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Functions with general monotone Fourier coefficients

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Abstract. This paper is a study of trigonometric series with general monotone coefficients in the class $\mathrm{GM}(p)$ with $p \geqslant 1$. Sharp estimates are proved for the Fourier coefficients of integrable and continuous functions. Also obtained are optimal results in terms of coefficients for various types of convergence of Fourier series. For $1 < p < \infty$ two-sided estimates are obtained for the L_p-moduli of smoothness of sums of series with $\mathrm{GM}(p)$-coefficients, as well as for the (quasi-)norms of such sums in Lebesgue, Lorentz, Besov, and Sobolev spaces in terms of Fourier coefficients.

Bibliography: 99 titles.

Keywords: functions with general monotone Fourier coefficients; estimates of Fourier coefficients; moduli of smoothness; Lebesgue, Lorentz, Besov, Sobolev spaces.

Contents

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1. Introduction

It is well known that monotonicity conditions, either on a signal spectrum or on a signal itself, are extremely useful in various problems in analysis, in particular, in the theory of Fourier series. For example, trigonometric series with monotonic coefficients have been well studied ([4], [99], [9]). Moreover, many of their properties can be completely characterized in terms of Fourier coefficients. To illustrate this point, we recall Parseval's theorem $\|f\|_2 = \|\{c_n\}\|_{l_2}$, which has no analogue in the general case for L_p, $p \neq 2$. However, in the case of monotonic coefficients, the corresponding equivalence can be written as follows: $\|f\|_p \asymp \bigl(\sum_n |c_n|^p n^{p-2}\bigr)^{1/p}$ for $1 < p < \infty$. This statement has been known already since the first half of the last century (the Hardy–Littlewood theorem). Due to their optimality, results of this type have important applications in Fourier analysis, approximation theory, and functional analysis. In particular, we mention the Paley–Wiener theorem on integrability of the function conjugate to an odd function ([67], [98]), Boas' conjecture on weighted integrability of the Fourier transform ([10], [40], [57], [74]), certain convergence and approximation problems for trigonometric series and transforms (see, for instance, [99], Chaps. 5 and 12, and also [25], [48], [9], [11], [12], [15], [45], [46], [50], [54], [58]).

At the same time, it is clear that the monotonicity condition is rather restrictive. Fairly recently it was noted that instead of the monotonicity condition for the coefficients one can consider regularity conditions of local variations, that is,

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \leqslant C\beta_n \quad \text{for all } n \in \mathbb{N},$$

where β_n is a suitable majorant (see [89]). Such sequences are called general monotone sequences with majorant β_n, written $\{a_n\} \in \mathrm{GM}(\beta_n)$. Let us consider some examples of such majorants.

We note that a maximal majorant is $\beta_n = 2\sum_{k=n}^{2n+1} |a_k|$, that is, in this case any given sequence lies in the class $\mathrm{GM}(\beta_n)$. In particular, this class contains highly oscillating sequences such as sequences of Rudin–Shapiro type, for example. A somewhat narrower class — we call it GM(max) — is the class $\mathrm{GM}(\beta_n)$ with the majorant $\beta_n = \max_{n/\gamma \leqslant k \leqslant \gamma n} |a_k|$ for some $\gamma > 1$. This class still contains both monotonic and lacunary sequences. It is too large for applications, since, for example, a criterion for the sums of lacunary series to belong to L_p for $1 < p < \infty$ is given by $\|f\|_p \asymp \|f\|_2 \asymp \|\{c_n\}\|_{l_2}$ (see [99]). This is fundamentally different from the case of series with monotonic coefficients.

A systematic study of suitable majorants and the corresponding function classes was begun in 2005 (see [87] and [89]). In particular, the class $\mathrm{GMS} := \mathrm{GM}(\beta_n)$ with $\beta_n = |a_n|$ and a larger class $\mathrm{GM}(1) := \mathrm{GM}(\beta_n)$ with $\beta_n = \frac{1}{n}\sum_{k=n/\gamma}^{\gamma n} |a_k|$, $\gamma > 1$, were considered.

In this paper we consider trigonometric series with coefficients in the following class: for $p > 1$,

$$\mathrm{GM}(p) = \left\{ a = \{a_n\}_{n\in\mathbb{N}} \colon a_n \in \mathbb{C}, \sum_{\nu=n}^{2n-1} |a_\nu - a_{\nu+1}| \leqslant C\biggl(\frac{1}{n}\sum_{k=n/\gamma}^{\gamma n} |a_k|^p\biggr)^{1/p} \right\} \quad (1.1)$$

for some $C > 0$ and $\gamma > 1$ depending on a sequence a. Here and further on, we denote by C and C_i positive constants which may depend on inessential parameters.

Firstly, we note that

$$\mathrm{GMS} \subsetneq \mathrm{GM}(1) \subsetneq \mathrm{GM}(p_1) \subsetneq \mathrm{GM}(p_2) \subsetneq \mathrm{GM}(\max) \subsetneq \bigl\{\{a_n\}_{n\in\mathbb{N}} \colon a_n \in \mathbb{C}\bigr\}$$

for $1 < p_1 < p_2$. The third embedding here and its optimality will be proved in § 2, the other embeddings being known. We also note that $\bigcup_p \mathrm{GM}(p) \neq \mathrm{GM}(\max)$.

Secondly, we show that $\{a_n\} \in \mathrm{GM}(p)$ if and only if $\{a_n\} \in \mathrm{GM}(\max) \cap \mathrm{WM}(p)$, where

$$\mathrm{WM}(p) = \left\{ a = \{a_n\}_{n\in\mathbb{N}} \colon a_n \in \mathbb{C}, |a_n| \leqslant C\biggl(\frac{1}{n}\sum_{k=n/\gamma}^{\gamma n} |a_k|^p\biggr)^{1/p} \right\}$$

for some $C > 0$ and $\gamma > 1$.

The main goal of this paper is to study trigonometric series with coefficients in the classes $\mathrm{GM}(p)$.

1.1. Convergence problems. Let

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{or} \quad \sum_{n=1}^{\infty} a_n \sin(nx) \tag{1.2}$$

be the Fourier expansion of a function f. For a sequence of coefficients tending to zero we will use the notation

$$a_n^{\#} = \max_{k \geqslant n} |a_k| \quad \text{for } n \geqslant 1.$$

Then $\{a_n^{\#}\}_{n=1}^{\infty}$ is a monotonic null sequence and $a_n^{\#} \geqslant |a_n|$ for any $n \geqslant 1$. As usual, for any $p \in [1,\infty)$ and any function $f \in L_p(\mathbb{T})$ we write

$$\|f\|_p = \|f\|_{L_p(\mathbb{T})} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p\,dt\right)^{1/p}.$$

For a function $f \in C(\mathbb{T})$ we write $\|f\|_\infty = \|f\|_{C(\mathbb{T})} = \max_{t\in\mathbb{T}} |f(t)|$.

Let us first discuss various types of convergence of the series (1.2). In this paper we obtain the following results.

1. (*Convergence almost everywhere.*) Let $\{a_n\} \in \mathrm{GM}(p)$ for some $p \geqslant 1$ and let

$$\sum_{n=1}^{\infty} \frac{a_n^2}{n} < \infty.$$

Then the series (1.2) converge almost everywhere. Moreover, this condition is sharp (see Theorem 4.1).

2. (*Uniform convergence.*) Let $a \in \mathrm{GM}(p)$ for some $p > 1$. Then the series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx)$ converges uniformly on $[0,2\pi]$ if and only if $na_n = o(1)$ and the series $\sum_n a_n$ is convergent. The series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $[0,2\pi]$ if and only if $na_n = o(1)$.

Similar results are obtained for uniform boundedness of the partial sums of these series if we replace $o(1)$ with $O(1)$ and convergence of the partial sums of $\sum_n a_n$ with their boundedness (see Theorem 4.2, and see also [81], [88], [29], [35], [37], [52], [65], [91], [89]).

3. (*Convergence in the mean and conditions for belonging to L_1.*) Let $\{a_n\} \in \mathrm{GM}(p)$ for some $p \geqslant 1$ and let

$$\sum_{n=1}^{\infty} \frac{\log n}{n}\,|a_n| < \infty.$$

Then the series of type (1.2) is the Fourier series of a function $f \in L_1$ and it converges in the mean, that is, converges in the L_1-norm (see Theorem 4.4).

A criterion for convergence in the mean of Fourier series of L_1-functions is given by the condition

$$|a_n| \log n = o(1) \quad \text{as } n \to \infty$$

(see Theorem 4.3; see also [5]–[7] and [92]).

4. (*Behaviour near the origin.*) In Theorem 4.9 we describe the behaviour of the Fourier series of an integrable function f with coefficients $\{a_n\}_{n=1}^\infty$ of type $\mathrm{GM}(p)$, $p\geqslant 1$. The conditions

$$a_n=O(n^{-\alpha})\quad\text{as } n\to\infty\quad\text{and}\quad f(x)=O(x^{\alpha-1})\quad\text{as } x\to 0$$

are equivalent for $0<\alpha<1$ in the case of the cosine series and for $0<\alpha<2$ in the case of the sine series (see also [14], [36], [38], [43], [44], [75], [76], [91], [89]).

5. (*Pointwise convergence and convergence in L_p, $0<p<1$.*) Let $\{a_n\}\in\mathrm{GM}(p_0)$ for some $p_0\geqslant 1$ and let $\sum_{n=1}^\infty|a_n|/n<\infty$. Then $\{a_n\}$ is a sequence of bounded variation and the series (1.2) converge on $(0,2\pi)$ and converge uniformly on $(\varepsilon,2\pi-\varepsilon)$. Moreover, $f\in L_p(\mathbb{T})$, $0<p<1$ (see Corollary 4.13).

6. (*Convergence in L_p, $1<p<\infty$.*) In Theorem 5.2 we obtain an analogue of the Hardy–Littlewood theorem. Let (1.2) be the Fourier expansion of a function $f\in L_1$ and let the Fourier coefficients $\{a_n\}$ be in $\mathrm{GM}(p_0)$ for some $p_0\geqslant 1$. Then

$$\|f\|_p^p\asymp|a_0|^p+\sum_{n=1}^\infty n^{p-2}(a_n^\#)^p\asymp|a_0|^p+\sum_{n=1}^\infty n^{p-2}|a_n|^p. \tag{1.3}$$

In particular, if $f\in L_p$, then $|a_n|n^{1/p'}=o(1)$ as $n\to\infty$, where, as usual, $1/p+1/p'=1$ (see also [27], [25], [28], [31], [47], [62], [93], [3], [9], [11], [12], [32], [33]–[35], [42], [50], [54], [73], [89], [97]). The relations (1.3) are valid for $p\geqslant 2$ in the case of positive coefficients (or coefficients changing sign a uniformly bounded number of times on dyadic intervals) $\{a_n\}_{n=1}^\infty\in\mathrm{WM}(p_0)$ with $p_0\geqslant 1$ (see Corollary 5.5).

7. (*Absolute convergence.*) It is shown in Corollary 4.11 that, for continuous functions with coefficients $\{a_n\}_{n=1}^\infty\in\mathrm{GM}(p)$, $p\geqslant 1$, we get that for any $\theta>0$ and any $\alpha\in\mathbb{R}$

$$\sum_{n=1}^\infty n^\alpha(n|a_n|)^\theta\leqslant C\sum_{n=1}^\infty n^\alpha E_{n-1}(f)_\infty^\theta.$$

This estimate supplements the classical results by Bernstein and Szász [4], and moreover it is optimal.

1.2. Estimates of Fourier coefficients and moduli of smoothness. Now we discuss estimates of the Fourier coefficients from above. In the general case an estimate of the Fourier coefficients in terms of the function itself has only the trivial form $|a_n|,|b_n|\leqslant\|f\|_{L_1(\mathbb{T})}$. We show in Theorem 3.3 that if (1.2) is the Fourier expansion of a function $f\in L_1$ and the Fourier coefficients $\{a_n\}_{n=1}^\infty$ are in $\mathrm{GM}(p)$ with $p\geqslant 1$, then for any positive integer n

$$|a_n|\leqslant a_n^\#\leqslant C\biggl(\int_0^{\pi/n}|f(t)|\,dt+\frac{\pi^2}{n^2}\int_{\pi/n}^\pi\frac{|f(t)|}{t^2}\,dt\biggr). \tag{1.4}$$

The same estimate also holds for positive coefficients (or coefficients changing sign a uniformly bounded number of times on dyadic intervals) $\{a_n\}_{n=1}^\infty\in\mathrm{WM}(p)$ with $p\geqslant 1$ (see Theorem 3.10 and Corollary 3.11).

Further, for any integrable function we have the Lebesgue type inequality

$$|a_n|, |b_n| \leqslant C E_{n-1}(f)_1 \leqslant C\omega_\beta\biggl(f, \frac{1}{n}\biggr)_1,$$

where $E_n(f)_p$ is the best approximation of a function f by trigonometric polynomials of degree at most n in the $L_p(\mathbb{T})$-norm, and $\omega_\beta(f,\delta)_p$ is the modulus of smoothness of f of order $\beta>0$ in the $L_p(\mathbb{T})$-norm, that is,

$$\omega_\beta(f,\delta)_p = \sup_{|h|\leqslant\delta}\biggl\|\sum_{k=0}^{\infty}(-1)^k\binom{\beta}{k}f(\,\cdot\,+(\beta-k)h)\biggr\|_p.$$

Theorem 3.4 below enables us to significantly improve this estimate for continuous functions with general monotone coefficients. If (1.2) is the Fourier expansion of a function $f\in C(\mathbb{T})$ with Fourier coefficients $\{a_n\}_{n=1}^{\infty}\in \mathrm{GM}(p)$, $p\geqslant 1$, then the Fourier series of f converges uniformly, and for any $q>0$

$$n|a_n| \leqslant n a_n^{\#} \leqslant C n^{-q}\max_{1\leqslant k\leqslant n} k^q E_{k-1}(f)_\infty, \qquad n\in\mathbb{N}. \tag{1.5}$$

The same estimate also holds for positive coefficients (or coefficients changing sign a uniformly bounded number of times on dyadic intervals) $\{a_n\}_{n=1}^{\infty}\in \mathrm{WM}(p)$ with $p\geqslant 1$ (see Theorem 3.10 and Corollary 3.11).

The estimate (1.5) immediately implies the following improvement of a Lebesgue type inequality:

$$n|a_n| \leqslant C\omega_\beta\biggl(f, \frac{\pi}{n}\biggr)_\infty$$

for $\beta=1$ (see [36]).

A natural question arises about estimates of L_p-moduli of smoothness in terms of Fourier coefficients. In § 6, we show that for functions $f\in L_p(\mathbb{T})$, $p\in(1,\infty)$, with Fourier coefficients $\{a_n\}_{n=1}^{\infty}\in\mathrm{GM}(p_0)$, $p_0\geqslant 1$, we get that for any $\delta>0$

$$\begin{aligned}\omega_\beta(f,\delta)_p &\asymp \biggl(\delta^{p\beta}\sum_{n=1}^{[\pi/\delta]} n^{p-2+p\beta}(a_n^{\#})^p + \sum_{n=1+[\pi/\delta]}^{\infty} n^{p-2}(a_n^{\#})^p\biggr)^{1/p}\\ &\asymp \biggl(\delta^{p\beta}\sum_{n=1}^{[\pi/\delta]} n^{p-2+p\beta}|a_n|^p + \sum_{n=1+[\pi/\delta]}^{\infty} n^{p-2}|a_n|^p\biggr)^{1/p}.\end{aligned} \tag{1.6}$$

In the case of series with monotone or quasi-monotone coefficients this result is known (see [47], [1], [2], [70], and see also [41] for some extensions). For the class GM(1) these equivalences were proved in [28].

We note that such results play an important role in functional analysis, in particular, for describing various function spaces (see, for instance, [22] and [23]). Such function classes are, in a way, 'borderline' for some smooth spaces (see [24], for example).

If (1.2) is the Fourier expansion of a function f, then we use the notation

$$f^{\#}(x) = a_0 + \sum_{n=1}^{\infty} a_n^{\#}\cos(nx) \quad\text{or}\quad f^{\#}(x) = \sum_{n=1}^{\infty} a_n^{\#}\sin(nx), \tag{1.7}$$

respectively.

Theorem 1.1. *If $p \in (1,\infty)$ and (1.2) is the Fourier expansion of a function $f \in L_p(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^{\infty} \in \mathrm{GM}(p_0)$, then $\|f\|_p \asymp \|f^{\#}\|_p$. Moreover, $\omega_\beta(f,\delta)_p \asymp \omega_\beta(f^{\#},\delta)_p$ for $\beta > 0$ for any $\delta > 0$. In addition, if $f \in L_1(\mathbb{T})$ and $\gamma \in (1-p,1)$, or $\gamma \in (1-p,1+p)$ in the case of a sine series, then the following order relation holds*:

$$\int_0^\pi \frac{1}{t^\gamma}\,|f(t)|^p\,dt \asymp \int_0^\pi \frac{1}{t^\gamma}\,|f^{\#}(t)|^p\,dt.$$

1.3. Fourier coefficients and Lorentz, Besov, and Sobolev spaces. Let $a_0/2+\sum_{n=1}^{\infty} a_n\cos(nx)+b_n\sin(nx)$ be the Fourier expansion of a function $f\in L(\mathbb{T})$ with coefficients $\mathbf{a}=\{a_n\}_{n=1}^{\infty}$ and $\mathbf{b}=\{b_n\}_{n=1}^{\infty}$ satisfying the $\mathrm{GM}(p_0)$-condition, $p_0>1$. Then the following two-sided estimates hold for the Lorentz space $L_{r,s}(\mathbb{T})$, the Besov space $B^{\alpha}_{p,\tau}(\mathbb{T})$, and the Sobolev space $W^r_p(\mathbb{T})$ (see the definitions in § 7), respectively:

(i) for any $1<r,s<\infty$

$$\|f\|_{L_{r,s}} \asymp \|f^{\#}\|_{L_{r,s}} \asymp \|\mathbf{a}\|_{l_{r',s}} + \|\mathbf{b}\|_{l_{r',s}} \asymp \|\mathbf{a}^{\#}\|_{l_{r',s}} + \|\mathbf{b}^{\#}\|_{l_{r',s}};$$

(ii) for any $0<\tau\leqslant\infty$ and $1<p\leqslant\infty$

$$\begin{aligned}\|f\|_{B^{\alpha}_{p,\tau}} \asymp \|f^{\#}\|_{B^{\alpha}_{p,\tau}} &\asymp \big\|n^{\alpha+1/p'-1/\tau}|a_n|\big\|_{l_\tau} + \big\|n^{\alpha+1/p'-1/\tau}|b_n|\big\|_{l_\tau}\\ &\asymp \|n^{\alpha+1/p'-1/\tau}a_n^{\#}\|_{l_\tau} + \|n^{\alpha+1/p'-1/\tau}b_n^{\#}\|_{l_\tau};\end{aligned}$$

(iii) for any $r>0$ and $1<p<\infty$

$$\begin{aligned}\|f\|_{W^r_p} \asymp \|f^{\#}\|_{W^r_p} &\asymp \big\|n^{r+1-2/p}|a_n|\big\|_{l_p} + \big\|n^{r+1-2/p}|b_n|\big\|_{l_p}\\ &\asymp \|n^{r+1-2/p}a_n^{\#}\|_{l_p} + \|n^{r+1-2/p}b_n^{\#}\|_{l_p}.\end{aligned}$$

The corresponding results are obtained in Theorems 7.1, 7.3, and 7.7. Particular cases of item (i) for the Lorentz space were derived in [11], [12], [24], [30], [42], [73], and for the Besov space see [28], [3], [41], [70], [64], [65], [73].

