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The normal derivative lemma and surrounding issues

D. E. Apushkinskaya and A. I. Nazarov

Abstract. In this survey we describe the history and current state of one of the key areas in the qualitative theory of elliptic partial differential equations related to the strong maximum principle and the boundary point principle (normal derivative lemma).

Bibliography: 234 titles.

Keywords: strong maximum principle, normal derivative lemma, Hopf–Oleinik lemma, Harnack inequality, Aleksandrov–Bakelman maximum principle.

Contents

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1. Introduction

The qualitative theory of partial differential equations has been expanding extensively in the last 100 years. Two of the main tools in the analysis of solutions of elliptic and parabolic equations are the normal derivative lemma (also known as the Hopf–Oleinik lemma or the boundary point principle) and the strong maximum principle. They play a key role in the proofs of uniqueness theorems for boundary-value problems, are used in investigations of the symmetry properties of solutions and their behaviour in unbounded domains (Phragmén–Lindelöf type theorems), and also have some other applications.

The first results in this area can be traced back to Gauss, who proved the strong maximum principle for harmonic functions in his famous paper [1], § 21 in 1840. In modern notation Gauss's statement looks like this:

Let u be a non-constant harmonic function in a domain $\Omega \subset \mathbb{R}^3$, that is, $\Delta u = 0$ in Ω. Then u cannot attain its maximum or minimum value at an interior point of Ω.

In what follows, by the strong maximum principle for a second-order linear elliptic operator $\mathbb{L}$ we mean the following result.

Strong maximum principle. *Let u be a superelliptic function in a domain $\Omega \subset \mathbb{R}^n$, that is,*[1] *$\mathbb{L}u \geqslant 0$ in Ω. If u attains its minimum value at an interior point of the domain, then $u \equiv \mathrm{const}$ and $\mathbb{L}u \equiv 0$.*

We also recall the statement of the weak maximum principle.

Weak maximum principle. *Let u be a superelliptic function in a bounded domain $\Omega \subset \mathbb{R}^n$. If u is non-negative on the boundary of this domain, then u is also non-negative in Ω.*

The boundary version of the strong maximum principle is the normal derivative lemma. It was originally stated in 1910 by Zaremba [2] for harmonic functions in a (three-dimensional, bounded) domain satisfying the interior ball condition.[2]

Normal derivative lemma. *Let u be a non-constant superelliptic function in a domain $\Omega \subset \mathbb{R}^n$. If u attains its minimum value at a boundary point $x^0 \in \partial\Omega$, then*

$$\liminf_{\varepsilon \to +0} \frac{u(x^0 + \varepsilon \mathbf{n}) - u(x^0)}{\varepsilon} > 0, \tag{1.1}$$

where $\mathbf{n}$ is the inward normal vector to the boundary of the domain at x^0.

In particular, if u has a derivative in the direction of $\mathbf{n}$ at x^0, then $\partial_{\mathbf{n}} u(x^0) > 0$.

It is noteworthy that, while the strong maximum principle is a property of the operator $\mathbb{L}$, the validity of the normal derivative lemma also depends on the behaviour of $\partial\Omega$ in a neighbourhood of x^0.

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[1] We assume that the leading coefficients of $\mathbb{L}$ form a *non-positive* matrix

[2] Note that Zaremba used this lemma to establish the uniqueness theorem for a mixed problem (when the boundary of the domain is divided into two parts, with Dirichlet conditions prescribed on one part and Neumann conditions on the other part). It is now called the Zaremba problem, although Zaremba himself mentioned in [2] that this problem had been proposed to him by Wirtinger.

Harnack's inequality, which can be regarded as a quantitative version of the strong maximum principle, is close to the main topic of this survey. It was originally proved in 1887 by Harnack[3] [3], § 19, for harmonic functions on the plane. Its classical formulation is as follows.

Harnack's inequality. *Let* $\mathbb{L}$ *be an elliptic operator in a domain* Ω*. If* u *is a non-negative solution of the equation* $\mathbb{L}u = 0$ *in* Ω*, then the following inequality holds in each bounded subdomain* Ω' *such that* $\overline{\Omega}' \subset \Omega$*:*

$$\sup_{\Omega'} u \leqslant C \inf_{\Omega'} u, \tag{1.2}$$

where C *is a constant independent of* u*.*

Remark 1.1. A compactness argument makes it clear that it is sufficient to prove (1.2) in the case when Ω and Ω' are concentric balls. However, it is important for applications that C does not depend on the radii of these balls (and only depends on their ratio) or that, at least, it remains bounded as the radii tend to zero so that their ratio stays fixed.

Some other *a priori* estimates for solutions, for instance, the Aleksandrov–Bakelman maximum principle, can also be regarded as qualitative versions of the strong maximum principle. On the other hand, it has only relatively recently become clear that an *a priori* estimate for the gradient of a solution on the boundary of the domain is dual to the normal derivative lemma.

The area in question is immense, so we focus on the elliptic case[4] in the present paper. The main body of the paper consists of three sections. In § 2 we discuss the properties of *classical* and *strong* (sub/super)solutions of equations in *non-divergence* form, and in § 3 the properties of *weak* (sub/super)solutions of equations in *divergence* form. Finally, § 4 is a mix of various generalizations and applications. We do not aspire to present a complete exposition, and the selection of topics reflects our personal interests.

Various aspects of the topic under consideration have been discussed in the monographs and surveys [5]–[14]. In this work we use information from these sources as well as from our papers [15] and [16], but we also make an attempt to go deeper into the history of the questions under consideration.

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[3]The mathematician Carl Gustav Axel Harnack had a twin brother Carl Gustav Adolf von Harnack, who was a historian and theologist, and was the founding President of the Kaiser Wilhelm Society for the Advancement of Science (now the Max Planck Society). The highest award of the Max Planck Society carries his name.

[4]In addition, we limit ourselves to scalar equations. In this connection we refer to the recent survey [4], which is devoted to the maximum principle for elliptic *systems*.

The main notation.

• $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ are points in $\mathbb{R}^n$, $n \geqslant 2$.

• $|x|$, $|x'|$ are the Euclidean norms in the relevant spaces.

• $\mathbb{R}_+ = [0, +\infty)$ is the *closed* half-axis.

• Ω is a domain (connected open set) in $\mathbb{R}^n$ with boundary $\partial\Omega$; unless otherwise stated (as in § 4.2), we assume that Ω is bounded; $\overline{\Omega}$ denotes the closure of Ω, $|\Omega|$ is the Lebesgue measure of Ω, and $\operatorname{diam}(\Omega)$ is its diameter.

• $\mathrm{d}(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance between x and $\partial\Omega$.

• $B^n_r(x^0) = \{x \in \mathbb{R}^n \colon |x - x^0| < r\}$ is the open ball in $\mathbb{R}^n$ with centre x^0 and radius r; we put $B^n_r = B^n_r(0)$; when the dimension is clear from the context, we simply write $B_r(x^0)$ and B_r.

• $Q_{r,h} = B^{n-1}_r \times (0, h)$.

The indices i and j range between 1 and n. We let D_i denote the operator of differentiation with respect to the variable x_i. Summation over repeated indices is implicit.

Given a function f, we put $f_\pm = \max\{\pm f, 0\}$ and

$$\fint_\Omega f\,dx = \frac{1}{|\Omega|} \int_\Omega f\,dx.$$

We denote various positive constants by C and N (with or without indices). The notation $C(\dots)$ indicates that C depends only on the parameters in parentheses.

Classes of functions and domains.

• $\mathcal{C}^k(\overline{\Omega})$ is the space of functions defined on $\overline{\Omega}$ that have continuous derivatives up to and including order k ($k \geqslant 0$). In place of $\mathcal{C}^0$ we normally write $\mathcal{C}$ for brevity.

• $L_p(\Omega)$, $W^k_p(\Omega)$, and $\overset{\circ}{W}{}^k_p(\Omega)$ are the standard Lebesgue and Sobolev spaces (for instance, see [17], § 4.2.1); $\|\cdot\|_{p,\Omega}$ is the norm in $L_p(\Omega)$. Furthermore, $f \in L_{p,\mathrm{loc}}(\Omega)$ if $f \in L_p(\Omega')$ for each subdomain Ω' such that $\overline{\Omega}' \subset \Omega$. The notation $f \in W^k_{p,\mathrm{loc}}(\Omega)$ has similar meaning.

• $L_{p,q}(\Omega)$ are the Lorentz spaces (for instance, see [17], § 1.18.6).

We say that $\sigma\colon [0,1] \to \mathbb{R}_+$ is a function of class $\mathcal{D}$ if

(i) σ is continuous and increasing, and $\sigma(0) = 0$;

(ii) $\sigma(\tau)/\tau$ is integrable and decreasing.

Remark 1.2. The assumption that $\sigma(\tau)/\tau$ is decreasing is not restrictive. Indeed, let $\sigma\colon [0,1] \to \mathbb{R}_+$ be an increasing function such that $\sigma(0) = 0$, and let $\sigma(\tau)/\tau$ be integrable. Then we put

$$\widetilde{\sigma}(t) = t \sup_{\tau \in [t,1]} \frac{\sigma(\tau)}{\tau}, \qquad t \in (0,1).$$

It is obvious that $\widetilde{\sigma}(t)/t$ is decreasing on $[0,1]$ and $\sigma(t) \leqslant \widetilde{\sigma}(t)$ on $(0,1]$. (In view of the last inequality, we can replace σ by $\widetilde{\sigma}$ in all estimates.) Now, the set of points at which $\sigma(t) < \widetilde{\sigma}(t)$ is an at most countable union of intervals (t_{1j}, t_{2j}). On each interval $\widetilde{\sigma}$ is increasing, so it is increasing on $[0,1]$.

Now we look at the integral

$$\int_0^1 \frac{\widetilde{\sigma}(\tau)}{\tau}\,d\tau = \int_{\{\widetilde{\sigma}=\sigma\}} \frac{\sigma(\tau)}{\tau}\,d\tau + \sum_j \int_{t_{1j}}^{t_{2j}} \frac{\widetilde{\sigma}(\tau)}{\tau}\,d\tau.$$

Note that on each interval (t_{1j}, t_{2j}) we have

$$\frac{\widetilde{\sigma}(t)}{t} \equiv \frac{\sigma(t_{1j})}{t_{1j}} = \frac{\sigma(t_{2j})}{t_{2j}}.$$

Since σ is monotone, from this we obtain

$$\int_0^1 \frac{\widetilde{\sigma}(\tau)}{\tau}\,d\tau = \int_{\{\widetilde{\sigma}=\sigma\}} \frac{\sigma(\tau)}{\tau}\,d\tau + \sum_j \big(\sigma(t_{2j}) - \sigma(t_{1j})\big) < \infty,$$

so that $\widetilde{\sigma} \in \mathcal{D}$.

Remark 1.3. There will be no loss of generality either if we assume that σ is continuously differentiable on $(0,1]$. In fact, for any $\sigma \in \mathcal{D}$, we put

$$\widehat{\sigma}(r) := 2\int_{r/2}^{r} \frac{\sigma(\tau)}{\tau}\,d\tau = 2\int_{1/2}^{1} \frac{\sigma(r\tau)}{\tau}\,d\tau, \qquad r \in (0,1]. \tag{1.3}$$

Then, as σ and $\sigma(\tau)/\tau$ are monotone, we can conclude from the second equality in (1.3) that $\widehat{\sigma}$ is also increasing, while $\widehat{\sigma}(r)/r$ is decreasing on $(0,1]$. Next, it is obvious from the first equality in (1.3) that $\widehat{\sigma} \in \mathcal{C}^1(0,1]$ and

$$\sigma(r) \leqslant \widehat{\sigma}(r) \leqslant 2\sigma\left(\frac{r}{2}\right), \qquad r \in (0,1]. \tag{1.4}$$

The second inequality in (1.4) yields $\widehat{\sigma} \in \mathcal{D}$. Finally, by using the first inequality in (1.4), we can replace σ by $\widehat{\sigma}$ in all estimates.

We say that a function $\zeta\colon \Omega \to \mathbb{R}$ satisfies

- the Hölder condition with exponent $\alpha \in (0,1]$ if

$$|\zeta(x) - \zeta(y)| \leqslant C|x-y|^{\alpha} \quad \text{for all } x, y \in \Omega;$$

- the Dini condition if

$$|\zeta(x) - \zeta(y)| \leqslant \sigma(|x-y|) \quad \text{for all } x, y \in \Omega$$

  and $\sigma \in \mathcal{D}$.

Next, $\mathcal{C}^{k,\alpha}(\Omega)$ and $\mathcal{C}^{k,\mathcal{D}}(\Omega)$, where $k \geqslant 0$, are the spaces of functions whose derivatives of order k satisfy the Hölder condition with exponent $\alpha \in (0,1]$ or the Dini condition, respectively. The functions in $\mathcal{C}^{0,1}(\Omega)$ are said to be Lipschitz.

We say that $\Omega \subset \mathbb{R}^n$ is a domain of *class* $\mathcal{C}^k$, $k \geqslant 0$, if there is an $r > 0$ such that for each $x^0 \in \partial\Omega$, the set $B_r(x^0) \cap \partial\Omega$ is the graph of a function[5] $x_n = f(x')$ in an appropriate Cartesian coordinate system, where $f \in \mathcal{C}^k(G)$ (and G is some domain in $\mathbb{R}^{n-1}$). We define domains of class $\mathcal{C}^{k,\alpha}$ and $\mathcal{C}^{k,\mathcal{D}}$ similarly.

Domains in $\mathcal{C}^{0,1}$ are said to be strictly Lipschitz.

Recall that the *interior ball condition* is that each point on the boundary $\partial\Omega$ can be attained by a ball of fixed radius that lies fully in Ω.

[5] Moreover, the set $B_r(x^0) \cap \Omega$ lies to one side of this graph.

Similarly, let $\mathfrak{T}(\phi, h)$ (here $h > 0$ and $\phi\colon [0, +\infty) \to [0, +\infty)$ is a convex function such that $\phi(0) = 0$) denote the following domain (body):

$$\mathfrak{T}(\phi, h) = \{x \in \mathbb{R}^n \colon \phi(|x'|) < x_n < h\}.$$

Assume that each point $x^0 \in \partial\Omega$ can be attained by a body congruent to $\mathfrak{T}(\phi, h)$ with vertex x^0 that lies fully in Ω, where ϕ and h are independent of x^0. Then we say that Ω satisfies

- the *interior* $\mathcal{C}^{1,\alpha}$*-paraboloid condition* with $\alpha \in (0, 1]$ if $\phi(s) = Cs^{1+\alpha}$ (for $\alpha = 1$, this is the same as the interior ball condition);
- the *interior* $\mathcal{C}^{1,\mathcal{D}}$*-paraboloid condition* if $\phi'(0+) = 0$ and ϕ' satisfies the Dini condition;
- the *interior cone condition* if $\phi(s) = Cs$.

The *exterior ball*, *exterior paraboloid*, and *exterior cone conditions* are defined similarly.

It is easy to see that a domain of class $\mathcal{C}^{1,1}$ satisfies both the interior and exterior ball conditions. (Moreover, the combination of these conditions is equivalent to the $\mathcal{C}^{1,1}$-regularity of the domain; for instance, see [18], Lemma 2.) Similarly, domains of class $\mathcal{C}^{1,\alpha}$ are precisely the ones that satisfy the interior and exterior $\mathcal{C}^{1,\alpha}$-paraboloid conditions. Domains in $\mathcal{C}^{1,\mathcal{D}}$ are those satisfying the interior and exterior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid conditions.[6] Strictly Lipschitz domains satisfy the interior and exterior cone conditions.[7]

2. Non-divergence form operators

In this section we look at operators with the following structure:

$$\mathcal{L} \equiv -a^{ij}(x)D_iD_j + b^i(x)D_i. \tag{2.1}$$

We put $\mathcal{A} = (a^{ij})$ and $\mathbf{b} = (b^i)$. If $\mathbf{b} \equiv 0$, then we write $\mathcal{L}_0$ in place of $\mathcal{L}$.

The matrix $\mathcal{A}$ of leading coefficients is symmetric and satisfies either the *degenerate ellipticity condition*

$$a^{ij}(x)\xi_i\xi_j \geqslant 0 \quad \text{for all } \xi \in \mathbb{R}^n \tag{2.2}$$

or the *uniform ellipticity condition*

$$\nu|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \nu^{-1}|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{2.3}$$

(Here $\nu \in (0, 1]$ is the so-called *ellipticity constant*.)

In §§ 2.1 and 2.2 we assume that condition (2.2) or (2.3) holds for all $x \in \Omega$. Starting from § 2.3, we assume that the entries of $\mathcal{A}$ are measurable functions and (2.2) or (2.3) holds for almost all $x \in \Omega$.

[6] These equivalences were proved in [14] without the *a priori* assumption that 'the boundary of the domain is locally the graph of a function'.

[7] However, contrary to an assertion in [14], we do not have equivalence here. A Lipschitz but not strictly Lipschitz domain formed by two 'bricks' can serve as a counterexample.

Remark 2.1. For operators of the more general form $\mathcal{L} + c(x)$, we obviously have neither the strong maximum principle nor the normal derivative lemma in the form presented in the Introduction. (The principal eigenfunction of the Dirichlet problem for the Laplace operator provides a counterexample even to the weak maximum principle.) In this case one normally imposes some condition on the sign of $c(x)$ in a neighbourhood of the minimum point. Here are two pairs of simple results.

1. Assume that the strong maximum principle holds for the operator $\mathcal{L}$.

(a) Let $c \geqslant 0$ with $c \not\equiv 0$. If $\mathcal{L}u + cu \geqslant 0$ in Ω, then u can have no *negative* minimum in Ω.

(b) Let $c \leqslant 0$ with $c \not\equiv 0$. If $\mathcal{L}u + cu \geqslant 0$ in Ω, then u can have no *non-negative* minimum in Ω unless $u \equiv 0$.

2. Assume that the normal derivative lemma holds for the operator $\mathcal{L}$ in a domain Ω.

(a) Let $\mathcal{L}u + cu \geqslant 0$ in Ω, and let $c \geqslant 0$ with $c \not\equiv 0$. If u attains a *negative* minimum value at a boundary point $x^0 \in \partial\Omega$, then $\partial_{\mathbf{n}} u(x^0) > 0$.

(b) Let $\mathcal{L}u + cu \geqslant 0$ in Ω, and let $c \leqslant 0$ with $c \not\equiv 0$. If u attains a *non-negative* minimum value at a boundary point $x^0 \in \partial\Omega$, then $\partial_{\mathbf{n}} u(x^0) > 0$ unless $u \equiv 0$.

All four assertions follow from the fact that the inequality $\mathcal{L}u + cu \geqslant 0$ yields $\mathcal{L}u \geqslant 0$ in a neighbourhood of a minimum point.

2.1. The classical era: from Gauss and Neumann to Hopf and Oleinik. Recall that the strong maximum principle for harmonic functions in a three-dimensional domain was established by Gauss [1] on the basis of his mean value theorem.[8] Since this theorem holds for harmonic functions in any space $\mathbb{R}^n$, Gauss's proof is obviously valid in any dimension. Moreover, it can easily be extended to superharmonic functions.

For uniformly elliptic operators of the more general form $\mathcal{L}+c(x)$ with $\mathcal{C}^2$-smooth coefficients (of the form indicated in Remark 2.1, part 1(a)), the proof was given by

- Paraf in 1892, for $c(x) > 0$ in the two-dimensional case;
- Moutard in 1894, for $c(x) > 0$ in the multidimensional case;
- Picard in 1905, for $c(x) \geqslant 0$ in the two-dimensional case.

The crucial step was made by Hopf[9] [20] in 1927. Although in this paper he only established the strong maximum principle for uniformly elliptic operators of the form (2.1) with continuous coefficients, Hopf's proof actually also extends word for word to operators with *bounded* coefficients.

A further important observation was made in [20] for operators of the form $\mathcal{L} + c(x)$. Apart from the obvious result in part 1(a) of Remark 2.1, Hopf showed[10] that if $\mathcal{L}u + cu \geqslant 0$ in Ω, then u cannot attain *zero* minimum value in Ω unless $u \equiv 0$, even without any condition on the sign of $c(x)$.

As mentioned in the Introduction, the normal derivative lemma was originally established by Zaremba [2] for harmonic functions under the interior ball condition on the boundary of a three-dimensional domain. Apart from the weak maximum

[8] The reader can find an extensive survey of mean value theorems for various function classes in [19].

[9] A similar idea can be found in a slightly earlier paper by Picone, though he did not prove the strong maximum principle there.

[10] In 1954 Aleksandrov presented another, geometric, proof of the same result.

principle, Zaremba's proof used only the Green's function for the Dirichlet problem for the Laplace's equation in the ball, so it is valid in any dimension and also for superharmonic functions.

