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Existence, uniqueness, and stability of best and near-best approximations

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Abstract. Existence and stability of ε-selections (selections of operators of near-best approximation) are studied. Results relating the existence of continuous ε-selections with other approximative and structural properties of approximating sets are given. Both abstract and concrete sets are considered — the latter include n-link piecewise linear functions, n-link r-polynomial functions and their generalizations, k-monotone functions, and generalized rational functions. Classical problems of the existence, uniqueness, and stability of best and near-best generalized rational approximations are considered.

Bibliography: 70 titles.

Keywords: generalized rational functions, ε-selection, near-best approximant, sun, monotone path-connected set, stability of approximation, piecewise-polynomial function.

Contents

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1. Introduction

The most important problems in geometric approximation theory are those dealing with the existence, uniqueness, and stability of best (or near-best) approximants, as well as problems of solarity and protosolarity. Stability of best approximation or the lack thereof are fundamental properties of approximating sets, which are related directly to their structural characteristics, from which, in turn, one can frequently derive other approximative properties. The stability of best approximation depends, *inter alia*, on the existence of continuous, uniformly continuous, or even smooth ε-selections to the approximating set (the corresponding definitions are given below in § 3). So it is important to see which structural characteristics of abstract approximating sets imply such stability or the lack of it. In this way the approximative properties of concrete objects of approximation can be ascertained. In the present paper we try to give an account of the stability of ε-selections, and demonstrate their relation to other approximative properties of some classical objects of approximation, among which we consider, in particular, algebraic rational functions, generalized rational functions, piecewise polynomial functions, and so on (some approximative properties of such sets are considered, for example, in the books [15], [46], [14], [38], and [5]). The above problems are considered both in normed linear spaces and in more general spaces with asymmetric norm or seminorm.

The paper is organized as follows. In § 2 basic definitions of the theory of asymmetric spaces and generalizations thereof are given. In § 3 we present general theorems on the existence of continuous selections to the set of near-best approximants. We also give results on the existence of continuous selections relative to cost functionals and their families. In § 4 we provide some results on the existence of continuous selections in normed and asymmetric normed spaces. The following examples of sets admitting continuous ε-selections for all $\varepsilon > 0$ are considered: n-link piecewise linear functions, n-link r-polynomial functions and their generalizations, k-monotone functions, and generalized rational functions. Some classical problems (the existence, uniqueness, stability, and characterization of best approximants) of generalized rational approximation are outlined in § 5. Problems of the existence of continuous selections to the set of generalized rational functions in L^p-spaces, $0 < p < \infty$, are considered in § 6. In § 7 some results on uniformly continuous selections to some classes of generalized rational functions in spaces of continuous functions are provided.

2. Spaces with asymmetric distance and their generalizations

In what follows, in parallel with usual normed and seminormed spaces, we consider spaces equipped with asymmetric (semi)norm, spaces with asymmetric (semi)metric, and spaces X equipped with a cost function $G\colon X \times X \to \mathbb{R}$.

An *asymmetric norm* $\|\cdot|$ on a real linear space X is defined by the following axioms:

1) $\|\alpha x| = \alpha\|x|$ for all $\alpha \geqslant 0$, $x \in X$;
2) $\|x + y| \leqslant \|x| + \|y|$ for all $x, y \in X$;
3) $\|x| \geqslant 0$ for all $x \in X$, and
3a) $\|x| = 0 \Leftrightarrow x = 0$.

The class of asymmetric spaces, which is an important and useful extension of the class of normed linear spaces, has numerous applications to problems of approximation, variational calculus, computer science, and mathematical economics. A functional $\|\cdot|$ satisfying axioms 1)–3) is called an *asymmetric seminorm*.

The theory of asymmetric normed spaces and their applications is in active development at the present time. For example, various topological and functional-analytic topics are considered in [17], [18], and [24], optimal location problems (with asymmetric norms) are studied, for instance, in [25], [47], and [43] (in such problems, an important role is also played by Chebyshev centres and Chebyshev nets relative to the asymmetric norms), and problems related to principal component analysis in statistics (one of the most popular methods of compact representation of data) are dealt with in [56]. For other applications, also see [17]. In geometric approximation theory, asymmetric norms proved to be useful in many problems (see, for example, [4], [6], [27], [29], and [68]). Asymmetric distances also appear naturally in approximation theory (in particular, in problems of best one-sided approximations), serving as a 'bridge' between best approximation and best one-sided approximation. For a survey of some results on the general theory of asymmetric normed subspaces and the problem of characterization of best approximants by convex sets in such spaces, we refer the reader to [18], [17] and [1].

Definition 2.1. A function $\rho\colon X\times X\to\mathbb{R}_+$ is called an *asymmetric semimetric* on a set X if:

1) $\rho(x,x)=0$ for all $x\in X$;

2) $\rho(x,z)\leqslant\rho(x,y)+\rho(y,z)$ for all $x,y,z\in X$.

The pair (X,ρ) is called an *asymmetric semimetric space*. The function

$$\sigma(x,y):=\max\{\rho(x,y),\rho(y,x)\}$$

is known as the *symmetrization semimetric*.

A space (X,ρ) is *complete* if it is complete with respect to the symmetrization semimetric σ.

Below, approximative and geometric properties of sets will also be considered relative to a general cost function

$$G\colon X\times X\to\mathbb{R};$$

an asymmetric (semi)norm $\|\cdot|$ on a linear space X is a particular case of $G(\,\cdot\,,\,\cdot\,)$.

Definition 2.2. Let (X,ρ) be a semimetric space. A mapping $G\colon X\times X\to\mathbb{R}$ is a *cost function* if $G\geqslant 0$ and $G(x,x)=0$ for all $x\in X$.

Definition 2.3. Let (X,ρ) be a semimetric space, $A\subset X$, and let $G\colon X\times X\to\mathbb{R}$ be a continuous (relative to the semimetric ρ) cost function. We set

$$\rho_G(x,A)=\inf_{z\in A}G(x,z), \tag{2.1}$$

$$\begin{aligned} B_G(x,r)&=\{y\in X\mid G(x,y)\leqslant r\},\\ \mathring{B}_G(x,r)&=\{y\in X\mid G(x,y)< r\}\end{aligned}\qquad (x\in X,\ r\in\mathbb{R}_+).$$

In the particular case where $G(x,y)=\|x-y|$, we have

$$B(x,r):=B_{\|\cdot|}(x,r)=\{y\in X\mid \|y-x|\leqslant r\}$$

and

$$\mathring{B}(x,r):=\mathring{B}_{\|\cdot|}(x,r)=\{y\in X\mid \|y-x|< r\};$$

$B(x,r)$ is a closed ball, and $\mathring{B}(x,r)$ is an open ball in a seminormed (or asymmetric seminormed) linear space $X=(X,\|\cdot\|)$ (respectively, $X=(X,\|\cdot|)$) with centre x and radius r.

Remark 2.1. Below the subscript G ($\|\cdot|$, or $\|\cdot\|$) is used to denote the cost function (or the asymmetric (semi)norm $\|\cdot|$, which is a particular case of a cost function) with respect to which the approximative and geometric properties of sets are studied. The subscript G (or $\|\cdot|$) is dropped if the context makes it clear which cost function G (or asymmetric (semi)norm $\|\cdot|$) is considered.

In what follows we work mostly with symmetric and asymmetric (semi)norms on X. We say that the original (semi)norm $\|\cdot\|$ is *equivalent* to an asymmetric norm $\|\cdot|$ if the topology generated by the open balls $\mathring{B}_{\|\cdot|}(x,r)$ coincides with that generated by the (semi)norm $\|\cdot\|$.

Given a subset M of an asymmetric (semi)normed space X, we let P_M and P_M^ε denote, respectively, the metric projection operator and the ε-metric projection operator defined by

$$\begin{aligned} P_Mx&:=\{y\in M\mid \|y-x|=\rho(x,M)\},\\ P_M^\varepsilon x&:=\{y\in M\mid \|y-x|\leqslant\rho(x,M)+\varepsilon\},\\ \mathring{P}_M^\varepsilon x&:=\{y\in M\mid \|y-x|<\rho(x,M)+\varepsilon\},\end{aligned} \tag{2.2}$$

where $\rho(y,M):=\inf_{z\in M}\|z-y|$ is the distance of $y\in X$ to M.

In a semimetric space (X,ρ), the distance of a point $y\in X$ to a non-empty set $M\subset X$ is defined by $\rho(y,M):=\inf_{z\in M}\rho(y,z)$. It is easily checked that for all $x,y\in X$:

1) $\rho(x,M)\leqslant\rho(y,M)+\rho(x,y)$;

2) $|\rho(x,M)-\rho(y,M)|\leqslant\max\{\rho(x,y),\rho(y,x)\}$.

The structural and topological properties of the sets of near-best approximants (2.2) are now actively studied; results of this kind also have applications to numerical mathematics. In addition to the papers cited above, we mention the following ones: [3], [7], [23], [34], [42], [62]–[67], and [70].

3. Definition of a continuous ε-selection. General theorems on continuous selections. Continuous selections relative to cost functionals and their families

It is well known (Stechkin, Cline, Maehly, Witzgall, Werner; see, for example, Remarks 2.6 and 15.17 in [5]) that in $C[a,b]$ the metric projection operator onto any finite-dimensional Chebyshev subspace of dimension $\geqslant 2$ is not stable (not uniformly continuous on $C[a,b]$ and not Lipschitz-continuous), and the metric projection onto the set of rational functions is discontinuous [39]. In this regard it is natural to study the stability properties of the operators of near-best approximation (ε-selections).

In this section we examine sets admitting ε-selection (that is, selections to the set of near-best approximants) for all $\varepsilon > 0$, and also selections of the metric projection operator (0-selections). We consider the structural and approximative properties of such sets. In particular, we characterize the closed subsets of Banach subspaces (and, more generally, complete asymmetric symmetrizable seminormed subspaces) which admit continuous ε-selections for any $\varepsilon > 0$. We also present some results on the existence of continuous ε-selections in the case where the sets (Theorem 4.3) and even the (semi)norms on the ambient space vary continuously (Theorem 3.2).

Definition 3.1. Let $G\colon X\times X\to\mathbb{R}$ be a continuous cost function on a semimetric space (X,ρ), and let $\varepsilon>0$ and $M\subset X$.

We say that $\varphi\colon X\to M$ is an *additive* (*multiplicative*) *ε-selection* (of the operator of near-best approximation) if, for all $x\in X$,

$$G(x,\varphi(x))\leqslant \rho_G(x,M)+\varepsilon$$

($G(x,\varphi(x))\leqslant (1+\varepsilon)\rho_G(x,M)$, respectively), where the G-distance $\rho_G(x,M)$ is defined as in (2.1).

The asymmetric norm $\|x-y|$ (or $\|y-x|$) is a particular case of a cost function $G(x,y)$.

For (semi)normed subspaces Definition 3.1 assumes the following form.

Definition 3.2. Let $\varepsilon>0$ and $M\subset X$. A mapping $\varphi\colon X\to M$ is an *additive* (*multiplicative*) *ε-selection* if, for all $x\in X$,

$$\|\varphi(x)-x\|\leqslant \rho(x,M)+\varepsilon$$
$$(\|\varphi(x)-x\|\leqslant (1+\varepsilon)\rho(x,M),\quad \text{respectively}).$$

Geometrically, these inequalities mean that, for all $x\in X$,

$$\varphi(x)\in B(x,\rho(x,M)+\varepsilon)\cap M=P_M^{\varepsilon}x$$
$$(\text{respectively}\ \ \varphi(x)\in B(x,(1+\varepsilon)\rho(x,M))\cap M=P_M^{\varepsilon\rho(x,M)}x).$$

The operator of near-best approximation was introduced by Wulbert. The stability of near-best approximation operators was studied by Berdyshev [12], [13], Liskovets, Marinov, Morozov, Wegmann, and others (see, for example, [40], [50]) with applications, in particular, to ill-posed problems. Various results on sets admitting stable ε-selections and on the stability of minimization problems can be found in

[3], [32], [8]–[11], [36], [52], [53], [60], [69], and [70]. The book [48] by Repovš and Semenov is also worth mentioning, which surveys results on continuous selections of set-valued mappings.

It is well known that the metric projection onto the (Chebyshev) set of classical rational functions $\mathscr{R}_{n,m}$, $m \geqslant 1$, is discontinuous in $C[0,1]$ (see (5.2) below), but according to Konyagin [32], for any $\varepsilon > 0$ there exists a continuous ε-selection to $\mathscr{R}_{n,m}$. Another example of a non-convex set admitting a continuous additive (multiplicative) ε-selection for all $\varepsilon > 0$ is the unit sphere of an arbitrary infinite-dimensional normed space. Note also that an arbitrary convex set has an additive ε-selection for any $\varepsilon > 0$, and if this set is also closed, then it also admits a multiplicative ε-selection. That there exists a continuous ε-selection to any subspace of a Banach space is a direct corollary of the classical Michael continuous selection theorem (see, for example, § 16.9 in [5]). Results due to Berdyshev and Marinov [40] and Al'brekht show that any finite-dimensional subspace admits an additive Lipschitz ε-selection with Lipschitz constant ε^{-1}. The differential properties of ε-selections to subspaces were studied by Al'brekht, who also found the order of the Lipschitz constants of ε-selections with respect to ε (as $\varepsilon \to 0$) in some classical spaces (see, for example, [50]). These results show that there are ε-selections which are more smooth than the metric projection operator. Tsar'kov showed that there exists a bounded convex closed set M in $C[0,1]$ and a number $\varepsilon > 0$ such that, for any $\delta > 0$, there is no uniformly continuous additive ε-selection to M in the neighbourhood $\mathscr{O}_\delta(M) = \{x \mid \rho(x,M) \leqslant \delta\}$.

Definition 3.3. A subset A of a (symmetric or asymmetric) semimetric space (X,ν) is called *infinitely connected* if for all $n \in \mathbb{N}$, any continuous mapping $\varphi\colon \operatorname{bd} B \to A$ of the boundary $\operatorname{bd} B$ of the unit ball $B \subset \mathbb{R}^n$ has a continuous extension $\widetilde{\varphi}\colon B \to A$ to the whole of the unit ball B (here and in what follows $\operatorname{bd} A$ is the boundary of A).

If Q denotes some property (for example, connectedness), then we say that a closed set M has the property

- P-Q if, for all $x \in X$, the set $P_M x$ is non-empty and has property Q;
- P_0-Q if $P_M x$ has property Q for all $x \in X$;
- B-Q if $M \cap B(x,r)$ has property Q for all $x \in X$ and $r > 0$;
- $\mathring{B}$-Q if $M \cap \mathring{B}(x,r)$ has property Q for all $x \in X$ and $r > 0$.

For example, a set $M \subset X$ is $\mathring{B}$-*infinitely connected* if its intersection with any open ball is infinitely connected.

Definition 3.4. Let $G\colon X \times X \to \mathbb{R}$ be a continuous cost function on a semimetric space (X,ρ). A set $M \subset X$ is called $\mathring{B}_G$-*infinitely connected* if, for all $x \in X$ and $r \in \mathbb{R}_+$, the set $\mathring{B}_G(x,r) \cap M$ is infinitely connected.

As above, the subscript G (or $\|\cdot|$) can be omitted when the cost function G (or asymmetric (semi)norm $\|\cdot|$) considered on X is clear from the context.

Remark 3.1. A $\mathring{B}$-infinitely connected set can fail to be B-infinitely connected, that is, its intersection with some closed ball can fail to be infinitely connected (and even connected; see Theorem 5 in [57]). Note also that the set of generalized rational functions in $C(Q)$ is $\mathring{B}$-infinitely connected (and even B-infinitely connected); see, for example, Theorem 5.2 below.

Definition 3.5. Let (X,ρ) be a semimetric space. A mapping $\vartheta\colon X\to\overline{\mathbb{R}}$ is *lower semicontinuous* on X if

$$\varliminf_{n\to\infty}\vartheta(x_n)\geqslant\vartheta(x)$$

for any point $x\in X$ and any sequence $(x_n)\subset X$ such that $\rho(x,x_n)\to 0$ as $n\to\infty$; this is equivalent to the condition

$$\forall\,\varepsilon>0\ \exists\,\delta>0\ \forall\,x'\in X\quad(\rho(x,x')<\delta\quad\Rightarrow\quad\vartheta(x')>\vartheta(x)-\varepsilon).$$

Theorem 3.1 (see [63]). *Let $(X,\|\cdot\|)$ be a seminormed linear space, $G\colon X\times X\to\mathbb{R}$ be a cost function uniformly continuous on any bounded set, and let $M\subset X$ be $\mathring{B}_G$-infinitely connected. Then for any lower semicontinuous function $\psi\colon X\to\overline{\mathbb{R}}$ such that $\rho_G(x,M)<\psi(x)$, $x\in X$, there exists a mapping $\varphi\in C(X,M)$ such that*

$$G(x,\varphi(x))<\psi(x)\qquad(x\in X).$$

In a similar way, given any non-empty open set $D\subset X$ and a lower semicontinuous function $\psi\colon D\to\overline{\mathbb{R}}$ such that $\rho_G(x,M)<\psi(x)$ $(x\in D)$, there exists a mapping $\varphi\in C(D,M)$ such that $G(x,\varphi(x))<\psi(x)$ $(x\in D)$.

Definition 3.6. Let (X,ρ) and (Y,g) be semimetric spaces, let $G\in C(Y\times X\times X,\mathbb{R})$, and let $\mathscr{G}=\{G_y=G(y,\,\cdot\,,\,\cdot\,)\}$ be a family of continuous cost functions. Set

$$\rho_{\mathscr{G}}(y,x,M):=\inf_{z\in M}G(y,x,z),\qquad y\in Y,\quad x\in X.$$

A mapping $F\colon Y\to 2^X$ is *$\mathscr{G}$-stable* (see [65]) if:

1) $F(y)\neq\varnothing$ for all $y\in Y$;
2) the function $\pi(x,y)=\rho_{\mathscr{G}}(y,x,F(y))\colon X\times Y\to R$ is continuous on $X\times Y$;
3) the modulus of $\mathscr{G}$-stability

$$\omega_E^{\mathscr{G}}(F,y_0,\delta):=\sup_{\substack{(y,x)\in E\\ \gamma(y_0,y_2)\leqslant\delta}}|\rho_{\mathscr{G}}(y,x,F(y_0))-\rho_{\mathscr{G}}(y,x,F(y_2))|,$$

where $E\subset Y\times X$ is an arbitrary bounded set and $y_0\in Y$, tends to zero as $\delta\to 0+$.