1.4. Structure of the paper. In § 2, we present several important properties of general monotone sequences. In particular, we completely describe the class $\mathrm{GM}(p)$ as $\mathrm{GM}(\max)\cap\mathrm{WM}(p)$, and we show that for a sequence $\{a_n\}_{n=1}^{\infty}\in\mathrm{GM}(p_0)$ with some $p_0\geqslant 1$ (in fact, even for $\mathrm{WM}(p_0)$) we find that for any $p,\alpha\in(0,\infty)$

$$\sum_{n=1}^{\infty}|a_n|^p n^{\alpha-1} \asymp \sum_{n=1}^{\infty}(a_n^{\#})^p n^{\alpha-1} \asymp \sum_{n=1}^{\infty}(a_n^{*})^p n^{\alpha-1},$$

where a_n^* is the non-increasing rearrangement of the sequence $\{a_n\}$. As usual, $f_n\asymp g_n$ means that $C_1 f_n\leqslant g_n\leqslant C_2 f_n$ for some positive constants C_1 and C_2 which may depend on inessential parameters.

Further, in § 3 we obtain upper estimates of Fourier coefficients: the inequalities (1.4) and (1.5).

In § 4, we derive sharp results on various types of convergence for the series (1.2). Section 5 is devoted to the proof of an analogue of the Hardy–Littlewood result for series with general monotone coefficients (see (1.3) and Theorem 1.1). The relation (1.6) for L_p-moduli of smoothness is proved in § 6. In § 7 we give applications of the results obtained to approximation theory and functional analysis. In particular, we characterize Lorentz, Besov, and Sobolev spaces in terms of Fourier coefficients (proof of the results given in § 1.3).

To conclude, we stress that $\mathrm{GM}(p)$ is, to a certain extent, the widest possible class so that one can still develop a meaningful and extensive theory of Fourier series in the sense that the results noted above can be stated in the form of criteria. This is not the case for the class GM(max) nor for $\mathrm{WM}(p)$.

In this paper we do not aim to give an extensive survey of the literature on trigonometric series with special coefficients (see [26] and [58], for example). Neither do we investigate the properties of Fourier transforms of general monotone functions (see [19], [18], [39]–[41], [59], [57], [63], [74]).

2. Properties of general monotone sequences

First we give the needed definitions.

2.1. Main notation. Henceforth, let ν be a natural number and $D, p \in [1, \infty)$. We say that a sequence of complex numbers $a = \{a_n\}_{n=1}^{\infty}$ is general monotone, that is, of type GM with parameters ν, D, p, and we write $a \in \mathrm{GM}(\nu, D, p)$, if it satisfies the condition

$$\sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant D\biggl(2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p\biggr)^{1/p} \quad \text{for any } n \geqslant \nu. \tag{2.1}$$

We say that a sequence a belongs to the class $\mathrm{GM}(p)$ if there exist an integer ν and a $D \in [1, \infty)$ such that $a \in \mathrm{GM}(\nu, D, p)$. In other words,

$$\mathrm{GM}(p) = \bigcup_{\nu, D \geqslant 1} \mathrm{GM}(\nu, D, p).$$

If a sequence of complex numbers $\{a_k\}_{k=1}^{\infty}$ is bounded, then we set

$$M_n = \max_{k=2^n, \dots, 2^{n+1}} |a_k| \quad \text{for } n \geqslant 0.$$

A sequence of complex numbers $a = \{a_n\}_{n=1}^{\infty}$ is a sequence of type $\mathrm{WM}(\nu, D, p)$ if the following condition holds for some integer ν and some positive D:

$$|a_j| \leqslant D \left(\frac{1}{j} \sum_{k=[j\,2^{-\nu}]+1}^{j\,2^{\nu}} |a_k|^p \right)^{1/p} \quad \text{for } j \geqslant 2^{\nu}. \tag{2.2}$$

We also define

$$\mathrm{WM}(p) = \bigcup_{\nu, D \geqslant 1} \mathrm{WM}(\nu, D, p).$$

2.2. Examples of p-general monotone sequences. We start by recalling several known extensions of the class M of sequences tending monotonically to zero. There are two types of such extensions.

The first consists of various quasi-monotone sequences. In [78] and [85], the class of classical quasi-monotone sequences was defined as follows:

$$\mathrm{QM}=\bigl\{a=\{a_n\}_{n\in\mathbb{N}}\colon a_n\in\mathbb{R}_+ \text{ and there is a } \tau>0 \text{ such that } n^{-\tau}a_n{\downarrow}\bigr\}.$$

The more general class of O-regularly varying quasi-monotone sequences (see [81], for example) is given by

$$\mathrm{ORVQM}=\biggl\{a=\{a_n\}_{n\in\mathbb{N}}\colon a_n\in\mathbb{R}_+ \text{ and there is a sequence } \{\lambda_n\}{\uparrow},\ \lambda_{2n}\leqslant C\lambda_n, \text{ such that } \frac{a_n}{\lambda_n}{\downarrow}\biggr\}.$$

The second way to generalize monotone sequences is to define so-called rest of bounded variation sequences:

$$\mathrm{RBVS}=\biggl\{a=\{a_n\}_{n\in\mathbb{N}}\colon a_n\in\mathbb{C},\ \sum_{\nu=n}^{\infty}|a_\nu-a_{\nu+1}|\leqslant C|a_n|\biggr\} \tag{2.3}$$

(see [52] and [71]).

The classes QM (or ORVQM) and RBVS are not comparable ([53], [89]).

In [89] one of the authors introduced the class of general monotone sequences:

$$\mathrm{GMS}=\biggl\{a=\{a_n\}_{n\in\mathbb{N}}\colon a_n\in\mathbb{C},\ \sum_{\nu=n}^{2n-1}|a_\nu-a_{\nu+1}|\leqslant C|a_n|\biggr\}. \tag{2.4}$$

It is known [89] that $\{a_n\}\in\mathrm{GMS}$ if and only if

$$\begin{cases} |a_k|\leqslant C|a_n| & \text{for any } n\leqslant k\leqslant 2n;\\ \displaystyle\sum_{s=n}^{N}|\Delta a_s|\leqslant C\biggl(|a_n|+\sum_{s=n+1}^{N}\frac{|a_s|}{s}\biggr) & \text{for any } n\leqslant N. \end{cases} \tag{2.5}$$

Here $\Delta a_s=a_s-a_{s+1}$. The interrelation between the classes ORVQM, RBVS, and GMS follows from the embeddings

$$M\subsetneq \mathrm{ORVQM}\cup\mathrm{RBVS}\subsetneq\mathrm{GMS}\subsetneq\mathrm{GM}(1) \tag{2.6}$$

(see [89], p. 725).

We also note that the following class was introduced in [7]. For any integers $n_1\leqslant n_2$ and any $A\geqslant 1$, the notation

$$\{a_n\}_{n=n_1}^{n_2}\in\mathrm{GM}(A)$$

means that either

$$|a_{n_1}|+\sum_{k=n_1}^{m-1}|a_k-a_{k+1}|\leqslant A|a_m| \quad \text{for any } m=n_1,\dots,n_2$$

or

$$|a_{n_2}| + \sum_{k=m}^{n_2-1} |a_k - a_{k+1}| \leqslant A|a_m| \quad \text{for any } m = n_1, \dots, n_2.$$

One can also consider sequences of complex numbers $\{a_n\}_{n=1}^{\infty}$ such that there exist a finitely lacunary sequence of positive integers $\{N_n\}_{n=1}^{\infty}$ and an $A \geqslant 1$ such that for any $k = 1, 2, \dots$

$$\{a_n\}_{n=N_{k-1}+1}^{N_k} \in \mathrm{GM}(A).$$

2.3. Sum and product of *p*-general monotone sequences. It is clear that the sum and product of two general monotone sequences are not necessarily general monotone, that is, for any $p_0, p_1, p_2 \geqslant 1$ there exist $a = \{a_n\}_{n=1}^{\infty} \in \mathrm{GM}(p_0)$ and $b = \{b_n\}_{n=1}^{\infty} \in \mathrm{GM}(p_1)$ such that $\{a_n + b_n\}_{n=1}^{\infty} \notin \mathrm{GM}(p_2)$ or $\{a_n b_n\}_{n=1}^{\infty} \notin \mathrm{GM}(p_2)$. For example, in the first case it suffices to consider the sequences

$$a_n = \begin{cases} \dfrac{1}{n} & \text{if } 2^k \leqslant n < 2^k + 2^{k-1},\ k \in \mathbb{N}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$b_n = \begin{cases} -a_n & \text{if } n \neq 2^k,\ k \in \mathbb{N}, \\ a_n & \text{otherwise,} \end{cases}$$

while in the second case one can take the same a_n and

$$b_n = \begin{cases} \dfrac{1}{n} - a_n & \text{if } n \neq 2^k,\ k \in \mathbb{N}, \\ a_n & \text{otherwise.} \end{cases}$$

However, under some additional conditions, one can assert that the sum and product of two general monotone sequences are also general monotone. For example, if $a, b \in \mathrm{GM}(p)$ are non-negative, then $\{a_n + b_n\}_{n=1}^{\infty} \in \mathrm{GM}(p)$.

Property 2.1. *For any $p_0, p_1 \geqslant 1$, if $a = \{a_n\}_{n=1}^{\infty} \in \mathrm{GM}(p_0)$, $b = \{b_n\}_{n=1}^{\infty} \in \mathrm{GM}(p_1)$ and for some $C_1, C_2 > 0$*

$$C_1 b_k \leqslant b_n \leqslant C_2 b_k \quad \textit{for any} \quad k \leqslant n \leqslant 2k, \quad k, n \geqslant 1, \tag{2.7}$$

then $\{a_n b_n\}_{n=1}^{\infty} \in \mathrm{GM}(p_0)$. In particular, $\{a_n n^{\gamma}\}_{n=1}^{\infty} \in \mathrm{GM}(p_0)$ for any $\gamma \in \mathbb{R}$.

Proof. Indeed, by Theorem 2.5 (see the inequality (2.10)),

$$\begin{aligned} \sum_{k=2^n}^{2^{n+1}} |\Delta(a_k b_k)| &\leqslant \sum_{k=2^n}^{2^{n+1}} |b_{k+1}\Delta(a_k)| + \sum_{k=2^n}^{2^{n+1}} |a_k \Delta(b_k)| \\ &\leqslant D\biggl(\sum_{k=2^{n-\nu_0}}^{2^{n+\nu_0}} \frac{|a_k|^{p_0}}{k}\biggr)^{1/p_0} \biggl(\sum_{k=2^{n-\nu_1}}^{2^{n+\nu_1}} \frac{|b_k|^{p_1}}{k}\biggr)^{1/p_1} \\ &\leqslant D|b_{2^n}|\biggl(\sum_{k=2^{n-\nu_0}}^{2^{n+\nu_0}} \frac{|a_k|^{p_0}}{k}\biggr)^{1/p_0} \leqslant D\biggl(\sum_{k=2^{n-\nu_0}}^{2^{n+\nu_0}} \frac{|a_k b_k|^{p_0}}{k}\biggr)^{1/p_0}. \quad \square \end{aligned}$$

Remark 2.2. Note that under the condition (2.7) the class of $b=\{b_n\}_{n=1}^{\infty}\in \mathrm{GM}(p_1)$ coincides with the class of $b=\{b_n\}_{n=1}^{\infty}\in$ GMS with the additional condition $b_n\leqslant Cb_{2n}$ for any $n\geqslant 1$. It is also clear that the condition $b_n\leqslant Cb_{2n}$ is essential. For example, let $b_n=1$ for $n\leqslant N$ and $=0$ otherwise. Then $\{b_n\}\in \mathrm{GMS}\subset \mathrm{GM}(1)$, but the condition $b_n\leqslant Cb_{2n}$ is not satisfied. At the same time, if $\{a_n\}$ is given by $a_k=1/\xi_n$ for $\xi_n\leqslant k\leqslant 2\xi_n$ and zero otherwise, with ξ_n increasing sufficiently fast (for example, $\xi_n=2^{2^n}$), then $\{a_nb_n\}\notin \mathrm{GM}(p)$ for any p.

In the case when $b_n=a_n$, additional conditions for $\{a_nb_n\}_{n=1}^{\infty}\in \mathrm{GM}(p)$ are not needed.

Property 2.3. (A) *Let $p,\gamma\geqslant 1$ and $a=\{a_n\}_{n=1}^{\infty}\in \mathrm{GM}(p)$. Then*

$$\{a_n|a_n|^{\gamma-1}\}_{n=1}^{\infty}\in \mathrm{GM}(p).$$

(B) *In particular, for a non-negative sequence $a=\{a_n\}_{n=1}^{\infty}\in \mathrm{GM}(p)$, $p\geqslant 1$, the sequence $\{a_n^{\gamma}\}_{n=1}^{\infty}$ is in $\mathrm{GM}(p)$ if and only if $\gamma\geqslant 1$.*

Proof. (A) Indeed, for $\gamma\geqslant 1$ we get by the mean value theorem that

$$\big|a_k|a_k|^{\gamma-1}-a_{k+1}|a_{k+1}|^{\gamma-1}\big|\leqslant \gamma|\Delta a_k|(|a_k|^{\gamma-1}+|a_{k+1}|^{\gamma-1}).$$

Then the inequality (2.10) implies that

$$\begin{aligned}\sum_{k=2^n}^{2^{n+1}}\big|a_k|a_k|^{\gamma-1}-a_{k+1}|a_{k+1}|^{\gamma-1}\big|&\leqslant \gamma\max_{2^n\leqslant k\leqslant 2^{n+1}+1}|a_k|^{\gamma-1}\sum_{k=2^n}^{2^{n+1}}|\Delta a_k|\\ &\leqslant DD_1\bigg(\sum_{k=2^{n-\nu_0}}^{2^{n+\nu_0}}\frac{|a_k|^p}{k}\bigg)^{1/p}\bigg(\sum_{k=2^{n-\nu_1}}^{2^{n+\nu_1}}\frac{|a_k|^p}{k}\bigg)^{(\gamma-1)/p}\leqslant D_2\bigg(\sum_{k=2^{n-\nu_2}}^{2^{n+\nu_2}}\frac{a_k^{\gamma p}}{k}\bigg)^{1/p},\end{aligned}$$

where we used Hölder's inequality in the last step.

(B) From (A) we have $\{a_n^{\gamma}\}_{n=1}^{\infty}\in \mathrm{GM}(p)$ for $\gamma\geqslant 1$. Let $0<\gamma\leqslant 1$. Let $c_n=2^{-2n}$ for $n=0,1,2,\dots$ and put

$$\begin{aligned}&a_k=c_n && \text{for}\quad 2^{2n}\leqslant k<2^{2n+1},\\ &a_k=c_n\,2^{-2n} && \text{for odd}\quad k\in[2^{2n+1},2^{2n+2}),\\ &a_k=0 && \text{for even}\quad k\in[2^{2n+1},2^{2n+2}).\end{aligned}$$

Note that if $\nu,p\geqslant 1$ are integers and $\gamma\in(0,1]$, then there are constants $C_1=C_1(\nu,p,\gamma)>0$ and $C_2=C_2(\nu,p,\gamma)>0$ such that for any m

$$C_1\,2^{-m\gamma}\leqslant\bigg(2^{-m}\sum_{k=2^{m-\nu}}^{2^{m+\nu}}|a_k|^{\gamma p}\bigg)^{1/p}\leqslant C_2\,2^{-m\gamma}. \tag{2.8}$$

Further, we obtain

$$\sum_{k=2^m}^{2^{m+1}}|a_k-a_{k+1}|\leqslant 2^{-m}$$

for both even and odd m. Thus, our sequence lies in the class $\mathrm{GM}(p)$. At the same time, for $\gamma \in (0,1)$ we have

$$\sum_{k=2^{2n+1}}^{2^{2n+2}} |a_k^\gamma - a_{k+1}^\gamma| \geqslant 2^{2n}\, 2^{-2n\gamma}\, 2^{-2n\gamma} = 2^{-2n\gamma}\, 2^{2n(1-\gamma)}$$

for any n. This and (2.8) imply that $\{a_k^\gamma\}_{k=1}^\infty \notin \mathrm{GM}(p)$.

Finally, if $\gamma < 0$, then it is sufficient to take $a_n = 2^{-n}$, $n \in \mathbb{N}$. ☐

It is interesting that for the class GMS of general monotone sequences given by (2.4), the result is fundamentally different. Namely,

a non-negative sequence $a = \{a_n\}_{n=1}^\infty \in \mathrm{GMS}$ *satisfies the condition* $\{a_n^\gamma\}_{n=1}^\infty \in \mathrm{GMS}$ *if and only if* $\gamma \geqslant 0$.

This follows, in particular, from the results of [94]. We give a simple proof of this fact. For $\gamma \geqslant 1$ we use the mean value theorem. Let $\gamma \in (0,1)$. For simplicity assume that $a_n > 0$. Then we note that

$$|\Delta(a_k^\gamma)| \leqslant \frac{|\Delta a_k|}{a_k^{1-\gamma}}\,.$$

For a positive integer n let $s_0 = n$ and let s_1 be the first index in the interval $n < s_1 \leqslant 2n-1$ such that $a_n^{1-\gamma} > 2a_{s_1}^{1-\gamma}$. Next, let s_2 be the first index in the interval $s_1 < s_2 \leqslant 2n-1$ such that $a_{s_1}^{1-\gamma} > 2a_{s_2}^{1-\gamma}$, and so on, up to the index $s_j \leqslant 2n-1$. We have

$$\sum_{k=n}^{2n} |\Delta(a_k^\gamma)| \leqslant \sum_{i=0}^{j} \sum_{k=s_i}^{s_{i+1}-1} \frac{|\Delta a_k|}{a_k^{1-\gamma}} \leqslant 2\sum_{i=0}^{j} \sum_{k=s_i}^{s_{i+1}-1} \frac{|\Delta a_k|}{a_{s_i}^{1-\gamma}}\,.$$

Further, using the definition of the class GMS, we establish that the sum on the right-hand side of the last inequality is less than or equal to

$$2C\sum_{i=0}^{j} a_{s_i}^\gamma \leqslant C_1 a_n^\gamma \sum_{i=0}^{j} 2^{-\gamma i/(1-\gamma)} \leqslant C_2 a_n^\gamma.$$

Thus, $\{a_n^\gamma\}_{n=1}^\infty \in \mathrm{GMS}$.