It should be noted that for the Laplace operator, there is an alternative (and equivalent) form of the normal derivative lemma:

Let $\mathcal{G}$ be the Green's function for Laplace's equation in a domain Ω. If $x \in \Omega$ and $x^0 \in \partial\Omega$, then $\partial_{\mathbf{n}}\mathcal{G}(x, x^0) > 0$.

This result was proved by Neumann for a two-dimensional $\mathcal{C}^2$-smooth convex domain as long ago as 1888. Then it was generalized

- in 1901 by Korn to a two-dimensional domain of class $\mathcal{C}^2$ strictly star-shaped with respect to a point;
- in 1909 by Lichtenstein for a general two-dimensional domain of class $\mathcal{C}^2$;
- in 1912 by Kellog for a two-dimensional domain of class $\mathcal{C}^{1,\alpha}$ with $\alpha \in (0, 1)$;
- in 1918 by Lichtenstein for a three-dimensional domain of class[11] $\mathcal{C}^{1,1}$.

For the operator $-\Delta + b^i(x)D_i + c(x)$ with $c(x) \geqslant 0$ in a two-dimensional domain of class $\mathcal{C}^{2,\alpha}$, where $\alpha \in (0, 1)$, this result was established by Lichtenstein in 1924. However, subsequently almost all results that we know of have been stated in the form of the conventional normal derivative lemma.[12]

In 1931 Brelot was the first to note (in the case of the operator $-\Delta + c(x)$ in a two-dimensional domain of class $\mathcal{C}^2$, where $c(x) \geqslant 0$) that the normal derivative lemma holds, in fact, for any derivative in a strictly inward direction (making an acute angle with the inward normal).

In 1932 Giraud [21], Chap. V, proved the normal derivative lemma[13] for uniformly elliptic operators $\mathcal{L} + c(x)$ with coefficients in $\mathcal{C}^{0,\alpha}$ with $\alpha \in (0, 1)$, where $c(x) \geqslant 0$, in an n-dimensional domain of class $\mathcal{C}^{1,1}$. In [22] this result was extended to the case when the lower-order coefficients can have singularities on a set $\mathfrak{M}$ which is a union of a finite number of $\mathcal{C}^{1,\alpha}$-smooth manifolds of codimension 1 and

$$|b^i(x)|, |c(x)| \leqslant C \cdot \operatorname{dist}^{\gamma-1}(x, \mathfrak{M}), \qquad \gamma \in (0, 1).$$

In 1937 the condition on the boundary of the domain was significantly relaxed for the first time: Keldysh and Lavrentiev proved the normal derivative lemma for the Laplace operator in a (three-dimensional) domain satisfying the interior $\mathcal{C}^{1,\alpha}$-paraboloid condition.[14]

Finally, Hopf [24] and Oleinik [25] made the decisive step by proving simultaneously and independently the normal derivative lemma for uniformly elliptic operators with continuous coefficients under the interior ball condition on the boundary

[11] Lichtenstein claimed a result for domains in $\mathcal{C}^{1,\alpha}$, $\alpha \in (0, 1)$. However, his proof was based on the following fact: for each point $x^0 \in \partial\Omega$, we can find a point $x \in \Omega$ such that x^0 is the boundary point closest to x. This does not hold for domains of class $\mathcal{C}^{1,\alpha}$ for $\alpha < 1$.

[12] Perhaps this is because for operators with variable leading coefficients the proof of the alternative statement is much more difficult, and in the general case of measurable leading coefficients the Green's function is not defined.

[13] In place of the normal $\mathbf{n}$, Giraud used the conormal $\mathbf{n}^{\mathcal{L}}$ with coordinates $\mathbf{n}_i^{\mathcal{L}} = a^{ij}\mathbf{n}_j$, which leads to an equivalent statement. It is essential that he also considered the case when $u(x^0) = 0$ and no condition is imposed on the sign of $c(x)$; cf. Remark 2.1, part 2(a).

[14] Some authors (for example, see [23] or [14]) say that this condition on the domain goes back to Giraud. Indeed, in [21] and [22] some results were proved for domains in $\mathcal{C}^{1,\alpha}$, but in the normal derivative lemma it was required that $\alpha = 1$.

of the domain. Their proofs in [25] and [24] were based on the same idea and, like in [20], they extend word for word to operators with bounded coefficients.[15]

Now we present the full proof of the classical results in [20] and [24], [25].

Theorem 2.1. A. *Let $\mathcal{L}$ be an operator of the form* (2.1), *let a^{ij}, b^i, and c be bounded functions in Ω, assume that* (2.3) *holds, let $u \in \mathcal{C}^2(\Omega)$, and let $\mathcal{L}u + cu \geqslant 0$ in Ω. Then*:

(A1) *the function u cannot attain zero minimum value in Ω unless $u \equiv 0$;*
(A2) *if $c \geqslant 0$, then u can attain no negative minimum in Ω unless $u \equiv \mathrm{const}$ and $c \equiv 0$;*
(A3) *if $c \leqslant 0$, then u can attain no positive minimum in Ω unless $u \equiv \mathrm{const}$ and $c \equiv 0$.*

B. *Assume that, in addition, Ω satisfies the interior ball condition, and let $u \not\equiv \mathrm{const}$ be a continuous function in $\overline{\Omega}$. Let x^0 denote a point on $\partial\Omega$ at which u takes its minimum value. Then inequality* (1.1) *holds, provided that any of the following conditions is satisfied:*[16]

(B1) $u(x^0) = 0$;
(B2) $u(x^0) < 0$ *and* $c \geqslant 0$;
(B3) $u(x^0) > 0$ *and* $c \leqslant 0$.

Furthermore, the normal $\mathbf{n}$ *can be replaced in* (1.1) *by any strictly inward direction* $\boldsymbol{\ell}$.

Proof. 1. First consider the case when $c \equiv 0$. We start by establishing the weak maximum principle for $\mathcal{L}$ in a domain π of sufficiently small diameter d.

Assume, on the contrary, that $\mathcal{L}u \geqslant 0$ in π and $u\big|_{\partial\pi} \geqslant 0$, but $u(x^0) = -A < 0$ for some $x^0 \in \pi$. Consider the function

$$u^\varepsilon(x) = u(x) - \varepsilon|x - x^0|^2.$$

It is obvious that for all sufficiently small ε, we have

$$u^\varepsilon\big|_{\partial\pi} \geqslant -\varepsilon d^2 > -A = u^\varepsilon(x^0).$$

Hence u^ε attains its minimum value at a point $x^1 \in \pi$. At the minimum point we have $Du^\varepsilon(x^1) = 0$ and the matrix $D^2u^\varepsilon(x^1)$ is non-negative definite, so $\mathcal{L}u^\varepsilon(x^1) \leqslant 0$.

However, since $\mathcal{L}u \geqslant 0$, we have

$$\mathcal{L}u^\varepsilon \geqslant 2\varepsilon\big(a^{ij}\delta_{ij} - b^i(x_i - x_i^0)\big) \geqslant 2\varepsilon(n\nu - d\sup|\mathbf{b}(x)|) > 0 \quad \text{in } \pi,$$

provided that $d < d_0 := n\nu/\sup|\mathbf{b}(x)|$. This contradiction proves the required result.

2. Now we prove the strong maximum principle for $\mathcal{L}$. Assume that, on the contrary, $\mathcal{L}u \geqslant 0$ in Ω and $u \not\equiv \mathrm{const}$, but the set

$$M = \Big\{x \in \Omega\colon u(x) = \inf_\Omega u\Big\} \tag{2.4}$$

[15] Hopf considered operators of the form (2.1), while Oleinik considered operators $\mathcal{L} + c(x)$ for $c(x) \geqslant 0$ under the assumption that $u(x^0) \leqslant 0$. In addition, in place of the normal, an arbitrary direction making an acute angle with $\mathbf{n}$ was considered in [25].

[16] In case (B1), when no condition is imposed on the sign of $c(x)$, this result was apparently first identified in [26].

is non-empty. The complement $\Omega \setminus M$ is open, so there is a ball in it such that its boundary contains a point in M. Let the centre of this ball be the origin and let r denote its radius. Let x^0 be a point in $\partial B_r \cap M$, and π be the spherical shell $B_r \setminus \overline{B}_{r/2}$. Without loss of generality, we can assume that $r < d_0/2$.

In π we consider the *barrier function*[17]

$$v_s(x) = |x|^{-s} - r^{-s}. \tag{2.5}$$

We estimate $\mathcal{L}v_s$ with (2.3) taken into account:

$$D_i v_s(x) = -s x_i |x|^{-s-2}; \qquad D_i D_j v_s(x) = s(s+2) x_i x_j |x|^{-s-4} - s\delta_{ij}|x|^{-s-2};$$

$$\mathcal{L}v_s(x) = |x|^{-s-2}\Big(-s(s+2)a^{ij}\,\frac{x_i}{|x|}\,\frac{x_j}{|x|} + s a^{ij}\delta_{ij} - s b^i x_i\Big)$$

$$\leqslant s|x|^{-s-2}\Big[-(s+2)\nu + n\nu^{-1} + r\sup_{\Omega}|\mathbf{b}(x)|\Big].$$

We take a sufficiently large s so that the expression in square brackets is negative. Then, for any $\varepsilon > 0$, the function $w^\varepsilon = u - \inf_\Omega u - \varepsilon v_s$ satisfies $\mathcal{L}w^\varepsilon \geqslant 0$ in π.

Now, $\partial\pi = \partial B_r \cup \partial B_{r/2}$. It is obvious that $w^\varepsilon\big|_{\partial B_r} \geqslant 0$. Since $B_r \subset \Omega \setminus M$ by construction, the function $u - \inf_\Omega u$ is separated from zero on $\partial B_{r/2}$, and if $\varepsilon > 0$ is sufficiently small, then $w^\varepsilon\big|_{\partial B_{r/2}} \geqslant 0$. Hence the weak maximum principle can be applied to w^ε in π, and $w^\varepsilon \geqslant 0$ in π.

However, $w^\varepsilon(x^0) = 0$, so for any vector $\boldsymbol{\ell}$ pointing inside of π, we have $\partial_{\boldsymbol{\ell}} w^\varepsilon(x^0) \geqslant 0$, that is,

$$\partial_{\boldsymbol{\ell}} u(x^0) \geqslant \varepsilon \partial_{\boldsymbol{\ell}} v_s(x^0) > 0,$$

which is impossible because $Du(x^0) = 0$ at the minimum point. This contradiction proves the required result.

3. Now we prove the normal derivative lemma for $\mathcal{L}$. By assumption, we can find a ball of radius r touching $\partial\Omega$ at a point x^0. Let the centre of this ball be the origin. By the strong maximum principle, $u > u(x^0)$ in B_r. Now, by repeating part 2 of this proof word for word, we obtain inequality (1.1), where $\mathbf{n}$ can be replaced by $\boldsymbol{\ell}$.

4. Finally, we drop the condition $c \equiv 0$. Then assertions (A2), (A3), (B2), and (B3) are direct consequences of Remark 2.1.

To prove (A1) and (B1) we represent u as a product $u = \psi v$, where $\psi > 0$ and $v \geqslant 0$ in $\overline{\Omega}$. Straightforward calculations yield

$$0 \leqslant \frac{\mathcal{L}u + cu}{\psi} = \widetilde{\mathcal{L}}v := -a^{ij} D_i D_j v + \widetilde{b}^i D_i v + \widetilde{c} v, \tag{2.6}$$

where

$$\widetilde{b}^i = b^i - \frac{2a^{ij} D_j \psi}{\psi} \quad \text{and} \quad \widetilde{c} = \frac{\mathcal{L}\psi + c\psi}{\psi}.$$

Now we put $\psi(x) = \exp\{\lambda x_1\}$. Then

$$\mathcal{L}\psi + c\psi = \psi(-a^{11}\lambda^2 + b^1\lambda + c) \leqslant \psi\Big(-\nu\lambda^2 + \sup_{\Omega} b^1(x)\lambda + \sup_{\Omega} c(x)\Big).$$

[17] Hopf and Oleinik used other barrier functions. The function (2.5) was apparently introduced to this end in [27] (also see [28], Chap. 1).

We choose a sufficiently large λ so that the last expression in parentheses is negative. Then parts 1(b) and 2(b) of Remark 2.1 are valid for the operator $\widetilde{\mathcal{L}}$ in (2.6). In particular, v does not vanish in the interior of the domain, which yields (A1). Since $u(x^0) = 0$ implies that $Du(x^0) = \psi(x^0)Dv(x^0)$, it follows that part 2(b) of Remark 2.1 for v implies (B1) for u. Theorem 2.1 is proved. □

2.2. Extending the classical results. Refining the assumptions about the boundary of the domain. Following the publication of the fundamental results in [24] and [25], many authors have contributed to the development of this area in a number of directions:

1) extending the class of differential operators, that is, relaxing the conditions imposed on their leading and lower-order coefficients;
2) extending the class of domains, that is, relaxing the assumptions about the boundary of the domain (in the case of the normal derivative lemma);
3) clarifying the range of applications of the relevant results by constructing various counterexamples.

We start the description of these results with Pucci [29], who proved the normal derivative lemma in the domain $\Omega = B_r$ for a wider class of operators than the ones in [24] and [25]. Namely, the operator can lose the property of being elliptic in directions tangent to $\partial\Omega$, and the lower-order coefficients must satisfy the conditions

$$|b^i(x)| \leqslant \frac{\sigma(\mathrm{d}(x))}{\mathrm{d}(x)} \quad \text{and} \quad 0 \leqslant c(x) \leqslant \frac{\sigma(\mathrm{d}(x))}{\mathrm{d}^2(x)}, \qquad \sigma \in \mathcal{D}. \tag{2.7}$$

Pucci's proof was based on the barrier function

$$\mathfrak{v}(x) = \int_0^{\mathrm{d}(x)} \int_0^{\tau} \frac{\sigma(t)}{t}\, dt\, d\tau + \kappa \mathrm{d}(x)$$

with a suitable choice of the constant κ. This function and various versions of it have been used by many authors since then.

If the condition of ellipticity is even more degenerate, then the strong maximum principle does not hold in its classical form. In the series of papers [30] Aleksandrov provided a description of the structure of the zero set of a non-negative function u such that $\mathcal{L}u + cu \geqslant 0$ in Ω for such operators.[18]

Výborný [31], [32] proved the normal derivative lemma for the operator $\mathcal{L} + c(x)$ in a domain of class[19] $\mathcal{C}^{1,\mathcal{D}}$. The assumptions on the coefficients of the operator were the same[20] as in [29]. Unfortunately, the results in [31] and [32] have not received the attention they deserve.

In [34], for the Laplace equation in a domain in $\mathcal{C}^{1,\mathcal{D}}$, Widman obtained sharp estimates for the derivatives of the Green's function in the Dirichlet problem.[21] In

[18] This problem was also discussed in many subsequent papers, starting with Ch. III of the well-known monograph by Oleinik and Radkevich.

[19] More precisely, Výborný assumed that there is a function $\rho \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ such that $\rho(x) = 0$ and $D\rho \neq 0$ on $\partial\Omega$, while $\rho > 0$ and $|D^2\rho(x)| \leqslant \sigma(\rho(x))/\rho(x)$ in Ω, where $\sigma \in \mathcal{D}$. That such a function does (locally) exist for a domain in $\mathcal{C}^{1,\mathcal{D}}$ was proved in [33].

[20] Výborný proved the assertion stated in Remark 2.1, part 2(a). In this case the upper bound for $c(x)$ in (2.7) is unnecessary.

[21] Some of these estimates had been established previously by Èidus and Solomentsev under more restrictive assumptions on the domain.

particular, he proved the normal derivative lemma in Neumann's form (the normal derivative of the Green's function on $\partial\Omega$ is positive) and presented a counterexample showing that the condition $\mathcal{C}^{1,\mathcal{D}}$ on the boundary of the domain cannot be relaxed to $\mathcal{C}^1$. Namely, if ϕ' fails the Dini condition at the origin, then $\partial_{\mathbf{n}}\mathcal{G}(x,0)=0$ in the paraboloid $\mathfrak{T}(\phi,h)$.

In the note [35], which was published simultaneously with [34], the authors found refined asymptotic formulae for harmonic functions near non-smooth boundary points. As a consequence, they showed that if u is a harmonic function in a paraboloid $\mathfrak{T}(\phi,h)$ that attains its minimum value at the vertex $x^0=0$, then $\partial_{\boldsymbol{\ell}}u(0)$ is positive for each strictly inward direction $\boldsymbol{\ell}$ if and only if ϕ' satisfies the Dini condition at the origin (this is equivalent to the result in [34]).

The behaviour of solutions to $\mathcal{L}u=0$ in a neighbourhood of $x^0\in\partial\Omega$ in the case when the boundary of the domain satisfies only the interior/exterior cone condition at x^0, subject to the condition $b^i(x)=o(|x-x^0|^{-1})$, was investigated by Oddson and Miller, respectively.

A large cycle of papers generalizing the normal derivative lemma was published by Khimchenko and Kamynin.

In [36] the normal derivative lemma for the Laplace operator was proved for domains satisfying the interior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition. That paper also contains an estimate for the normal derivative at $\partial\Omega$ of the solution of the problem

$$-\Delta u=f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega}=0,$$

in the case when Ω satisfies the exterior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition[22] and f satisfies $|f(x)|\leqslant C\mathrm{d}^{\gamma-1}(x)$, $\gamma\in(0,1)$. Finally, [36] contains examples showing that the assumptions on the boundary cannot be improved significantly (these are in fact quite similar to the corresponding counterexamples in [34] and [35]).

In [37] the results of [36] were extended to uniformly elliptic operators of the form $\mathcal{L}+c(x)$ with bounded coefficients $b^i(x)$. The normal derivative lemma was stated (for an arbitrary strictly inward direction) under the assumption that 'the maximum principle holds' (which should perhaps mean that $c(x)\geqslant 0$), and an estimate for the gradient on $\partial\Omega$ of the solution of the problem

$$\mathcal{L}u+cu=f \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega}=g,$$

was given under the conditions

$$|c(x)|,|f(x)|\leqslant C\mathrm{d}^{\gamma-1}(x), \quad \gamma\in(0,1); \qquad g\in\mathcal{C}^{1,\mathcal{D}}(\partial\Omega).$$

In [23] the normal derivative lemma was extended to elliptic-parabolic operators of the form

$$-a^{ij}(x,y)D_{x_i}D_{x_j}-\tilde{a}^{kl}(x,y)D_{y_k}D_{y_l}+b^i(x,y)D_{x_i}+\tilde{b}^k(x,y)D_{y_k}+c(x,y)$$

with bounded coefficients under the following assumptions: the matrix $\mathcal{A}$ is uniformly elliptic, the matrix $\widetilde{\mathcal{A}}$ is non-negative definite, $c(x)\geqslant 0$, and the domain Ω

[22] Here the reader can (perhaps for the first time in the literature) see the duality between an estimate for the gradient of a solution on $\partial\Omega$ and the normal derivative lemma.

satisfies the condition of the interior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid whose axis is not orthogonal to the plane $y = 0$.

In [38] the results of [37] were generalized to the class of weakly degenerate operators with leading coefficients satisfying conditions close to the ones in [29] and [31] (the lower-order coefficients are bounded).[23]

Finally, in a series of papers published in 1978–1980 Kamynin and Khimchenko presented sophisticated generalizations of the results in [30].

A rather interesting 'weakened' form of the normal derivative lemma was established by Nadirashvili [40] in a domain Ω satisfying the interior cone condition. Namely, let $\mathcal{L}$ be a uniformly elliptic operator of the form (2.1) and let $c(x) \geqslant 0$. If u is a non-constant function such that $\mathcal{L}u + cu \geqslant 0$ and u attains a non-positive minimum value at a point $x^0 \in \partial\Omega$, then *in any neighbourhood of* x^0 *there is an* $x^* \in \partial\Omega$ such that

$$\liminf_{\varepsilon \to +0} \frac{u(x^* + \varepsilon \boldsymbol{\ell}) - u(x^*)}{\varepsilon} > 0$$

for any strictly inward direction $\boldsymbol{\ell}$. In [41] this result was extended to a certain class of domains with outward 'peaks' and weakly degenerate (in the spirit of [38]) non-divergence operators.