In Theorem 3.2 we prove the existence of a continuous ε-selection in the case where both the set and the norm of the space containing this set change continuously.

Theorem 3.2 (see [63]). *Let $(X,\|\cdot\|)$ be a complete seminormed linear space and $(Y,\|\cdot\|)$ be a seminormed linear space, G be a real-valued function uniformly continuous on every bounded subset of $Y\times X\times X$, let $\mathscr{G}=\{G_y=G(y,\,\cdot\,,\,\cdot\,)\}=\{\|\cdot-\cdot\|_y\}$, where $\{\|\cdot\|_y\}$ is a family of seminorms on X, each of which is equivalent to the original seminorm $\|\cdot\|$, and let $F\colon Y\to 2^X$ be $\mathscr{G}$-stable, and for all $y\in Y$ assume that the set $M_y=F(y)$ is closed and $\mathring{B}_{G_y}$-infinitely connected (that is, is $\mathring{B}$-infinitely connected with respect to the semi-norm $\|\cdot\|_y$).*

Then there exists a mapping $f\in C(X\times Y\times(0,+\infty);X)$ such that $\varphi(\,\cdot\,)=f(\,\cdot\,,y,\varepsilon)$ is a continuous additive (multiplicative) ε-selection to the set $M_y=F(y)$ in the space $X_y=(X,\|\cdot\|_y)$, that is,

$$\|x-\varphi(x)\|_y\leqslant\rho_{\mathscr{G}}(y,x,F(y))+\varepsilon\|x-\varphi(x)\|_y\leqslant(1+\varepsilon)\rho_{\mathscr{G}}(y,x,F(y)),\quad x\in X.$$

Definition 3.7. Let (X,θ) be a semimetric space. We say that a set $M \subset X$ *possesses the* UV^ℓ*-property* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that each continuous mapping f of the unit sphere $S^{\ell-1} \subset \mathbb{R}^\ell$ to the δ-neighbourhood of M admits a continuous extension $f_0\colon B^\ell \to O_\varepsilon(M)$ to the unit ball $B^\ell \subset \mathbb{R}^\ell$ (that is, $f_0\big|_{S^{\ell-1}} \equiv f$). A set M *has the* UV^∞*-property* if it has the UV^ℓ-property for all $\ell \in \mathbb{N}$.

Below, $\mathrm{AC}(M)$ is the set of points of *approximative compactness* for M (that is, the set of all points x such that in any sequence $(y_n)_{n\in\mathbb{N}} \subset M$ satisfying $\|x-y_n\| \to \rho(x,M)$ there is a subsequence converging to some point in M).

Theorem 3.3 (see [67]). *Let M be a $\mathring{B}$-infinitely connected subset of a Banach space X, and let $x \in \mathrm{AC}(M)$. Then the set P_Mx has the UV^∞-property, and, in addition, P_Mx is a cell-like compact set.*

Definition 3.8 (see [67]). A subset M of a normed linear space X is called *stably monotone path-connected* if there exists a continuous mapping $p\colon M\times M\times[0,1] \to M$ such that $p(x,y,\,\cdot\,)$ is a monotone path connecting the points $x, y \in M$.

Remark 3.2. The set of generalized rational functions $\mathscr{R}_U$ (with convex U; see (5.3) below) in $C(Q)$ is stably monotone path-connected and therefore admits a continuous additive ε-selection for any $\varepsilon > 0$ (see Theorem 2.3 in [8]).

According to [60], any approximatively compact monotone path-connected set admits a continuous additive (multiplicative) ε-selection for all $\varepsilon > 0$ and therefore is $\mathring{B}$-infinitely connected. Hence for any point $x \in X$ the set P_Mx is $\mathring{B}$-infinitely connected. It was shown in [62] that a stably monotone path-connected set (and therefore its intersection with any open or closed ball) is $\mathring{B}$-infinitely connected, and even $\mathring{B}$-contractible (B-contractible). Now, by Theorem 4.4 (see below) we have the following result.

Corollary 3.1 (see [64]). *Let X be a Banach space and $M \subset X$ be a stably monotone path-connected set* (*or P-compact monotone path-connected set*) *with Hausdorff continuous metric projection. Then M admits a continuous selection of the metric projection operator.*

Let us mention some results on the structural properties of sets in a Hilbert space.

Definition 3.9. Let $n \geqslant 2$. A set $M \subset X$ is $\mathring{B}^n$*-infinitely connected* if its intersection with any open ball is either empty or $\mathring{B}^{n-1}$-infinitely connected. By definition, a set is $\mathring{B}^1$*-infinitely connected* if it is $\mathring{B}$-infinitely connected.

The following results hold (see [61]).

Theorem 3.4. *In a Hilbert space the classes of all $\mathring{B}$-infinitely connected sets and all $\mathring{B}^n$-infinitely connected sets coincide.*

Theorem 3.5. *In a Hilbert space the intersection of a $\mathring{B}$-infinitely connected set with a finite number of open balls is $\mathring{B}$-infinitely connected.*

4. Continuous selections in normed and asymmetric normed spaces

4.1. Sufficient and necessary conditions for the existence of continuous 0-selections and ε-selections. We explore which structural and approximative properties of a given set guarantee the existence of a continuous selection to this set. In particular, we present conditions on the structure of the set of best approximants (or on the set itself, or on the set of near-best approximants) guaranteeing the existence of such a selection. We also characterize the sets admitting continuous ε-selections for all $\varepsilon > 0$.

The following result is a consequence of Theorem 5 in [65].

Theorem 4.1. *Let $X = (X, \|\cdot|)$ be an asymmetric normed space, and let $M \subset X$ be $\mathring{B}$-infinitely connected. Then for any lower semicontinuous function $\psi\colon X \to \overline{\mathbb{R}}$ such that $\rho(x, M) < \psi(x)$, $x \in X$, there exists a continuous mapping $\varphi\colon X \to M$ (the preimage is equipped with the symmetrization topology and the range, with the original asymmetric topology) such that $\|\varphi(x) - x| < \psi(x)$ $(x \in X)$.*

Definition 4.1. Let (Y, ρ) and (X, θ) be semimetric spaces. A mapping $F\colon Y \to 2^X$ is said to be *stable* if

$$\omega_E(F, y_0, \delta) := \sup_{\substack{x \in E \\ \rho(y_0, y) \leqslant \delta}} |\rho(x, F(y_0)) - \rho(x, F(y))| \to 0 \qquad (\delta \to 0+)$$

for each $y_0 \in Y$ and an arbitrary bounded set $E \subset X$.

The following result was proved in [63].

Theorem 4.2. *Let $(X, \|\cdot\|)$ be a complete seminormed linear space, $(Y, \|\cdot\|)$ be a seminormed linear space, and let the mapping $F\colon Y \to 2^X$ be stable, and, for all $y \in Y$, the set $M_y = F(y)$ be closed and $\mathring{B}$-infinitely connected. Then there exists a mapping $f \in C(X \times Y \times (0, +\infty); X)$ such that $f(\,\cdot\,, y, \varepsilon)$ is a continuous additive (multiplicative) ε-selection of $M_y = F(y)$ in $(X, \|\cdot\|)$.*

Definition 4.2. Let (X, ρ) and (Y, γ) be semimetric spaces, let $M \subset X$, and let $G\colon X \times X \to \mathbb{R}$ be a continuous cost function. A mapping $F\colon Y \to 2^M$ is said to be *G-stable* if:

1) $F(y) \neq \varnothing$ for all $y \in Y$;

2) the function

$$\pi(x, y) = \rho_G(x, F(y))\colon X \times Y \to \mathbb{R}$$

is continuous on $X \times Y$;

3) the modulus of G-stability

$$\omega^G(F, y_0, \delta) = \omega^G_E(F, y_0, \delta) := \sup_{\substack{x \in E \\ \gamma(y_0, y) \leqslant \delta}} |\rho_G(x, F(y_0)) - \rho_G(x, F(y))|,$$

where $E \subset X$ is an arbitrary bounded set and $y_0 \in Y$, tends to zero as $\delta \to 0+$.

Theorem 4.3 (see [64]). *Let $(X, \|\cdot|)$ be an asymmetric seminormed space whose asymmetric seminorm is equivalent to some seminorm on X with respect to which X is complete, let $F\colon X \to 2^X$ be a $\|\cdot|$-stable mapping, and let, for all $y \in X$, the* *set* $M_y = F(y)$ *be closed and* $\mathring{B}$*-infinitely connected* (*that is,* $\mathring{B}$*-infinitely connected relative to the seminorm* $\|\cdot|$).

Then there exists a mapping $f \in C(X \times X \times (0,+\infty); X)$ *such that* $\varphi(\,\cdot\,) = f(\,\cdot\,, y, \varepsilon)$ *is a continuous additive* (*multiplicative*) ε*-selection of* $M_y = F(y)$ *in* $X = (X, \|\cdot|)$, *that is,*

$$\|\varphi(x)-x| \leqslant \rho_G(x,F(y))+\varepsilon \quad (\textit{respectively}\ ,(\|\varphi(x)-x| \leqslant (1+\varepsilon)\rho_G(x,F(y)))\ \textit{on}\ X.$$

Definition 4.3. Given non-empty sets $M, N \subset Y$, we denote by

$$d(M,N) := \sup_{y\in N} \rho(y,M)$$

the *directed* (*one-sided*) *Hausdorff distance* between M and N; the *Hausdorff distance* is defined by

$$h(M,N) := \max\{d(M,N), d(N,M)\}.$$

Recall that a mapping $F\colon X \to 2^Y$ between the metric spaces X and Y is said to be *Hausdorff continuous* (*upper semicontinuous*) if

$$h(F(x), F(x_n)) \to 0 \quad (d(F(x), F(x_n)) \to 0) \quad \text{as} \quad x_n \to x \quad (n \to \infty).$$

Theorem 4.4 (see [64]). *Let* X *be a Banach space and* $M \subset X$ *be an existence set such that the metric projection operator onto* M *is Hausdorff continuous and has* $\mathring{B}$*-infinitely connected values. Then* M *is* P*-contractible and has continuous selection of the metric projection operator.*

The ε-neighbourhood of a set $E \subset X$ is defined by

$$\{x \in X \mid \rho(x,E) < \varepsilon\}.$$

Theorem 4.5 (see [64]). *Let* $(X, \|\cdot|)$ *be an asymmetric seminormed space whose asymmetric seminorm* $\|\cdot|$ *is equivalent to some seminorm of* X, *and let a non-empty set* $A \subset X$ *be* $\mathring{B}$*-infinitely connected. Then the set* A *admits a continuous additive* ε*-selection for any* $\varepsilon > 0$, *and any* r*-neighbourhood* $(r > 0)$ *of* A *is* $\mathring{B}$*-infinitely connected.*

Definition 4.4. A compact set Y is said to be *cell-like* if there exists an absolute neighbourhood retract Z and an embedding $i\colon Y \to Z$ such that the range $i(Y)$ is contractible in each neighbourhood $U \subset Z$ of itself.

Definition 4.5. Let $G\colon X \times X \to \mathbb{R}$ be a cost function. A set $M \subset X$ is B*-cell-like* (respectively, P*-cell-like*) if its intersection with any closed ball is either empty or cell-like (respectively if $P_M x$ is cell-like for all $x \in X$) relative to the function G.

Given a cost function $G(x,y) = \|y-x|$, we write $P_{\|\cdot|}$ and $B_{\|\cdot|}$ or simply P and B for P_G and B_G, respectively, if the (asymmetric) seminorm $\|\cdot|$ is clear from the context.

Definition 4.6. A subset A of a semimetric space (Y, ν) is called *approximatively infinitely connected* (see § 7 in [3]) if, for any $n \in \mathbb{N}$, any continuous mapping $\varphi\colon \operatorname{bd} B \to A$ of the boundary of the unit ball $B \subset \mathbb{R}^n$, and any $\varepsilon > 0$ there exists an ε-extension of φ to B, that is, there exists a continuous mapping $\varphi_\varepsilon\colon B \to A$ such that $\|\varphi(x) - \varphi_\varepsilon(x)\| < \varepsilon$ $(x \in \operatorname{bd} B)$.

Theorem 4.6 (see [64]). *Let $(X, \|\cdot|)$ be an asymmetric seminormed linear space whose asymmetric seminorm $\|\cdot|$ is equivalent to some seminorm $\|\cdot\|$ with respect to which $(X, \|\cdot\|)$ is complete, and let $M \subset X$ be $\mathring{B}$-approximatively infinitely connected and closed. Then M is $\mathring{B}$-infinitely connected.*

Theorem 4.7 (see [64]). *Let $(X, \|\cdot|)$ be an asymmetric seminormed linear space whose asymmetric seminorm $\|\cdot|$ is equivalent to some seminorm $\|\cdot\|$ with respect to which the space $(X, \|\cdot\|)$ is complete, and let $M \subset X$ be a P-cell-like existence set with Hausdorff upper semicontinuous metric projection. Then the set M is $\mathring{B}$-infinitely connected and admits a $\|\cdot|$-continuous additive (multiplicative) ε-selection for any $\varepsilon > 0$.*

Theorem 4.8 (see [64]). *Let $(X, \|\cdot|)$ be an asymmetric seminormed linear space whose asymmetric seminorm $\|\cdot|$ is equivalent to some seminorm $\|\cdot\|$ with respect to which the space $(X, \|\cdot\|)$ is complete, and let $M \subset X$ be closed and admit, for any $\varepsilon > 0$, a continuous additive (multiplicative) ε-selection. Assume that $M \cap \mathring{B}(x_0, R) \neq \varnothing$. Then the set $M \cap \mathring{B}(x_0, R)$ is a retract of the ball $\mathring{B}(x_0, R)$.*

Theorem 4.9 (see [64]). *Let $(X, \|\cdot|)$ be an asymmetric seminormed linear space whose asymmetric seminorm $\|\cdot|$ is equivalent to some seminorm $\|\cdot\|$ with respect to which the space $(X, \|\cdot\|)$ is complete, and let $M \subset X$ be $\mathring{B}$-infinitely connected and closed. Then M is $\mathring{B}$-contractible.*

Theorem 4.10 (see [64]). *Let $(X, \|\cdot|)$ be an asymmetric seminormed finite-dimensional space whose asymmetric seminorm $\|\cdot|$ is equivalent to some seminorm $\|\cdot\|$, and let $M \subset X$ be an existence set with $\|\cdot|$-lower semicontinuous metric projection. Then M is B-cell-like.*

Definition 4.7. An seminormed space $X = (X, \|\cdot|)$ is called *symmetrizable* if there exists a number $K \geqslant 1$ such that

$$\frac{1}{K}\|x\| \leqslant \|x| \leqslant \|x\| \quad \text{for all} \quad x \in X,$$

where $\|x\| := \max\{\|x|, \|-x|\}$.

Theorem 4.11 (characterization of sets admitting a continuous ε-selection for all $\varepsilon > 0$; see [65]). *Let $(X, \|\cdot|)$ be a complete symmetrizable asymmetric seminormed linear space, and let $M \subset X$ be non-empty and closed.*

Then the following conditions are equivalent:

a) *for all $x \in X$ and $\delta > 0$ the set $\mathring{P}^{\delta}_M x$ is a retract of a ball*;

b) *for all $x \in X$ and $\delta > 0$ the set $\mathring{P}^{\delta}_M x$ is contractible to a point*;

c) *M is $\mathring{B}$-infinitely connected*;

d) *for each $\varepsilon > 0$ there exists a continuous* (*additive multiplicative*) *ε-selection to M*;

e) *for each lower semicontinuous function $\psi\colon X \to (0, +\infty)$ such that $\psi(x) > \rho(x, M)$, $x \in X$, there exists a mapping $\varphi \in C(X, M)$ such that $\|\varphi(x) - x\| < \psi(x)$ for all $x \in X$*;

f) *for each lower semicontinuous function $\theta\colon X \to (1, +\infty)$ there exists a mapping $\varphi \in C(X, M)$ such that $\|\varphi(x) - x\| \leqslant \theta(x)\rho(x, M)$ for all $x \in X$.*

Example 4.1. In the space of all continuous functions on a compact set Q consider the asymmetric norm

$$\|f|_{\psi_+,\psi_-} := \max_{x\in Q}\left\{\frac{f_+}{\psi_+}, \frac{f_-}{\psi_-}\right\},$$

where

$$f_+ := \max\{f,0\} \quad \text{and} \quad f_- := \max\{-f,0\}$$

where $f \in C(Q,\mathbb{R})$, and ψ_+ and ψ_- are fixed positive continuous functions. Such norms were introduced in § 4 of [26]; solar and other approximative and geometric properties of sets in spaces with this norm, and also in more general spaces $C_{0;\psi_+,\psi_-}$, were studied in [2]. The asymmetric ball $B(0,R)$ consists of all functions f lying between the functions $R\psi_+$ and $-R\psi_-$, that is, satisfying $-R\psi_-(x) \leqslant f(x) \leqslant R\psi_+(x)$ for all $x \in Q$. Hence, the ball $B(g,R)$ consists of all f for which the function $f-g$ lies between $R\psi_+$ and $-R\psi_-$. The space of continuous functions on a compact set Q with asymmetric norm $\|\cdot|_{\psi_+,\psi_-}$ will be denoted by $C_{\psi_+,\psi_-}(Q)$.