Property 2.4. *Let* $p_0 \geqslant 1$, $\alpha > 0$, *and* $0 < p < q < \infty$. *If a sequence* $a = \{a_n\}_{n=1}^\infty \in \mathrm{GM}(p_0)$ *is such that* $\sum_{n=1}^\infty |a_n|^p n^{\alpha-1} < \infty$, *then*

(A) $a_n n^{\alpha/p} \to 0$,

(B) $\sum_{n=1}^\infty |a_n|^q n^{\frac{\alpha q}{p}-1} < \infty$,

(C) *the relations*

$$\sum_{n=1}^\infty |a_n|^p n^{\alpha-1} \asymp \sum_{n=1}^\infty (a_n^{\#})^p n^{\alpha-1} \asymp \sum_{n=1}^\infty (a_n^*)^p n^{\alpha-1} \asymp \sum_{n=1}^\infty M_n^p\, 2^{n\alpha} \tag{2.9}$$

are valid.

Note that for $\alpha = p = 1$ the results in (A) and (C) extend, respectively, the well-known Abel–Olivier and Cauchy tests for monotonic series. Item (A) with $\alpha = p = 1$ was proved in [17] for the class GM(1), while (C) was proved in [11] for the class GMS.

In the general case without the condition of general monotonicity, one can only claim in (B) that $\sum_{n=1}^{\infty} |a_n|^q n^{(\alpha-1)q/p} < \infty$.

Proof. Item (A) follows immediately from the fact that

$$\sum_{n=1}^{\infty} (a_n^{\#})^p n^{\alpha-1} < \infty$$

(see Theorem 2.9). From (A) it follows that $|a_n|^q n^{\alpha q/p-1} = o(|a_n|^p n^{\alpha-1})$ as $n \to \infty$, which gives (B). The first two equivalences in (C) follow from Theorems 2.9 and 2.12 while the last one follows from Lemma 5.1. □

2.4. Criteria for *p*-general monotone sequences. The main goal of this subsection is to prove the representation $\mathrm{GM}(p) = \mathrm{GM}(\max) \cap \mathrm{WM}(p)$.

Theorem 2.5. *A sequence of complex numbers $a = \{a_n\}_{n=1}^{\infty}$ is a sequence of type* $\mathrm{GM}(\nu, D, p)$ *if and only if for some D_1, D_2 and $\nu \geqslant 1$*

$$\max_{k=2^n,\dots,2^{n+1}} |a_k| \leqslant D_1 \left(2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p \right)^{1/p} \quad \text{for any } n \geqslant \nu \tag{2.10}$$

and

$$\sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant D_2 \max_{k=2^{n-\nu},\dots,2^{n+\nu}} |a_k| \quad \text{for any } n \geqslant \nu. \tag{2.11}$$

Note that one can easily construct examples showing that the conditions (2.10) and (2.11) are independent.

Proof of Theorem 2.5. By the definition of the class $\mathrm{GM}(\nu, D, p)$, the condition (2.1) is satisfied. Let

$$M_n = \max_{k=2^n,\dots,2^{n+1}} |a_k| \quad \text{for } n \geqslant 0$$

and let $q_n \in \{2^n, \dots, 2^{n+1}\}$ be such that $M_n = |a_{q_n}|$. Then for $n \geqslant \nu$ and any $j \in \{2^n, \dots, 2^{n+1}\}$

$$|a_{q_n}| - |a_j| \leqslant \sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant D \left(2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p \right)^{1/p}.$$

Hence

$$\begin{aligned} |a_{q_n}| &\leqslant \frac{1}{2^n+1} \sum_{k=2^n}^{2^{n+1}} |a_k| + D \left(2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p \right)^{1/p} \\ &\leqslant \left(\frac{1}{2^n+1} \sum_{k=2^n}^{2^{n+1}} |a_k|^p \right)^{1/p} + D \left(2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p \right)^{1/p}. \end{aligned}$$

Thus, (2.10) holds with the constant $D_1 = D + 1 \geqslant 1$.

From (2.1) we also have the estimate

$$\sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant D\, 2^{\nu/p} \max_{k=2^{n-\nu},\dots,2^{n+\nu}} |a_k|,$$

that is, (2.11) holds with $D_2 = D\, 2^{\nu/p} \geqslant 1$. In other words, if a sequence $a = \{a_n\}_{n=1}^{\infty}$ is of type $\mathrm{GM}(\nu, D, p)$, then the conditions (2.10) and (2.11) hold, where $D_1 \geqslant 1$ and $D_2 \geqslant 1$ depend only on the parameters ν, D, and p.

To prove the converse statement, suppose that the conditions (2.10) and (2.11) hold for an integer ν and some numbers $D_1, D_2 \in [1, \infty)$. Then from (2.10) with $n \geqslant 2\nu$ we have

$$\begin{aligned}\max_{k=2^{n-\nu},\dots,2^{n+\nu}} |a_k| &= \max_{j=n-\nu,\dots,n+\nu-1} M_j \\ &\leqslant D_1 \max_{j=n-\nu,\dots,n+\nu-1} \biggl(2^{-j} \sum_{k=2^{j-\nu}}^{2^{j+\nu}} |a_k|^p\biggr)^{1/p} \\ &\leqslant D_1 \biggl(2^{-n+\nu} \sum_{k=2^{n-2\nu}}^{2^{n+2\nu-1}} |a_k|^p\biggr)^{1/p}.\end{aligned}$$

This and (2.11) give us that

$$\sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant D_2 D_1 \biggl(2^{\nu}\, 2^{-n} \sum_{k=2^{n-2\nu}}^{2^{n+2\nu}} |a_k|^p\biggr)^{1/p} \quad \text{for any } n \geqslant 2\nu,$$

that is, $a \in \mathrm{GM}(2\nu, D^*, p)$, where $D^* = D_1 D_2\, 2^{\nu/p}$. Thus, the conditions (2.10) and (2.11) mean that the sequence a is general monotone with parameters 2ν, D^*, and p. □

The following analogue of Theorem 2.5 also holds.

Theorem 2.6. *A sequence of complex numbers $a = \{a_n\}_{n=1}^{\infty}$ is of type $\mathrm{GM}(\nu, D, p)$ if and only if, for some integers ν_1 and ν_2 and some $H_1, H_2 \in [1, \infty)$, the conditions*

$$j\, |a_j|^p \leqslant H_1^p \sum_{k=[j\, 2^{-\nu_1}]+1}^{j\, 2^{\nu_1}} |a_k|^p \quad \text{for } j \geqslant 2^{\nu_1} \tag{2.12}$$

and

$$\sum_{k=j}^{2j} |a_k - a_{k+1}| \leqslant H_2 \max_{j\, 2^{-\nu_2} \leqslant k \leqslant j\, 2^{\nu_2}} |a_k| \quad \text{for } j \geqslant 2^{\nu_2} \tag{2.13}$$

are valid.

Proof. Let us carefully analyze the condition (2.10). If $j \geqslant 2^{\nu}$, then $2^n \leqslant j < 2^{n+1}$ for some $n \geqslant \nu$. Hence, it follows from (2.10) that

$$|a_j|^p \leqslant M_n^p \leqslant D_1^p\, 2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^p \leqslant 2 D_1^p\, \frac{1}{j} \sum_{k=[j\, 2^{-\nu-1}]+1}^{j\, 2^{\nu}} |a_k|^p.$$

In particular, (2.12) holds with $\nu_1 = \nu + 1$ and $H_1 = D_1\, 2^{1/p}$. To prove the converse statement, assume that the condition (2.12) holds for some $H_1 \in [1, \infty)$ and an integer ν_1. Then for $n \geqslant \nu_1$

$$2^n M_n^p \leqslant \max_{j=2^n,\dots,2^{n+1}} j|a_j|^p \leqslant H_1^p \sum_{k=2^{n-\nu_1}}^{2^{n+1+\nu_1}} |a_k|^p.$$

Hence (2.10) is satisfied for $\nu = \nu_1 + 1$ and $D_1 = H_1$. Thus, the condition (2.10) holds for some integer ν and some $D_1 \in [1, \infty)$ if and only if (2.12) is satisfied for some integer ν_1 and some $H_1 \in [1, \infty)$.

Let us now analyze the condition (2.11). If $j \geqslant 2^\nu$, then $2^n \leqslant j < 2^{n+1}$ for some $n \geqslant \nu$. Hence (2.11) implies that

$$\begin{aligned}\sum_{k=j}^{2j} |a_k - a_{k+1}| &\leqslant \sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| + \sum_{k=2^{n+1}}^{2^{n+2}} |a_k - a_{k+1}| \\ &\leqslant 2D_2 \max_{k=2^{n-\nu},\dots,2^{n+1+\nu}} |a_k| \leqslant 2D_2 \max_{j\,2^{-\nu-1} < k \leqslant j\,2^{1+\nu}} |a_k|,\end{aligned}$$

that is, (2.13) holds for $\nu_2 = \nu + 1$ and $H_2 = 2D_2$. To prove the converse statement, assume that the condition (2.13) holds for some $H_2 \in [1, \infty)$ and some integer ν_2. Then for $n \geqslant \nu_2$ and $j = 2^n$

$$\sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant H_2 \max_{2^{n-\nu_2} \leqslant k \leqslant 2^{n+\nu_2}} |a_k|,$$

that is, (2.11) is satisfied for $\nu = \nu_2$ and $D_2 = H_2$. Hence, (2.11) holds for some integer ν and some $D_2 \in [1, \infty)$ if and only if (2.13) holds for some integer ν_2 and some $H_2 \in [1, \infty)$.

We note that if (2.10) holds for some $\nu = \nu_1$ and (2.11) holds for some $\nu = \nu_2$, then both (2.10) and (2.11) hold for $\nu = \max\{\nu_1, \nu_2\}$. □

Corollary 2.7. *A sequence of complex numbers $a = \{a_n\}_{n=1}^\infty$ is of type $\mathrm{GM}(\nu, D, p)$ if and only if, for some $H_4, H_5 \in [1, \infty)$ and some integer ν_4, the condition*

$$\sum_{k=j}^{2j} |a_k - a_{k+1}| \leqslant H_4 \biggl(\frac{1}{j} \sum_{k=[j\,2^{-\nu_4}]}^{j\,2^{\nu_4}} |a_k|^p\biggr)^{1/p} + H_5 \sum_{s=1-2^{\nu_4}}^{2^{\nu_4}} (|a_{j+s}| + |a_{2j+s}|) \quad (2.14)$$

holds for any $j \geqslant 2^{\nu_4}$.

Proof. If $2^n \leqslant j < 2^{n+1}$ for some $n \geqslant \nu$, then (2.1) implies that

$$\sum_{k=j}^{2j} |a_k - a_{k+1}| \leqslant 2D \biggl(2^{-n} \sum_{k=2^{n-\nu}}^{2^{n+\nu+1}} |a_k|^p\biggr)^{1/p} \leqslant 2D \biggl(\frac{2}{j} \sum_{k=[j\,2^{-\nu-1}]}^{j\,2^{\nu+1}} |a_k|^p\biggr)^{1/p},$$

that is, (2.14) is satisfied.

On the other hand, by (2.14) with $j \geqslant 2^{\nu_4+1}$ one has

$$\sum_{k=j}^{2j} |a_k - a_{k+1}| \leqslant H_4\, 2^{\nu_4/p} \max_{j\, 2^{-\nu_4} < k \leqslant j\, 2^{\nu_4}} |a_k| + H_5\, 2^{\nu_4+1} \max_{j-2^{\nu_4} < k \leqslant 2j+2^{\nu_4}} |a_k|$$
$$\leqslant H_4\, 2^{\nu_4/p} \max_{j\, 2^{-\nu_4} < k \leqslant j\, 2^{\nu_4}} |a_k| + H_5\, 2^{\nu_4+1} \max_{j/2 < k \leqslant 3j} |a_k|$$
$$\leqslant (H_4\, 2^{\nu_4/p} + H_5\, 2^{\nu_4+1}) \max_{k=[j\, 2^{-\nu_4}]+1,\dots,j\, 2^{\nu_4+1}} |a_k| \quad \text{for any } j \geqslant 2^{\nu_4+1}.$$

Hence (2.13) holds for $\nu_2 = \nu_4 + 1$ and $H_2 = H_4\, 2^{\nu_4/p} + H_5\, 2^{\nu_4+1}$. Therefore, the condition (2.11) is satisfied for $\nu = \nu_4 + 1$ and $D_2 = H_4\, 2^{\nu_4/p} + H_5\, 2^{\nu_4+1}$. Since

$$\max_{k=j,\dots,2j} |a_k| - \frac{1}{j+1} \sum_{k=j}^{2j} |a_k| \leqslant \sum_{k=j}^{2j} |a_k - a_{k+1}|,$$

for $j \geqslant 2^{\nu_4}$ we have

$$\max_{k=j,\dots,2j} |a_k| \leqslant \left(\frac{1}{j+1} \sum_{k=j}^{2j} |a_k|^p \right)^{1/p}$$
$$+ H_4 \left(\frac{1}{j} \sum_{k=[j\, 2^{-\nu_4}]+1}^{j\, 2^{\nu_4}} |a_k|^p \right)^{1/p} + H_5 \sum_{s=1-2^{\nu_4}}^{2^{\nu_4}} (|a_{j+s}| + |a_{2j+s}|)$$
$$\leqslant (1 + H_4) \left(\frac{1}{j} \sum_{k=[j\, 2^{-\nu_4}]+1}^{j\, 2^{\nu_4}} |a_k|^p \right)^{1/p} + H_5 \sum_{s=1-2^{\nu_4}}^{2^{\nu_4}} (|a_{j+s}| + |a_{2j+s}|).$$

Thus,

$$\max_{k=j,\dots,4j} |a_k| \leqslant (1 + H_4) \left(\frac{1}{j} \sum_{k=[j\, 2^{-\nu_4}]+1}^{j\, 2^{\nu_4+1}} |a_k|^p \right)^{1/p}$$
$$+ H_5 \sum_{s=1-2^{\nu_4}}^{2^{\nu_4}} (|a_{j+s}| + |a_{2j+s}| + |a_{4j+s}|).$$

Let $n \geqslant \nu_4 + 1$ and let $M_n = |a_q|$, with $2^n \leqslant q \leqslant 2^{n+1}$. For $j = 2^{n-1}, \dots, 2^n$ one has $j \leqslant q \leqslant 4j$ and

$$|a_q| \leqslant (1+H_4) \left(\frac{1}{2^{n-1}} \sum_{k=[2^{n-1-\nu_4}]+1}^{2^{n+\nu_4+1}} |a_k|^p \right)^{1/p} + H_5 \sum_{s=1-2^{\nu_4}}^{2^{\nu_4}} (|a_{j+s}| + |a_{2j+s}| + |a_{4j+s}|).$$

Therefore,

$$M_n \leqslant (1+H_4)\biggl(\frac{1}{2^{n-1}}\sum_{k=2^{n-1-\nu_4}+1}^{2^{n+\nu_4+1}}|a_k|^p\biggr)^{1/p}$$
$$+\frac{H_5}{2^{n-1}+1}\sum_{s=1-2^{\nu_4}}^{2^{\nu_4}}\sum_{j=2^{n-1}}^{2^n}(|a_{j+s}|+|a_{2j+s}|+|a_{4j+s}|)$$
$$\leqslant (1+H_4)\biggl(\frac{2}{2^n}\sum_{k=2^{n-1-\nu_4}+1}^{2^{n+\nu_4+1}}|a_k|^p\biggr)^{1/p}+\frac{H_5\,2^{\nu_4+1}}{(2^{n-1})^{1/p}}\biggl(\biggl(\sum_{k=2^{n-1}+1-2^{\nu_4}}^{2^n+2^{\nu_4}}|a_k|^p\biggr)^{1/p}$$
$$+\biggl(\sum_{k=2^n+1-2^{\nu_4}}^{2^{n+1}+2^{\nu_4}}|a_k|^p\biggr)^{1/p}+\biggl(\sum_{k=2^{n+1}+1-2^{\nu_4}}^{2^{n+2}+2^{\nu_4}}|a_k|^p\biggr)^{1/p}\biggr).$$

If $n \geqslant \nu_4+2$, then this implies the inequality

$$M_n \leqslant (1+H_4)\biggl(\frac{2}{2^n}\sum_{k=2^{n-1-\nu_4}+1}^{2^{n+\nu_4+1}}|a_k|^p\biggr)^{1/p}+3H_5\,2^{\nu_4+1}\biggl(\frac{2}{2^n}\sum_{k=2^{n-2}}^{2^{n+3}}|a_k|^p\biggr)^{1/p}.$$

Consequently, (2.10) holds for $\nu=\nu_4+2$ and $D_1=2^{1/p}(H_4+1+3H_5\,2^{\nu_4+1})$. Finally, (2.10) and (2.11) hold for $\nu=\nu_4+2$, and therefore (2.1) holds for $\nu=2\nu_4+4$. ☐

2.5. Embeddings of the classes GM($\boldsymbol{p}$). Let us show that the parameter p_0 is essential in the definition (2.1) of general monotone sequences, that is, $\mathrm{GM}(p_1)\subsetneq \mathrm{GM}(p_0)$ for $p_1<p_0$.