In [33] Lieberman introduced the important concept of *regularized distance*.[24] In particular, he showed that in any domain Ω of class $\mathcal{C}^1$ there is a function $\rho \in \mathcal{C}^2(\mathbb{R}^n \setminus \partial\Omega) \cap \mathcal{C}^1(\mathbb{R}^n)$ such that the following estimates hold (in which the $+$ and $-$ signs relate to points $x \in \overline{\Omega}$ and $x \in \mathbb{R}^n \setminus \Omega$, respectively):

$$\begin{gathered} C^{-1}\mathrm{d}(x) \leqslant \pm\rho(x) \leqslant C\,\mathrm{d}(x), \\ |D\rho(x) - D\rho(y)| \leqslant C\sigma(|x-y|), \\ |D^2\rho(x)| \leqslant C\,\frac{\sigma(|\rho(x)|)}{|\rho(x)|}\,. \end{gathered}$$

Here σ is the common modulus of continuity of the gradients of the functions defining $\partial\Omega$ in local coordinates.

As a consequence, the normal derivative lemma was proved in [33] for a domain of class $\mathcal{C}^{1,\mathcal{D}}$ when the (leading and lower-order) coefficients satisfy conditions close to the ones in [29] and [31]. Next, in [42] estimates for the gradient on $\partial\Omega$ of the solution of the Dirichlet problem in a domain in $\mathcal{C}^{1,\mathcal{D}}$ with boundary data $g \in \mathcal{C}^{1,\mathcal{D}}(\partial\Omega)$ were established, and the boundary regularity of the solution was analysed in the case when $Dg \in \mathcal{C}(\partial\Omega)$ fails the Dini condition.

Finally, we note the monumental paper [14], where the assumptions on the coefficients ensuring the validity of the normal derivative lemma and the strong maximum principle are slightly relaxed in comparison with the work mentioned above, although they are significantly harder to verify. That paper also contains some new counterexamples, which show that these assumptions are sharp.

2.3. The Aleksandrov–Bakelman maximum principle. This subsection concerns one of the most beautiful geometric ideas in the theory of partial differential

[23]This topic has been developed further, for instance, in [39].

[24]This construction had also appeared previously in some special cases (for instance, see [31] and [32]).

equations, the maximum principle of Aleksandrov and Bakelman. This is the common name for *a priori* maximum estimates for the solutions of non-divergence equations, which have a great number of applications. In particular, they play a key role in the proof of the strong maximum principle and the normal derivative lemma for equations with unbounded lower-order coefficients belonging to Lebesgue spaces.

The first estimates of this type were published in [43] and [44].[25] An estimate for the solution of the Dirichlet problem in the general case was obtained in [48]. In addition, it was proved in [48] that the resulting estimates are sharp.[26] In 1963 Aleksandrov gave a cycle of talks in Italy, where he presented his method. These talks were published in Rome two years later.

To prove the Aleksandrov–Bakelman estimate we introduce a few definitions.

Let u be a continuous function in a domain Ω such that $u\big|_{\partial\Omega} < 0$. Let $\widetilde{\Omega} = \operatorname{conv}(\Omega)$ denote the convex hull of Ω. In what follows we assume that u_+ is extended to $\widetilde{\Omega} \setminus \Omega$ by zero.

By the *convex hull* of u_+ we mean the smallest upper convex function that majorizes u_+ in $\widetilde{\Omega}$. We denote it by z. It is obvious that $z\big|_{\partial\widetilde{\Omega}} = 0$ and the subgraph of z is a convex set (the convex hull of the subgraph of u_+). We can also show (see [18]) that if Ω is a domain in $\mathcal{C}^{1,1}$ and $u \in \mathcal{C}^{1,1}(\Omega)$, then[27] $z \in \mathcal{C}^{1,1}(\widetilde{\Omega})$. We also introduce the so-called *contact set*

$$\mathcal{Z} = \{x \in \Omega \colon z(x) = u(x)\}.$$

Now we define the (generally speaking, multivalued) *normal* (or *hodograph*) *map* $\Phi\colon \widetilde{\Omega} \to \mathbb{R}^n$ induced by z. To each point $x^0 \in \widetilde{\Omega}$ it assigns all vectors $p \in \mathbb{R}^n$ such that the graph of $\pi(x) = p\cdot(x-x^0) + z(x^0)$ is a support plane of the subgraph of z at x^0. It is obvious that if $z \in C^1(\widetilde{\Omega})$, then Φ is single valued in $\widetilde{\Omega}$ (but not in its closure!) and can be expressed by $\Phi(x) = Dz(x)$.

First we consider an operator $\mathcal{L}_0$ with measurable coefficients.

Lemma 2.1. *Let Ω be a domain of class $\mathcal{C}^{1,1}$, and let $u \in \mathcal{C}^{1,1}(\Omega)$ with $u\big|_{\partial\Omega} < 0$. Assume that the uniform ellipticity condition (2.3) is satisfied. Then for each non-negative function $\mathfrak{g}$,*

$$\int_{\Phi(\widetilde{\Omega})} \mathfrak{g}(p)\,dp \leqslant \frac{1}{n^n}\int_{\mathcal{Z}} \mathfrak{g}(Du)\frac{(\mathcal{L}_0 u)^n}{\det(\mathcal{A})}\,dx. \tag{2.8}$$

[25] This result has complicated history. The paper [44] was published later than the short note [43], but it was submitted for publication slightly earlier. It is written in [45], § 28.1, that "the first version of these maximum principles was obtained by Bakelman [46], [47] in 1959". However, those papers did not yet contain the estimates under consideration, although the idea that normal images can be used to estimate solutions had been developed by both Bakelman and Aleksandrov in their previous work. On the other hand, the importance of [44] was reflected incorrectly in the survey [11].

[26] The results in [48] were subsequently re-discovered in [49] and [50]. In this connection, the term *Aleksandrov–Bakelman–Pucci (ABP) maximum principle* is common in English language literature.

[27] Note that this does not hold if the condition $u\big|_{\partial\Omega} < 0$ is relaxed to $u\big|_{\partial\Omega} \leqslant 0$.

Proof. Note that under the hypotheses of the lemma, Φ satisfies the Lipschitz condition. By the formula for a change of variables under the integral sign,

$$\int_{\Phi(\widetilde{\Omega})} \mathfrak{g}(p)\,dp = \int_{\widetilde{\Omega}} \mathfrak{g}(Dz)|\det(D^2z)|\,dx = \int_{\widetilde{\Omega}} \mathfrak{g}(Dz)\det(-D^2z)\,dx. \tag{2.9}$$

(The last equality holds because the matrix $(-D^2z)$ is non-negative definite.)

If $x \notin \mathcal{Z}$, then by Carathéodory's theorem, $(x, z(x))$ is an interior point of a simplex[28] which lies entirely on the graph of z. Hence the second derivative of z vanishes in some direction. However, as $D^2z(x)$ is sign definite, this must be a principal direction, so that $\det(-D^2z(x)) = 0$.

On the other hand, if $x \in \mathcal{Z}$, then the condition of tangency at x yields

$$Dz(x) = Du(x) \quad \text{and} \quad -D^2z(x) \leqslant -D^2u(x).$$

(The second relation holds in the sense of quadratic forms for almost all x.) Hence it follows from (2.9) that

$$\int_{\Phi(\widetilde{\Omega})} \mathfrak{g}(p)\,dp \leqslant \int_{\mathcal{Z}} \mathfrak{g}(Du)\det(-D^2u)\,dx.$$

Now, since the matrices $\mathcal{A}$ and $-D^2u$ are non-negative definite on $\mathcal{Z}$, the eigenvalues of $-\mathcal{A}\cdot D^2u$ are non-negative. By the inequality between the arithmetic and geometric means, we have (throughout, Tr denotes the trace of a matrix)

$$\det(-D^2u) = \frac{\det(-\mathcal{A}\cdot D^2u)}{\det(\mathcal{A})} \leqslant \frac{1}{n^n}\,\frac{(\mathrm{Tr}(-\mathcal{A}\cdot D^2u))^n}{\det(\mathcal{A})} = \frac{1}{n^n}\,\frac{(\mathcal{L}_0u)^n}{\det(\mathcal{A})},$$

which immediately yields (2.8). □

Remark 2.2. Since $u > 0$ and $\mathcal{L}_0u \geqslant 0$ on $\mathcal{Z}$, the more convenient inequality

$$\int_{\Phi(\widetilde{\Omega})} \mathfrak{g}(p)\,dp \leqslant \frac{1}{n^n}\int_{\{u>0\}} \mathfrak{g}(Du)\,\frac{(\mathcal{L}_0u)_+^n}{\det(\mathcal{A})}\,dx \tag{2.10}$$

is often used in place of (2.8).

Theorem 2.2. *Under assumption* (2.2), *let* $\mathrm{Tr}(\mathcal{A}) > 0$ *almost everywhere in* Ω. *Then the following estimate holds for each function* $u \in W^2_{n,\mathrm{loc}}(\Omega)$ *such that*[29] $u\big|_{\partial\Omega} \leqslant 0$:

$$\big(\max_{\overline{\Omega}} u_+\big)^n \leqslant \frac{\mathrm{diam}^n(\Omega)}{n^n|B_1|}\int_{\mathcal{Z}} \frac{(\mathcal{L}_0u)^n}{\det(\mathcal{A})}\,dx. \tag{2.11}$$

(*Here and throughout*, *we put* $0/0 = 0$ *when this uncertainty arises*.)

Proof. First suppose that the matrix $\mathcal{A}$, the function u, and the domain Ω satisfy the hypotheses of Lemma 2.1. It is sufficient to look at the case when $M = \max_{\overline{\Omega}} u = \max_{\widetilde{\Omega}} z > 0$.

[28] In this case the simplex can have any dimension between 1 and n.

[29] This means that for each $\varepsilon > 0$, the inequality $u - \varepsilon < 0$ holds in a neighbourhood of $\partial\Omega$.

We put $d = \operatorname{diam}(\Omega) = \operatorname{diam}(\widetilde{\Omega})$ and show that the set $\Phi(\widetilde{\Omega})$ contains the ball $B_{M/d}$. Indeed, let $p \in B_{M/d}$. Consider the graph of $\pi(x) = p \cdot x + h$, which is a plane. By selecting h appropriately, we can ensure that it is a support plane for the subgraph of z at some point x^0, so that we can write $\pi(x) = p\cdot(x-x^0)+z(x^0)$.

If $x^0 \in \partial\widetilde{\Omega}$, then $z(x^0) = 0$ and at a maximum point of z we have

$$M = z(x) \leqslant p \cdot (x - x^0) \leqslant |p| \cdot d < M,$$

which is impossible. Hence $x^0 \in \widetilde{\Omega}$, which shows that

$$p = Dz(x^0) = \Phi(x^0) \in \Phi(\widetilde{\Omega})$$

and proves the required result.

Using (2.8) with $\mathfrak{g} \equiv 1$, we obtain

$$|B_1| \cdot \left(\frac{M}{d}\right)^n = |B_{M/d}| \leqslant |\Phi(\widetilde{\Omega})| \leqslant \frac{1}{n^n} \int_{\mathcal{Z}} \frac{(\mathcal{L}_0 u)^n}{\det(\mathcal{A})}\, dx,$$

which yields (2.11) directly.

Now consider the general case. The integrand in (2.11) does not change after multiplying the matrix $\mathcal{A}$ by a positive function, so we can assume without loss of generality that $\operatorname{Tr}(\mathcal{A}) \equiv 1$. We consider the function $u^\varepsilon = u - \varepsilon$ and approximate Ω from inside by domains with smooth boundaries. Since (2.11) survives a passage to the limit in W^2_n, we can assume that u^ε is a smooth function. We apply (2.11) to u^ε and the uniformly elliptic operator $\mathcal{L}_0 - \nu\Delta$. Next we let $\nu \to 0$, and then $\varepsilon \to 0$. □

Theorem 2.3. *Let $\mathcal{L}$ be an operator of the general form* (2.1) *and assume that* (2.2) *holds and* $\operatorname{Tr}(\mathcal{A}) > 0$ *almost everywhere in* Ω. *Let*

$$\mathfrak{h} \equiv \frac{|\mathbf{b}|}{\det^{1/n}(\mathcal{A})} \in L_n(\Omega). \tag{2.12}$$

Then

$$\max_{\overline{\Omega}} u_+ \leqslant N(n, \|\mathfrak{h}\|_{n,\{u>0\}})\operatorname{diam}(\Omega) \left\| \frac{(\mathcal{L}u)_+}{\det^{1/n}(\mathcal{A})} \right\|_{n,\{u>0\}} \tag{2.13}$$

for any function u satisfying the assumptions of Theorem 2.2.

Proof. Suppose that the matrix $\mathcal{A}$, the function u, and the domain Ω satisfy the hypotheses of Lemma 2.1. We can derive the general case from this as in the second part of the proof of Theorem 2.2.

Let $\mathfrak{g} = \mathfrak{g}(|p|)$. Bearing in mind that $B_{M/d} \subset \Phi(\widetilde{\Omega})$, from (2.10) we obtain

$$n|B_1| \int_0^{M/d} \mathfrak{g}(\rho)\rho^{n-1}\, d\rho \leqslant \frac{1}{n^n} \int_{\{u>0\}} \mathfrak{g}(|Du|)\, \frac{(\mathcal{L}u - b^i D_i u)^n_+}{\det(\mathcal{A})}\, dx. \tag{2.14}$$

We put

$$F = \left\| \frac{(\mathcal{L}u)_+}{\det^{1/n}(\mathcal{A})} \right\|_{n,\{u>0\}} + \varepsilon, \qquad \varepsilon > 0.$$

Then we can estimate the ratio on the right-hand side of (2.14) using Hölder's inequality:

$$\frac{(\mathcal{L}u - b^i D_i u)_+^n}{\det(\mathcal{A})} \leqslant (F^{n/(n-1)} + |Du|^{n/(n-1)})^{n-1}\biggl(\frac{(\mathcal{L}u)_+^n}{\det(\mathcal{A})F^n} + \mathfrak{h}^n\biggr).$$

We put $\mathfrak{g}(\rho) = (F^n + \rho^n)^{-1}$. Then, from (2.14) we obtain

$$\begin{aligned} n|B_1| \int_0^{M/d} \frac{\rho^{n-1}}{F^n + \rho^n}\, d\rho \\ \leqslant \frac{1}{n^n} \int_{\{u>0\}} \frac{(F^{n/(n-1)} + |Du|^{n/(n-1)})^{n-1}}{F^n + |Du|^n} \biggl(\frac{(\mathcal{L}u)_+^n}{\det(\mathcal{A})F^n} + \mathfrak{h}^n\biggr)\, dx. \end{aligned}$$

Now, using the elementary inequality $(x + y)^{n-1} \leqslant 2^{n-2}(x^{n-1} + y^{n-1})$, we can deduce that

$$\ln\biggl(1 + \frac{M^n}{d^n F^n}\biggr) \leqslant \frac{2^{n-2}}{n^n|B_1|}(1 + \|\mathfrak{h}\|^n_{n,\{u>0\}})$$

or

$$M \leqslant d \cdot F \cdot \biggl(\exp\biggl\{\frac{2^{n-2}}{n^n|B_1|}(1 + \|\mathfrak{h}\|^n_{n,\{u>0\}})\biggr\} - 1\biggr)^{1/n}.$$

Letting $\varepsilon \to 0$ in the expression for F, we arrive at (2.13). □

Remark 2.3. If the uniform ellipticity condition (2.3) is satisfied, then taking Remark 2.2 into account, we obtain a simpler estimate from (2.13):

$$\max_{\overline{\Omega}} u_+ \leqslant N\biggl(n, \frac{\|\mathbf{b}\|_{n,\{u>0\}}}{\nu}\biggr) \frac{\operatorname{diam}(\Omega)}{\nu} \|(\mathcal{L}u)_+\|_{n,\{u>0\}}. \tag{2.15}$$

Now we explain the difference between Theorem 2.3 and some other maximum estimates.

Hopf's maximum estimate is well known for uniformly elliptic operators of the form (2.1) (for instance, see [51], Theorem 3.7):

$$\max_{\overline{\Omega}} u_+ \leqslant C\biggl(\operatorname{diam}(\Omega), \frac{\|\mathbf{b}\|_{\infty,\{u>0\}}}{\nu}\biggr) \frac{\|(\mathcal{L}u)_+\|_{\infty,\{u>0\}}}{\nu}.$$

Here the maximum of the solution is estimated in terms of the L_∞-norm of the right-hand side, which turns out to be insufficient for applications.

On the other hand, it follows from coercive estimates in L_r (see [51], Theorem 9.13) and Sobolev's embedding theorem that

$$\max_{\overline{\Omega}} u_+ \leqslant C\|(\mathcal{L}_0 u)_+\|_{r,\Omega}, \qquad r > \frac{n}{2}. \tag{2.16}$$

However, here the constant C *depends on the moduli of continuity of the coefficients* a^{ij}. Therefore, for instance, for quasilinear equations in which the coefficients a^{ij} depend on the solution u itself and on its derivatives, the estimate (2.16) is of little use.

The Aleksandrov–Bakelman estimate is distinguished by the fact that it requires neither continuous leading coefficients nor bounded lower-order coefficients and right-hand side.

In connection with Theorem 2.3, we recall the so-called *maximum principle in the Bony form*:

Let $\mathcal{L}$ be an operator of the form (2.1) and assume that (2.2) holds. If a function u attains its minimum value at a point $x^0 \in \Omega$, then

$$\operatorname{ess}\liminf_{x\to x^0} \mathcal{L}u \leqslant 0.$$

For operators with bounded coefficients, this was proved in 1967 by Bony for $u \in W_q^2(\Omega)$, $q > n$, and in 1983 by P.-L. Lions[30] for $u \in W_n^2(\Omega)$. We prove a version of this result for operators with unbounded lower-order coefficients.

Corollary 2.1. *Assume that the coefficients of the operator $\mathcal{L}$ satisfy the hypotheses of Theorem 2.3. If a function $u \in W^2_{n,\mathrm{loc}}(\Omega)$ attains its minimum value at a point $x^0 \in \Omega$, then*

$$\operatorname{ess}\liminf_{x\to x^0} \frac{\mathcal{L}u}{\operatorname{Tr}(\mathcal{A})} \leqslant 0. \tag{2.17}$$

Proof. As in Theorem 2.2, we can assume without loss of generality that $\operatorname{Tr}(\mathcal{A}) \equiv 1$. Let x^0 be the origin.

Assume that $\mathcal{L}u \geqslant \delta > 0$ almost everywhere in a neighbourhood of the origin. In the ball B_r we consider the function

$$w^\varepsilon(x) = \varepsilon\left(1 - \frac{|x|^2}{r^2}\right) - u(x) + u(0).$$

Then $w^\varepsilon(0) = \varepsilon$ and for any sufficiently small r, we have $w^\varepsilon\big|_{\partial B_r} \leqslant 0$. Applying estimate (2.15) to w^ε in B_r, we obtain

$$\varepsilon \leqslant N(n, \|\mathfrak{h}\|_{n,B_r}) \cdot 2r \cdot \left\| \frac{(\mathcal{L}w^\varepsilon)_+}{\det^{1/n}(\mathcal{A})} \right\|_{n,B_r}.$$

Since

$$\mathcal{L}w^\varepsilon = \frac{2\varepsilon}{r^2}\big(\operatorname{Tr}(\mathcal{A}) + b^i x_i\big) - \mathcal{L}u \leqslant \frac{2\varepsilon}{r^2}(1 + r|\mathbf{b}|) - \delta,$$

for $\varepsilon < \delta r^2/4$ this yields

$$\varepsilon \leqslant N \left\| \frac{(4\varepsilon|\mathbf{b}| - r\delta)_+}{\det^{1/n}(\mathcal{A})} \right\|_{n,B_r} \overset{(*)}{\leqslant} 4\varepsilon N \left\| \left(\mathfrak{h} - \frac{r\delta}{4\varepsilon}\right)_+ \right\|_{n,B_r} = o(\varepsilon) \quad \text{as } \varepsilon \to 0.$$

(Inequality $(*)$ holds because $\det^{1/n}(\mathcal{A}) \leqslant \operatorname{Tr}(\mathcal{A}) = 1$.) This contradiction proves (2.17). □

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[30] Lions proved a stronger property (the second relation is treated in the sense of quadratic forms):

$$\operatorname{ess}\liminf_{x\to x^0} |Du| = 0 \quad \text{and} \quad \operatorname{ess}\liminf_{x\to x^0} D^2u \geqslant 0.$$

However, for operators with unbounded coefficients, this does not yield (2.17) directly.

Aleksandrov developed and refined the results in [48] repeatedly. In [52] he obtained *pointwise* estimates for the solution of the Dirichlet problem in terms of the distance to the boundary of the domain, and in [53] these were extended to a wider class of equations. The paper [54] was devoted to showing that the bounds obtained are attainable, and in the short note [55] Aleksandrov proved that *in general* the assumptions on the right-hand side of the equation cannot be relaxed. Finally, in [56] pointwise estimates for the solution in terms of some delicate characteristics of the domain Ω were obtained, and an estimate on $\partial\Omega$ for the gradient of the solution was found on the basis of this result in some special cases.