We need the following simple inequality

$$\frac{b}{d} \leqslant \frac{a+b}{c+d} \leqslant \frac{a}{c},$$

where $c, d > 0$ and $b/d \leqslant a/c$. This inequality implies, in particular, that

$$\frac{b}{d} \leqslant \frac{\lambda a + (1-\lambda)b}{\lambda c + (1-\lambda)d} \leqslant \frac{a}{c}$$

for all $\lambda \in [0,1]$ under the same assumptions. Consequently, for any $w \in \mathbb{R}$,

$$\left|\frac{\lambda a + (1-\lambda)b}{\lambda c + (1-\lambda)d} - w\right| \leqslant \max\left\{\left|\frac{b}{d} - w\right|, \left|\frac{a}{c} - w\right|\right\}.$$

Given convex sets $U, V \subset C(Q,\mathbb{R})$, consider the set of generalized rational functions

$$\mathscr{R}_{U,V} = \left\{\frac{p(x)}{q(x)} \;\middle|\; p \in U,\ q \in V,\ q > 0\right\}.$$

The above inequality implies that

$$\frac{(1-\lambda)p_0(x) + \lambda p(x)}{(1-\lambda)q_0(x) + \lambda q(x)} \in \left[\frac{p_0(x)}{q_0(x)}, \frac{p(x)}{q(x)}\right]$$

for all $p_0(x)/q_0(x), p(x)/q(x) \in \mathscr{R}_{U,V}$ and $\lambda \in [0,1]$ (in this formula the left-hand endpoint of the interval is not necessarily smaller than the right-hand endpoint). It follows that, for all generalized rational functions $p_0(x)/q_0(x)$, $p(x)/q(x)$ in $\mathscr{R}_{U,V} \cap B(g,R)$ or $\mathscr{R}_{U,V} \cap \mathring{B}(g,R)$, the rational function

$$\frac{(1-\lambda)p_0(x) + \lambda p(x)}{(1-\lambda)q_0(x) + \lambda q(x)}, \qquad \lambda \in [0,1],$$

also lies in $\mathscr{R}_{U,V} \cap B(g,R)$ or $\mathscr{R}_{U,V} \cap \mathring{B}(g,R)$, respectively. Therefore, the set $\mathscr{R}_{U,V}$ is B-, and $\mathring{B}$-contractible in the asymmetric space $C_{\psi_+,\psi_-}(Q)$. Hence by Theorem 3.1 the set $\mathscr{R}_{U,V}$ admits a continuous additive ε-selection for all $\varepsilon > 0$.

In a similar way one can consider the set

$$\mathscr{P}_U = \{p(x)q(x) \mid p \in U,\ q > 0\}$$

(here U is an arbitrary convex subset of $C(Q)$). The set $\mathscr{P}_U$ is a subset of the set of generalized rational functions $\mathscr{R}_{U,V}$, namely

$$\mathscr{P}_U = \mathscr{R}_{U,V}, \quad \text{where } V = \left\{ \frac{1}{q} \in C_{\psi_+,\psi_-}(Q) \;\middle|\; q > 0 \right\}$$

(because $p(x)q(x) = p(x)/(1/q(x))$). This shows that the set $\mathscr{P}_U$ is B- and $\mathring{B}$-contractible in the asymmetric space $C_{\psi_+,\psi_-}(Q)$ and therefore, is $\mathring{B}$-infinitely connected. Hence $\mathscr{P}_U$ admits a continuous additive ε-selection for all $\varepsilon > 0$ (for further details, see [8] and § 5 below).

The following result is a direct consequence of Theorems 4.11 and 3.5.

Theorem 4.12. *Let the closed subset M of a Hilbert space admit a continuous additive ε-selection for any $\varepsilon > 0$. Then the intersection of M with any number of open balls (if non-empty) admits a continuous additive ε-selection for any $\varepsilon > 0$. The closure of this intersection admits a continuous multiplicative ε-selection for all $\varepsilon > 0$.*

4.2. Examples of sets admitting a continuous ε-selection for all $\varepsilon > 0$. The results in this section are due to Tsar'kov [66].

Definition 4.8. Let $n \in \mathbb{N}$, and let $K > 0$ and $a < b$. We denote by $S(n, K) = S(n, K, [a, b])$ the set of all *n-link K-Lipschitz piecewise linear functions* $s \in C[a, b]$ for which there exists a partition $T = \{t_i\}_{i=0}^{k}$, $1 \leqslant k \leqslant n$, of $[a, b]$ such that $s\big|_{\Delta_j} = A_j t + B_j$, where $\Delta_j = [t_{j-1}, t_j]$ are non-degenerate subintervals of the partition, and $A_j, B_j \in \mathbb{R}$, $|A_j| \leqslant K$, $j = 1, \dots, k$.

A continuous curve $k(\tau)$, $0 \leqslant \tau \leqslant 1$, in a normed linear space X is called *monotone* if $f(k(\tau))$ is a monotone function in τ for any $f \in \operatorname{ext} S^*$ (here and in what follows $\operatorname{ext} S^*$ is the set of extreme points of the dual unit sphere S^*).

Definition 4.9. A set $M \subset X$ is *monotone path-connected* if any two points in M can be connected by a continuous monotone curve (arc) $k(\,\cdot\,) \subset M$.

Note that any monotone path-connected set is always B-connected (that is, its intersection with any closed — and therefore any open — ball is connected). For further details, see § 7.7 in [5]).

Theorem 4.13. *The set of all n-link piecewise linear functions $S_n = \bigcup_{m=1}^{\infty} S(n, m)$ is monotone path-connected in $C[a, b]$.*

Since the class of all closed sets admitting local ε-selections for all $\varepsilon > 0$ coincides with the class of all closed sets admitting global ε-selections for all $\varepsilon > 0$, it follows from Theorem 4.13 that S_n admits a continuous additive (multiplicative) ε-selection in $C[a, b]$ for all $\varepsilon > 0$. This result was proved earlier by Livshits [37].

According to [60], each approximatively compact monotone path-connected set admits a continuous additive (multiplicative) ε-selection for any $\varepsilon > 0$. Hence we have the following result.

Theorem 4.14. *The set $S(n,K)$ has a continuous additive (multiplicative) ε-selection in $C[a,b]$ for any $\varepsilon>0$.*

Remark 4.1. In actual fact, the sets $S(n,K)$ and S_n are monotone path-connected in the space $L^\infty[a,b]$, $a<b$, with respect to the set $\mathscr{E}=\{\delta_t\}_t\subset(L^\infty[a,b])^*$, where δ_t is the delta function on $L^\infty[a,b]$, which is defined as follows: if t is a Lebesgue point of f. then δ_t is the essential value of f at this point, otherwise δ_t is zero. The set $\mathscr{E}$ is 1-defining in $L^\infty[a,b]$ (see § 2 in [66]). Hence for $M=S_n\vee S(n,K)$ and all $\varepsilon>0$ there exists a continuous mapping $g\colon L^\infty[a,b]\to M$ such that

$$\|g(x)-x\|_{L^\infty[a,b]}<\rho(x,M)+\varepsilon.$$

Let $F\colon[a,b]\to 2^{\mathbb{R}}$ be a set-valued mapping such that $F(t)$ is an interval of $\mathbb{R}$, $t\in[a,b]$. Consider the sets

$$S^F(n,K)=\{s\in S(n,K)\mid s(t)\in F(t)\ \forall\, t\in[a,b]\}$$

and

$$S_n^F=\{s\in S_n\mid s(t)\in F(t)\ \forall\, t\in[a,b]\}=\bigcup_{m\in\mathbb{N}}S^F(n,m).$$

Note that $S^F(n,K)$ is the set of all K-Lipschitz functions in S_n^F. Hence $S^F(n,K)$ is boundedly compact if S_n^F closed.

Theorem 4.15. *If the set S_n^F (or $S^F(n,K)$) is closed and non-empty, then it is monotone path-connected in $C[a,b]$ and admits a continuous additive (multiplicative) ε-selection in $C[a,b]$ for any $\varepsilon>0$.*

Proof. That $S^F(n,K)$ is monotone path-connected follows because $S(n,K)$ is monotone path-connected. Therefore, S_n^F is monotone path-connected, because

$$S_n^F=\bigcup_{m\in\mathbb{N}}S^F(n,m).$$

By Theorem 3 in [60] the bounded compactness of $S^F(n,K)$ implies the existence of a continuous additive (multiplicative) ε-selection in $C[a,b]$ to this set for all $\varepsilon>0$. Given an arbitrary compact set $N\subset C[a,b]$, we assume that for any $\varepsilon>0$ there exists $m\in\mathbb{N}$ such that

$$\rho(x,S^F(n,m))\leqslant\rho(x,S_n^F)+\frac{\varepsilon}{2},\qquad x\in N.$$

Next, there exists a continuous mapping $g\colon N\to S^F(n,m)$ such that

$$\|g(x)-x\|\leqslant\rho(x,S^F(n,m))+\frac{\varepsilon}{2}\leqslant\rho(x,S_n^F)+\varepsilon.$$

By Proposition 1 in [60], since the compact set N and $\varepsilon>0$ are arbitrary, the set S_n^F admits a continuous additive (multiplicative) ε-selection in $C[a,b]$ for all $\varepsilon>0$. This proves Theorem 4.15.

Now consider the problem of the existence of a continuous ε-selections to the set of n-link piecewise linear functions with localized knots [66].

Consider a family $\mathscr{T}_n = \{[a_k, b_k]\}_{k=0}^n$ consisting of $n+1$ closed intervals, where $a = a_0 = b_0 \leqslant a_1 \leqslant b_1 \leqslant \dots \leqslant a_k \leqslant b_k \leqslant \dots \leqslant a_n = b_n = b$. We let $S(\mathscr{T}_n)$ denote the set of piecewise linear functions s with knots $\{t_k\}_{k=0}^n$ such that $t_k \in [a_k, b_k]$, $k = 0, \dots, n$. We also set

$$\kappa(\mathscr{T}_n) := \min_{k=0,\dots,n-1} \{a_{k+1} - b_k\}.$$

Definition 4.10. The *Banach–Mazur hull* $\mathrm{m}(M)$ (also called the *cover* or the *ball hull*) of a bounded set $\varnothing \neq M \subset X$ is defined as the intersection of *all* closed balls containing M.

A set $M \subset X$ is m-*connected* (or *Menger-connected* [16]) if

$$\mathrm{m}(\{x, y\}) \cap M \neq \{x, y\}.$$

Theorem 4.16. *The set $S(\mathscr{T}_n)$ is Menger-connected in $C[a, b]$; if $\mathscr{T}_n$ consists of disjoint intervals, then $S(\mathscr{T}_n)$ is a boundedly compact monotone path-connected subset of $C[a, b]$.*

Proof. Let $s_1, s_2 \in S(\mathscr{T}_n)$, and let

$$T_1 = \{t_k^1\}_{k=0}^n \quad \text{and} \quad T_2 = \{t_k^2\}_{k=0}^n$$

be the knots of the piecewise linear functions s_1 and s_2, respectively. We denote by

$$X_1 = \{x_k^1 = (t_k^1, s_1(t_k^1))\}_{k=0}^n \quad \text{and} \quad X_2 = \{x_k^2 = (t_k^2, s_2(t_k^2))\}_{k=0}^n,$$

respectively, the vertex sets of these piecewise linear functions. Let us construct recursively vertices $\{x_k = (t_k, s(t_k))\}$ and thus a piecewise linear function $s \in \mathrm{m}(s_1, s_2)$ distinct from s_1 and s_2.

Given $\mathscr{A} \subset S^*$, set

$$\begin{aligned} [\![x, y]\!]_{\mathscr{A}} &= \bigl\{z \in X \mid \min\{x^*(x), x^*(y)\} \leqslant x^*(z) \leqslant \max\{x^*(x), x^*(y)\} \ \forall\, x^* \in \mathscr{A}\bigr\} \\ &= \bigl\{z \in X \mid x^*(z) \in [x^*(x), x^*(y)] \ \forall\, x^* \in \mathscr{A}\bigr\}. \end{aligned}$$

In our case $\mathrm{m}(x, y) = [\![x, y]\!]_{\mathscr{A}}$, where $\mathscr{A} = \{\delta(\cdot - t_0)\}_{t_0 \in [a,b]}$ is the family of delta functions, which, together with $-\mathscr{A}$, comprises the set of all norm-one extreme continuous functionals on $C[a, b]$.

We take the point $(x_0^1 + x_0^2)/2$ as x_0. If $x_0^1 \neq x_0^2$, then we go over to the next step, $k = 1$, of the construction. If $x_0^1 = x_0^2$, then there exists a minimum number $m \in \mathbb{N}$ such that $x_m^1 \neq x_m^2$. We set $x_i = x_i^1 = x_i^2$ for $i = 0, \dots, m-1$, put $x_m = (x_m^1 + x_m^2)/2$, and then proceed with the next step $k = m + 1$ (of course, if $k < n$).

Assuming that a vertex $x_k \in (x_k^1, x_k^2)$, $1 \leqslant k < n$. has already been constructed, we define the next vertex x_{k+1}. If $x_{k+1}^1 = x_{k+1}^2$, then we set $x_{i+1} = x_{i+1}^1$ for all $i \geqslant k$. We thus define all vertices of the piecewise linear function s, and thus the function s itself. It can easily be checked that s is the required piecewise linear function.

Now assume that $x^1_{k+1} \neq x^2_{k+1}$ and that the intervals $[x^1_k, x^1_{k+1}]$ and $[x^2_k, x^2_{k+1}]$ are disjoint. Assume that the interval $[x_k, x^1_{k+1}]$ ($[x_k, x^2_{k+1}]$) has common points with the relative interior of $[x^2_k, x^2_{k+1}]$ (of $[x^1_k, x^1_{k+1}]$). Note that in this case the interval $[x_k, x^1_{k+1}]$ (respectively, $[x_k, x^2_{k+1}]$) does not meet the relative interior of the interval $[x^1_k, x^1_{k+1}]$ (of $[x^2_k, x^2_{k+1}]$). We set $x_{i+1} = x^2_{i+1}$ ($x_{i+1} = x^2_{i+1}$) for all $i \geqslant k$. Thus, $s \equiv s_2$ (respectively, $s \equiv s_1$) on the interval $[t^1_{k+1}, b]$, and s is the required piecewise linear function. In the case where $[x_k, x^1_{k+1}]$ is disjoint from the relative interior of $[x^2_k, x^2_{k+1}]$, we set $x_{i+1} = x^1_{i+1}$ for all $i \geqslant k$. As a result, $s \equiv s_1$ on the interval $[t^1_{k+1}, b]$, and s is the required piecewise linear function.

Now, if $x^1_{k+1} \neq x^2_{k+1}$, then we assume that the intervals $[x^1_k, x^1_{k+1}]$ and $[x^2_k, x^2_{k+1}]$ intersect. Let ℓ_k be the straight line passing through their point of intersection and x_k. Let x_{k+1} denote the point of intersection of ℓ_k with the interval $[x^1_{k+1}, x^2_{k+1}]$. It is easily seen that $x_{k+1} \in (x^1_{k+1}, x^2_{k+1})$. Next we proceed with step $k+1$ (if, of course, $k+1 < n$).

So at the nth step or earlier we defined all the vertices $X = \{x_k\}_{k=0}^n$ of the piecewise linear function s, and therefore the function s with the required properties itself. Since s_1 and s_2 are arbitrary piecewise linear functions, the set $S(\mathscr{T}_n)$ is Menger-connected in the space $C[a,b]$.

If different closed intervals in the family $\mathscr{T}_n$ are disjoint, then $\kappa(\mathscr{T}_n) > 0$, and therefore any function in $S(\mathscr{T}_n) \cap [\![s_1, s_2]\!]_{\mathscr{A}}$ is $\|s_1 - s_2\|/\kappa(\mathscr{T}_n)$-Lipschitz continuous in $C[a,b]$. Hence the set $S(\mathscr{T}_n) \cap [\![s_1, s_2]\!]_{\mathscr{A}}$ is compact. By the above, this set is also Menger-connected, and therefore monotone path-connected. Hence the functions s_1 and s_2 can be connected by a monotone path in $S(\mathscr{T}_n)$. Thus, $S(\mathscr{T}_n)$ is monotone path-connected. That this set is boundedly connected follows from the fact that all functions $s \in S(\mathscr{T}_n)$ with $\|s\| \leqslant C$, are $2C/\kappa(\mathscr{T}_n)$-Lipschitz. This proves Theorem 4.16.

Let $F\colon [a,b] \to 2^{\mathbb{R}}$ be a set-valued mapping such that $F(t)$ is an interval in $\mathbb{R}$, $t \in [a,b]$. Consider the sets

$$S^F(\mathscr{T}_n) = \{s \in S(\mathscr{T}_n) \mid s(t) \in F(t)\ \forall\, t \in [a,b]\}.$$

Theorem 4.17. *The set $S^F(\mathscr{T}_n)$ has a continuous additive (multiplicative) ε-selection in $C[a,b]$ for all $\varepsilon > 0$.*

Proof. Let $\mathscr{T}_n = \{[a_k, b_k]\}_{k=0}^n$. For each interval $[a_k, b_k]$ consider the nested sequence of intervals $\{[a^j_k, b^j_k]\}_{j\in\mathbb{N}}$ such that $(a_k, b_k) = \bigcup\{[a^j_k, b^j_k] \mid j \in \mathbb{N}\}$. Let $\mathscr{T}_{n,j}$ be the family of closed intervals $\{[a^j_k, b^j_k]\}_{k=0}^n$. Note that $\kappa(\mathscr{T}_{n,j}) > 0$. In addition, $\bigcup_j S^F(\mathscr{T}_{n,j})$ is dense in $S^F(\mathscr{T}_n)$.

Since $S^F(\mathscr{T}_{n,j})$ is boundedly compact and monotone path-connected, it follows from Theorem 3 in [60] that $S^F(\mathscr{T}_{n,j})$ admits a continuous additive (multiplicative) ε-selection for any $\varepsilon > 0$ in $C[a,b]$. Let $N \subset C[a,b]$ be an arbitrary compact set. Assume that for an arbitrary $\varepsilon > 0$ there exists $j \in \mathbb{N}$ such that

$$\rho(x, S^F(\mathscr{T}_{n,j})) \leqslant \rho(x, S^F(\mathscr{T}_{n,j})) + \frac{\varepsilon}{2}, \qquad x \in N.$$

There exists a continuous mapping $g\colon N \to S^F(\mathscr{T}_{n,j}) \subset S^F(\mathscr{T}_n)$ such that

$$\|g(x) - x\| \leqslant \rho(x, S^F(\mathscr{T}_{n,j})) + \frac{\varepsilon}{2} \leqslant \rho(x, S^F(\mathscr{T}_{n,j})) + \varepsilon.$$

By Proposition 1 in [60], since the compact set K and $\varepsilon > 0$ are arbitrary, the closed set S_n^F admits a continuous additive (multiplicative) ε-selection in $C[a,b]$ for all $\varepsilon > 0$. Theorem 4.17 is proved.

Now consider the problem of the existence of continuous ε-selections to the sets of n-link r-polynomial functions and their generalizations in the spaces $W_p^r[a,b]$ (see [66]).

Let $S_n^r = S_n^r[a,b]$, $a < b$, denote the set of all n-link r-polynomial functions which are r-primitives of n-link piecewise constant functions s in the set $S_n^0 = S_n^0[a,b]$ defined above — these are functions $s\colon [a,b] \to \mathbb{R}$ for which there exists a partition $T = \{t_i\}_{i=0}^n$ such that $a = t_0 \leqslant t_1 \leqslant \dots \leqslant t_n = b$, and $s|_{(t_{i-1},t_i)}$ is constant.