Theorem 2.8. *For any $1\leqslant p_1<p_0<\infty$, every class $\mathrm{GM}(\nu,D_1,p_1)$ is contained in some class $\mathrm{GM}(\nu,D,p_0)$. Moreover, there exists a non-negative sequence $a=\{a_n\}_{n=1}^{\infty}$ of type $\mathrm{GM}(4,4,p_0)$ such that, for any integer ν_1 and any $D_1\in[1,\infty)$, the sequence a does not belong to the class $\mathrm{GM}(\nu_1,D_1,p_1)$.*

Proof. It is clear that by Hölder's inequality, for $1\leqslant p_1<p_0<\infty$ we have

$$\mathrm{GM}(\nu,D,p_1)\subset \mathrm{GM}(\nu,D_1,p_0),\quad \text{where}\quad D_1=2^{\nu(1/p_1-1/p_0)}D.$$

To construct a counterexample, we first present an auxiliary construction. Let n and τ be natural numbers with $\tau+1\leqslant n$. Put $\lambda=p_0/(p_0-p_1)$. Take the largest integer l such that $l\leqslant n/2$, $l+2\leqslant n$, and $\lambda^l\tau\leqslant n$. Let $M>0$, $\tau_{n+k}=[\lambda^k\tau]$, and

$$M_{n+k}=M\,2^{(\tau_{n+k}+\dots+\tau_{n+1})/p_0}\quad \text{for } k=0,\dots,l. \tag{2.15}$$

Then $M_n=M$ and $M_{n+1}=M2^{[\lambda\tau]/p_0}$. Taking any $k=0,\dots,l$, we put $a_j=M_{n+k}$ for $j=2^{n+k}+1,\dots,2^{n+k}+2^{n+k-\tau_{n+k}}$ and $a_j=0$ for $j=2^{n+k}$ as well as for any $j=2^{n+k}+2^{n+k-\tau_{n+k}}+1,\dots,2^{n+k+1}$. Then the numbers a_j are defined for any $j=2^n,\dots,2^{n+l+1}$, and moreover $M_{n+k}=\max\{a_j\colon j=2^{n+k},\dots,2^{n+k+1}\}$

and $a_j = 0$ for $j = 2^{n+k}$ with $k = 0, \dots, l+1$. From (2.15) with $k = 0, \dots, l-1$ we get that $(M_{n+k+1}/M_{n+k})^{p_0} = 2^{\tau_{n+k+1}}$, and therefore

$$2^{n+k+1} M_{n+k}^{p_0} = \sum_{j=2^{n+k+1}}^{2^{n+k+2}} a_j^{p_0}.$$

We set $M_{n+l+1} = M_{n+l}$, $a_j = M_{n+l}$ for $j = 2^{n+l+1}+1, \dots, 2^{n+l+1}+2^{n+l}$, and $a_j = 0$ for $j = 2^{n+l+1}+2^{n+l}+1, \dots, 2^{n+l+2}$ as well as for any $j = 2^{n+l+2}, \dots, 2^{2n+l}$. Then

$$2^{n+l} M_{n+l+1}^{p_0} = \sum_{j=2^{n+l+1}}^{2^{n+l+2}} a_j^{p_0}$$

and

$$\sum_{j=2^n}^{2^{2n}} a_j = \sum_{k=0}^{l} M_{n+k}\, 2^{n+k-\tau_{n+k}} + M_{n+l}\, 2^{n+l}.$$

In other words,

$$\sum_{j=2^n}^{2^{2n}} a_j = M\biggl(\sum_{k=0}^{l} 2^{n+k-\tau_{n+k}}\, 2^{(\tau_{n+k}+\dots+\tau_{n+1})/p_0} + 2^{n+l}\, 2^{(\tau_{n+l}+\dots+\tau_{n+1})/p_0}\biggr),$$

and $a_{2^k} = 0$ for $k = n, \dots, 2n$.

For definiteness let M be chosen so that

$$\sum_{j=2^n}^{2^{2n}} a_j = 2^{-n},$$

and $\tau = [(n+1)^{1/2}]$. We assume that $n \geqslant 4$ and $M_{n+k} = 0$ for $k = l+2, \dots, n$. Then $M_{n+k} = \max\{a_j : j = 2^{n+k}, \dots, 2^{n+k+1}\}$ for $k = 0, \dots, n$,

$$2^{n+k} M_{n+k}^{p_0} \leqslant \sum_{j=2^{n+k}}^{2^{n+k+2}} a_j^{p_0} \quad \text{for } k = 0, \dots, n-1, \tag{2.16}$$

and $M_{2n} = 0$. We note that in this case $l = l_n$ is the largest integer such that $l_n \leqslant n/2$ and $l_n \leqslant \log(n/\tau)/\log\lambda$. Then $l_n \to +\infty$ as $n \to +\infty$. Therefore, the sequence $E(n) = \{a_j : j = 2^n, \dots, 2^{2n+1}\}$ constructed is completely determined by the number n.

For $k = 0, \dots, l-1$ we have $(M_{n+k+1}/M_{n+k})^{p_1} = 2^{(p_1/p_0)\tau_{n+k+1}}$,

$$\tau_{n+k} - \tau_{n+k+1}\biggl(1 - \frac{p_1}{p_0}\biggr) < \lambda^k \tau - \lambda^{k+1}\tau\biggl(1 - \frac{p_1}{p_0}\biggr) + 1 = 1,$$

and

$$\tau_{n+k} - \tau_{n+k+1}\biggl(1 - \frac{p_1}{p_0}\biggr) > \lambda^k \tau - \lambda^{k+1}\tau\biggl(1 - \frac{p_1}{p_0}\biggr) - 1 = -1.$$

Therefore,

$$\left(\frac{M_{n+k+1}}{M_{n+k}}\right)^{p_1} > 2^{\tau_{n+k+1}-\tau_{n+k}-1} \quad \text{and} \quad \left(\frac{M_{n+k+1}}{M_{n+k}}\right)^{p_1} < 2^{\tau_{n+k+1}-\tau_{n+k}+1}.$$

Hence, for $k = 0, \dots, l_n - 1$ the following inequalities hold:

$$\sum_{j=2^{n+k+1}}^{2^{n+k+2}} a_j^{p_1} = 2^{n+1+k-\tau_{n+1+k}} M_{n+k+1}^{p_1} > 2^{n+k-\tau_{n+k}} M_{n+k}^{p_1} = \sum_{j=2^{n+k}}^{2^{n+k+1}} a_j^{p_1}$$

and

$$\sum_{j=2^{n+k+1}}^{2^{n+k+2}} a_j^{p_1} = 2^{n+1+k-\tau_{n+1+k}} M_{n+k+1}^{p_1} < 4 \cdot 2^{n+k-\tau_{n+k}} M_{n+k}^{p_1} = 4 \sum_{j=2^{n+k}}^{2^{n+k+1}} a_j^{p_1}.$$

Thus,

$$\sum_{j=2^{n+k}}^{2^{n+k+1}} a_j^{p_1} \leqslant 4^k \sum_{j=2^n}^{2^{n+1}} a_j^{p_1} = 4^k\, 2^{n-\tau_n} M_n^{p_1} = 4^k\, 2^n M_n^{p_1}\, 2^{-\tau}.$$

Moreover, if $0 \leqslant \nu \leqslant l_n$, then

$$2^{-n} \sum_{j=2^n}^{2^{n+\nu+1}} a_j^{p_1} < 2 \cdot 4^\nu\, 2^{-\tau(n)} M_n^{p_1}, \tag{2.17}$$

where $\tau(n) = \tau > (n+1)^{1/2} - 1 \to +\infty$ as $n \to +\infty$.

After presenting the above construction, we can now build the needed sequence $a = \{a_n\}_{n=1}^\infty$. For this, take any sequence of integers $\{n_s\}_{s=1}^\infty$ such that $n_1 \geqslant 4$ and $n_{s+1} \geqslant 2n_s+1$. For example, one can take $n_s = 4^s$. Let $\{a_j : j = 2^{n_s}, \dots, 2^{2n_s+1}\} = E(n_s)$ for any $s = 1, 2, \dots$. For other j put $a_j = 0$. This non-negative sequence $a = \{a_n\}_{n=1}^\infty$ has the property that $\sum_{j=1}^\infty a_j < 1$. In view of (2.16), it satisfies the condition (2.10) with $\nu = 2$, $D_1 = 1$ and the condition (2.11) with $D_2 = 2$. Therefore, $a \in \mathrm{GM}(4, 4, p_0)$. However, due to (2.17), for any integer ν_1 and any $D_1 \in [1, \infty)$ the condition (2.10) with $n = n_s$ and p_1 in place of p_0 fails for sufficiently large s. In other words, the sequence a does not belong to the class $\mathrm{GM}(\nu_1, D_1, p_1)$. □

2.6. Equivalence of series with a_n and $a_n^\#$.

Theorem 2.9. *Let a null sequence $\{a_n\}_{n=1}^\infty$ belong to the class $\mathrm{WM}(\nu, D, p_0)$ for some $p_0 \geqslant 1$. If $a_n^\# = \max_{l \geqslant n} |a_l|$ for $n \geqslant 1$, then for $p, \alpha \in (0, \infty)$*

$$\sum_{n=1}^\infty |a_n|^p n^{\alpha-1} \leqslant \sum_{n=1}^\infty (a_n^\#)^p n^{\alpha-1} \leqslant C \sum_{n=1}^\infty |a_n|^p n^{\alpha-1} \tag{2.18}$$

with some positive $C = C(\alpha, p, p_0, \nu, D)$.

First we prove the following auxiliary result.

Lemma 2.10. *Let $\{d_i\}_{i=1}^N$ be a sequence of non-negative numbers such that the following two conditions hold for some $A > 0$ and $C \in (0,1)$: $d_i \leqslant A$ for any i and $\frac{1}{N}\sum_{i=1}^N d_i \geqslant CA$. Then at least $[NC/2]$ numbers in this sequence satisfy the condition $d_i \geqslant AC/2$.*

Proof. Assume not. Then for

$$\Omega = \left\{ i \in [1,N] \cap \mathbb{Z} \colon d_i \geqslant \frac{AC}{2} \right\}$$

we have

$$\sum_{i=1}^N d_i \leqslant \sum_{i\in\Omega} d_i + \sum_{i\in[1,N]\cap\mathbb{Z}\setminus\Omega} d_i < |\Omega| A + \frac{AC}{2} N = CAN,$$

a contradiction. □

Proof of Theorem 2.9. Without loss of generality we may assume that the sum on the right-hand side of (2.18) is finite and $\nu \geqslant 2$. For any integer $k \geqslant 0$ we define

$$A_k := \max_{2^k \leqslant n \leqslant 2^{k+1}-1} |a_n|, \qquad B_k := \max_{2^{k-\nu} \leqslant n \leqslant 2^{k+\nu}-1} |a_n|,$$

and

$$\alpha_k := \max_{2^k \leqslant n \leqslant 2^{k+1}-1} a_n^{\#} = \max_{2^k \leqslant n \leqslant 2^{k+1}-1} \max_{l \geqslant n} |a_l| = a_{2^k}^{\#}.$$

We say that an integer $l \geqslant 0$ is dominated by an integer $r \geqslant 0$ if $r \geqslant l+2$ and $\alpha_l = |a_i|$, where $2^r \leqslant i \leqslant 2^{r+1}-1$. Here and below, in the case of equality of several numbers with absolute values equal to α_l we take a_i to be the one with the smallest index. The integers not dominated by other integers will be called basic. The set of basic numbers is denoted by Ω. Moreover, the set of basic numbers such that $\alpha_k = A_k$ is denoted by Ω_1 while the set of basic numbers such that $\alpha_k = A_{k+1}$ is denoted by Ω_2. It is clear that

$$\Omega = \Omega_1 \sqcup \Omega_2.$$

If r is a basic number and the set Q_r of integers dominated by r is not empty, then $r \in \Omega_1$ and $Q_r = [k_0, r-2] \cap \mathbb{Z}$. We note that for any $l \in Q_r$ and any $n \in [2^l, 2^{l+1}-1] \cap \mathbb{Z}$ we have

$$a_n^{\#} = \alpha_{k_0} = \alpha_{k_0+1} = \cdots = \alpha_l = \cdots = \alpha_{r-1} = \alpha_r.$$

Furthermore, for $j \in [2^{r-1}, 2^r - 1] \cap \mathbb{Z}$ the equality $a_j^{\#} = \alpha_r$ holds.

Hence,

$$\sum_{l\in Q_r} \sum_{n=2^l}^{2^{l+1}-1} (a_n^{\#})^p n^{\alpha-1} = \alpha_r^p \sum_{l\in Q_r} \sum_{n=2^l}^{2^{l+1}-1} n^{\alpha-1} \leqslant \alpha_r^p \sum_{l=1}^{2^r} n^{\alpha-1}$$
$$\leqslant C(\alpha) \sum_{n=2^{r-1}}^{2^r-1} (a_n^{\#})^p n^{\alpha-1}. \tag{2.19}$$

A number $r \in \Omega$ such that

$$A_r \geqslant \frac{B_r}{2^{2(\alpha/p+1)\nu}}$$

is said to be good. Otherwise, r is said to be bad. We note that if a number $r \in \Omega_2$ is good, then $r+1 \in \Omega_1$ is also good.

Suppose that a good number k is in Ω_1. Then

$$\alpha_k = A_k = |a_i|, \quad \text{where} \quad 2^k \leqslant i \leqslant 2^{k+1}-1.$$

By assumption, $\{a_n\} \in \mathrm{WM}(\nu, D, p_0)$, which implies that

$$2^{-k} \sum_{n=2^{k-\nu}}^{2^{k+\nu}} |a_n|^{p_0} \geqslant \frac{\alpha_k^{p_0}}{D^{p_0}} = \frac{A_k^{p_0}}{D^{p_0}} \geqslant \frac{B_k^{p_0}}{D^{p_0}\, 2^{2(\alpha/p+1)\nu p_0}}\,.$$

Taking into account that all the numbers in the sum on the left are less than or equal to $B_k^{p_0}$ and that the number of terms in this sum is $2^{k+\nu} - 2^{k-\nu} + 1$, which with regard to order is comparable with 2^k, we conclude by Lemma 2.10 that if

$$S_k = \left\{ n \in [2^{k-\nu}, 2^{k+\nu}] \cap \mathbb{Z} \colon |a_n|^{p_0} > \frac{B_k^{p_0}}{2D^{p_0}\, 2^{2(\alpha/p+1)\nu p_0}} \right\},$$

then

$$|S_k| \geqslant \frac{2^k}{C_1}\,,$$

where the positive constant C_1 depends only on p, p_0, α, ν, and D. But then

$$\begin{aligned} \sum_{n=2^{k-\nu}}^{2^{k+\nu}} |a_n|^p n^{\alpha-1} &\geqslant \sum_{n\in S_k} |a_n|^p n^{\alpha-1} \geqslant \frac{B_k^p}{C_2} \min\{2^{(\alpha-1)(k-\nu)}, 2^{(\alpha-1)(k+\nu)}\}\, \frac{2^k}{C_1} \\ &\geqslant C_3 \sum_{n=2^k}^{2^{k+1}-1} (a_n^{\#})^p n^{\alpha-1}, \end{aligned} \tag{2.20}$$

where the positive constants C_2 and C_3 depend only on p, p_0, α, ν, and D. At the same time, if $k-1 \in \Omega_2$, then

$$\begin{aligned} \sum_{n=2^{k-1}}^{2^k-1} (a_n^{\#})^p n^{\alpha-1} &= \alpha_k^p \sum_{n=2^{k-1}}^{2^k-1} n^{\alpha-1} \leqslant C(\alpha)\alpha_k^p \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha-1} \\ &\leqslant C(\alpha, p_0, \nu, D) \sum_{n=2^{k-\nu}}^{2^{k+\nu}} |a_n|^p n^{\alpha-1}. \end{aligned} \tag{2.21}$$

Now consider the case when a bad number l_0 is in Ω_1. Then

$$A_{l_0} \leqslant \frac{B_{l_0}}{2^{2(\alpha/p+1)\nu}}\,.$$

We note that $B_{l_0} = A_{l_1}$, $l_1 < l_0$, and $l_1 \in \Omega_1$. If l_1 is a good number, then we finish our construction. Otherwise we have

$$A_{l_1} \leqslant \frac{B_{l_1}}{2^{2(\alpha/p+1)\nu}} \equiv \frac{A_{l_2}}{2^{2(\alpha/p+1)\nu}}$$

and $l_2 < l_1$, $l_2 \in \Omega_1$. Continuing this process, we arrive at a finite sequence $l_0 > l_1 > l_2 > \cdots > l_{j_s}$, where $k, l_1, \ldots, l_{j_s} \in \Omega_1$, such that l_{j_s} is a good number and the rest are bad. Moreover, $l_r - l_{r+1} \leqslant 2\nu$ and

$$A_{l_r} \leqslant \frac{A_{l_{r+1}}}{2^{2(\alpha/p+1)\nu}}$$

for any r. Thus, any integer $k_0 \in \Omega_1$ generates a finite or infinite sequence $k_0 < k_1 < k_2 < \cdots$, where all the k_i are in Ω_1, the k_i are bad numbers for $i \geqslant 1$, and

$$A_{k_i} \leqslant \frac{A_{k_{i-1}}}{2^{2(\alpha/p+1)}} \quad \text{and} \quad k_i - k_{i+1} \leqslant 2\nu,$$

for $i \geqslant 1$. But since $\sum_{n=2^{k-1}}^{2^{k+1}-1} n^{\alpha-1} \asymp 2^{k\alpha}$, we obtain

$$\begin{aligned} \sum_{i\geqslant 1} \sum_{n=2^{k_i}}^{2^{k_i+1}-1} (a_n^{\#})^p n^{\alpha-1} &\leqslant \sum_{i\geqslant 1} A_{k_i}^p \sum_{n=2^{k_i-1}}^{2^{k_i+1}-1} n^{\alpha-1} \leqslant C(\alpha) A_{k_0}^p \sum_{i\geqslant 1} \frac{2^{\alpha k_i}}{2^{2(\alpha/p+1)\nu i p}} \\ &\leqslant C(\alpha) A_{k_0}^p\, 2^{\alpha k_0} \sum_{i\geqslant 1} \frac{2^{2\alpha\nu i}}{2^{2(\alpha/p+1)\nu i p}} \leqslant C_5 A_{k_0}^p \sum_{n=2^{k_0}}^{2^{k_0+1}-1} n^{\alpha-1}, \end{aligned} \tag{2.22}$$

where the positive constant C_5 depends only on p, α, and ν. Similarly,

$$\sum_{i\geqslant 1:\ k_i-1\in\Omega_2} \sum_{n=2^{k_i-1}}^{2^{k_i}-1} (a_n^{\#})^p n^{\alpha-1} \leqslant \sum_{i\geqslant 1} A_{k_i}^p \sum_{n=2^{k_i-1}}^{2^{k_i+1}-1} n^{\alpha-1} \leqslant C_5 A_{k_0}^p \sum_{n=2^{k_0}}^{2^{k_0+1}-1} n^{\alpha-1}. \tag{2.23}$$

Using the inequality (2.19), we obtain

$$\sum_{n=1}^{\infty} (a_n^{\#})^p n^{\alpha-1} = \sum_{k=0}^{\infty} \sum_{n=2^k}^{2^{k+1}-1} (a_n^{\#})^p n^{\alpha-1} \leqslant C(\alpha) \sum_{k\in\Omega} \sum_{n=2^k}^{2^{k+1}-1} (a_n^{\#})^p n^{\alpha-1}.$$

For simplicity, we denote by $\Omega_{1,+}$ the set of good numbers in Ω_1. Applying (2.22) and (2.23), we continue the estimate as follows:

$$\begin{aligned} &\leqslant C \sum_{k\in\Omega_{1,+}} A_k^p \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha-1} + C \sum_{k\in\Omega_2:k+1\in\Omega_{1,+}} A_{k+1}^p \sum_{n=2^k}^{2^{k+1}-1} n^{\alpha-1} \\ &\leqslant C \sum_{k\in\Omega_{1,+}} \sum_{n=2^{k-\nu}}^{2^{k+\nu}} |a_n|^p n^{\alpha-1}, \end{aligned}$$

where the last inequality follows from (2.19)–(2.21). Finally,

$$\sum_{n=1}^{\infty}(a_n^{\#})^p n^{\alpha-1} \leqslant C(p,\alpha,p_0,D,\nu)\sum_{n=1}^{\infty}|a_n|^p n^{\alpha-1}. \qquad \square$$

Corollary 2.11. *Let $\{a_n\}_{n=1}^{\infty}$ be a null sequence of type* WM(p_0) *for some $p_0 \geqslant 1$. Then for any $\gamma \in (-\infty,\infty)$ and $p \in (0,\infty)$*

$$\sum_{n=1}^{\infty} 2^{n\gamma} M_n^p \asymp \sum_{n=1}^{\infty} n^{\gamma-1}|a_n|^p.$$

Proof. Theorem 2.9 implies that

$$\sum_{n=1}^{\infty} 2^n M_n^p \leqslant C\sum_{n=1}^{\infty} 2^n (a_{2^n}^{\#})^p \leqslant C\sum_{n=1}^{\infty}(a_n^{\#})^p \leqslant C\sum_{n=1}^{\infty}|a_n|^p.$$

Since the sequence $\{n^{(\gamma-1)/p}a_n\}_{n=1}^{\infty}$ is of type WM(p_0), application to it of the previous estimate gives us the desired result. $\square$

2.7. Equivalence of series with a_n and a_n^*.

Theorem 2.12. *Let $\{a_n\}_{n=1}^{\infty}$ be a null sequence of type* WM(p_0) *for some $p_0 \geqslant 1$. Then*

$$\sum_{n=1}^{\infty}|a_n|^p n^{\alpha-1} \asymp \sum_{n=1}^{\infty}(a_n^*)^p n^{\alpha-1}$$

for $p,\alpha \in (0,\infty)$.

