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Two-dimensional Dirac operator and the theory of surfaces

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Abstract. A survey is given of the Weierstrass representations of surfaces in three- and four-dimensional spaces, their applications to the theory of the Willmore functional, and related problems in the spectral theory of the two-dimensional Dirac operator with periodic coefficients.

Contents

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§ 1. Introduction

In this paper we survey some results and problems relating to global representations of surfaces in three- and four-dimensional spaces in terms of solutions of the Dirac equation and to the Willmore functional and its generalizations.

This activity started ten years ago [1]. Under this approach, the Gauss map of a surface is represented in terms of solutions ψ of the Dirac equation

$$\mathscr{D}\psi = 0,$$

where $\mathscr{D}$ stands for the Dirac operator with potentials:

$$\mathscr{D} = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

This representation has different forms for surfaces in $\mathbb{R}^3$. However, in the above explicit form involving the Dirac equation, the representation was first written out in [2] to construct surfaces admitting soliton deformations (also introduced in [2]).

The appearance of an operator with a well-developed spectral theory made it possible to use this theory to study problems in the global theory of surfaces. Moreover, this approach explains the importance of the Willmore functional, since, up to a factor, this functional is the squared L_2-norm of the potential $U = V = \overline{U}$ of the operator $\mathscr{D}$ for surfaces in $\mathbb{R}^3$ [1].

The approach to the proof of the Willmore conjecture for tori proposed in [1] and [3] and based on the theory of spectral curves (at some energy level) [4] led to a very interesting paper by Schmidt [5], where substantial progress was achieved. However, the conjecture remained unproved.

In this case a spectral curve of the operator $\mathscr{D}$ with doubly periodic potentials leads to the notion of spectral curve of a torus in $\mathbb{R}^3$ [3], and the latter curve encodes a lot of information about the geometry of the surface.

Another approach to obtaining lower bounds for the Willmore functional involved methods of the inverse scattering problem and algebraic geometry of curves and led to estimates that are quadratic with respect to the dimension of the kernel of $\mathscr{D}$. Estimates of this kind were first obtained for spheres of revolution and some of their generalizations and conjectured for all spheres in [6] using the inverse scattering problem. These estimates were proved in full generality for surfaces of all genera in [7], where the theory of algebraic curves was applied to the theory of surfaces in a surprising and unusual way.

This representation was later extended to surfaces in $\mathbb{R}^4$ ([8], [9]) and to Lie groups of dimension three ([10], [11]). In [12] and [7] it was proposed to consider the representations of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ in the conformal setting from the very beginning. However, the analogues of the Willmore functional for non-commutative non-compact three-dimensional groups acquire the form

$$\int \left(\alpha H^2 + \beta \widehat{K} + \gamma\right) d\mu,$$

where H is the mean curvature, $\widehat{K}$ is the sectional curvature of the ambient space along the tangent plane to the surface, and $d\mu$ is the induced measure on the surface. We note that functionals of similar form,

$$\int \bigl(\alpha H^2 + \beta K + \gamma\bigr)\, d\mu,$$

are widely known in physics as Helfrich functionals [13] (see also, for instance, [14] and [15]). Even for surfaces in $\mathbb{R}^3$, these functionals are not conformally invariant for values of α, β, γ in general position. Here the term containing the Gaussian curvature K cannot be reduced to a topological term for surfaces with boundaries, which are of interest for physical applications.

Although up to now these representations have been mainly applied to problems related to the Willmore functional and its generalizations, we are sure that the representations can be effectively used to study other problems of the global theory of surfaces.

§ 2. Representations of surfaces in three- and four-dimensional spaces

2.1. Generalization of the Weierstrass formula for surfaces in $\mathbb{R}^3$. The Grassmann manifolds (Grassmannians) of oriented two-dimensional planes in $\mathbb{R}^n$ are diffeomorphic to quadrics in $\mathbb{C}P^{n-1}$.

Indeed, let us take a two-dimensional plane and choose a positively oriented orthonormal basis $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ in it, that is, $|u| = |v|$, $(u, v) = 0$. This basis is determined by a vector $y = u + iv \in \mathbb{C}^n$ such that

$$y_1^2 + \dots + y_n^2 = [(u, u) - (v, v)] + 2i(u, v) = 0.$$

The plane determines such a basis up to a rotation of the plane by an angle φ, $0 \leqslant \varphi \leqslant 2\pi$, and this results in the transformation $y \to re^{i\varphi}y$. Therefore, the Grassmannian $\widetilde{G}_{n,2}$ of oriented two-dimensional planes in $\mathbb{R}^n$ is diffeomorphic to the quadric

$$y_1^2 + \dots + y_n^2 = 0, \qquad (y_1 : \ldots : y_n) \in \mathbb{C}P^{n-1},$$

where $(y_1 : \ldots : y_n)$ are homogeneous coordinates in $\mathbb{C}P^{n-1}$. The Grassmannian $G_{n,2}$ of unoriented two-dimensional planes in $\mathbb{R}^n$ is the quotient space of the manifold $\widetilde{G}_{n,2}$ with respect to the free action of the antiholomorphic involution $y \to \overline{y}$.

For a given immersed surface

$$f \colon \Sigma \to \mathbb{R}^n$$

with a (local) conformal parameter z the Gauss map of this surface is

$$\Sigma \to \widetilde{G}_{n,2} \colon P \to (x_z^1(P) : \ldots : x_z^n(P)),$$

where $x^1, \dots, x^n$ are the Euclidean coordinates in $\mathbb{R}^n$ and $P \in \Sigma$.

There are only two cases in which the Grassmannian admits a rational parametrization, namely, the manifolds

$$\widetilde{G}_{3,2} = \mathbb{C}P^1 \quad \text{and} \quad \widetilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1$$

admit Weierstrass representations of surfaces.

Let us first consider surfaces in $\mathbb{R}^3$.

The Grassmannian $\widetilde{G}_{3,2}$ is the quadric

$$y_1^2 + y_2^2 + y_3^2 = 0,$$

which admits the following rational parametrization:[1]

$$y_1 = \frac{i}{2}(b^2 + a^2), \quad y_2 = \frac{1}{2}(b^2 - a^2), \quad y_3 = ab, \qquad (a:b) \in \mathbb{C}P^1.$$

We write

$$\psi_1 = a, \qquad \psi_2 = \bar{b},$$

and substitute these expressions into the formulae for $x_z^k = y_k$, $k = 1,2,3$. Since $x^k \in \mathbb{R}$ for any k, we have

$$\operatorname{Im} x^k_{z\bar{z}} = 0, \qquad k = 1,2,3.$$

In terms of ψ this condition becomes the Dirac equation

$$\mathscr{D}\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}\right]\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0, \qquad U = \overline{U}. \tag{1}$$

Moreover, if for a complex-valued function f we have $\operatorname{Im} f_{\bar{z}} = 0$, then locally we have $f = g_z$, where g is a real-valued function of the form

$$g = \int [\operatorname{Re} f\, dx - \operatorname{Im} f\, dy].$$

We have the following theorem.

Theorem 1. 1) [2] *If ψ satisfies the Dirac equation* (1), *then the formulae*

$$x^k = x^k(0) + \int (x^k_z\, dz + \bar{x}^k_z\, d\bar{z}), \qquad k = 1,2,3, \tag{2}$$

where

$$x^1_z = \frac{i}{2}(\overline{\psi}_2^2 + \psi_1^2), \qquad x^2_z = \frac{1}{2}(\overline{\psi}_2^2 - \psi_1^2), \qquad x^3_z = \psi_1\overline{\psi}_2 \tag{3}$$

define a surface in $\mathbb{R}^3$.

2) [1] *Every smooth surface in* $\mathbb{R}^3$ *is locally defined by formulae of the form* (2) *and* (3).

The proof of the second statement is given above, and the proof of the first statement is as follows: by the Dirac equation, the integrands in (2) are closed forms and, by the Stokes theorem, the values of the integrals do not depend on the choice of a path in a simply connected domain in $\mathbb{C}$.

This representation of a surface is called *a Weierstrass representation.* For $U = 0$ it reduces to the classical Weierstrass (or Weierstrass–Enneper) representation of minimal surfaces.

The following proposition is derived by straightforward computations.

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[1] This is well known in number theory as the Lagrange representation of all integer solutions of the equation $x^2 + y^2 = z^2$.

Proposition 1. *If Σ is a surface defined by the formulae (2) and (3), then:*

1) *z is a conformal parameter on the surface, and the induced metric has the form*

$$ds^2 = e^{2\alpha}\,dz\,d\bar z, \qquad e^{\alpha} = |\psi_1|^2 + |\psi_2|^2;$$

2) *the potential U of the Dirac operator is equal to*

$$U = \frac{He^{\alpha}}{2},$$

where H is the mean curvature,[2] that is, $H = \dfrac{\varkappa_1 + \varkappa_2}{2}$, where $\varkappa_1$ and $\varkappa_2$ are the principal curvatures;

3) *the Hopf differential is equal to $A\,dz^2 = (f_{zz}, N)\,dz^2$ and*

$$|A|^2 = \frac{(\varkappa_1 - \varkappa_2)^2 e^{4\alpha}}{16}, \qquad A = \overline{\psi}_2\partial\psi_1 - \psi_1\partial\overline{\psi}_2;$$

4) *the Gauss–Weingarten equations become*

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = \left[\frac{\partial}{\partial \bar z} - \begin{pmatrix} 0 & U \\ -\overline{A}e^{-\alpha} & \alpha_{\bar z} \end{pmatrix}\right]\psi = 0;$$

5) *the Gauss–Codazzi equations of the form*

$$A_{\bar z} = (U_z - \alpha_z U)e^{\alpha}, \qquad \alpha_{z\bar z} + U^2 - A\overline{A}e^{-2\alpha} = 0$$

are the compatibility conditions for the Gauss–Weingarten equations, and the Gaussian curvature is $K = -4e^{-2\alpha}\alpha_{z\bar z}$.

It can readily be seen that if φ satisfies the Dirac equation (1), then the vector function φ^* defined by the formula

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \varphi^* = \begin{pmatrix} -\overline{\varphi}_2 \\ \overline{\varphi}_1 \end{pmatrix}, \tag{4}$$

also satisfies the Dirac equation.

Let us identify the space $\mathbb{R}^3$ with the linear space of 2×2 matrices spanned (over the field $\mathbb{R}$) by the matrices

$$e_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad e_3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We have the following orthogonal representation of $SU(2)$ on $\mathbb{R}^3$:

$$e_k \to \rho(S)(e_k) = \overline{S}^{\top} e_i S = S^* e_k S, \qquad k = 1, 2, 3,$$
$$S = \begin{pmatrix} \lambda & \mu \\ -\overline{\mu} & \overline{\lambda} \end{pmatrix} \in SU(2), \quad \text{that is,} \quad |\lambda|^2 + |\mu|^2 = 1,$$

which can be factorized through $SO(3) = SU(2)/\{\pm 1\}$. The following lemma is proved by straightforward computations.

[2] We note that the normal vector N satisfies the equation

$$\Delta f = 2HN,$$

where $\Delta = 4e^{-2\alpha}\partial\overline{\partial}$ is the corresponding Laplace–Beltrami operator on the surface.

Lemma 1. *If a surface Σ is given by a vector function ψ by the Weierstrass representation, then:*

1) *the function $\lambda\psi + \mu\psi^*$ defines the surface obtained from Σ by the transformation $\rho(S)$ of the ambient space $\mathbb{R}^3$;*
2) *the function $\lambda\psi$ with $\lambda \in \mathbb{R}$ defines the image of Σ under the homothety $x \to \lambda x$.*

Remark. The formulae (1) and (3) were introduced in [2] to construct surfaces admitting soliton deformations described by the modified Novikov–Veselov equation (the mNV equation). These formulae originate from complex-valued formulae derived for other reasons by Eisenhart [16]. A similar representation for CMC surfaces (surfaces of *constant mean curvature*) in terms of the Dirac operator was proposed in 1989 by Abresch (at his talk in Luminy). It was very soon understood that these formulae give a local representation of a general surface (see [1]; while in the above proof we followed the paper [10], another proof was later given in [17], and from the physical point of view the representation was also described in [18]). Moreover, this representation turned out to be equivalent to the Kenmotsu representation [19], which does not involve the Dirac operator explicitly.

2.2. Global Weierstrass representation. The global Weierstrass representation of closed surfaces was introduced in [1]. For this it was necessary to use special ψ_1-bundles over surfaces and consider the Dirac operator defined on sections of the bundles. Furthermore:

a) the Willmore functional arises as the integral of the squared norm of the potential U, and the conformal geometry of a surface is connected with the spectral properties of the corresponding Dirac operator;
b) as was proved in [1], the tori are deformed into tori by the flow determined by the modified Novikov–Veselov equation, and this flow preserves the Willmore functional, therefore, the moduli space of immersed tori can be embedded in the phase space of an integrable system for which the Willmore functional is an integral of motion.

By the uniformization theorem, any closed oriented surface Σ is conformally equivalent to a surface Σ_0 of constant sectional curvature, and any choice of a conformal parameter z on Σ determines an equivalence $\Sigma_0 \to \Sigma$ of this kind.

Since the quantities

$$\overline{\psi}_2^2\, dz, \quad \psi_1^2 dz, \quad \psi_1\overline{\psi}_2 dz, \quad e^{2\alpha}\, dz\, d\bar{z}, \quad H = 2Ue^{-\alpha}$$

are defined globally on the surface Σ_0, this leads to the following description.

Theorem 2 ([1], [6]). *Every oriented closed surface Σ immersed in $\mathbb{R}^3$ admits a Weierstrass representation of the form* (2), (3), *where ψ is a section of some bundle E over a surface Σ_0 conformally equivalent to Σ which has constant sectional curvature, and $\mathscr{D}\psi = 0$. Moreover:*

1) *if $\Sigma = \mathbb{C} \cup \{\infty\}$ is a sphere, then ψ and U, which are defined on $\mathbb{C}$, can be extended to a neighbourhood of the point at infinity by the formulae*

$$(\psi_1, \overline{\psi}_2) \to (z\psi_1, z\overline{\psi}_2), \quad U \to |z|^2 U \quad \text{for} \quad z \to -z^{-1}, \tag{5}$$

and U has at infinity the asymptotic behaviour

$$U=\frac{\text{const}}{|z|^2}+O\biggl(\frac{1}{|z|^3}\biggr) \quad \text{as} \quad z\to\infty;$$

2) *if Σ is conformally equivalent to a torus $\Sigma_0=\mathbb{R}^2/\Lambda$, then*

$$U(z+\gamma,\bar z+\overline{\gamma})=U(z,\bar z), \quad \psi(z+\gamma,\bar z+\overline{\gamma})=\mu(\gamma)\psi(z,\bar z) \quad \text{for any} \quad \gamma\in\Lambda,$$

where $\mu\in\{\pm1\}$ is the group character $\Lambda\to\{\pm1\}$ giving the bundle

$$E\xrightarrow{\mathbb{C}^2}\Sigma_0$$

of which the vector function $\psi=(\psi_1,\psi_2)^{\perp}$ is a section;

3) *if Σ is a surface of genus $g\geqslant 2$, then $\Sigma_0=\mathscr{H}/\Lambda$, where $\mathscr{H}$ is the Lobachevskii upper half-plane and Λ is a discrete subgroup of $PSL(2,\mathbb{R})$ that acts on $\mathscr{H}=\{\operatorname{Im} z>0\}\subset\mathbb{C}$ by the formula*

$$z\to\gamma(z)=\frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbb{R}),$$

and here the ψ-bundle

$$E\xrightarrow{\mathbb{C}^2}\Sigma_0$$

is given by the monodromy rules

$$\gamma\colon (\psi_1,\overline{\psi}_2)\to(cz+d)(\psi_1,\overline{\psi}_2) \tag{6}$$

and

$$U(\gamma(z),\overline{\gamma(z)}\,)=|cz+d|^2U(z,\bar z).$$

The bundle E splits into a sum $E=E_0\oplus\overline{E}_0$ of two conjugate bundles, and ψ_1 and ψ_2 are sections of E_0 and $\overline{E}_0$, respectively.

Since $PSL(2,\mathbb{R})=SL(2,\mathbb{R})/\{\pm1\}$, any element $\gamma\in PSL(2,\mathbb{R})$ defines a monodromy up to a sign. A similar situation holds for a torus. For this reason, the bundles E are called *spin bundles.*

The potential U is fixed if a conformal parameter on Σ is given, and U gives the potential of the representation. Moreover, we have

$$\mathscr{W}(\Sigma)=4\int_\Sigma U^2\,dx\wedge dy.$$

Every section $\psi\in\Gamma(E)$ such that $\mathscr{D}\psi=0$ determines a surface which is non-closed in the general case and has only a periodic Gaussian map. In this case the Weierstrass formulae define an immersion of the universal covering space $\widetilde{\Sigma}$ of the surface Σ. The following proposition lists the cases in which this immersion can be factorized through an immersion of the corresponding compact surface.

Proposition 2. *The Weierstrass representation defines an immersion of the compact surface* Σ *if and only if*

$$\int_{\Sigma_0} \overline{\psi}_1^2 \, d\bar{z} \wedge \omega = \int_{\Sigma_0} \psi_2^2 \, d\bar{z} \wedge \omega = \int_{\Sigma_0} \overline{\psi}_1 \psi_2 \, d\bar{z} \wedge \omega = 0 \tag{7}$$

for any holomorphic differential ω *on* Σ_0.

We see that corresponding to any immersed torus $\Sigma \subset \mathbb{R}^3$ with a fixed conformal parameter z is the Dirac operator $\mathscr{D}$ with doubly periodic potential

$$U = V = \frac{He^{\alpha}}{2},$$

where H is the mean curvature and $e^{2\alpha}\, dz\, d\bar{z}$ is the induced metric.

2.3. Surfaces in three-dimensional Lie groups. For surfaces in Lie groups of dimension three the Weierstrass representation can be generalized as follows.

Let G be a three-dimensional Lie group with a left-invariant metric and let

$$f \colon \Sigma \to G$$

be an immersion of a surface Σ in G. We denote by $\mathscr{G}$ the Lie algebra of G. Let $z = x + iy$ be a conformal parameter on the surface.

We consider the pullback of the tangent bundle TG to a $\mathscr{G}$-bundle over Σ, $\mathscr{G} \to E = f^{-1}(TG) \xrightarrow{\pi} \Sigma$, and the differential

$$d_{\mathscr{A}} \colon \Omega^1(\Sigma; E) \to \Omega^2(\Sigma; E),$$

which acts on the E-valued 1-forms,

$$d_{\mathscr{A}}\omega = d'_{\mathscr{A}}\omega + d''_{\mathscr{A}}\omega,$$

where $\omega = u\, dz + u^*\, d\bar{z}$ and

$$d'_{\mathscr{A}}\omega = -\nabla_{\overline{\partial} f} u \, dz \wedge d\bar{z}, \qquad d''_{\mathscr{A}}\omega = \nabla_{\partial f} u^* \, dz \wedge d\bar{z}.$$

By straightforward computations we obtain the first derivational equation

$$d_{\mathscr{A}}(df) = 0. \tag{8}$$

The tension vector $\tau(f)$ is determined from the equation

$$d_{\mathscr{A}}(*df) = f \cdot (e^{2\alpha}\tau(f))\, dx \wedge dy = \frac{i}{2} f \cdot (e^{2\alpha}\tau(f))\, dz \wedge d\bar{z},$$

where $f \cdot \tau(f) = 2HN$, N is the normal vector, and H is the mean curvature. This gives the second derivational equation,

$$d_{\mathscr{A}}(*df) = ie^{2\alpha} HN \, dz \wedge d\bar{z}. \tag{9}$$

Since the metric is left invariant, we can rewrite the derivational equations in terms of the functions

$$\Psi = f^{-1}\partial f, \qquad \Psi^* = f^{-1}\overline{\partial} f$$

as follows:

$$\partial\Psi^* - \overline{\partial}\Psi + \nabla_\Psi \Psi^* - \nabla_{\Psi^*}\Psi = 0, \tag{10}$$

$$\partial\Psi^* + \overline{\partial}\Psi + \nabla_\Psi \Psi^* + \nabla_{\Psi^*}\Psi = e^{2\alpha} H f^{-1}(N). \tag{11}$$

The equation (10) is equivalent to the equation (8) and the equation (11) to the equation (9).

Let us take an orthonormal basis e_1, e_2, e_3 in the Lie algebra $\mathscr{G}$ of the group G and decompose the functions Ψ and Ψ^* with respect to this basis:

$$\Psi = \sum_{k=1}^{3} Z_k e_k, \qquad \Psi^* = \sum_{k=1}^{3} \overline{Z}_k e_k.$$

The equations (10) and (11) become

$$\sum_j (\partial\overline{Z}_j - \overline{\partial}Z_j)e_j + \sum_{j,k}(Z_j\overline{Z}_k - \overline{Z}_j Z_k)\nabla_{e_j}e_k = 0, \tag{12}$$

$$\sum_j (\partial\overline{Z}_j + \overline{\partial}Z_j)e_j + \sum_{j,k}(Z_j\overline{Z}_k + \overline{Z}_j Z_k)\nabla_{e_j}e_k$$
$$= 2iH\big[(\overline{Z}_2 Z_3 - Z_2\overline{Z}_3)e_1 + (\overline{Z}_3 Z_1 - Z_3\overline{Z}_1)e_2 + (\overline{Z}_1 Z_2 - Z_1\overline{Z}_2)e_3\big]. \tag{13}$$

We assume here that the basis $\{e_1, e_2, e_3\}$ is positively oriented, and therefore

$$f^{-1}(N) = 2ie^{-2\alpha}\big[(\overline{Z}_2 Z_3 - Z_2\overline{Z}_3)e_1 + (\overline{Z}_3 Z_1 - Z_3\overline{Z}_1)e_2 + (\overline{Z}_1 Z_2 - Z_1\overline{Z}_2)e_3\big]$$

(this formula becomes $f^{-1}(N) = 2ie^{-2\alpha}[\Psi^*, \Psi]$ for $G = SU(2)$ with the Killing metric). Since the parameter z is conformal, we obtain

$$\langle\Psi, \Psi\rangle = \langle\Psi^*, \Psi^*\rangle = 0, \qquad \langle\Psi, \Psi^*\rangle = \frac{1}{2}e^{2\alpha},$$

which can be rewritten as

$$Z_1^2 + Z_2^2 + Z_3^2 = 0, \qquad |Z_1|^2 + |Z_2|^2 + |Z_3|^2 = \frac{1}{2}e^{2\alpha}.$$

Hence, as in the case of surfaces in $\mathbb{R}^3$, the vector Z is parametrized in terms of ψ as follows:

$$Z_1 = \frac{i}{2}(\overline{\psi}_2^2 + \psi_1^2), \qquad Z_2 = \frac{1}{2}(\overline{\psi}_2^2 - \psi_1^2), \qquad Z_3 = \psi_1\overline{\psi}_2. \tag{14}$$

We show how to recover a surface from a vector function ψ satisfying the derivational equations (10) and (11). In the case of non-commutative Lie groups this recovery cannot be achieved by using integral Weierstrass formulae.

Let ψ be defined on a surface Σ with a complex parameter z and let Ψ be constructed from ψ and satisfy the equations (10) and (11). Let $P \in M$. We substitute ψ into the formula (14) for the components Z_1, Z_2, Z_3 of the vector $\Psi = \sum_{k=1}^{3} Z_k e_k = f^{-1}\partial f$ and solve the linear equation

$$f_z = f\Psi$$

in the Lie group G with the initial data $f(P) = g \in G$. Thus, we obtain the desired surface as the map

$$f\colon \Sigma \to G.$$

For the group $\mathbb{R}^3$ a solution of the equation is given by the Weierstrass formulae (2) and (3).

It is clear from the proof of the equations (10) and (11) that every surface Σ in G can be constructed by this procedure, which is just the generalized Weierstrass representation for surfaces in Lie groups. In this case we say that the function ψ generates the surface Σ.

We write the derivational equations (10) and (11) in terms of ψ. These equations can be represented as the Dirac equation,

$$\mathscr{D}\psi = \left[\begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}\right]\psi = 0, \tag{15}$$

the induced metric is given by the same formula,

$$ds^2 = e^{2\alpha}\,dz\,d\bar z, \qquad e^\alpha = |\psi_1|^2 + |\psi_2|^2,$$

and the Hopf differential $A\,dz^2$ is equal to

$$A = (\overline{\psi}_2 \partial \psi_1 - \psi_1 \partial \overline{\psi}_2) + \left(\sum_{j,k} Z_j Z_k \nabla_{e_j} e_k, N\right).$$

For any compact Lie group equipped with the Killing metric (in particular, for $G = SU(2)$) we have $\nabla_{e_j} e_k = -\nabla_{e_k} e_j$, and the Hopf differential has the same form as for surfaces in $\mathbb{R}^3$, that is, $A = \overline{\psi}_2 \partial \psi_1 - \psi_1 \partial \overline{\psi}_2$.

We consider three-dimensional Lie groups with Thurston's geometries. Let us recall that, by Thurston's theorem ([20], [21]), the three-dimensional maximal simply connected geometries $(X, \operatorname{Isom} X)$ admitting compact quotients can be listed as follows:

1) the geometries of constant sectional curvature, $X = \mathbb{R}^3$, S^3, and H^3;
2) two product geometries, $X = S^2 \times \mathbb{R}$ and $H^2 \times \mathbb{R}$;
3) three geometries modelled on Lie groups, Nil, Sol, and $\widetilde{SL}_2$, with some left-invariant metrics.

The group $\mathbb{R}^3$ with Euclidean metric was already treated above. Hence, it remains to consider the following four groups:

$$SU(2) = S^3, \qquad \mathrm{Nil}, \qquad \mathrm{Sol}, \qquad \widetilde{SL}_2,$$

where Nil is a nilpotent group, Sol is a soluble group, and $\widetilde{SL}_2$ is the universal covering group of $SL_2(\mathbb{R})$,

$$\mathrm{Nil} = \left\{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}\right\}, \qquad \mathrm{Sol} = \left\{\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix}\right\},$$

$x, y, z \in \mathbb{R}$.