Krylov [57] was the first to obtain Aleksandrov-type estimates for parabolic operators in the mid 1970s. Following this, investigations of elliptic and parabolic problems proceeded almost in parallel, but the discussion of results for non-stationary equations lies outside the scope of our survey.

Subsequently, the techniques using normal images were also applied to boundary value problems other than the Dirichlet problem. An Aleksandrov-type local maximum estimate for a *problem with oblique derivative* (where the derivative along the direction of a non-tangential vector field is prescribed on part of the boundary of the domain) was established in [58] for bounded coefficients b^i and in [59] in the general case (see also [60]).

For the *Venttsel problem*, where an operator of the second order in the tangential variables

$$\mathcal{L}' \equiv -\alpha^{ij}(x)\mathfrak{d}_i\mathfrak{d}_j + \beta^i(x)D_i, \qquad \mathfrak{d}_i \equiv D_i - \mathbf{n}_i\mathbf{n}_k D_k, \qquad \beta^i(x)\mathbf{n}_i \leqslant 0,$$

is given on the boundary, the corresponding estimates were obtained in [61] and [62] in two cases, the non-degenerate case, when $\mathcal{L}'$ is uniformly elliptic in the tangential variables, and the degenerate case when the second-order terms in the boundary operator can vanish on a subset of positive measure, but the vector field (β^i) is not tangential to $\partial\Omega$. Subsequently, these estimates were generalized to the case when the operators $\mathcal{L}$ and $\mathcal{L}'$ have unbounded lower-order coefficients [63], [64]. In [65] local estimates of Aleksandrov type were established for solutions of the so-called *two-phase* Venttsel problem. In all these cases such estimates were the starting point for the derivation of a series of *a priori* estimates needed in the proofs of existence theorems for solutions of quasilinear or non-linear boundary-value problems.

Another direction in which Aleksandrov's ideas have been developed involves transferring maximum estimates to equations whose lower-order coefficients and right-hand sides belong to other function classes. In [66], [60], and [67] various classes of operators with 'compound' coefficients were considered. The paper [68] was devoted to an Aleksandrov-type estimate in terms of the norms of the right-hand side in weighted Lebesgue spaces. Each of these results extended accordingly a class of non-linear equations for which one can prove the solvability of the main boundary-value problems.

Caffarelli [69] established the Aleksandrov–Bakelman estimate for the so-called *viscosity solutions* of elliptic equations. Subsequently, this idea was repeatedly applied to various classes of non-linear equations (for instance, see Chap. 3 of [70] and the references therein; also see a number of more recent papers).

Yet another group of papers was devoted to relaxing the conditions on the right-hand side of the equation for *certain classes* of operators $\mathcal{L}$. In 1984 Fabes and Stroock [71] obtained the estimate (2.16) for operators with measurable leading coefficients under the assumption $r > r_0$, where $r_0 < n$ is an exponent depending on the ellipticity constant of the operator. In [72] and [73] this estimate was also shown for the problem with oblique derivative. On the other hand, Pucci [74] introduced the concept of *maximal* and *minimal operators*, and used this to find a lower bound for those r_0 for which such an estimate is possible (in this connection, see [75] and the literature cited therein). Necessary and sufficient conditions for (2.16) to hold have only been obtained in the two-dimensional case [76]. In some papers (see [77] and the literature therein) the results in [71] were extended to viscosity solutions of non-linear equations.

In [78] several estimates for the maximum of the solution in terms of the L_m-norm of the right-hand side were established (here $m \in (n/2, n]$ is an integer) under the condition that for almost all $x \in \Omega$, the matrix of leading coefficients of the equation belongs to a certain special convex cone in the space of matrices. Among the most recent achievements in this direction, we also refer to Trudinger's paper [79]. All these investigations are certainly far from being complete.

We must also mention the paper [80], which looked at the dependence of an estimate for the maximum on the characteristics of the domain. In particular, an estimate in terms of $|\Omega|^{1/n}$ in place of the diameter of the domain was obtained (note that for *convex domains*, this result goes back to [48]).

We also refer to a 2000 paper by Kuo and Trudinger, where they obtained a discrete analogue of the Aleksandrov–Bakelman estimate for difference operators.

2.4. Results for operators with coefficients $b^i(x)$ in Lebesgue spaces. A simple consequence of the Aleksandrov–Bakelman estimate is the weak maximum principle for operators of the form (2.1) with coefficients $b^i \in L_n(\Omega)$ and for $u \in W^2_n(\Omega)$. Moreover, it was already observed in [48] that this estimate allows one to consider operators $\mathcal{L} + c(x)$ with $c(x)$ 'of wrong sign'.

Corollary 2.2. *Assume that the coefficients of the operator $\mathcal{L}$ satisfy the hypotheses of Theorem 2.3. Then there is a positive constant δ which only depends on n,* $\operatorname{diam}\Omega$, *and $\|\mathfrak{h}\|_{n,\Omega}$ (the function $\mathfrak{h}$ was introduced in (2.12)) such that if*

$$h \equiv \frac{c_-}{\det^{1/n}(\mathcal{A})} \in L_n(\Omega) \quad \textit{and} \quad \|h\|_{n,\Omega} < \delta$$

(*recall that we put $0/0 = 0$ when this indeterminacy arises*), *then the weak maximum principle holds for $\mathcal{L} + c(x)$ and $u \in W^2_{n,\mathrm{loc}}(\Omega)$.*

Proof. Suppose that, on the contrary, $\mathcal{L}u + cu \geqslant 0$ in Ω and $u \geqslant 0$ on $\partial\Omega$, but $\min_\Omega u = -A < 0$. Consider $u^\varepsilon = -u - \varepsilon$ and apply (2.13) to it. Since $\mathcal{L}u^\varepsilon = -\mathcal{L}u \leqslant cu \leqslant Ac_-$ on $\{u^\varepsilon > 0\}$, we obtain

$$(A - \varepsilon)_+ \leqslant N(n, \|\mathfrak{h}\|_{n,\Omega}) \operatorname{diam}(\Omega)\, \|h\|_{n,\Omega}\, A,$$

which is impossible in the case when $N(n, \|\mathfrak{h}\|_{n,\Omega}) \operatorname{diam}(\Omega) \|h\|_{n,\Omega} < 1$ and $\varepsilon > 0$ is sufficiently small. □

It is easy to see that now the proof of Theorem 2.1 works without any changes also for the so-called *strong supersolutions*, that is, those $u \in W_n^2(\Omega)$ that satisfy $\mathcal{L}u + cu \geqslant 0$ almost everywhere in Ω (the operator $\mathcal{L}$ has bounded measurable coefficients). However, new ideas were needed to extend the arguments to lower-order coefficients in Lebesgue spaces.

Note that we cannot relax the condition $b^i \in L_n(\Omega)$ to $b^i \in L_p(\Omega)$ for $p < n$: the function $u(x) = |x|^2$ satisfies the equation

$$-\Delta u + \frac{nx_i}{|x|^2} D_i u = 0 \quad \text{in } B_1,$$

but it does not satisfy the maximum principle. Here the coefficients $b^i(x) = nx_i/|x|^2$ lie in the spaces $L_p(B_1)$ for any $p < n$ and even in the weak space L_n (the Lorentz space $L_{n,\infty}(B_1)$), but not in $L_n(B_1)$.

For operators with $b^i \in L_n(\Omega)$, the strong maximum principle was established in [30], Part VI. We prove the simplest version of this result.[31]

Theorem 2.4. *Let $\mathcal{L}$ be an operator of the form* (2.1), *assume that* (2.3) *is satisfied, and let $b^i \in L_{n,\mathrm{loc}}(\Omega)$. Also assume that $u \in W^2_{n,\mathrm{loc}}(\Omega)$ and $\mathcal{L}u \geqslant 0$ almost everywhere in Ω. If u takes its minimum value at an interior point of the domain, then $u \equiv \mathrm{const}$ and $\mathcal{L}u \equiv 0$.*

Proof. Suppose that $u \not\equiv \mathrm{const}$, but the set (2.4) is non-empty. As in the proof of Theorem 2.1, there is a ball in $\Omega \setminus M$ whose boundary contains a point $x^0 \in M$. Let $r/2$ be the radius of this ball and assume without loss of generality that $B_r \subset \Omega$. Taking $\pi = B_r \setminus \overline{B}_{r/4}$, we consider the barrier function (2.5) in π.

Then we have

$$\mathcal{L}v_s(x) \leqslant s|x|^{-s-2}\big(-(s+2)\nu + n\nu^{-1} + r|\mathbf{b}(x)|\big).$$

In contrast to Theorem 2.1, here we cannot show that $\mathcal{L}v_s \leqslant 0$. However, taking $s = n\nu^{-2}$, we obtain

$$\mathcal{L}v_s(x) \leqslant s|x|^{-s-2}|\mathbf{b}(x)| \leqslant 4^{s+2}sr^{-s-1}|\mathbf{b}(x)| \quad \text{in } \pi.$$

By construction, $u(x) - u(x^0) > 0$ on $\partial B_{r/4}$. Hence, if $\varepsilon > 0$ is small enough, then $w^\varepsilon(x) = \varepsilon v_s(x) - u(x) + u(x_0)$ is non-positive on the whole boundary of π.

Applying (2.15) to w^ε in π, we obtain

$$\begin{aligned} w^\varepsilon(x) &\leqslant C(n,\nu,\|\mathbf{b}\|_{n,\pi}) r\varepsilon \|(\mathcal{L}v_s(x))_+\|_{n,\pi} \\ &\leqslant C(n,\nu,s,\|\mathbf{b}\|_{n,\pi})\varepsilon r^{-s}\|\mathbf{b}\|_{n,\pi}, \end{aligned}$$

so that

$$u(x) - u(x^0) \geqslant \varepsilon\big(|x|^{-s} - r^{-s} - C(n,\nu,s,\|\mathbf{b}\|_{n,\pi})\|\mathbf{b}\|_{n,\pi} r^{-s}\big). \tag{2.18}$$

[31] In [30], Part VI, operators of the form $\mathcal{L}+c(x)$ were considered such that $c(x) \leqslant h(x)/|x-x^0|$, where x^0 is a (zero-)minimum point of u and $h \in L_n(\Omega)$. In addition, in this paper the conditions for the coefficients can depend on the direction.

By Lebesgue's theorem, for each $\delta > 0$ we can take a sufficiently small r such that $\|\mathbf{b}\|_{n,\pi} \leqslant \delta$. Then inequality (2.18) at x^0 yields

$$0 \geqslant \varepsilon r^{-s}\big(2^s - 1 - C(n,\nu,s,\delta)\delta\big),$$

which is impossible if δ is small enough. □

As a consequence, the following result was proved in [30], Part VI.[32]

Corollary 2.3. *Suppose that $\mathcal{L}$ and u satisfy the hypotheses of Theorem 2.4. Assume that Ω satisfies the interior ball condition in a neighbourhood U of a point $x^0 \in \partial\Omega$. Let*

$$u\big|_{\partial\Omega\cap U} \equiv \inf_{\Omega} u \quad \textit{and} \quad Du\big|_{\partial\Omega\cap U} \equiv 0. \tag{2.19}$$

Then $u \equiv \text{const}$ in Ω.

Proof. By extending u by a constant outside Ω in a neighbourhood of x^0, we make sure that the hypotheses of Theorem 2.4 are satisfied. □

Corollary 2.3 is easily seen to be significantly weaker than the normal derivative lemma because conditions (2.19) must hold on a whole piece of the boundary. However, in contrast to the case of bounded lower-order coefficients (when the strong maximum principle and the normal derivative lemma have virtually identical proofs), the normal derivative lemma fails under the assumptions of Theorem 2.4! Here is a counterexample (see [81]–[83]).

Let $u(x) = x_n \ln^{\alpha}(|x|^{-1})$ in the half-ball $B_r^+ = B_r \cap \{x_n > 0\}$. Then it is easy to see that $u \in W_n^2(B_r^+)$ for $r \leqslant 1/2$ and $\alpha < (n-1)/n$. Next, direct calculations show that u solves

$$-\Delta u + b^n(x) D_n u = 0,$$

where

$$|b^n| \leqslant \frac{C(\alpha)}{|x|\ln(|x|^{-1})} \in L_n(B_r^+).$$

Finally, $u > 0$ in B_r^+ and u takes its minimum value at the boundary point 0. However, for $\alpha < 0$ it is obvious that $D_n u(0) = 0$.

Remark 2.4. A weak form of the normal derivative lemma (see [40]) holds in this example. We believe that this result also holds for a general uniformly elliptic operator $\mathcal{L}$ such that $b^i \in L_n(\Omega)$, but to our knowledge the question is still open.

Remark 2.5. The above counterexample also shows that the condition $b^i \in L_n(\Omega)$ is insufficient for the gradient estimates of the solution of the Dirichlet problem on $\partial\Omega$ because $D_n u(0) = +\infty$ for $\alpha > 0$.

The paper [84] by Ladyzhenskaya and Uraltseva plays an important role. (A short note was published in *Doklady Akademii Nauk SSSR*[33] three years earlier.) It is where the iterative method for estimating a solution in a neighbourhood of the boundary was used for the first time. In the simplest case it is as follows.

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[32] We also state a simplified version of this result.

[33] Translated as *Soviet Mathematics. Doklady*.

Let u be a function in a cylinder $Q_{1,1}$ such that $\mathcal{L}u = f$ and $u\big|_{x_n=0} = 0$. Consider the sequence of cylinders Q_{r_k,h_k}, where $r_k = 2^{-k}$ and h_k is an appropriately selected sequence such that $h_k = o(r_k)$ as $k \to \infty$. We put

$$M_k = \sup_{Q_{r_k,h_k}} \frac{u(x)}{h_k}$$

and apply the Aleksandrov–Bakelman estimate to the difference

$$u(x) - M_k h_k \cdot \mathfrak{v}\left(\frac{x'}{r_k}, \frac{x_n}{h_k}\right),$$

where $\mathfrak{v}$ is a certain special barrier function.

The resulting estimate at $x \in Q_{r_{k+1},h_{k+1}}$ is a recurrence relation linking M_{k+1} and M_k. Iterating it, we obtain $\limsup_k M_k < \infty$, which provides an upper bound for $D_n u(0)$ in terms of $\sup_{Q_{1,1}} u$ and a certain integral norm of the right-hand side.

In [84] this approach was applied to the equation $\mathcal{L}u = f$ with a uniformly elliptic operator $\mathcal{L}$ under the assumptions

$$u \in W_n^2(\Omega), \quad b^i \in L_q(\Omega), \quad f_+ \in L_q(\Omega), \qquad q > n, \tag{2.20}$$

when the domain belongs to one of the following two classes:

1) convex domains;
2) domains in the class[34] W_q^2.

As mentioned in § 2.3, an Aleksandrov-type estimate in $\Omega \subset Q_{R,R}$ was established in [66] for operators of the form (2.1) with 'composite' lower-order coefficients $b^i = b^i_{(1)} + b^i_{(2)}$ under the assumptions that

$$b^i_{(1)} \in L_n(\Omega) \quad \text{and} \quad |b^i_{(2)}(x)| \leqslant C x_n^{\gamma-1}, \quad \gamma \in (0,1). \tag{2.21}$$

On the basis of this result, in [66] an estimate for $\operatorname{ess\,sup} \partial_{\mathbf{n}} u$ on $\partial\Omega$ was established in domains of class W_q^2, $q > n$, under the assumptions

$$\begin{aligned} b^i &= b^i_{(1)} + b^i_{(2)}, & b^i_{(1)} &\in L_q(\Omega), & |b^i_{(2)}(x)| &\leqslant C x_n^{\gamma-1}, \\ \mathcal{L}u &= f^{(1)} + f^{(2)}, & f^{(1)}_+ &\in L_q(\Omega), & f^{(2)}_+(x) &\leqslant C x_n^{\gamma-1}, \end{aligned} \qquad \gamma \in (0,1).$$

Safonov [85] (see also [86]) developed a new approach to this problem, based on the boundary Harnack inequality (see § 4.3). This approach was used to prove in a uniform way

1) the normal derivative lemma under the assumption that $\mathcal{L}_0 u \geqslant 0$ in a domain satisfying the interior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition;[35]
2) an upper bound for $\partial_{\mathbf{n}} u(0)$ under the assumptions that $\mathcal{L}_0 u \leqslant 0$ and $u\big|_{\partial\Omega \cap B_r} = 0$ in a domain satisfying the exterior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition.[35]

[34] This means that each point $x^0 \in \partial\Omega$ has a neighbourhood U such that $U \cap \Omega$ is mapped onto $Q_{1,1}$ by a diffeomorphism of class W_q^2 so that the norms of the direct and inverse diffeomorphisms have uniform estimates with respect to x^0. This condition ensures that conditions (2.20) are invariant under local flattening of the boundary.

[35] In [85] the condition $\displaystyle\int_0^\varepsilon \tau^{-2}\phi(\tau)\,d\tau < \infty$ on the function ϕ defining the interior or exterior paraboloid is formally more general, but it was shown in Lemma 2.4 in [83] that the resulting condition for the domain is, in essence, equivalent to the standard one.

In [82] the (slightly refined) iterative method due to Ladyzhenskaya and Uraltseva was applied[36] to deduce the normal derivative lemma in $\Omega = Q_{R,R}$ under the following conditions:

$$u \in W^2_{n,\mathrm{loc}}(\Omega) \cap \mathcal{C}(\overline{\Omega}), \qquad \min_{\overline{\Omega}} u = u(0);$$
$$b^i \in L_n(\Omega), \qquad b^n \in L_q(\Omega), \quad q > n.$$

Thus, it turned out that, in comparison with the case when $b^i \in L_n(\Omega)$, it is sufficient to impose a stronger condition only on the normal component of the vector $\mathbf{b}$.

As of today, the sharpest conditions ensuring the validity of both the normal derivative lemma and the gradient estimate for the solution of the Dirichlet problem on the boundary of the domain are those obtained in [83]. It was also shown explicitly that these results are dual to one another. These results were obtained by combining the technique due to Ladyzhenskaya, Uraltseva, and Safonov with an Aleksandrov-type estimate from [60], where the condition on $b^i_{(2)}$ in (2.21) was improved to $|b^i_{(2)}(x)| \leqslant \sigma(x_n)/x_n$, $\sigma \in \mathcal{D}$.

Here is the statement of this result.

Theorem 2.5. *Let $\mathcal{L}$ be a uniformly elliptic operator of the form* (2.1) *in the domain $\Omega = Q_{R,R}$. Let $b^i = b^i_{(1)} + b^i_{(2)}$ and assume that*

$$b^i_{(1)} \in L_n(\Omega), \qquad \|b^n_{(1)}\|_{n,Q_{r,r}} \leqslant \sigma(r) \quad \text{for } r \leqslant R,$$
$$\text{and} \quad |b^i_{(2)}(x)| \leqslant \frac{\sigma(x_n)}{x_n}, \qquad \sigma \in \mathcal{D}.$$

Also let $u \in W^2_{n,\mathrm{loc}}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Then the following assertions hold:

1. *If $u > 0$ in $Q_{R,R}$, $u(0) = 0$, and $\mathcal{L}u \geqslant 0$, then*

$$\inf_{0<x_n<R} \frac{u(0,x_n)}{x_n} > 0.$$

2. *If $u\big|_{x_n=0} \leqslant 0$, $u(0) = 0$, and $\mathcal{L}u = f^{(1)} + f^{(2)}$, where*

$$\|f^{(1)}_+\|_{n,Q_{r,r}} \leqslant \sigma(r) \quad \text{for } r \leqslant R, \quad \text{and} \quad f^{(2)}_+(x) \leqslant \frac{\sigma(x_n)}{x_n},$$

then

$$\sup_{0<x_n<R} \frac{u(0,x_n)}{x_n} \leqslant C,$$

where the value of $C < \infty$ is determined by known quantities.

It is important to note that due to the presence of $b^i_{(2)}$ one can perform a coordinate change involving the regularized distance in a neighbourhood of an insufficiently smooth boundary. In this way we can reduce to Theorem 2.5 the corresponding results in domains satisfying the exterior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition.

[36] It was noted in [82] that for $b^i \in L_q(\Omega)$ with $q > n$, the normal derivative lemma was in fact already established in [84], Lemma 4.4. This had remained unnoticed for more than 20 years!

A new counterexample was constructed in [15], which shows that the interior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition is sharp for the normal derivative lemma. We formulate it in the simplest case.