Remark 2.13. Using the Hardy–Littlewood inequality for rearrangements [8], for any sequence we have $\sum_{n=1}^{\infty}|a_n|^p n^{\alpha-1} \leqslant \sum_{n=1}^{\infty}(a_n^*)^p n^{\alpha-1}$ for $0<\alpha\leqslant 1$, and we have the reverse inequality for $\alpha \geqslant 1$.

Proof of Theorem 2.12. Let $\alpha>1$ and $\{a_n\}_{n=1}^{\infty} \in$ GM(p_0). Let $\lambda = 2^{\nu}$ be as in the definition of WM(p_0). For any $n \geqslant 0$ we define

$$A_n = \max_{k\in[2^n,2^{n+1}-1]}|a_k| \quad \text{and} \quad B_n = \max_{k\in[2^{n-\nu},2^{n+\nu+1}-1]}|a_k|.$$

As above, an integer n is said to be good if either it is sufficiently small or $B_n \leqslant C_1 A_n$ (here C_1 does not depend on n). We proved in Theorem 2.9 that if Ω is the set of good numbers n, then

$$\sum_{k=1}^{\infty} k^{\alpha-1}|a_k|^p = \sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1} k^{\alpha-1}|a_k|^p \leqslant C_2\sum_{n\in\Omega}\sum_{k=2^n}^{2^{n+1}-1} k^{\alpha-1}|a_k|^p =: C_2\sum_{n\in\Omega} I_n. \tag{2.24}$$

Note that

$$I_n \leqslant A_n^p\, 2^{(n+1)\alpha}. \tag{2.25}$$

Moreover, we established that there exist $C_3, C_4 > 0$ such that for any $n \in \Omega$ there is a set of integers $T_n \subset [2^{n-\nu}, 2^{n+\nu}]$ containing at least $C_3\, 2^n$ elements and such that $|a_k| > C_4 A_n$ for $k \in T_n$.

It is also clear that any fixed l is contained in at most $2\nu+2$ different sets T_n.

Fix some $n \in \Omega$. Let $l(k)$ be the index of the kth term of the non-increasing rearrangement of the original sequence. Also, let r_n be the cardinality of T_n. Then since $\alpha > 1$, we obtain

$$\sum_{k\in T_n} l(k)^{\alpha-1}|a_k|^p \geqslant C_4^p A_n^p \sum_{k\in T_n} l(k)^{\alpha-1} \geqslant C_4^p A_n^p \sum_{q=1}^{r_n} q^{\alpha-1}$$
$$\geqslant C_5 A_n^p r_n^\alpha \geqslant C_6 A_n^p\, 2^{n\alpha}. \tag{2.26}$$

From (2.25) and (2.26) it follows that

$$\sum_{n\in\Omega} I_n \leqslant C_7 \sum_{n\in\Omega}\sum_{k\in T_n} l(k)^{\alpha-1}|a_k|^p \leqslant C_7(2\nu+2)\sum_{k=1}^{\infty} l(k)^{\alpha-1}|a_k|^p.$$

This and (2.24) imply that $\sum_{n=1}^\infty |a_n|^p n^{\alpha-1} \leqslant C\sum_{n=1}^\infty (a_n^*)^p n^{\alpha-1}$ for $\alpha>1$.

For $\alpha=1$ the required estimate is trivial, while for $0<\alpha<1$ it follows from Remark 2.13.

The reverse inequality follows from Theorem 2.9 and the property $a_n^* \leqslant a_n^\#$ for any n. $\square$

3. Estimates of Fourier coefficients

3.1. Lemma on a local and a global majorant. In this subsection we prove a lemma which we use in the next two subsections.

Lemma 3.1. *Let a sequence $a=\{a_k\}_{k=1}^\infty$ of complex numbers be bounded and let $M_n = \max_{k=2^n,\dots,2^{n+1}}|a_k|$ for $n\geqslant 0$. Let the positive sequence $\{\beta_k\}_{k=1}^\infty$ and the positive numbers γ and K be such that the sequence $\{k^{-\gamma}\beta_k\}_{k=1}^\infty$ is non-increasing while the sequence $\{k^{\gamma}\beta_k\}_{k=1}^\infty$ is non-decreasing. Suppose that for some positive integer m*

$$|a_k| \leqslant K\beta_k \quad \text{for all } k=1,\dots,2^m \tag{3.1}$$

and for any positive integer $n\geqslant m$ satisfying the condition

$$\max_{k=n-m,\dots,n+m} M_k \leqslant 2^{m\gamma} M_n \tag{3.2}$$

the estimate

$$|a_k| \leqslant K\beta_k \quad \text{for all } k=2^n,\dots,2^{n+1} \tag{3.3}$$

holds. Then

$$|a_k| \leqslant 2^\gamma K\beta_k \quad \text{for all } k=1,2,\dots\,. \tag{3.4}$$

Proof. Let $Y_n = \max_{k=2^n,\dots,2^{n+1}}\beta_k$ for $n\geqslant 0$. We show that

$$2^{-\gamma}Y_n \leqslant \min_{k=2^n,\dots,2^{n+1}}\beta_k \leqslant Y_{n+1} \leqslant 2^\gamma Y_n \quad \text{for all } n\geqslant 0. \tag{3.5}$$

Indeed, if $2^n \leqslant q \leqslant 2^{n+1}$, then $k^\gamma\beta_k \leqslant q^\gamma\beta_q$ for $k=2^n,\dots,q$ and $k^{-\gamma}\beta_k \leqslant q^{-\gamma}\beta_q$ for $k=q,\dots,2^{n+1}$. Therefore,

$$\max_{k=2^n,\dots,q}\beta_k \leqslant \left(\frac{q}{2^n}\right)^\gamma \beta_q \quad\text{and}\quad \max_{k=q,\dots,2^{n+1}}\beta_k \leqslant \left(\frac{2^{n+1}}{q}\right)^\gamma\beta_q.$$

Thus, $Y_n \leqslant 2^\gamma \beta_q$, which yields the first inequality in (3.5). The second is clear since $\beta_{2^{n+1}} \leqslant Y_{n+1}$. Replacing n by $n+1$ in the first inequality in (3.5), we obtain $2^{-\gamma} Y_{n+1} \leqslant \beta_{2^{n+1}} \leqslant Y_n$. This completes the proof of (3.5).

Now we prove that

$$M_n \leqslant K Y_n \quad \text{for all } n \geqslant 0. \tag{3.6}$$

For $n = 0, \dots, m-1$, the estimates (3.6) hold due to (3.1). Note that according to (3.3) the condition (3.2) implies (3.6). Suppose that (3.6) does not hold. Then there exists a smallest positive integer n_0 such that $M_{n_0} > K Y_{n_0}$. It follows from the argument above that $n_0 \geqslant m$ and $2^{m\gamma} M_{n_0} < \max_{k=n_0-m,\dots,n_0+m} M_k$. There is a positive integer n_1 such that $n_0 - m \leqslant n_1 \leqslant n_0 + m$, $M_{n_1} = \max_{k=n_0-m,\dots,n_0+m} M_k$, and

$$M_n < M_{n_1} \quad \text{for all } n_0 - m \leqslant n < n_1. \tag{3.7}$$

Since $M_{n_1} > 2^{m\gamma} M_{n_0}$, we have $n_1 \neq n_0$, and using (3.5) for all $s = 1, \dots, m$, we conclude that

$$M_{n_0-s} \leqslant K Y_{n_0-s} \leqslant 2^{s\gamma} K Y_{n_0} \leqslant 2^{m\gamma} K Y_{n_0} < 2^{m\gamma} M_{n_0} < M_{n_1}.$$

Therefore, $n_1 > n_0$, and by (3.5)

$$K Y_{n_1} \leqslant 2^{(n_1-n_0)\gamma} K Y_{n_0} \leqslant 2^{m\gamma} K Y_{n_0} < 2^{m\gamma} M_{n_0} < M_{n_1}.$$

Finally, $n_0 + m \geqslant n_1 > n_0$, $K Y_{n_1} < 2^{m\gamma} M_{n_0} < M_{n_1}$, and (3.7) holds.

There exists a positive integer n_2 such that $n_1 - m \leqslant n_2 \leqslant n_1 + m$, $M_{n_2} = \max_{k=n_1-m,\dots,n_1+m} M_k$, and $M_n < M_{n_2}$ for all $n_1 - m \leqslant n < n_2$. The fact that the condition (3.6) does not hold for $n = n_1$ implies that (3.2) also does not hold, that is, $M_{n_2} > 2^{m\gamma} M_{n_1}$. In view of (3.7) we have $n_1 + m \geqslant n_2 > n_1$. Therefore, by (3.5)

$$K Y_{n_2} \leqslant 2^{(n_2-n_1)\gamma} K Y_{n_1} \leqslant 2^{m\gamma} K Y_{n_1} < 2^{m\gamma} M_{n_1} < M_{n_2}.$$

Thus, the previous conditions were repeated with n_1, n_2 in place of n_0, n_1. Repeating the above construction, we obtain a sequence of positive integers $\{n_k\}_{k=0}^\infty$ such that $n_k + m \geqslant n_{k+1} > n_k$ and $K Y_{n_{k+1}} < 2^{m\gamma} M_{n_k} < M_{n_{k+1}}$ for all $k \geqslant 0$. This leads us to a contradiction with the boundedness of $\{a_k\}_{k=1}^\infty$, so (3.6) is proved.

If $n \geqslant m$ and $2^n \leqslant k \leqslant 2^{n+1}$, then (3.6) together with (3.5) give us that $|a_k| \leqslant M_n \leqslant K Y_n \leqslant K 2^\gamma \beta_k$, that is, the estimate (3.4) is proved, and with it Lemma 3.1. □

We note that the idea of the proof of Lemma 3.1 was also used in [36], though in a slightly different form.

3.2. Estimates for Fourier coefficients in the general case. As usual, for a function $f \in L_1(\mathbb{T})$,

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt$$

are its Fourier coefficients, and

$$S_n(f, x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t)\, dt, \qquad n \geqslant 0,$$

are the partial sums of its Fourier series, where $D_n(t)$ is the Dirichlet kernel. For all integers $0 \leqslant n_1 \leqslant n_2$ let

$$V_{n_1,n_2}(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)K_{n_1,n_2}(t)\,dt = \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} S_n(f,x)$$

denote the de la Vallée-Poussin sums, let

$$K_{n_1,n_2}(t) = \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} D_n(t),$$

denote the de la Vallée-Poussin kernel, and let $\sigma_n(f,x) = V_{0,n}(f,x)$ denote the Fejér means. Since

$$V_{n_1,n_2}(f,0) = \sum_{k=-n_2}^{n_2} c_k(f) - \sum_{k=n_1+1}^{n_2} \frac{k-n_1}{n_2-n_1+1}(c_k(f)+c_{-k}(f)),$$

we have

$$\begin{aligned}\Bigl|\sum_{k=-n_2}^{n_2} c_k(f)\Bigr| - \sum_{k=n_1+1}^{n_2} \frac{k-n_1}{n_2-n_1+1}\bigl(|c_k(f)|+|c_{-k}(f)|\bigr)\\ \leqslant |V_{n_1,n_2}(f,0)| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|\,|K_{n_1,n_2}(t)|\,dt \qquad (3.8)\end{aligned}$$

and

$$\begin{aligned}|V_{n_1,n_2}(f,0)| &= \frac{1}{n_2-n_1+1}\,|(n_2+1)\sigma_{n_2}(f,0) - n_1\sigma_{n_1-1}(f,0)|\\ &\leqslant \frac{n_2+1}{n_2-n_1+1}\,|\sigma_{n_2}(f,0)| + \frac{n_1}{n_2-n_1+1}\,|\sigma_{n_1-1}(f,0)|. \qquad (3.9)\end{aligned}$$

Note that

$$\sum_{k=n_1+1}^{n_2} \frac{k-n_1}{n_2-n_1+1}\bigl(|c_k(f)|+|c_{-k}(f)|\bigr) \leqslant \max_{k=-n_2,\dots,n_2} |c_k(f)|\cdot(n_2-n_1).$$

It is known that

$$|K_{n_1,n_2}(t)| \leqslant \frac{1}{2} + \frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} n = \frac{n_2+n_1+1}{2} \quad \text{for all } t.$$

Since for the Fejér kernel we have

$$K_{0,n}(t) = \frac{\sin^2((n+1)t/2)}{2(n+1)\sin^2(t/2)} \leqslant \frac{1}{2(n+1)\sin^2(t/2)}\,,$$

we get that

$$\begin{aligned}|K_{n_1,n_2}(t)| &= \frac{1}{n_2-n_1+1}\,|(n_2+1)K_{0,n_2}(t) - n_1K_{0,n_1-1}(t)|\\ &\leqslant \frac{1}{(n_2-n_1+1)\,2\sin^2(t/2)}\\ &\leqslant \frac{\pi^2}{(n_2-n_1+1)\,2t^2} \quad \text{for all } |t|\in(0,\pi].\end{aligned}$$

Therefore, for any function $f \in L_1(\mathbb{T})$, from (3.8) we have

$$\Big|\sum_{k=-n_2}^{n_2} c_k(f)\Big| - (n_2 - n_1) \max_{k=-n_2,\dots,n_2} |c_k(f)| \\ \leqslant \frac{1}{2\pi}\int_0^\pi \big(|f(t)| + |f(-t)|\big) \min\Big\{n_2 + n_1 + 1, \frac{\pi^2}{(n_2 - n_1 + 1)t^2}\Big\}\,dt. \qquad (3.10)$$

Since $|\sigma_n(f,0)| \leqslant \|f\|_\infty$, it follows from (3.8) and (3.9) that for any $f \in C(\mathbb{T})$

$$\Big|\sum_{k=-n_2}^{n_2} c_k(f)\Big| - (n_2 - n_1) \max_{k=-n_2,\dots,n_2} |c_k(f)| \leqslant \frac{n_2 + n_1 + 1}{n_2 - n_1 + 1}\|f\|_\infty. \qquad (3.11)$$

Lemma 3.2. *Let m_1 and m_2 be positive integers with $m_1 < m_2$ and let a non-negative integer s be such that $m_2 - m_1$ is even and $2s \leqslant m_2 - m_1 - 2$. Then:*

(a) *for any $f \in L_1(\mathbb{T})$*

$$\Big|\sum_{k=m_1}^{m_2} c_k(f)\Big| - s \max_{k=m_1,\dots,m_2} |c_k(f)| \\ \leqslant \frac{1}{2\pi}\int_0^\pi \big(|f(t)| + |f(-t)|\big) \min\Big\{m_2 - m_1 + 1 - s, \frac{\pi^2}{(s+1)t^2}\Big\}\,dt; \qquad (3.12)$$

(b) *for any $f \in C(\mathbb{T})$*

$$\Big|\sum_{k=m_1}^{m_2} c_k(f)\Big| - s \max_{k=m_1,\dots,m_2} |c_k(f)| \leqslant \frac{m_2 - m_1 + 1 - s}{s+1} E_{m_1-1}(f)_\infty. \qquad (3.13)$$

Proof. Let $n_1 = (m_2 - m_1 - 2s)/2$, $n_2 = (m_2 - m_1)/2$, and $q = (m_2 + m_1)/2$. Applying (3.10) and (3.11) to the function $f(t)e^{-iqt}$, we get that

$$\Big|\sum_{k=-n_2}^{n_2} c_{k+q}(f)\Big| - s \max_{k=-n_2,\dots,n_2} |c_{k+q}(f)| \\ \leqslant \frac{1}{2\pi}\int_0^\pi \big(|f(t)| + |f(-t)|\big) \min\Big\{m_2 - m_1 - s + 1, \frac{\pi^2}{(s+1)t^2}\Big\}\,dt,$$

so that (3.12) is proved. Similarly, the inequality (3.11) implies that

$$\Big|\sum_{k=m_1}^{m_2} c_k(f)\Big| - s \max_{k=m_1,\dots,m_2} |c_k(f)| \leqslant \frac{n_2 + n_1 + 1}{n_2 - n_1 + 1}\|f\|_\infty = \frac{m_2 - m_1 + 1 - s}{s+1}\|f\|_\infty.$$

The last inequality contains an arbitrary $f \in C(\mathbb{T})$, while its left-hand side involves only the coefficients $\{c_k(f)\}_{k=m_1}^{m_2}$. Let us change the coefficients $\{c_k(f)\}_{k=-(m_1-1)}^{m_1-1}$ of f in such a way that $\|f\|_\infty = E_{m_1-1}(f)_\infty$. Then we get (3.13). $\square$.

3.3. Estimates for Fourier coefficients of type GM(p). The following two theorems are the main results of this subsection.