By $W_p^r = W_p^r[a,b]$, $1 \leqslant p < \infty$, we mean the space of all functions $f\colon [a,b] \to \mathbb{R}$ with absolutely continuous $(r-1)$st derivative $f^{(r-1)}$ and such that $f^{(r)} \in L^p = L^p[a,b]$. We equip this space with the seminorm $\|f\|_{W_p^r} := \|f^{(r)}\|_{L^p}$. We set $W_p^0[a,b] = L^p[a,b]$ by definition.

Theorem 4.18 (see [66]). *The set S_n^r admits a continuous multiplicative* (*additive*) *ε-selection in the seminormed space W_p^r for all $\varepsilon > 0$.*

Consider the following generalization of piecewise-polynomial functions.

Let Φ be a non-empty convex closed subset of $L^p[0,1]$, $1 \leqslant p < \infty$, and let $\varepsilon \in (0,1)$. Below we assume that Φ is boundedly compact in $L^p[0,1]$.

We denote by $\Phi_{[c,d]}$ the restriction of Φ to $[c,d] \subset [0,1]$. Sometimes, for convenience we assume that $\Phi_{[c,d]}$ is naturally embedded in $L^p[0,1]$ (in this case each function in $\Phi_{[c,d]}$ is extended by zero outside $[c,d]$). Since $\Phi_{[c,d]}$ is convex and closed and Φ is boundedly compact, the mapping $F\colon \mathbb{R}^2 \to 2^{L^p[0,1]}$, as defined by $F(c,d) := \Phi_{[\widehat{c},\widehat{d}]}$, where $[\widehat{c},\widehat{d}] = [c,d] \cap [0,1]$, is stable (see Definition 4.1). By Theorem 4.2 there exists a continuous mapping $\tau = \tau_{c,d}(\cdot)\colon L^p[0,1] \times \mathbb{R}^2 \to \Phi_{[\widehat{c},\widehat{d}]}$ such that

$$\|\tau(f) - f\|_{L^p[c,d]} \leqslant \left(1 + \frac{\varepsilon}{4}\right)\rho(f, \Phi_{[c,d]})$$

for any $f \in L^p[c,d]$. Here, when applying Theorem 4.2, we assume that $[c,d] \subset [0,1]$, and the functions $\tau(f), f \in L^p[c,d]$ are considered as elements of $L^p[0,1]$ which coincide with $\tau(f)$ and f, respectively, on $[c,d]$ and which vanish outside this interval. However, in what follows we assume that $\tau(f)$ is an element of $L^p[c,d]$. We denote by $S_n^\Phi = S_n^\Phi[0,1]$ the set of n-piecewise Φ-functions, that is, functions $s\colon [0,1] \to \mathbb{R}$ such that there exists a partition

$$T = \{t_i\}_{i=0}^n, \qquad 0 = t_0 \leqslant t_1 \leqslant \dots \leqslant t_n = 1,$$

for which $s|_{(t_{i-1},t_i)}$ is some $\varphi \in \Phi_{[t_{i-1},t_i]}$. We denote the r-primitives of functions in S_n^Φ by $S_n^{\Phi,r} = S_n^{\Phi,r}[0,1]$.

Theorem 4.19. *The set $S_n^{\Phi,r}$ admits a continuous multiplicative* (*additive*) *ε-selection in the seminormed space W_p^r for any $\varepsilon > 0$.*

Remark 4.2. The conclusion of Theorem 4.19 remains true if the assumption of bounded compactness is replaced by the weaker condition that Φ is closed.

The proof of Theorem 4.19 is based on the following result from [60].

Proposition 4.1. *Let M be a closed subset of a complete symmetrizable asymmetric linear seminormed space such that, for any $\varepsilon > 0$ and an arbitrary compact set $K \subset X$, there exists a continuous mapping $\psi\colon K \to O_\varepsilon(M)$ such that*

$$\|\psi(x) - x| \leqslant \rho(x, M) + \varepsilon.$$

Then the set M is $\mathring{B}$-infinitely connected and admits a continuous additive (multiplicative) ε-selection for all $\varepsilon > 0$.

Now consider the problem of the existence of continuous ε-selections to sets of k-monotone functions (see [62]).

Below we give a positive answer to the question, raised by Kashin, as to whether the set of k-monotone functions in $C[a,b]$ admits a continuous ε-selection for any $\varepsilon > 0$? To this end we solve the analogous problem for k-monotone vectors in the space ℓ^∞_n; this being the space of all n-dimensional vectors $x = (x_1, \dots, x_n)$, $x_l \in \mathbb{R}$, $l = 1, \dots, n$, with the norm

$$\|x\| = \|x\|_{\ell^\infty_n} = \max_{l=1,\dots,n} |x_l|.$$

Let $\mathscr{L}^k$ be the set of all k-monotone vectors $x = (x_1, \dots, x_n) \in \ell^\infty_n$. This means that there exists a set of numbers

$$1 = n_0 < n_1 < n_2 < \dots < n_l < n_{l+1} = n, \qquad l \leqslant k,$$

such that either

$$x_{n_i} \leqslant \dots \leqslant x_{n_{i+1}} \quad \text{and} \quad x_{n_{i+1}} \geqslant \dots \geqslant x_{n_{i+2}}$$

or

$$x_{n_i} \geqslant \dots \geqslant x_{n_{i+1}} \quad \text{and} \quad x_{n_{i+1}} \leqslant \dots \leqslant x_{n_{i+2}}$$

for all $i = 0, \dots, l-1$. The numbers $n_1, \dots, n_l$ are called *monotonicity change moments* of x, and the number l will be referred to as the *number of monotonicity changes.*

Lemma 4.1 (see [62]). *For all pairs of distinct $x, y \in \ell^\infty_n$ there exists a vector $z \in \ell^\infty_n$, $z \in \mathrm{m}(x, y) \setminus \{x, y\}$, such that the number of monotonicity changes of z is majorized by the maximum number of monotonicity changes of x and y.*

Proof. Let $m_0 < n$ be the maximum number such that

$$x_1 = y_1, \quad \dots, \quad x_{m_0} = y_{m_0}.$$

If such a number does not exist, then we set $m_0 = 0$. Let i be the smallest number such that $n_i > m_0$ (here n_i is a monotonicity change moment for x; if there is no such i, then we set $n_i = m_0 + 1$). It can be assumed without loss of generality that the number of monotonicity changes of y (which is larger than m_0) is not smaller than n_i. We assume first that $m_0 \geqslant 1$. Let us investigate various variants of monotonicity changes for the tuples $\{x_{m_0}, \dots, x_{n_i}\}$ and $\{y_{m_0}, \dots, y_{n_i}\}$.

1) The case of similar monotonicity (the coordinates are increasing):

$$x_{m_0} \leqslant \dots \leqslant x_{n_i} \quad \text{and} \quad y_{m_0} \leqslant \dots \leqslant y_{n_i}.$$

We define the vector $z \in \ell_n^\infty$ by

$$z_1 = x_1 = y_1, \quad \dots, \quad z_{m_0} = x_{m_0} = y_{m_0},$$
$$z_j = \frac{1}{2}(x_j + y_j) \quad \text{for all} \quad j = (m_0+1), \dots, n_i.$$

For any $j > n_i$ we set $z_j = x_j$. The number of monotonicity changes of z is majorized by that of x. In addition, we have $z_j \in [x_j, y_j]$ for all j, and, as is easily seen, $z \in \mathrm{m}(x,y) \setminus \{x,y\}$.

2) Consider one case of different monotonicity

$$x_{m_0} \leqslant \dots \leqslant x_{n_i} \quad \text{and} \quad y_{m_0} \geqslant \dots \geqslant y_{n_i}.$$

In this case we set

$$z_1 = x_1 = y_1, \quad \dots, \quad z_{m_0} = x_{m_0} = y_{m_0}.$$

By the definition of m_0,

$$y_{n_i} \leqslant \dots \leqslant y_{m_0+2} \leqslant y_{m_0+1} \leqslant y_{m_0} = x_{m_0} \leqslant x_{m_0+1} \leqslant x_{m_0+2} \leqslant \dots \leqslant x_{n_i} =: z_{n_i}.$$

Here $y_{m_0+1} < x_{m_0+1}$. If $x_{m_0} < x_{m_0+1}$, then we set

$$z_{m_0+1} = \frac{1}{2}(x_{m_0} + x_{m_0+1}) \quad \text{and} \quad z_j = \begin{cases} x_{m_0+1}, & j = m_0+2, \dots, n_i, \\ x_j, & j > n_i. \end{cases}$$

In this case the number of monotonicity changes of z is dominated by that of x. If $x_{m_0} = x_{m_0+1}$, then we set

$$z_{m_0+1} = \frac{1}{2}(y_{m_0} + y_{m_0+1}) = \frac{1}{2}(x_{m_0+1} + y_{m_0+1}) \quad \text{and} \quad z_j = y_j, \quad j > m_0+1.$$

The number of monotonicity changes of z is bounded above by that of y. By construction, $z \neq x, y$ and $z_j \in [x_j, y_j]$ for all j. As a consequence, $z \in \mathrm{m}(x,y) \setminus \{x,y\}$.

The cases

3) $x_{m_0} \geqslant \dots \geqslant x_{n_i}$, $y_{m_0} \geqslant \dots \geqslant y_{n_i}$, and

4) $x_{m_0} \geqslant \dots \geqslant x_{n_i}$, $y_{m_0} \leqslant \dots \leqslant y_{n_i}$

are dealt with similarly to cases 1) and 2), respectively.

Now assume that $m_0 = 0$. We consider different variants of monotonicity of the tuples $\{x_{m_0}, \dots, x_{n_i}\}$ and $\{y_{m_0}, \dots, y_{n_i}\}$.

I) The case of similar monotonicity (the coordinates are increasing):

$$x_1 \leqslant \dots \leqslant x_{n_i} \quad \text{and} \quad y_1 \leqslant \dots \leqslant y_{n_i}.$$

We set

$$z_j = \frac{1}{2}(x_j + y_j) \quad \text{for all} \quad j = 1, \dots, n_i.$$

For all $j > n_i$, we set $z_j = x_j$. The number of monotonicity changes of z is majorized by that of x. In addition, $z_j \in [x_j, y_j]$ for all j, and, as is easily seen, $z \in \mathrm{m}(x,y) \setminus \{x,y\}$.

II) Let $x_1 \leqslant \dots \leqslant x_{n_i}$ and $y_1 \geqslant \dots \geqslant y_{n_i}$. If $y_1 < x_1$, then we set

$$z_1 = \frac{1}{2}(x_1 + y_1) \quad \text{and} \quad z_j = x_j, \quad j \geqslant 2.$$

In this case the number of monotonicity changes of z is majorized by that of x, and $z_j \in [x_j, y_j]$ for all j. Hence $z \in \mathrm{m}(x,y) \setminus \{x,y\}$.

Consider the case $y_1 > x_1$. Let m be the largest number in the range $1, \dots, n_i$ such that $x_1 = \dots = x_m$ and $y_1 = \dots = y_m$. If $m = n_i$, then we set

$$z_j = \frac{1}{2}(x_j + y_j), \quad j = 1, \dots, n_i, \quad \text{and} \quad z_j = x_j, \quad j > n_i.$$

In this case $z_j \in [x_j, y_j]$ for all j and $z \in \mathrm{m}(x,y) \setminus \{x,y\}$. If $m < n_i$, then we either have $y_m > y_{m+1}$ or $x_m < x_{m+1}$. In the first case we set

$$z_j = \max\biggl\{y_{m+1}, \frac{1}{2}(x_1 + y_1)\biggr\} \ \text{ if } j \leqslant m+1, \quad \text{and} \quad z_j = y_j \ \text{ if } j > m+1.$$

In this case $z_j \in [x_j, y_j]$ for all j, and $z \in \mathrm{m}(x,y) \setminus \{x,y\}$. In the second case we put

$$z_j = \min\biggl\{x_{m+1}, \frac{1}{2}(x_1 + y_1)\biggr\} \ \text{ if } j \leqslant m+1, \quad \text{and} \quad z_j = x_j \ \text{ if } j > m+1.$$

Here $z_j \in [x_j, y_j]$ for all j and $z \in \mathrm{m}(x,y) \setminus \{x,y\}$.

In all above cases the number of monotonicity changes of z is not greater than that of the vectors x and y.

The cases

III) $x_1 \geqslant \dots \geqslant x_{n_i}$, $y_1 \geqslant \dots \geqslant y_{n_i}$, and

IV) $x_1 \geqslant \dots \geqslant x_{n_i}$, $y_1 \leqslant \dots \leqslant y_{n_i}$

are dealt with similarly to cases I) and II), respectively. Lemma 4.1 is proved.

The segment $[\![x,y]\!]$ is defined by

$$[\![x,y]\!] = \bigl\{z \in X \mid \min\{x^*(x), x^*(y)\} \leqslant x^*(z) \leqslant \max\{x^*(x), x^*(y)\} \ \forall\, x^* \in \operatorname{ext} S_{X^*}\bigr\}.$$

Remark 4.3. In finite-dimensional spaces X_n Menger-connectedness and monotone path-connectedness are equivalent for closed sets (see, for example, § 7.7 in [5]); in addition, any closed Menger-connect (monotone path-connected) set in a finite-dimensional normed space admits an ε-selection for any $\varepsilon > 0$ (see, for example, [3]). It is also worth noting that in any separable or reflexive space $\mathrm{m}(x,y) = [\![x,y]\!]$ (see, for example, § 7.7.1 in [5]).

Corollary 4.1. *The set of all k-monotone vectors $\mathscr{L}^k$ is Menger-connected and monotone path-connected in ℓ^∞_n.*

Proof. That $\mathscr{L}^k$ is Menger-connected (m-connected) follows from Lemma 4.1, and since ℓ^∞_n is finite-dimensional, $\mathscr{L}^k$ is monotone path-connected as a convex set.

Definition 4.11. A function $f \in C[a,b]$ is called *k-monotone* if $[a,b]$ can be split into at most k intervals on each of which the function f is monotone.

Theorem 4.20. *The set $\mathscr{M}_k$ of all k-monotone functions in $C[a,b]$ admits a continuous additive (multiplicative) ε-selection for all $\varepsilon > 0$.*

Proof. Since $\mathscr{M}_k$ is closed, it suffices to prove the theorem for an additive selection. Consider an arbitrary compact set $K \subset C[a,b]$. Let $N \subset K$ be an $(\varepsilon/6)$-net for K. Given any $g \in N$, consider the function

$$f = f_g \in \mathscr{M}_k \quad \text{such that} \quad \|f - g\| \leqslant \rho(g, \mathscr{M}_k) + \frac{\varepsilon}{12}.$$

We set $F := \{f_g\}_{g \in N}$. Let $T = T_M = \{t_j\}_{j=0}^{M-1}$ be a partition of $[a,b]$ such that

$$\|\psi - \ell_\psi\| \leqslant \frac{\varepsilon}{12}$$

for any function $\psi \in F \cup N$ and any piecewise linear function ℓ_ψ with vertices $\{(t_j, \psi(t_j))\}_{j=0}^{M-1}$. Note that $\ell_\psi \in \mathscr{M}_k$ for any function $\psi \in \mathscr{M}_k$.

Let the mapping $\Phi\colon C[a,b] \to \ell^\infty_M$ be defined by $\Phi(f) = \{f(t_j)\}_{j=0}^{M-1}$. Then the image of the set $\mathscr{M}_k$ under Φ is $\mathscr{L}^k \subset \ell^\infty_M$. In addition, we have

$$\Phi(\ell_\psi) = \Phi(\psi) \quad \text{and} \quad \|\Phi(\psi_1) - \Phi(\psi_2)\|_{\ell^\infty_M} = \|\ell_{\psi_1} - \ell_{\psi_2}\|_{C[a,b]}.$$

Next we define the mapping $\widehat{\Phi}$ by associating with any point $x = (x_1, \dots, x_M) \in \ell^\infty_M$ the piecewise linear function with vertices $\{(t_j, x_{j+1})\}_{j=0}^{M-1}$. Note that $\widehat{\Phi}(\Phi(\psi)) = \ell_\psi$ for any function $\psi \in C[a,b]$.

We set

$$\widehat{N} = \Phi(N), \qquad \widehat{K} = \Phi(K), \quad \text{and} \quad \widehat{F} = \Phi(F).$$

Given any function $\psi \in N$, we have

$$\rho_{\ell^\infty_M}(\Phi(\psi), \mathscr{L}^k) \leqslant \rho_{\ell^\infty_M}(\Phi(\psi), \widehat{F}) = \rho_{C[a,b]}(\ell_\psi, F) \leqslant \rho_{C[a,b]}(\ell_\psi, \mathscr{M}_k) + \frac{\varepsilon}{6}.$$

Further, for any function $\psi \in K$ there exists $\psi_0 \in N$ such that $\|\psi - \psi_0\| \leqslant \varepsilon/6$. As a consequence,

$$\rho_{C[a,b]}(\psi, \mathscr{M}_k) \leqslant \rho_{C[a,b]}(\psi_0, \mathscr{M}_k) + \frac{\varepsilon}{6},$$

which gives

$$\rho_{\ell^\infty_M}(\Phi(\psi), \mathscr{L}^k) \leqslant \rho_{C[a,b]}(\ell_{\psi_0}, \mathscr{M}_k) + \frac{\varepsilon}{3}.$$

Let Ψ be a continuous $(\varepsilon/6)$-selection to the set $\mathscr{L}^k$ in ℓ^∞_M (this selection exists by Corollary 4.1 and Remark 4.3). Then we have

$$\begin{aligned}\|\widehat{\Phi}(\Psi(u)) - \widehat{\Phi}(u)\|_{C[a,b]} &= \|\Psi(u) - u\|_{\ell^\infty_M} \leqslant \rho_{\ell^\infty_M}(u, \mathscr{L}^k) + \frac{\varepsilon}{6} \\ &\leqslant \rho_{C[a,b]}(\widehat{\Phi}(u), \mathscr{M}_k) + \frac{\varepsilon}{2} \quad \text{for all} \quad u \in \widehat{K}.\end{aligned}$$

As a result, $G = \widehat{\Phi} \circ \Psi \circ \Phi$ is a continuous additive ε-selection to the compact set K. Indeed, for all $\psi \in K$ we have

$$\|\psi - \ell_\psi\| \leqslant \frac{\varepsilon}{6}$$

and

$$\begin{aligned}\|G(\psi)-\psi\| &\leqslant \|G(\psi)-\ell_\psi\|+\frac{\varepsilon}{6}=\|\widehat{\Phi}(\Psi(\Phi(\psi)))-\widehat{\Phi}(\Phi(\ell_\psi))\|+\frac{\varepsilon}{6}\\ &=\|\widehat{\Phi}(\Psi(\Phi(\psi)))-\widehat{\Phi}(\Phi(\psi))\|+\frac{\varepsilon}{6}\leqslant \rho_{C[a,b]}(\widehat{\Phi}(\Phi(\psi)),\mathscr{M}_k)+\frac{4\varepsilon}{6}\\ &\leqslant \rho_{C[a,b]}(\psi,\mathscr{M}_k)+\|\widehat{\Phi}(\Phi(\psi))-\psi\|+\frac{4\varepsilon}{6}\leqslant \rho_{C[a,b]}(\psi,\mathscr{M}_k)+\frac{5\varepsilon}{6}.\end{aligned}$$

By Proposition 4.1 the set $\mathscr{M}_k$ admits a continuous additive ε-selection in the space $C[a,b]$. This proves Theorem 4.20.