Theorem 2.6. *Let Ω be a domain that is locally convex in a neighbourhood of the origin, that is,*

$$\Omega \cap B_R = \bigl\{x \in \mathbb{R}^n \colon F(x') < x_n < \sqrt{R^2 - |x'|^2}\,\bigr\}$$

for some $R > 0$, where F is a convex function, $F \geqslant 0$, and $F(0) = 0$.

Next, let $u \in W^2_{n,\mathrm{loc}}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be a solution of the equation $\mathcal{L}_0 u = 0$ with the uniformly elliptic operator $\mathcal{L}_0$, and let $u\big|_{\partial\Omega \cap B_R} = 0$.

If the function

$$\delta(r) = \sup_{|x'| \leqslant r} \frac{F(x')}{|x'|}$$

fails the Dini condition at zero, then

$$\lim_{\varepsilon \to +0} \frac{u(\varepsilon x_n)}{\varepsilon} = 0.$$

Note that if $\delta(r)$ satisfies the Dini condition at zero, then Ω satisfies the interior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition at the origin. Thus, in the case of a *locally convex* domain the Dini condition for $\delta(r)$ at zero is necessary and sufficient for the normal derivative lemma to hold.

We stress that in all previous counterexamples of this type (see [34], [35], [37], and [85]) it is assumed that the function $\inf\limits_{|x'| \leqslant r} \dfrac{F(x')}{|x'|}$ fails the Dini condition. Roughly speaking, in these counterexamples the Dini condition must fail in all directions, whereas in Theorem 2.6 it is sufficient for it to be violated in a single direction.

For domains of general form, a more refined counterexample was constructed in [86]. However, it is too complicated to be described here.

2.5. Harnack's inequality. As already mentioned in the Introduction, Harnack's inequality, which can be regarded as a qualitative version of the strong maximum principle, was first proved by Harnack [3] for harmonic functions on the plane. Since Harnack's proof is based on Poisson's formula, it clearly works in all dimensions. Harnack's formulation is presented in the majority of textbooks:

If $u \geqslant 0$ is a harmonic function in $B_R \subset \mathbb{R}^n$, then

$$u(0)\frac{(R - |x|)R^{n-2}}{(R + |x|)^{n-1}} \leqslant u(x) \leqslant u(0)\frac{(R + |x|)R^{n-2}}{(R - |x|)^{n-1}}\,. \tag{2.22}$$

Hence, for $\Omega = B_R$ and $\Omega' = B_{\theta R}$ with $\theta < 1$, we obtain inequality (1.2) with $C = \left(\dfrac{1+\theta}{1-\theta}\right)^n$ directly.

Throughout this subsection we assume that the uniform ellipticity condition (2.3) is satisfied.

In 1912 Lichtenstein proved inequality (1.2) for general operators $\mathcal{L}+c(x)$, $c \geqslant 0$, with $\mathcal{C}^2$-smooth coefficients (also in dimension two).

In 1955 Serrin established Harnack's inequality for $n = 2$ and operators $\mathcal{L}+c(x)$, $c \geqslant 0$, with *bounded* coefficients. This result was also proved independently by Bers and Nirenberg at the same time. For $n \geqslant 3$, Serrin also proved (1.2) under the assumption that[37] $a^{ij} \in \mathcal{C}^{0,\mathcal{D}}(\Omega)$.

Landis [87] (see also [28], Ch. 1) made a significant improvement. Using his '*growth lemma*', Landis proved Harnack's inequality in any dimension for the operator $\mathcal{L}_0$ with bounded coefficients under the additional condition that the eigenvalues of the matrix $\mathcal{A}$ have sufficiently small dispersion.[38] Namely, he assumed that the following relations hold (after multiplying $\mathcal{A}$ by a suitable positive function):

$$\operatorname{Tr}(\mathcal{A}) \equiv 1 \quad \text{and} \quad \nu > \frac{1}{n+2}. \tag{2.23}$$

(Clearly, we always have $\nu \leqslant 1/n$, where equality is only attained for the Laplace operator.)

Note that all the above results were established for classical solutions $u \in \mathcal{C}^2(\Omega)$.

Finally, Krylov and Safonov [88], [89] made the decisive step. By combining Landis's method with the estimates due to Aleksandrov–Bakelman (in the elliptic case) and Krylov [57] (in the parabolic case), they were able to obtain inequality (1.2) for *strong* solutions of elliptic [89] and parabolic [88] equations with operators of the general form $\mathcal{L} + c(x)$, $c \geqslant 0$ (with bounded coefficients) without assuming that the matrix $\mathcal{A}$ is continuous or that the dispersion of its eigenvalues is small.[39]

For operators $\mathcal{L}$ such that $b^i \in L_n(\Omega)$, Harnack's inequality was proved in [82] (see also [59]). In [91] and [92] a unified approach to the proof of Harnack's inequality for both divergence and non-divergence operators was presented. At the same time, it was shown in [91][40] that Harnack's inequality can fail even when $n = 1$ for operators of mixed (divergence-non-divergence) form

$$-D_i(a^{ij}(x)D_j) - \tilde{a}^{ij}(x)D_iD_j$$

(the matrices of the leading coefficients $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ satisfy the uniform ellipticity condition).

We also refer to [94], where Harnack's inequality and the Hölder continuity of solutions were treated in the 'abstract' context of metric and quasimetric spaces.

[37] More precisely, the leading coefficients of the operator must satisfy the Dini condition in a neighbourhood of $\partial\Omega$.

[38] Such conditions were introduced for the first time by Cordes in 1956, so Landis called (2.23) a Cordes-type condition.

[39] Note that if $c \equiv 0$, then Harnack's inequality yields easily an *a priori* estimate for the Hölder norm of the solution. By extending this estimate (also proved in [88] and [89]) to quasilinear equations, Ladyzhenskaya and Uraltseva showed that the Dirichlet problem for non-divergence quasilinear equations is solvable under natural structure conditions only (see the survey [90]). Subsequently, this result was extended to other boundary-value problems for quasilinear and fully non-linear equations.

[40] See also [93] in this connection.

3. Divergence form operators

In this section we look at operators with the structure

$$\mathfrak{L} \equiv -D_i(a^{ij}(x)D_j) + b^i(x)D_i \tag{3.1}$$

(if $\mathbf{b} \equiv 0$, then we write $\mathfrak{L}_0$ in place of $\mathfrak{L}$), and also operators of the more general form

$$\widehat{\mathfrak{L}} \equiv -D_i(a^{ij}(x)D_j + d^i) + b^i(x)D_i + c(x). \tag{3.2}$$

The matrix of leading coefficients $\mathcal{A}$ is symmetric and satisfies the ellipticity condition

$$\nu(x)|\xi|^2 \leqslant a^{ij}(x)\xi_i\xi_j \leqslant \mathcal{V}(x)|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \tag{3.3}$$

or the uniform ellipticity condition (2.3) for almost all $x \in \Omega$. The functions $\nu(x)$ and $\mathcal{V}(x)$ in (3.3) are positive and finite[41] almost everywhere in Ω.

Here, by a solution of the equation $\widehat{\mathfrak{L}}u = 0$ we mean a *weak solution*, that is, a function $u \in W^1_{2,\mathrm{loc}}(\Omega)$ such that the *integral identity*

$$\langle \widehat{\mathfrak{L}}u, \eta\rangle := \int_\Omega (a^{ij}D_juD_i\eta + b^iD_iu\,\eta + d^iuD_i\eta + cu\eta)\,dx = 0$$

holds for each test function $\eta \in \mathcal{C}_0^\infty(\Omega)$. Correspondingly, a *weak supersolution* ($\widehat{\mathfrak{L}}u \geqslant 0$) is a function $u \in W^1_{2,\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega (a^{ij}D_juD_i\eta + b^iD_iu\,\eta + d^iuD_i\eta + cu\eta)\,dx \geqslant 0 \tag{3.4}$$

for each *non-negative* test function $\eta \in \mathcal{C}_0^\infty(\Omega)$. In a similar way, we define a weak subsolution ($\widehat{\mathfrak{L}}u \leqslant 0$).

For the operator $\widehat{\mathfrak{L}}$ we prove the weak maximum principle under the simplest conditions on its coefficients.

Theorem 3.1. *For $n \geqslant 3$, let $\widehat{\mathfrak{L}}$ be an operator of the form (3.2) in a domain $\Omega \subset \mathbb{R}^n$, assume that (2.3) is satisfied,*

$$b^i, d^i \in L_n(\Omega), \quad \textit{and} \quad c \in L_{n/2}(\Omega),$$

and let $\mathfrak{u} \equiv 1$ be a weak supersolution of the equation $\widehat{\mathfrak{L}}u = 0$ in Ω.

Let $u \in W^1_{2,\mathrm{loc}}(\Omega)$, and let $\widehat{\mathfrak{L}}u \geqslant 0$ in Ω and $u \geqslant 0$ on[42] $\partial\Omega$. Then $u \geqslant 0$ in Ω.

Proof. 1. To begin with, note that the bilinear form $\langle \widehat{\mathfrak{L}}u, \eta\rangle$ is continuous on $W^1_{2,\mathrm{loc}}(\Omega) \times \overset{\circ}{W}{}^1_2(\Omega')$ when $\overline{\Omega}' \subset \Omega$. Indeed, using the Hölder and Sobolev inequalities, we obtain

$$\begin{aligned} |\langle \widehat{\mathfrak{L}}u, \eta\rangle| &\leqslant \nu^{-1}\|Du\|_{2,\Omega'}\|D\eta\|_{2,\Omega'} + \|\mathbf{b}\|_{n,\Omega}\|Du\|_{2,\Omega'}\|\eta\|_{2^*,\Omega'} \\ &\qquad + \|\mathbf{d}\|_{n,\Omega}\|D\eta\|_{2,\Omega'}\|u\|_{2^*,\Omega'} + \|c\|_{n/2,\Omega}\|u\|_{2^*,\Omega'}\|\eta\|_{2^*,\Omega'} \\ &\leqslant C\big(\|Du\|_{2,\Omega'} + \|u\|_{2,\Omega'}\big)\|D\eta\|_{2,\Omega'}. \end{aligned}$$

[41]Note that, by contrast to non-divergence form operators, multiplication by an arbitrary positive function does not preserve the properties of $\mathfrak{L}$. Hence the behaviour of $\nu(x)$ and $\mathcal{V}(x)$ must be treated separately.

[42]Just as in footnote 29, this means that for each $\varepsilon > 0$, we have $u + \varepsilon > 0$ in a neighbourhood of $\partial\Omega$.

(Here and in what follows $2^* = 2n/(n-2)$ is the critical Sobolev exponent.) So we can take arbitrary test functions $\eta \in \overset{\circ}{W}{}_2^1(\Omega)$ with compact support in the definition of a weak (sub/super)solution.

2. By contrast, assume that $\operatorname{ess\,inf}_\Omega u = -A < 0$ (the case when $A = \infty$ is not ruled out). Then, for $0 < k < A$, the function $\eta = (u+k)_- \in \overset{\circ}{W}{}_2^1(\Omega)$ is non-negative and has compact support in Ω, so that (3.4) holds. Since $D(u+k)_- = -Du \cdot \chi_{\{u<-k\}}$, this yields

$$\begin{aligned}\int_{\{u<-k\}} a^{ij} D_j u D_i u \, dx &\leqslant \int_{\{u<-k\}} (b^i D_i u \, \eta + d^i u D_i \eta + cu\eta) \, dx \\ &= \int_{\{u<-k\}} (b^i - d^i) D_i u \, \eta \, dx \\ &\quad + \int_{\{u<-k\}} (d^i D_i(u\eta) + c(u\eta)) \, dx.\end{aligned}$$

Here the last term is non-positive because $\mathfrak{u} \equiv 1$ is a weak supersolution. Using condition (2.3) on the left-hand side and the Hölder and Sobolev inequalities on the right-hand side, we obtain

$$\nu \|Du\|^2_{2,\{u<-k\}} \leqslant \big(\|\mathbf{b}\|_{n,\{u<-k\}} + \|\mathbf{d}\|_{n,\{u<-k\}} \big) \|Du\|^2_{2,\{u<-k\}}. \tag{3.5}$$

If $A = \infty$, then the first factor on the right-hand side tends to zero as $k \to \infty$, which yields a contradiction.

On the other hand, if $A < \infty$, then $Du = 0$ almost everywhere on the set $\{u = -A\}$, and we can write (3.5) as

$$\nu \leqslant \|\mathbf{b}\|_{n,\mathscr{A}_k} + \|\mathbf{d}\|_{n,\mathscr{A}_k},$$

where

$$\mathscr{A}_k = \{x \in \Omega \colon -A < u(x) < -k, \ Du(x) \neq 0\}.$$

It is obvious that $|\mathscr{A}_k| \to 0$ as $k \to A$. Therefore,

$$\|\mathbf{b}\|_{n,\mathscr{A}_k} + \|\mathbf{d}\|_{n,\mathscr{A}_k} \to 0,$$

and we arrive at a contradiction once again. □

Remark 3.1. Recently, the weak maximum principle was proved in [95] for functions $u \in W_2^1(\Omega)$ in a domain of John class such that $\widehat{\mathfrak{L}} u \geqslant 0$ in Ω and in place of the condition $u \geqslant 0$ on $\partial\Omega$ we have the condition with conormal derivative $(a^{ij} D_j u + d^i u)\mathbf{n}_i \geqslant 0$, that is, (3.4) holds for all non-negative functions $\eta \in W_2^1(\Omega)$.

3.1. Harnack's inequality and the strong maximum principle. By contrast to non-divergence operators,[43] in the divergence case almost all results on the strong

[43] Compare the chronology of the original results as presented in the table:

	Strong maximum principle	Harnack's inequality
Laplace operator	1839/40	1887
Operators with smooth coefficients	1892	1912
Operators with discontinuous coefficients	1927	1955

maximum principle were obtained as consequences of the corresponding Harnack inequalities. In this connection, we present the history of these results in parallel.

Two papers of 1959 and 1963 by Littman stand slightly apart. He considered operators

$$\mathcal{L}^* \equiv -D_i D_j a^{ij}(x) - D_i b^i(x), \tag{3.6}$$

which are the formal adjoints of operators of the form (2.1). By a weak supersolution of the equation $\mathcal{L}^* u + cu = 0$ we mean a function $u \in L_{1,\mathrm{loc}}(\Omega)$ such that

$$\langle \mathcal{L}^* u + cu, \eta \rangle := \int_\Omega u(\mathcal{L}\eta + c\eta)\, dx \geqslant 0$$

for each non-negative test function $\eta \in \mathcal{C}_0^\infty(\Omega)$. In the first of these papers the operator has smooth coefficients, but in the other one the assumptions are significantly relaxed. Here is the statement of the result.

Let $\mathcal{L}$ be an operator of the form (2.1), *let a^{ij}, b^i, and c be functions in $\mathcal{C}^{0,\alpha}(\Omega)$ with $\alpha \in (0,1)$, assume that* (2.3) *holds, and let u be a weak supersolution of the equation $\mathcal{L}^* u + cu = 0$ in Ω. Then the following conditions hold*:

1. *The function u can have no zero minimum in Ω unless $u \equiv 0$.*
2. *If the function $\mathfrak{u} \equiv 1$ is a weak supersolution of the equation $\mathcal{L}^* u + cu = 0$ in Ω,*[44] *then u can have no negative minimum in Ω unless $u \equiv$ const (in this case u is a weak solution).*
3. *If the function $-\mathfrak{u}$ is a weak supersolution of the equation $\mathcal{L}^* u + cu = 0$ in Ω, then u can have no positive minimum in Ω unless $u \equiv$ const (in this case u is a weak solution).*

Concerning further development of these results for operators of the form (3.6), the reader can consult [96] and [97] (see also the literature cited therein).

We return to divergence equations. For a uniformly elliptic operator $\mathfrak{L}_0$ with measurable coefficients, Harnack's inequality was first proved by Moser [98].[45] Stampacchia [101] generalized this to operators of the form (3.2) under the assumption that

$$b^i \in L_n(\Omega), \quad d^i \in L_q(\Omega), \quad \text{and} \quad c \in L_{q/2}(\Omega), \qquad q > n. \tag{3.7}$$

A similar result can be extracted from the paper [102] on quasilinear equations.

Two versions of the strong maximum principle were proved as corollaries in [101]:

1) for the operator $\widehat{\mathfrak{L}}$ when $\operatorname{ess\,inf}_\Omega u = 0$;
2) for the operator $\mathfrak{L}$.

We present a slightly simplified proof of the second result, which is based on an idea due to Moser [103], but does not rely on Harnack's inequality.

Theorem 3.2. *Let $\mathfrak{L}$ be a uniformly elliptic operator of the form* (3.1) *in a domain $\Omega \subset \mathbb{R}^n$, where $n \geqslant 3$, and let $b^i \in L_n(\Omega)$. Let $u \in W^1_{2,\mathrm{loc}}(\Omega)$ and $\mathfrak{L}u \geqslant 0$ in Ω. If u attains its minimum value at a point*[46] *$x^0 \in \Omega$, then $u \equiv$ const.*

[44] Here it means that $-D_i D_j(a^{ij}) - D_i(b^i) + c \geqslant 0$ in the sense of distributions.

[45] As shown in [99], (1.2) can also be derived from De Giorgi's proof [100] of the Hölder continuity of weak solutions of the equation $\mathfrak{L}_0 u = 0$.

[46] In the following sense: $\operatorname{ess\,lim\,inf}_{x \to x^0} u = \operatorname{ess\,inf}_\Omega u$.

Proof. 1. Similarly as in part 1 of the proof of Theorem 3.1, we can show that it is possible to take arbitrary test functions $\eta \in \overset{\circ}{W}{}_2^1(\Omega)$ with compact support in the definition of a weak (sub/super)solution.

2. Now let v be a weak subsolution, so that $\mathfrak{L}v \leqslant 0$ in Ω. In the inequality $\langle \mathfrak{L}v, \eta\rangle \leqslant 0$ we take the test function $\eta = \varphi'(v)\cdot\varsigma$, where ς is a non-negative Lipschitz function with support in $\overline{B}_{2R} \subset \Omega$ and φ is a convex Lipschitz function on $\mathbb{R}$ that is equal to zero on the negative half-axis. This yields

$$\int_{B_{2R}\cap\{u>0\}} \left(a^{ij}D_jVD_i\varsigma + \frac{\varphi''(v)}{(\varphi'(v))^2}a^{ij}D_jVD_iV\varsigma + b^iD_iV\varsigma\right)dx \leqslant 0, \tag{3.8}$$

where $V = \varphi(v) \in W^1_{2,\mathrm{loc}}(\Omega)$. In particular, since the second term in (3.8) is non-negative, V is also a weak subsolution.

In (3.8) we put[47] $\varphi(\tau) = \tau_+^p$, $p > 1$, and $\varsigma = V\zeta^2$, where ζ is a smooth cut-off function in B_{2R}. Then we obtain

$$\int_{B_{2R}} \frac{2p-1}{p}\, a^{ij}D_jVD_iV\,\zeta^2\,dx \leqslant -\int_{B_{2R}} (2a^{ij}D_jV\,VD_i\zeta\,\zeta + b^iD_iV\,V\zeta^2)\,dx. \tag{3.9}$$

We estimate the left-hand side of (3.9) from below by using (2.3), and the right-hand side from above by using the Hölder and Sobolev inequalities:

$$\begin{aligned}\nu\|DV\,\zeta\|^2_{2,B_{2R}} &\leqslant 2\nu^{-1}\|DV\,\zeta\|_{2,B_{2R}}\|VD\zeta\|_{2,B_{2R}}\\ &\qquad + \|\mathbf{b}\|_{n,B_{2R}}\|DV\,\zeta\|_{2,B_{2R}}\|V\,\zeta\|_{2^*,B_{2R}}\\ &\leqslant N(n)\|\mathbf{b}\|_{n,B_{2R}}\|DV\,\zeta\|^2_{2,B_{2R}} + C\|DV\,\zeta\|^2_{2,B_{2R}}\|VD\zeta\|^2_{2,B_{2R}}.\end{aligned}$$

By Lebesgue's theorem, for any sufficiently small R_*,

$$N(n)\|\mathbf{b}\|_{n,B_{2R_*}} \leqslant \frac{\nu}{2}\,.$$

This shows that

$$\|DV\,\zeta\|_{2,B_{2R}} \leqslant C(n,\nu,\|\mathbf{b}\|_{n,\Omega})\|VD\zeta\|_{2,B_{2R}} \tag{3.10}$$

for any $R \leqslant R_*$.