Theorem 3.3. *The following estimate holds for a function $f \in L_1(\mathbb{T})$ with Fourier series of the form (1.2) and with coefficients $\{a_n\}_{n=1}^{\infty}$ of type* $\mathrm{GM}(\nu, D, p_0)$:

$$|a_n| \leqslant a_n^{\#} \leqslant C_1\biggl(\int_0^{\pi/n} |f(t)|\,dt + \frac{\pi^2}{n^2}\int_{\pi/n}^{\pi} \frac{|f(t)|}{t^2}\,dt\biggr) \tag{3.14}$$

for all $n \geqslant 1$, where $C_1 > 0$ depends only on ν, D, and p_0.

Theorem 3.4. *For any $q > 0$ there exists a positive constant C_2 depending only on q and the parameters ν, D, p_0, such that the following condition holds. If a function $f \in C(\mathbb{T})$ has a Fourier series expansion of the form (1.2) with coefficients $\{a_n\}_{n=1}^{\infty}$ of type* $\mathrm{GM}(\nu, D, p_0)$, *then the Fourier series of f converges uniformly, and for any positive integer n*

$$n|a_n| \leqslant n a_n^{\#} \leqslant C_2 n^{-q} \max_{1\leqslant k\leqslant n} k^q E_{k-1}(f)_\infty, \tag{3.15}$$

where $E_{k-1}(f)_\infty$ is the best approximation of f by trigonometric polynomials of degree less than k in the space $C(\mathbb{T})$.

Proof of Theorem 3.3. The complex sequence $a = \{a_n\}_{n=1}^{\infty}$ is of type $\mathrm{GM}(\nu, D, p_0)$, so Theorem 2.5 implies the conditions (2.10) and (2.11).

Let $m = 2\nu$. Suppose that the condition (3.2) is satisfied for some $n \geqslant m$ and some $\gamma > 0$, that is,

$$2^{m\gamma} M_n \geqslant \max_{k=n-m,\dots,n+m} M_k = \max_{k=2^{n-m},\dots,2^{n+m+1}} |a_k|. \tag{3.16}$$

First assume that $M_n > 0$. Note that

$$\begin{aligned}\biggl(\sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^{p_0}\biggr)^{1/p_0} &\leqslant \biggl(\sum_{k=2^{n-\nu}}^{2^{n+\nu}} (|\operatorname{Re} a_k| + |\operatorname{Im} a_k|)^{p_0}\biggr)^{1/p_0} \\ &\leqslant \biggl(\sum_{k=2^{n-\nu}}^{2^{n+\nu}} |\operatorname{Re} a_k|^{p_0}\biggr)^{1/p_0} + \biggl(\sum_{k=2^{n-\nu}}^{2^{n+\nu}} |\operatorname{Im} a_k|^{p_0}\biggr)^{1/p_0}.\end{aligned}$$

Denote by $\{b_k\}_{k=2^{n-\nu}}^{2^{n+\nu}}$ any of the sequences $\{\operatorname{Re} a_k\}_{k=2^{n-\nu}}^{2^{n+\nu}}$ or $\{\operatorname{Im} a_k\}_{k=2^{n-\nu}}^{2^{n+\nu}}$ for which

$$\biggl(\sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^{p_0}\biggr)^{1/p_0} \leqslant 2\biggl(\sum_{k=2^{n-\nu}}^{2^{n+\nu}} |b_k|^{p_0}\biggr)^{1/p_0}.$$

Further, we assume that the first non-zero term of the sequence $\{b_k\}_{k=2^{n-\nu}}^{2^{n+\nu}}$ is positive, otherwise we can replace b_k by $-b_k$ for all k. Let

$$\varepsilon = \frac{1}{2D_1}\,2^{-(\nu+1)/p_0} \tag{3.17}$$

and note that $\varepsilon > 0$ depends only on ν, D_1, p_0 and that $\varepsilon \leqslant 1/2$. Let

$$E_n = \{k \in \{2^{n-\nu}, \dots, 2^{n+\nu}\} \colon |b_k| \geqslant \varepsilon M_n\},$$

and as usual, let $|E_n|$ denote the cardinality of the set E_n. Then from (2.10), (3.16), and (3.17) we get that

$$2^n \left(\frac{M_n}{D_1}\right)^{p_0} \leqslant 2^{p_0} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |b_k|^{p_0} \leqslant 2^{p_0} \sum_{k \in E_n} |a_k|^{p_0} + 2^{n+\nu}(2\varepsilon M_n)^{p_0}$$
$$\leqslant (2 \cdot 2^{m\gamma} M_n)^{p_0} |E_n| + \frac{1}{2}\, 2^n \left(\frac{M_n}{D_1}\right)^{p_0}.$$

Thus,

$$|E_n| \geqslant 4D_3\, 2^n, \tag{3.18}$$

where $D_3 = (2D_1\, 2^{m\gamma})^{-p_0}/8$ is a positive constant depending only on ν, D_1, p_0, and γ. We consider a partition

$$2^{n-\nu} = j_0 < j_1 < \dots < j_{\tau_n} = 2^{n+\nu} + 1$$

such that if $I_l = \{j_{l-1}, \dots, j_l - 1\}$, then for $k \in I_l$ we have $(-1)^{l-1} b_k \geqslant 0$ and

$$\max_{k \in I_l} |b_k| > 0 \quad \text{for all } l = 1, \dots, \tau_n.$$

Let

$$L_n = \Big\{ l = 1, \dots, \tau_n \colon \max_{k \in I_l} |b_k| \geqslant \varepsilon M_n \Big\},$$

and let $|L_n|$ denote the cardinality of L_n, as usual. For each $l \in \{1, \dots, \tau_n\}$ we find a number $k(l) \in I_l$ such that $|b_{k(l)}| = \max_{k \in I_l} |b_k|$. Consider two successive elements $l_1 < l_2$ of the set L_n arranged in increasing order. If $l_2 - l_1$ is odd, then $b_{k(l_1)}$ and $b_{k(l_2)}$ have opposite signs and

$$|a_{k(l_1)} - a_{k(l_2)}| \geqslant |b_{k(l_1)} - b_{k(l_2)}| = |b_{k(l_1)}| + |b_{k(l_2)}| \geqslant 2\varepsilon M_n.$$

Otherwise, if $l_2 - l_1$ is even, then we call the integer $k(l_1 + 1)$ additional. Thus,

$$|a_{k(l_1)} - a_{k(l_1+1)}| \geqslant |b_{k(l_1)} - b_{k(l_1+1)}| = |b_{k(l_1)}| + |b_{k(l_1+1)}| > \varepsilon M_n$$

and

$$|a_{k(l_1+1)} - a_{k(l_2)}| \geqslant |b_{k(l_1+1)} - b_{k(l_2)}| = |b_{k(l_1+1)}| + |b_{k(l_2)}| > \varepsilon M_n.$$

We apply a similar procedure to all pairs of successive elements in L_n. Further, we consider all the numbers $\{k(l) \colon l \in L_n\}$ together with the additional numbers and enumerate them in increasing order as $k_1, \dots, k_\tau$. Then

$$\tau \geqslant |L_n|, \quad 2^{n-\nu} \leqslant k_1 < \dots < k_\tau \leqslant 2^{n+\nu}, \tag{3.19}$$

the signs of non-zero terms $b_{k_1}, \dots, b_{k_\tau}$ alternate, and

$$|a_{k_j} - a_{k_{j+1}}| \geqslant |b_{k_j} - b_{k_{j+1}}| > \varepsilon M_n \quad \text{for all } j = 1, \dots, \tau - 1. \tag{3.20}$$

At this point we use the condition (2.11) or, as in the next subsection, a certain modification of (2.11). Since $n \geqslant m = 2\nu$, we deduce from (2.11), (3.19), (3.20), and (3.16) that

$$(\tau-1)\varepsilon M_n \leqslant \sum_{j=1}^{\tau-1} |a_{k_j} - a_{k_{j+1}}| \leqslant \sum_{j=1-\nu}^{\nu} \sum_{k=2^{n-j}}^{2^{n-j+1}} |a_k - a_{k+1}|$$
$$\leqslant 2\nu D_2 \max_{k=2^{n-2\nu},\dots,2^{n+2\nu}} |a_k| \leqslant D_2 m\, 2^{m\gamma} M_n, \tag{3.21}$$

and hence

$$\tau \leqslant G, \tag{3.22}$$

where the constant $G = 1 + D_2 m 2^{m\gamma}/\varepsilon \geqslant 2$ depends only on ν, D_1, p_0, D_2, and γ, but, importantly, not on n. Let N be the smallest positive integer such that

$$N \geqslant m = 2\nu \quad \text{and} \quad 2^N \geqslant \frac{6G}{D_3}. \tag{3.23}$$

Note that N depends only on ν, D_1, p_0, G, and γ. Assume that

$$n \geqslant N. \tag{3.24}$$

Then

$$n \geqslant m = 2\nu \quad \text{and} \quad 2^n \geqslant \frac{6G}{D_3}. \tag{3.25}$$

Since $|E_n| = \sum_{l\in L_n} |E_n \cap I_l|$, we can find an $l \in L_n$ such that in view of (3.19), (3.22), (3.18), and (3.25) we have

$$|E_n \cap I_l| \geqslant \frac{|E_n|}{|L_n|} \geqslant \frac{|E_n|}{\tau} \geqslant \frac{|E_n|}{G} \geqslant \frac{4D_3}{G} 2^n \geqslant 24. \tag{3.26}$$

From now on we fix such an $l \in L_n$. If $j_l - 1 - j_{l-1}$ is an even integer, then we set $m_1 = j_{l-1}$ and $m_2 = j_l - 1$. Otherwise, if $j_l - 1 - j_{l-1}$ is odd, then for $j_{l-1} \notin E_n$ we set $m_1 = j_{l-1} + 1$ and $m_2 = j_l - 1$, and for $j_l - 1 \notin E_n$ we set $m_1 = j_{l-1}$ and $m_2 = j_l - 2$. Thus, $m_2 - m_1$ is even, and by (3.26)

$$|E_n \cap \{m_1, \dots, m_2\}| = |E_n \cap I_l| \geqslant 24.$$

If $j_l - 1 - j_{l-1}$ is an odd integer and $j_{l-1} \in E_n$, $j_l - 1 \in E_n$, then we set $m_1 = j_{l-1} + 1$ and $m_2 = j_l - 1$. In this case

$$|E_n \cap \{m_1, \dots, m_2\}| = |E_n \cap I_l| - 1 \geqslant 23.$$

Thus, we have found numbers

$$2^{n-\nu} \leqslant j_{l-1} \leqslant m_1 < m_2 \leqslant j_l - 1 \leqslant 2^{n+\nu}$$

such that $m_2 - m_1$ is even,

$$m_2 - m_1 + 1 \geqslant |E_n \cap \{m_1, \dots, m_2\}| \geqslant 23,$$

and by (3.26) and (3.25)

$$m_2 - m_1 + 1 \geqslant |E_n \cap \{m_1, \dots, m_2\}| \geqslant |E_n \cap I_l| - 1 \geqslant \frac{4D_3}{G}\, 2^n - 1 \geqslant \frac{23D_3}{6G}\, 2^n.$$

In particular,

$$m_2 - m_1 - 2 \geqslant \frac{20}{23}(m_2 - m_1 + 1) \geqslant \frac{20D_3}{6G}\, 2^n.$$

Since $(-1)^{l-1} b_k \geqslant 0$ for $k = m_1, \dots, m_2$, it follows that

$$\left| \sum_{k=m_1}^{m_2} a_k \right| \geqslant \left| \sum_{k=m_1}^{m_2} b_k \right| = \sum_{k=m_1}^{m_2} |b_k| \geqslant |E_n \cap \{m_1, \dots, m_2\}| \varepsilon M_n \geqslant \frac{23D_3}{6G}\, 2^n \varepsilon M_n.$$

Let

$$s = \left[\frac{\varepsilon D_3\, 2^n}{G\, 2^{m\gamma}} \right]. \tag{3.27}$$

Next, by (3.16)

$$\left| \sum_{k=m_1}^{m_2} a_k \right| - s \max_{k=m_1,\dots,m_2} |a_k| \geqslant \frac{23\varepsilon D_3}{6G}\, 2^n M_n - s\, 2^{m\gamma} M_n \geqslant \frac{17\varepsilon D_3}{6G}\, 2^n M_n$$

and by (3.27)

$$2s \leqslant \frac{2\varepsilon D_3\, 2^n}{G\, 2^{m\gamma}} \leqslant \frac{D_3\, 2^n}{G} < m_2 - m_1 - 2.$$

Note that we have $|c_k(f)| = |a_k|/2$ for all $k \geqslant 1$ in view of (1.2). Therefore,

$$\left| \sum_{k=m_1}^{m_2} c_k(f) \right| - s \max_{k=m_1,\dots,m_2} |c_k(f)| \geqslant \frac{17\varepsilon D_3}{12G}\, 2^n M_n.$$

Thus, all the conditions of Lemma 3.2 are satisfied. Note that by (3.27)

$$s + 1 > \frac{\varepsilon D_3\, 2^n}{G\, 2^{m\gamma}}$$

and

$$m_2 - m_1 + 1 \leqslant 2^{n+\nu} - 2^{n-\nu} + 1 \leqslant 2^{n+\nu}.$$

Therefore, from (3.12) we have

$$\frac{17\varepsilon D_3}{12G}\, 2^n M_n \leqslant \frac{1}{\pi} \int_0^\pi |f(t)| \min\left\{ 2^{n+\nu}, \frac{G\, 2^{m\gamma} \pi^2}{\varepsilon D_3\, 2^n t^2} \right\} dt,$$

that is,

$$M_n \leqslant D_4 \frac{2}{\pi} \int_0^\pi |f(t)| \min\left\{ 1, \frac{\pi^2}{2^{2n} t^2} \right\} dt, \tag{3.28}$$

where the positive constant

$$D_4 = \frac{6G}{17\varepsilon D_3} \max\left\{ 2^\nu, \frac{G\, 2^{m\gamma}}{\varepsilon D_3} \right\}$$

depends only on ν, D_1, p_0, G, and γ.

If $f \in C(\mathbb{T})$, then we get from (3.13) that

$$\frac{17\varepsilon D_3}{12G}\, 2^n M_n \leqslant \frac{2^\nu G\, 2^{m\gamma}}{\varepsilon D_3} E_{2^{n-\nu}-1}(f)_\infty,$$

so that

$$2^n M_n \leqslant D_5 E_{2^{n-\nu}-1}(f)_\infty, \tag{3.29}$$

where the positive constant

$$D_5 = \frac{12G}{17\varepsilon D_3}\,\frac{2^\nu G\, 2^{m\gamma}}{\varepsilon D_3}$$

depends only on ν, D_1, p_0, G, and γ.

We point out that the estimates (3.28) and (3.29) are valid when the conditions (3.16) and (3.22)–(3.24) hold, and they are clearly valid if $M_n = 0$.

Up to this point, the proofs of Theorems 3.3 and 3.4 are the same. Now we focus specifically on the proof of Theorem 3.3.

Let (1.2) be the Fourier series of a function $f \in L_1(\mathbb{T})$, $\|f\|_1 > 0$, and let

$$\beta_k = \frac{2}{\pi}\biggl(\int_0^{\pi/k} |f(t)|\, dt + \frac{\pi^2}{k^2}\int_{\pi/k}^{\pi} \frac{|f(t)|}{t^2}\, dt\biggr) = \frac{2}{\pi}\int_0^{\pi} |f(t)| \min\biggl\{1, \frac{\pi^2}{k^2 t^2}\biggr\}\, dt \tag{3.30}$$

for positive integers k. Then $\beta_k > 0$, $\beta_{k+1} \leqslant \beta_k$, and $(k+1)^2\beta_{k+1} \geqslant k^2\beta_k$ for all $k = 1, 2, \dots$. Assume that $\gamma = 2$ in (3.16) and further on. It follows from (3.14) that

$$|a_k| \leqslant \frac{1}{\pi}\int_{-\pi}^{\pi} |f(t)|\, dt = \frac{2}{\pi}\int_0^{\pi} |f(t)|\, dt \leqslant k^2\beta_k \quad \text{for all } k = 1, 2, \dots . \tag{3.31}$$

In particular,

$$|a_k| \leqslant 2^{2N}\beta_k \quad \text{for all } k = 1, 2, \dots, 2^N. \tag{3.32}$$

If $n \geqslant N$, in other words, if (3.24) holds, then under the condition (3.16) we have (3.28), that is, $M_n \leqslant D_4\beta_{2^n}$. Hence, for $k = 2^n, \dots, 2^{n+1}$ we have

$$|a_k| \leqslant M_n \leqslant D_4\beta_{2^n} \leqslant D_4(k^2\, 2^{-2n})\beta_k \leqslant 4D_4\beta_k.$$

Thus,

$$|a_k| \leqslant 4D_4\beta_k \quad \text{for all } k \geqslant 2^N. \tag{3.33}$$

Let $K = \max\{2^{2N}, 4D_4\}$. Then (3.32) and (3.33) imply (3.1), and under the condition (3.2) the condition (3.3) also holds. By Lemma 3.1 the estimate (3.4) is valid, and hence in light of (3.30) also the estimate (3.14) with $C_1 = 8K/\pi$. The proof of Theorem 3.3 is complete. □

Proof of Theorem 3.4. Let (1.2) be the Fourier series of a non-constant function $f \in C(\mathbb{T})$ and let

$$\beta_k = k^{-q-1} \max_{1 \leqslant j \leqslant k} j^q E_{j-1}(f)_\infty \tag{3.34}$$

for positive integers k. Then

$$\beta_k > 0, \quad k\beta_k \geqslant E_{k-1}(f)_\infty \geqslant E_k(f)_\infty, \qquad (k+1)^{q+1}\beta_{k+1} \geqslant k^{q+1}\beta_k,$$

and

$$(k+1)\beta_{k+1} = \max_{1\leqslant j\leqslant k+1}\left(\frac{j}{k+1}\right)^q E_{j-1}(f)_\infty \leqslant \max\{k\beta_k, E_k(f)_\infty\} \leqslant k\beta_k$$

for all $k = 1, 2, \dots$. Assume that $\gamma = q+1$ in the relations (3.16) to (3.29). It follows from (1.2) that

$$|a_k| \leqslant 2E_{k-1}(f)_\infty \leqslant 2k\beta_k \quad \text{for all } k = 1, 2, \dots . \tag{3.35}$$

In particular,

$$|a_k| \leqslant 2^{N+1}\beta_k \quad \text{for all } k = 1, 2, \dots, 2^N. \tag{3.36}$$

If $n \geqslant N$, that is, the condition (3.24) holds, then under the condition (3.16) we have (3.29), that is, $2^{n(q+1)}M_n \leqslant D_5\, 2^{\nu q}\, 2^{(n-\nu)q}E_{2^{n-\nu}-1}(f)_\infty$. Hence, $M_n \leqslant D_5\, 2^{\nu q}\beta_{2^n}$. Thus, for $k = 2^n, \dots, 2^{n+1}$ we have

$$|a_k| \leqslant M_n \leqslant D_5\, 2^{\nu q}\beta_{2^n} \leqslant D_5\, 2^{\nu q}(k^{q+1}\, 2^{-(q+1)n})\beta_k \leqslant D_5\, 2^{\nu q}\, 2^{q+1}\beta_k.$$

Therefore,

$$|a_k| \leqslant D_5\, 2^{\nu q+q+1}\beta_k \quad \text{for all } k \geqslant 2^N. \tag{3.37}$$

Let $K = \max\{2^{N+1}, 2^{\nu q+q+1}D_5\}$. Then (3.36) and (3.37) imply the inequality (3.1), and in light of the condition (3.2), (3.3) holds as well. By Lemma 3.1, (3.4) is valid, which by (3.34) yields (3.15) with $C_2 = 2^{q+1}K$. It follows from (3.15) that $|a_n| = o(n^{-1})$. Therefore, the Fourier series of a continuous function f converges uniformly. In particular, it is convergent at $x = 0$. □

3.4. Lebesgue inequalities for Fourier coefficients.

Corollary 3.5. *If (1.2) is the Fourier expansion of a function $f \in C(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^\infty$ of type* GM(p), *$p \geqslant 1$, then for all positive integers n*

$$n|a_n| \leqslant na_n^{\#} \leqslant C\omega_\beta\left(f, \frac{\pi}{n}\right)_\infty. \tag{3.38}$$

The estimate (3.38) is a Lebesgue type inequality (for $\beta = 1$, see [36]).