In relation to Theorem 4.20, it is interesting to consider the problem of the existence of continuous selection of the metric projection operator to the set of all k-monotone functions in $C[a,b]$ as a function of k.

Let $\mathscr{M}_{k,n}$ be the set of all continuous functions

$$f=(f_1,\dots,f_n)\colon [a,b]\to \ell_n^\infty$$

that are coordinatewise k-monotone functions, that is, $f_m\in\mathscr{M}_k$ for all $m=1,\dots,n$.

Theorem 4.21. *The set $\mathscr{M}_{k,n}\subset C([a,b],\ell_n^\infty)$ admits a continuous additive (multiplicative) ε-selection for all $\varepsilon>0$.*

Proof. The arguments for additive and multiplicative selections are similar. Let us prove the required result for an additive selection.

Let $\varphi\colon C[a,b]\to\mathscr{M}_k$ be the continuous ε-selection constructed in the previous theorem. Then the required ε-selection $\Phi\colon C([a,b],\ell_n^\infty)\to\mathscr{M}_{k,n}$ is given by

$$\Phi(f)=(\varphi(f_1),\dots,\varphi(f_n))\quad\text{for all}\quad f=(f_1,\dots,f_n).$$

Indeed,

$$\begin{aligned}\max_{x\in[a,b]}\|\Phi(f)(x)-f(x)\|_{\ell_n^\infty} &= \max_{x\in[a,b]}\max_{m=1,\dots,n}|\varphi(f_m)(x)-f_m(x)|\\ &=\max_{m=1,\dots,n}\|\varphi(f_m)(x)-f_m(x)\|_{C[a,b]}\leqslant \max_{m=1,\dots,n}\{\rho_{C[a,b]}(f_m,\mathscr{M}_k)+\varepsilon\}\\ &=\max_{m=1,\dots,n}\Big\{\inf_{g_m\in\mathscr{M}_k}\|g_m-f_m\|_{C[a,b]}\Big\}+\varepsilon\\ &=\max_{m=1,\dots,n}\Big\{\inf_{g=(g_1,\dots,g_n)\in\mathscr{M}_{k,n}}\|g_m-f_m\|_{C[a,b]}\Big\}+\varepsilon\\ &\leqslant \inf_{g\in\mathscr{M}_{k,n}}\|g-f\|_{C([a,b],\ell_n^\infty)}+\varepsilon=\inf_{g\in\mathscr{M}_{k,n}}\|g-f\|_{C([a,b],\ell_n^\infty)}+\varepsilon.\end{aligned}$$

This proves Theorem 4.21.

5. Classical problems of generalized rational approximation: existence, uniqueness, stability, and characterization of best approximants

By classical problems of rational approximation we mean problems of the existence, uniqueness, and stability of best or near-best approximants, as well as various types of solarity (characterization of best approximants). The importance of existence and solarity problems for generalized rational functions stems from numerous applications of the latter in approximation theory and numerical mathematics (see, for example, [28], [45], [14], [55]).

Given a set $\varnothing \neq M \subset X$, we say that $x \in X \setminus M$ is a *solar point* (see, for example, § 10.2 in [5], and § 2 in [3]) if there exists a point $y \in P_M x \neq \varnothing$ (called a *luminosity point*) such that

$$y \in P_M((1-\lambda)y + \lambda x) \quad \text{for all} \quad \lambda \geqslant 0 \tag{5.1}$$

(geometrically, this means that there is a 'solar' ray from y through x such that y is a nearest point in M for any point on this ray).

A point $x \in X \setminus M$ is a *strict solar point* if $P_M x \neq \varnothing$ and (5.1) holds for any point $y \in P_M x$ (that if, if any best approximant from M to x is a luminosity point). Further, if for $x \in X \setminus M$ condition (5.1) is satisfied for each $y \in P_M x$, then x is called a *strict protosolar point* (unlike the case of strict solar points, a nearest point y to x may fail to exist).

A closed set $M \subset X$ is called a *sun* if any point $x \in X \setminus M$ is a solar point. A set $M \subset X$ is called a *strict protosun* if each point $x \in X \setminus M$ is a strict protosolar point. A Chebyshev set (a set of existence and uniqueness) which is a sun is called a *Chebyshev sun*.

Strict protosuns are the most general objects satisfying the generalized Kolmogorov criterion for an element of best approximation (see, for example, [7]) — namely, a point not lying in a strict protosun can be separated from it by a support cone constructed from any best approximant in the set (if, of course, such a nearest point exists). As in the case of convex sets, this separation property characterizes suns (strict (proto)suns); see § 5 in [7]. Accordingly, strict protosuns are sometimes called Kolmogorov sets.

Consider the following classical family of rational functions in $C[a,b]$:

$$\mathscr{R}_{n,m} = \mathscr{R}_{n,m}[a,b] := \left\{ \frac{p}{q} \;\middle|\; p \in \mathscr{P}_n,\ q \in \mathscr{P}_m,\ q \neq 0 \right\}, \tag{5.2}$$

where $\mathscr{P}_n$ is the subspace of algebraic polynomials of degree at most n. It is well known that $\mathscr{R}_{n,m}$ is a Chebyshev sun in $C[a,b]$ (see, for example, § 2 in [8]). However in $L^p[a,b]$, $1 \leqslant p < \infty$, Efimov and Stechkin showed, with the help of general theorems of geometric approximation theory, that $\mathscr{R}_{n,m}$, $m \geqslant 1$, is an existence set but not a uniqueness set. They also showed that the class $\mathscr{R}_{0,2}$ is not a uniqueness set in $L^1[a,b]$. The same result, but for all classes $\mathscr{R}_{n,m}$, $m \geqslant 1$, was established by Tsar'kov [69] with the help of general methods of geometric approximation theory. We also consider the following more general class of rational functions:

$$\mathscr{R}_W^V := \left\{ r = \frac{v}{w} \;\middle|\; v \in V,\ w \in W \right\},$$

here Q is a metrizable compact set, $V, W \subset C(Q)$ are convex sets, and W consists of positive functions. It is well known (see, for example, [8]) that $\mathscr{R}_W^V$ is a strict protosun in $C(Q)$.

Let us consider the following generalizations of the classes $\mathscr{R}_{n,m}$ and $\mathscr{R}_W^V$. Let $V, W \subset C(Q)$, and let $U \subset V \times W$ be a non-empty convex set. Consider the class of generalized rational functions

$$\mathscr{R}_U := \{r \in C(Q) \mid rw = v,\ w \not\equiv 0,\ (v, w) \in U\}. \tag{5.3}$$

Theorem 5.1 (see [8]). *The set of generalized rational functions $\mathscr{R}_U$ is a strict protosun in $C(Q)$.*

This result means that the best rational approximants in the class $\mathscr{R}_U$ are characterized in terms of the Kolmogorov criterion for an element of best approximation. In turn, this result paves the way for the construction of algorithms for finding best rational approximants [14], [21], [22], [44], [55].

The stability of elements of (near-) best approximation is traditionally related to the properties of approximative compactness or existence of continuous ε-selections. It is well known that in non-degenerate cases (that is, for $m \geqslant 1$) the metric projection onto the (Chebyshev) set $\mathscr{R}_{n,m}$ has points of discontinuity in $C[a, b]$, but, according to Konyagin, for any $\varepsilon > 0$ there exists a continuous ε-selection to $\mathscr{R}_{n,m}$ (see [32], and also § 3 bellow). The next result extends and generalizes Konyagin's result (see also [50]).

Theorem 5.2 (see [8]). *The set of generalized rational functions $\mathscr{R}_U$ (with convex set U; see (5.3)) in $C(Q)$ is a stably monotone path-connected set and therefore admits a continuous additive ε-selection for any $\varepsilon > 0$. In addition, if $\mathscr{R}_U$ is closed, then $\mathscr{R}_U$ admits a continuous multiplicative ε-selection for any $\varepsilon > 0$. Moreover, $\mathscr{R}_U$ has contractible intersections with closed and open balls in $C(Q)$.*

First results on generalized rational approximation date back to Cheney, Loeb, Rubinshtein, Boehm, Dunham, and other authors (see § 11.1 in [5]). By contrast with the classical case of approximation by the class $\mathscr{R}_{n,m}$ in $C[a, b]$, an element of best uniform generalized rational approximation can fail to exist or be unique.

Definition 5.1. Let Q be a Hausdorff compact set, and let $U \subset V \times W$, where $V, W \subset C(Q)$. We say that

$$\mathscr{R}_U := \{r \in C(Q) \mid rw = v,\ w \not\equiv 0,\ (v, w) \in U\}$$

is *algebraically complete* (see [8]) if the conditions:

(a) $(v_k, w_k) \to (v, w)$ in $C(Q) \times C(Q)$, where $(v_k, w_k) \in U$, $w \not\equiv 0$,

(b) there exists a function $r \in C(Q)$ such that $r(t) = v(t)/w(t)$ for all $t \in Q \setminus \mathrm{Z}(w)$, where $\mathrm{Z}(w)$ is the set of zeros of w,

are equivalent to the condition $(v, w) \in U$.

Definition 5.2. A net (x_δ) is said to Δ-*converge* to $x \in C(Q)$ (written $x_\delta \overset{\Delta}{\to} x$) if there exists a dense subset $Q_0 \subset Q$ such that $x_\delta(t) \to x(t)$ for any $t \in Q_0$ (see [20]). A set $M \subset C(Q)$ is called *boundedly* Δ-*compact* if any bounded net from M contains a subnet Δ-converging to a point in M (see [20] and § 4.3 in [5]).

Let Q be a compact set, $V, W \subset C(Q)$ be boundedly compact sets, and let $U \subset V \times W$ be a non-empty set. Consider the class of generalized rational functions

$$\mathscr{R}_U := \{r \in C(Q) \mid rw = v,\ w \not\equiv 0,\ (v, w) \in U\}.$$

Theorem 5.3 (see [8]). *Let $\mathscr{R}_U$ be algebraically complete and, for each non-zero function in W, let the complement of its zero set in Q be dense in Q. Then $\mathscr{R}_U$ is boundedly Δ-compact in $C(Q)$; as a corollary, $\mathscr{R}_U$ is an existence set in $C(Q)$.*

The conclusion of Theorem 5.3 also holds in $L^\infty(Q, \mu)$, where Q is the unit element of the σ-algebra of Borel sets and μ is a σ-additive Borel measure on Q.

Let D be a compact domain in $\mathbb{R}^n$, let $V, W \subset C(D)$ be non-empty boundedly compact sets consisting of real analytic functions, and let $\varnothing \neq U \subset V \times W$. Consider the class of generalized rational functions

$$\mathscr{R}_U(D) := \{r \in C(D) \mid rw = v,\ w \not\equiv 0,\ (v, w) \in U\}.$$

Corollary 5.1 (see [8]). *If $\mathscr{R}_U(D)$ is algebraically complete, then $\mathscr{R}_U(D)$ is boundedly Δ-compact in $C(D)$. As a consequence, $\mathscr{R}_U(D)$ is an existence set in $C(D)$, and the set $P_{\mathscr{R}_U} x$ is Δ-compact for any $x \in C(D)$.*

As a corollary to Theorem 5.3, we obtain a result of Deutsch [20]: the set

$$\mathscr{R}_V^W := \{r \in C[a, b] \mid rw = v,\ w \in W,\ w \not\equiv 0,\ v \in V\}$$

is an existence set in $C[a, b]$, where V and W are finite-dimensional subspaces of $C[a, b]$ which consist of analytic functions.

The case considered in Corollary 5.1 includes that of algebraically complete multivariate algebraic rational functions $\mathscr{R}_U$, where $U = V \times W$ is algebraically complete.

From Theorem 5.3 we also have the following classical result (in which the proximinality of $\mathscr{R}_{n,m}$ was proved independently by Akhiezer and Walsh): the set of rational functions $\mathscr{R}_{n,m}$ is boundedly Δ-compact in $C[a, b]$. In particular, $\mathscr{R}_{n,m}$ is an existence set, and the set $P_{\mathscr{R}_{n,m}} x$ is Δ-compact for any $x \in C[a, b]$.

Now consider the problem of the existence of best rational approximation in L^p, $1 \leqslant p < \infty$. It is well known (see, for example, § 11.3 in [5]) that the set $\mathscr{R}_{n,m}$ is approximatively compact in $L^p[a, b]$, $1 \leqslant p < \infty$, and therefore is an existence set.

Definition 5.3. Let Σ be a σ-algebra on Ω and μ be a σ-finite measure on Σ. We say that a sequence of functions $x_n \colon \Omega \to \mathbb{R}$ aes-*converges*[1] to a function $x \colon \Omega \to \mathbb{R}$ if, for any set $A \in \Sigma$, $\mu(A) < \infty$, there exists a subsequence (n_k) such that (x_{n_k}) converges to x almost everywhere on A.

A set M is aes-*compact* if any sequence $(x_n) \subset M$ contains a subsequence aes-converging to an element $x \in M$. A set is *boundedly* aes-*compact* if its intersection with any closed ball is aes-compact.

Let μ be a σ-finite measure on Ω, and let $L^p = L^p(\Omega, \Sigma, \mu)$, $1 \leqslant p < \infty$. Next, let $V \subset L^1$ and $W \subset L^q$ be finite-dimensional subspaces ($1/p + 1/q = 1$, $1 < p, q < \infty$; for $p = 1$ we set $q = \infty$), and let $U \subset V \times W$ be a non-empty set.

[1]Here 'aes' stands for '*a*lmost *e*verywhere convergence of a *s*ubsequence'.

Definition 5.4. A set

$$\mathscr{R}_U(\Omega) := \{r \in L^p \mid rw = v,\ w \not\equiv 0,\ (v,w) \in U\} \tag{5.4}$$

is said to be *algebraically complete* if the conditions

(a) $(v_k, w_k) \to (v, w)$ in $L^1 \times L^q$, where $(v_k, w_k) \in U$, $w \not\equiv 0$, and

(b) there exists a function $r \in L^p$ such that $r(t) = v(t)/w(t)$ holds for all $t \in \Omega \setminus \mathrm{Z}(w)$, where $\mathrm{Z}(w)$ is the set of zeros of w,

are equivalent to the inclusion $(v, w) \in U$.

Theorem 5.4 (see [8]). *Let $\mathscr{R}_U(\Omega)$ be algebraically complete, and for each non-zero function from W let its zero set in Ω be a nullset. Then $\mathscr{R}_U(\Omega)$ is boundedly aes-compact in $L^p(\Omega)$ for any $1 \leqslant p < \infty$ and is approximatively compact. As a consequence, $\mathscr{R}_U(\Omega)$ is an existence set.*

Let μ be the Lebesgue measure on D, where D is a bounded domain in $\mathbb{R}^n$ whose boundary is a Lebesgue nullset. Next consider the space $L^p = L^p(D) = L^p(D, \mu)$, $1 \leqslant p < \infty$, and finite-dimensional subspaces $V \subset L^1$ and $W \subset L^q$ consisting of real analytic functions (where $1/p + 1/q = 1$ and $1 < p, q < \infty$; if $p = 1$, then we set $q = \infty$). Let $U \subset V \times W$ be a non-empty set. Consider the following class of generalized rational functions

$$\mathscr{R}_U(D) := \{r \in L^p(D) \mid rw = v,\ w \not\equiv 0,\ (v,w) \in U\}. \tag{5.5}$$

Corollary 5.2 (see [8]). *Let $\mathscr{R}_U(D)$ be algebraically complete. Then $\mathscr{R}_U(D)$ is boundedly aes-compact. As a consequence, $\mathscr{R}_U(D)$ is approximatively compact and is an existence set in $L^p(D)$ for any $1 \leqslant p < \infty$.*

Note that if $D = [a, b] \subset \mathbb{R}$, then the class of rational functions $\mathscr{R}_U(D)$, where $D = [a, b]$, coincides with the class

$$\mathscr{R}^0_U[a,b] := \{r \in C[a,b] \mid rw = v,\ w \not\equiv 0,\ (v,w) \in U\}.$$

In view of this remark, using Corollary 5.2 we obtain the following well-known result due to Deutsch and Huff [20]: the set $\mathscr{R}^W_V[a, b]$ is approximatively compact in $L^p[a, b]$ for any $1 \leqslant p < \infty$ and, as a consequence, is an existence set; here V and W are finite-dimensional subspaces of $L^p[a, b]$ consisting of real analytic functions.

Definition 5.5 (see [69]). We say that $A \subset M \subset X$ is *not woven from closed intervals* if, for any interval $[a, b]$, where $a \in A$ and $b \in M$, $[a, b] \not\subset M$, and for any number $\delta > 0$ there exists a point $c \in A \cap O_\delta(a)$ such that $(c, b) \cap O_\delta(a) \not\subset A$.