The case $G = SU(2)$ was studied in [10] and the surfaces in the other groups were treated in [11], with the following results.

a) $G = SU(2)$:

$$U = \overline{V} = \frac{1}{2}(H - i)(|\psi_1|^2 + |\psi_2|^2),$$

the Gauss–Weingarten equations are

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = \left[\frac{\partial}{\partial \bar z} - \begin{pmatrix} 0 & \overline{U} \\ -\overline{A}e^{-\alpha} & \alpha_{\bar z} \end{pmatrix}\right]\psi = 0,$$

and their compatibility conditions (that is, the Gauss–Codazzi equations) become

$$\alpha_{z\bar z} + |U|^2 - |A|^2 e^{-2\alpha} = 0, \qquad A_{\bar z} = (\overline{U}_z - \alpha_z \overline{U})e^{\alpha},$$

where $A\,dz^2$ is the Hopf differential,

$$A = \overline{\psi}_2 \partial\psi_1 - \psi_1 \partial\overline{\psi}_2.$$

b) $G = \mathrm{Nil}$:

$$U = V = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + \frac{i}{4}(|\psi_2|^2 - |\psi_1|^2),$$

the Gauss–Weingarten equations are

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z - \frac{i}{2}\psi_1\overline{\psi}_2 & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = 0,$$
$$\left[\frac{\partial}{\partial \bar z} - \begin{pmatrix} 0 & U \\ -\overline{A}e^{-\alpha} & \alpha_{\bar z} - \frac{i}{2}\overline{\psi}_1\psi_2 \end{pmatrix}\right]\psi = 0,$$

and the Gauss–Codazzi equations become

$$\alpha_{z\bar z} - |A|^2 e^{-2\alpha} + \frac{H^2}{4}e^{2\alpha} = \frac{1}{16}(3|\psi_1|^4 + 3|\psi_2|^4 - 10|\psi_1|^2|\psi_2|^2),$$
$$A_{\bar z} - \frac{H_z}{2}e^{2\alpha} + \frac{1}{2}(|\psi_2|^4 - |\psi_1|^4)\psi_1\overline{\psi}_2 = 0,$$

where the Hopf differential is

$$A = (\overline{\psi}_2\partial\psi_1 - \psi_1\partial\overline{\psi}_2) + i\psi_1^2\overline{\psi}_2^2.$$

c) $G = \widetilde{SL}_2$:

$$U = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + i\left(\frac{1}{2}|\psi_1|^2 - \frac{3}{4}|\psi_2|^2\right),$$
$$V = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + i\left(\frac{3}{4}|\psi_1|^2 - \frac{1}{2}|\psi_2|^2\right),$$

the Gauss–Weingarten equations are

$$\left[\frac{\partial}{\partial z} - \begin{pmatrix} \alpha_z + \frac{5i}{4}\psi_1\overline{\psi}_2 & Ae^{-\alpha} \\ -U & 0 \end{pmatrix}\right]\psi = 0,$$
$$\left[\frac{\partial}{\partial \bar z} - \begin{pmatrix} 0 & V \\ -\overline{A}e^{-\alpha} & \alpha_{\bar z} + \frac{5i}{4}\overline{\psi}_1\psi_2 \end{pmatrix}\right]\psi = 0,$$

and the Gauss–Codazzi equations become

$$\alpha_{z\bar z} - e^{-2\alpha}|A|^2 + \frac{1}{4}e^{2\alpha}H^2 = e^{2\alpha} - 5|Z_3|^2,$$
$$\bar\partial\left(A + \frac{5Z_3^2}{2(H-i)}\right) = \frac{1}{2}H_z e^{2\alpha} + \bar\partial\left(\frac{5}{2(H-i)}\right){Z_3}^2,$$

where the Hopf differential is

$$A = (\overline{\psi}_2\partial\psi_1 - \psi_1\partial\overline{\psi}_2) - \frac{5i}{2}\psi_1^2\overline{\psi}_2^2.$$

d) $G = \mathrm{Sol}$: we consider only domains such that $Z_3 = \psi_1\overline{\psi}_2$ and for which

$$U = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + \frac{1}{2}\overline{\psi}_2^2\frac{\overline{\psi}_1}{\psi_1},$$
$$V = \frac{H}{2}(|\psi_1|^2 + |\psi_2|^2) + \frac{1}{2}\overline{\psi}_1^2\frac{\overline{\psi}_2}{\psi_2};$$

the Gauss–Weingarten equations are formed by the Dirac equation and the system

$$\partial\psi_1 = \alpha_z\psi_1 + Ae^{-\alpha}\psi_2 - \frac{1}{2}\overline{\psi}_2^3, \qquad \bar\partial\psi_2 = -\overline{A}e^{-\alpha}\psi_1 + \alpha_{\bar z}\psi_2 - \frac{1}{2}\overline{\psi}_1^3,$$

and the Gauss–Codazzi equations become

$$\alpha_{z\bar z} - e^{-2\alpha}|A|^2 + \frac{1}{4}e^{2\alpha}H^2 = \frac{1}{4}(6|\psi_1|^2|\psi_2|^2 - (|\psi_1|^4 + |\psi_2|^4)),$$
$$A_{\bar z} - \frac{1}{2}H_z e^{2\alpha} = (|\psi_2|^4 - |\psi_1|^4)\psi_1\overline{\psi}_2,$$

where

$$A = (\overline{\psi}_2\partial\psi_1 - \psi_1\partial\overline{\psi}_2) + \frac{1}{2}(\overline{\psi}_2^4 - \psi_1^4).$$

We must make several explanatory remarks.

1) The formulae for the last three groups contain the term Z_3. The direction of the vector e_3 has a different meaning for these groups.

1a) The groups Nil and $\widetilde{SL}_2$ admit an S^1-symmetry given by the rotations about the geodesic drawn in the direction of e_3. These rotations, together with the left translations, generate the isometry group $\mathrm{Isom}\,G$.

1b) For the group Sol the vectors e_1 and e_2 commute. Hence, the equation $Z_3 = \psi_1\overline{\psi}_2 = 0$ can hold on an open subset B of the surface, and the Dirac equation cannot be extended by continuity to the entire surface. Since $H = 0$ in B, we set

$$U = V = 0 \quad \text{for} \quad \psi_1\overline{\psi}_2 = 0 \quad \text{and} \quad G = \mathrm{Sol}\,.$$

However, the potentials U_{Sol} and V_{Sol} are not always correctly defined on the boundary ∂B of the set $\{Z_3 \neq 0\}$, because the expression $\dfrac{\overline{\psi}_1}{\psi_1}$ is undefined for $\psi_1 = 0$, and the Dirac equation with the given potentials is satisfied outside ∂B.

2) For $G=\mathbb{R}^3$ or $SU(2)$ the Gauss–Codazzi equations can be derived as follows. We have

$$R\psi = (\partial-\mathscr{A})(\bar\partial-\mathscr{B})\psi-(\bar\partial-\mathscr{B})(\partial-\mathscr{A})\psi = (\mathscr{A}_{\bar z}-\mathscr{B}_z+[\mathscr{A},\mathscr{B}])\psi=0,$$

where $(\partial-\mathscr{A})\psi=(\bar\partial-\mathscr{B})\psi=0$ are the Gauss–Weingarten equations and the vector function ψ^* (see (4)) satisfies the same equation $R\psi^*=0$, which, together with the condition $R\psi=0$, implies that $R=\mathscr{A}_{\bar z}-\mathscr{B}_z+[\mathscr{A},\mathscr{B}]=0$. For the other groups the equations $\mathscr{D}\psi^*=0$ and $R\psi^*=0$ fail to hold and, in particular, the kernel of the Dirac operator cannot be treated as a vector space over the quaternions. For this reason, the Gauss–Codazzi equations are derived in [11] in another way.

3) In fact, the Dirac equations for non-commutative Lie groups are non-linear with respect to ψ due to constraints on the potentials. Hence, if a function ψ determines a surface, then $\lambda\psi$ need not determine any surface for $|\lambda|\neq 1$, since these groups do not admit homotheties. For the group $SU(2)$ the map (4) takes a solution of the Dirac equation into another solution of the equation, and the following analogue of part 1) of Lemma 1 holds: the sum $\lambda\psi+\mu\psi^*$, where $|\lambda|^2+|\mu|^2=1$, defines the image of the initial surface under some inner automorphism of $SU(2)$ corresponding to a rotation of the Lie algebra.

We present some corollaries. Since the case $G=SU(2)$ has been thoroughly studied,[3] we consider only the other groups.

Theorem 3. 1) *If a function ψ generates a minimal surface in a Lie group of dimension three, then the following equations hold:*

$$\bar\partial\psi_1=\frac{i}{4}(|\psi_2|^2-|\psi_1|^2)\psi_2,\qquad \partial\psi_2=-\frac{i}{4}(|\psi_2|^2-|\psi_1|^2)\psi_1\qquad\text{for}\quad G=\mathrm{Nil},$$
$$\bar\partial\psi_1=i\biggl(\frac34|\psi_1|^2-\frac12|\psi_2|^2\biggr)\psi_2,\quad \partial\psi_2=-i\biggl(\frac12|\psi_1|^2-\frac34|\psi_2|^2\biggr)\psi_1\quad\text{for}\quad G=\widetilde{SL}_2,$$
$$\bar\partial\psi_1=\frac12\bar\psi_1^2\bar\psi_2,\qquad \partial\psi_2=-\frac12\bar\psi_1\bar\psi_2^2\qquad\text{for}\quad G=\mathrm{Sol}\,.$$

2) (*Abresch* [22]) *If a surface has constant mean curvature, then the following quadratic differential $\widetilde A\,dz^2$ is holomorphic*:

$$\widetilde A\,dz^2=\biggl(A+\frac{{Z_3}^2}{2H+i}\biggr)dz^2\qquad\text{for}\quad G=\mathrm{Nil},$$
$$\widetilde A\,dz^2=\biggl(A+\frac{5}{2(H-i)}Z_3^2\biggr)dz^2\quad\text{for}\quad G=\widetilde{SL}_2.$$

3) *If the differential $\widetilde A\,dz^2$ is holomorphic for a surface in $G=\mathrm{Nil}$, then the surface has constant mean curvature.*

[3] The equations of the minimal surfaces for $SU(2)$ are

$$\bar\partial\psi_1=-\frac{i}{2}(|\psi_1|^2+|\psi_2|^2)\psi_2,\qquad \partial\psi_2=\frac{i}{2}(|\psi_1|^2+|\psi_2|^2)\psi_1,$$

and the CMC surfaces are distinguished by the condition $A_{\bar z}=0$.

It would be of interest to understand the relationship between the formulae for surfaces of constant mean curvature and soliton equations. Relationships of this kind are well known for surfaces in $\mathbb{R}^3$ and $SU(2)$.

Analogues of the assertion 2) are also known for surfaces in $S^2\times\mathbb{R}$ and $H^2\times\mathbb{R}$ [23]. However, the converse assertion (the assertion 3) has been proved only for surfaces in Nil.[4]

We note that an analogue of the Weierstrass representation for minimal surfaces in Nil and Sol was obtained in another way in [25] and [26]. Other approaches to the study of surfaces in Lie groups were used in [27] and [28].

2.4. Quaternion language and quaternionic function theory. Pedit and Pinkall [8] wrote out the Weierstrass representation for surfaces in $\mathbb{R}^3$ in the language of quaternions and then extended this representation to surfaces in $\mathbb{R}^4$ (see some preliminary results in [29]–[31]).

Indeed, the idea of using quaternions comes from the symmetry of the kernel of the Dirac operator under the transformation (4) (we note that this symmetry holds for surfaces in $\mathbb{R}^3$ and $SU(2)$ if $U=\overline{V}$ and fails for surfaces in the other three-dimensional Lie groups).

We identify $\mathbb{C}^2$ with the space $\mathbb{H}$ of quaternions,

$$(z_1,z_2)\to z_1+\mathbf{j}z_2=\begin{pmatrix} z_1 & -\bar z_2\\ z_2 & \bar z_1\end{pmatrix},$$

and we consider the two matrix operators

$$\overline{\partial}=\begin{pmatrix}\overline{\partial} & 0\\ 0 & \partial\end{pmatrix},\qquad \mathbf{j}U=\mathbf{j}\begin{pmatrix}U & 0\\ 0 & \overline{U}\end{pmatrix}=\begin{pmatrix}0 & -\overline{U}\\ U & 0\end{pmatrix}.$$

Here $\mathbf{j}$ is one of the standard generators of the quaternion algebra, and we have

$$\mathbf{j}^2=-1,\qquad z\mathbf{j}=\mathbf{j}\bar z,\qquad \overline{\partial}\mathbf{j}=\mathbf{j}\partial.$$

The Dirac equation becomes

$$(\overline{\partial}+\mathbf{j}U)(\psi_1+\mathbf{j}\psi_2)=(\overline{\partial}\psi_1-\overline{U}\psi_2)+\mathbf{j}(\partial\psi_2+U\psi_1)=0.$$

Since, by (6), both ψ_1 and $\overline{\psi}_2$ are sections of the same bundle E_0, it is more reasonable to rewrite the Dirac equation in terms of quaternions in the form

$$(\overline{\partial}+\mathbf{j}U)(\psi_1+\overline{\psi}_2\mathbf{j})=0.$$

One can regard $L=E_0\oplus E_0$ as a quaternionic line bundle whose sections are of the form $\psi_1+\overline{\psi}_2\mathbf{j}$ and which is endowed with a quaternionic linear endomorphism J such that $J^2=-1$. In our case J simply acts as the right multiplication by $\mathbf{j}$:

$$J\colon(\psi_1,\overline{\psi}_2)\to(-\overline{\psi}_2,\overline{\psi}_1)\quad\text{or}\quad \psi_1+\overline{\psi}_2\mathbf{j}\to(\psi_1+\overline{\psi}_2\mathbf{j})\mathbf{j}=-\overline{\psi_2}+\psi_1\mathbf{j}.$$

[4] After the present paper was submitted for publication, it was proved that an analogue of this statement fails for surfaces in $\widetilde{SL}_2$ and $H^2\times\mathbb{R}$, that is, there are surfaces for which the quadratic differential $\widetilde{A}\,dz^2$ is holomorphic and the mean curvature is not constant [24].

This map J determines a canonical splitting of any quaternion fibre into the sum $\mathbb{C}\oplus\mathbb{C}$ (in our case, this is a splitting into ψ_1 and $\overline{\psi}_2$). In [8] and [12] such a bundle is called a *complex quaternionic line bundle.*

The Dirac operator in these terms is

$$\mathscr{D}\psi = (\overline{\partial}+\mathbf{j}U)(\psi_1+\overline{\psi}_2\mathbf{j}) = (\overline{\partial}\psi_1-\overline{U}\psi_2)+(\overline{\partial\psi}_2+\overline{U\psi}_1)\mathbf{j},$$

and we see that the kernel of this operator is invariant with respect to right multiplications by constant quaternions (see Lemma 1), and hence the kernel can be regarded as a vector space over the skew field $\mathbb{H}$ of quaternions. According to (6), we have the operator

$$\mathscr{D}\colon \Gamma(L)\to\Gamma(\overline{K}L),$$

where for a given bundle V the symbol $\Gamma(V)$ stands for the space of sections of V and $\overline{K}$ is the bundle of 1-forms of type $(0,1)$ (that is, of the form $f\,d\bar{z}$) over the surface Σ_0.

This operator is certainly non-linear with respect to right multiplications by quaternion-valued functions, and the following obvious formula holds:

$$\mathscr{D}(\psi\lambda) = (\mathscr{D}\psi)\lambda+\psi_1(\overline{\mu}+\mathbf{j}\partial\eta)+\overline{\psi}_2(-\overline{\partial}\eta+\mathbf{j}\partial\mu),$$

where $\lambda=\mu+\mathbf{j}\eta=\mu+\overline{\eta}\mathbf{j}$. In [8] this formula is represented in the coordinate-free form as

$$\mathscr{D}(\psi\lambda) = (\mathscr{D}\psi)\lambda+\frac{1}{2}(\psi\,d\lambda+J\psi * d\lambda),$$

the potential U multiplied from the left by $\mathbf{j}$ is called the *Hopf field* $Q=\mathbf{j}U$ of the connection $\mathscr{D}$ on L, and the quantity

$$\mathscr{W} = \int_{\Sigma_0}|U|^2\,dx\wedge dy$$

is called the *Willmore energy* of $\mathscr{D}$.

Although the quaternion language seemed at first to be very artificial (at least to the author), it led to an extension of the Weierstrass representation for surfaces in $\mathbb{R}^4$ [8]. Later on it was transformed into an investigative tool for developing analogies between complex algebraic geometry and the theory of complex quaternionic line bundles. It turns out that this tool can be effectively applied to study special types of surfaces and Bäcklund transforms in the framework of the conformal approach when $\mathbb{R}^4$ and S^4 are not distinguished ([12], [32]). Finally, this approach had led to a remarkable extension of Plücker-type relations from complex algebraic geometry to the geometry of complex quaternionic line bundles and to their application to the proof of lower bounds for the Willmore functional [7] (see § 5.4). Moreover, this theory deals with general bundles L, which need not come from the theory of surfaces [7]. The bundles connected with surfaces are distinguished by their degrees, namely, it follows from (5) and (6) that

$$\deg E_0 = \mathrm{genus}(\Sigma_0)-1 = g-1.$$

2.5. Surfaces in $\mathbb{R}^4$. The Grassmannian of oriented two-dimensional planes in $\mathbb{R}^4$ is diffeomorphic to the quadric

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0, \qquad y \in \mathbb{C}P^3.$$

Let us consider other coordinates y_1', y_2', y_3', y_4' in $\mathbb{C}^4$:

$$y_1 = \frac{i}{2}(y_1' + y_2'), \quad y_2 = \frac{1}{2}(y_1' - y_2'), \quad y_3 = \frac{1}{2}(y_3' + y_4'), \quad y_4 = \frac{i}{2}(y_3' - y_4').$$

In these coordinates the manifold $\widetilde{G}_{4,2}$ is given by the equation

$$y_1' y_2' = y_3' y_4'.$$

Obviously, there is a diffeomorphism

$$\mathbb{C}P^1 \times \mathbb{C}P^1 \to \widetilde{G}_{4,2}$$

defined by the Segre map

$$y_1' = a_2 b_2, \quad y_2' = a_1 b_1, \quad y_3' = a_2 b_1, \quad y_4' = a_1 b_2,$$

where $(a_1 : a_2)$ and $(b_1 : b_2)$ are homogeneous coordinates on different copies of $\mathbb{C}P^1$.

We parameterize the coordinates x_z^k, $k = 1, 2, 3, 4$, in terms of these coordinates and we set

$$a_1 = \varphi_1, \quad a_2 = \overline{\varphi}_2, \quad b_1 = \psi_1, \quad b_2 = \overline{\psi}_2.$$

In contrast to the three-dimensional situation, this parametrization is not unique, not even up to multiplication by ± 1, and the vector functions ψ and φ are determined up to gauge transformations of the form

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^f \psi_1 \\ e^{\overline{f}} \psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-f} \varphi_1 \\ e^{-\overline{f}} \varphi_2 \end{pmatrix}, \tag{16}$$

where f is an arbitrary function. However, the maps

$$G_\psi = (\psi_1 : \overline{\psi}_2), \qquad G_\varphi = (\varphi_1 : \overline{\varphi}_2)$$

into $\mathbb{C}P^1$ are well defined and split the Gauss map

$$G = (G_\psi, G_\varphi) \colon \Sigma \to \widetilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1.$$

We have the following formulae for an immersion of the surface:

$$x^k = x^k(0) + \int (x_z^k \, dz + \overline{x}_z^k \, d\bar{z}), \qquad k = 1, 2, 3, 4, \tag{17}$$

where

$$\begin{aligned} x_z^1 &= \frac{i}{2}(\overline{\varphi}_2 \overline{\psi}_2 + \varphi_1 \psi_1), \qquad & x_z^2 &= \frac{1}{2}(\overline{\varphi}_2 \overline{\psi}_2 - \varphi_1 \psi_1), \\ x_z^3 &= \frac{1}{2}(\overline{\varphi}_2 \psi_1 + \varphi_1 \overline{\psi}_2), \qquad & x_z^4 &= \frac{i}{2}(\overline{\varphi}_2 \psi_1 - \varphi_1 \overline{\psi}_2). \end{aligned} \tag{18}$$

Of course, as in the three-dimensional case, these formulae define a surface if and only if the integrands are closed forms, or equivalently,

$$\operatorname{Im} x^k_{z\bar z} = 0, \qquad k = 1, 2, 3, 4.$$

This condition can be rewritten as

$$(\overline{\varphi}_2\psi_1)_{\bar z} = (\overline{\varphi}_1\psi_2)_z, \qquad (\overline{\varphi}_2\overline{\psi}_2)_{\bar z} = -(\overline{\varphi}_1\overline{\psi}_1)_z. \tag{19}$$

These conditions cannot be represented in terms of Dirac equations for generic vector functions φ and ψ.

However, the following assertion holds.

Theorem 4. *Let $r\colon W \to \mathbb{R}^4$ be an immersed surface with a conformal parameter z and let $G_\psi = (e^{i\theta}\cos\eta : \sin\eta)$ be one of the components of the corresponding Gauss map.*

There is another representative ψ of the map $G_\psi = (\psi_1 : \overline{\psi}_2)$ that satisfies the Dirac equation

$$\mathscr{D}\psi = 0, \qquad \mathscr{D} = \begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & \overline{U} \end{pmatrix} \tag{20}$$

with some potential U.

The vector function $\psi = (e^{g+i\theta}\cos\eta, e^{\overline{g}}\sin\eta)$ is determined from the equation

$$g_{\bar z} = -i\theta_{\bar z}\cos^2\eta \tag{21}$$

whose solution g is determined up to addition of an arbitrary holomorphic function h, and the corresponding potential U is given by

$$U = -e^{\overline{g}-g-i\theta}(i\theta_z \sin\eta\cos\eta + \eta_z)$$

up to multiplication by $e^{\overline{h}-h}$.

If a vector function ψ is given, then any function φ representing another component G_φ of the Gauss map satisfies the equation

$$\mathscr{D}^\vee\varphi = 0, \qquad \mathscr{D}^\vee = \begin{pmatrix} 0 & \partial \\ -\overline{\partial} & 0 \end{pmatrix} + \begin{pmatrix} \overline{U} & 0 \\ 0 & U \end{pmatrix}. \tag{22}$$

Different lifts to $\mathbb{C}^2 \times \mathbb{C}^2$ of the Gauss map $G\colon \Sigma \to \mathbb{C}P^1 \times \mathbb{C}P^1$ are connected by the gauge transformations

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^{h}\psi_1 \\ e^{\overline{h}}\psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-h}\varphi_1 \\ e^{-\overline{h}}\varphi_2 \end{pmatrix}, \qquad U \to \exp(\overline{h} - h)U, \tag{23}$$

where h is an arbitrary holomorphic function on W.

Corollary 1. *Every oriented surface in $\mathbb{R}^4$ is defined by formulae (17) and (18) with the vector functions ψ and φ satisfying the Dirac-type equations (20) and (22):*

$$\mathscr{D}\psi = \mathscr{D}^\vee\varphi = 0.$$

The induced metric is

$$e^{2\alpha}\,dz\,d\bar z = (|\psi_1|^2 + |\psi_2|^2)(|\varphi_1|^2 + |\varphi_2|^2)\,dz\,d\bar z,$$

and the norm of the mean curvature vector $\mathbf{H} = 2x_{z\bar z}/e^{2\alpha}$ *satisfies the equality*

$$|U| = \frac{|\mathbf{H}|e^{\alpha}}{2}\,.$$

Let us consider the diagonal embedding

$$\widetilde G_{3,2} = \mathbb{C}P^1 \to \widetilde G_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1.$$

If φ and ψ generate a surface and lie on the diagonal, $\varphi = \pm\psi$, then $x^4 = 0$, and we obtain a Weierstrass representation of the surface in $\mathbb{R}^3$.

The formulae (17) and (18) appeared in [9] in the construction of surfaces. This corollary shows that they have a general nature; however, this fact must follow from [8], where such a representation was first indicated in the language of quaternions.

We indicate two specific features of the representation of surfaces in $\mathbb{R}^4$ that were not discussed in the previous papers:

a) for a given surface a representation need not be unique, and different representations are connected by non-trivial gauge transformations;
b) a Weierstrass representation of some domain need not be extendable to the entire surface and, in contrast to the three-dimensional case, to obtain a representation of the entire surface, one must solve the $\bar\partial$-problem (21) on the surface.

Indeed, let us take ψ and φ that generate a surface Σ, a domain $W \subset \Sigma$, and a holomorphic function f on W that admits no analytic continuation outside W. Using (16), we then construct from ψ, φ, and f another representation of W which also cannot be continued outside W.

Example. Lagrangian surfaces in $\mathbb{R}^4$. We present the Weierstrass representation for Lagrangian surfaces in $\mathbb{R}^4$ that was obtained by Hélein and Romon [33]. The reduction of the formulae (18) to formulae in [33] was indicated by Hélein in [34].

Let us consider the following symplectic form on $\mathbb{R}^4$:

$$\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4.$$

We recall that an n-dimensional submanifold Σ of a $2n$-dimensional symplectic manifold M^{2n} with symplectic form ω is said to be *Lagrangian* if the restriction of ω to Σ vanishes:

$$\omega\big|_{\Sigma} = 0.$$

This means that at any point $x \in \Sigma$ the restriction of the form ω to the tangent space $T_x\Sigma$ vanishes, that is, $T_x\Sigma$ is a Lagrangian n-plane in $\mathbb{R}^{2n}$.

The condition that a two-dimensional plane is Lagrangian in $\mathbb{R}^4$ can be represented in the form

$$\mathrm{Im}(y_1\overline{y}_2 + y_3\overline{y}_4) = 0,$$

or

$$|y_1'|^2 - |y_2'|^2 - |y_3'|^2 + |y_4'|^2 = 0.$$

In terms of a_1, a_2, b_1, and b_2, this condition becomes

$$|b_1|^2 = |b_2|^2.$$

Hence, the Grassmannian of Lagrangian two-dimensional planes in $\mathbb{R}^4$ is the product of manifolds

$$G_{4,2}^{\mathrm{Lag}} = \mathbb{C}P^1 \times S^1,$$

where the space $\mathbb{C}P^1$ is parameterized by $(a_1 : a_2)$ and the circle S^1 is parametrized by the quantity

$$\beta = \frac{1}{i} \log \frac{b_1}{b_2} \mod 2\pi,$$

the so-called *Lagrangian angle*.

We conclude that a surface is Lagrangian if and only if

$$|\psi_1| = |\psi_2|$$

in its Weierstrass representation. Let us set

$$s = \left(\frac{e^{i\beta}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \qquad (s_1 : s_2) = (\psi_1 : \overline{\psi}_2) \in \mathbb{C}P^1,$$

and apply Theorem 4. We obtain the following formulae:

$$g = -\frac{i\beta}{2}, \qquad U = -\frac{1}{2}\beta_z, \qquad \psi_1 = \psi_2 = \frac{1}{\sqrt{2}} e^{i\beta/2}.$$

For any solution φ of the equation $\mathscr{D}^\vee \varphi = 0$ we obtain a Lagrangian surface determined by the vector functions ψ and φ by the formulae (18). Moreover, all Lagrangian surfaces can be represented in this form.

Let

$$f \colon \Sigma \to \mathbb{R}^4$$

be an immersion of an oriented closed surface into $\mathbb{R}^4$. By Theorem 4, this surface is locally defined by the formulae (17) and (18). A globalization of this representation is similar to the case of surfaces in $\mathbb{R}^3$ and was described in [8] and [12] in the language of quaternions; however, to obtain the globalization, one must solve the $\overline{\partial}$-problem on the surface [35]. The following assertion holds.