Proof. By Jackson's inequality we have

$$E_{n-1}(f)_\infty \leqslant C_\beta\omega_\beta\left(f, \frac{\pi}{n}\right).$$

Therefore, we deduce from (3.15) that for $q = \beta$ and any positive integer n

$$n|a_n| \leqslant C_\beta n^{-\beta}\max_{1\leqslant k\leqslant n} k^\beta\omega_\beta\left(f, \frac{\pi}{k}\right) \leqslant C_\beta\omega_\beta\left(f, \frac{\pi}{n}\right),$$

so that the estimate (3.38) is valid. □

3.5. Approximation by partial sums of Fourier series. In [51] (see also [99], Chap. II, § 10) Lebesgue proved that for a function h in the Lipschitz class

$$\operatorname{Lip}\alpha = \bigl\{f \in C(\mathbb{T})\colon \omega(f,\delta)_C = O(\delta^\alpha)\bigr\}$$

one has

$$\|h - S_n(h)\|_{C(\mathbb{T})} = O\Bigl(\frac{\log n}{n^\alpha}\Bigr). \tag{3.39}$$

Here $\omega(f,\delta)_C = \sup_{|h|\leqslant\delta} \|f(\,\cdot+h) - f(\,\cdot\,)\|_C$ is the modulus of continuity of a function f in C. Salem and Zygmund [77] showed that the logarithm cannot be suppressed, even if in addition to the condition $h \in \operatorname{Lip}\alpha$ we assume that h is of bounded variation. However, they showed that if $h \in \operatorname{Lip}\alpha$ is of monotone type, then the logarithmic factor in (3.39) can be omitted.

Theorem 3.6 (see [77]). *Let h be a continuous function of monotone type, that is, there exists a real number K such that the function $h(x) + Kx$ is either non-increasing or non-decreasing on $(-\infty,\infty)$. Let $h \in \operatorname{Lip}\alpha$, where $0 < \alpha < 1$. Then*

$$\|h - S_n(h)\|_{C(\mathbb{T})} = O\biggl(\frac{1}{n^\alpha}\biggr). \tag{3.40}$$

We show that this estimate still holds for functions in $\operatorname{Lip}\alpha$ with coefficients $\{a_n\}_{n=1}^\infty$ of type $\mathrm{GM}(\nu, D, p_0)$.

Corollary 3.7. *If (1.2) is the Fourier expansion of a function $f \in C(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^\infty$ of type $\mathrm{GM}(p)$, $p \geqslant 1$, then the following conditions are equivalent for $\alpha > 0$:*

(i) $|a_n| = O\biggl(\dfrac{1}{n^{\alpha+1}}\biggr)$;

(ii) $\|f - S_n(f)\|_C = O\biggl(\dfrac{1}{n^\alpha}\biggr)$;

(iii) $E_n(f)_C = O\biggl(\dfrac{1}{n^\alpha}\biggr)$;

(iv) $f \in \operatorname{Lip}\alpha$, *where $\alpha < 1$ in the case of a sine series and $\alpha \leqslant 1$ in the case of a cosine series.*

Proof. Note that the inequality (3.15) implies that

$$n|a_n| \leqslant CE_{n-1}(f)_C \leqslant C\|f - S_{n-1}(f)\|_C \leqslant C\sum_{k=n}^\infty |a_k|.$$

Therefore, for any positive α, we have (i) ⇔ (ii) ⇔ (iii). By Jackson's inequality, (iv) ⇒ (iii). The relation (i) ⇒ (iv) follows in the same way as the estimates in Theorem 2.2 of [36]. □

Remark 3.8. 1. For the series

$$f(x) = \sum_k k^{-2}\sin(kx),$$

one has

$$n|a_n| \leqslant CE_n(f)_C \leqslant \|f - S_n(f)\|_C = O(n^{-1})$$

but $f \notin \operatorname{Lip}1$. This shows the sharpness of the conditions of Corollary 3.7.

2. For monotone coefficients, see [9] and [60]. The result presented gives a significant improvement of results in the paper [29].

3. One can obtain similar results for moduli of smoothness of higher order (see [36]) and for the spaces

$$\operatorname{Lip}\alpha = \bigl\{f \in C(\mathbb{T})\colon \omega(f,\delta)_C = o(\delta^\alpha)\bigr\}.$$

3.6. Estimates for Fourier coefficients under certain conditions involving constant signs. Let $b = \{b_n\}_{n=1}^\infty$ be a sequence of real numbers and let $n_1 \leqslant n_2$ be positive integers. We discard all the zeros from $\{b_n\}_{n=n_1}^{n_2}$ and then group together all successive numbers with the same sign. We define the number of groups obtained by $\mathrm{SC}(n_1,n_2)$. Thus, $\mathrm{SC}(n_1,n_2)-1$ is the number of sign changes in the sequence $\{b_n\}_{n=n_1}^{n_2}$.

Definition 3.9. For a positive integer ξ, we say that a sequence b is of type SC_ξ (written $b \in \mathrm{SC}_\xi$) if

$$\mathrm{SC}(2^n,2^{n+1}) \leqslant \xi \quad \text{for all } n \geqslant 0. \tag{3.41}$$

A sequence $a = \{a_n\}_{n=1}^\infty$ of complex numbers is said to be of type SC_ξ if the sequences $\{\operatorname{Re} a_n\}_{n=1}^\infty$ and $\{\operatorname{Im} a_n\}_{n=1}^\infty$ are of type SC_ξ.

Comparing the conditions of type GM and those of type SC_ξ, we can see that in the first case we estimate the variation of the sequence $\{a_n\}$ on the intervals $(2^n,2^{n+1})$, and in the second case the variation of the sequence of signs $\{\operatorname{sgn}(a_n)\}$.

The following theorem is a modification of Theorems 3.3 and 3.4.

Theorem 3.10. *Assume that* (1.2) *is the Fourier expansion of a function* $f \in L(\mathbb{T})$ *with coefficients* $\{a_n\}_{n=1}^\infty$ *of type* SC_ξ *for some* $\xi \in \mathbb{N}$ *and that* (2.10) *is satisfied. Then the following assertions hold.*

(A) *For all positive integers* n

$$|a_n| \leqslant a_n^{\#} \leqslant C_1'\biggl(\int_0^{\pi/n} |f(t)|\,dt + \frac{\pi^2}{n^2}\int_{\pi/n}^{\pi} \frac{|f(t)|}{t^2}\,dt\biggr), \tag{3.42}$$

where the constant $C_1' > 0$ *depends only on* ν, D_1, p_0, *and* ξ.

(B) *There exists a constant* $C_2' > 0$ *depending only on a positive number* q *and on the parameters* ν, D_1, p_0, ξ *such that if* $f \in C(\mathbb{T})$, *then its Fourier series converges uniformly, and for any positive integer* n

$$n|a_n| \leqslant na_n^{\#} \leqslant C_2' n^{-q} \max_{1\leqslant k\leqslant n} k^q E_{k-1}(f)_\infty. \tag{3.43}$$

Proof. We repeat the part of the proof of Theorem 3.3 from (3.16) up to (3.20). Since $2^{n-\nu} \leqslant k_1 < \dots < k_\tau \leqslant 2^{n+\nu}$ and the signs of the non-zero terms $b_{k_1},\dots,b_{k_\tau}$ alternate, we have in light of (3.41) that $\tau \leqslant \mathrm{SC}(2^{n-\nu},2^{n+\nu}) \leqslant 2\nu\xi$. Thus, if $G = 2\nu\xi$, then the condition (3.22) is satisfied. After that we repeat the proofs of Theorems 3.3 and 3.4. Finally, we arrive at (3.42) and (3.43) with $C_1' = 8K/\pi$ and $C_2' = 2^{q+1}K$. □

Corollary 3.11. *If* (1.2) *is the Fourier expansion of a function* $f \in L(\mathbb{T})$ *with positive coefficients* $\{a_n\}_{n=1}^\infty \in \mathrm{WM}(p)$, $p \geqslant 1$, *then* (3.42) *holds. If, in addition,* $f \in C(\mathbb{T})$, *then* (3.43) *holds.*

4. Different types of convergence of series with GM-coefficients

4.1. Convergence almost everywhere and uniform convergence.

Theorem 4.1. (A) *Let* $\{a_n\} \in \mathrm{GM}(p)$ *for some* $p \geqslant 1$ *and let*

$$\sum_{n=1}^{\infty} \frac{a_n^2}{n} < \infty. \tag{4.1}$$

Then the series (1.2) *converge almost everywhere.*

(B) *For any decreasing sequence* $\{\gamma_n\}$ *satisfying the condition*

$$\sum_{n=1}^{\infty} \frac{\gamma_n^2}{n} = \infty \tag{4.2}$$

there exists a sequence $\{a_n\} \in \mathrm{GM}(p)$ *such that* $|a_n| \leqslant C\gamma_n$ *and the series* (1.2) *diverge almost everywhere.*

Proof. (A) We provide arguments for cosine series. Using the Abel transformation, we have (for $a_0 = 0$)

$$S_N(x) = \sum_{n=0}^{N} a_n \cos(nx) = \frac{1}{2\sin(x/2)} \left(\sin\left(\frac{x}{2}\right) \sum_{n=0}^{N-1} \Delta a_n \cos(nx) \right.$$
$$\left. + \cos\left(\frac{x}{2}\right) \sum_{n=0}^{N-1} \Delta a_n \sin(nx) + a_N \sin\left(\left(N + \frac{1}{2}\right)x\right)\right). \tag{4.3}$$

Note that

$$\sum_{n=2^{k-1}}^{2^k-1} |\Delta a_n|^2 \leqslant 2 \max_{2^{k-1}\leqslant n\leqslant 2^k} |a_n| \sum_{n=2^{k-1}}^{2^k-1} |\Delta a_n|$$
$$\leqslant C \max_{2^{k-1}\leqslant n\leqslant 2^k} |a_n| \left(\sum_{n=[(2^{k-1})/\nu]}^{[2^k\nu]} \frac{|a_n|^p}{n} \right)^{1/p} \leqslant C \max_{2^{k-\nu}\leqslant n\leqslant 2^{k+\nu}} |a_n|^2.$$

Since (4.1) implies that $\sum_{n=1}^{\infty} \max_{2^k\leqslant n\leqslant 2^{k+1}} |a_n|^2 < \infty$ (see Corollary 2.11), we get that

$$\sum_{n=0}^{\infty} |\Delta a_n|^2 < \infty. \tag{4.4}$$

The representation (4.3) along with Carleson's theorem and the condition (4.4) ensure that the series $\sum_{n=0}^{\infty} a_n \cos(nx)$ converges almost everywhere.

(B) For any decreasing null sequence $\{\gamma_n\}$ we construct the sequence

$$a_n = \begin{cases} \gamma_n, & n \neq 2^k, \\ 0, & n = 2^k. \end{cases}$$

Then we have

$$\sum_{n=m}^{2m} |\Delta a_n| \leqslant C(a_m + a_{m+1}) \leqslant C \left(\sum_{n=[m/\nu]}^{[m\nu]} \frac{|a_n|^p}{n} \right)^{1/p}.$$

Moreover,

$$\sum_{n=1}^{\infty} a_n \cos(nx) = \sum_{n=1}^{\infty} \gamma_n \cos(nx) - \sum_{k=1}^{\infty} \gamma_{2^k} \cos(2^k x),$$

where the first series converges everywhere on $(0, 2\pi)$, while the second is almost everywhere divergent due to the fact that $\sum_{k=1}^{\infty} \gamma_{2^k}^2 \asymp \sum_{n=1}^{\infty} \frac{\gamma_n^2}{n} = \infty$ (see [99]). □

Now we give necessary and sufficient conditions for uniform convergence of series of the form (1.2) with GM(p)-coefficients.

Theorem 4.2. *Let $a \in \mathrm{GM}(p)$ for some $p > 1$.*

(A) *The series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx)$ converges uniformly on $[0, 2\pi]$ if and only if $na_n = o(1)$ and $\sum_n a_n$ converges.*

(A′) *The sequence of partial sums of the series $a_0/2 + \sum_{n=1}^{\infty} a_n \cos(nx)$ is uniformly bounded on $[0, 2\pi]$ if and only if $na_n = O(1)$ and the sequence of partial sums of the series $\sum_n a_n$ is bounded.*

(B) *The series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges uniformly on $[0, 2\pi]$ if and only if $na_n = o(1)$.*

(B′) *The sequence of partial sums of $\sum_{n=1}^{\infty} a_n \sin(nx)$ is uniformly bounded on $[0, 2\pi]$ if and only if $na_n = O(1)$.*

The proof follows from results in [29], [34], [88].

4.2. Convergence in the mean. Let f be a 2π-periodic L_1-integrable function and let (1.2) be its Fourier series. As usual, we define the partial sums of the series (1.2) by $S_n(f,x) = a_0/2 + \sum_{k=1}^{n} a_k \cos(kx)$ or $S_n(f,x) = \sum_{k=1}^{n} a_k \sin(kx)$, respectively. We say that the series (1.2) converge in the mean, that is, in the L_1-norm, if $\|f(x) - S_n(f,x)\|_1 = o(1)$ as $n \to \infty$.

Theorem 4.3. *A series*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \textit{or} \quad \sum_{n=1}^{\infty} a_n \sin(nx) \tag{4.5}$$

with coefficients $\{a_n\} \in \mathrm{GM}(p)$ for some $p \geqslant 1$ converges in the mean if and only if it is the Fourier series of some $f \in L_1(\mathbb{T})$ and

$$|a_n| \log n \to 0 \quad \textit{as } n \to \infty. \tag{4.6}$$

Proof. Sufficiency. For a series of the form (4.5), assume that (4.6) holds. Let $S_n(x)$ be the partial sums of (4.5) and let $D_n(x)$ be the Dirichlet kernel (or the conjugate Dirichlet kernel). Then

$$\sum_{k=n}^{2n-1} |a_k - a_{k+1}| = o\left(\frac{1}{\log n}\right)$$

and for $1 < n \leqslant m \leqslant 2n$ we have

$$\begin{aligned}
\|S_m(x) - S_{n-1}(x)\|_1 &= \left\| \sum_{k=n}^{m} a_k(D_k(x) - D_{k-1}(x)) \right\|_1 \\
&= \left\| \sum_{k=n}^{m-1} \Delta a_k(D_k(x) - D_{n-1}(x)) + a_m(D_m(x) - D_{n-1}(x)) \right\|_1 \\
&\leqslant \sum_{k=n}^{m-1} |\Delta a_k| \left\|D_k(x) - D_{n-1}(x)\right\|_1 + |a_m| \left\|D_m(x) - D_{n-1}(x)\right\|_1 \\
&\leqslant 2 \max_{k\leqslant 2n} \|D_k\|_1 \left(\sum_{k=n}^{m-1} |\Delta a_k| + |a_m| \right) \\
&\leqslant 2 \max_{k\leqslant 2n} \|D_k\|_1 \left(\sum_{k=n}^{2n-1} |\Delta a_k| + \max_{n\leqslant m\leqslant 2n} |a_m| \right) \leqslant C \log n \cdot o\left(\frac{1}{\log n}\right) = o(1).
\end{aligned}$$

Hence,

$$\max_{1\leqslant n\leqslant m\leqslant 2n} \|S_m - S_{n-1}\|_1 = o(1),$$

and the Fourier series (4.5) converges in the mean (see [5], [7]).

