA J. Appl. Math. **28** (1982), 287-301. MR **83i**:35021.

[321] D.G. Schaeffer, A new proof of the infinite differentiability of the free boundary in the Stefan problem, J. Differential Equations **20** (1976), 266-269. MR **52** # 11325.

[322] ———, Some examples of singularities in a free boundary, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), 133-144. MR **58** # 24345.

[323] A. Schatz, Free boundary problems of Stefan type with prescribed flux, J. Math. Anal. Appl. **28** (1969), 569-579. MR **42** # 2187.

[324] B. Sherman, General one-phase Stefan problems and free boundary problems for the heat equation with Cauchy data prescribed on the free boundary, SIAM J. Appl. Math. **20** (1971), 555-570. MR **44** # 4400.

[325] ———, Free boundary problems for the heat equation in which the moving interface coincides initially with the fixed face, J. Math. Anal. Appl. **33** (1971), 449-466. MR **43** # 733.

[326] J. Spruck, Regularity in elliptic free boundary problems, Proc. Internat. Meeting "Recent methods in Non-linear Analysis" (Rome 1978), Pitagora, Bologna 1979, 73-82. MR **82k**:35052.

[327] J. Stefan, Über einige Probleme der Theorie der Wärmeleitung, Sitzungsber. Wien. Akad. Mat. Natur. **98** (1889), 473-484.

[328] ———, Über die Diffusion von Säuren und Basen gegen einander, Sitzungsber. Wien. Akad. Mat. Natur. **98** (1889), 614-634.

[329] ———, Über die Theorie der Eisbildung, insbesondere über die Eisbildung in Polarmeere, Sitzungsber. Wien Akad. Math. Natur. **98** (1889), 965-983.

[330] ———, Über die Verdampfung und die Auflösung als Vorgänge der Diffusion, Sitzungsber. Wien. Akad. Mat. Natur. **98** (1889), 1418-1442.

[331] L.N. Tao, The Stefan problem with arbitrary initial and boundary conditions, Quart. Appl. Math. **36** (1978), 223-233. MR **80b**:80003.

[332] ———, The analyticity of solutions of the Stefan problem, Arch. Rat. Mech. Anal. 72 (1980), 285-301.

[333] D.A. Tarzio, Sur le problème de Stefan à deux phase, C.R. Acad. Sci. Paris **288** (1970), 941-944.
[334] A.B. Tayler, The mathematical formulation of the Stefan problem, in: Moving boundary problems, Clarendon Press, Oxford 1975, 120-138.
[335] A. Torelli, Existence and uniqueness of the solution for a non steady free boundary problem, Boll. Un. Mat. Ital. B (5) **14** (1977), 423-466. MR **58** # 29243.
[336] D.L. Turcotte, Geophysical problems with moving phase change boundaries and heat flow, in: Moving boundary problems, Clarendon Press, Oxford 1975, 91-101.
[337] A. Visintin, Sur le problème de Stefan avec flux non linéare, Boll. Un. Mat. Ital. C**18** (1981), 63-86.
[338] ———, General free boundary evolution problems in several space dimensions, Math. Anal. Appl. **95** (1983), 117-143. MR **84g**:35168.
[339] H. Weber, Die partiellen Differentialgleichungen der mathematischen Physik, Vol. 2, 6th ed., Braunschweig 1919.

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