In (3.10) we put $R_k = R(1+2^{-k})$, $k \in \mathbb{N}\cup\{0\}$, and take $\zeta = \zeta_k$ such that

$$\zeta_k \equiv 1 \quad \text{in } B_{R_{k+1}}, \qquad \zeta_k \equiv 0 \quad \text{outside } B_{R_k}, \qquad |D\zeta_k| \leqslant \frac{2^{k+2}}{R}\,.$$

Then we obtain

$$\|DV\,\zeta_k\|_{2,B_{R_k}} \leqslant \frac{C(n,\nu,\|\mathbf{b}\|_{n,\Omega})}{R}\cdot 2^k\|V\|_{2,B_{R_k}}. \tag{3.11}$$

[47] More rigorously, we must consider $\varphi'(v) = p(\min\{v_+, N\})^{p-1}$, $N > 0$, and $\varsigma = \varphi(v)\zeta^2$, followed by taking the limit as $N \to \infty$ in (3.10).

Now, for $p = p_k \equiv (2^*/2)^k$, from Sobolev's inequality and (3.11) we can deduce that

$$\begin{aligned}\Big(\fint_{B_{R_{k+1}}} v_+^{2p_{k+1}}\,dx\Big)^{1/(2p_{k+1})} &\leqslant \Big(N(n)\fint_{B_{R_k}} (V\zeta_k)^{2^*}\,dx\Big)^{1/(2^*p_k)}\\ &\leqslant \Big(4^k C\fint_{B_{R_k}} V^2\,dx\Big)^{1/(2p_k)}\\ &= \Big(4^k C\fint_{B_{R_k}} v_+^{2p_k}\,dx\Big)^{1/(2p_k)}, \end{aligned}\tag{3.12}$$

where C depends only on n, ν, and $\|\mathbf{b}\|_{n,\Omega}$.

Iterating (3.12), we can see that the following estimate holds for any (weak) subsolution v:

$$\operatorname*{ess\,sup}_{B_R} v_+ \leqslant C(n,\nu,\|\mathbf{b}\|_{n,\Omega})\Big(\fint_{B_{2R}} v_+^2\,dx\Big)^{1/2}, \qquad R\leqslant R_*. \tag{3.13}$$

3. Let us turn to the proof of the theorem. We can assume without loss of generality that $\operatorname{ess\,inf}_\Omega u = 0$.

Suppose that the theorem does not hold. Then there is an x^0 in Ω such that $\operatorname{ess\,lim\,inf}_{x\to x^0} u = 0$, but for some $k>0$, $\delta>0$, and $R\leqslant R_*$, we have

$$|\{u\geqslant k\}\cap B_R(x^0)| \geqslant \delta|B_R|. \tag{3.14}$$

Without loss of generality, we can assume that $\overline{B}_{2R}(x^0)\subset\Omega$. We place the origin at x^0 and consider the function $v_\varepsilon(x) = 1-\varepsilon-u/k$, $\varepsilon>0$. It is obvious that v_ε is a subsolution.

Now we use (3.8) for $V = \varphi(v^\varepsilon) \equiv \Big(\ln\dfrac{1}{1-v^\varepsilon}\Big)_+$ (this is possible because $v^\varepsilon < 1$) and $\varsigma = \zeta^2$, where ζ is a smooth cut-off function equal to one in B_R. Since $\varphi''/\varphi'^2\equiv 1$, using (2.3) and the Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}\nu\|DV\,\zeta\|^2_{2,B_{2R}} &\leqslant \int_{B_{2R}} a^{ij}D_jV D_iV\,\zeta^2\,dx\\ &\leqslant -\int_{B_{2R}} (2a^{ij}D_jV\,\zeta D_i\zeta + b^iD_iV\,\zeta^2)\,dx\\ &\leqslant C(n,\nu,\|\mathbf{b}\|_{n,\Omega})\|DV\,\zeta\|_{2,B_{2R}}\|D\zeta\|_{2,B_{2R}}, \end{aligned}$$

or

$$\|DV\,\zeta\|_{2,B_{2R}} \leqslant C(n,\nu,\|\mathbf{b}\|_{n,\Omega})R^{n/2-1}. \tag{3.15}$$

Now note that V vanishes on $\{u\geqslant k\}\cap B_R$ and $\zeta\equiv 1$ on this set. Hence it follows from the proof of Lemma 5.1 in [104], Chap. II, that

$$|\{u\geqslant k\}\cap B_R|\cdot V(x)\,\zeta(x) \leqslant \frac{(4R)^n}{n}\int_{B_{2R}}\frac{|DV(y)|\,\zeta(y)}{|y-x|^{n-1}}\,dy.$$

We use the Hardy–Littlewood–Sobolev inequality (for instance, see [17], Theorem 1.18.9/3) to estimate the right-hand side, taking (3.14) and (3.15) into account:

$$\begin{aligned}\|V\|_{2,B_R} &\leqslant C(n)R\|V\zeta\|_{2^*,B_{2R}} \leqslant \frac{C(n)}{\delta}\,R\|DV\,\zeta\|_{2,B_{2R}}\\ &\leqslant C(n,\nu,\delta,\|\mathbf{b}\|_{n,\Omega})R^{n/2},\end{aligned}$$

or

$$\biggl(\fint_{B_R} V^2\,dx\biggr)^{1/2} \leqslant C(n,\nu,\delta,\|\mathbf{b}\|_{n,\Omega}).$$

Finally, as V is a subsolution, we can use (3.13). This yields $\operatorname{ess\,sup}_{B_{R/2}} V_+ \leqslant C$, which is equivalent to

$$\operatorname*{ess\,inf}_{B_{R/2}} u \geqslant k(\exp\{-C\}-\varepsilon).$$

Since the constant C is independent of ε, we arrive at a contradiction with the assumption that $\operatorname{ess\,lim\,inf}_{x\to 0} u = 0$. □

Remark 3.2. If we use the estimate

$$\int_{B_{2R}} |b^i D_i V\,V\zeta^2|\,dx \leqslant \|\mathbf{b}\|_{L_{n,\infty}(B_{2R})}\|DV\,\zeta\|_{2,B_{2R}}\|V\,\zeta\|_{L_{2^*,2}(B_{2R})}$$

for the last term in (3.9) (recall that the $L_{p,q}$ are Lorentz spaces) and use the strengthened Sobolev embedding theorem $\overset{\circ}{W}{}^1_2(\Omega) \hookrightarrow L_{2^*,2}(\Omega)$, then we can relax the condition $b^i \in L_n(\Omega)$ to $b^i \in L_{n,q}(B_{2R})$ for any $q<\infty$. As shown by the counterexample at the beginning of § 2.4, we cannot take $q=\infty$. However, if the norm $\|\mathbf{b}\|_{L_{n,\infty}(\Omega)}$ is sufficiently small, then the proof goes through unaltered.

Harnack's inequality holds under the same conditions (the proof of Theorem 2.5′ in [105] transfers fully to this case).

Remark 3.3. In the two-dimensional case Theorem 3.2 (and even Theorem 3.1) fails.[48] Here is a counterexample from [106].

For $n=2$, we put $u(x)=\ln^{-1}(|x|^{-1})$. It is obvious that for $r\leqslant 1/2$, the function $u\in W^1_2(B_r)$ is a weak solution of the equation

$$-\Delta u + b^i(x)D_i u = 0,$$

where

$$b^i(x) = \frac{2x_i}{|x|^2\ln(|x|^{-1})} \in L_2(B_r).$$

However, u takes its minimum value at 0.

Thus, for $n=2$, the condition on the b^i must be stronger. For example, the last term in (3.9) can be estimated as follows (cf. [107], Theorem 3.1):

$$\int_{B_{2R}} |b^i D_i V\,V\zeta^2|\,dx \leqslant \|\mathbf{b}\|_{L_{\Phi_1}(B_{2R})}\|DV\,\zeta\|_{2,B_{2R}}\|V\,\zeta\|_{L_{\Phi_2}(B_{2R})},$$

[48] This is not mentioned in [101].

where L_Φ is the Orlicz space generated by the N-function Φ (for instance, see [108], § 10) with

$$\Phi_1(t) = t^2 \ln(1+t) \quad \text{and} \quad \Phi_2(t) = \exp\{t^2\} - 1,$$

and the Yudovich–Pohozaev embedding theorem $\overset{\circ}{W}{}^1_2(\Omega) \hookrightarrow L_{\Phi_2}(\Omega)$ (for instance, see [108], § 10.6) can be used. This yields the strong maximum principle under the assumption that $\mathbf{b} \ln^{1/2}(1+|\mathbf{b}|) \in L_2(\Omega)$, which was introduced in [105]. Harnack's inequality also holds under this assumption (see [105], Theorem 2.5′). The above example shows that the exponent $1/2$ of the logarithm cannot be reduced.

The number of publications devoted to Harnack's inequality for divergence equations (even linear ones) has been growing rapidly since the second half of the 1960s. We focus on three important lines of development of this subject.

I. *Non-uniformly elliptic operators.* Operators satisfying the ellipticity condition (3.3) under different assumptions about $\nu(x)$ and $\mathcal{V}(x)$ have been studied in a number of papers.

Trudinger [109] proved Harnack's inequality for operators $\mathfrak{L}_0$ such that

$$\nu^{-1} \in L_q(\Omega) \quad \text{and} \quad \nu^{-1}\mathcal{V}^2 \in L_r(\Omega), \qquad \frac{1}{q} + \frac{1}{r} < \frac{2}{n}\,.$$

In [110] he considered operators of the general form (3.2) under the weaker condition

$$\nu^{-1} \in L_q(\Omega) \quad \text{and} \quad \mathcal{V} \in L_r(\Omega), \qquad \frac{1}{q} + \frac{1}{r} < \frac{2}{n}\,, \tag{3.16}$$

where the lower-order coefficients were subject to certain integrability conditions with weight determined by the matrix $\mathcal{A}$.[49]

Under these conditions, Harnack's inequality was proved in [110], as was the strong maximum principle in the following form:

Let u be a weak supersolution of the equation $\widehat{\mathfrak{L}}u = 0$ in Ω. If $\mathfrak{u} \equiv 1$ is another supersolution, then u can attain no negative minimum in Ω unless $u \equiv$ const (in this case u is a weak solution).

For operators of the simplest form $\mathfrak{L}_0$, the condition on the exponents in (3.16) was recently relaxed in [111] to $\dfrac{1}{q} + \dfrac{1}{r} < \dfrac{2}{n-1}$. On the other hand, a counterexample in [112] shows that if $n \geqslant 4$, then for $\dfrac{1}{q} + \dfrac{1}{r} > \dfrac{2}{n-1}$, the equation $\mathfrak{L}_0 u = 0$ can have a weak solution in B_R which is unbounded in $B_{R/2}$. Whether or not Harnack's inequality is valid in the borderline case $\dfrac{1}{q} + \dfrac{1}{r} = \dfrac{2}{n-1}$ is still an open question.

In [113] operators $\mathfrak{L}_0$ were considered under the following conditions:[50]

1) there is an $N \geqslant 1$ such that $\mathcal{V}(x) \leqslant N\nu(x)$ for almost all $x \in \Omega$;
2) ν belongs to the Muckenhoupt class A_2, that is,

$$\sup_{x\in\mathbb{R}^n,\, r>0} \biggl(\fint_{B_r(x)} \nu(y)\,dy \cdot \fint_{B_r(x)} \nu^{-1}(y)\,dy \biggr) < \infty. \tag{3.17}$$

[49] For a uniformly elliptic operator, these are close to Stampacchia's conditions (3.7).

[50] These go back to a 1972 paper on quasilinear equations by Edmunds and Peletier. However, the constraints (3.16) were additionally imposed on $\nu(x)$ and $\mathcal{V}(x)$ in this paper.

Under these assumptions, Harnack's inequality and the strong maximum principle were proved in [113]. In addition, a counterexample in [113] shows that relaxing the condition $\nu \in A_2$ to $\nu \in \bigcup_{p>2} A_p$ does not ensure Harnack's inequality.[51]

In [114] the results of [113] were generalized to operators of the general form (3.2), with the conditions

$$\frac{b^i}{\nu} \in L_m(\Omega), \quad \frac{d^i}{\nu} \in L_q, \quad \text{and} \quad \frac{c}{\nu} \in L_{q/2}, \qquad q > m, \tag{3.18}$$

imposed on the lower-order coefficients. (Here m, called the 'intrinsic dimension' in [114], is generated by the behaviour of the weight ν. For uniformly elliptic operators, $m = n$ and these conditions turn into (3.7).)

We also refer to [115] and [116], where Harnack's inequality was proved for the operator $\mathfrak{L}_0$ with functions $\nu(x)$ and $\mathcal{V}(x)$ satisfying 'abstract' conditions, namely, certain weighted Sobolev and Poincaré inequalities.

II. *Lower-order coefficients in Kato classes.* Lebesgue spaces (as well as Lorentz and Orlicz spaces) are rearrangement invariant: the norm of a function f in such a space depends only on the behaviour of the measure of the set $\{x \in \Omega \colon |f(x)| > N\}$ as $N \to \infty$. A more refined description of the singularities of the coefficients can be given in terms of Kato classes.

Recall that the class $\mathcal{K}_{n,\beta}$, $\beta \in (0, n)$, consists of the functions $f \in L_1(\Omega)$ such that

$$\omega_\beta(r) := \sup_{x\in\Omega} \int_{\Omega\cap B_r(x)} \frac{|f(y)|}{|x-y|^{n-\beta}}\, dy \to 0 \quad \text{as } r \to 0. \tag{3.19}$$

Correspondingly, $f \in \mathcal{K}_{n,\beta,\mathrm{loc}}$ means that $f\chi_{\Omega'} \in \mathcal{K}_{n,\beta}$ for each subdomain Ω' such that $\overline{\Omega}' \subset \Omega$.

The functionals $\omega_\beta(r)$ and spaces defined in terms of these were introduced by Schechter in [117] and thoroughly investigated in [118].[52] Information on further development of this theory and references can be found in [121].

All the results in this subsection concern the case $n \geqslant 3$.

In [122] Harnack's inequality was obtained for the operator $-\Delta + c(x)$ with $c \in \mathcal{K}_{n,2}$. In [123] this result was extended to uniformly elliptic operators of the form $\mathfrak{L}_0 + c(x)$ under the same assumption.[53]

In [125] Harnack's inequality was proved for uniformly elliptic operators of the more general form $\mathfrak{L} + c(x)$ under the assumption[54]

$$(b^i)^2, c \in \mathcal{K}_{n,2,\mathrm{loc}}. \tag{3.20}$$

Finally, in [127] the two lines of research described above were combined. Namely, Harnack's inequality was proved for operators (3.2) such that the functions $\nu(x)$ and $\mathcal{V}(x)$ in (3.3) satisfy $\mathcal{V}(x) \leqslant N\nu(x)$ and (3.17), while $(b^i)^2$, $(d^i)^2$, and c belong

[51] The strong maximum principle is not violated in this counterexample.

[52] For some particular values of β, condition (3.19) was used in [119] and [120]. In this connection, the classes $\mathcal{K}_{n,\beta}$ are usually called the Kato or Kato–Stummel classes (which is another manifestation of Arnold's principle). Some generalizations of the classes $\mathcal{K}_{n,\beta}$ were presented in a 2005 paper by Eridani and Gunawan.

[53] See also [124] in this connection.

[54] In the earlier paper [126] the operator $-\Delta + b^i(x)D_i$ was considered under the more restrictive conditions $(b^i)^2 \in \mathcal{K}_{n,2,\mathrm{loc}}$ and $b^i \in \mathcal{K}_{n,1,\mathrm{loc}}$.

to a weighted analogue of the Kato class $\mathcal{K}_{n,2}$ satisfying the following additional condition:[55] the counterpart of ω_2 in (3.19) has an estimate $O(r^\gamma)$ for some $\gamma > 0$ as $r \to 0$.

In general, condition (3.20) is very close to being optimal. Some variations are possible under certain additional assumptions about the matrix $\mathcal{A}$.

In [128] a uniformly elliptic operator of the form (3.1), where $a^{ij} \in \mathcal{C}^{0,\alpha}(\Omega)$ with $\alpha \in (0,1)$, was considered. This restriction made it possible to prove Harnack's inequality for $b^i \in \mathcal{K}_{n,1}$.

Note that the Hölder condition on the leading coefficients in [128] is superfluous: using the estimates from [129] for the Green's function and its derivatives, one can derive the same result for $a^{ij} \in \mathcal{C}^{0,\mathcal{D}}(\Omega)$.

In the recent paper [130] an intermediate (in a certain sense) case was examined. In this paper the leading coefficients of the uniformly elliptic operator $\mathfrak{L}$ belong to the Sarason space VMO(Ω). This means that $\omega^{ij}(\rho) \to 0$ as $\rho \to 0$, where

$$\omega^{ij}(\rho) := \sup_{x\in\Omega} \sup_{r\leqslant\rho} \fint_{\Omega\cap B_r(x)} \left| a^{ij}(y) - \fint_{\Omega\cap B_r(x)} a^{ij}(z)\, dz \right| dy. \tag{3.21}$$

In addition, the condition $|b^i|^\beta \in \mathcal{K}_{n,\beta}$, $\beta > 1$, with the additional constraint

$$\sup_{x\in\Omega} \int_{\Omega\cap B_r(x)\setminus B_{r/2}(x)} \frac{|b^i(y)|^\beta}{|x-y|^{n-\beta}}\, dy \leqslant \sigma^\beta(r), \qquad \sigma \in \mathcal{D}, \tag{3.22}$$

is imposed on the lower-order coefficients. The strong maximum principle for such operators was proved in [130].[56] Note that Harnack's inequality can also be established under such assumptions. It is still unclear if the constraint (3.22) can be dropped or relaxed.

III. *Operators with* $\operatorname{div}(\mathbf{b}) \leqslant 0$. In investigations of problems in hydrodynamics one often encounters (for instance, see [131] or [132]) operators $-\Delta + b^i(x)D_i$ (or, more generally, operators of the form (3.1)) satisfying the additional structure condition $D_i(b^i) = 0$ or $D_i(b^i) \leqslant 0$ in the sense of distributions. Recall that this means that

$$\int_\Omega b^i D_i\eta\, dx = 0 \quad \text{for all} \quad \eta \in \mathcal{C}_0^\infty(\Omega)$$

or

$$\int_\Omega b^i D_i\eta\, dx \geqslant 0 \quad \text{for all} \quad \eta \in \mathcal{C}_0^\infty(\Omega), \quad \eta \geqslant 0,$$

respectively.

Using this condition, we can significantly relax the regularity assumptions on the coefficients b^i.

In [133] Harnack's inequality was established for the operator $-\Delta + b^i(x)D_i$, where $D_i(b^i) = 0$, under the assumption that $b^i \in \mathrm{BMO}^{-1}(\Omega)$. This means that $b^i = D_j(B^{ij})$ in the sense of distributions, where $B^{ij} \in \mathrm{BMO}(\Omega)$, that is, the functions $\omega^{ij}(\rho)$ defined in (3.21) (for B^{ij} in place of a^{ij}) are bounded.[57] The

[55] We believe that this condition is technical, but the question is still open to our knowledge.

[56] In the case when $n = 2$, also considered in [130], condition (3.22) is slightly modified.

[57] It is obvious that $L_n(\Omega) \subset \mathrm{BMO}^{-1}(\Omega)$ because of the embedding $W^1_n(\Omega) \hookrightarrow \mathrm{BMO}(\Omega)$.

equality $D_i(b^i) = 0$ is ensured by the additional condition $B^{ij}(x) = -B^{ji}(x)$ for almost all $x \in \Omega$.

In [105] uniformly elliptic operators of the form (3.1) were investigated in the case when $D_i(b^i) \leqslant 0$. Furthermore, conditions on the lower-order coefficients were expressed in terms of Morrey spaces.

Recall that the Morrey space $\mathbb{M}_p^\alpha(\Omega)$, $1 \leqslant p < \infty$, $\alpha \in (0, n)$, consists of functions $f \in L_p(\Omega)$ such that

$$\|f\|_{\mathbb{M}_p^\alpha(\Omega)} := \sup_{B_r(x)\subset\Omega} r^{-\alpha}\|f\|_{p,B_r(x)} < \infty.$$

In particular, Harnack's inequality was proved in [105] under the assumption that[58] $b^i \in \mathbb{M}_q^{n/q-1}(\Omega)$, $n/2 < q < n$. Filonov produced a very delicate counterexample (Theorem 1.6 in [106]), which shows that not even when $D_i(b^i) = 0$ can we take an exponent α smaller than $n/q - 1$.