Given two boundedly compact sets $V \subset L^1$ and $W \subset L^\infty$ (not necessarily convex), consider the following class of generalized rational functions:

$$\widehat{R}^V_W := \left\{ r = \frac{v}{w} \in L^1 \;\middle|\; v \in V,\ w \in W,\ w > 0 \right\}.$$

Theorem 5.5 (Tsar'kov [69]). *Let $\widehat{R}^V_W$ be a boundedly aes-compact Chebyshev set in L^1, and let $\widehat{R}^V_W$ lie in a linear manifold in which the zero set of each nontrivial function is a nullset. Assume further that, for any neighbourhood $O(x_0)$ of any point $x_0 \in \widehat{R}^V_W$, the set $O(x_0) \cap \widehat{R}^V_W$ is not woven from closed intervals. Then $\widehat{R}^V_W$ is an approximatively compact convex set.*

Let D be a bounded domain in $\mathbb{R}^n$ whose boundary has zero Lebesgue measure, let μ be the Lebesgue measure on D, and set $L^1 = L^1(D,\mu)$. Next, let $V \subset L^1$ be a finite-dimensional subspace of dimension $n \in \mathbb{N}$, let $W \subset L^1$, and let $U \subset V \times W$ be a non-empty set. Consider the following class of generalized rational functions:

$$R^1_U(D) := \bigl\{r \in L^1(D) \mid rw = v,\ w \not\equiv 0,\ (v,w) \in U\bigr\}.$$

Here, in addition, we assume that, for all distinct $w_1, w_2 \in W \setminus \{0\}$ and any function $r \in R^1_U(D)$ such that $rw_1, +rw_2 \in V$, the functions rw_1 and rw_2 are equal. In this case, we say that $R^1_U(D)$ is *irreducible.*

Theorem 5.6 (Tsar'kov [69]). *Let $R^1_U(D)$ be a boundedly aes-compact (or approximatively compact) irreducible subset of L^1 such that its linear hull is dense in $L^1(D)$. Then for any $(n+1)$-dimensional subspace $L \subset L^1[D]$ consisting of real analytic functions there exists a function in L with at least two nearest fractions in $R^1_U(D)$.*

For subspaces of L^p, $1 < p < \infty$, we have the following analogues of Theorem 5.6 (see [8]).

Theorem 5.7. *If the set of rational functions $\mathscr{R}_U(D)$ (see (5.5)) is algebraically complete and not convex, then it is not a Chebyshev set in $L^p(D)$, $1 < p < \infty$. Moreover, in this case, for any convex dense subset H of $L^p(D)$, $1 < p < \infty$, there exists a point $x \in H$ such that the set $P_{\mathscr{R}_U(D)}x$ is not acyclic.*

Theorem 5.8. *If the set of rational functions $\mathscr{R}_{n,m}$ (see (5.2)) is not convex (this set is not convex if and only if $m \geqslant 1$), then it is not a Chebyshev set in $L^p[a,b]$, $1 < p < \infty$. Moreover, in this case, for any convex dense subset H of $L^p[a,b]$, $1 < p < \infty$, there exists a point $x \in H$ such that the set $P_{\mathscr{R}_{n,m}}x$ is not acyclic.*

It is interesting to compare Theorem 5.6–5.8 with the following well-known result of Braess (see, for example, Theorem 11.6 in [5]). *Let $1 < p < \infty$, $n \geqslant 0$ and $m \geqslant 1$. Then any $(n+2)$-dimensional subspace E of $L^p[a,b]$ such that $E \cap \mathscr{R}_{n,m} = \{0\}$ contains a function which has at least two best approximants in $\mathscr{R}_{n,m}$.* It is worth pointing out here that in Theorem 5.6 the result on 'bad' properties of values of the metric projection operator onto the set $\mathscr{R}_U(D)$ (and, in particular, onto $\mathscr{R}_{n,m}$) was obtained by general methods of geometric approximation theory.

6. Existence of continuous selections to the set of generalized rational functions in L^p, $0 < p < \infty$

The first investigations of continuous selections to the set of near-best approximants for rational functions was carried out in spaces with Chebyshev norm (see § 3 and § 7).

In the case of subspaces of $L^p[0,1]$, $1 < p < \infty$, Tsar'kov [58] showed that, for sufficiently small $\varepsilon > 0$, any additive ε-selection of $\mathscr{R}_{n,m}$, $m \geqslant 1$, is discontinuous. A similar problem for generalized rational functions in $L^p[0,1]$ was studied by Ryutin [51], [53].

In this section, we assume that $X = L^p[0,1]$, $0 < p < \infty$; the norm of $L^p[0,1]$ is denoted by $\|\cdot\|$ (if not otherwise specified). Recall that the functional $\|\cdot\| = \|\cdot\|_{L^p[0,1]}$ is a norm for $p \geqslant 1$; for $0 < p < 1$, $\|\cdot\|^p = \|\cdot\|^p_{L^p[0,1]}$ is a metric.

Definition 6.1. Let V and W be subspaces of X. Consider the following class of *generalized rational functions*:

$$\mathscr{R}_{V,W} = \left\{ r = \frac{v}{w} \;\middle|\; v \in V,\ w \in W; \operatorname{ess\,inf} w > 0 \text{ on } [0,1] \right\};$$

the closure of this set in X is denoted by $\overline{\mathscr{R}}_{V,W}$.

In what follows we assume that the sets $\mathscr{R}_{V,W}$ and $\overline{\mathscr{R}}_{V,W}$ are non-empty. Note that for $W = \operatorname{span} 1$ each of the sets $\mathscr{R}_{V,W}$ and $\overline{\mathscr{R}}_{V,W}$ coincides with the subspace V. If $W = \operatorname{span} g$ ($\operatorname{ess\,inf} g > 0$), then the problem reduces to the one for the subspace V/g. Throughout this section $\operatorname{span}\{f_1, f_2, \dots, f_k\}$ denotes the linear hull of the set $(f_j)_{j=1}^k \subset X$ ($k \in \mathbb{N}$). We will generally assume that (f_j) is linearly independent. The characteristic function of a set A is denoted by $\chi_A(\cdot)$.

Definition 6.2. We say that V and W form an *admissible pair* in $L^p[0,1]$, $0 < p < \infty$, if $V, W \subset L^p[0,1]$ are finite-dimensional subspaces of measurable functions on $[0,1]$ such that:

1) $\dim V \geqslant 1$ and $\dim W \geqslant 1$;

2) there exists a function $w_0 \in W$ such that $\operatorname{ess\,inf} w_0 > 0$ on $[0,1]$.

Note that if V, W is an admissible pair in $L^p[0,1]$, $0 < p < \infty$, then $\overline{\mathscr{R}}_{V,W}$ is a non-empty closed subset of L^p.

We define the function $C(m,n,p)$ as follows: $C(m,n,p) := n^{1/p}$ for $1 \leqslant p < \infty$, $m, n \in \mathbb{N}$, and $C(m,n,p) := n(m+1)^{1-p}$ for $0 < p \leqslant 1$, $m, n \in \mathbb{N}$.

Theorem 6.1. *Let V, W be an admissible pair in $X = L^p[0,1]$, $0 < p < \infty$, let $\dim V = m$ and $\dim W = n$, where $m, n < \infty$, and let $R = \overline{\mathscr{R}}_{V,W}$. Then for any constant $C > C(m,n,p)$ there exists a continuous mapping $\Phi\colon X \to R$ such that*

$$\|\Phi(f) - f\| \leqslant C\rho(f, R) \quad \forall\, f \in X.$$

Note that this result is well known in the particular case where $1 \leqslant p < \infty$, V is a subspace in L^p, and $W = \operatorname{span} 1$, that is, where $\overline{\mathscr{R}}_{V,W} = V$ (this case will be referred to as the subspace case); in this case the required result follows easily from Michael's continuous selection theorem. If $0 < p < 1$, then L^p is not a normed space, and in this setting even the subspace case requires a proof. We will employ the method of averaging used by Al'brecht (see, for example, [36]).

Let $\Delta \subset \mathbb{R}^{m+n}$ be a convex polyhedron of dimension $m+n$, and let Σ be the set of all simplexes σ such that each vertex of σ is a vertex of Δ, and $\dim \sigma \leqslant n$.

The proof of Theorem 6.1 depends on the following lemma.

Lemma 6.1. *Let $\Delta \subset \mathbb{R}^{m+n}$ be a convex polyhedron with non-empty interior, and let π be an affine subspace of $\mathbb{R}^{m+n}$, where $\dim \pi = m$. Then*

$$\Delta \cap \pi = \operatorname{conv}\left\{ \bigcup_{\sigma \in \Sigma} (\pi \cap \sigma) \right\}.$$

Proof of Theorem 6.1. We set $\rho(x) := \rho(x, R)$ and define

$$P^\alpha(x) := \{ r \in R \mid \|x - r\| \leqslant (1+\alpha)\rho(x) \},$$

where $x \in X$ and $\alpha > 0$. Consider

$$W^+ := \{w \in W \mid \operatorname{ess\,inf} w \geqslant 0 \text{ on } [0,1]\}.$$

It is clear that W^+ is a closed convex cone with non-empty interior in W. By the definition of W^+, $W^+ \cap (-W^+) = \{0\}$. Hence there exists an affine hyperplane Γ in W that does not pass through 0 and intersects any ray from 0 lying in the cone W^+ in a unique point. We set $Q = W^+ \cap \Gamma$. It is easily seen that Q is a compact subset of Γ. Let $U = V \times \Gamma$. In what follows we put $u = (v, w)$, where $v \in V$ and $w \in \Gamma$. We also assume that V and Γ are equipped with the Lebesgue measures dv and dw, respectively.

Consider the two mappings $r\colon V \times Q \to \overline{\mathscr{R}}_{V,W}$ and $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$r(u) := \frac{v}{w} \quad \text{and} \quad f(t) = f_x(t) := \max\{0, (1+\alpha)\rho(x) - t\}$$

(r is in general not defined on the whole of $V \times Q$). It is easily seen that any rational function $y \in R$ can be written as $y = v/w$, where $v \in V$ and $w \in Q$. With each point $x \in X$ we associate the bounded set

$$\Omega(x) = \{u \in U \mid r(u) \in P^\alpha(x),\ w \in Q\}, \qquad \Omega = \Omega(x) \subset V \times Q.$$

We also set

$$\Psi(x) = \biggl(\int_\Omega u f(\|x(u) - r(u)\|)\,du\biggr) \cdot \biggl(\int_\Omega f(\|x(u) - r(u)\|)\,du\biggr)^{-1},$$

where $du = dv\,dw$. We claim that $\Phi = r \circ \Psi$ is the required mapping. To show this we estimate the norm $\|x - \Phi(x)\|$ from above. Let $\widehat{u} = (\widehat{v}, \widehat{w}) := \Psi(x)$. We show below that $r(\widehat{u})$ is well defined, that is, $r(\widehat{u}) \in R \subset X$. Note that $\Psi(x) \in \operatorname{conv}\Omega(x)$. By Carathéodory's theorem (see, for example, [5], Appendix B), there exist points

$$u_0 = (v_0, w_0),\ u_1 = (v_1, w_1),\ \dots,\ u_{m+n} = (v_{m+n-1}, w_{m+n-1}) \in V \times Q$$

and numbers

$$\mu_0, \dots, \mu_{m+n-1} \in [0,1]$$

such that

$$\widehat{u} = \sum_{j=0}^{m+n-1} \mu_j u_j \quad \text{and} \quad \sum_{j=0}^{m+n-1} \mu_j = 1$$

and $r(u_j) \in P^\alpha(x)$ for any $0 \leqslant j \leqslant m+n-1$. Consider the set

$$Y(\widehat{w}) := \biggl\{\tilde{v} = \sum_{k=0}^{n-1} \xi_k v_{j_k} \biggm| \sum_{k=0}^{n-1} \xi_k = 1,\ \xi_k \geqslant 0,\ \sum_{k=0}^{n-1} \xi_k w_{j_k} = \widehat{w}\biggr\} \subset V.$$

Given any function $\tilde{v} \in Y(\widehat{w})$, we have

$$\biggl\|\frac{\tilde{v}}{\widehat{w}} - x\biggr\| \leqslant \gamma(n,p)(1+\alpha)\rho(x), \tag{6.1}$$

where $\gamma(n,p) := n^{1/p}$ for $1 \leqslant p < \infty$ and $\gamma(n,p) := n$ for $0 < p < 1$. Let us prove (6.1) for $0 < p < 1$. First of all, we have

$$\left|\frac{\tilde{v}}{\widehat{w}} - x\right|^p = \left|\frac{\sum_{k=0}^{n-1} \xi_k (v_{j_k} - x w_{j_k})}{\sum_{k=0}^{n-1} \xi_k w_{j_k}}\right|^p \leqslant \sum_{k=0}^{n-1} \left|\frac{v_{j_k}}{w_{j_k}} - x\right|^p.$$

Therefore,

$$\left\|\frac{\tilde{v}}{\widehat{w}} - x\right\| \leqslant \sum_{k=0}^{n-1} \left\|\frac{v_{j_k}}{w_{j_k}} - x\right\| \leqslant n(1+\alpha)\rho(x) = \gamma(n,p)(1+\alpha)\rho(x).$$

Inequality (6.1) for $1 \leqslant p < \infty$ is proved similarly.

Using Lemma 6.1 for $\Delta = \operatorname{conv}\{u_0, \dots, u_{m+n-1}\}$ and $\pi = \{(v,w) \in U \mid w = \widehat{w}\}$, we find that $\widehat{v} \in \operatorname{conv} Y(\widehat{w})$. Hence there exist numbers $\eta_0, \eta_1, \dots, \eta_m \in [0,1]$ and points $\tilde{v}_0, \dots, \tilde{v}_m \in Y(\widehat{w})$ such that $\sum_{j=0}^m \eta_j = 1$ and $\widehat{v} = \sum_{j=0}^m \eta_j \tilde{v}_j$.

We have

$$\|\Phi(x) - x\| = \left\|\frac{\widehat{v}}{\widehat{w}} - x\right\| = \left\|\sum_{j=0}^m \eta_j \left(\frac{\tilde{v}_j}{\widehat{w}} - x\right)\right\|.$$

An application of inequality (6.1) shows that

$$\|\Phi(x) - x\| \leqslant \sum_{j=0}^m \eta_j^p n(1+\alpha)\rho(x) \leqslant (m+1)^{1-p} n(1+\alpha)\rho(x)$$

for $0 < p < 1$. A similar argument for $1 \leqslant p < \infty$ shows that

$$\|\Phi(x) - x\| \leqslant n^{1/p}(1+\alpha)\rho(x).$$

Thus, for all $0 < p < \infty$ and $\delta > 0$, there exists $\alpha > 0$ such that $\|x - \Phi(x)\| \leqslant C\rho(x)$ for any $x \in X$.

Note that $\Omega(x)$ and both integrals involved in the definition of Ψ depend continuously on x, and the denominator of Ψ does not vanish. Therefore, Φ is continuous. This proves Theorem 6.1.

We show below that in $X = L^p[0,1]$ $(0 < p < 1)$ there are no continuous ε-selections for all sufficiently small $\varepsilon > 0$ in some simple cases of approximation by generalized rational functions (and, in particular, in the subspace case).

Theorem 6.2. *Consider the following two cases.*

A. *V is an arbitrary subspace of $L^p[0,1]$, $0 < p < 1$, $\dim V < \infty$, and $W = \operatorname{span} 1$.*

B. *V and W are subspaces of $L^p[0,1]$, $0 < p < 1$, such that $V = \operatorname{span} 1$, $\dim W < \infty$, and there exists a function $w_0 \in W$ such that $\operatorname{ess\,inf} w_0 > 0$ on $[0,1]$.*

In both cases A *and* B, *there is no continuous multiplicative ε-selection $\Phi\colon L^p \to \overline{\mathscr{R}}_{V,W}$ for any $0 \leqslant \varepsilon < 2^{1-p} - 1$.*

Proof. First we verify Theorem 6.2 in case A, that is, where $\overline{\mathscr{R}}_{V,W}$ is a subspace of V. There exist a hyperplane Γ in V and a function $\psi \in V$ such that $\|\psi\| = 1$ and $\rho(\psi, \Gamma) = 1$. Indeed, the required function ψ can be constructed as follows: we take an arbitrary point $q \in V \setminus \Gamma$, and let q_0 be a nearest point from Γ for q. It is easily seen that there exists $c \in \mathbb{R}$ such that $\|\psi\| = 1$, where $\psi := c(q - q_0)$. For arbitrary $\sigma > 0$ one can construct a curve $\gamma = \{\gamma(s)\} \subset X$, $s \in [0,1]$, with the following properties:

a) $\gamma(0) = \psi$, $\gamma(1) = -\psi$;

b) $(2^{1-p} - \sigma)\rho(\gamma(s), V) < \rho(\gamma(s), \Gamma)$ for any $s \in [0,1]$.

If $E \subset [0,1]$, then by $V[E]$ we denote the set of restrictions to E of elements of V. We use an idea due to Kamuntavichius [30]. For any $N > 1$ one can (see [30]) construct a partition of $[0,1]$ (up to a nullset) into a countable set of measurable subsets E_j ($j \in J$) with the following properties:

1) $0 < \mu(E_j)$ and $V[E_j] \subset L^\infty(E_j)$ for any $j \in J$;

2) $\operatorname{ess\,sup}_{\tau \in E_j} f(\tau) - \operatorname{ess\,inf}_{\tau \in E_j} f(\tau) \leqslant \|f\|^{1/p}/N$ for all $f \in V$, $j \in J$.

We fix some (sufficiently large) number N and the corresponding partition E_j ($j \in J$). We indicate below how N should be chosen.

For each $j \in J$ we fix a family $\{H_j(\lambda)\}_{\lambda \in [0,1]}$ of measurable subsets of E_j such that $\mu(H_j(\lambda)) = \lambda\mu(E_j)$ and $H_j(\lambda) \subset H_j(\mu)$ if $\lambda \leqslant \mu$. We also set $G_j(\lambda) := E_j \setminus H_j(\lambda)$.

The required curve $\{\gamma(s)\}$ is defined by

$$\gamma(s) = \sum_J \psi \cdot \chi_{G_j(s)} - \sum_J \psi \cdot \chi_{H_j(s)}.$$

We claim that $\gamma(s)$ satisfies condition b). First we note that

$$\rho(\gamma(s), V) \leqslant \min\bigl\{\|\gamma(s) - \psi\|, \|\gamma(s) + \psi\|\bigr\}.$$

We set $H(s) := \bigcup_J H_j(s)$ and $G(s) := \bigcup_J G_j(s)$. It is clear that

$$\min\bigl\{\|\gamma(s) - \psi\|, \|\gamma(s) + \psi\|\bigr\} = \min\biggl\{\int_{H(s)} |2\psi|^p \, d\mu, \int_{G(s)} |2\psi|^p \, d\mu\biggr\},$$

which is at most 2^{p-1}.