Proposition 3. *For any Weierstrass representation of an immersion of an oriented closed surface Σ into $\mathbb{R}^4$ the corresponding functions ψ and φ are sections of $\mathbb{C}^2$-bundles E and $E^\vee$ over Σ that have the following form*:

1) *E and $E^\vee$ split into sums of pairwise conjugate line bundles,*

$$E = E_0 \oplus \overline{E}_0, \qquad E^\vee = E_0^\vee \oplus \overline{E}_0^\vee,$$

in such a way that ψ_1 and $\overline{\psi}_2$ are sections of E_0 and φ_1 and $\overline{\varphi}_2$ are sections of $E_0^\vee$;

2) *the pairing of sections of* E_0 *and* $E_0^\vee$ *defines a* $(1,0)$ *form on* Σ, *that is, if*

$$\alpha \in \Gamma(E_0), \qquad \beta \in \Gamma(E_0^\vee),$$

then

$$\alpha\beta\, dz$$

is a well-defined 1*-form on* Σ;

3) *the Dirac equation* $\mathscr{D}\psi = 0$ *implies that* U *is a section of the same line bundle* E_U *as the section*

$$\frac{\partial\gamma}{\alpha} \in \Gamma(E_U) \quad \text{for} \quad \alpha \in \Gamma(E_0), \quad \gamma \in \Gamma(\overline{E}_0),$$

and $U\overline{U}\, dz \wedge d\bar{z}$ *is a well-defined* $(1,1)$*-form on* Σ *whose integral over the surface is equal to*

$$\int_\Sigma U\overline{U}\, dz \wedge d\bar{z} = -\frac{i}{2}\mathscr{W}(\Sigma),$$

where $\mathscr{W}(\Sigma) = \displaystyle\int_\Sigma |\mathbf{H}|^2\, d\mu$ *is the Willmore functional.*

The gauge transformation (23) shows that, in contrast to the three-dimensional case, the vector functions ψ are not necessarily sections of spin bundles.

For tori we can derive the following result from Theorem 4.

Theorem 5 [35]. *Let* Σ *be a torus in* $\mathbb{R}^4$ *conformally equivalent to* $\mathbb{C}/\Lambda$ *and let* z *be a conformal parameter on the torus.*

Then there are vector functions ψ *and* φ *and a function* U *on* $\mathbb{C}$ *such that*:

1) ψ *and* φ *give a Weierstrass representation of* Σ;

2) *the potential* U *of this representation is* Λ*-periodic*;

3) *the functions* ψ, φ, *and* U *satisfying the conditions* 1) *and* 2) *are determined up to gauge transformations*

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} \to \begin{pmatrix} e^{h}\psi_1 \\ e^{\bar{h}}\psi_2\end{pmatrix}, \qquad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix} e^{-h}\varphi_1 \\ e^{-\bar{h}}\varphi_2\end{pmatrix}, \qquad U \to e^{\bar{h}-h}U, \tag{24}$$

where

$$h(z) = a + bz, \qquad \operatorname{Im}(b\gamma) \in \pi\mathbb{Z} \quad \text{for any} \quad \gamma \in \Lambda.$$

As in the case of surfaces in $\mathbb{R}^3$, the vector functions ψ and φ determine in general an immersion of the universal covering surface $\widetilde{\Sigma}$ of Σ into $\mathbb{R}^4$.

Proposition 4. *The immersion of* $\widetilde{\Sigma}$ *can be factorized through an immersion of* Σ *if and only if*

$$\int_\Sigma \overline{\psi}_1\overline{\varphi}_1\, d\bar{z}\wedge\omega = \int_\Sigma \overline{\psi}_1\varphi_2\, d\bar{z}\wedge\omega = \int_\Sigma \psi_2\overline{\varphi}_1\, d\bar{z}\wedge\omega = \int_\Sigma \psi_2\varphi_2\, d\bar{z}\wedge\omega = 0 \tag{25}$$

for any holomorphic differential ω *on* Σ.

For $\psi_1 = \pm\varphi_1$ and $\psi_2 = \pm\varphi_2$ the formula (25) is reduced to (7).

§ 3. Integrable deformations of surfaces

3.1. The modified Veselov–Novikov equation. The hierarchy of modified Veselov–Novikov (mVN) equations was introduced by Bogdanov ([36], [37]), and the equations of this hierarchy have the form of L, A, B-triples

$$\frac{\partial L}{\partial t_n} = [L, A_n] - B_n L,$$

where $L = \mathscr{D}$ is the Dirac operator

$$L = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$$

and A_n and B_n are matrix differential operators such that the leading term of the operator A_n is given by

$$A_n = \begin{pmatrix} \partial^{2n+1} + \bar\partial^{2n+1} & 0 \\ 0 & \partial^{2n+1} + \bar\partial^{2n+1} \end{pmatrix} + \cdots .$$

In contrast to L, A-pairs, any L, A, B-triple preserves only the zero energy level of the operator L, deforming the corresponding eigenfunctions. Indeed, we have

$$\frac{\partial L\psi}{\partial t} = L_t\psi + L\psi_t = L[(A + \partial_t)\psi] - (A + B)[L\psi].$$

Hence, if ψ satisfies the equation

$$\frac{\partial \psi}{\partial t} + A\psi = 0 \tag{26}$$

and if $L\psi_0 = 0$ for the initial data $\psi_0 = \psi\big|_{t=t_0}$ of this evolution equation, then

$$L\psi = 0$$

for any $t \geqslant t_0$.

For $n = 1$ we have the original mVN equation

$$U_t = \left(U_{zzz} + 3U_z V + \frac{3}{2} U V_z\right) + \left(U_{\bar z\bar z\bar z} + 3U_{\bar z}\overline{V} + \frac{3}{2} U \overline{V}_{\bar z}\right), \tag{27}$$

where

$$V_{\bar z} = (U^2)_z. \tag{28}$$

We see that if the initial data $U\big|_{t=0}$ of the Cauchy problem are given by a real-valued function, then the solution is also real-valued. If the function $U\big|_{t=0}$ depends only on the variable x, then we have $U = U(x, t)$, and the mVN equation reduces to the modified Korteweg–de Vries equation

$$U_t = \frac{1}{4} U_{xxx} + 6U_x U^2 \tag{29}$$

(here $V = U^2$).

This reduction explains the name of the equation, since Novikov and Veselov introduced in [38] and [39] a hierarchy of (2+1)-dimensional soliton equations which have the form of L, A, B-triples for the scalar operator $L = \partial\bar{\partial}+U$ (two-dimensional potential Schrödinger operator) and pass into the Korteweg–de Vries equation in the $(1+1)$-limit. The original Novikov–Veselov equation is

$$U_t = U_{zzz} + U_{\bar{z}\bar{z}\bar{z}} + (VU)_z + (\overline{V}U)_{\bar{z}}, \qquad V_{\bar{z}} = 3U_z$$

and its proof was later modified by Bogdanov to derive the mNV equation.

It follows from the formulae (2) and (3) of the Weierstrass representation that it is the zero energy level of the Dirac operator that corresponds to surfaces in $\mathbb{R}^3$. This leads to the following assertion.

Theorem 6 [2]. *Let $U(z,\bar{z},t)$ be a real-valued solution of the mVN equation (27). Let Σ be a surface constructed using the Weierstrass representation (2), (3) from a vector function ψ_0 satisfying the Dirac equation $\mathscr{D}\psi_0 = 0$ with the potential $U = U(z,\bar{z},0)$. Let $\psi(z,\bar{z},t)$ be a solution of the equation (26) with the initial data $\psi\big|_{t=0} = \psi_0$.*

Then the surfaces $\Sigma(t)$ constructed from $\psi(z,\bar{z},t)$ using the Weierstrass representation determine a soliton deformation of the surface Σ.

The deformation given by this theorem is called the *mNV deformation of a surface*.

Of course, this theorem holds for all equations of the mNV hierarchy. No recursion formula for these functions is known, and the equations presented below were not written out explicitly until recently, except for the case $n = 2$ [1]. Finite-gap solutions of the mNV equations were constructed in [40] (see also [41]).

As was proved in [1], this deformation has a global character for tori and preserves the Willmore functional.

Theorem 7 [1]. *The mNV deformation takes tori into tori and preserves both their conformal classes and the values of the Willmore functional.*

The proof of this theorem is as follows. To correctly define the deformation, we must solve (28), and this can indeed be done for tori, as was shown in [1]. We must take a solution V of (28) normalized by the condition

$$\int_\Sigma V\,dz \wedge d\bar{z} = 0.$$

The form $(U^2)_t\,dz \wedge d\bar{z}$ is an exact form on the torus Σ,

$$UU_t = \left(UU_{zz} - \frac{U_z^2}{2} + \frac{3}{2}\,U^2V\right)_z + \left(UU_{\bar{z}\bar{z}} - \frac{U_{\bar{z}}^2}{2} + \frac{3}{2}\,U^2\overline{V}\right)_{\bar{z}},$$

and hence the Willmore functional is preserved,

$$\frac{d}{dt}\int_\Sigma U^2\,dz \wedge d\bar{z} = \int_\Sigma (U^2)_t\,dz \wedge d\bar{z} = 0.$$

The flat structure on the torus enables us to identify differentials with periodic functions. For example, formally, $U^2\,dz\,d\bar{z}$ is a $(1,1)$-differential and $V\,dz^2$ is a quadratic differential. This is impossible for surfaces of higher genera, and this therefore prevents a global definition of mNV deformations of these surfaces.

An attempt to redefine soliton deformations in completely geometric terms was made in [42]. In the end it was impossible to avoid introducing a conformal parameter on the surface; however, some interesting geometric properties of deformations were revealed.

After the papers [2] and [1], some other soliton deformations of surfaces with geometric conservation laws were introduced and studied in [43]–[45] in the framework of affine geometry and Lie spherical geometry.

3.2. The modified Korteweg–de Vries equation. If the potential U depends only on $x = \operatorname{Re} z$, then the Dirac equation $\mathscr{D}\psi = 0$ for functions of the form

$$\psi(z, \bar{z}) = \varphi(x) \exp\biggl(\frac{iy}{2}\biggr) \tag{30}$$

reduces to the Zakharov–Shabat problem

$$L\varphi = 0, \qquad L = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} q & -ik \\ -ik & q \end{pmatrix}\right], \qquad q = 2U,$$

for $k = i/2$.

We note that for surfaces of revolution the function ψ acquires the form (30) with respect to some conformal coordinate $z = x + iy$, where y stands for the angle of rotation. However, there are many other surfaces with intrinsic S^1-symmetry for which the potential U depends only on x. The function φ is periodic for tori of revolution and is rapidly decreasing for spheres of revolution [6].

The operator L is associated with the soliton hierarchy of modified Korteweg–de Vries equations that admit a representation in the form of an L, A pair

$$\frac{dL}{dt} = [L, A_n].$$

The simplest of these equations are

$$q_t = q_{xxx} + \frac{3}{2} q^2 q_x, \qquad n = 1,$$
$$q_t = q_{xxxxx} + \frac{5}{2} q^2 q_{xxx} + 10 q q_x q_{xx} + \frac{5}{2} q_x^3 + \frac{15}{8} q^4 q_x, \qquad n = 2.$$

The first equation here coincides with the reduction (29) of the mNV equation after the substitution $q \to 4U$ and after rescaling the time parameter $t \to 4t$. In fact, the mKdV hierarchy is a reduction of the mNV hierarchy for $U = U(x)$.

We see that, in the case of the mKdV equations, we have no conditions of the form (28), and we can readily define mKdV deformations of surfaces of revolution. Moreover, in this case there is a recursion formula for the higher equations:

$$\frac{\partial q}{\partial t_n} = D^n q_x, \qquad D = \partial_x^2 + q^2 + q_x \partial_x^{-1} q.$$

We introduce the Kruskal–Miura integrals. Their densities R_k are defined by the following recursion procedure:

$$R_1 = \frac{iq_x}{2} - \frac{q^2}{4}, \qquad R_{n+1} = -R_{nx} - \sum_{k=1}^{n-1} R_k R_{n-k}.$$

One can show that the expressions R_{2n} are total derivatives if and only if the integrals

$$H_k = \int R_{2k-1}\,dx$$

do not vanish identically.

Theorem 8 [46]. *For every $n \geqslant 1$ the nth mKdV equation (as the reduction of the mNV deformation) transforms the tori of revolution into tori of revolution and preserves their conformal types and the values of the functionals H_k, $k \geqslant 1$.*

The proof of the analogous theorem for spheres of revolution (they are studied in [6]) is in fact the same as for tori.

We note that the preservation of tori is a non-trivial assertion. The operator L also appears in the L, A-pair for the sine-Gordon equation, which thus also induces deformations of surfaces of revolution. However, this deformation develops tori into cylinders.

We see that

$$H_1 = -\frac{1}{4}\int q^2\,dx = -4\int U^2\,dx = -\frac{2}{\pi}\int U^2\,dx \wedge dy,$$

and hence the first Kruskal–Miura integral is proportional to the Willmore functional. The next integrals are

$$H_2 = \frac{1}{16}\int (q^2 - 4q_x^2)\,dx, \qquad H_3 = \frac{1}{32}\int (q^6 - 20q^2q_x^2 + 8q_x x^2)\,dx.$$

It would be of interest to answer the following question.

> *What is the geometric meaning of the functionals H_k, and what are the extremals of these functionals among the compact surfaces of revolution?*

The mKdV deformations of surfaces of revolution determine deformations of curves in the upper half-plane that generate the surfaces upon revolution. Both the geometry of these deformations and the relationship between the recursion relations and the geometry of curves were studied in [47], [48].

3.3. The Davey–Stewartson equation. The mNV equations are themselves reductions (for $U = -p = \overline{q}$) of the Davey–Stewartson (DS) equations represented by L, A, B-triples with

$$L = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} -p & 0 \\ 0 & q \end{pmatrix}.$$

In fact, this reduction of the Davey–Stewartson equations gives more equations of the form

$$U_t = i(\partial^{2n} U + \overline{\partial}^{2n} U) + \cdots$$

or

$$U_t = \partial^{2n+1} U + \overline{\partial}^{2n+1} U + \cdots$$

for $n \geqslant 1$. The equations in the first series do not preserve the 'realness condition' $U = \overline{U}$, and the second series reduces to the mNV hierarchy for $U = \overline{U}$.

The first two equations here are the DS_2 equation

$$U_t = i(U_{zz} + U_{\bar{z}\bar{z}} + 2(V + \overline{V})U), \tag{31}$$

where

$$V_{\bar{z}} = \partial(|U|^2), \tag{32}$$

and the DS_3 equation (sometimes called the Davey–Stewartson equation I):

$$U_t = U_{zzz} + U_{\bar{z}\bar{z}\bar{z}} + 3(VU_z + \overline{V}U_{\bar{z}}) + 3(W + W')U, \tag{33}$$

where

$$V_{\bar{z}} = (|U|^2)_z, \qquad W_{\bar{z}} = (\overline{U}U_z)_z, \qquad W'_z = (\overline{U}U_{\bar{z}})_{\bar{z}}. \tag{34}$$

The Davey–Stewartson equations determine soliton deformations of surfaces in $\mathbb{R}^4$. As in the case of surfaces in $\mathbb{R}^3$, these deformations were introduced by Konopel'chenko, who proved the corresponding analogue of Theorem 6 in [9].

However, in this case we face two specific problems.

1) As was already noted in § 2.5, a Weierstrass representation of a surface in $\mathbb{R}^4$ is not unique. Is it true that deformations of the Davey–Stewartson equations are geometrically different for different representations?

2) The constraints for the Davey–Stewartson equations are more complicated. How must one solve the conditions (32) and (34) in order to obtain global deformations of closed surfaces?

We have considered these problems in [35].

The answer to the first question shows a significant difference from mVN deformations, namely:

> *Deformations of the Davey–Stewartson equations are well defined only for surfaces with given potentials U of their Weierstrass representations, and these deformations are geometrically different for different choices of the potential.*

It would be of interest to understand the geometric meaning of the different deformations of the same surface.

The answer to the second question is given by the following analogue of Theorem 7.

Theorem 9. 1) *For the function V uniquely determined by the equation (32) and the normalization $\int V\,dz \wedge d\bar{z} = 0$ the DS_2 equation determines a deformation of tori into tori that preserves their conformal classes and the values of the Willmore functional.*

2) *For*

$$V_{\bar z} = (|U|^2)_z, \quad \int V\, dz \wedge d\bar z = 0, \quad W = \partial \bar\partial^{-1}(\bar u u_z), \quad W' = \bar\partial \partial^{-1}(\bar u u_{\bar z}) \qquad (35)$$

the DS_3 equation determines a deformation of tori into tori that preserves their conformal classes and the Willmore functional.

The surface is deformed by deformations of the vector functions ψ and φ, and these deformations are connected with the operator A in the L, A, B-triple. There are many other additional potentials appearing in A and the DS equations, as was explained in [9]. We do not explain here the reductions in the formula for A that are necessary to keep the surface closed in the course of the deformations. We note only that the formula (34) determines periodic potentials W and W' up to constants, and the formula (35) normalizes these constants. This normalization is necessary to preserve the Willmore functional. The resolution of all these constraints is presented in [35], and we refer to [35] for details.

§ 4. Spectral curves

4.1. Some facts from functional analysis. For a given domain $\Omega \subset \mathbb{R}^n$ we denote by $L_p(\Omega)$ and W_p^k the Sobolev spaces that are the closures of the space of compactly supported closed functions on Ω with respect to the norms

$$\|f\|_p = \int_\Omega |f(x)|^p\, dx_1 \cdots dx_n$$

and

$$\|f\|_{k,p} = \sum_{0 \leqslant l_1 + \cdots + l_n = l \leqslant k} \int_\Omega \left| \frac{\partial^l f}{\partial^{l_1} x_1 \cdots \partial^{l_n} x_n} \right|^p dx_1 \cdots dx_n,$$

respectively. For a torus $T^n = \mathbb{R}^n/\Lambda$ we denote by $L_p(T^n)$ and $W_p^k(T^n)$ the analogous Sobolev spaces of Λ-periodic functions. Here the integrals in the definitions of the norms are taken over some compact fundamental domain of the translation group Λ.

Proposition 5. *Let Ω be a compact closed domain in $\mathbb{R}^n$ or a torus. In this case:*

a) (*Rellich*) *there is a natural continuous embedding $W_p^k(\Omega) \to L_p(\Omega)$, which is compact for $k > 0$;*

b) (*Hölder*) *multiplication by any function $u \in L_p$ is a bounded operator from L_q to L_r with $\|uv\|_r \leqslant \|u\|_p \|v\|_q$, $1/p + 1/q = 1/r$;*

c) (*Sobolev*) *there is a continuous embedding $W_p^1(\Omega) \to L_q(\Omega)$, $q \leqslant np/(n-p)$, whose norm is called the Sobolev constant;*

d) (*Kondrashov*) *the Sobolev embedding is compact for $q < np/(n-p)$.*

Let us denote the spaces of two-component vector functions on a torus $M = \mathbb{R}^2/\Lambda$ by

$$\mathbf{L}_p = L_p(M) \times L_p(M) \quad \text{and} \quad \mathbf{W}_p^k = W_p^k(M) \times W_p^k(M),$$

respectively, to distinguish them from the spaces $L_2(M)$ and $W_p^1(M)$ of scalar-valued functions.

Let H be a Hilbert space. An operator $A\colon H \to H$ is said to be *compact* if the closure of the image $A(B)$ of the unit ball $B = \{|x| < 1 : x \in H\}$ is compact. The spectrum $\operatorname{Spec} A$ of any compact operator A is bounded and can have a limit point only at the origin.

For a given Hilbert space H and an operator A (not necessarily bounded) we denote by $R(\lambda)$ the resolvent of A. This is an operator pencil of the form

$$R(\lambda) = (A - \lambda)^{-1},$$

which has poles at the non-zero points of $\operatorname{Spec} A$ and is holomorphic with respect to λ outside $\operatorname{Spec} A$.

The Hilbert identity says that

$$R(\mu)R(\lambda) = \frac{1}{\mu - \lambda}(R(\lambda) - R(\mu)), \tag{36}$$

or, otherwise expressed,

$$\frac{1}{A-\mu}\,\frac{1}{A-\lambda} = \frac{1}{\mu-\lambda}\left(\frac{1}{A-\lambda} - \frac{1}{A-\mu}\right).$$

If a resolvent is given on some domain in $\mathbb{C}$, then it can be meromorphically continued to the entire plane $\mathbb{C}$ by using the following corollary to the Hilbert identity:

$$R(\mu) = R(\lambda)((\mu - \lambda)R(\lambda) + 1)^{-1}$$

(we note that $R(\lambda)R(\mu) = R(\mu)R(\lambda)$).

Proposition 6. *If the operator $R(\lambda)$ is compact at $\lambda = \lambda_0$ and the operator function is holomorphic with respect to λ near λ_0, then*:

1) *the resolvent $R(\mu)$ is compact for any $\mu \in \mathbb{C} \setminus \operatorname{Spec} A$ and has poles only at the points of* $\operatorname{Spec} A$;

2) *$R(\lambda)$ is holomorphic on* $\mathbb{C} \setminus \operatorname{Spec} A$.

4.2. Spectral curve of the Dirac operator with bounded potentials. In this section we explain a scheme for proving the existence of a spectral curve for a differential operator with periodic coefficients. We used this scheme in [3] for Dirac operators with bounded potentials. This case covers all Dirac operators corresponding to tori in $\mathbb{R}^3$.

Let

$$\mathscr{D} = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \mathscr{D}_0 + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}.$$

Here we denote by $\mathscr{D}_0$ the free Dirac operator,

$$\mathscr{D}_0 = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix}. \tag{37}$$

A *Floquet eigenfunction* ψ of the operator $\mathscr{D}$ with the eigenvalue (or *energy*) E is a formal solution of the equation

$$\mathscr{D}\psi = E\psi$$

satisfying the periodicity conditions

$$\psi(z+\gamma_j,\bar z+\overline{\gamma}_j)=e^{2\pi i(k,\gamma_j)}\psi(z,\bar z)=\mu(\gamma_j)\psi(z,\bar z),\qquad j=1,2,$$

where

$$(k,\gamma_j)=k_1\gamma_j^1+k_2\gamma_j^2,\qquad \gamma_j=\gamma_j^1+i\gamma_j^2\in\mathbb{C}=\mathbb{R}^2,\qquad k=(k_1,k_2).$$

The quantities k_1,k_2 are called *quasi-momenta* of the function ψ and the pairs $(\mu_1,\mu_2)=(\mu(\gamma_1),\,\mu(\gamma_2))$ are called *multipliers* of ψ.

We represent a Floquet eigenfunction ψ as the product

$$\psi(z,\bar z)=e^{2\pi i(k_1x+k_2y)}\varphi(z,\bar z),\qquad z=x+iy,\quad x,y\in\mathbb{R},$$

where the function $\varphi(z,\bar z)$ is Λ-periodic. The equation $\mathscr{D}\psi=E\psi$ becomes

$$\left[\begin{pmatrix}0&\partial\\-\bar\partial&0\end{pmatrix}+\begin{pmatrix}U&\pi i(k_1-ik_2)\\-\pi i(k_1+ik_2)&V\end{pmatrix}\right]\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}=E\begin{pmatrix}\varphi_1\\\varphi_2\end{pmatrix}.$$

We have an operator pencil of the form

$$\mathscr{D}(k)=\mathscr{D}+T_k,\tag{38}$$

where

$$T_k=\begin{pmatrix}0&\pi i(k_1-ik_2)\\-\pi i(k_1+ik_2)&0\end{pmatrix},\tag{39}$$

and this pencil is analytic with respect to k_1 and k_2.

We see that finding a Floquet eigenfunction ψ with some quasi-momenta k_1, k_2 and some energy E is the same as finding a periodic solution φ of the equation

$$\mathscr{D}(k)\varphi=E\varphi.$$

Let us consider the solutions of this equation that belong to L_2.

We choose an E_0 such that the operator $(\mathscr{D}_0-E_0)$ is invertible on L_2, that is, the inverse operator

$$(\mathscr{D}_0-E_0)^{-1}\colon L_2\to W_2^1$$

exists. We represent the function φ in the form

$$\varphi=(\mathscr{D}_0-E_0)^{-1}f,$$

substitute this expression into the equation

$$(\mathscr{D}(k)-E)\varphi=0,$$

and arrive at the equation

$$(1+A(k,E))f=0,\qquad f\in L_2,$$

where

$$A(k,E)=\begin{pmatrix}U+(E_0-E)&\pi i(k_1-ik_2)\\-\pi i(k_1+ik_2)&V+(E_0-E)\end{pmatrix}(\mathscr{D}_0-E_0)^{-1}=B(k,E)(\mathscr{D}_0-E_0)^{-1}.$$

Finally, the existence problem for Floquet functions with given quasi-momenta k and energy E reduces to the problem of solving the equation

$$(1 + A(k, E))f = 0$$

in L_2. We note that the operator $A(k, E)$ can be decomposed into the following chain of operators:

$$L_2 \overset{(\mathscr{D}_0 - E)^{-1}}{\longrightarrow} W_2^1 \overset{\text{embedding}}{\longrightarrow} L_2 \overset{\text{multiplication}}{\longrightarrow} L_2. \tag{40}$$

The first map is continuous, the second map is compact, and if the potentials U and V are assumed to be bounded, then the third map (the multiplication by $B(k, E)$) is continuous. Hence, the following assertion holds.

Proposition 7. *For any given bounded potentials U and V the analytic pencil of operators $A(k, E)\colon L_2 \to L_2$ consists of compact operators.*

We can now use the Keldysh theorem ([49], [50]), which is the Fredholm alternative for analytic operator pencils of the form $[1 + A(\mu)]$, where the operator $A(\mu)$ is compact for every μ. This theorem asserts the following.

> The resolvent of the pencil $[1 + A(\mu)]\colon H \to H$, where $A(\mu)$ is an analytic pencil of compact operators, is a meromorphic function of μ. Its singularities correspond to solutions of the equation $(1 + A(\mu))f = 0$ and form an analytic subset Q in the space of parameters μ.

In what follows, we consider only Floquet functions with $E = 0$.

For an operator $\mathscr{D}$ with potentials U and V we have $\mu = (k, E) \in \mathbb{C}^3$. We write

$$Q_0(U, V) = Q \cap \{E = 0\}. \tag{41}$$

This set is invariant with respect to translations by the vectors in the dual lattice $\Lambda^* \subset \mathbb{R}^2 = \mathbb{C}$:

$$k_1 \to k_1 + \eta_1, \qquad k_2 \to k_2 + \eta_2.$$

We recall that the *dual lattice* consists of the vectors $\eta = \eta_1 + i\eta_2$ such that $(\eta, \gamma) = \eta_1\gamma^1 + \eta_2\gamma^2 \in \mathbb{Z}$ for any $\gamma = \gamma^1 + i\gamma^2 \in \Lambda$.

The spectral curve is defined as

$$\Gamma = Q_0(U, V)/\Lambda^*.$$

Remark. One can readily see that the composition of the operator

$$(\mathscr{D}(k) - E)^{-1} = (\mathscr{D}_0 - E_0)^{-1}(1 + A(k, E))^{-1}\colon L_2 \to W_2^1$$

with the canonical embedding $W_2^1 \to L_2$ is the resolvent $R(k, E)$ of the operator

$$\mathscr{D}(k) = \mathscr{D} + \begin{pmatrix} 0 & \pi i(k_1 - ik_2) \\ -\pi i(k_1 + ik_2) & 0 \end{pmatrix}.$$

The intersection of the set of poles of the resolvent $R(k, E)$ with the plane $E = 0$ is the set $Q_0(U, V)$.

We arrive at the following definitions.

a) The *spectral curve* Γ of the operator $\mathscr{D}$ with potentials U and V is the complex curve $Q(U,V)/\Lambda^*$ regarded up to biholomorphic equivalence.
b) A *multiplier map* is defined on Γ: a local embedding at any generic point, namely,

$$\mathscr{M}\colon \Gamma \to \mathbb{C}^2 \ : \ \mathscr{M}(k) = (\mu_1, \mu_2) = (e^{2\pi i(k,\gamma_1)}, e^{2\pi (k,\gamma_2)}),$$

where γ_1 and γ_2 are the generators of $\Lambda \subset \mathbb{C}$ and $(k,\gamma_j) = k_1 \operatorname{Re}\gamma_j + k_2 \operatorname{Im}\gamma_j$, $j = 1, 2$.[5]
c) To each point of Γ one assigns the space of Floquet functions with the given multipliers. The dimension of these spaces increases in general at singular points of Γ.