The strong maximum principle for Lipschitz supersolutions[59] was established in [105] in the case when $b^i \in L_q(\Omega)$, $q > n/2$. However, by approximation ([134], Theorem 3.1), we can partially generalize this result as follows:

Let $\Omega \subset \mathbb{R}^n$ with $n \geqslant 3$. Let $u \in W^1_{2,\mathrm{loc}}(\Omega)$ be a weak solution of the equation $-\Delta u + b^i(x)D_i u = 0$ in Ω, where

$$D_i(b^i) = 0, \quad b^i \in L_q(\Omega), \quad q > \frac{n}{2} \quad \text{for } n \geqslant 4, \quad \text{and} \quad q = 2 \quad \text{for } n = 3.$$

If u attains its minimum value at a point $x^0 \in \Omega$, then $u \equiv \mathrm{const}$.

On the other hand, the following counterexample was presented in [134].

Let $n \geqslant 4$ and let $u(x) = \ln^{-1}(|x'|^{-1})$. Then $u \in W_2^1(B_r)$ for $r \leqslant 1/2$. Direct calculations show that u is a weak solution of the equation $-\Delta u + b^i(x)D_i u = 0$ for[60]

$$b^i(x) = \begin{cases} \left(\dfrac{n-3}{|x'|} + \dfrac{2}{|x'|\ln(|x'|^{-1})}\right)\dfrac{x_i}{|x'|}, & i < n; \\ -\left(\dfrac{(n-3)^2}{|x'|} + \dfrac{2(n-3)}{|x'|\ln(|x'|^{-1})} + \dfrac{2}{|x'|\ln^2(|x'|^{-1})}\right)\dfrac{x_n}{|x'|}, & i = n. \end{cases}$$

It is easy to see that $D_i(b^i) = 0$ and $b^i \in L_q(B_r)$ for all $q < (n-1)/2$. However, there is no strong maximum principle. The recent paper [135] contains an example of a vector field $\mathbf{b} \in L_{(n-1)/2}(B_r)$ with $D_i(b^i) = 0$ such that the equation $-\Delta u + b^i(x)D_i u = 0$ has a weak solution that is unbounded in $B_{r/2}$. This can also be regarded as a violation of the strong maximum principle. The question of whether or not the strong maximum principle holds for $(n-1)/2 < q \leqslant n/2$ in the case when $D_i(b^i) = 0$ is open.

3.2. The normal derivative lemma. The normal derivative lemma for weak (super)solutions of the equation $\mathfrak{L}u = 0$ has a fairly short history. The first result was due to Finn and Gilbarg [136] in 1957. They considered uniformly elliptic

[58] It is obvious that $L_n(\Omega) \subset \mathbb{M}_q^{n/q-1}(\Omega)$ in view of Hölder's inequality.

[59] For weak supersolutions, the assumptions on b^i in [105] are slightly stronger.

[60] The formula for b^n in [134] contains a typo.

operators of the form (3.1) for $a^{ij} \in \mathcal{C}^{0,\alpha}(\Omega)$ and $b^i \in \mathcal{C}(\overline{\Omega})$ and a two-dimensional domain of class $\mathcal{C}^{1,\alpha}$, $\alpha \in (0,1)$.

Only in 2015 was this generalized to n dimensions by Sabina de Lis, who assumed that the domain has a smooth boundary.[61] In [137] the normal derivative lemma was proved for all $n \geqslant 3$ under the same assumptions about the a^{ij} and $\partial\Omega$ as in [136] for $b^i \in L^q(\Omega)$, $q > n$.

As long ago as 1959, Gilbarg constructed a counterexample[62] showing that the assumption about the leading coefficients cannot be relaxed to $a^{ij} \in \mathcal{C}(\overline{\Omega})$. Here is an example of a more general form (see [83]).

Let Ω be a domain in $\mathbb{R}^n$ such that $\Omega \cap \{x_n < h\} = \mathfrak{T}(\phi, h)$, where $\phi \in \mathcal{C}^1$, but the Dini condition at the origin fails for ϕ'. As mentioned in § 2.2, it was shown in [35] that the normal derivative lemma does not hold for the Laplace operator in such a domain.

Next we flatten the boundary in a neighbourhood of the origin. This produces an operator $\mathfrak{L}_0$ with *continuous* leading coefficients, for which the normal derivative lemma does not hold in a *smooth* domain.

We can see from this example that the Dini condition is a natural condition on the leading coefficients of the operator. In this connection, we refer to the paper [138] by Kozlov and Maz'ya, who found a more elaborate condition on the coefficients a^{ij} of the operator $\mathfrak{L}_0$, which ensures an estimate for the gradient of the solution at points on the (smooth) boundary $\partial\Omega$. From the asymptotic formula for the solution obtained in [138] one can perhaps also derive a condition for the normal derivative lemma to hold that is sharper than the Dini condition.

To present the central idea we prove the normal derivative lemma for the simplest operator $\mathfrak{L}_0$ with coefficients[63] $a^{ij} \in \mathcal{C}^{0,\mathcal{D}}(\Omega)$ under certain minimal conditions on the boundary of the domain.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the interior $\mathcal{C}^{1,\mathcal{D}}$-paraboloid condition. Assume that the coefficients of the operator $\mathfrak{L}_0$ satisfy (2.3) and the conditions $a^{ij} \in \mathcal{C}^{0,\mathcal{D}}(\Omega)$. Let $u \not\equiv$ const be a weak supersolution of the equation $\mathfrak{L}_0 u = 0$ in Ω.*

If u is a continuous function in $\overline{\Omega}$ that attains its minimum at a point $x^0 \in \partial\Omega$, then for any strictly inward direction $\boldsymbol{\ell}$,

$$\liminf_{\varepsilon \to +0} \frac{u(x^0 + \varepsilon\boldsymbol{\ell}) - u(x^0)}{\varepsilon} > 0.$$

Proof. We can assume without loss of generality that $x^0 = 0$ and $\Omega = \mathfrak{T}(\phi, h)$, where $\phi \in \mathcal{C}^{1,\mathcal{D}}$. Next, the assumptions on a^{ij} survive coordinate transformations of class $\mathcal{C}^{1,\mathcal{D}}$. Hence we can flatten $\partial\Omega$ in a neighbourhood of x^0 and assume that $B_R \cap \{x_n > 0\} \subset \Omega$ for some $R > 0$.

For $0 < r < R/2$, let $x^r = (0, \dots, 0, r)$ and consider the spherical shell $\pi = B_r(x^r) \setminus \overline{B}_{r/2}(x^r) \subset \Omega$.

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[61] Presented in this work there are also examples of papers where the normal derivative lemma for weak solutions was used incorrectly.

[62] In different forms it can be found in [51], Chap. 3, and [13], Chap 2.

[63] It is obviously sufficient that this condition should hold only in a neighbourhood of $\partial\Omega$. One could perhaps even manage with this condition on $\partial\Omega$ only (see [138]).

The condition $a^{ij} \in \mathcal{C}^{0,\mathcal{D}}(\Omega)$ yields

$$|a^{ij}(x) - a^{ij}(y)| \leqslant \sigma(|x-y|), \qquad x, y \in \overline{\pi}, \quad \sigma \in \mathcal{D}. \tag{3.23}$$

Let x^* be a point in $\overline{\pi}$. Following [136], we define a barrier function $\mathfrak{V}$ and an auxiliary function Ψ_{x^*} to be the solutions of the following boundary value problems:

$$\begin{cases} \mathfrak{L}_0 \mathfrak{V} = 0 & \text{in } \pi, \\ \mathfrak{V} = 1 & \text{on } \partial B_{r/2}(x^r), \\ \mathfrak{V} = 0 & \text{on } \partial B_r(x^r), \end{cases} \qquad \begin{cases} \mathfrak{L}_0^{x^*} \Psi_{x^*} = 0 & \text{in } \pi, \\ \Psi_{x^*} = 1 & \text{on } \partial B_{r/2}(x^r), \\ \Psi_{x^*} = 0 & \text{on } \partial B_r(x^r), \end{cases}$$

where $\mathfrak{L}_0^{x^*}$ is an operator with constant coefficients:

$$\mathfrak{L}_0^{x^*} \Psi_{x^*} := -D_i(a^{ij}(x^*) D_j \Psi_{x^*}).$$

It is well known that $\Psi_{x^*} \in \mathcal{C}^\infty(\overline{\pi})$. On the other hand, the existence of a (unique) weak solution $\mathfrak{V}$ follows from the general linear theory. Moreover, Lemma 3.2 in [129] shows that $\mathfrak{V} \in \mathcal{C}^1(\overline{\pi})$ and the following estimate holds for $y \in \overline{\pi}$:

$$|D\mathfrak{V}(y)| \leqslant \frac{N_1(n, \nu, \sigma)}{r}. \tag{3.24}$$

We put $\mathfrak{w} = \mathfrak{V} - \Psi_{x^*}$ and observe that $\mathfrak{w} = 0$ on $\partial\pi$. Hence we have a representation for $\mathfrak{w}$ in terms of the Green's function $\mathcal{G}_{x^*}$ of the operator $\mathfrak{L}_0^{x^*}$ in π:

$$\mathfrak{w}(x) = \int_\pi \mathcal{G}_{x^*}(x,y) \mathfrak{L}_0^{x^*} \mathfrak{w}(y)\, dy \overset{(\star)}{=} \int_\pi \mathcal{G}_{x^*}(x,y) \big(\mathfrak{L}_0^{x^*} \mathfrak{V}(y) - \mathfrak{L}_0 \mathfrak{V}(y)\big)\, dy$$

(equality $(\star)$ holds because $\mathfrak{L}_0^{x^*} \Psi_{x^*} = \mathfrak{L}_0 \mathfrak{V} = 0$).

Integrating by parts, we obtain

$$\mathfrak{w}(x) = \int_\pi D_{y_i} \mathcal{G}_{x^*}(x,y) \big(a^{ij}(x^*) - a^{ij}(y)\big) D_j \mathfrak{V}(y)\, dy. \tag{3.25}$$

Differentiating both sides of (3.25) with respect to x_k, we have

$$D_k \mathfrak{w}(x^*) = \int_\pi D_{x_k} D_{y_i} \mathcal{G}_{x^*}(x^*, y) \big(a^{ij}(x^*) - a^{ij}(y)\big) D_j \mathfrak{V}(y)\, dy, \qquad k = 1, \dots, n. \tag{3.26}$$

The derivatives of the Green's function $\mathcal{G}_{x^*}(x,y)$ have the following estimate (for instance, see [129], Theorem 3.3):

$$|D_x D_y \mathcal{G}_{x^*}(x,y)| \leqslant \frac{N_2(n,\nu)}{|x-y|^n}, \qquad x, y \in \overline{\pi}. \tag{3.27}$$

The substitution of (3.24), (3.27), and (3.23) into (3.26) yields

$$|D\mathfrak{w}(x^*)| \leqslant \frac{N_1 N_2}{r} \int_{B_{2r}(x^*)} \frac{\sigma(|x^*-y|)}{|x^*-y|^n}\, dy,$$

so that

$$|D\mathfrak{V}(x^*) - D\Psi_{x^*}(x^*)| \leqslant \frac{N_3(n,\nu,\sigma)}{r} \int_0^{2r} \frac{\sigma(\tau)}{\tau}\, d\tau, \qquad x^* \in \overline{\pi}. \tag{3.28}$$

Since the normal derivative lemma holds for operators with constant coefficients, for any strictly inward direction $\boldsymbol{\ell}$ we have

$$\partial_{\boldsymbol{\ell}}\Psi_0(0) \geqslant \frac{N_4(n,\nu,\boldsymbol{\ell})}{r} > 0.$$

In view of (3.28),

$$\partial_{\boldsymbol{\ell}}\mathfrak{V}(0) \geqslant \partial_{\boldsymbol{\ell}}\Psi_0(0) - |D\mathfrak{V}(0) - D\Psi_0(0)| \geqslant \frac{N_4}{r} - \frac{N_3}{r}\int_0^{2r} \frac{\sigma(\tau)}{\tau}\, d\tau \geqslant \frac{N_4}{2r}$$

for any sufficiently small $r > 0$. Fix such a value of r. Since $u \not\equiv \text{const}$, the strong maximum principle yields $u - u(0) > 0$ on $\partial B_{r/2}(x^r)$. Hence for any sufficiently small $\varkappa > 0$, we have

$$\mathfrak{L}_0(u - u(0) - \varkappa\mathfrak{V}) \geqslant 0 \quad \text{in } \pi; \qquad u - u(0) - \varkappa\mathfrak{V} \geqslant 0 \quad \text{on } \partial\pi.$$

Now the weak maximum principle yields $u - u(0) \geqslant \varkappa\mathfrak{V}$ in π. As equality holds at the origin, we have

$$\liminf_{\varepsilon\to+0} \frac{u(\varepsilon\boldsymbol{\ell}) - u(0)}{\varepsilon} \geqslant \varkappa\partial_{\boldsymbol{\ell}}\mathfrak{V}(0),$$

and the proof is complete. □

We present the statement of a more general result from [16]. The conditions on the lower-order coefficients in that paper which ensure the normal derivative lemma are currently the sharpest ones.

Theorem 3.4. *Assume that a domain $\Omega \subset \mathbb{R}^n$ and the leading coefficients of the operator $\mathfrak{L}$ satisfy the hypotheses of Theorem 3.3. Also let*

$$\sup_{x\in\Omega} \int_{\Omega\cap B_r(x)} \frac{|\mathbf{b}(y)|}{|x-y|^{n-1}} \frac{\mathrm{d}(y)}{\mathrm{d}(y) + |x-y|}\, dy \to 0 \quad \textit{as } r \to 0. \tag{3.29}$$

Let $u \in W_2^1(\Omega)$ be a weak non-constant supersolution of the equation $\mathfrak{L}u = 0$ in Ω, and let $b^i D_i u \in L_1(\Omega)$. Then the result of Theorem 3.3 is valid.

Remark 3.4. In any subdomain Ω' such that $\overline{\Omega}' \subset \Omega$, condition (3.29) coincides with $b^i \in \mathcal{K}_{n,1}$ (cf. (3.22)). Hence it follows from (3.29), in particular, that $b^i \in \mathcal{K}_{n,1,\mathrm{loc}}$. On the other hand, it was shown in [16] that the assumptions on b^i stated in Theorem 2.5 imply (3.29).

Remark 3.5. The normal derivative lemma for divergence-type operators is directly connected with the properties of the Green's functions for these operators.

The Green's function for a uniformly elliptic operator $\mathfrak{L}_0$ with measurable coefficients was first constructed in the historic paper [139]. Among the other results in that paper we name the bound[64]

$$\frac{C^{-1}}{|x-y|^{n-2}} \leqslant \mathcal{G}(x,y) \leqslant \frac{C}{|x-y|^{n-2}}$$

(here C depends only on n and ν), which holds in $\mathbb{R}^n$ for $n \geqslant 3$.

The role of [129] was no less important. Among other results, the paper contained the following estimates for the Green's function of a uniformly elliptic operator $\mathfrak{L}_0$ with coefficients satisfying the Dini condition in a domain $\Omega \subset \mathbb{R}^n$ with $n \geqslant 3$ satisfying the exterior ball condition:

$$\begin{aligned} \mathcal{G}(x,y) &\leqslant \frac{C}{|x-y|^{n-2}}\,\frac{\mathrm{d}(x)}{\mathrm{d}(x)+|x-y|}\,\frac{\mathrm{d}(y)}{\mathrm{d}(y)+|x-y|}\,, \\ |D_x\mathcal{G}(x,y)| &\leqslant \frac{C}{|x-y|^{n-1}}\,\frac{\mathrm{d}(y)}{\mathrm{d}(y)+|x-y|}\,, \\ |D_xD_y\mathcal{G}(x,y)| &\leqslant \frac{C}{|x-y|^{n}} \end{aligned}$$

(the constant C depends on n, ν, on the function σ in the Dini condition for the coefficients, and on the domain Ω).

Thus, condition (3.29) means, roughly speaking, that the function $|\mathbf{b}(y)| \times |D_x\mathcal{G}(x,y)|$ is uniformly integrable with respect to x.

4. Some generalizations and applications

As already mentioned in the Introduction, in this section we give a brief presentation of a few topics that either generalize the main results in this survey or rely on them directly.

4.1. The symmetry of solutions of non-linear boundary-value problems. We start with the celebrated *moving plane method*. It was first applied by Aleksandrov [141], Part V, to the problem of characterising the sphere by the property that its mean curvature (or some other function of the principal curvatures) is constant.[65] In 1971 Serrin re-discovered this method in his solution of the overdetermined problem

$$-\Delta u = 1 \quad \text{in } \Omega, \qquad u\big|_{\partial\Omega} = 0, \qquad \partial_{\mathbf{n}}u\big|_{\partial\Omega} = \mathrm{const}$$

in an unknown domain Ω of class $\mathcal{C}^2$. Serrin showed that such a problem is solvable if and only if Ω is a ball.

The method owes its popularity to [26], where the problem

$$-\Delta u = f(u) \quad \text{in } B_R, \qquad u\big|_{\partial B_R} = 0 \tag{4.1}$$

and some generalizations of it were considered. Here is the core result of that paper.

[64]Subsequently, this bound was extended to more general operators of the form (3.1). For recent results in this area and a historical survey, see [140].

[65]The statement of the problem and its history can be found in [141], Part I; see also [141], Part III. For generalizations of this result see, for instance, [142].

Theorem 4.1. *Let $f \in \mathcal{C}^1_{\mathrm{loc}}(\mathbb{R}_+)$, and let $u \in \mathcal{C}^2(\overline{B}_R)$ be a solution of problem (4.1) that is positive in B_R. Then $u = u(r)$ (that is, u is a radially symmetric function), and $u'(r) < 0$ for $0 < r < R$.*

Let us sketch the proof of Theorem 4.1. Clearly, it is sufficient to show that u is an even function of x_n and $D_n u(x) < 0$ for $x_n > 0$.

For $0 < \lambda < R$, the plane $\Pi_\lambda = \{x \colon x_n = \lambda\}$ cuts a segment Σ_λ off the ball. For any $x \in \overline{\Sigma}_\lambda$, let $\widehat{x}_\lambda = (x', 2\lambda - x_n)$ denote the point symmetric to x relative to Π_λ.

Consider the function $v_\lambda(x) = u(\widehat{x}_\lambda) - u(x)$ in $\overline{\Sigma}_\lambda$. It solves the equation

$$-\Delta v_\lambda + c(x) v_\lambda = 0,$$

where

$$c(x) = \frac{f(u(\widehat{x}_\lambda)) - f(u(x))}{u(x) - u(\widehat{x}_\lambda)} \in L_\infty(\Sigma_\lambda).$$

When λ is sufficiently close to one, v_λ is positive in Σ_λ (the graph of the 'reflected' function lies above the original graph) and attains its *zero* minimum on Π_λ. By the normal derivative lemma (part (B1) of Theorem 2.1), we have $\partial_{\mathbf{n}} v_\lambda(x) = 2D_n u(x) < 0$ on[66] Π_λ. Hence we can reduce λ slightly (so that Π_λ moves closer to the centre of the ball) while preserving the inequality $v_\lambda > 0$ in Σ_λ.

Let λ_0 denote the infimum of λ such that $v_\lambda > 0$ in Σ_λ. If we assume that $\lambda_0 > 0$, then $v_{\lambda_0} > 0$ on the 'round' part of $\partial\Sigma_{\lambda_0}$. By the strong maximum principle (part (A1) of Theorem 2.1), we have $v_{\lambda_0} > 0$ in Σ_{λ_0}. However, then we can repeat the above argument and show that Π_{λ_0} can be moved even closer to the centre, which is impossible. Thus $\lambda_0 = 0$ and $v_0 \equiv 0$, that is, $u(x', -x_n) \equiv u(x)$. The proof is complete.

As mentioned in [26], for $f(0) \geqslant 0$ the *a priori* assumption that u is positive can be replaced by the condition $u \geqslant 0$ with $u \not\equiv 0$. It is also obvious that the condition $f \in \mathcal{C}^1_{\mathrm{loc}}(\mathbb{R}_+)$ can be replaced by the local Lipschitz condition. The following example from [26] shows that a Hölder condition on f is insufficient, in general.

Let $p > 2$ and let $u(x) = (1 - |x - x^0|^2)_+^p$. Then direct calculations show that u is a solution of (4.1) for $R > |x^0| + 1$, provided that[67]

$$f(u) = 2p(n - 2 + 2p) u^{1-1/p} - 4p(p-1) u^{1-2/p} \in \mathcal{C}^{0,1-2/p}_{\mathrm{loc}}(\mathbb{R}_+).$$

The Hölder exponent can be made arbitrarily close to one by a suitable choice of p. However, the result of the theorem fails.[68]

The paper [26] (and also [144], where equations of the form (4.1) were considered in the whole space) gave rise to a great number of refinements and generalizations. Among these we can point out [145] by Berestycki and Nirenberg. Using the Aleksandrov–Bakelman maximum principle, the authors of [145] extended the results in [26] to strong solutions of a rather wide class of uniformly elliptic

[66] Recall that the sign of $c(x)$ is not important here. Note also that if $f(0) < 0$, then $D_n u$ can vanish at points $x \in \Pi_\lambda \cap \partial B_R$, but it was shown in [26] that $D_n D_n u(x) > 0$ in this case. This is sufficient for the argument that follows.