We claim that for any $\delta > 0$ there exists $N > 0$ (from condition 2)) such that $\|\gamma(s) - f\| \geqslant 1 - \delta$ for all $s \in [0,1]$ and $f \in \Gamma$. Assume on the contrary that there exists a number $\delta_0 > 0$ such that for any $N > 0$ there are $f \in \Gamma$ and $s \in [0,1]$ such that $\|\gamma(s) - f\| \leqslant 1 - \delta_0$. For $v \in V$ we set $v_j := \operatorname{ess\,inf}_{\tau \in E_j} v(\tau)$ and consider the function $\widehat{v} := \sum_J v_j \chi_{E_j}$. By condition 2), $\widehat{v}$ tends to v in the norm of L^∞ as $N \to \infty$. For sufficiently large N we have

(i) $\|\widehat{\gamma}(s) - \widehat{f}\| \leqslant 1 - (2/3)\delta_0$;

(ii) $\|\widehat{\psi} - \widehat{f}\| \geqslant 1 - \delta_0/3$, $\|\widehat{\psi} + \widehat{f}\| \geqslant 1 - \delta_0/3$,

where $\widehat{\gamma}(s) := \sum_J \widehat{\psi}\chi_{G_j(s)} - \sum_J \widehat{\psi}\chi_{H_j(s)}$. Let us show that inequalities (i) and (ii) are inconsistent. We set

$$A_j := |f_j - \psi_j|^p, \qquad B_j := |f_j + \psi_j|^p, \quad \text{and} \quad \mu_j := \mu(E_j).$$

Now inequalities (i) and (ii) can be written as

$$(1 - s)\sum_J A_j\mu_j + s\sum_J B_j\mu_j \leqslant 1 - \frac{2}{3}\,\delta_0,$$

$$\sum_J A_j\mu_j \geqslant 1 - \frac{\delta_0}{3}, \quad \text{and} \quad \sum_J B_j\mu_j \geqslant 1 - \frac{\delta_0}{3},$$

which is a contradiction. So we have shown that the curve $\{\gamma(s)\}$ satisfies condition b), while condition a) is satisfied by the construction of $\{\gamma(s)\}$.

To complete the proof of the theorem in case A, we assume that $\Phi\colon X \to V$ is a continuous ε-selection and $1+\varepsilon < 2^{1-p}$. Consider the curve $\{\gamma(s)\}$ corresponding to $\sigma > 0$ such that $1+\varepsilon < 2^{1-p} - \sigma$. We claim that no ε-selection Φ can even be continuous on $\{\gamma(s)\}$. Indeed, it is easily seen that $\Phi(\gamma(0)) = \psi$, $\Phi(\gamma(1)) = -\psi$, and $\{\Phi(\gamma(s))\} \cap \Gamma = \varnothing$. But then the set $\{\Phi(\gamma(s))\}$ cannot be connected. This proves Theorem 6.2 in case A.

To prove the theorem in case B, consider a curve $\{\widetilde{\gamma}(s)\}$ such that $\widetilde{\gamma}(0) = 1/w_0$, $\widetilde{\gamma}(1) = -1/w_0$, and for all $s \in [0,1]$,

$$(2^{1-p} - \sigma)\rho(\widetilde{\gamma}(s), \overline{\mathscr{R}}_{V,W}) < \|\widetilde{\gamma}(s)\|.$$

Such a curve $\widetilde{\gamma}$ is constructed similarly to γ. To complete the proof in case B, it remains to take $\widetilde{\gamma}$ as γ, replace Γ by $\{0\}$, and note that the set $\{\Phi(\widetilde{\gamma}(s))\}$ cannot be connected. Theorem 6.2 is proved.

Remark 6.1. In the actual fact, Theorem 6.2 establishes slightly more than we have claimed. Namely, we showed that there are no continuous ε-selections not only on the whole of L^p, but also on any subset M of L^p containing the curve $\{\gamma(s)\}$ (or $\{\widetilde{\gamma}(s)\}$). An important example of such a set M is the unit ball of L^p.

Now consider the space $X = L^1[0,1]$. We claim that for small $\varepsilon > 0$ there is no continuous multiplicative ε-selection to the set of generalized rational functions in $L^1[0,1]$.

Here we consider only the case where V is a subspace of $L^1[0,1]$ and W is a subspace of $C[0,1]$. Since W contains a positive function, the set $\mathscr{R}_{V,W}$ is non-empty.

We recall (see [35], § II.1, and [5], Appendix A) that a linearly independent system of functions $\{p_0,\dots,p_{k-1}\} \subset C^{k-1}[0,1]$ ($k \in \mathbb{N}$) is an ET-system on $[0,1]$ if any function $p \not\equiv 0$ in the linear hull of $p_0,\dots,p_{k-1}$ has at most $k-1$ zeros on $[0,1]$, where each zero is counted according to its algebraic multiplicity. By an ET_{k-1}-space we mean the liner hull of an ET-system of cardinality k.

We assume that V and W satisfy the following conditions:

1) V is a subspace in $L^1[0,1]$, $1 \leqslant \dim V < \infty$;
2) W is an ET_{n-1}-space, $n \geqslant 3$.

Note that there exists a function $w_0 \in W$ such that $w_0(\tau) > 0$ on $[0,1]$ (see [35], Theorem 1.4, § II.1). In addition, it can be assumed that W contains the constant functions. Indeed, we can change from the pair of subspaces (V,W) to the pair

$$(\widetilde{V},\widetilde{W}), \quad \text{where} \quad \widetilde{W} = \frac{1}{w_0}W \quad \text{and} \quad \widetilde{V} = \frac{1}{w_0}V.$$

As before, the pair $(\widetilde{V},\widetilde{W})$ satisfies 1) and 2). We set

$$W^+ = \{w \in W \mid w(\tau) \geqslant 0 \text{ on } [0,1]\}.$$

Note that for each function $r \in \overline{\mathscr{R}}_{V,W}$ one can find a pair $v \in V$, $w \in W^+$ such that $r(\tau) = v(\tau)/w(\tau)$ almost everywhere on $[0,1]$.

Theorem 6.3. *Let the pair* (V,W) *satisfy conditions* 1) *and* 2). *Then there exists* $\varepsilon_0=\varepsilon_0(V,W)>0$ *such that* $\varepsilon\geqslant\varepsilon_0$ *for any continuous multiplicative* ε*-selection* $\Phi\colon L^1[0,1]\to\overline{\mathscr{R}}_{V,W}$.

Let $\|\cdot\|_C$ be the Chebyshev norm on W, and let $\|\cdot\|$ be the usual L^1-norm on W. For brevity we will sometimes write $\mathscr{R}$ in place of $\overline{\mathscr{R}}_{V,W}$. Further, we let $I(m)$ denote the set $\{1,\dots,m\}$, where $m\in\mathbb{N}$, and let

$$\Delta=\biggl\{\lambda=(\lambda_1,\dots,\lambda_N)\in\mathbb{R}^N\biggm|\sum_{j=1}^N\lambda_j=1,\ \lambda_j\geqslant 0\ \forall\, j\biggr\}$$

be the 'standard' simplex. By a neighbourhood of a point $\tau\in[0,1]$ we mean a closed interval with centre τ.

We need some auxiliary results. Let $v\in V$, $v\neq 0$. Then there exist a number $d=d(v)\in(0,1)$ and a measurable set $T=T(v)\subset[d/10,1-d/10]$ such that $\mu(T)=d$ and $v(\tau)\neq 0$ on T.

Lemma 6.2 (see [53]). *Let a pair* (V,W) *satisfy conditions* 1) *and* 2). *Then for any* $\delta>0$ *and almost all* $\widetilde{\tau}\in T$ *there exist a function* $r(\tau)\in\overline{\mathscr{R}}_{V,W}$ *and a neighbourhood* U *of* $\widetilde{\tau}$ *such that*

$$\mu(U)\leqslant\delta,\qquad\int_U|r(\tau)|\,d\tau=1,\qquad\textit{and}\qquad\int_{[0,1]}|r(\tau)|\,d\tau\leqslant 1+\delta.$$

Corollary 6.1. *For all* $N\in\mathbb{N}$ *and* $\delta>0$ *there exist points* $\tau_1,\dots,\tau_N\in T$, *functions* $e_1(\tau),\dots,e_N(\tau)\in\overline{\mathscr{R}}_{V,W}$, *and neighbourhoods* $U_1,\dots,U_N$ *of* $\tau_1,\dots,\tau_N$ *such that, for any* $j\in I(N)$,

$$\min_{i\neq j}|\tau_i-\tau_j|\geqslant\frac{d}{3N},\qquad\mu(U_j)\leqslant\delta,$$
$$\int_{U_j}|e_j(\tau)|\,d\tau=1,\qquad\textit{and}\qquad\int_{[0,1]}|e_j(\tau)|\,d\tau\leqslant 1+\delta.$$

Lemma 6.3 (see [53]). *Let the pair* (V,W) *satisfy conditions* 1) *and* 2), *let* $h>0$, $K>n/2$, *and let* $\tau_j\in(0,1)$, $j\in I(K)$, *be distinct points. Then for any* $C>0$ *there exists* $\delta>0$ *such that if* $r(\tau)\in\overline{\mathscr{R}}_{V,W}$, U_j *are neighbourhoods of* τ_j, $\mu(U_j)\leqslant\delta$, *and if* $\displaystyle\int_{U_j}|r(\tau)|\,d\tau\geqslant h$ *for each* $j\in I(K)$, *then*

$$\int_{[0,1]}|r(\tau)|\,d\tau\geqslant C.$$

We set $F(\tau,a):=[0,1]\setminus[\tau-a,\tau+a]$, where $a\in(0,1/10)$ and $\tau\in[2a,1-2a]$, and we set $S(W):=\{w\in W^+\mid\|w\|=1\}$.

Lemma 6.4 (see [53]). *Let* W *be an* ET_{n-1}*-space,* $n\geqslant 3$, *and let* $a\in(0,1/10)$. *Then there exist* $\delta_0=\delta_0(a,W)>0$ *and* $k=k(a,W)>0$ *such that for all* $\tau_0\in[2a,1-2a]$, $\theta>0$ *and* $w\in S(W)$, $w(\tau_0)\leqslant k\theta$, *there exists a function* $f\in W^+$ *such that*

$$f(\tau)\geqslant w(\tau)\qquad\textit{on}\ \ U=[\tau_0-\delta_0,\tau_0+\delta_0]\tag{6.2}$$

and

$$f(\tau) \leqslant \theta w(\tau) \quad \text{on} \quad F = F(\tau_0, a). \tag{6.3}$$

Consider a number $\delta > 0$ and the functions e_j from Corollary 6.1 ($j \in I(N)$). Let $J \subset I(N)$ be a set of cardinality $\leqslant N - 1$. Then we set

$$G(J) := \operatorname{conv}\{(e_j)_{j \in J}\} \quad \text{and} \quad \gamma := \operatorname{conv}\{(e_j)_{j \in I(N)}\}.$$

Assume that $\delta < \eta$ (the conditions on the parameter $\eta > 0$ will be specified in the proof of Lemma 6.5). Then the following result holds.

Lemma 6.5. 1) *$\rho(g, R) \leqslant (N-1)(1+\delta)/N$ for any point $g \in \gamma$.*

2) *For all $g \in G(J)$, $r \in R$ and $\varepsilon \in [0, 1)$ the conditions $\|g - r\| \leqslant (1+\varepsilon)\rho(g, R)$ and $j_0 \in I(N) \setminus J$ imply that $\displaystyle\int_{U_{j_0}} |r(\tau)|\, d\tau \leqslant 5\varepsilon$.*

Proof. Let us check 1). If $g = \sum_{j=1}^N \mu_j e_j \in \gamma$ ($\mu \in \Delta$), then

$$\rho(g, R) \leqslant \min_j \|g - \mu_j e_j\| \leqslant \Bigl(1 - \max_j \mu_j\Bigr)(1+\delta) \leqslant \frac{N-1}{N}(1+\delta).$$

Now let us verify 2). Let $r(\tau) = \dfrac{v(\tau)}{w(\tau)}$, $\displaystyle\int_{U_{j_0}} |r(\tau)|\, d\tau = \sigma > 4\varepsilon$, where $v \in V$, $w \in W^+$, and $\|w\|_C = 1$. Let $\eta < a := \delta/(10N)$, and let $O = [-a + \tau_{j_0}, \tau_{j_0} + a]$ be a neighbourhood of τ_{j_0}. We have $U_{j_0} \subset O$ and $O \cap U_j = \varnothing$ for $j \neq j_0$. Next, we define $U = U_{j_0}$, $E = O \setminus U$, and $F = [0, 1] \setminus O$. Using assertion 1) of the lemma (for $\varepsilon < 1$ and $\delta < 1/3$) we have $\|r\| \leqslant 4$. As a result, $\displaystyle\int_O |v(\tau)|\, d\tau \leqslant 4$. For each $q > 0$ there exists $\delta_1 > 0$ such that

$$\int_U |v(\tau)|\, d\tau \leqslant q$$

whenever $\mu(U) = \delta \leqslant \delta_1$ and $\displaystyle\int_O |v(\tau)|\, d\tau \leqslant 4$. As a result, $\min_U w(\tau) \leqslant q/\sigma$. We set $\theta = q/(2k\varepsilon)$, where the number k is defined as in Lemma 6.4 and q is such that $4\theta/(1+\theta) \leqslant \varepsilon/4$. There exists $\delta_1 > 0$ such that we have $w(\tau_{j_0}) \leqslant 2q/\sigma \leqslant k\theta$. An application of Lemma 6.4 produces a function f. Consider the rational function $\tilde{r} = v/(w + f)$. Let us give the necessary estimates.

First, we have

$$\int_E \Bigl| g(\tau) - \tilde{r}(\tau) \Bigr|\, d\tau \leqslant \delta + \int_E \frac{|v(\tau)|}{w(\tau) + f(\tau)}\, d\tau \leqslant \delta + \int_E \frac{|v(\tau)|}{w(\tau)}\, d\tau$$
$$\leqslant \int_E \left| g(\tau) - \frac{v(\tau)}{w(\tau)} \right| d\tau + 2\delta,$$

where we have used the fact that $\int_O |g(\tau)|\,d\tau \leqslant \delta$. Second, we have

$$\int_U \left| g(\tau) - \frac{v(\tau)}{w(\tau)+f(\tau)} \right| d\tau \leqslant \delta + \int_U \left| \frac{v(\tau)}{w(\tau)+f(\tau)} \right| d\tau$$
$$\leqslant \delta + \frac{1}{2}\int_U \left| \frac{v(\tau)}{w(\tau)} \right| d\tau = \delta + \frac{\sigma}{2} \leqslant \int_U \left| g(\tau) - \frac{v(\tau)}{w(\tau)} \right| d\tau - \frac{\sigma}{2} + 2\delta.$$

Further, we have

$$\int_F \left| g(\tau) - \frac{v(\tau)}{w(\tau)+f(\tau)} \right| d\tau \leqslant \int_F \left| g(\tau) - \frac{v(\tau)}{w(\tau)} \right| d\tau + \int_F \left| \frac{v(\tau)}{w(\tau)+f(\tau)} - \frac{v(\tau)}{w(\tau)} \right| d\tau.$$

Since $\|r\| \leqslant 4$, the last integral is estimated as follows:

$$\int_F \left| \frac{v(\tau)}{w(\tau)+f(\tau)} - \frac{v(\tau)}{w(\tau)} \right| d\tau \leqslant \int_F \left| \frac{v(\tau)}{w(\tau)} \right| \left(1 - \frac{w(\tau)}{w(\tau)+f(\tau)}\right) d\tau$$
$$\leqslant \frac{\theta}{1+\theta}\int_F \left| \frac{v(\tau)}{w(\tau)} \right| d\tau \leqslant \frac{4\theta}{1+\theta}.$$

So we have $\rho(g,R) \leqslant \|g-\tilde{r}\| \leqslant \|g-r\| - \sigma/2 + 4\delta + 4\theta/(1+\theta)$. Now using the inequality $\|g-r\| \leqslant (1+\varepsilon)\rho(g,R)$ we obtain

$$\frac{\sigma}{2} \leqslant \varepsilon\rho(g,R) + 4\delta + \frac{4\theta}{1+\theta} \leqslant 2\varepsilon + \frac{\varepsilon}{2}$$

(here we assume that $\eta \leqslant \varepsilon/16$). Setting

$$\eta = \min\left\{\delta_0, \delta_1, \frac{d}{10N}, \frac{1}{3}, \frac{\varepsilon}{16}\right\},$$

we have $\sigma \leqslant 5\varepsilon$, which proves Lemma 6.5.

Lemma 6.6. *Let $\sigma > 0$ be sufficiently small, and let $\psi\colon \Delta \to \Delta$ be a continuous mapping such that $\sup_{a\in\psi(G)} \rho(a,G) \leqslant \sigma$ for any face G of the simplex Δ ($0 \leqslant \dim G < \dim \Delta$). Then*

$$p \in \psi(\Delta),$$

where $p = (1/N,\dots,1/N)$ is the centre of the simplex Δ.

Proof. Assume for a contradiction that $p \notin \psi(\Delta)$. Consider the mapping $f\colon \Delta \to \Delta$ defined by $f(x) = l(\psi(x);p) \cap \partial\Delta$, where $l(a,b)$ is the ray from a to b. Note that f has no fixed points. However, this contradicts Brouwer's fixed point theorem. Lemma 6.6 is proved.

Now we proceed with the proof of Theorem 6.3.