Proposition 8. *Let $k = (k_1, k_2)$ be the quasi-momenta of a Floquet function of the operator $\mathscr{D}$.*

1) *If $U = \overline{V}$, then Γ admits the antiholomorphic involution $\tau\colon k \to -\overline{k}$.*
2) *If $U = \overline{U}$ and $V = \overline{V}$, then Γ admits the antiholomorphic involution $k \to \overline{k}$.*
3) *If $U = \overline{U} = V$, then the composition of involutions in* 1) *and* 2) *gives the holomorphic involution $\sigma\colon k \to -k$.*

These conditions are standard for spectral curves (see, for instance, the case of a potential Schrödinger operator in [38] and [39]) and are explained for the Dirac operator in [5], [40], [41]. The simplest of these conditions is the first one, which is proved by the following obvious lemma.

Lemma 2. *If $U = \overline{V}$, then the transformation $\varphi \to \varphi^*$ in* (4) *takes any Floquet function to a Floquet function and modifies the multipliers by the rule $k \to -\overline{k}$.*

We denote by $\Gamma_{\rm nm}$ the normalization of Γ. The Riemann surface Γ is not algebraic, but is a complex space for which the existence of a normalization was proved in [51]. Since we are in a one-dimensional situation, all singular points are isolated and the normalization has the following form:

1) if a point $P \in \Gamma$ is reducible, that is, several branches of Γ meet at P, then these branches are unglued;

2) for an irreducible singular point P the normalization $\Gamma_{\rm nm} \to \Gamma$ is a local homeomorphism near P and can be represented in terms of series in local parameters:

$$k_1 = t^a + \cdots, \qquad k_2 = t^b + \cdots, \qquad a > 1, \quad b > 1.$$

Here t is a local coordinate near P on $\Gamma_{\rm nm}$.

If there are no reducible singular points, then the normalization map $\Gamma_{\rm nm} \to \Gamma$ is a homeomorphism.

The genus of the complex curve $\Gamma_{\rm nm}$ is called the *geometric genus* of Γ and is denoted by $p_g(\Gamma)$. An operator is called a *finite-gap* operator (at the zero energy level) if $p_g(\Gamma) < \infty$.

[5] This map depends on the choice of generators γ_1, γ_2. If the basis γ_1, γ_2 is replaced by another basis $\widetilde{\gamma}_1 = a\gamma_1 + b\gamma_2$, $\widetilde{\gamma}_2 = c\gamma_1 + d\gamma_2$, then the quantity $\mathscr{M} = (\mu_1, \mu_2)$ is transformed according to the formula

$$\mathscr{M} \to \widetilde{\mathscr{M}} = (\mu_1^a\,\mu_2^b, \mu_1^c\,\mu_2^d). \tag{42}$$

The analogue of the arithmetic genus for Γ (which appears in theorems of Riemann–Roch type) is always infinite, $p_a(\Gamma) = \infty$.

We have the following assertion.

> The non-singular points of the normalized spectral curve Γ_{nm} parametrize (up to factors) the Floquet functions ψ, $\mathscr{D}\psi = 0$. In contrast to Γ, the one-to-one correspondence of this parametrization is violated at only finitely many singular points.[6]

In § 4.7 we shall show that if the genus of Γ_{nm} is finite, then it is better to replace the curve Γ_{nm} by the curve Γ_ψ whose definition involves the Baker–Akhiezer function of the operator $\mathscr{D}$.

Example. The spectral curve for $U = V = 0$ (the free operator). For simplicity, we assume that $\Lambda = \mathbb{Z} + i\mathbb{Z}$. The Floquet functions are

$$\psi^+ = (e^{\lambda_+ z}, 0), \qquad \psi^- = (0, e^{\lambda_- \bar{z}})$$

and are parametrized by a pair of complex lines with parameters λ_+ and λ_-. These complex lines form the normalized spectral curve Γ_{nm}. Since the curve is of finite genus, we compactify it by two points at infinity such that ψ has exponential singularities at these points. The quasi-momenta of these functions are

$$k_1 = \frac{\lambda_+}{2\pi i} + n_1, \qquad k_2 = \frac{\lambda_+}{2\pi} + n_2 \quad \text{for} \quad \psi^+,$$
$$k_1 = \frac{\lambda_-}{2\pi i} + m_1, \qquad k_2 = -\frac{\lambda_-}{2\pi} + m_2 \quad \text{for} \quad \psi^-,$$

where $m_j, n_j \in \mathbb{Z}$. The functions ψ^+ and ψ^- have the same multipliers at the points

$$\lambda_+^{m,n} = \pi(n + im), \quad \lambda_-^{m,n} = \pi(n - im), \qquad m, n \in \mathbb{Z},$$

which form resonance pairs. The complex curve Γ is obtained from the two complex lines by the pairwise identification of points in resonance pairs.

Remark. The spectral curve and the Kadomtsev–Petviashvili equation. We presented above the scheme which we used in 1985 to define spectral curves of differential operators with periodic coefficients (this paper was never published, though a reference to it can be found in [52]). We found out later that a very similar scheme was used by Kuchment [53] (see also [54]). However, we should mention an important observation made at the time concerning the Kadomtsev–Petviashvili equations. There are two Kadomtsev–Petviashvili (KP) equations,

$$\partial_x(u_t + 6uu_x + u_{xxx}) = -3\varepsilon^2 u_{yy},$$

where $\varepsilon^2 = \pm 1$. This equation is called the *KPI equation* for $\varepsilon = i$ and the *KPII equation* for $\varepsilon = 1$. From the point of view of physics, these equations are drastically different. The two equations admit similar representations in the form of an L, A-pair $\dot{L} = [L, A]$ with the operator L of the form

$$L = \varepsilon\partial_y + \partial_x^2 + u.$$

[6] This follows from the asymptotic behaviour of the spectral curve (see § 4.3).

Here the potential u is doubly periodic or, which is the same, is defined on some torus $\mathbb{R}^2/\Lambda$. The free operator is equal to $L_0 = \varepsilon\partial_y - \partial^2$, and to prove the existence of the spectral curve by the above scheme, we must consider the inverse operator

$$(L_0 - E_0)^{-1}\colon L_2 \to W_2^{2,1},$$

where $W_2^{2,1}$ is the space of functions u on the torus such that u, u_x, u_{xx}, u_y belong to L_2. To simplify the calculations, we consider the case in which the lattice Λ is spanned by the vectors $(2\pi, 0)$ and $(0, 2\pi\tau^{-1})$. In this case, the Fourier basis in L_2 is formed by the functions

$$e^{i(kx+l\tau y)}, \qquad k, l \in \mathbb{Z}.$$

The operator $(L_0 - E_0)$ is diagonal in this basis, and

$$(L_0 - E_0)e^{i(kx+l\tau y)} = (i\varepsilon l\tau - k^2 - E_0)e^{i(kx+l\tau y)}.$$

Since $\varepsilon = 1$ for the KPII equation, we obtain a bounded operator for $E_0 > 0$,

$$(L_0 - E_0)^{-1}e^{i(kx+l\tau y)} = \frac{1}{il\tau - k^2 - E_0}e^{i(kx+l\tau y)}.$$

One can readily see that if $\varepsilon = i$, then for any E_0 either the operator $(L_0 - E_0)$ is not invertible or its inverse is unbounded. This holds for any lattice Λ. One can derive from these considerations that there is no spectral curve for the operator $L = i\partial_y + \partial_x^2 + u$. For the heat operator $L = \partial_y + \partial_x^2 + u$ the spectral curve exists and is preserved by the KPII equation.

The spectral curve of a two-dimensional periodic differential operator L on the zero energy level was first introduced in the paper [4] of Dubrovin, Krichever, and Novikov in the case of the Schrödinger operator. It was shown in [4] that:

1) the periodic operator, which is a finite-gap operator on the zero energy level, can be recovered from some algebraic data including this curve;[7]

2) this curve is a first integral of the deformations of the operator L that are determined by the L, A, B-triples.

Proposition 9 [4]. *Let L be a two-dimensional periodic operator, let Γ be its spectral curve, and let $\mathscr{M}$ be the multiplier map.*

Let an evolution equation

$$\frac{\partial L}{\partial t} = [L, A] - BL$$

be given, where the operator A is also periodic. In this case the deformation of the operator L preserves the curve Γ and the map $\mathscr{M}$.

This result generalizes the conservation law for the spectral curve of a one-dimensional operator L under a deformation of this operator determined by an L, A-pair $\dfrac{\partial L}{\partial t} = [L, A]$ (this fact was first established by Novikov for the periodic KdV equation in [56]).

[7] For the Dirac operator $\mathscr{D}$ one can find the recovery formula (51), together with the proof, in [55] and in § 4.7.

This assertion follows from the deformation equation $\psi_t + A\psi = 0$ for the Floquet functions, which preserves the multipliers (see § 3.1 and the equation (26)). The preservation of the zero-level spectrum for general operators was first indicated by Manakov in [57], where the L, A, B-triples were introduced.

Corollary 2. *The spectral curve* Γ *and the multiplier map* $\mathscr{M}$ *of the periodic Dirac operator* $\mathscr{D}$ *are preserved by the modified Novikov–Veselov equations and by the Davey–Stewartson equations.*

For the mKdV equations we have two spectral curves, namely, the curve Γ defined for the two-dimensional Dirac operator and the curve Γ' defined for the one-dimensional operator L_{mKdV} in both the Zakharov–Shabat problem (see § 3.2) and the representation in the form of an L, A-pair for the mKdV equation. These complex curves are related by the canonical branched two-sheeted covering $\Gamma \to \Gamma_0$ [40], and both the curves are preserved by the mKdV equation. The complex curve Γ_0 is uniquely recovered from the Kruskal–Miura integrals H_k, $k = 1, 2, \dots$, which are thus also first integrals of the mKdV equation.

4.3. Asymptotic behaviour of the spectral curve. The spectral curve of the operator $\mathscr{D}$ is a perturbation of the spectral curve of the free operator $\mathscr{D}_0$. Although this perturbation could be rather strong in a bounded domain $|k| \leqslant C$, it reduces to a transformation of double points corresponding to resonance pairs into handles outside the domain. Moreover, the size of handles decreases as $|k| \to \infty$ and can be estimated in terms of the perturbation.

Thus, we have:

1) a compact part of the form $\Gamma_0 = Q_0 \cap \{|k| \leqslant C\}$ whose boundary consists of a pair of circles;

2) a complex curve Γ_∞ obtained from the planes $k_1 = ik_2$ and $k_1 = -ik_2$ by removing the domains $\{|k| \leqslant C\}$ from these planes and by transforming some double points corresponding to resonance pairs into handles;

3) Γ_0 and Γ_∞ are glued together along their boundaries;

4) Γ has two ends at which the image $\mathscr{M}(\Gamma)$ has the same asymptotic behaviour as the free operator.

This complex curve is obtained from the spectral curve Γ by ungluing the double points that corresponded to resonance pairs and were preserved under the perturbation. We denote this curve again by Γ.

The operator is called a *finite-gap operator* (on the zero energy level) if only finitely many double points are transformed into handles under the perturbation $\mathscr{D}_0 \to \mathscr{D}$.

This picture is typical in soliton theory, where the spectral curve of some operator with potentials is a perturbation of the spectral curve of the corresponding free operator and the perturbation is small for large values of quasi-momenta. The picture was rigorously justified for the two-dimensional Schrödinger operator by Krichever [52], who used asymptotic methods. In [10] we proposed a justification of this geometric picture for the Dirac operator by using the same methods, and we formulated the desired statement as a ‘pretheorem’.

The theory of spectral curves initiated the development of the analytic theory of Riemann surfaces (not only hyperelliptic) of infinite genus ([58], [59]).

In [5] Schmidt proposed another approach to the justification of this asymptotic behaviour of the spectral curve. His approach is based on his result on the existence of spectral curves for the Dirac operators with L_2-potentials and on the continuous behaviour of these curves with respect to weakly convergent sequences of potentials.

Theorem 10 [5]. *For $U, V \in L_2(T^2)$ the equation*

$$\mathscr{D}(k)\varphi = (\mathscr{D} + T_k)\varphi = E\varphi,$$

where $k \in \mathbb{C}^2$, $E \in \mathbb{C}$, has a solution in L_2 if and only if $(k, E) \in Q$, where Q is an analytic subset of $\mathbb{C}^3$. This subset Q is formed by the poles of the operator pencil

$$(1 + A_{U,V}(k, E))^{-1}\, L_2 \to L_2,$$

where the operator $A_{U,V}(k, E)$ is a polynomial in k and E. Moreover, if

$$U_n, V_n \overset{\text{weakly}}{\longrightarrow} U_\infty, V_\infty$$

in $\{\|U\|_{2;\varepsilon} \leqslant C, \|V\|_{2;\varepsilon} \leqslant C\}$,[8] *then*

$$\|A_{U_n,V_n}(k, E) - A_{U_\infty,V_\infty}(k, E)\|_2 \to 0$$

uniformly near every point $k \in \mathbb{C}^2$.

We present the proof of this theorem in Appendix 1. Let us return to the asymptotic behaviour of the spectral curve.

We first note the following identity, which can be verified by straightforward computations:

$$\begin{pmatrix} e^{-a} & 0 \\ 0 & e^{-b} \end{pmatrix} \left(\mathscr{D}_0 + \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} + T_k \right) \begin{pmatrix} e^{b} & 0 \\ 0 & e^{a} \end{pmatrix}$$
$$= \mathscr{D}_0 + \begin{pmatrix} e^{b-a}U & 0 \\ 0 & e^{a-b}V \end{pmatrix} + T_k + \begin{pmatrix} 0 & a_z \\ -b_{\bar z} & 0 \end{pmatrix} \qquad (43)$$

for any smooth functions $a, b\colon \mathbb{C} \to \mathbb{C}$.

For any $\kappa = (\kappa_1, \kappa_2) \in \Lambda^* \subset \mathbb{C}$ we define Λ-periodic functions

$$\psi_{\pm\kappa}(z, \bar z) = e^{\pm 2\pi i(\kappa_1 x + \kappa_2 y)},$$

and we choose functions $a(z, \bar z)$ and $b(z, \bar z)$ of the form

$$a(z, \bar z) = 2\pi i(\alpha_1 x + \alpha_2 y), \qquad b(z, \bar z) = 2\pi i((\alpha_1 - \kappa_1)x + (\alpha_2 - \kappa_2)y),$$

where

$$\alpha(\kappa) = (\alpha_1, \alpha_2) = \left(\frac{\kappa_1 + i\kappa_2}{2}, \frac{-i\kappa_1 + \kappa_2}{2} \right).$$

The following equalities are clear: $e^{b-a} = \psi_{-\kappa}$ and $a_z = b_{\bar z} = 0$. This, together with (43), implies the following assertion.

[8] We recall that a sequence $\{u_n\}$ in a Hilbert space H is said to be *weakly convergent* to u_∞, $u_n \overset{\text{weakly}}{\longrightarrow} u_\infty$, if $\lim_{n\to\infty}\langle u_n, v\rangle = \langle u_\infty, v\rangle$ for any $v \in H$, where $\langle u, v\rangle$ is the inner product in H.

Proposition 10 [5]. *If a function $\varphi \in L_2$ satisfies the equation*

$$\left[\mathscr{D}_0 + \begin{pmatrix} \psi_{-\kappa} U & 0 \\ 0 & \psi_\kappa V \end{pmatrix} + T_k\right] \varphi = 0,$$

then the function $\varphi' = \begin{pmatrix} \psi_{-\kappa} & 0 \\ 0 & 1 \end{pmatrix} \varphi \in L_2$ *satisfies the equation*

$$(\mathscr{D} + T_{k+\alpha})\varphi' = 0.$$

Hence,

$$Q_0(\psi_{-\kappa}U, \psi_\kappa V) = Q_0(U,V) + \alpha(\kappa) \quad \text{for any} \quad \kappa \in \Lambda^*$$

(*where the right-hand side stands for the set $Q_0(U,V)$ translated by α*).

The functions $\psi_\kappa, \kappa \in \Lambda^*$ stand for the Fourier basis in L_2. The map $U \to \widehat{U} = \psi_\kappa U$, $U = \sum_{\nu \in \lambda^*} U_\nu \psi_\nu$, shifts the Fourier coefficients of the function U, namely, $\widehat{U}_\nu = U_{\nu-\kappa}$. Hence, we have

$$\psi_\kappa U \overset{\text{weakly}}{\longrightarrow} 0 \quad \text{as} \quad |\kappa| \to \infty.$$

Theorem 10 (see Appendix 1) and Proposition 10 imply that the intersection of $Q_0(U,V)$ with $O(k) + \alpha(\kappa)$ is very close to the intersection of $Q_0(0,0)$ with $O(k)$ in small bounded neighbourhoods $O(k)$ of the points $k \in \mathbb{C}^2$ for large values of $|\kappa|$:

$$Q_0(U,V) \cap [O(k) + \alpha(\kappa)] \approx Q_0(0,0) \cap O(k) \quad \text{as} \quad |\kappa| \to \infty.$$

We conclude that, asymptotically as $|k| \to \infty$, the behaviour of the spectral curve for the operator $\mathscr{D}$ is similar to that of the spectral curve of the free operator $\mathscr{D}_0$ on L_2.

For $U = V = 0$ the spectral curve Γ is biholomorphically equivalent to a pair of two planes (complex lines) defined in $\mathbb{C}^2$ by the equations

$$k_2 = ik_1, \qquad k_2 = -ik_1,$$

and glued together at infinitely many pairs of points corresponding to the so-called resonance pairs,

$$\left(k_1 = \frac{\overline{\gamma}_1 n - \overline{\gamma}_2 m}{\overline{\gamma}_1 \gamma_2 - \gamma_1 \overline{\gamma}_2}, k_2 = ik_1\right) \leftrightarrow \left(k_1 = \frac{\gamma_1 n - \gamma_2 m}{\overline{\gamma}_1 \gamma_2 - \gamma_1 \overline{\gamma}_2}, k_2 = -ik_1\right),$$

where $m, n \in \mathbb{Z}$. Moreover, these planes are naturally completed by a pair of points at infinity $\infty_\pm$ obtained as the limits $(k_1, \pm ik_1) \to \infty_\pm$ as $k_1 \to \infty$. A double covering $\Gamma \to \mathbb{C}$: $(k_1, k_2) \to k_1$ is defined near any generic point. By Proposition 10, we have the following assertion.

Corollary 3. *For any Dirac operator with L_2-potentials the image $\mathscr{M}(\Gamma)$ has the asymptotic behaviour*

$$k_2 \approx \pm ik_1$$

for sufficiently large values of $|k|$. Hence, $\mathscr{M}(\Gamma)$ has at most two irreducible components such that every component contains at least one of the asymptotic ends.

The bound for the number of irreducible components is clear, because the other components must be localized in a bounded domain of $\mathbb{C}^2$, which is impossible for one-dimensional analytic sets.

We thus arrive at the following definition compatible with that used in the theory of finite-gap integration ([4], [60]).

> If the spectral curve Γ of an operator $\mathscr{D}$ has finite genus, then $\mathscr{D}$ is a finite-gap operator, and the completion of Γ by a pair $\infty_\pm$ of points at infinity is called the *spectral curve of the finite-gap operator*.

We conclude with a procedure for recovering the value of

$$\int_{\mathbb{C}/\Lambda} UV\,dx\wedge dy$$

from $(\Gamma,\mathscr{M})$ provided that Γ is of finite genus. Near the asymptotic end at which $k_2\approx ik_1$, we introduce a local parameter λ_+^{-1} such that the multipliers have the following behaviour:

$$\mu(\gamma)=\lambda_+\gamma+\frac{C_0\overline{\gamma}}{\lambda_+}+O(\lambda_+^{-2}).$$

Then

$$\int_{\mathbb{C}/\Lambda} UV\,dx\wedge dy=-C_0\cdot(\mathrm{Area}(\mathbb{C}/\Lambda)) \tag{44}$$

(see [61] and [10] in the case $U=V$).

A similar formula for the area of minimal tori in S^3 was derived by Hitchin in [62].

This formula gives a reason to treat the pair $(\Gamma,\mathscr{M})$ as a generalization of the Willmore functional. This was first discussed for tori of revolution in [46]. In this case the spectral curve is recovered from infinitely many quantities known as Kruskal–Miura integrals.

4.4. Spectral curves of tori. For a torus Σ immersed in one of the three-dimensional Lie groups $G=\mathbb{R}^3$, $SU(2)=S^3$, Nil, or $\widetilde{SL}_2$ and for a Weierstrass representation of Σ we consider the spectral curve Γ of the operator $\mathscr{D}$ in this representation.

We refer to this curve as the *spectral curve of the torus* Σ.

This curve is defined for all smooth tori and not just for integrable tori (see § 4.6). This definition was originally introduced in [3] for tori in $\mathbb{R}^3$ and in [6] for tori in S^3 in connection with the physical explanation of the Willmore conjecture. The formula (44) shows that the Willmore functional can be recovered from Γ and the multiplier map $\mathscr{M}$ (at least if Γ is of finite genus).

This definition does not depend on the choice of a conformal parameter on the torus $\Sigma=\mathbb{R}^2/\Lambda$. The multiplier map $\mathscr{M}$ depends on the choice of a basis in Λ, and any change of the basis leads to a simple algebraic transformation of $\mathscr{M}$ (see (42)).

We define the spectral curve for tori in $\mathbb{R}^4$.

As we have explained in [35], a Weierstrass representation is not unique for surfaces in $\mathbb{R}^4$. The potentials of different representations of a torus are connected by the formula

$$U\to U\exp\left(\overline{a}+\overline{bz}-a-bz\right), \tag{45}$$

where $\operatorname{Im} b\gamma \in \pi\mathbb{Z}$ for any $\gamma \in \Lambda$. The multiplier map $\mathscr{M}$ depends on the choice of U and changes as follows under the transformation (45):

$$\mu(\gamma) \to e^{b\gamma}\mu(\gamma), \qquad \gamma \in \Lambda.$$

As in the case of tori in $\mathbb{R}^3$, the integral of the squared norm of the potential U can be recovered from $(\Gamma, \mathscr{M})$ by the same formula (44).

The conformal invariance of the Willmore functional led us to the conjecture which we justified by numerical experiments in [46] and which was confirmed in [61] (soon after its formulation) as follows.

Theorem 11. *For any torus in $\mathbb{R}^3$ the spectral curve Γ and the multiplier map $\mathscr{M}$ of the torus are invariant with respect to conformal transformations of $\overline{\mathbb{R}}^3$.*

The proof in [61] works rigorously only for spectral curves of finite genus and is as follows. We consider the generators of the conformal group $SO(4,1)$ and write out the deformation equations for the Floquet functions φ; these equations are

$$\mathscr{D}\delta\varphi + \delta U \cdot \varphi. \tag{46}$$

It suffices to verify the invariance only for inversions and even only for one inversion, because any two inversions are conjugate by orthogonal transformations. For an inversion let us choose the generator

$$\delta x^1 = -2x^1x^3, \qquad \delta x^2 = -2x^2x^3, \qquad \delta x^3 = (x^1)^2 + (x^2)^2 - (x^3)^2,$$

and compute the corresponding variation of the potential,

$$\delta U = |\psi_2|^2 - |\psi_1|^2,$$

where ψ generates the torus. An explicit formula for this variation was found in [61] for a solution of the equation (46) in terms of functions meromorphic on the spectral curve. It follows from this explicit formula that the multipliers are preserved. One can readily find these meromorphic functions for any spectral curve of finite genus. If the spectral curve in question is of infinite genus, then one must clarify some analytic details, which in our opinion is really possible and depends on a rigorous and careful study of the asymptotic behaviour of the spectral curve.

Another proof of Theorem 11 for isothermal tori was given in [10]. It is geometric and works for spectral curves of arbitrary genus.

4.5. Examples of spectral curves.

Products of circles in $\mathbb{R}^4$. We consider the tori $\Sigma_{r,R}$ defined by the equations

$$(x^1)^2 + (x^2)^2 = r^2, \qquad (x^3)^2 + (x^4)^2 = R^2.$$

These tori are parametrized by angle variables x, y defined modulo 2π, namely, $x^1 = r\cos x$, $x^2 = r\sin x$, $x^3 = R\cos y$, $x^4 = R\sin y$. The conformal parameter, the period lattice, and the induced metric are of the form

$$z = x + i\frac{R}{r}y, \qquad \Lambda = \left\{2\pi m + i2\pi\frac{r}{R}n : m, n \in \mathbb{Z}\right\}, \qquad ds^2 = r^2\,dz\,d\bar{z},$$

respectively. A formula for the Gauss map can be obtained by simple computations,

$$a_1/a_2 = -e^{i(y-x)}, \qquad b_1/b_2 = e^{-i(y+x)}.$$

Let us apply Theorem 4 to the map

$$\Sigma_{r,R} \to (b_1 : b_2) = \left(\frac{e^{-i(x+y)}}{\sqrt{2}} : \frac{1}{\sqrt{2}}\right) \in \mathbb{C}P^1.$$

We have $g = \dfrac{i(x+y)}{2}$,

$$U = \frac{1}{4}\left(\frac{r}{R} + i\right),$$

and the torus $\Sigma_{r,R}$ is defined in terms of the Weierstrass representation by the vector functions

$$\psi_1 = \psi_2 = \frac{1}{\sqrt{2}} \exp\left(-\frac{i(x+y)}{2}\right), \qquad \varphi_1 = -\varphi_2 = -\frac{r}{\sqrt{2}} \exp\left(\frac{i(y-x)}{2}\right).$$

The values of the Willmore functional on these tori are given by the formula

$$\mathscr{W}(\Sigma_{r,R}) = 4\int_{\Sigma_{r,R}} |U|^2\, dx \wedge dy = \pi^2\left(\frac{r}{R} + \frac{R}{r}\right)$$

and attain their minimum at the Clifford torus Σ in $\mathbb{R}^4$, that is, $\mathscr{W}(\Sigma_{r,r}) = 2\pi^2$.

The spectral curve $\Gamma(u)$ of the Dirac operator

$$\mathscr{D} = \begin{pmatrix} 0 & \partial \\ -\bar\partial & 0 \end{pmatrix} + \begin{pmatrix} u & 0 \\ 0 & \bar u \end{pmatrix}, \qquad u = \mathrm{const},$$

with the constant potential $U = u$ is the complex sphere with a pair of distinguished points ('points at infinity'), namely, $\lambda = 0$ and $\lambda = \infty$,

$$\Gamma(u) = \mathbb{C}P^1.$$

The normalized Baker–Akhiezer function (or Floquet function) is

$$\psi(z, \bar z, \lambda) = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{\lambda}{\lambda - u} \exp\left(\lambda z - \frac{|u|^2}{\lambda}\bar z\right)\begin{pmatrix} 1 \\ -u/\lambda \end{pmatrix}.$$

The normalization means that the following two asymptotic formulae hold:

$$\psi \approx \begin{pmatrix} e^{\lambda_+ z} \\ 0 \end{pmatrix} \quad \text{as} \quad \lambda_+ \to \infty, \qquad \psi \approx \begin{pmatrix} 0 \\ e^{\lambda_- \bar z} \end{pmatrix} \quad \text{as} \quad \lambda_- \to 0$$

with the local parameters given by the formulae $\lambda_+ = \lambda$ near $\lambda = \infty$ and $\lambda_- = -|u|^2/\lambda$ near $\lambda = \infty$.