[67] In [26] this formula was published with a typo.

[68] Nonetheless, for $f > 0$ the Lipschitz condition on f can be relaxed (for instance, see [143]).

non-linear equations. For applications of the moving plane method to degenerate operators of p-Laplacian type, the reader can consult [146], [147], and the literature cited therein.

A number of authors have used the *moving sphere method*, a combination of the moving plane method with conformal transformations (for instance, see [148]).

For other applications of the strong maximum principle and the normal derivative lemma to establishing symmetry in geometric problems, the reader can, for instance, consult [149]–[152] (also see [6]). Applications of the Aleksandrov–Bakelman maximum principle and versions thereof to the investigation of the symmetry properties of solutions of non-linear boundary value problems and the proofs of isoperimetric inequalities are the subject of [153]–[155] (also see [156]).

4.2. Phragmén–Lindelöf type theorems. In its original form, the Phragmén–Lindelöf principle [157] describes the behaviour at infinity of an analytic function in an unbounded domain.

For solutions of general uniformly elliptic (non-divergence) equations, results of this type were first proved by Landis [158], [159] (the short note [160] had been published even earlier). The leading coefficients of the operator in [159] satisfy the Dini condition, and the behaviour of the domain at infinity is described in terms of a measure.

We obtain sharper theorems of Phragmén–Lindelöf type by describing domains in terms of capacity. The first results of this kind were obtained in [161] and [162] for divergence equations with measurable leading coefficients and in [162] for non-divergence equations with leading coefficients satisfying the Hölder condition.

Finally, Landis [27] made the decisive step (see also [28], Chap. 1). Using the concept of s-capacity introduced by himself, he proved Phragmén–Lindelöf type theorems for non-divergence equations with measurable leading coefficients.

As an example, we present one result from [28], Chap. 1, § 6.

Theorem 4.2. *Let Ω be an unbounded domain inside an infinite layer*:

$$\Omega \subset \{x \in \mathbb{R}^n \colon |x_n| < h\}.$$

Assume that an operator $\mathcal{L}_0$ satisfies (2.3), *and let $u \in \mathcal{C}^2(\Omega)$ be a classical subsolution*[69] *of the equation $\mathcal{L}_0 u = 0$ such that $u\big|_{\partial\Omega} \leqslant 0$.*

If $u(x) > 0$ at some point $x \in \Omega$, then

$$\liminf_{R\to\infty} \frac{\max_{|x|=R} u(x)}{\exp\{(C/h)R\}} > 0,$$

where the positive constant C depends only on n and ν.

We also refer to Maz'ya's paper [163], where related questions were investigated for quasilinear operators of p-Laplacian type.

In the case when the derivative in a non-tangential direction is prescribed on a part of $\partial\Omega$, theorems of Phragmén–Lindelöf type were proved in [164] and [165] for

[69] With the aid of the Aleksandrov–Bakelman maximum principle, this result can also be transferred to strong subsolutions $u \in W^2_{n,\mathrm{loc}}(\Omega)$.

divergence equations and in [166] for non-divergence ones. Note that a 'weakened' form of the normal derivative lemma [40] was used in the last paper.

Landis' conjecture is close to the results mentioned above. This is the problem of the maximum possible rate of convergence to zero of a non-trivial solution of a uniformly elliptic equation in $\Omega = \mathbb{R}^n \setminus B_R$. It was stated originally in [7] for the equation

$$-\Delta u + c(x)u = 0 \tag{4.2}$$

with bounded coefficient $c(x)$. (In this case the expected answer is exponential decay: if $|u(x)| = O(\exp(-N|x|))$ as $|x| \to \infty$ for every $N > 0$, then $u \equiv 0$.) Not even in the case of the simplest equation (4.2) has this problem been fully solved yet. For recent results in this area and a historical survey, see [167] (and also [168]).

4.3. The boundary Harnack inequality. If the normal derivative lemma fails, then the following result can play the role of a weak version of this lemma.

Boundary Harnack inequality. *Let $\mathbb{L}$ be an elliptic operator in a domain Ω such that $0 \in \Omega$. If u_1 and u_2 are positive solutions of $\mathbb{L}u = 0$ in Ω such that*

$$u_1\big|_{\partial\Omega\cap B_R} = u_2\big|_{\partial\Omega\cap B_R} = 0,$$

then

$$C^{-1}\frac{u_1(0)}{u_2(0)} \leqslant \frac{u_1(x)}{u_2(x)} \leqslant C\frac{u_1(0)}{u_2(0)} \tag{4.3}$$

in $\Omega \cap B_{R/2}$, where C is a constant independent of u_1 and u_2.

Remark 4.1. For example, if Ω is a domain in $\mathcal{C}^{1,\mathcal{D}}$ and $\mathcal{L}$ is a uniformly bounded operator of the form (2.1) with bounded coefficients, then (4.3) is an easy consequence of the normal derivative lemma, an estimate for the gradient of the solution on $\partial\Omega$, and the standard Harnack inequality.

Remark 4.2. In the important special case of flat boundary $x_n = 0$ and the operator $\mathcal{L}_0$, when we can take $u_2(x) = x_n$, the boundary Harnack inequality was first established by Krylov [169] in order to derive boundary estimates in $\mathcal{C}^{2,\alpha}$ for solutions of *non-linear* equations.

To describe the results in this subsection we need some new classes of domains:

- non-tangentially accessible domains (NTA domains);
- uniform domains (UD);
- domains satisfying John's λ-condition with $\lambda \geqslant 1$; for $\lambda = 1$ these are simply called John domains (JD);
- twisted Hölder domains (THD); if needed, the phrase 'of order $\alpha \in (0,1]$' (THD-α) can be added.

The precise definitions of these classes can be found in the papers listed in Table 1. For the reader's convenience we simply indicate the relations between these classes (for instance, see [170][70]):

$$\mathcal{C}^{0,1} \subset \text{NTA} \subset \text{UD} \subset \text{JD} \;=\; \text{THD-1};$$
$$\mathcal{C}^{0,\alpha} \subset \tfrac{1}{\alpha}\text{-JD} \overset{(\triangle)}{=} \text{THD-}\alpha.$$

[70] The relation $(\triangle)$ is not explicitly stated in [170], but follows from Remark 2.5 therein.

In Table 1 we assume by default that the leading coefficients of the operators are measurable and satisfy (2.3).

Table 1. The boundary Harnack inequality for various classes of domains

Operator	$\mathcal{C}^{0,1}$	NTA	UD	JD	$\mathcal{C}^{0,\alpha}$	THD[71]
$-\Delta$	[173]	[174]	[175]			
$\mathcal{L}+c(x)$ [72]	[176]					
$\mathfrak{L}_0$	[177]				[178]	[171]
$-\Delta+b^i(x)D_i$ [73]	[126]					
$\mathcal{L}_0$	[179][74]					
$\mathfrak{L}_0+c(x), \quad c\in\mathcal{K}_{n,2}$	[124]					
$\widehat{\mathfrak{L}}$	[114][75]					
$\mathcal{L}, \quad b^i\in L_\infty(\Omega)$					[182][76]	
$\mathcal{L}, \quad b^i\in L_n(\Omega)$	[82]			[183]		[170]

The recent papers [184] and [185] present a unified approach to the boundary Harnack inequality for divergence and non-divergence operators.[77]

A variant of the boundary Harnack inequality for supersolutions and 'almost supersolutions' of the equation $\mathfrak{L}u+cu=0$ with bounded coefficients was obtained in [186].[78]

Results of weak Harnack inequality type for the ratio $u(x)/\mathrm{d}(x)$ (see [188] and the references therein) are close to the boundary Harnack inequality. Here is one of the results from [188].

Theorem 4.3. *Let u be a non-negative weak supersolution of the equation*[79] $\mathfrak{L}u=f$ *in a domain of class $\mathcal{C}^{1,1}$. Assume that (2.3) and the following conditions hold:*

$$a^{ij}\in W^1_q(\Omega),\quad b^i\in L_q(\Omega)\quad \textit{and}\quad f_-\in L_q(\Omega),\qquad q>n.$$

[71] The results have been obtained for $\alpha>1/2$. The counterexamples were constructed in [171] for $\alpha<1/2$ and in [172] for $\alpha=1/2$.

[72] The coefficients satisfy the Hölder condition.

[73] For the conditions on the b^i, see footnote 54.

[74] See also [180]. A slightly more general condition on the domain was considered in [181].

[75] The leading coefficients satisfy the conditions (3.3), $\mathcal{V}(x)\leqslant N\nu(x)$, and (3.17), and the lower-order ones satisfy (3.18).

[76] The result has been obtained for $\alpha>1/2$. A counterexample has been constructed for $\alpha<1/2$. If $\partial\Omega$ also satisfies condition (A) due to Ladyzhenskaya and Uraltseva (for instance, see [90]), then the result has been established for all $\alpha>0$.

[77] Similar ideas appeared earlier in Safonov's work (see [91], [82], and [170]).

[78] In this connection, see [187], where an estimate in terms of the principal eigenfunction of the Dirichlet Laplacian was found for a superharmonic function satisfying the homogeneous Dirichlet boundary condition in a plane domain with corners.

[79] Importantly, no assumption is made on the behaviour of u on $\partial\Omega$.

Then

$$\left(\int_\Omega \left(\frac{u(x)}{\mathrm{d}(x)}\right)^s dx\right)^{1/s} \leqslant C\left(\inf_{x\in\Omega}\frac{u(x)}{\mathrm{d}(x)} + \|f_-\|_{q,\Omega}\right)$$

for each $s<1$. *The constant* C *depends on* n, ν, s, q, *on the norms of the* a^{ij} *and* b^i *in the corresponding spaces*, *on* $\operatorname{diam}(\Omega)$, *and on the properties of* $\partial\Omega$.

The example of the harmonic function $x_n|x|^{-n}$ in the half-ball $B_r^+ = B_r \cap \{x_n>0\}$ shows that the condition $s<1$ is sharp.

We can also mention some publications (for instance, see [189] and the literature cited therein) where the boundary Harnack inequality was obtained in the 'abstract' context of metric spaces.

4.4. Other results for linear operators. In [190] and [191] a generalized strong maximum principle was established for operators of the form $-\Delta + c(x)$, where $c \in L_1(\Omega)$. In this context the solutions are treated in the sense of measures. For further results in this direction, see [192] and [193].

It is well known that for a second-order elliptic operator, the weak maximum principle is equivalent to the principal eigenvalue of the corresponding Dirichlet problem being positive. In [194] the generalized principal eigenvalue was defined for uniformly elliptic operators $\mathcal{L}+c(x)$ with bounded coefficients in an arbitrary bounded domain:[80]

$$\lambda_1 = \sup_\phi \inf_{x\in\Omega} \frac{\mathcal{L}\phi(x)+c(x)\phi(x)}{\phi(x)} \tag{4.4}$$

(the supremum is taken over $\phi \in W^2_{n,\mathrm{loc}}(\Omega)$ with $\phi>0$ in Ω), and the weak maximum principle (as well as the 'improved' weak maximum principle established in this paper) for the operator $\mathcal{L}+c(x)$ was shown to be equivalent to $\lambda_1>0$.

The study of partial differential equations on complicated structures has become very popular in recent decades. Some authors (for instance, see [196], [197], and the references therein) studied conditions ensuring the validity of the strong maximum principle, Harnack's inequality, the normal derivative lemma, and the boundary Harnack inequality for *subelliptic* operators, including sub-Laplacians on homogeneous Carnot groups.

In [198] and [199] the strong maximum principle and the normal derivative lemma were considered for the simplest elliptic operators on *stratified sets*, that is, cell complexes with certain special properties.[81]

4.5. Non-linear operators. Even a simple keyword search shows that during the recent years dozens of papers treating non-linear operators were published annually on the topic of our survey. It means that this subsection can only be very sketchy, and it does nor even claim to be minimally comprehensive.

For quasilinear operators of divergence type, Harnack's inequality was first proved in [102] and then, for wider operator classes, in [200] and [201]. These are

[80] For operators with smooth coefficients in smooth domains, this formula does indeed produce the principal eigenvalue. In this case we can take the supremum over positive smooth functions ϕ in Ω. For the Laplace operator, formula (4.4) was apparently first identified in [195]. Subsequently, it was generalized to various operator classes (see [194] and the literature cited therein).

[81] The simplest examples are the operators of the Venttsel problem and the two-phase Venttsel problem.

now regarded as classic papers. Note also [99], where Harnack's inequality was established for *quasiminimisers* in variational problems.

In [202] the normal derivative lemma from [31] and [32] was generalized to the quasilinear case.

For operators of the type of the p-Laplacian

$$\Delta_p u \equiv D_i(|Du|^{p-2} D_i u), \qquad p > 1, \tag{4.5}$$

the normal derivative lemma was first proved in [203]. Among recent generalizations we can mention [204].

In [205] and [206] the authors found sharp conditions for the strong maximum principle and the normal derivative lemma to hold for the minimisers of the functional

$$J[u] = \int_\Omega f(Du)\,dx.$$

Vázquez [207] considered the equation

$$-\Delta_p u + f(u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \quad n \geqslant 2, \tag{4.6}$$

and proved the following result.

Theorem 4.4. *Let $f \in \mathcal{C}(\mathbb{R}_+)$ be a non-decreasing function with $f(0) = 0$. Then a necessary and sufficient condition for any (non-trivial) non-negative supersolution of equation (4.6) not to vanish in Ω is that*

$$\int_0^\delta \frac{dt}{(F(t))^{1/p}} = \infty, \quad \text{where} \quad F(t) = \int_0^t f(s)\,ds. \tag{4.7}$$

For a generalization of this result to wider classes of quasilinear operators, see [208], [209], and [167]. The corresponding Harnack inequality was established (for $p = 2$) in [210].

Theorem 4.5. *Let $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function, and let $u \in W_2^1(B_{2R})$ be a solution of the equation $\mathfrak{L}_0 u + f(u) = 0$, where $\mathfrak{L}_0$ is a uniformly elliptic (divergence) operator with measurable coefficients. Put $M = \sup_{B_R} u$ and $m = \inf_{B_R} u$. Then*[82]

$$\int_m^M \frac{dt}{(F(t))^{1/2} + t} \leqslant C,$$

where F is defined in (4.7) and the constant C depends only on n and ν (in particular, it is independent of f!).

The boundary Harnack inequality for operators of p-Laplacian type in domains of class $\mathcal{C}^2$ was proved in [211]. Subsequently, it was also established for the wider classes of domains discussed in § 4.3 (see [212] and the literature cited therein). For Pucci's maximal and minimal operators, the boundary Harnack inequality was proved in [213].

Among the popular objects currently investigated are also $p(x)$-Laplacians, which are operators of the form (4.5) in which p is a function of x. For such operators,

[82] Note that for $f \equiv 0$, we obtain the classical Harnack inequality.

Harnack's inequality was first proved in [214] (concerning recent generalizations, see [215] and [216], for instance). In [217] the boundary Harnack inequality was established in a domain of class $\mathcal{C}^{1,1}$.

4.6. Non-local operators. In recent decades interest in the study of non-local (integro-differential) operators has increased significantly. Among these operators we should distinguish *fractional Laplacians*. The simplest (and historically first) is the fractional Laplacian of order s in $\mathbb{R}^n$. It is defined in terms of the Fourier transform:[83]

$$(-\Delta)^s u = \mathcal{F}^{-1}\bigl(|\xi|^{2s}(\mathcal{F}u)(\xi)\bigr), \qquad s>0.$$

For $s\in(0,1)$, this operator can be defined in terms of a hypersingular integral:

$$\bigl((-\Delta)^s u\bigr)(x) = C_{n,s}\cdot \mathrm{PV}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy, \qquad C_{n,s} = \frac{s\cdot 2^{2s}\Gamma(n/2+s)}{\pi^{n/2}\Gamma(1-s)}\,.$$

It was Riesz [218] who proved a direct analogue of Harnack's inequality (2.22) for $(-\Delta)^s$, where $s\in(0,1)$:

Let u be a non-negative function in $\mathbb{R}^n$ that satisfies the equation $(-\Delta)^s u=0$ in B_R. Then

$$u(0)\frac{(R-|x|)^s R^{n-2s}}{(R+|x|)^{n-s}} \leqslant u(x) \leqslant u(0)\frac{(R+|x|)^s R^{n-2s}}{(R-|x|)^{n-s}}$$

for any $x\in B_R$.

By contrast to the case of the whole space, fractional Laplacians in domains $\Omega\subset\mathbb{R}^n$ depend, of course, on the boundary conditions (authors distinguish between Dirichlet, Neumann, and other fractional Laplacians). Moreover, even when the type of the boundary condition is fixed, there are several significantly distinct definitions of fractional Laplacians: *restricted* ones, *spectral* ones, and so on. Note that in [219] and [220] the classical normal derivative lemma for weakly degenerate operators (see [38] and [14]) was used to compare the restricted and spectral Dirichlet Laplacians.

For various fractional Laplacians of order $s\in(0,1)$ in Ω, the proofs of the strong maximum principle can be found in [221]–[223]. In [224] a unified approach was proposed for a large family of fractional Laplacians and more general non-local operators. At the same time, it was shown in [225] that even the weak maximum principle fails for the restricted fractional Dirichlet Laplacian with $s>1$ in a domain of general type.[84]

In a Lipschitz domain the boundary Harnack inequality for $(-\Delta)^s$, $s\in(0,1)$, was proved in [226]. Due to the non-locality of the operator, its formulation differs from the standard one (see § 4.3):

Let $0\in\Omega$. If u_1 and u_2 are non-negative functions in $\mathbb{R}^n$ that are continuous in the ball B_R and satisfy the equation $(-\Delta)^s u=0$ in $\Omega\cap B_R$ and the condition

[83] For a rigorous definition of this and some similar operators and for the concept of a weak (sub/super)solution, we would need to introduce the Sobolev–Slobodetsky spaces ([17], Chaps. 2–4). However, we do not do this out of compassion for the reader.

[84] Note that in both $\mathbb{R}^n$ and the ball $\Omega=B_R$ the strong maximum principle holds for each $s>0$. This was also shown in [225].

$u_1\big|_{B_R\setminus\Omega} = u_2|_{B_R\setminus\Omega} = 0$, *then inequality*[85] (4.3) *with constant* C, *which only depends on* n, s, Ω, *and* R, *holds in the subdomain* $\Omega \cap B_{R/2}$.

Subsequently, this result was extended to *arbitrary* domains Ω and a wide class of integro-differential operators (see [227] and the references therein).

In [228] the authors constructed a barrier, which is sufficient to prove an analogue of the normal derivative lemma in the following form:

Let Ω *be a domain of class* $\mathcal{C}^{1,1}$ *and let* $s \in (0,1)$. *Let* u *be a weak supersolution of* $(-\Delta)^s u = 0$ *in* Ω *and let* $u = 0$ *in* $\mathbb{R}^n \setminus \Omega$. *If* $u \not\equiv 0$, *then*

$$\inf_{x\in\Omega} \frac{u(x)}{\mathrm{d}^s(x)} > 0. \tag{4.8}$$

Further generalizations of this result can be found in [229], for instance. For operators of fractional p-Laplacian type, a similar result was proved in [230].

For the *spectral* fractional Laplacian, we have $\inf_{x\in\Omega} u(x)/\mathrm{d}(x) > 0$ in place of (4.8) under the same assumptions (see Theorem 1.2 in [231], where more general functions of the Laplace operator with Dirichlet conditions were also considered). An analogue of the normal derivative lemma for the *regional* fractional Laplacian with $s \in (1/2, 1)$ was obtained in the recent preprint [232].

In [233] a generalization of the Aleksandrov–Bakelman maximum principle to non-local analogues of Pucci's maximal and minimal operators was obtained.

The reader can find applications of the moving plane method to problems involving fractional Laplacians in [234] and the references cited therein.

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[85] If $u_2(0) = 0$, then $u_2 \equiv 0$, so we can assume that $u_2(0) > 0$.

We have significantly reduced the bibliography at the request of the Editorial Board. The complete list of literature can be found in an e-print via the link http://arxiv.org/abs/2206.08043.

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Darya E. Apushkinskaya
RUDN University;
St Petersburg State University
E-mail: apushkinskaya@gmail.com

Alexander I. Nazarov
St Petersburg Department
of the Steklov Mathematical Institute
of Russian Academy of Sciences;
St Petersburg State University
E-mail: al.il.nazarov@gmail.com

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