Proof. Let $N = [n/2]+1$. We claim that for any $\varepsilon \in (0,\varepsilon_0)$ there exist $\delta > 0$, functions $e_1,\dots,e_N$ (from Corollary 6.1), and a simplex

$$\gamma = \left\{ f_\lambda \in L^1[0,1] \;\middle|\; f_\lambda = \sum_{j=1}^{N} \lambda_j e_j,\ \lambda \in \Delta \right\}$$

such that Φ has points of discontinuity even on γ (the constraints on ε_0 will be clear in the course of the proof). Let $r_\lambda = \Phi(f_\lambda)$. Consider the mapping $F\colon \gamma \to \mathbb{R}^N$ defined by

$$F(f_\lambda) := \frac{1}{S(f_\lambda)}(F_1(r_\lambda), \dots, F_N(r_\lambda))$$

where

$$F_j(g) := \int_{U_j} |g(\tau)|\,d\tau, \quad F_j\colon L^1[U_j] \to \mathbb{R}, \ \ j \in I(N),$$

and $S(f_\lambda) := \sum_{j=1}^N F_j(r_\lambda)$ (we show below that $S(f_\lambda) > 0$). Note that $F(\gamma) \subset \Delta$. It suffices to verify that the above mapping F cannot be continuous on the whole of γ. For each $j \in I(N)$ we have $F_j(f_\mu) \geqslant 1 - \delta$. By the definition of an ε-selection,

$$(1+\varepsilon)\rho(f_\mu, R) \geqslant \|f_\mu - r_\mu\| \geqslant \sum_{j=1}^N F_j(f_\mu - r_\mu) \geqslant 1 - N\delta - \sum_{j=1}^N F_j(r_\mu).$$

Now, by the first assertion of Lemma 6.5, for any $\mu \in \Delta$ we have

$$S(f_\mu) \geqslant 1 - N\delta - (1+\varepsilon)(1+\delta)\frac{N-1}{N}.$$

So, $S(f_\mu) \geqslant 1/(2N)$ for sufficiently small $\varepsilon, \delta > 0$. From this estimate and the second part of Lemma 6.5 it follows that Lemma 6.6 can be used in our setting. Hence there exists a point $f_\lambda \in \gamma$ such that $F(f_\lambda) = (1/N, \dots, 1/N)$. From Lemma 6.3 for $C = 2$ we obtain the inequality $\|r_\lambda\| \geqslant 2$, which contradicts the estimate

$$\|r_\lambda\| \leqslant (1+\varepsilon)\rho(f_\lambda, R) + \|f_\lambda\| \leqslant \frac{N-1}{N}(1+\delta)(1+\varepsilon) + 1 + \delta.$$

Theorem 6.3 is proved.

Remark 6.2. (i) Theorem 6.3 applies to the classical set of algebraic rational functions $\mathscr{R}_{m,n}$, $n \geqslant 2$. There is also an analogue of this result for $n = 1$; however, the proof in this case uses a different construction.

(ii) The above results show that $\varepsilon_0(V, W)$ is $\varepsilon_0(n)$, that is, this quantity depends only on $\dim W = n$.

(iii) In fact, in Theorem 6.3 we have established that for small $\varepsilon > 0$ any ε-selection is discontinuous on any set containing the above simplex γ. An important example of such a set is the intersection of the unit ball of $L^1[0,1]$ with the linear hull of $\overline{\mathscr{R}}_{V,W}$.

To conclude this section, we give some comments.

1. Theorems 6.1 and 6.2 can be obtained for subspaces of $L^p(Z, \mu)$, where (Z, μ) is an arbitrary set with non-atomic measure μ, $\mu(Z) < \infty$. The proofs require only minor modifications.

2. From the results in this section it follows that, in L^p, $0 < p < \infty$, for some sets of generalized rational functions $\overline{\mathscr{R}}$ there exists a constant $\varepsilon_0 = \varepsilon_0[\overline{\mathscr{R}}] \in [0, \infty)$ such that for $\varepsilon \in [0, \varepsilon_0)$ no multiplicative ε-selection to $\overline{\mathscr{R}}$ is continuous on the whole of L^p (and even on the unit ball of L^p) but, still, for any $\varepsilon > \varepsilon_0$ one can construct a multiplicative ε-selection to $\overline{\mathscr{R}}$ which is continuous on the entire L^p.

The quantity $\varepsilon_0[\overline{\mathscr{R}}]$ coincides with the infimum of $\varepsilon \geqslant 0$ such that there exists a continuous ε-selection from the unit ball of L^p to the set $\overline{\mathscr{R}}$. This follows from our constructions and the following simple observation. *Let X be a normed linear space, B be the unit ball of X, and K be a cone in X. If, for any $\varepsilon > 0$, there exists a continuous multiplicative ε-selection $\Phi\colon B \to K$, then there also exists a continuous multiplicative ε-selection $\widetilde{\Phi}\colon X \to K$.*

The required selection is constructed as follows:

$$\widetilde{\Phi}(x) := \|x\|\Phi\biggl(\frac{x}{\|x\|}\biggr) \quad \text{for } \|x\| > 1 \quad \text{and} \quad \widetilde{\Phi}(x) := \Phi(x) \quad \text{for} \quad \|x\| \leqslant 1.$$

A similar construction can also be carried out in L^p, $0 < p < 1$, and so the above fact also holds in this setting.

3. Theorem 6.1 and 6.2 imply the equality $\varepsilon_0[V] = 2^{1-p}-1$ for any $0 < p < 1$ and any one-dimensional subspace V in $L^p[0,1]$ (the quantity $\varepsilon_0[V]$ has been defined above).

4. For some subspaces one can obtain better results than those provided by Theorem 6.1. We define $L^{(n)} := \operatorname{span}(\chi^{(j)})$, where $\chi^{(j)} := \chi_{[(j-1)/n,j/n]}$, and $1 \leqslant j \leqslant n$. We claim that $\varepsilon_0\bigl[L^{(n)}\bigr] = 2^{1-p} - 1$ for all $n \in \mathbb{N}$ and $0 < p < 1$. Let $\pi_j\colon L^p[0,1] \to L^p[0,1]$, $1 \leqslant j \leqslant n$, be the projection defined by $\pi_j(f) = \chi^{(j)}f$, where $f \in L^p$. We set $V_j = \pi_j(L^{(n)})$ and $X_j = \pi_j(L^p[0,1])$, $1 \leqslant j \leqslant n$. It is easily seen that V_j is a one-dimensional space, and X_j can be identified with $L^p[(j-1)/n, j/n]$. From Theorem 6.1 it follows that, for all $\sigma > 0$ and $1 \leqslant j \leqslant n$, there exists a continuous multiplicative $(2^{1-p}-1+\sigma)$-selection $\Phi_j\colon X_j \to V_j$. Given $x \in L^p[0,1]$, we set $\Phi(x) := \sum_{j=1}^n \Phi_j(\pi_j(x))$. We claim that, for each $\sigma > 0$, Φ is a continuous $(2^{1-p} - 1 + \sigma)$-selection from $L^p[0,1]$ to $L^{(n)}$. Indeed,

$$\|\Phi(x) - x\| = \sum_{j=1}^{n} \|\Phi_j(\pi_j(x)) - \pi_j(x)\| \leqslant (2^{1-p} + \sigma)\sum_{j=1}^{n} \rho(\pi_j(x), V_j).$$

Further, for any $x \in L^p[0,1]$, we have $\sum_{j=1}^n \rho(\pi_j(x), V_j) = \rho(x, L^{(n)})$, which gives, as a result, $\|\Phi(x) - x\| \leqslant (2^{1-p} + \sigma)\rho(x, L^{(n)})$.

7. Stability of near-best approximation by generalized rational functions in the uniform norm

Kirchberger [31] showed that in the space $C[0,1]$ the (single-valued) metric projection operator onto the subspace $\mathscr{P}_n$ of all polynomials of degree at most n is continuous and locally Lipschitz continuous. This result has been generalized extensively — we mention one such extension. A well-known result due to Newman, Shapiro, and Chebotarev (see Theorem 2.13 in [5]) states that any finite-dimensional Chebyshev subspace L in $C(Q)$ has the strong uniqueness property, and therefore the metric projection operator P_L onto it is locally Lipschitz continuous (see Theorem 2.12 in [5]) and, in addition, locally uniformly Lipschitz continuous on $C[a,b]\setminus L$

(see Theorem 2.10 in [5]). Marinov extended this result to subspaces of $C(Q)$ (Q is a metrizable compact set) and obtained several non-trivial results on the stability of the operator of near-best approximation by convex subsets of normed linear spaces in terms of moduli of convexity and smoothness of the space.

Stechkin was the first to point out the lack of stability (uniform continuity, or, equivalently, Lipschitz continuity) of the metric projection operator onto the subspace of algebraic polynomials of degree at most n in the uniform norm (see Remark 2.6 and Example 5.3 in [5]). Namely, for any $\varepsilon > 0$, he constructed functions $x, y \in C[-1,1]$ such that $\|x-y\| < \varepsilon$, but $\|P_L x - P_L y\| \geqslant 1$, where $L := \operatorname{span}\{1, t\}$ is a (Chebyshev) subspace in $C[-1,1]$. A similar general result for an arbitrary finite-dimensional Chebyshev subspace $L \subset C(Q)$, $\dim L \geqslant 2$ (Q is an infinite compact set), was established subsequently by Cline[2] (see § 2.2 in [5]). These results show that the metric projection onto a finite-dimensional Chebyshev subspace of dimension $\geqslant 2$ in $C(Q)$ on an infinite compact set can fail to be (globally) Lipschitz continuous (or, equivalently, uniformly continuous). However, in ℓ^∞_n the metric projection onto any Chebyshev subspace is globally Lipschitz continuous on the whole space (see Remark 2.6 in [5]).

Tsar'kov [59] showed that the estimates in [40] for the modulus of continuity of an ε-selection are sharp in the order of the dimension, and established that there exist a bounded convex closed subset Y of $C[0,1]$ and a number $\varepsilon > 0$ such that, for any $\delta > 0$ no uniformly continuous additive ε-selection exists from the neighbourhood $U_\delta(Y) = \{x \mid \rho(x, Y) \leqslant \delta\}$ on Y.

Konyagin announced the following fact in [33]: $\mathscr{R}_{0,1}$ does not admit a uniformly continuous ε-selection in $C[0,1]$ for any $\varepsilon \in (0,2)$. Marinov [41] estimated the Lipschitz constant of a locally Lipschitz selection to the set of generalized rational functions in $C[0,1]$.

Ryutin [50] found sufficient conditions for the existence of a Lipschitz retraction from a neighbourhood of a manifold lying in an arbitrary normed space onto this manifold. The construction of this retraction is related to that of a Lipschitz multiplicative ε-selection. As a corollary, he constructed a Lipschitz multiplicative ε-selection (for large $\varepsilon > 0$) from $C[0,1]$ to $\mathscr{R}_{0,1}$.

Let $X = (X, \|\cdot\|)$ be a real normed linear space, $M \subset X$, and let $\delta_0, \xi_1, \xi_2 > 0$, $n \in \mathbb{N}$. We set for brevity $B^n = \{x \in \mathbb{R}^n \mid |x| \leqslant 1\}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

Definition 7.1. A set M is called an *L-surface* with parameters $(n, \delta_0, \xi_1, \xi_2)$ if:

1) there is a maximal set $\Sigma = \{m_\alpha\}_{\alpha \in \mathscr{A}} \subset M$ such that $\inf_{\beta \neq \alpha} \|m_\beta - m_\alpha\| \geqslant \delta$ for any $\alpha \in \mathscr{A}$, where $\delta = \delta_0/20$;

[2]Cline's result is sometimes ascribed to Bernstein, which is incorrect, because Bernstein proved only that, for any ε, there exist a number n (depending on ε) and two functions f and g (continuous on the interval) such that $\|f-g\| < \varepsilon$, but the best approximants to f and g lie at a distance $\geqslant 2$. However, this result of Bernstein's does not prove the lack of uniform continuity of the metric projection operator even onto the subspace of polynomials of degree $\leqslant n$ on the closed interval, because in Bernstein's result the parameter n (the degree of the best polynomials approximant) depends on ε.

2) for any point $m \in \Sigma$ there exist a neighbourhood $V_m \subset M$ of it and a homeomorphism $q = q(m)\colon V_m \to B^n$ such that

a) $B(m, \delta_0) \cap M \subset V_m$,

b) for all $l_1, l_2 \in V_m$, $\xi_1|q(l_1) - q(l_2)| \leqslant \|l_1 - l_2\| \leqslant \xi_2|q(l_1) - q(l_2)|$;

3) for all $m_1, m_2 \in \Sigma$ such that $V_{m_1} \cap V_{m_2} \neq \varnothing$, the mapping $q_{12} := q(m_1) \circ q(m_2)^{-1}\colon B^n \to B^n$ is affine; hence there exist a vector $v_{12} \in \mathbb{R}^n$ and a linear operator T_{12} such that $q_{12}(u) = T_{12}u + v_{12}$.

We mention the following result [53] on the existence of a Lipschitz retraction.

Theorem 7.1. *Let $M \subset X$ be an L-surface with parameters $(n, \delta_0, \xi_1, \xi_2)$. Then there exists a K-Lipschitz retraction $\Phi\colon \mathscr{O}_\eta(M) \to M$ from $\mathscr{O}_\eta(M) = \{x \in X \mid \rho(x, M) < \eta\}$ to M, where $K = K(n, \delta_0, \xi_1, \xi_2) > 0$ and $\eta = \eta(n, \delta_0, \xi_1, \xi_2) > 0$.*

Some subclasses of L-surfaces (Lipschitz and near-Lipschitz surfaces) important for applications were also studied.

The geometry of the set $\mathscr{R}_{n,m}$ is an interesting topic for research. The topology of some subspaces of algebraic rational functions was considered in [19]. It turns out that $\mathscr{R}_{0,m}$, $m \in \mathbb{N}$, is a cone, its intersection with the sphere $S(0,1)$ is $\gamma_m \sqcup (-\gamma_m)$, and γ_m is homeomorphic to $\mathbb{R}^m$.

In addition, γ_1 is a near-Lipschitz curve, that is, there exists a parametrization $\varphi\colon \mathbb{R} \to \gamma_1$ with the following property: there exist numbers $c_1, c_2, \mu > 0$ such that, for all points $t_1, t_2 \in \mathbb{R}$,

$$\min\{c_1|t_1 - t_2|, \mu\} \leqslant \|\varphi(t_1) - \varphi(t_2)\| \leqslant c_2|t_1 - t_2|.$$

It can be derived from the above results that there exists a Lipschitz retraction from $C[0,1]$ to $\mathscr{R}_{0,1}$ (see [49]).

Unfortunately, it does not seem possible to apply Theorem 7.1 to the sets $\mathscr{R}_{n,m}$ for $n \in \mathbb{Z}_+$ and $m \geqslant 3$ (as in the case of $\mathscr{R}_{0,1}$). The principal obstacle is as follows (see [50]): *if* $m \geqslant 3$ and $n \in \mathbb{Z}_+$, *then for all* $N \in \mathbb{N}$ and $\delta \in (0, 1/100)$ *there exists a set* $\{r_j\}_{1 \leqslant j \leqslant N} \subset \mathscr{R}_{n,m} \cap S(0,1)$ *such that*:

1) $\min_{1 \leqslant i \neq j \leqslant N} \|r_i - r_j\| \geqslant \delta/10$, and

2) $\max_{1 \leqslant i \neq j \leqslant N} \|r_i - r_j\| \leqslant 2\delta$.

The existence of uniformly continuous (on the ball of the space) ε-selections of the set of generalized rational functions for small $\varepsilon > 0$ in $C(K)$, where K is a metrizable compact set, was examined in [54].

Let V and W be subspaces in $C(K)$ such that $\dim V = m$ and $\dim W = n$, where $m, n \in \mathbb{N}$, and let there exist a function $w_0 \in W$ such that $w_0 > 0$ on K. Consider the following set of generalized rational functions:

$$\mathscr{R}_{V,W} = \mathscr{R}_{V,W}(K) = \operatorname{Cl}\Big\{r = \frac{v}{w} \;\Big|\; v \in V,\ w \in W,\ w > 0 \text{ on } K\Big\},$$

where Cl is the closure in $C(K)$.

Theorem 7.2 (see [54]). *Let B be the unit ball in $C(K)$, let $\mathscr{R}_{V,W}\cap B$ be compact, and let* $\operatorname{int}\{\tau\in K \mid w(\tau)=0\}=\varnothing$ *for any* $w\in W\setminus\{0\}$. *Then for any $\varepsilon>0$ there exists a uniformly continuous multiplicative ε-selection $\Phi\colon B\to\mathscr{R}_{V,W}$.*

The next theorem shows that examples of 'compact families of rational functions' (satisfying the assumptions of Theorem 7.2) can be found for subspaces V and W of any finite dimension.

Theorem 7.3 (see [54]). *Let B be the unit ball in $C[0,1]$. Then for all $m,n\in\mathbb{N}$ there exists a pair of subspaces $V,W\subset C^\infty[0,1]\subset C[0,1]$ such that* $\dim V=m$, $\dim W=n$, *and the set $\mathscr{R}_{V,W}\cap B$ is non-empty and compact.*

However, for small $\varepsilon>0$, for some sets of generalized rational functions, there are no uniformly continuous (on the unit ball) ε-selections.

Let V and W be subspaces in $C[0,1]$, where $V=\operatorname{span}\varphi_0$ and $W=\operatorname{span}\{w_1,\dots,w_n\}$, $n\geqslant 2$, and let $\{w_j\}_{j=1}^n$ be a basis for W. Consider the following set of generalized rational functions in $C[0,1]$:

$$\mathscr{R}_{V,W}=\operatorname{Cl}\Bigl\{r=\frac{v}{w}\Bigm| v\in V,\ w\in W,\ w\neq 0 \text{ on } [0,1]\Bigr\},$$

where Cl denotes closure.

Assume that V and W are such that:

1) $\varphi_0(\tau)>0$ for all $\tau\in[0,1]$;

2) there exists a function $f\in W$ such that $f(\tau)>0$ for all $\tau\in[0,1]$;

3) the zero set of any non-zero function $f\in W$ is nowhere dense on $[0,1]$.

Theorem 7.4 (see [52]). *Let V and W satisfy* 1)–3). *Then for any $\varepsilon\in(0,2)$, there is no uniformly continuous multiplicative ε-selection from the unit ball in $C[0,1]$ to $\mathscr{R}_{V,W}$.*

The assumptions on the subspaces V and W in Theorem 7.4 are not necessary. One can find a corresponding example with relaxed condition 1) for a subspace V, but still one cannot eliminate this condition altogether. An analogue of Theorem 7.4 for complex-valued functions (with $\varepsilon\in(0,1)$) was obtained in [52].

In connection with Theorems 7.2 and 7.4, Konyagin made the following conjecture. *Let V* and *W* be *finite-dimensional subspaces of $C(K)$, and let B be the unit ball in $C(K)$. Then the following conditions are equivalent*:

1) *the set $\mathscr{R}_{V,W}(K)\cap B$ is non-empty and compact*;

2) *for each $\varepsilon>0$ there exists a uniformly continuous multiplicative ε-selection from B to $\mathscr{R}_{V,W}(K)$.*

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