For the tori $\Sigma_{r,R}$ we have the following assertions:

a) the function ψ generating the torus under the representation (18) is equal to $\psi(z, \bar z, -u)$, where $u = \frac{1}{4}(\frac{r}{R} + i)$, and has the monodromy

$$\psi(z + 2\pi, \bar z - 2\pi i, -u) = \psi\left(z + i2\pi\frac{R}{r}, \bar z - i2\pi\frac{R}{r}, -u\right) = -\psi(z, \bar z, -u);$$

b) there are exactly four points on the spectral curve $\Gamma(u)$ for which the function $\psi(z,\bar z,\lambda)$ has the same monodromy as the function $\psi(z,\bar z,-\lambda)$, namely, the points $\lambda=\pm u,\pm\overline{u}$, and moreover,

$$\begin{pmatrix}\psi_1(z,\bar z,-u)\\ \psi_2(z,\bar z,-u)\end{pmatrix}=\begin{pmatrix}-\overline{\psi}_2(z,\bar z,u)\\ \overline{\psi}_1(z,\bar z,u)\end{pmatrix};$$

c) the spectral curve $\Gamma(u)$ is smooth.

Here k_1 and k_2 are the quasi-momenta of the Floquet functions $\psi(z,\bar z,\lambda)$.

A periodic potential U is determined up to gauge transformations of the form (24). For $b=0$ and for $e^{\overline{a}-a}=-\frac{1+i}{\sqrt{2}}$ this transformation takes the potential U of the Clifford torus to the potential

$$\frac{1}{4}(1+i)\to\frac{e^{\overline{a}-a}}{4}(1+i)=-\frac{i}{2\sqrt{2}},$$

which coincides with the potential of the same torus regarded as a torus in the three-dimensional sphere $S^3\subset\mathbb{R}^4$ [10]. This leads to the following questions.

1) *Do the spectral curves of a torus in $S^3\subset\mathbb{R}^4$ as a torus in S^3 and as a torus in $\mathbb{R}^4$ always coincide*?

2) *Is it true that for any torus in $S^3\subset\mathbb{R}^4$ the potential U of a Weierstrass representation of it in $\mathbb{R}^4$ is gauge equivalent to the potential of its Weierstrass representation in S^3: $U=\frac{(H-i)e^{\alpha}}{2}$, where H stands for the mean curvature of the torus in S^3*?

A positive answer to the second question implies a positive answer to the first. We think that the answers to both the questions are affirmative.

The Clifford torus in $\mathbb{R}^3$. The Clifford torus in $\mathbb{R}^3$ is the image of the Clifford torus in $S^3\subset\mathbb{R}^4$ under the stereographic projection

$$(x^1,x^2,x^3,x^4)\to\left(\frac{x^1}{1-x^4},\frac{x^2}{1-x^4},\frac{x^3}{1-x^4}\right),\qquad\sum_k(x^k)^2=1.$$

The torus is regarded up to conformal transformations of $\overline{\mathbb{R}}^3$, and hence can be obtained as the following torus of revolution: if we take a circle of radius $r=1$ in the x^1x^3 plane with the distance between the centre of the circle and the x^1 axis equal to $R=\sqrt{2}$, then the Clifford torus can be obtained by rotating this circle about the x^1 axis.

Theorem 12 [55]. *The Baker–Akhiezer function of the Dirac operator $\mathscr{D}$ with the potential*

$$U=\frac{\sin y}{2\sqrt{2}(\sin y-\sqrt{2})}\tag{47}$$

is a vector function $\psi(z,\bar z,P)$, $z\in\mathbb{C}$, $P\in\Gamma$, such that:

1) *the complex curve Γ is the sphere $\mathbb{C}P^1=\overline{\mathbb{C}}$ with two distinguished points $\infty_+=(\lambda=\infty)$ and $\infty_-=(\lambda=0)$, where λ is an affine parameter on $\mathbb{C}\subset\mathbb{C}P^1$, and with two double points obtained by identifying the points in the pairs*

$$\left(\frac{1+i}{4},\frac{-1+i}{4}\right)\quad\text{and}\quad\left(-\frac{1+i}{4},\frac{1-i}{4}\right);$$

2) *the function* ψ *is meromorphic on* $\Gamma \setminus \{\infty_\pm\}$ *and has at the distinguished points ('at infinity') the asymptotic behaviour*

$$\psi \approx \begin{pmatrix} e^{k_+ z} \\ 0 \end{pmatrix} \text{ as } k_+ = \lambda \to \infty, \quad \psi \approx \begin{pmatrix} 0 \\ e^{k_- \bar{z}} \end{pmatrix} \text{ as } k_- = -\frac{|u|^2}{\lambda} \to \infty,$$

where $u = \dfrac{1+i}{4}$ *and* $k_\pm^{-1}$ *are local parameters near* $\infty_\pm$;

3) ψ *has three poles* $\Gamma \setminus \{\infty_\pm\}$ *which do not depend on* z *and are located at the points*

$$p_1 = \frac{-1+i+\sqrt{-2i-4}}{4\sqrt{2}}, \qquad p_2 = \frac{-1+i-\sqrt{-2i-4}}{4\sqrt{2}}, \qquad p_3 = \frac{1}{\sqrt{8}}.$$

The geometric genus $p_g(\Gamma)$ *and the arithmetic genus* $p_a(\Gamma)$ *of the curve* Γ *are*

$$p_g(\Gamma) = 0, \qquad p_a(\Gamma) = 2.$$

The Baker–Akhiezer function satisfies the Dirac equation $\mathscr{D}\psi = 0$ *with the potential* U *given by* (47) *at any point of* $\Gamma \setminus \{\infty_+, \infty_-, p_1, p_2, p_3\}$.

The Clifford torus can be constructed by using the Weierstrass representation (2) *and* (3) *from the function*

$$\psi = \psi\left(z, \bar{z}, \frac{1-i}{4}\right).$$

As was shown, the function ψ is of the form

$$\psi_1(z,\bar{z},\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar{z}}\left(q_1\frac{\lambda}{\lambda-p_1} + q_2\frac{\lambda}{\lambda-p_2} + (1-q_1-q_2)\frac{\lambda}{\lambda-p_3}\right),$$

$$\psi_2(z,\bar{z},\lambda) = e^{\lambda z - \frac{|u|^2}{\lambda}\bar{z}}\left(t_1\frac{p_1}{p_1-\lambda} + t_2\frac{p_2}{p_2-\lambda} + (1-t_1-t_2)\frac{p_3}{p_3-\lambda}\right),$$

where $u = \dfrac{1+i}{4}$ and the functions q_1, q_2, t_1, t_2 depend only on y and are 2π-periodic with respect to y. These functions can be found from the following conditions:

$$\psi\left(z,\bar{z},\frac{1+i}{4}\right) = \psi\left(z,\bar{z},\frac{-1+i}{4}\right), \qquad \psi\left(z,\bar{z},-\frac{1+i}{4}\right) = \psi\left(z,\bar{z},\frac{1-i}{4}\right).$$

4.6. Spectral curves of integrable tori. A surface is said to be *integrable* if the Gauss–Codazzi equations are the compatibility conditions,

$$[\partial_x - A(\lambda), \partial_y - B(\lambda)] = 0, \tag{48}$$

for the linear problems

$$\partial_x\varphi = A(\lambda)\varphi, \qquad \partial_y\varphi = B(\lambda)\varphi,$$

where A and B are Laurent series in the spectral parameter λ. It is also assumed that λ appears in this representation non-trivially. To obtain explicit solutions of the *zero curvature equation* (48), one can use the whole machinery of soliton theory and, in particular, of the theory of integrable harmonic maps originating from the papers [63]–[65] and intensively developed during the last thirty years (the current state of this theory is presented in [66]–[68]). The most complete list of integrable surfaces in $\mathbb{R}^3$ is given in [69] (see also [70]).

This theory works well for spheres, where it suffices to use algebraic geometry of complex rational curves, and for tori, where explicit formulae for surfaces are derived in terms of theta functions on certain Riemann surfaces. However, the theory of integrable systems has not led to substantial progress for surfaces of higher genera. This probably has serious reasons related to the fact that tori are the only closed surfaces admitting flat metrics.

The spectral curves of integrable tori arise as spectral curves of operators in these auxiliary linear problems. The complex curves (Riemann surfaces) serve to construct explicit formulae for tori in terms of theta functions of these Riemann surfaces.

It turns out that this property is not accidental, and these spectral curves of integrable tori are special cases of the general spectral curves defined in § 4.4 for all tori (and not only integrable tori).

In [10] we proved this coincidence (modulo additional irreducible components) for tori of constant mean curvature and for isothermal tori in $\mathbb{R}^3$ and for minimal tori in S^3. Corollary 3 excludes the existence of additional components.

A) Tori of constant mean curvature (CMC tori) in $\mathbb{R}^3$. The Ruh–Vilms theorem gives us that the Gauss map of a surface in $\mathbb{R}^3$ is harmonic if and only if this surface is of constant mean curvature [71]. By the Gauss–Codazzi equations, this is equivalent to the condition that the Hopf differential $A\,dz^2$ is holomorphic:

$$A_{\bar{z}} = 0.$$

Every holomorphic quadratic differential on the sphere vanishes, and therefore by the Hopf theorem the CMC spheres in $\mathbb{R}^3$ are exactly the round spheres (that is, spheres of constant curvature) [72].

It was also conjectured by Hopf that the immersed closed CMC surfaces in $\mathbb{R}^3$ are exactly the round spheres. Although this conjecture was confirmed for embedded surfaces by Alexandrov [73], it was disproved for immersed surfaces of higher genera. The existence of CMC tori was established in the early 1980s by Wente by using the implicit function theorem for Banach spaces [74]. The first explicit examples were found by Abresch in [75], and the investigation of these examples in [76] indicated a relationship between this problem and integrable systems. As was proved later, for a CMC torus the complex curve Γ is of finite genus [77], which enabled one to apply the Baker–Akhiezer functions to derive explicit formulae for tori of this kind in terms of theta functions of Γ (this programme was realized by Bobenko in [78], [79]). The existence of CMC surfaces of genera exceeding one was established by Kapouleas, also by implicit methods [80], [81], and the problem of explicit description of these surfaces remains open. We note that another interpretation of CMC surfaces in terms of an infinite-dimensional integrable system was proposed in [82] and is based on the Weierstrass representation.

Every holomorphic quadratic differential on the torus has constant coefficients (with respect to a conformal parameter z). For a given CMC torus one can reduce everything to the situation with

$$A\,dz^2 = \frac{1}{2}d\,z^2, \qquad H = 1$$

by a homothety of the surface and a linear transformation $z \to az$ of the conformal parameter. In this case the Gauss–Codazzi equations become

$$u_{z\bar{z}} + \sinh u = 0,$$

where $u = 2\alpha$ and $e^{2\alpha}dzd\bar{z}$ is the metric on the torus. This equation is the compatibility condition for the system

$$\left[\frac{\partial}{\partial z} - \frac{1}{2}\begin{pmatrix} -u_z & -\lambda \\ -\lambda & u_z \end{pmatrix}\right]\psi = 0, \qquad \left[\frac{\partial}{\partial \bar{z}} - \frac{1}{2\lambda}\begin{pmatrix} 0 & e^{-u} \\ e^{u} & 0 \end{pmatrix}\right]\psi = 0. \tag{49}$$

Let Λ be the period lattice for the torus. We consider the linear problem

$$L\psi = \partial_z\psi - \frac{1}{2}\begin{pmatrix} -u_z & 0 \\ 0 & u_z \end{pmatrix}\psi = \frac{1}{2}\begin{pmatrix} 0 & -\lambda \\ -\lambda & 0 \end{pmatrix}\psi.$$

Since the operator L is a first-order 2×2 matrix operator, the system (49) has a two-dimensional space V_λ of solutions for any $\lambda \in \mathbb{C}$, and these spaces are invariant with respect to the translation operators of the form

$$\widehat{T}_j\varphi(z) = \varphi(z + \gamma_j), \qquad j = 1, 2,$$

where γ_1 and γ_2 are the generators of the lattice Λ. The operators $\widehat{T}_1$, $\widehat{T}_2$, and L commute, and hence have common eigenvectors, which are glued together into a meromorphic function $\psi(z, \bar{z}, P)$ on the two-sheeted covering

$$\widehat{\Gamma} \to \mathbb{C}\colon P \in \widehat{\Gamma} \to \lambda \in \mathbb{C},$$

branched at the points at which the operators $\widehat{T}_j$ and L cannot be diagonalized simultaneously. This is the standard procedure of constructing spectral curves of periodic operators [56].

Corresponding to each point $P \in \widehat{\Gamma}$ is a unique (up to a constant factor) Floquet function $\psi(z, \bar{z}, P)$ with multipliers $\mu(\gamma_1, P)$ and $\mu(\gamma_2, P)$. The complex curve $\widehat{\Gamma}$ can be compactified by four points $\infty^1_\pm, \infty^2_\pm$ 'at infinity' in such a way that the points $\infty^1_\pm$ project to $\lambda = \infty$ and the points $\infty^2_\pm$ project to $\lambda = 0$, and we can take a function ψ meromorphic on $\widehat{\Gamma}$ with the following essential singularities at the points 'at infinity':

$$\psi(z, \bar{z}, P) \approx \exp\left(\mp\frac{\lambda z}{2}\right)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \quad \text{as} \quad P \to \infty^1_\pm,$$
$$\psi(z, \bar{z}, P) \approx \exp\left(\mp\frac{\bar{z}}{2\lambda}\right)\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \quad \text{as} \quad P \to \infty^2_\pm.$$

The multipliers tend to ∞ as $\lambda \to 0, \infty$.

The complex curve Γ admits an involution preserving the multipliers, namely,

$$\sigma(\lambda) = -\lambda, \qquad \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \to \begin{pmatrix}\varphi_1\\ -\varphi_2\end{pmatrix}, \qquad \sigma(\infty^1_\pm) = \infty^1_\mp, \qquad \sigma(\infty^2_\pm) = \infty^2_\mp.$$

The complex quotient curve $\widehat{\Gamma}/\sigma$ is called the *spectral curve of the torus of constant mean curvature*.

The following assertion holds.

Proposition 11 [10]. *The vector function φ satisfies the equation* (49) *if and only if the function $\psi = (\lambda\varphi_2, e^\alpha\varphi_1)^\top$ satisfies the Dirac equation $\mathscr{D}\psi = 0$ with $U = \dfrac{He^\alpha}{2} = \dfrac{e^\alpha}{2}$.*

We thus have an analytic map of Γ onto the spectral curve (defined in § 4.4) of the generic torus, and the map preserves the values of the multipliers. This implies that the complex curves coincide up to irreducible components. This, together with Corollary 3, implies the next proposition.

Proposition 12. *The spectral curve of any CMC torus in $\mathbb{R}^3$ coincides with the* (*generic*) *spectral curve of the torus as defined in* § 4.4.

B) Minimal tori in S^3. Let us regard the unit sphere in $\mathbb{R}^4$ as the Lie group $SU(2)$. For minimal surfaces in $SU(2)$ the derivational equations (10) and (11) can be simplified, and we obtain the Hitchin system [62]

$$\overline{\partial}\Psi - \partial\Psi^* + [\Psi^*, \Psi] = 0, \qquad \overline{\partial}\Psi + \partial\Psi^* = 0. \tag{50}$$

It follows from the first equation that the SL_2-connection $\mathscr{A} = (\partial + \Psi, \overline{\partial} + \Psi^*)$ on $f^{-1}(TG)$ is flat, and it follows from the second equation that this connection can be extended to an analytic family of flat connections,

$$\mathscr{A}_\lambda = \left(\partial + \frac{1+\lambda^{-1}}{2}\Psi, \overline{\partial} + \frac{1+\lambda}{2}\Psi^*\right),$$

where $\mathscr{A} = \mathscr{A}_1$ and $\lambda \in \mathbb{C}\setminus\{0\}$. We thus obtain an L, A-pair with a spectral parameter and can conclude that this system is integrable. This trick is standard for integrable harmonic maps.

We define the spectral curve.

Suppose that Σ is a minimal torus in $SU(2)$ and $\{\gamma_1, \gamma_2\}$ is a basis for the lattice Λ. Let us define matrices $H(\lambda)$ and $\widetilde{H}(\lambda) \in SL(2,\mathbb{C})$ that describe the monodromy of the connection $\mathscr{A}_\lambda$ along a closed loop realizing γ_1 and γ_2, respectively. These matrices commute and hence have common eigenvectors $\varphi(\lambda,\mu)$, where μ is a root of the characteristic equation for $H(\lambda)$:

$$\mu^2 - \operatorname{Tr} H(\lambda)\mu + 1 = 0.$$

The eigenvalues

$$\mu_{1,2} = \frac{1}{2}\left(\operatorname{Tr} H(\lambda) \pm \sqrt{\operatorname{Tr}^2 H(\lambda) - 4}\right)$$

are defined on a Riemann surface Γ which is a two-sheeted covering of $\mathbb{C}P^1$ branched at the simple zeros of the function $(\mathrm{Tr}^2 H(\lambda) - 4)$ and also at 0 and ∞ (the multiple zeros are removed under the normalization). The complex curve Γ is called the *spectral curve of the minimal torus in* $SU(2)$ and it has finite genus.

Above we presented Hitchin's results that remain valid for any harmonic torus in S^3 (this includes both cases of minimal tori in S^3 and harmonic Gauss maps into $S^2 \subset S^3$) [62]. Let us now restrict ourselves to minimal tori in S^3.

Let $\mathscr{D}$ be the Dirac operator associated with this torus and let a spinor ψ' generate the torus by means of the Weierstrass representation. Let

$$L = \frac{1}{\sqrt{2}} \begin{pmatrix} \overline{a} & -\overline{b} \\ b & a \end{pmatrix}, \qquad a = -i\overline{\psi}'_1 + \psi'_2, \quad b = -i\psi'_1 + \overline{\psi}'_2.$$

The following assertion holds.

Proposition 13 [10]. *The Hitchin eigenfunctions φ are transformed according to the map*

$$\varphi \to \psi = e^{\alpha} \begin{pmatrix} 0 & i\lambda \\ 1 & 0 \end{pmatrix} \cdot L^{-1}\varphi$$

into solutions of the Dirac equation $\mathscr{D}\psi = 0$ corresponding to the torus Σ in S^3.

As in the case of CMC tori in $\mathbb{R}^3$ (see above), this proposition, together with Corollary 3, implies the following assertion.

Proposition 14. *The spectral curve of a minimal torus in S^3 coincides with the* (*generic*) *spectral curve of the torus as defined in* § 4.4.

4.7. Singular spectral curves. A perturbation of the free operator can be so strong that the spectral curve Γ can acquire other singularities (besides resonance pairs). If Γ_{nm} is an algebraic curve, then we write out the corresponding Baker–Akhiezer function $\psi(z, \bar{z}, P)$ such that:

1) $\mathscr{D}\psi = 0$;
2) ψ is meromorphic on Γ and has at the points at infinity the asymptotic behaviour

$$\psi \approx \begin{pmatrix} e^{\lambda_+ z} \\ 0 \end{pmatrix} \quad \text{as} \quad P \to \infty_+, \qquad \psi \approx \begin{pmatrix} 0 \\ e^{\lambda_- \bar{z}} \end{pmatrix} \quad \text{as} \quad P \to \infty_-,$$

where $\lambda_\pm^{-1}$ are local coordinates near $\infty_\pm$, $\lambda_\pm^{-1}(\infty_\pm) = 0$ and one can set $\lambda_\pm = 2\pi i k_1$.

The function ψ is formed by Floquet functions $\psi(z, \bar{z}, P)$ taken at different points of the spectral curve so that ψ is meromorphic and has the above asymptotic behaviour. The function ψ already 'draws' the complex curve Γ_ψ on which it is defined in such a way that no Floquet function is taken into account twice at different points of Γ_ψ. There is a chain of maps

$$\Gamma_{\mathrm{nm}} \to \Gamma_\psi \to \Gamma$$

such that their composition is a normalization of Γ and the first map in the chain is a normalization of the curve Γ_ψ. We have the obvious inequalities

$$p_g(\Gamma) = p_g(\Gamma_\psi) \leqslant p_a(\Gamma_\psi) < \infty,$$

where $p_a(\Gamma_\psi)$ is the arithmetic genus of the curve Γ_ψ (this genus differs from the geometric genus of Γ_ψ owing to the contribution of singular points).

The function ψ can be pulled back to the non-singular curve Γ, where it has exactly $p_a(\Gamma_\psi)+1$ poles (this follows from finite-gap integration theory). For the Dirac operator the number $p_a(\Gamma_\psi)$ is one less than 'the number of poles of its normalized Baker–Akhiezer function'.

We arrive at the following conclusion.

a) The Baker–Akhiezer function ψ determines the Riemann surface Γ_ψ in the classical spirit of Riemann's work, as a surface on which the given function ψ is naturally defined. This surface can be obtained from Γ by normalizing the singularities only if the normalization reduces the dimension of the space of Floquet functions at the point (for instance, this is the case for resonance pairs).

b) In contrast to Γ_{nm}, the complex curve Γ_ψ gives a one-to-one parametrization of all Floquet functions (up to constant multiples).

This situation is explained in detail for minimal tori in S^3 in [62].

If we want to construct a torus of finite spectral genus in terms of theta functions, then we must again work with the curve Γ_ψ, as was shown in § 4.5 in the case of the Clifford torus.

The following definition of Γ_ψ comes from finite-gap integration theory.

Let $\mathscr{D}$ be a Dirac operator with doubly periodic potentials U and V and let Γ_ψ be a Riemann surface (possibly singular) of finite arithmetic genus $p_a(\Gamma_\psi)=g$ with two distinguished non-singular points $\infty_\pm$ and local coordinates $k_\pm^{-1}$ near these points such that $k_\pm^{-1}(\infty_\pm)=0$. Let $\psi(z,\bar z,P)$ be a Baker–Akhiezer function ψ defined on $\mathbb{C}\times\Gamma_\psi\setminus\{\infty_\pm\}$, and:

1) let ψ be meromorphic with respect to P outside the points $\infty_\pm\in\Gamma$ and have poles at $g+1$ non-singular points $P_1+\dots+P_{g+1}$;

2) let ψ have at $\infty_\pm$ the asymptotic behaviour

$$\psi\approx e^{k_+z}\left[\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}\xi_1^+\\ \xi_2^+\end{pmatrix}k_+^{-1}+O(k_+^{-2})\right]\quad\text{as}\quad P\to\infty_+,$$
$$\psi\approx e^{k_-\bar z}\left[\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}\xi_1^-\\ \xi_2^-\end{pmatrix}k_-^{-1}+O(k_-^{-2})\right]\quad\text{as}\quad P\to\infty_-$$

and let it satisfy the Dirac equation $\mathscr{D}\psi=0$ everywhere on Γ_ψ except for the 'points at infinity' $\infty_\pm$ and the poles $P_1,\dots,P_{g+1}$.

We say that Γ_ψ is the *spectral curve of the finite-gap operator* $\mathscr{D}$.

Such a function is unique for any generic divisor $P_1+\dots+P_{g+1}$, and the potentials can be recovered by the formulae

$$U=-\xi_2^+,\qquad V=\xi_1^-. \tag{51}$$

An attempt to define a Riemann surface of this kind in the case $p_g(\Gamma)=\infty$ faces many analytic complications.

We refer to [55] for a more detailed exposition of some facts relating to singular spectral curves.

As we saw in § 4.5, for the Clifford torus $\Sigma_{1,1} \subset \mathbb{R}^4$ the potential is constant and the spectral curve is the sphere. Moreover,

$$p_g(\Gamma) = p_a(\Gamma_\psi) = 0.$$

However, the potential of its stereographic projection, which is the Clifford torus in $\mathbb{R}^3$, is equal to

$$U = \frac{\sin x}{2\sqrt{2}(\sin x - \sqrt{2})},$$

where x is one of the angle variables, and by Theorem 12, we have for the operator with this potential that

$$p_g(\Gamma) = 0, \qquad p_a(\Gamma_\psi) = 2.$$

Hence, stereographic projection of the Clifford torus in S^3 to $\mathbb{R}^3$ leads to the appearance of singularities for the curve Γ_ψ.

This leads to an interesting problem:

What is the relationship between the spectral curve of a torus in the unit sphere $S^3 \subset \mathbb{R}^4$ and the spectral curve of its stereographic projection?

In our opinion the answer to this question is as follows: the potentials are connected by some Bäcklund transformation leading to a transformation of the spectral curve. Possibly there is an analogy with a similar transformation (presented in [83]) for the one-dimensional Schrödinger operator. We also conjecture that the answer to the following question is positive:

Is it true that the images $\mathscr{M}(\Gamma)$ of the multiplier map for a torus in S^3 and for its stereographic projection coincide?

There is another interesting problem:

Characterize the spectral curves of tori in $\mathbb{R}^3$ and $\mathbb{R}^4$.

The answers must differ for tori in $\mathbb{R}^3$ and $\mathbb{R}^4$. Indeed, as was already mentioned in [78], the spectral curves for CMC tori in $\mathbb{R}^3$ must be singular (for these curves this means that there are multiple branch points that are transformed by the normalization into pairs of points transposed by the hyperelliptic involution).[9] However, the spectral curve for the Clifford torus in $\mathbb{R}^4$ is non-singular.

§ 5. The Willmore functional

5.1. Willmore surfaces and the Willmore conjecture. The Willmore functional for closed surfaces in $\mathbb{R}^3$ is defined by the formula

$$\mathscr{W}(\Sigma) = \int_\Sigma H^2 \, d\mu, \tag{52}$$

where $d\mu$ is the induced area form on the surface. It was introduced by Willmore in the context of variational problems [84]. Willmore was the first to pose a global

[9] As was shown in [5], for any torus in $\mathbb{R}^3$ the image $\mathscr{M}(\Gamma)$ contains a point of multiplicity at least four or a pair of double points at which the differentials dk_1 and dk_2 vanish (here k_1 and k_2 are the quasi-momenta). We note that this does not mean that Γ_ψ satisfies the same conditions; for instance, for the Clifford torus in $\mathbb{R}^3$ the spectral curve Γ_ψ has a pair of double points at which dk_1 and dk_2 do not vanish and contains no other singular points.

problem of conformal geometry of surfaces, the so-called Willmore conjecture, which we discuss below. The Euler–Lagrange equation for this functional is

$$\Delta H + 2H(H^2 - K) = 0,$$

where Δ is the Laplace–Beltrami operator on the surface. Any surface satisfying this equation is called a *Willmore surface*.

We note that $H = \dfrac{\varkappa_1 + \varkappa_2}{2}$, and by the Gauss–Bonnet theorem, for a compact oriented surface Σ without boundary we have

$$\int_\Sigma K\,d\mu = \int_\Sigma \varkappa_1\varkappa_2\,d\mu = 2\pi\chi(\Sigma),$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. By adding a topological term to $\mathscr{W}$, we obtain a functional with the same extremals among closed surfaces, and we can simplify the variational problem. This is the case for spheres. Namely, by considering the functional

$$\widehat{\mathscr{W}}(\Sigma) = \int (H^2 - K)\,d\mu = \mathscr{W}(\Sigma) - 2\pi\chi(\Sigma),$$

we see that[10]

$$\widehat{\mathscr{W}} = \frac{1}{4}\int_\Sigma (\varkappa_1 - \varkappa_2)^2\,d\mu.$$

We recall that a point on a surface is said to be *umbilical* if $\varkappa_1 = \varkappa_2$ at the point. A surface is said to be *totally umbilical* if any point of the surface is umbilical. By the Hopf theorem, a totally umbilical surface in $\mathbb{R}^3$ is a domain either in a round sphere or in a plane. For spheres this gives a lower bound for the Willmore functional and a description of all its minima:

For spheres we have

$$\mathscr{W}(\Sigma) \geqslant 4\pi,$$

and $\mathscr{W}(\Sigma) = 4\pi$ if and only if Σ is a round sphere.

This trick does not work for surfaces of higher genera.

The functional $\widehat{\mathscr{W}}$ was introduced by Thomsen [85] and Blaschke [86], who called it the *conformal area* for the following reasons:

1) the quantity $(H^2 - K)d\mu$ is invariant with respect to the conformal transformations of the ambient space, and hence for any compact oriented surface $\Sigma \subset \mathbb{R}^3$ and any conformal transformation $G\colon \overline{\mathbb{R}}^3 \to \overline{\mathbb{R}}^3$ taking Σ into a compact surface we have

$$\widehat{\mathscr{W}}(\Sigma) = \widehat{\mathscr{W}}(G(\Sigma));$$

2) if Σ is a minimal surface in S^3 and $\pi\colon S^3 \to \overline{\mathbb{R}}^3$ is the stereographic projection taking Σ into $\mathbb{R}^3$, then $\pi(\Sigma)$ is a Willmore surface.

Moreover, as was proved in [87],

3) there is a quartic differential $\widehat{A}\,(dz)^4$ defined outside the umbilical points, and it is holomorphic for Willmore surfaces.

[10] This trick is similar to the instanton trick which helped to discover self-dual connections.

We present these results in Appendix 2.

By 2), there are examples of compact closed Willmore surfaces. We note that a compact Willmore surface need not be the stereographic projection of a minimal surface in S^3 (this was first proved for tori in [88]).

It follows from 3) that outside the umbilical points Willmore surfaces admit a good description similar to the description of CMC surfaces in terms of the holomorphic property of the quadratic Hopf differential. However, there are examples of compact Willmore surfaces that contain whole curves consisting of umbilical points [89].

By 1), the minimum of the Willmore functional in each topological class of surfaces is conformally invariant, and hence degenerate. We note that the existence of a minimum which is a real-analytic surface was proved for tori by Simon [90] and for surfaces of genus $g \geqslant 2$ by Bauer and Kuwert [91]. Recently, Schmidt presented a proof of the following result: for a given genus and a given conformal class of oriented surfaces the Willmore functional achieves its minimum on some surface which *a priori* can have branch points or can be a branched covering of an immersed surface [92]. Schmidt's technique uses the Weierstrass representation and some ideas from [5].[11]

Bryant initiated a programme for classifying the Willmore spheres based on the fact that a holomorphic differential of degree four vanishes identically on every sphere, and hence Willmore spheres admit a description in terms of algebro-geometric data [87]. The following assertion holds.

> The image of any minimal surface in $\mathbb{R}^3$ under any Möbius transformation $(x - x_0) \to (x - x_0)/|x - x_0|^2$ is a Willmore surface, and any minimal surface Σ with planar ends is taken by every Möbius transformation with centre x_0 outside the surface into a smooth compact Willmore surface Σ' such that
>
> $$\mathscr{W}(\Sigma') = 4\pi n,$$
>
> where n is the number of planar ends of Σ.

Bryant proved that all Willmore spheres are Möbius images of minimal surfaces with planar ends, the case $n = 1$ corresponds to round spheres, and there are no such spheres with $n = 2$ and 3. He also described the Willmore spheres with $n = 4$. Later on, it was proved in [93] that Willmore spheres exist for any even $n \geqslant 6$ and any odd $n \geqslant 9$. The remaining cases $n = 5$ and 7 were finally excluded in [94].

The Willmore conjecture asserts the following.

> *For tori one has*
>
> $$\mathscr{W} \geqslant 2\pi^2,$$
>
> *and the Willmore functional attains its minimum on the Clifford torus and its images under conformal transformations of* $\overline{\mathbb{R}}^3$.

The Clifford torus was already introduced in § 4.5.

Since the Willmore functional is conformally invariant and the stereographic projection $\pi\colon S^3 \to \overline{\mathbb{R}}^3$ is conformal, we do not distinguish between the original

[11] See Appendix 1.

Willmore conjecture for tori and its version for tori in S^3, for which the Willmore functional is replaced by

$$\mathscr{W}_{S^3} = \int (H^2 + 1)\, d\mu, \qquad \mathscr{W}_{S^3}(\Sigma) = \mathscr{W}(\pi(\Sigma)). \tag{53}$$

Willmore expressed his conjecture in [84], where he verified it for round tori of revolution. This conjecture has been proved for many special cases:

1) for tube tori, that is, for tori formed by carrying a circle centred on a closed curve along this curve such that the circle always lies in the normal plane, by Shiohama and Takagi [95] and by Willmore [96] (if a modification of the radius of the circle is admitted, then we obtain channel tori, for which the conjecture was established in [97]);
2) for tori of revolution by Langer and Singer [98];
3) for tori conformally equivalent to $\mathbb{R}^2/\Gamma(a,b)$, where $0 \leqslant a \leqslant 1/2$, $\sqrt{1-a^2} \leqslant b \leqslant 1$, and the lattice $\Gamma(a,b)$ is generated by the vectors $(1,0)$ and (a,b) (Li–Yau [99]);
4) the previous result of Li and Yau was improved by Montiel and Ros, who extended it to the closed domain $\left(a-\frac{1}{2}\right)^2 + (b-1)^2 \leqslant \frac{1}{4}$ [100];
5) for tori in S^3 which are invariant under the antipodal map (Ros [101]);
6) Li and Yau [99] also proved that if a surface has a self-intersection point of multiplicity n, then $\mathscr{W} \geqslant 4\pi n$, and hence the conjecture is proved for tori with self-intersections.

Some other partial results were obtained in [102] and [103].

In the paper [104] a formula for the second variation of the functional $\mathscr{W}$ was computed for the Clifford torus and it was proved that this form is non-negative. A formula for the second variation for general Willmore surfaces was obtained in [105].

The Willmore conjecture remains open in the general case.

In the next section we discuss a new approach used in [5].

According to (53), the following conjecture is a special case of the Willmore conjecture and is also open:

For the minimal tori in S^3 the volume is bounded below by $2\pi^2$ and attains its minimal value on the Clifford torus in S^3.

By the Li–Yau theorem on surfaces with self-intersections this conjecture is implied by the following conjecture of Hsiang and Lawson:

The Clifford torus is the only minimal torus embedded in S^3.

Since any holomorphic differential of order four on a torus has constant coefficients, there are two possibilities: the differential either vanishes or is equal to $c(dz)^4$, $c = \text{const} \neq 0$.

In the first case the torus is obtained as the Möbius image of a minimal torus with planar ends. For obvious reasons, it is clear that there are no such tori with $n = 1$ or 2 ends. The case $n = 3$ was excluded by Kusner and Schmitt, who also constructed examples with $n = 4$ [106]. The first examples of minimal rectangular tori with four planar ends were constructed by Costa [107]. Recently, Shamaev constructed examples of tori of this kind for any even $n \geqslant 6$ [108]. Although it is clear from the construction that these tori have no branch points, this has been rigorously proved only for $n = 6$, 8, and 10.

In the second case Codazzi-type equations for Willmore tori without umbilical points coincide with the four-particle Toda lattice [109], [110].[12] Theta formulae for Willmore tori of this kind were derived in [110] by using Baker–Akhiezer functions connected with this Toda lattice.

Another construction of Willmore tori using methods of the theory of integrable systems was proposed in [111].

Candidates for minima of the Willmore functional for surfaces of higher genera were suggested by Kusner [112].

There is a conjecture that the Willmore functional $\int |H|^2\,d\mu$ for tori in $\mathbb{R}^4$ attains its minimum on the Clifford torus in $\mathbb{R}^4$, that is, on the product of two circles of the same radius (see [113], [114]). Since this is a Lagrangian torus, the conjecture can be weakened as follows: the Clifford torus is the minimum of the Willmore functional on the smaller class of Lagrangian tori. This conjecture is discussed in [115], where it is proved that the functional $\mathscr{W}$ attains its minimum among Lagrangian tori at some real-analytic torus.

We do not discuss generalizations of the Willmore functional for surfaces in arbitrary Riemannian manifolds,

$$\int (|H|^2 + \widehat{K})\,d\mu,$$

where $\widehat{K}$ is the sectional curvature of the ambient space along the tangent plane to the surface. The quantity $(|H|^2 - K + \widehat{K}\,)\,d\mu$ is invariant with respect to conformal transformations of the ambient space [116].

In [11] another generalization of the Willmore functional for surfaces in Lie groups of dimension three is proposed. It is based on the spectral theory of Dirac operators in Weierstrass representations (see also § 5.5).

We should also mention the Willmore flow, which is similar to the mean curvature flow and reduces the value of $\mathscr{W}$ (see the paper [117] and the references therein).

We conclude this subsection by a remark on constrained Willmore surfaces. By definition, these are critical points of the Willmore functional restricted to the space of surfaces of a given conformal type. It was first observed by Langer that compact surfaces of constant mean curvature in $\mathbb{R}^3$ are constrained Willmore surfaces, since the Gauss map is harmonic for the compact CMC surfaces [118]. For the fundamentals of the theory of these surfaces, see [119].

5.2. Spectral curves and the Willmore conjecture. As was shown in [1], in terms of the potential U of the Weierstrass representation of a torus in $\mathbb{R}^3$, the Willmore functional becomes

$$\mathscr{W} = 4\int_M U^2\,dx\,dy.$$

Thus, it measures the perturbation of the free operator.

We recall that the Willmore conjecture claims that this functional for tori attains its minimum at the Clifford torus, for which the value of the functional is $2\pi^2$.

[12] See Appendix 2.

Starting from the observation that the Willmore functional is the first integral of the mNV flow deforming tori into tori and preserving the conformal class (see § 3.1), we made the following conjecture in 1995 (see [1]).

A torus non-stationary with respect to the mNV flow cannot be a local minimum of the Willmore functional.

This conjecture was based on the assumption that the minimum of a variational problem of this kind is non-degenerate, and thus must be stable with respect to soliton deformations determined by equations in the mNV hierarchy and preserving the value of the Willmore functional. As is known from soliton theory, these equations are linearized on the Jacobi variety of the normalized spectral curve, and generically these linear flows span the whole Jacobi variety, which is an Abelian variety of complex dimension $p_g(\Gamma)$, or a Prym submanifold of the Jacobi variety.

A geometric analogue of the conjecture was formulated in [3], where we introduced a notion of spectral genus of a torus as the number $p_g(\Gamma)$:

For a given conformal class of tori in $\mathbb{R}^3$ the minima of the Willmore functional are attained at tori of minimal spectral genus.

In [3] we proposed the following explanation of the lower bounds for $\mathscr{W}$: for small perturbations of the zero potential $U = 0$ the Weierstrass representation gives planes that cannot be converted into tori, and since the Willmore functional for surfaces in $\mathbb{R}^3$ is the squared L_2-norm of U, the lower bound shows how large a perturbation of the zero potential must be in order to convert planes into tori.

The strategy of the proof of the Willmore conjecture after proving the last conjecture is to compute the values of the Willmore functional for tori of minimal spectral genus (by using the formula (44) or in another way) and to verify the Willmore conjecture.

We have already mentioned the paper [5] by Schmidt. This paper contains a series of interesting results.[13] For our purposes, we present only those relating to the asymptotic behaviour of the spectral curve. Although we did not go through the details of [5] until recently, we must say the following.

In fact, the paper [5] proposes a proof only of our last conjecture (see above); the value of $p_a(\Gamma_\psi)$ is a priori unbounded, but the computations of the Willmore functional in [5] are carried out only for the minimal possible values of $p_g(\Gamma)$ and $p_a(\Gamma_\psi)$.

In the spirit of the above hypotheses, it is natural to make the following conjecture.

For a given conformal class of tori in $\mathbb{R}^3$ and a given spectral genus, the minima of the Willmore functional are attained at tori with the minimal value of $p_a(\Gamma_\psi)$.

This conjecture also agrees with the soliton approach, because the additional degrees of freedom corresponding to the difference $p_a(\Gamma)-p_g(\Gamma)$ (or a part of it if the flows are linearized on the Prym manifold) also correspond to soliton deformations.

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[13] This paper also proposes a proof of the fact that the spectral genus of a constrained Willmore torus in $\mathbb{R}^3$ is finite. Another proof was proposed by Krichever (unpublished). This fact is non-trivial even for Willmore tori, because in this setting the trick based on soliton theory and used in [62] and [77] for harmonic tori in S^3 and tori of constant mean curvature in $\mathbb{R}^3$ (see also [79]) fails to work for arbitrary tori. It deals only with tori described by the four-particle Toda lattice without umbilical points at which $e^\beta = 0$ (see Appendix 2).

In our opinion these conjectures are of independent interest. We note that proofs of the last two conjectures, together with computations of the Willmore functional for tori with minimal possible values of $p_g(\Gamma)$ and $p_a(\Gamma_\psi)$, would lead to a verification of the Willmore conjecture.

We would also like to note another interesting problem:

How can this theory of spectral curves be generalized to compact immersed surfaces of higher genera?

5.3. Lower bounds for the Willmore functional. In [6] we established (in a special case) a lower bound for the Willmore functional. This bound was quadratic with respect to the dimension of the kernel of the Dirac operator.

Let us represent the sphere as an infinite cylinder Z compactified by a pair of points in such a way that $z = x + iy$ is a conformal parameter on Z, y is defined modulo 2π, $x \in \mathbb{R}$, and these two points at infinity are obtained upon passing to the limits $x \to \pm\infty$.

Lemma 3 [6]. *For a sphere in $\mathbb{R}^3$ the function ψ and the potential U have the following asymptotic behaviour*:

$$|\psi_1|^2 + |\psi_2|^2 = C_\pm e^{-|x|} + O(e^{-2|x|}), \quad U = U_\pm e^{-|x|} + O(e^{-2|x|}) \quad \text{as} \quad x \to \pm\infty,$$

where $C_\pm$ and $U_\pm$ are constants. If $C_+ = 0$ or $C_- = 0$, then the corresponding distinguished point $x = +\infty$ or $x = -\infty$ is a branch point.

The kernel of the operator $\mathscr{D}$ on the sphere is formed by the solutions ψ of the equation $\mathscr{D}\psi = 0$ on the cylinder such that $|\psi_1|^2 + |\psi_2|^2 = O(e^{-|x|})$ as $x \to \pm\infty$.

Suppose that the potential U of the Dirac operator depends only on x. For example, a situation of this kind arises for a sphere of revolution, where y is the angle of rotation. However, this is the case for many other surfaces with intrinsic S^1-symmetry reflected by the potential of the Weierstrass representation, and not only for spheres of revolution.

Theorem 13 [6]. *Let $\mathscr{D}$ be a Dirac operator on $M = S^2$ with a real-valued potential $U = V$ depending only on x. Then*

$$\int_M U^2 \, dx \wedge dy \geqslant \pi N^2, \tag{54}$$

where $N = \dim_{\mathbb{H}} \operatorname{Ker} \mathscr{D} = \frac{1}{2} \dim_{\mathbb{C}} \operatorname{Ker} \mathscr{D}$. These minima are attained at the potentials

$$U_N(x) = \frac{N}{2\cosh x}\,.$$

The proof of this theorem is based on the method of the inverse scattering problem applied to a one-dimensional Dirac operator. The quadratic estimate arises from the Faddeev–Takhtadzhyan trace formulae [120].

Along with the proof of Theorem 13, we expressed the following conjecture.

Conjecture 1 [6]. *For any Dirac operator on the two-dimensional sphere the estimate (54) holds.*

Before passing to the proof of Theorem 13, we present one of the corollaries of the conjecture, namely, Theorem 14.

Soon after the electronic publication [6], Friedrich indicated the following corollary to the conjecture.[14]

Theorem 14 [121]. *Suppose that Conjecture 1 is true. Let λ be an eigenvalue of the Dirac operator on a two-dimensional spin-manifold homeomorphic to the two-dimensional sphere S^2. The following inequality holds*:

$$\lambda^2 \operatorname{Area}(M) \geqslant \pi m^2(\lambda), \tag{55}$$

where $m(\lambda)$ is the multiplicity of λ.

We note that, due to the symmetry (4) of the operator $\operatorname{Ker} D$, the multiplicity of an eigenvalue is always even. In the case $m(\lambda) = 2$ the inequality (55) was proved by Bär [122].

Proof of Theorem 14. We first recall the definition of the Dirac operator on a spin-manifold (see [123] and [124] for detailed expositions).

A *spin n-manifold M* is a Riemannian manifold with a spin bundle E over M such that a Clifford multiplication

$$T_pM \times E_p \to E_p$$

is defined at each point $p \in M$ such that

$$v \cdot w \cdot \varphi + w \cdot v \cdot \varphi = -2(v, w)\varphi, \qquad v, w \in T_pM, \quad \psi \in E_p.$$

We also assume that there is a Riemannian connection ∇ inducing a connection on E. In this case the Dirac operator is defined at every point $p \in M$ by the formula

$$D\varphi = \sum_{k=1}^{n} e_k \cdot \nabla_{e_k} \varphi,$$

where $e_1, \dots, e_n$ is an orthonormal basis in T_pM and φ is a section of the bundle E.

For example, let us consider a two-dimensional spin manifold M with a flat metric. The Clifford algebra $\mathcal{C}l_2$ is isomorphic to $\mathbb{H}$. Thus, we have a $\mathbb{C}^2$-spin bundle over M (here we identify $\mathbb{H}$ with $\mathbb{C} \oplus \mathbb{C}$). For the flat metric on M the Clifford multiplication is represented by the matrices

$$e_1 = e_x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad e_2 = e_y = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

One can readily see that

$$e_x e_y + e_y e_x = 0, \qquad e_x^2 = e_y^2 = -1.$$

[14] The conjecture was finally proved by Ferus, Leschke, Pedit, and Pinkall in [7] together with a generalization of the formula (54) in the form of the so-called Plücker formula for surfaces of higher genera, for all $g \geqslant 0$ (we present this proof in § 5.4).

The Dirac operator D_0 is given by the formula

$$D_0 = e_x \cdot \partial_x + e_y \cdot \partial_y = 2 \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} = 2\mathscr{D}_0,$$

and its square is equal to the Laplace operator (up to a sign):

$$D_0^2 = -\partial_x^2 - \partial_y^2.$$

For a conformally Euclidean metric $e^{\sigma}\, dz\, d\bar{z}$ the Dirac operator becomes

$$D = e^{-3\sigma/4} D_0 e^{\sigma/4}$$

(see [125]). Hence, the eigenvalue problem

$$D\varphi = \lambda\varphi$$

for the Dirac operator associated with a metric of this kind becomes

$$D_0[e^{\sigma/4}\varphi] - \lambda e^{\sigma/2}[e^{\sigma/4}\varphi] = 0,$$

which can be rewritten as

$$(\mathscr{D}_0 + U)\psi = 0, \qquad U = -\frac{\lambda e^{\sigma/2}}{2}, \quad \psi = e^{\sigma/4}\varphi.$$

If Conjecture 1 holds, then we have the inequality

$$\int_M U^2\, dx \wedge dy = \frac{\lambda^2}{4} \operatorname{Area}(M) \geqslant \pi \left(\frac{\dim_{\mathbb{C}} \operatorname{Ker}(D_0 + U)}{2} \right)^2 = \pi \frac{m^2(\lambda)}{4}.$$

This proves Theorem 14.

Proof of Theorem 13. If the potential U depends only on x, then the linear space of solutions of the equation $\mathscr{D}\psi = 0$ on the sphere $S^2 = Z \cup \pm\infty = \mathbb{R}_x \times S^1_y \cup \infty$ is spanned by functions of the form $\psi(x,y) = \varphi(x) e^{\varkappa y}$ such that

$$L\varphi := \left[\begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} 2U & 0 \\ 0 & 2U \end{pmatrix} \right] \varphi = \begin{pmatrix} 0 & i\varkappa \\ i\varkappa & 0 \end{pmatrix} \varphi,$$

where $e^{2\pi\varkappa} = -1$ (this condition defines the spin bundle over the sphere; see [45]) and φ is exponentially decaying as $x \to \pm\infty$. This means that φ is a bound state of L, or equivalently, $\varkappa$ belongs to the discrete spectrum, which is invariant with respect to the complex conjugation $\varkappa \to \overline{\varkappa}$. Hence, $\dim_{\mathbb{C}} \mathscr{D} = 2N$ is the doubled number of bound states satisfying the condition $\operatorname{Im} \varkappa > 0$.

The trace formula (76) (see Appendix 3) for $p = q = 2U$ becomes

$$\int_{-\infty}^{\infty} U^2(x)\, dx = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \log(1 - |b(k)|^2)\, dk + \sum_{j=1}^{N} \operatorname{Im} \varkappa_j.$$

For a given $\dim \operatorname{Ker} \mathscr{D} = N$ the functional $\int_M U^2(x)\,dx \wedge dy = 2\pi \int_{-\infty}^{\infty} U^2(x)\,dx$ attains its minimum on the potential with the following spectral data:

$$b(k) \equiv 0, \quad \varkappa_k = \frac{i(2k-1)}{2}, \qquad k = 1, \dots, N,$$

and we have

$$\int_{S^2} U^2(x)\,dx \wedge dy \geqslant 2\pi \left(\frac{1}{2} + \frac{3}{2} + \dots + \frac{N}{2} \right) = \pi N^2.$$

In fact, there is an N-dimensional family of potentials parameterized by $\lambda_1, \dots, \lambda_N$, and moreover, this family is invariant with respect to the mKdV equations. One can readily show that each such family contains the potential $U_N = N/(2\cosh x)$, and $p_N(x) = 2U_N(x) = N/\cosh x$ is the famous N-soliton potential of the Dirac operator.

This completes the proof of Theorem 13.

We see that equality in (54) is attained on some special spheres, which are particular cases of the so-called *soliton spheres*. By definition, these are spheres for which the potential of the Dirac operator $\mathscr{D}$ is a soliton (reflectionless) potential $U(x)$. It is reasonable to distinguish a special subclass of soliton spheres defined by the condition that all poles $\varkappa_1, \dots, \varkappa_N$, $\operatorname{Im} \varkappa_k > 0$, of the transition coefficient $T(k)$ are of the form $\frac{(2m+1)i}{2}$, $m \in \mathbb{N}$.

Soliton spheres can readily be constructed from the spectral data by using the inverse scattering method (see (77) in Appendix 3).

We proved in [6] that:

a) the lower estimate (54) becomes an equality on the soliton spheres corresponding to the potentials $U_N = \frac{N}{2\cosh x}$;
b) generally, a soliton sphere is not a surface of revolution;[15]
c) the class of soliton spheres is preserved by the mKdV deformations (we note that these deformations are given by $1+1$-equations) for which the Kruskal–Miura integrals are integrals of motion;
d) the soliton spheres corresponding to the potentials $U_N = N/2\cosh x$ are described in terms of rational functions,[16] that is, these spheres can be called *rational spheres*;
e) the soliton spheres such that each pole $\varkappa_j$ is of the form $(2m_j+1)i/2$ are critical points of the Willmore functional restricted to the class of spheres with one-dimensional potentials.

[15] Indeed, denote by $f_1 = \varphi_1(x)e^{\varkappa_1 y}$, ..., $f_n = \varphi_N e^{\varkappa_N y}$ the distinct generators of $\operatorname{Ker} \mathscr{D}$. Then any linear combination $f = \alpha_1 f_1 + \dots + \alpha_N f_N$ determines a sphere in $\mathbb{R}^3$ by means of the Weierstrass representation. If there is a pair of non-zero coefficients α_j and α_k such that $\operatorname{Im} \varkappa_j \neq \operatorname{Im} \varkappa_k$, then the sphere is not a surface of revolution.

[16] It is clear from the recovery formulae (77) that this holds for all reflectionless potentials.

5.4. Plücker formula. Our attempts to prove Conjecture 1 had failed because of the absence of a well-developed inverse scattering method for two-dimensional operators. However, in the remarkable paper [7] this conjecture was proved together with its generalization to surfaces of arbitrary genera by using methods of algebraic geometry.

As was mentioned in [8], the following statement can be derived from the results of [126] (see also [127]).

Proposition 15. *Let E be a C^2-bundle over a surface M and let ψ be a non-trivial section of E such that $\mathscr{D}\psi = 0$. Then the zeros of ψ are isolated, and*

$$\psi = z^k \varphi + O(|z|^{k+1})$$

for any local complex coordinate z on M with origin located at some zero p of the function ψ (that is, if $\psi(p) = 0$ and $z(p) = 0$), where φ is a local section of E that does not vanish in some neighbourhood of p. The integer k is well defined and does not depend on the choice of the coordinate z.

The integer k is called the *order* of the zero p:

$$\operatorname{ord}_p \psi = k.$$

Let us now recall the equation (see Proposition 1 in § 2.1)

$$\alpha_{z\bar z} + U^2 - |A|^2 e^{-2\alpha} = 0, \qquad e^\alpha = |\psi_1|^2 + |\psi_2|^2. \tag{56}$$

We assume for simplicity that M is a sphere and E is a spin bundle. If the section ψ vanishes nowhere, then it determines a surface in $\mathbb{R}^3$, and, integrating the left-hand side of (56) over M, we obtain

$$\int_M \alpha_{z\bar z}\, dx\wedge dy + \int_M U^2\, dx\wedge dy - \int_M |A|^2 e^{-2\alpha}\, dx\wedge dy = 0. \tag{57}$$

By the Gauss theorem, the first term is equal to

$$-\frac{1}{4}\int_M (-4\alpha_{z\bar z} e^{-2\alpha}) e^{2\alpha}\, dx\wedge dy = -\frac{1}{4}\int_M K\, d\mu = -\pi,$$

where K stands for the Gaussian curvature and $d\mu$ for the measure corresponding to the induced metric. Thus,[17]

$$\int_M U^2\, dx\wedge dy = \pi + \int_M |A|^2 e^{-2\alpha}\, dx\wedge dy \geqslant -\int_M \alpha_{z\bar z}\, dx\wedge dy = \pi.$$

In the general case, for any surface and for any section ψ satisfying the equation $\mathscr{D}\psi = 0$ (that is, we do not assume here that ψ vanishes nowhere) we have

$$\int_M U^2\, dx\wedge dy = \pi\Bigl(-\deg E_0 + \sum_p \operatorname{ord}_p \psi\Bigr) + \int_M |A|^2 e^{-2\alpha}\, dx\wedge dy$$
$$\geqslant \pi(-\deg E_0 + \sum_p \operatorname{ord}_p \psi)$$

[17] For general complex quaternionic line bundles $L = E_0 \oplus E_0$ we have $\int_M \alpha_{z\bar z}\, dx\wedge dy = \pi \deg E_0 = \pi d$.

(see [8]). The integrand $|A|^2 e^{-2\alpha}$ has singularities at the zeros of ψ, but the integral converges and is non-negative.

Returning to the case of spin bundles over spheres ($\deg E_0 = g - 1 = -1$) and assuming that $\dim_{\mathbb{H}} \operatorname{Ker} \mathscr{D} = N$, we choose a point p and a function $\psi \in \operatorname{Ker} \mathscr{D}$ in such a way that $\operatorname{ord}_p \psi = \dim \operatorname{Ker}_{\mathbb{H}} \mathscr{D} - 1 = N - 1$. We now substitute ψ into (56) and obtain

$$\int_M U^2 \, dx \wedge dy = \pi(1 + N - 1) + \int_M |A|^2 e^{-2\alpha} \, dx \wedge dy \geqslant \pi N.$$

However, this estimate is too rough, because we can see from the proof of Theorem 13 that it is not just the function in $\operatorname{Ker} \mathscr{D}$ with maximal order of zeros that contributes to lower bounds for the Willmore functional, and one must consider the flag of functions.

In [7] a deep analogy was discovered between this problem and the Plücker relations connecting the degrees and the ramification indices of curves associated with some algebraic curve in $\mathbb{C}P^n$. This enabled one to write out the flag and to consider the contribution of the entire kernel of the operator $\mathscr{D}$ to the Willmore functional. This led finally to establishing lower bounds for the Willmore functional that are quadratic in $\dim_{\mathbb{H}} \operatorname{Ker} \mathscr{D}$.

To formulate the main result of [7], we introduce some definitions. Let H be a subspace of $\operatorname{Ker} \mathscr{D}$. For any point p we write

$$n_0(p) = \min \operatorname{ord}_p \psi \quad \text{for} \quad \psi \in H.$$

Then we write, step by step,

$$n_k(p) = \min \operatorname{ord}_p \psi \quad \text{for } \psi \in H \text{ such that } \operatorname{ord}_p \psi > n_{k-1}(p).$$

We have the Weierstrass gap sequence

$$n_0(p) < n_1(p) < \cdots < n_{N-1}(p), \qquad N = \dim_{\mathbb{H}} H,$$

and a chain of embeddings

$$H = H_0 \supset H_1 \supset \cdots \supset H_{N-1},$$

where H_k consists of the functions ψ such that $\operatorname{ord}_p \psi \geqslant n_k(p)$. The *order of the linear system H at the point p* is then defined as

$$\operatorname{ord}_p H = \sum_{k=0}^{N-1} (n_k(p) - k) = \sum_{k=0}^{N-1} n_k(p) - \frac{1}{2} N(N-1).$$

We say that p is a *Weierstrass point* if $\operatorname{ord}_p H \neq 0$.

It is now possible to formulate the main result of the theory.

Theorem 15 [7]. *Let $H \subset \operatorname{Ker} \mathscr{D}$ and $\dim_{\mathbb{H}} H = N$. Then*

$$\int_M |U|^2 \, dx \wedge dy = \pi(N^2(1-g) + \operatorname{ord} H) + \mathscr{A}(M), \tag{58}$$

where the summand $\mathscr{A}(M)$ is non-negative and reduces to $\int_M |A| e^{-2\alpha} \, dx \wedge dy$ in the case of (57).

In fact, the main result of [7] remains valid for the Dirac operators with complex-conjugate potentials, $U = \overline{V}$,[18] on arbitrary complex quaternionic line bundles of arbitrary degree d (this follows immediately from the proof) and explains the summand $\mathscr{A}(M)$ in terms of dual curves,

$$\int_M |U|^2\,dx\wedge dy = \pi(N((N-1)(1-g)-d)+\operatorname{ord}H)+\mathscr{A}(M),$$

where g is the genus of the surface M and $d = \deg L = \deg E_0$. In Theorem 15 we set $d = g-1$, that is, we consider the case which is of interest for the theory of surfaces.

For $U = 0$ we have $\mathscr{A}(M) = 0$, and the Plücker formula (58) reduces to the original Plücker relation for algebraic curves (see, for instance, [128]),

$$\operatorname{ord}H = N((N-1)(g-1)+d).$$

For $g = 0$ we have the following result.

Corollary 4 [7]. *Conjecture 1 is true*: $\displaystyle\int U^2\,dx\wedge dy \geqslant \pi N^2$.

For $g \geqslant 1$ we obtain an effective lower bound, in terms of $\operatorname{ord}H$ only, because the term quadratic in N vanishes for $g = 1$ and is negative for $g > 1$.

To obtain effective lower bounds for the Willmore functional, it was proposed in [7] to use some special linear systems H. Let $\dim\operatorname{Ker}\mathscr{D} = N$. We take in $\operatorname{Ker}\mathscr{D}$ a k-dimensional linear system H distinguished by the condition that $\operatorname{ord}_p\psi \geqslant N-k$ for some fixed point p and for any $\psi\in H$. The corresponding Weierstrass gap sequence satisfies the inequality

$$n_l(p) \geqslant N-k+l, \qquad l = 0,1,\dots,k-1,$$

and hence $\operatorname{ord}_p H \geqslant k(N-k)$. It follows from (58) that

$$\int U^2\,dx\wedge dy \geqslant \pi(k^2(1-g)+k(N-k)) = kN-k^2g. \tag{59}$$

If $g = 0$, then the right-hand side attains its maximum at $k = N$, and we obtain the estimate (54).

For $g \geqslant 1$ the function $f(x) = xN - x^2g$ attains its maximum at the point $x_{\max} = N/(2g)$. Hence, the right-hand side of (59) attains its maximum either at $k = [N/(2g)]$ or at $[N/(2g)]+1$, that is, at an integer point closest to the point $x_{\max}$. This readily implies a rough lower bound which holds for any g. Of course, this bound can be improved in special cases; for example, for $g = 1$.

Corollary 5 [7]. *The following inequalities hold*:

$$\int U^2\,dx\wedge dy \geqslant \frac{\pi}{4g}(N^2-g^2) \tag{60}$$

[18] In this case $\dfrac{1}{4}\mathscr{W} = \displaystyle\int UV\,dx\wedge dy = \int |U|^2\,dx\wedge dy$, where $\mathscr{W}$ is the Willmore functional.

for $g > 1$, *and*

$$\int U^2\,dx\wedge dy \geqslant \begin{cases} \dfrac{\pi N^2}{4} & \text{if } N \text{ is even,} \\ \dfrac{\pi(N^2-1)}{4} & \text{if } N \text{ is odd} \end{cases} \tag{61}$$

for $g = 1$.

The proof of Theorem 14 can be directly applied to derive the following corollary.

Corollary 6 [7]. *The following inequalities hold for a given eigenvalue* λ *of the Dirac operator on a two-dimensional spin-manifold of genus* g :

$$\lambda^2\,\mathrm{Area}(M) \geqslant \begin{cases} \pi m^2(\lambda) & \text{for } g=0, \\ \dfrac{\pi}{g}(m^2(\lambda)-g^2) & \text{for } g\geqslant 1, \end{cases}$$

where $m(\lambda)$ *stands for the multiplicity of the eigenvalue* λ.

Another important application of the formula (61) involves lower bounds for the area of CMC tori in $\mathbb{R}^3$ and minimal tori in S^3. One can see from the explicit construction of spectral curves (see § 4.6) that, in both cases, the normalized spectral curves are the hyperelliptic curves

$$\mu^2 = P(\lambda)$$

such that a pair of branch points correspond to the 'points at infinity' $\infty_\pm$. There are also $2g$ other branch points (here g stands for the genus of this hyperelliptic curve) at which the multipliers of Floquet functions are equal to ±1 (by the construction of spectral curves). Moreover, there is also a pair of points which are transposed by the hyperelliptic involution and at which the multipliers are also equal to ±1 (the tori are constructed in terms of these Floquet functions, as was shown in [62] and [78]). Thus, we have the space F with $\dim_{\mathbb{C}} F = 2g+2$ that consists of solutions of the equation $\mathscr{D}\psi = 0$ with the multipliers ±1. Let us take a four-sheeted covering $\widehat{M}$ of a torus M which doubles both the periods. The pull-backs of the functions in F to this covering are doubly periodic functions, that is, they are sections of the same spin bundle over $\widehat{M}$. The complex dimension of the kernel of the operator $\mathscr{D}$ acting on this spin bundle is at least $2g+2$, and thus $\dim_{\mathbb{H}} \operatorname{Ker}\mathscr{D} \geqslant g+1$. Applying (61), we obtain lower bounds for $\displaystyle\int |U|^2\,dx\wedge dy$.

For CMC tori we have $H=1$ and $U = \dfrac{e^\alpha}{2}$. Hence,

$$\int_{\widehat{M}} U^2\,dx\wedge dy = \frac{1}{4}\,\mathrm{Area}(\widehat{M}) = \mathrm{Area}(M).$$

For minimal tori in S^3 we have $U = -\dfrac{ie^\alpha}{2}$, and thus $\displaystyle\int_{\widehat{M}} |U|^2\,dx\wedge dy = \mathrm{Area}(M)$. We obtain the following assertion.

Corollary 7. *The following lower bounds for the area hold for minimal tori in* S^3 *and CMC tori in* $\mathbb{R}^3$ *of spectral genus* g:

$$\text{Area} \geqslant \begin{cases} \dfrac{\pi(g+1)^2}{4} & \text{if } g \text{ is odd}, \\ \dfrac{\pi((g+1)^2-1)}{4} & \text{if } g \text{ is even}. \end{cases}$$

As was noted in [7], it follows from [62] that for minimal tori in S^3 the bound can be improved by replacing $g+1$ by $g+2$. Moreover, the bound remains valid for the energy of all harmonic tori in S^3.

The genus of the spectral curve was recently applied by Haskins to a completely different problem, namely, in the study of special Lagrangian T^2-cones in $\mathbb{C}^3$ [129]. Haskins obtained linear (in the genus) lower bounds for some quantities characterizing the geometric complexity of these cones and conjectured that these bounds can be improved to quadratic estimates. We note that the approach of the paper [129] totally differs from the methods used in [7].

The contribution of the term $\operatorname{ord} H$ can readily be shown by the example of soliton spheres for which the poles of the transition coefficient are of the form $(2l+1)i/2$. In this case the number $\operatorname{ord}\operatorname{Ker}\mathscr{D}$ counts the gaps in filling these energy levels.

Recently, the definition of soliton spheres was generalized in the spirit of lower estimates for the Willmore functional: a sphere is called a *soliton sphere* if the 'Plücker inequality'

$$\int_M |U|^2\,dx\wedge dy \geqslant \pi(N^2(1-g)+\operatorname{ord} H)$$

becomes an equality for it, that is, $\mathscr{A}(M)=0$ [130].

As was shown by Bohle and Peters [131], this class of spheres contains many other interesting examples.

Before formulating their result, we recall that Bryant surfaces are exactly the surfaces of unit constant mean curvature in the hyperbolic three-dimensional space [132]. By [131], a Bryant surface M in the three-dimensional ball $B^3\subset\mathbb{R}^3$, which gives the Poincaré model of hyperbolic space, is a smooth Bryant end if there is a point p_∞ on the asymptotic boundary ∂B^3 such that $M\cup p_\infty$ is a conformally immersed open disc in $\mathbb{R}^3$. Generally, a Bryant surface is called a *compact Bryant surface with smooth ends* if it is conformally equivalent to a compact surface with finitely many punctured points each having open neighbourhoods isometric to a smooth Bryant end.

This is a clear generalization of minimal surfaces with planar ends.

The following assertion holds.

Theorem 16 [131]. *The Bryant spheres with smooth ends are soliton spheres. The possible values of the Willmore functional for these spheres are* $4\pi N$, *where* N *is a positive integer different from* 2, 3, 5, *and* 7.

As was noted by Bohle and Peters, they obtained this theorem by using the observation that the simplest soliton spheres corresponding to the potentials

$U_N = \dfrac{N}{2\cosh x}$ can be treated as Bryant spheres with smooth ends. Bohle and Peters also announced that all Willmore spheres are soliton spheres (we note that, according to the results of Bryant and Peng, the possible values of the Willmore functional on the Willmore spheres coincide with those for the Bryant spheres with smooth ends; see [87], [94], [93]).

5.5. Willmore-type functionals for surfaces in Lie groups of dimension three. The formula (44) shows that it is reasonable to consider the functional

$$E(\Sigma) = \int_\Sigma UV\,dx \wedge dy.$$

for surfaces. For tori this functional measures the asymptotic flatness of the spectral curve, and for surfaces in $\mathbb{R}^3$ it is equal to $E = \frac{1}{4}\mathscr{W}$ [1]. In [11] this functional was considered for surfaces in other Lie groups and was called the *energy* of a surface. Although the product UV is not always real-valued for closed surfaces, the functional is real-valued and takes the following values:

for $SU(2)$ [10],

$$E = \frac{1}{4}\int (H^2+1)\,d\mu,$$

that is, E is a multiple of the Willmore functional;

for Nil [11],

$$E = \frac{1}{4}\int \biggl(H^2 + \frac{\widehat{K}}{4} - \frac{1}{16}\biggr)\,d\mu;$$

for $\widetilde{SL}_2$ [11],

$$E(M) = \frac{1}{4}\int_M \biggl(H^2 + \frac{5}{16}\widehat{K} - \frac{1}{4}\biggr)\,d\mu;$$

the energy E is well defined for surfaces in Sol, because the potentials are undetermined only on zero-measure sets; however, we still do not know the geometric meaning of the energy.

We recall that the symbol $\widehat{K}$ stands here for the sectional curvature of the ambient space along the tangent plane to a surface.

These functionals have not been studied, and many problems remain open:

1) Are they bounded below (this is confirmed by some numerical experiments)?
2) What are their extremals?
3) What are the analogues of the Willmore conjecture for these functionals?

Appendix 1. Existence of a spectral curve for the Dirac operator with L_2-potentials

In this appendix we present a proof of Theorem 10 following the lines of [5], where the exposition is too brief in our opinion.

Moreover, the ideas of the proof of this theorem are important for the proof of the main result in [92] claiming that the minimum of the Willmore functional on a given conformal class of surfaces is constructed as follows. Let us consider the infimum of the Willmore functional on this class and take a sequence of surfaces (or, more precisely, of Weierstrass representations of surfaces) in the class for which the values of the Willmore functional converge to the infimum. Then there is a weakly convergent sequence of potentials of the corresponding Dirac operators. The Dirac operator with the limit potential also has non-trivial kernel (this follows from the convergence of the resolvents), and the desired minimizing surface is constructed from a function in this kernel by the Weierstrass representation. Of course, it is necessary to control the smoothness, which is possible. However, as was mentioned in [92], one cannot say that there are no branch points on the limit surface.

An analogue of the decomposition (40) is the following sequence:

$$L_p \overset{(\mathscr{D}_0-E_0)^{-1}}{\longrightarrow} W^1_p \overset{\text{Sobolev embedding}}{\longrightarrow} L_{\frac{2p}{2-p}} \overset{\text{multiplication}}{\longrightarrow} L_p, \qquad p<2. \tag{62}$$

The operators in the sequence are only continuous, and we cannot argue as in § 4.2.

Let $M=\mathbb{C}/\Lambda$ be a torus and let z be a linear complex coordinate on M defined modulo the lattice Λ. We denote by $\rho(z_1,z_2)$ the distance between points $z_1,z_2\in M$ with respect to the metric induced by the Euclidean metric on $\mathbb{C}$ under the covering $\mathbb{C}\to\mathbb{C}/\Lambda$.

The following proposition can be derived from the definition of the resolvent

$$(\mathscr{D}_0-E)R_0(E)=\delta(z-z')$$

by straightforward computations.

Proposition 16. *The resolvent*

$$R_0(E)=(\mathscr{D}_0-E)^{-1}\colon L_2\to W^1_2\to L_2$$

of the free operator $\mathscr{D}_0\colon L_2\to L_2$ is an integral matrix operator of the form

$$f(z,\bar z)\to[R_0(E)f](z,\bar z)=\int_M K_0(z,z',E)f(z',\bar z')\,dx'\,dy', \qquad z'=x'+iy',$$

with the kernel $K_0(z,\bar z,z',\bar z',E)=\begin{pmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{pmatrix}$, $r_{ik}=r_{ik}(z,\bar z,z',\bar z',E)$, where

$$r_{12}=\frac{1}{E}\partial r_{22}, \quad r_{21}=-\frac{1}{E}\bar\partial r_{11}, \quad \frac{1}{E}(\partial\bar\partial+E^2)r_{11}=\frac{1}{E}(\partial\bar\partial+E^2)r_{22}=-\delta(z-z').$$

Corollary 8. *The integral kernel of the resolvent $R_0(E)$ is equal to*

$$K_0(z,z',E)=\begin{pmatrix} -EG & -\partial G\\ \bar\partial G & -EG\end{pmatrix},$$

where G stands for the (modified) Green function of the Laplace operator on the torus M:

$$(\partial\bar\partial+E^2)G(z,z',E)=\delta(z-z').$$

Example. For the torus $M=\mathbb{C}/\{2\pi\mathbb{Z}+2\pi i\mathbb{Z}\}$ we have

$$\delta(z-z')=\sum_{k,l\in\mathbb{Z}} e^{i(k(x-x')+l(y-y'))}, \qquad z=x+iy, \quad z'=x'+iy',$$

$$G(z,z',E)=-4\sum_{k,l\in\mathbb{Z}}\frac{1}{k^2+l^2-4E^2}e^{i(k(x-x')+l(y-y'))}. \tag{63}$$

For other period lattices Λ the analogue of the series (63) for G has almost the same form and very similar analytic properties. We do not write it out and always refer to (63) when considering the analytic properties of the series.

The following proposition is obvious.

Proposition 17. *The series* (63) *converges for* $E = i\lambda$, *where* $\lambda \in \mathbb{R}$ *and* $\lambda \gg 0$ (*that is*, λ *is sufficiently large*).

To compute the operator

$$R_0(k,E) = (\mathscr{D}_0 + T_k - E)^{-1} \colon L_2 \to W_2^1 \overset{\text{embedding}}{\longrightarrow} L_2,$$

we use the identity

$$\mathscr{D}_0 + T_k - E = (1 + T_k(\mathscr{D}_0 - E)^{-1})(\mathscr{D}_0 - E) = (1 + T_k R_0(E))(\mathscr{D}_0 - E),$$

which implies the following formula for the resolvent:

$$R_0(k,E) = R_0(E)(1 + T_k R_0(E))^{-1} = R_0(E) \sum_{l=0}^{\infty} [-T_k R_0(E)]^l, \tag{64}$$

provided that the series on the right-hand side converges.

Remark. For a given p, $1 < p < 2$, the symbol

$$R_0(E) = (\mathscr{D}_0 - E)^{-1} \quad \text{or} \quad R_0(k,E) = (\mathscr{D}_0 + T_k - E)^{-1}$$

means one of the following objects in diverse situations:

a) the operator $A \colon L_p \to W_p^1$;
b) the composition $B \colon L_p \to L_q$, $q = 2p/(2-p)$, of the above operator A and the Sobolev embedding $W_p^1 \to L_q$;
c) the composition $C \colon L_p \to L_p$ of the above operator A and the natural embedding $W_p^1 \to L_p$.

The action of these operators is the same on the space of smooth functions, which can be regarded as the space embedded in W_p^1 or in L_q (the ambient spaces are the closures of the space of smooth functions with respect to the different norms). Hence, it suffices to prove the necessary estimates for smooth functions only, which can be done by using explicit formulae for the resolvents.

We decompose resolvents into sums of integral operators as follows.

Let χ_ε be the function $\chi_\varepsilon(r) = \begin{Bmatrix} 0 & \text{for } r > \varepsilon \\ 1 & \text{for } r \leqslant \varepsilon \end{Bmatrix}$, defined for $r \geqslant 0$, $r \in \mathbb{R}$. For a given $\delta > 0$ we decompose the resolvent $R_0(k,E)$ into the sum of two integral operators

$$R_0(k,E) = R_0^{\leqslant\varepsilon}(k,E) + R_0^{>\varepsilon}(k,E) \colon L_p \to L_q,$$

where the 'near' part $R_0^{\leqslant\varepsilon}(k,E)$ is determined by the kernel

$$K_0^{\leqslant\varepsilon}(z,\bar z,z',\bar z',E) = K_0(z,\bar z,z',\bar z',E)\chi_\varepsilon(\rho(z,z'))$$

and the 'distant' part $R_0^{>\varepsilon}(k,E)$ has the kernel

$$K_0^{>\varepsilon}(z,\bar z,z',\bar z',E) = K_0(z,\bar z,z',\bar z',E)(1 - \chi_\varepsilon(\rho(z,z'))).$$

Proposition 18. *For given p with $1 < p < 2$, $\hat{k} = (\hat{k}_1, \hat{k}_2) \in \mathbb{C}^2$, and δ with $0 < \delta < 1$ there is a real constant $\lambda_0 \gg 0$ such that*

$$\|T_k R_0(i\lambda)\|_{L_p \to L_p} < \delta$$

for any $\lambda > \lambda_0$ and for any k sufficiently close to $\hat{k}$.

Hence, for such λ and k:

1) *the series in (64) converges and defines a bounded operator from L_p to W^1_p, L_q, or L_p (depending on the meaning of the symbol $R_0(E)$ multiplied from the left by the series);*
2) *the norm of the operator*

$$R_0(k, i\lambda)\colon L_p \to W^1_p$$

is bounded by some constant r_p;

3) *for a given $\varepsilon > 0$*

$$\lim_{\lambda \to \infty} \|R_0^{>\varepsilon}(k, i\lambda)\|_{L_p \to L_q} = 0, \qquad q = \frac{2p}{2-p}\,.$$

This proposition follows from the explicit formula (63) for the kernel of the resolvent.

We denote by r_{inj} the injectivity radius of the metric on M, and we introduce the norms $\|\cdot\|_{2;\varepsilon}$ defined for $0 < \varepsilon < r_{\text{inj}}$ as follows. For any $U \in L_2(M)$ we denote by $U|_{B(z,\varepsilon)}$ the restriction of U to the ball $B(z,\varepsilon) = \{w \in M : \rho(z,w) < \varepsilon\}$ and define $\|U\|_{2;\varepsilon}$ by

$$\|U\|_{2;\varepsilon} = \max_{z \in M} \|U|_{B(z,\varepsilon)}\|_2.$$

Proposition 19. 1) *For any $U \in L_2(M)$*

$$\sqrt{\frac{\pi\varepsilon^2}{\operatorname{vol}(M)}} \leqslant \|U\|_{2;\varepsilon} \leqslant \|U\|_2.$$

2) *For any $C > 0$ and ε the sets $\{\|U\|_{2;\varepsilon} \leqslant C\}$ are closed, and hence compact with respect to both the weak topology and the topology of weak convergence on $L_2(M)$.*

Proof. Obviously, $\|U\|_{2;\varepsilon} \leqslant \|U\|_2$. Moreover, we have

$$\begin{aligned}\|U\|^2_{2;\varepsilon} \operatorname{vol}(M) &\geqslant \int_M \int_{B(z,\varepsilon)} |U(z+z', \bar{z}+\bar{z}')|^2 \, dz' \, dz \\ &= \int_M \int_{B(0,\varepsilon)} |U(z+z', \bar{z}+\bar{z}')|^2 \, dz' \, dz \\ &= \int_{B(0,\varepsilon)} \left[\int_M |U(z,\bar{z})|^2 \, dz\right] dz' = \pi\varepsilon^2 \|U\|^2_2,\end{aligned}$$

where $dz = dx \wedge dy$, $dz' = dx' \wedge dy'$. The second assertion is well known from functional analysis.

Let us consider the resolvent

$$R(k,E)=(\mathscr{D}+T_k-E)^{-1}\colon L_p\to L_p.$$

We again use the identity

$$\mathscr{D}+T_k-E=\left[1+\begin{pmatrix}U&0\\0&V\end{pmatrix}(\mathscr{D}_0+T_k-E)^{-1}\right](\mathscr{D}_0+T_k-E),$$

which implies the equality

$$R(k,E)=R_0(k,E)\sum_{l=0}^{\infty}\left[-\begin{pmatrix}U&0\\0&V\end{pmatrix}R_0(k,E)\right]^l.$$

Proposition 20. *Let $1<p<2$, let $\hat{k}=(\hat{k}_1,\hat{k}_2)\in\mathbb{C}^2$, let ε be sufficiently small, and let $0<\delta<1$. There is a $\gamma>0$ such that*

$$\left\|\begin{pmatrix}U&0\\0&V\end{pmatrix}R_0(k,i\lambda)\right\|_{L_p\to L_p}<\delta$$

for any $\lambda>\lambda_0$, for k sufficiently close to $\hat{k}$, and for U and V such that $\|U\|_{2;\varepsilon}<\gamma$ and $\|V\|_{2;\varepsilon}<\gamma$.

Proof. The following inequality is obvious:

$$\|R_0^{\leqslant\varepsilon}(k,E)\|\leqslant\|R_0(k,E)\|\quad\text{for any}\quad\varepsilon.$$

Let S_p be the Sobolev constant for the embedding $W_p^1\to L_q$ (see Proposition 5). For $\lambda>\lambda_0$ we have

$$\|R_0(k,i\lambda)\|_{L_p\to W_p^1}\leqslant r_p$$

(see Proposition 18). Let us now consider the composition of maps

$$L_p\overset{R_0^{\leqslant\varepsilon}(k,E)}{\longrightarrow}W_p^1\overset{\text{embedding}}{\longrightarrow}L_q\overset{\times\left(\begin{smallmatrix}U&0\\0&V\end{smallmatrix}\right)}{\longrightarrow}L_p,$$

where the norm of the first map is bounded above by the constant r_p and the norm of the second by the constant S_p. We compute the norm of the third map.

Since the integral kernel of the operator $R_0^{\leqslant\varepsilon}(k,E)$ is localized within the closed domain $\{\rho(z,z')\leqslant\varepsilon\}$, it follows that

$$\left[\begin{pmatrix}U&0\\0&V\end{pmatrix}R_0^{\leqslant\varepsilon}(k,E)f\right]\Big|_{B(x,\alpha)}=\left[\begin{pmatrix}U&0\\0&V\end{pmatrix}R_0^{\leqslant\varepsilon}(k,E)\right]\left(f\big|_{B(x,\alpha+\varepsilon)}\right)$$

for any ball $B(x,\alpha)$. Applying the Hölder inequality to the right-hand side of the last formula, we obtain

$$\left\|\left[\begin{pmatrix}U&0\\0&V\end{pmatrix}R_0^{\leqslant\varepsilon}(k,E)f\right]\Big|_{B(x,\alpha)}\right\|_p$$
$$\leqslant m\|R_0^{\leqslant\varepsilon}(k,E)\|_{L_p\to L_q}\|\left(f\big|_{B(x,\alpha+\varepsilon)}\right)\|_p\leqslant mr_pS_p\left\|\left(f\big|_{B(x,\alpha+\varepsilon)}\right)\right\|_p,$$

where $m = \max(\|U\|_{2;\varepsilon}, \|V\|_{2;\varepsilon})$. Let us now recall the identity

$$\int_M \|g_{B(x,\alpha)}\|_p^p \, dx = \operatorname{vol} B(x,\alpha)\|g\|_p^p = \pi\alpha^2\|g\|_p^p$$

and apply it to the previous inequality. We obtain

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\leqslant\varepsilon}(k,E) \right\|_p \leqslant m r_p S_p \left(1 + \frac{\varepsilon}{\alpha}\right)^{2/p}.$$

Since we use the Sobolev constant S_p for the torus, we must assume that $(\alpha+\varepsilon) < r_{\mathrm{inj}}$. If

$$m = \max\bigl(\|U\|_{2;\varepsilon}, \|V\|_{2;\varepsilon}\bigr) < \frac{\delta}{r_p S_p \sqrt[p]{4}} \tag{65}$$

and $\alpha = \varepsilon < r_{\mathrm{inj}}/2$, then

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\leqslant\varepsilon}(k,E) \right\|_p < \delta.$$

This completes the proof of the proposition.

Proposition 21 [5]. *Let p and $\hat{k} \in \mathbb{C}^2$ be the same as in Proposition 20, let $\gamma < (r_p S_p \sqrt[p]{4})^{-1}$, and let $\lambda \gg 0$, that is, λ is sufficiently large. Let $\varepsilon > 0$ be sufficiently small. For operators U and V such that $\|U\|_{2;\varepsilon} \leqslant C \leqslant \gamma$ and $\|V\|_{2;\varepsilon} \leqslant C \leqslant \gamma$ the series*

$$R(k,i\lambda) = R_0(k,i\lambda) \sum_{l=0}^{\infty} \left[-\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k,i\lambda) \right]^l \tag{66}$$

is uniformly convergent near $\hat{k}$ and defines the resolvent of the operator

$$\mathscr{D} + T_k \colon L_p \to L_p.$$

The action of this resolvent on smooth functions can be extended to the resolvent of the operator $\mathscr{D} + T_k$ on the space L_2,

$$(\mathscr{D} + T_k - E)^{-1} \colon L_2 \to W_2^1 \xrightarrow{\text{embedding}} L_2.$$

This pencil of compact operators is holomorphic with respect to k in a neighbourhood of $\hat{k}$. If $(U_n, V_n) \xrightarrow{\text{weakly}} (U_\infty, V_\infty)$ in $\{\|U\|_{2;\varepsilon} \leqslant C,\ \|V\|_{2;\varepsilon} < C\}$, then the corresponding resolvents are norm convergent to the resolvent of the operator with potentials (U_∞, V_∞).

Proof. By Proposition 20 and (65),

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{\leqslant\varepsilon}(k,i\lambda) \right\|_p < \sigma = \gamma r_p S_p \sqrt[p]{4} < 1$$

near $\hat{k}$ for $\lambda > \lambda_0$. By Proposition 18, for sufficiently large real values of λ we have

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0^{>\varepsilon}(k,i\lambda) \right\|_p < 1 - \sigma,$$

since the norm of the embedding $L_q \to L_p$ is finite. This means that the inequality

$$\left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k, i\lambda) \right\|_p \leqslant \left\| \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \left(R_0^{\leqslant \varepsilon}(k, i\lambda) + R_0^{> \varepsilon}(k, i\lambda) \right) \right\|_p < 1$$

holds for $\lambda \gg 0$, and the series (66) is uniformly convergent near $\hat{k}$ and defines the resolvent of the operator $\mathscr{D} + T_k \colon L_p \to L_p$.

The action of $R(k, i\lambda)$ on smooth functions is given by the formula (66). We extend this action to a compact operator on L_2 as follows. Let

$$B = \sum_{l=0}^{\infty} \left[- \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k, i\lambda) \right]^l$$

and consider the following composition of operators:

$$L_2 \xrightarrow{\text{embedding}} L_p \xrightarrow{B} L_p \xrightarrow{(\mathscr{D}+T_k-E)^{-1}} W_p^1 \xrightarrow{\text{embedding}} L_2,$$

where all operators are bounded and the embedding $W_p^1 \to L_2$ is compact by the Kondrashov theorem (see Proposition 5). This shows that the action of $R(k, i\lambda)$ on smooth functions can be extended to a compact operator on L_2. Since the series (66) is holomorphic with respect to k, the resolvent $R(k, i\lambda)$ is also holomorphic with respect to k.

It remains to prove that the resolvent is continuous with respect to U and V. Every occurrence of the matrix $\begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ in some term of the series (66) is bracketed by resolvents of the form $R_0(k, i\lambda)$, which are bounded integral operators. Let $l = 1$ and let $K(z, z', k, i\lambda)$ be the integral kernel of one of these operators. Then the composition

$$R_0(k, i\lambda) \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} R_0(k, i\lambda)$$

acts on smooth functions as the integral operator with kernel

$$F(z, z'') = K(z, z', k, i\lambda) \begin{pmatrix} U(z') & 0 \\ 0 & V(z') \end{pmatrix} K(z', z'', k, i\lambda).$$

It is clear that an integral operator of this kind is continuous with respect to weak convergence of potentials $U, V \in L_2(M)$. The proof is similar for other values of l. By Proposition 19, every term of the series (66) is continuous with respect to weak convergence of potentials in $\{\|U\|_{2;\varepsilon} \leqslant C,\ \|V\|_{2;\varepsilon} \leqslant C\}$. Since the series (66) is uniformly convergent, the same continuity property holds for the sum of the series. This proves the proposition.

This proposition establishes the existence of the resolvent only for large values of λ, where $E = i\lambda$. The resolvent can be extended to a meromorphic function on the E-plane by using the Hilbert formula (see Proposition 6).

Proof of Theorem 10. By Proposition 21, there exist $k_0 \in \mathbb{C}^2$ and $E \in \mathbb{C}$ such that the operator

$$(\mathscr{D} + T_{k_0} - E_0)^{-1} \colon L_2 \to W_2^1 \xrightarrow{\text{embedding}} L_2$$

is correctly defined. We substitute the expression $\varphi = (\mathscr{D} + T_{k_0} - E)^{-1} f$ into the equation

$$(\mathscr{D} + T_k - E)\varphi = 0$$

and rewrite this equation in the form

$$(\mathscr{D} + T_{k_0} - T_{k_0} + T_k - E_0 + E_0 - E)(\mathscr{D} + T_{k_0} - E)^{-1} f = [1 + A_{U,V}(k, E)]f = 0,$$

where

$$A_{U,V}(k, E) = (T_k - T_{k_0} + E_0 - E)(\mathscr{D} + T_{k_0} - E)^{-1}.$$

Since the first factor in this formula is a bounded operator for any k and E and since the second factor is a compact operator, the family $A_{U,V}(k, E)$ is a pencil of compact operators that are polynomials in k and E. By applying the Keldysh theorem as in § 4.2, we complete the proof of the theorem.

The spectral curve can now be defined as usual by the formula

$$\Gamma = Q_0(U, V)/\Lambda^*.$$

Remark. The resolvents of operators on non-compact spaces do not depend continuously on the potentials under weak convergence of them. Indeed, let us consider the Schrödinger operator

$$L = -\frac{d^2}{dx^2} + U(x),$$

where $U(x)$ is a soliton potential (and hence the operator does have bound states). The isospectral sequence of potentials $U_N(x) = U(x + N)$ converges weakly to the zero potential $U_\infty = 0$, for which the Schrödinger operator has no bound states. The same holds for the one-dimensional Dirac operator.

Appendix 2. The conformal Gauss map and the conformal area

In this appendix we present known results on the Gauss conformal map, mainly following the lines of [87], [133], [109].

We denote by $S_{q,r}$ the round sphere of radius r in $\mathbb{R}^3$ centred at a point q and by $\Pi_{p,N}$ the plane in $\mathbb{R}^3$ passing through a point p and with normal vector N. All these spheres and planes are parametrized by a quadric $Q^4 \subset \mathbb{R}^{4,1}$. Indeed, let

$$\langle x, y\rangle = x_1y_1 + \dots + x_4y_4 - x_5y_5$$

be the inner product in $\mathbb{R}^{4,1}$. We set

$$Q^4 = \{\langle x, x\rangle = 1\} \subset \mathbb{R}^{4,1},$$
$$S_{q,r} \to \frac{1}{r}\left(q, \frac{1}{2}(|q|^2 - r^2 - 1), \frac{1}{2}(|q|^2 - r^2 + 1)\right),$$
$$\Pi_{q,N} \to (N, \langle q, N\rangle, \langle q, N\rangle).$$

For a surface $f\colon \Sigma \to \mathbb{R}^3$ its *conformal Gauss map*

$$G^c\colon \Sigma \to Q^4$$

assigns to a point $p \in \Sigma$ the sphere of radius $1/H$ tangent to the surface at p for $H \neq 0$,

$$G^c(p) = S_{p+N/H,1/H},$$

and the tangent plane at p for $H = 0$. In the coordinates on Q^4, this map has the form

$$G^c(p) = H \cdot X + T,$$

where

$$X = \left(f, \frac{(f,f)-1}{2}, \frac{(f,f)+1}{2}\right), \qquad T = (N,(N,f),(N,f)).$$

This map is a special case of the so-called *sphere congruences*, one of the main topics in conformal geometry (the modern state of this theory is presented in [134]).

We have $\langle X, X\rangle = 0$, $\langle T, T\rangle = 1$ and $\langle X, T\rangle = 0$, which implies that $\langle dX, X\rangle = \langle dT, T\rangle = 0$, $\langle dT, X\rangle = \langle -dX, T\rangle$. One can readily see that

$$\langle dX, T\rangle = (df, N) = 0, \qquad \langle dX, dX\rangle = (df, df) = \mathbf{I},$$
$$\langle dX, dT\rangle = (df, dN) = -\mathbf{II}, \qquad \langle dT, dT\rangle = (dN, dN) = \mathbf{III},$$

where the third fundamental form of a surface (we denote it by $\mathbf{III}$) measures the lengths of images of curves under the Gauss map and satisfies the identity

$$K \cdot \mathbf{I} - 2H \cdot \mathbf{II} + \mathbf{III} = 0$$

connecting the third form with $\mathbf{I}$ and $\mathbf{II}$, the first and the second fundamental forms of a surface. This implies that

$$\langle Y_z, Y_z\rangle = \langle Y_{\bar z}, Y_{\bar z}\rangle = 0, \qquad \langle Y_z, Y_{\bar z}\rangle = e^{\beta} = \frac{(H^2-K)e^{2\alpha}}{2} = (H^2-K)(f_z, f_{\bar z}),$$

where Y denotes G^c for brevity, z is a conformal parameter on the surface, and $\mathbf{I} = e^{2\alpha}\, dz\, d\bar z$ is the induced metric on the surface. We conclude that

the conformal Gauss map is regular and conformal outside umbilical points.

It is clear that X and Y are linearly independent vectors. Outside umbilical points, the set of vectors Y, Y_z, $Y_{\bar z}$, X is uniquely completed by a vector $Z \in \mathbb{R}^5$ to form a basis

$$\sigma = (Y, Y_z, Y_{\bar z}, X, Z)^T$$

for the complexification $\mathbb{C}^5$ of $\mathbb{R}^5$ such that the inner product in $\mathbb{R}^{4,1}$ takes the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{\beta} & 0 & 0 \\ 0 & e^{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Analogues of the Gauss–Weingarten equations are as follows:

$$\sigma_z = \mathbf{U}\sigma, \qquad \sigma_{\bar z} = \mathbf{V}\sigma,$$

$$\mathbf{U} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \beta_z & 0 & C_2 & C_1 \\ -e^{\beta} & 0 & 0 & C_4 & C_3 \\ 0 & -e^{-\beta}C_3 & -e^{-\beta}C_1 & C_5 & 0 \\ 0 & -e^{-\beta}C_4 & -e^{-\beta}C_2 & 0 & -C_5 \end{pmatrix},$$

$$\mathbf{V} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -e^{\beta} & 0 & 0 & C_4 & C_3 \\ 0 & 0 & \beta_{\bar z} & \overline{C}_2 & \overline{C}_1 \\ 0 & -e^{-\beta}\overline{C}_1 & -e^{-\beta}\overline{C}_3 & \overline{C}_5 & 0 \\ 0 & -e^{-\beta}\overline{C}_2 & -e^{-\beta}\overline{C}_4 & 0 & -\overline{C}_5 \end{pmatrix},$$

where

$$C_1 = \langle Y_{zz}, X\rangle_{4,1}, \qquad C_2 = \langle Y_{zz}, Z\rangle_{4,1}, \qquad C_3 = \langle Y_{z\bar z}, X\rangle_{4,1},$$
$$C_4 = \langle Y_{z\bar z}, Z\rangle_{4,1}, \qquad C_5 = \langle X_z, Z\rangle_{4,1}.$$

The identity

$$\Delta Y + 2(H^2 - K)Y = (\Delta H + 2H(H^2 - K))X$$

can be verified directly. In particular, this identity implies that

$$C_3 = 0, \qquad C_4 = \frac{e^{2\alpha}}{4}(\Delta H + 2H(H^2 - K)).$$

Here $\Delta = 4e^{-2\alpha}\partial\bar\partial$ stands for the Laplace–Beltrami operator on the surface. Taking this into account and keeping in mind that the quantity C_4 is real-valued, we derive the Codazzi equations for the conformal Gauss map:

$$\begin{gathered} \beta_{z\bar z} + e^{\beta} - (\overline{C}_1 C_2 + C_1\overline{C}_2)e^{-\beta} = 0, \qquad C_{1\bar z} = C_1\overline{C}_5, \\ C_{2\bar z} + C_2\overline{C}_5 = C_{4z} - \beta_z C_4 + C_4 C_5, \qquad C_{5\bar z} - \overline{C}_{5z} = e^{-\beta}(C_1\overline{C}_2 - \overline{C}_1 C_2). \end{gathered} \tag{67}$$

Straightforward calculations show that

$$C_1 = A = \langle N, f_{zz}\rangle, \qquad e^{\beta} = 2|A|^2 e^{-2\alpha}.$$

The *conformal area* V^c of a surface Σ is the area of the image of Σ in Q^4,

$$V^c(\Sigma) = \int_{\Sigma} (H^2 - K)\, d\mu,$$

where $d\mu$ is the area form on Σ. The Euler–Lagrange equations for V^c take the form

$$\Delta H + 2H(H^2 - K) = 0.$$

A surface in $\mathbb{R}^3$ is said to be *conformally minimal* (or a *Willmore surface*) if it satisfies the above equation. We conclude that

conformally minimal surfaces are exactly the surfaces whose images under the conformal Gauss map G^c are minimal surfaces in Q^4.

For any non-umbilical point $p \in \Sigma$ the tangent space to Q^4 at the point $Y(p)$ is spanned by the vectors Y_z, $Y_{\bar z}$, X, and Z. We see that the map Y is *conformally harmonic* (that is, ΔY is everywhere orthogonal to the tangent plane to Q^4) if and only if the surface is conformally minimal.

It follows from the Gauss–Weingarten equations for G^c and the Euler–Lagrange equations for V^c that if Σ is conformally minimal, that is, $C_4 = 0$, then the quartic differential

$$\omega = \langle Y_{zz}, Y_{zz}\rangle (dz)^4 = C_1 C_2 (dz)^4$$

is holomorphic.

We recall that a holomorphic quartic differential vanishes on any two-dimensional sphere ($\omega = 0$) and has constant coefficients on any torus ($\omega = \mathrm{const}\cdot(dz)^4$).

A minimal surface in Q^4 is said to be *superminimal* if $\omega = 0$.

Let

$$\varphi = \log \frac{\overline{C}_1}{C_1}\,.$$

We note that the condition $C_1 \equiv 0$ holds only for totally umbilical surfaces, and by the Hopf theorem these are domains either in a round sphere in $\mathbb{R}^3$ or in the plane.

If $\omega \equiv 0$ and $C_1 \neq 0$, then $C_2 \equiv 0$, and the Gauss–Codazzi equations for the conformal Gauss map reduces to

$$\beta_{z\bar z} + e^{\beta} = 0, \qquad \varphi_{z\bar z} = 0.$$

The first of these equations is the Liouville equation and the second is the Laplace equation. These equations describe superminimal not totally umbilical surfaces.

Let us consider the case of a conformally minimal surface that is not superminimal. Locally, by changing the conformal parameter, we reduce everything to the case

$$\frac{1}{2}\langle Y_{zz}, Y_{zz}\rangle_{4,1} = C_1 C_2 = \frac{1}{2}\,.$$

Then the Gauss–Codazzi equations acquire the form of the four-particle Toda lattice

$$\beta_{z\bar z} + e^{\beta} - e^{-\beta}\cosh\varphi = 0, \qquad \varphi_{z\bar z} + e^{-\beta}\sinh\varphi = 0.$$

Appendix 3. Inverse scattering problem for the Dirac operator on the line and the trace formulae

Here, addressing mainly geometers, we present some facts needed to prove Theorem 13 and to introduce soliton spheres in § 5.4. This is a subset of the appendix to the electronic version of the paper [6] (which can be found on the Internet: see `http://arxiv.org/math.DG/9801022`), where Theorem 13 was originally proved.

The inverse scattering problem for the Dirac operator on the line was solved in [135] like the same problem for the Schrödinger operator $-\partial_x^2 + u(x)$ [136] (see also [137]).

We consider the following spectral problem (well known as the Zakharov–Shabat problem):

$$L\psi = \begin{pmatrix} 0 & ik \\ ik & 0 \end{pmatrix} \psi, \tag{68}$$

where

$$L = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix} + \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}. \tag{69}$$

We assume that the potentials p and q are rapidly decreasing as $x \to \pm\infty$. It is clear from the proofs that it suffices to assume that $p(x)$ and $q(x)$ decay exponentially.

If $p = q = 0$, then for each $k \in \mathbb{R} \setminus \{0\}$ we have a two-dimensional space of solutions (free waves) spanned by the columns of the matrix

$$\Phi_0(x,k) = \begin{pmatrix} 0 & e^{-ikx} \\ e^{ikx} & 0 \end{pmatrix}.$$

If p and q are non-trivial, then for any $k \in \mathbb{R} \setminus \{0\}$ we again have a two-dimensional space of solutions that behave asymptotically like free waves as $x \to \pm\infty$. These spaces are spanned by the so-called Jost functions $\varphi_l^\pm$, $l = 1, 2$. To define these functions, we consider the matrices $\Phi^+(x,k)$ and $\Phi^-(x,k)$ satisfying the integral equations

$$\Phi^+(x,k) = \Phi_0(x,k) + \int_x^{+\infty} \Phi_0(x - x', k) \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \Phi^+(x',k)\, dx',$$
$$\Phi^-(x,k) = \Phi_0(x,k) + \int_{-\infty}^{x} \Phi_0(x - x', k) \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \Phi^-(x',k)\, dx'.$$

These equations have the form $\Phi^\pm = \Phi_0 + A^\pm \Phi^\pm$, where $A^\pm$ are Volterra operators, and hence each equation has a unique solution given by the Neumann series $\Phi^\pm(x,k) = \sum_{l=0}^\infty (A^\pm)^l \Phi_0(x,k)$. The columns of the matrix $\Phi^\pm$ are the Jost functions $\varphi_l^\pm$, $l = 1, 2$. We see that, by construction, the Jost functions behave like free waves:

$$\varphi_1^\pm \approx \begin{pmatrix} 0 \\ e^{ikx} \end{pmatrix}, \qquad \varphi_2^\pm \approx \begin{pmatrix} e^{-ikx} \\ 0 \end{pmatrix} \quad \text{as} \quad x \to \pm\infty.$$

Straightforward computations show that for a pair of solutions $\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$ and $\tau = \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix}$ of the equation (68) the Wronskian $W = \theta_1\tau_2 - \theta_2\tau_1$ is constant; in particular, we have

$$\det \Phi^\pm(x,k) = -1. \tag{70}$$

In what follows we assume that the potentials p and q are complex conjugate:

$$p = \overline{q}.$$

One can also show by straightforward computations that the transformation

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \psi^* = \begin{pmatrix} -\bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix} \tag{71}$$

takes solutions of the equation (68) to solutions of the same equation. In particular, it follows from the asymptotics of the Jost functions that these functions are transformed as follows:

$$\varphi_1^{\pm} \xrightarrow{\ *\ } -\varphi_2^{\pm}, \qquad \varphi_2^{\pm} \xrightarrow{\ *\ } \varphi_1^{\pm}. \tag{72}$$

Since the Jost functions φ_l^+, $l = 1, 2$, and φ_l^-, $l = 1, 2$, give bases for the same space, they are connected by a linear transform

$$\begin{pmatrix} \varphi_1^- \\ \varphi_2^- \end{pmatrix} = S(k) \begin{pmatrix} \varphi_1^+ \\ \varphi_2^+ \end{pmatrix}.$$

It follows from (70) that $\det S(k) = 1$, and we can see from (72) that the *scattering matrix* $S(k)$ is unitary, $S(k) \in SU(2)$, that is,

$$S(k) = \begin{pmatrix} \overline{a(k)} & -\overline{b(k)} \\ b(k) & a(k) \end{pmatrix}, \qquad |a(k)|^2 + |b(k)|^2 = 1.$$

The respective quantities

$$T(k) = \frac{1}{a(k)}, \qquad R(k) = \frac{b(k)}{a(k)}$$

are called the *transmission coefficient* and the *reflection coefficient*. The operator L is said to be reflectionless if its reflection coefficient vanishes: $R(k) \equiv 0$.

The vector functions $\varphi_1^- e^{-ikx}$ and $\varphi_2^+ e^{ikx}$ can be analytically continued to the lower half-plane $\operatorname{Im} k < 0$, and the vector functions $\varphi_2^- e^{ikx}$ and $\varphi_1^+ e^{-ikx}$ can be analytically continued to the upper half-plane $\operatorname{Im} k > 0$.

Without loss of generality, it suffices to prove this fact for $\varphi_1^- e^{-ikx}$. This function satisfies the Volterra-type equation

$$f(x, k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \int_{-\infty}^{x} \begin{pmatrix} 0 & -e^{-2ik(x-x')} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} f(x', k)\, dx'$$

and, since the integral kernel decays exponentially for $\operatorname{Im} k < 0$, the Neumann series for the solution of the equation converges in this half-plane.

This implies that $T(k)$ can be analytically continued to the upper half-plane $\operatorname{Im} k \geqslant 0$.

It can be shown that:

a) $a(k)$ is nowhere zero on $\mathbb{R} \setminus \{0\}$;

b) the poles of $T(k)$ correspond to *bound states*, that is, to solutions of (68) that decay exponentially as $x \to \pm\infty$; these solutions are $\varphi_1^+(x, \varkappa)$ and $\varphi_2^-(x, \varkappa)$, where $a(\varkappa) = 0$, and hence

$$\varphi_2^-(x, \varkappa) = \mu(\varkappa)\varphi_1^+(x, \varkappa), \qquad \mu(\varkappa) \in \mathbb{C}, \tag{73}$$

and the multiplicity of each eigenvalue $\varkappa$ is equal to 1;

c) $T(k)$ has only simple poles in the half-plane $\operatorname{Im} k > 0$, and for exponentially decaying potentials there are only finitely many such poles;
d) since the set of solutions of the equation (68) is invariant under (71), the discrete spectrum of the operator L is preserved by the complex conjugation $\varkappa \to \overline{\varkappa}$ and is formed by the poles of $T(k)$ and their complex conjugates.

The *spectral data* of the operator L consist of:
1) the reflection coefficient $R(k)$, $k \neq 0$;
2) the poles $\varkappa_1, \dots, \varkappa_N$ of the function $T(k)$ in the upper-half plane $\operatorname{Im} \varkappa > 0$;
3) the quantities $\lambda_j = i\gamma_j\mu_j$, $j = 1, \dots, N$, where $\gamma_j = \gamma(\varkappa_j)$ is the residue of $T(k)$ at the pole $\varkappa_j$ and $\mu_j = \mu(\varkappa_j)$ (see (73)).

If the potential $p = \overline{q}$ is real-valued, then

$$\varphi_j^{\pm}(x, -k) = \overline{\varphi_j^{\pm}(x, k)} \quad \text{for} \quad k \in \mathbb{R} \setminus \{0\},$$

and this implies that

$$a(k) = \overline{a(-k)}, \qquad R(k) = \overline{R(-k)}, \qquad T(k) = \overline{T(-\overline{k})},$$

the poles of $T(k)$ are symmetric with respect to the imaginary axis, and

$$\lambda_j = \overline{\lambda}_k \quad \text{for} \quad \varkappa_j = -\overline{\varkappa}_k.$$

Let us now apply the Fourier transform (with respect to k) to the equality

$$T(k)\varphi_2^- = R(k)\varphi_1^+ + \varphi_2^+. \tag{74}$$

After some substitutions, we can represent the equations (74) for the components of the vector functions in the form of the Gel'fand–Levitan–Marchenko equations

$$B_2(x, y) + \int_x^{+\infty} B_1(x, x')\Omega(x' + y)\, dx' = 0,$$

$$\Omega(x + y) - B_1(x, y) + \int_x^{+\infty} B_2(x, x')\Omega(x' + y)\, dx' = 0$$

for B_1 and B_2, where

$$\Omega(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R(k)e^{-ikz}\, dk - \sum_{j=1}^{N} \lambda_j e^{i\varkappa_j z}$$

for $y > x$, and the following limits exist:

$$\lim_{y\to\infty} B_k(x, y) = 0, \quad \lim_{y\to x+} B_k(x, y) = B_k(x, x), \qquad k = 1, 2.$$

These are Volterra-type equations, and they can be solved uniquely. The recovery formulae for the potentials are

$$p(x) = -2B_1(x, x), \qquad p(x)q(x) = p(x)\overline{p(x)} = 2\frac{dB_2(x, x)}{dx}\,. \tag{75}$$

For the detailed derivation of these formulae from [135], see, for instance, [138].

In what follows, we assume for simplicity that the potential $p(x)$ takes only real values.

A series of formulae derived in [120] expresses the Kruskal integrals in terms of the spectral data; that is, these are the so-called *trace formulae*. We mention only the formula for the first non-trivial integral,

$$\int_{-\infty}^{\infty} p^2(x)\,dx = -\frac{1}{\pi}\int_{-\infty}^{\infty}\log(1-|b(k)|^2)\,dk + 4\sum_{j=1}^{N}\operatorname{Im}\varkappa_j. \tag{76}$$

For reflectionless operators the recovery procedure reduces to algebraic equations (for details, see, for instance, [135], [138], [6]). The spectral data consist of the poles $\varkappa_k$ and the corresponding quantities λ_j, $j=1,\dots,N$. Let

$$\Psi(x) = (-\lambda_1 e^{i\varkappa_1 x},\dots,-\lambda_N e^{i\varkappa_N x}),$$
$$M_{jk}(x) = \frac{\lambda_k}{i(\varkappa_j+\varkappa_k)}e^{i(\varkappa_j+\varkappa_k)x}, \qquad j,k=1,\dots,N.$$

We have

$$\begin{aligned} p(x) &= 2\frac{d}{dx}\arctan\frac{\operatorname{Im}\det(1+iM(x))}{\operatorname{Re}\det(1+iM(x))}, \\ \varphi_1^+(x,k) &= \begin{pmatrix} \langle\Psi(x)\cdot(1+M^2(x))^{-1}|W(x,k)\rangle \\ e^{ikx} - \langle\Psi(x)\cdot(1+M^2(x))^{-1}M(x)|W(x,k)\rangle \end{pmatrix}, \end{aligned} \tag{77}$$

where $\langle u|v\rangle = u_1v_1+\dots+u_Nv_N$ and

$$W(x,k) = \left(\frac{i}{\varkappa_1+k}e^{i(\varkappa_1+k)x},\dots,\frac{i}{\varkappa_N+k}e^{i(\varkappa_N+k)x}\right).$$

Concluding remark. This research was started and a significant part of it was written during a stay of the author in the Max-Planck-Institut für Mathematik between October 2003 and January 2004, and the final proofreading was carried out in the same institute in December 2005.

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