Necessity. Assume that the series (4.5) converges in the mean. Then it is a Fourier series and

$$\sum_{k=1}^{n} |a_k| = o\left(\frac{n}{1+\log n}\right)$$

(see [5]). For $n \geqslant 1$ let

$$v_n = \left(\sum_{k=1}^{n} |a_k|\right)(1+\log n)$$

and

$$w_n = \max_{k\leqslant n} v_k.$$

We have $w_n = o(n)$ and $w_n \leqslant w_{n+1}$. For $n \geqslant 1$ let

$$\varepsilon_n = \max_{k\geqslant n} \frac{w_k}{k}\,.$$

Then $\varepsilon_n = o(1)$, $\varepsilon_n \geqslant \varepsilon_{n+1}$, and $n\varepsilon_n \geqslant w_n \geqslant v_n$. Thus, $(n+1)\varepsilon_{n+1} \geqslant w_{n+1} \geqslant w_n$, and therefore

$$\varepsilon_n = \max\left\{\varepsilon_{n+1}, \frac{w_n}{n}\right\} \leqslant \max\left\{\varepsilon_{n+1}, \frac{(n+1)\varepsilon_{n+1}}{n}\right\} \leqslant \frac{(n+1)\varepsilon_{n+1}}{n}\,.$$

Hence, $n\varepsilon_n \leqslant (n+1)\varepsilon_{n+1}$. Let

$$\beta_n = \varepsilon_n \frac{1}{1+\log n}\,.$$

Note that $\beta_n \geqslant \beta_{n+1}$ and

$$n(1+\log n)\beta_n \leqslant (n+1)(1+\log{(n+1)})\beta_{n+1}.$$

Since $(1+\log(n+1)) \leqslant (1+1/n)(1+\log n)$,

$$n^2\beta_n \leqslant (n+1)^2\beta_{n+1}.$$

For any positive integer $n \geqslant \nu$ such that

$$2^{2\nu}M_n \geqslant \max_{k=n-\nu,\dots,n+\nu} M_k \tag{4.7}$$

we have

$$\begin{aligned} 2^n M_n^{p_0} &\leqslant (D+1)^{p_0} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k|^{p_0} \leqslant \Big(\max_{k=n-\nu,\dots,n+\nu} M_k\Big)^{p_0-1} \sum_{k=2^{n-\nu}}^{2^{n+\nu}} |a_k| \\ &\leqslant (2^{2\nu}M_n)^{p_0-1} w_{2^{n+\nu}} \frac{1}{1+(n+\nu)\log 2} \end{aligned}$$

(see (2.10)). Thus,

$$\begin{aligned} 2^n M_n &\leqslant (2^{2\nu})^{p_0-1} w_{2^{n+\nu}} \frac{1}{1+(n+\nu)\log 2} \\ &\leqslant (2^{2\nu})^{p_0-1} \varepsilon_{2^{n+\nu}} \frac{2^{n+\nu}}{1+(n+\nu)\log 2} \leqslant (2^{2\nu})^{p_0-1} \varepsilon_{2^{n+1}} \frac{2^{n+\nu}}{1+(n+\nu)\log 2}. \end{aligned}$$

We then conclude that

$$M_n \leqslant (2^{2\nu})^{p_0-1} \varepsilon_{2^{n+1}} \frac{2^{\nu}}{1+\log 2^{n+1}},$$

which implies that $|a_k| \leqslant K_1\beta_k$ for all $k = 2^n, \dots, 2^{n+1}$, where $K_1 = 2^{2\nu(p_0-1)}\, 2^{\nu}$. Let

$$K_2 = \max_{k=1,\dots,2^m} \frac{|a_k|}{\beta_k}$$

and $K = \max\{K_1, K_2\}$. Then for $m = \nu$ and $\gamma = 2$ all the conditions in Lemma 3.1 are satisfied and

$$|a_k| \leqslant 4K\beta_k \quad \text{for all } k = 1, 2, \dots .$$

Hence,

$$(1+\log k)|a_k| \leqslant 4K\varepsilon_k \quad \text{for all } k = 1, 2, \dots$$

and $\varepsilon_k = o(1)$. □

Further, we obtain sufficient conditions for (4.5) to be the Fourier series of an integrable function and to converge in the mean.

Theorem 4.4. *If*

$$\sum_{n=1}^{\infty} \frac{\log n}{n} |a_n| < \infty,$$

then a series of the form (4.5) with coefficients $\{a_n\} \in \mathrm{GM}(p)$ *for some* $p \geqslant 1$ *is the Fourier series of some* $f \in L_1$ *and converges in the mean.*

Proof. First, we show that

$$\sum_{n=1}^{\infty} nM_n \leqslant C \sum_{n=1}^{\infty} \frac{\log n}{n} |a_n| < \infty. \tag{4.8}$$

Indeed, Corollary 2.11 implies that

$$\sum_{n=1}^{\infty} M_n \leqslant C \sum_{n=1}^{\infty} n^{-1}|a_n|.$$

Applying this inequality to the sequence $\{(1+\log n)a_n\}_{n=1}^{\infty} \in \mathrm{GM}(p)$, we obtain (4.8).

Then $nM_n \to 0$ as $n \to \infty$. Hence, the condition (4.6) holds, and

$$\sum_{n=1}^{\infty} \log(n)|\Delta a_n| \leqslant C \sum_{n=1}^{\infty} nM_n + C \sum_{n=m}^{\infty} \sum_{k=-m}^{m} (n+k)M_{n+k} \leqslant C \sum_{n=1}^{\infty} nM_n$$
$$\leqslant C \sum_{n=1}^{\infty} \frac{1+\log n}{n} |a_n| < \infty.$$

Thus, the series (4.5) is the Fourier series of some $f \in L_1$ (see [4]) and converges in the mean according to Theorem (4.3). □

4.3. Continuously differentiable functions and the classes $\mathrm{GM}_k(p)$. In order to study the derivatives of the sums of series with coefficients of special type we consider subclasses of $\mathrm{GM}(p)$.

Definition 4.5. Let k be a positive integer, let $p \in [1, \infty)$, and let $\mathbf{a} = \{a_n\}_{n=0}^{\infty}$ be a sequence such that $a_n \to 0$ as $n \to \infty$. We say that $\mathbf{a} \in \mathrm{GM}_k(p)$ if there exist a $C > 0$ and a $\nu \geqslant 2$ such that for all m

$$\sum_{n=m}^{2m} n^{k-1}|\Delta^k a_n| \leqslant C\biggl(\sum_{n=[m/\nu]}^{[m\nu]} \frac{|a_n|^p}{n}\biggr)^{1/p},$$

where $\Delta^1 a_n = a_n - a_{n+1}$ and $\Delta^{k+1} a_n = \Delta^k a_n - \Delta^k a_{n+1}$ for $k \geqslant 2$.

Lemma 4.6. *Let $k \geqslant 2$, $p \in [1, \infty)$, and $\mathbf{a} \in \mathrm{GM}_k(p)$. Then $\mathbf{a} \in \mathrm{GM}_{k-1}(p)$.*

Proof. Assume that m is large enough that all the integer intervals below are non-degenerate. Let

$$A_m = \max_{m \leqslant n \leqslant 2m} |\Delta^{k-1} a_n| \equiv |\Delta^{k-1} a_{n_1}|$$

and let

$$B_m = \min_{m \leqslant n \leqslant 2m} \Delta^{k-1} a_n \equiv \Delta^{k-1} a_{n_2} \quad \text{if} \quad \Delta^{k-1} a_{n_1} \geqslant 0,$$
$$B_m = \max_{m \leqslant n \leqslant 2m} \Delta^{k-1} a_n \equiv \Delta^{k-1} a_{n_2} \quad \text{otherwise.}$$

Since the cases are symmetric, we assume for definiteness that $\Delta^{k-1}a_{n_1} > 0$ and $n_1 < n_2$. If $B_m \leqslant A_m/2$, then

$$\sum_{n=m}^{2m} |\Delta^k a_n| \geqslant \sum_{n=n_1}^{n_2-1} |\Delta^k a_n| \geqslant \left|\Delta^{k-1}a_{n_1} - \Delta^{k-1}a_{n_2}\right| \geqslant \frac{A_m}{2}\,.$$

Therefore,

$$\sum_{n=m}^{2m} n^{k-2}|\Delta^{k-1}a_n| \leqslant 2m(2m)^{k-2}A_m \leqslant 2^k m^{k-1}\sum_{n=m}^{2m}|\Delta^k a_n| \leqslant 2^k \sum_{n=m}^{2m} n^{k-1}|\Delta^k a_n| \leqslant C_1(k)\biggl(\sum_{n=[m/\nu]}^{[m\nu]}\frac{|a_n|^p}{n}\biggr)^{1/p}.$$

Now assume that $B_m > A_m/2$. Suppose that $\Delta^{k-2}a_m > mA_m/4$. Then for $n \in [m, m+[m/8]]$

$$\Delta^{k-2}a_n > \frac{mA_m}{4} - A_m\frac{m}{8} > \frac{mA_m}{8}\,.$$

But if $\Delta^{k-2}a_m < mA_m/4$, then for $n \in [2m-[m/8], 2m]$

$$\Delta^{k-2}a_n = \Delta^{k-2}a_m - \sum_{r=m}^{n-1}\Delta^{k-1}a_r < \frac{mA_m}{4} - B_m(n-m) < \frac{mA_m}{4} - B_m\frac{7m}{8} < \frac{mA_m}{4} - \frac{7mA_m}{16} < -\frac{mA_m}{8}\,.$$

Thus, there is an integer interval in $[m, 2m]$ of length at least $m/8$ on which all $|\Delta^{k-2}a_n|$ are $> mA_m/8$ and all these differences have the same sign.

Suppose that for some integer $j \in [1, k-2]$ we have already proved the existence of an integer interval $[n_1, n_2] \subset [m, 2m]$ of length at least m/r_j such that $|\Delta^j a_n| > m^{k-j-1}A_m/q_j$ for $n \in [n_1, n_2]$ and all these differences have the same sign. Assume for definiteness that they are all positive. Moreover, increasing r_j if necessary, we can make $n_2 - n_1$ divisible by 4. Let $\Delta^{j-1}a_{(n_1+n_2)/2} \geqslant 0$ (the negative case is similar). Then for $n \in [(n_1+3n_2)/4, n_2]$

$$\Delta^{j-1}a_n < -\frac{m^{k-j}A_m}{4q_jr_j}\,,$$

that is, taking $r_{j-1} = 4r_j$ and $q_{j-1} = 4q_jr_j$, we see that there is an integer interval of length at least m/r_{j-1} in $[m, 2m]$ on which all $|\Delta^{j-1}a_n|$ are $> m^{k-j}A_m/q_{j-1}$ and all these differences have the same sign.

Repeating the same argument $k-1$ times, we establish that there exists an integer interval J in $[m, 2m]$ of length at least m/r_0 on which all the numbers $|a_n|$ are $> m^{k-1}A_m/q_0$. Furthermore, one can see from the proof that r_0 and q_0 depend only on k. But then

$$\biggl(\sum_{n=[m/\nu]}^{[m\nu]}\frac{|a_n|^p}{n}\biggr)^{1/p}>\biggl(\sum_{n\in J}\frac{|a_n|^p}{n}\biggr)^{1/p}\geqslant C(k,p)m^{k-1}A_m$$
$$\geqslant C_1(k,p)\sum_{n=m}^{2m}n^{k-2}|\Delta^{k-1}a_n|. \qquad \square$$

For a function $f(x)\sim a_0/2+\sum_{n=1}^\infty a_n\cos(nx)$ it is well known that in order to have $f,f',\dots,f^{(k-1)}\in C(\mathbb{T})$ it is sufficient that $\sum_{n=1}^\infty n^{k-1}|a_n|<\infty$. We show that for $\mathrm{GM}_k(p)$-coefficients this condition can be relaxed significantly.

Theorem 4.7. *Let $k\geqslant 2$ be an integer, let $p\in[1,\infty)$, and let $\mathbf{a}\in\mathrm{GM}_k(p)$, with*

$$\sum_{n=1}^\infty\frac{|a_n|}{n}<\infty. \tag{4.9}$$

Then the series $a_0/2+\sum_{n=1}^\infty a_n\cos(nt)$ converges on $(0,2\pi)$, and its sum $f(t)$ is $(k-1)$ times continuously differentiable on this interval.

Proof. By Lemma 4.6, for $1\leqslant s\leqslant k$

$$\sum_{n=1}^\infty n^{s-1}|\Delta^s a_n|\leqslant C\sum_{m=1}^\infty\frac{1}{m}\sum_{n=m}^{2m}n^{s-1}|\Delta^s a_n|\leqslant C\sum_{m=1}^\infty\frac{1}{m}\biggl(\sum_{n=[m/\nu]}^{m\nu}\frac{|a_n|^p}{n}\biggr)^{1/p}$$
$$\leqslant C(\nu,p)\sum_{m=1}^\infty\frac{1}{m}\biggl(\sum_{n=m}^{2m}\frac{|a_n|^p}{n}\biggr)^{1/p}<\infty.$$

In view of Corollary 2.11 the conditions

$$\sum_{m=1}^\infty\frac{1}{m}\biggl(\sum_{n=m}^{2m}\frac{|a_n|^p}{n}\biggr)^{1/p}<\infty,\qquad \sum_{m=1}^\infty\frac{\max_{m\leqslant n\leqslant 2m}|a_n|}{m}<\infty$$

and (4.9) are equivalent. Finally, the assertion of the theorem follows from [66] and [96]. $\square$

Remark 4.8. The condition $\mathbf{a}\in\mathrm{GM}_k(p)$ without (4.9) does not ensure even the convergence of the series $\sum_{n=1}^\infty a_n\cos(nx)$. Namely, for any $p\geqslant 1$ and any integer $k\geqslant 2$, there exists a sequence $\mathbf{a}\in\mathrm{GM}_k(p)$ such that the series $\sum_{n=1}^\infty a_n\cos(nx)$ diverges almost everywhere. Indeed, we can consider the function in part (B) of Theorem 4.1 for sufficiently convex γ_n.

4.4. Asymptotic behaviour of series near the origin. First we formulate several basic results on the asymptotic behaviour of trigonometric series. Salem ([4], [75], [76]) proved the following result on trigonometric series with convex coefficients:

$$g(x)=\sum_{n=1}^\infty a(n)\sin(nx)\asymp\frac{a(1/x)}{x}\quad\text{as } x\to 0+$$

if $a(t)$ is convex, $a(t) \to 0$ as $t \to \infty$, and $ta(t)$ is increasing (see also [69] and the references therein). Here and below, $\xi_n \asymp \nu_n$ if $C_1\xi_n \leqslant \nu_n \leqslant C_2\xi_n$ and $\xi_n \sim \nu_n$ if $\xi_n/\nu_n \to 1$ as $n \to \infty$.

Hardy (see [43] and [99], Vol. 1, Chap. V, § 2) proved the following result: if $0 < \alpha < 1$, $a_n \geqslant a_{n+1} \geqslant \cdots$ and $a_n \to 0$, then

$$n^\alpha a_n \to A > 0 \quad \text{as } n \to \infty$$

if and only if

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx) \sim A \sin\biggl(\frac{\pi\alpha}{2}\biggr)\Gamma(1-\alpha)x^{\alpha-1} \quad \text{as } x \to 0+ \tag{4.10}$$

or

$$g(x) = \sum_{n=1}^{\infty} a_n \sin(nx) \sim A \cos\biggl(\frac{\pi\alpha}{2}\biggr)\Gamma(1-\alpha)x^{\alpha-1} \quad \text{as } x \to 0+. \tag{4.11}$$

Later, Heywood [44] extended the last statement to $1 \leqslant \alpha < 2$ in the case of sine series. Boas noted (see [9], p. 5, Theorem 8) that if $0 < \alpha < 1$, $a_n \geqslant a_{n+1} \geqslant \cdots$, and $a_n \to 0$, then

$$a_n = O(n^{-\alpha}) \quad \Longleftrightarrow \quad f(x) = O(x^{\alpha-1}) \quad \Longleftrightarrow \quad g(x) = O(x^{\alpha-1}).$$

These results were extended to more general majorants and classes of sequences in [14] and [91], Theorem 5.4 (see also [38] for similar results for sine series). In [36] these results were obtained for the class GM(1). The goal of this subsection is to prove analogues of these results for trigonometric series with GM(p)-coefficients, $p \geqslant 1$, that are not necessarily non-negative.

Let $\beta > 0$ and let φ be a majorant in the class Φ, that is, φ is a non-negative non-decreasing function on $[0,1]$ such that $\varphi(0) = 0$. We define the Bari–Stechkin conditions for the majorant φ:

$$\int_0^u \varphi(t)\,\frac{dt}{t} = O(\varphi(u)) \quad \text{as } u \to 0, \tag{B}$$

$$u^\beta \int_u^1 \frac{\varphi(t)}{t^\beta}\,\frac{dt}{t} = O(\varphi(u)) \quad \text{as } u \to 0. \tag{B_β}$$

Theorem 4.9. *Let* (1.2) *be the Fourier series of an* $f \in L_1(\mathbb{T})$ *with coefficients* $\{a_n\}_{n=1}^\infty$ *of type* GM(p), $p \geqslant 1$.

(A) *If* $\varphi \in \Phi \cap B \cap B_1$, *then the conditions*

$$a_n = O\biggl(\varphi\biggl(\frac{1}{n}\biggr)\biggr) \quad \textit{as } n \to \infty \tag{4.12}$$

and

$$f(x) = \sum_{n=1}^{\infty} a_n \cos(nx) = O\biggl(\frac{\varphi(x)}{x}\biggr) \quad \textit{as } x \to 0 \tag{4.13}$$

are equivalent.

(B) *If* $\varphi \in \Phi \cap \mathrm{B} \cap \mathrm{B}_2$, *then the condition* (4.12) *and the condition*

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx) = O\left(\frac{\varphi(x)}{x}\right) \quad \text{as } x \to 0 \tag{4.14}$$

are equivalent.

Remark 4.10. The examples

$$f(x) = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n} \sim \log\frac{1}{x} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} \sim x \log\frac{1}{x}$$

show that the conditions $\varphi \in \mathrm{B}_1$ and $\varphi \in \mathrm{B}_2$ are optimal.

We point out that estimates of the form (4.10) and (4.11) do not hold even for series with GMS-coefficients [89].

Proof of Theorem 4.9. Take $\xi := 1$ in the case of sine series, and $\xi := 0$ in the case of cosine series. Using the Abel transformation (see (5.8)) and (2.13), for $x \in [\pi\, 2^{-n-1}, \pi\, 2^{-n}]$ we have

$$\begin{aligned} |f(x)| &\leqslant C\left(|a_0| + 2^{-n\xi} \sum_{k=1}^{2^n-1} k^{\xi}|a_k| + \frac{1}{x}\sum_{k=2^n}^{\infty} |a_k - a_{k+1}|\right) \\ &\leqslant C\left(2^{-n\xi} \sum_{k=0}^{2^n-1} (k+1)^{\xi}\varphi\left(\frac{1}{k}\right) + \frac{1}{x}\sum_{k=2^n}^{\infty} \frac{\varphi(1/k)}{k}\right) \leqslant C\,\frac{\varphi(x)}{x}, \end{aligned}$$

where in the last inequality we have used the conditions on φ.

Conversely, for any $\varphi \in \Phi \cap \mathrm{B} \cap \mathrm{B}_2$, Theorem 3.3 implies that

$$|a_n| \leqslant C\left(\int_0^{\pi/n} \varphi(t)\,\frac{dt}{t} + \frac{1}{n^2}\int_{\pi/n}^{\pi} \varphi(t)\,\frac{dt}{t^3}\right) \leqslant C\varphi\left(\frac{1}{n}\right). \qquad \square$$

4.5. Absolute convergence. Theorems 3.4 and 3.10 provide sufficient conditions for estimates of type (3.15) to hold. Thus, it is of interest to obtain some corollaries of this estimate.

Corollary 4.11. *If* (1.2) *is the Fourier series of a function* $f \in C(\mathbb{T})$ *with coefficients* $\{a_n\}_{n=1}^{\infty}$ *of type* $\mathrm{GM}(\nu, D, p_0)$, $p_0 \geqslant 1$, *then for any* $\theta > 0$ *and any* $\alpha \in \mathbb{R}$ *there exists a positive constant* $C_{\theta,\alpha,\nu,D,p_0}$ *such that*

$$\sum_{n=1}^{\infty} n^{\alpha}(n|a_n|)^{\theta} \leqslant C_{\theta,\alpha,p_0} \sum_{n=1}^{\infty} n^{\alpha} E_{n-1}(f)_{\infty}^{\theta}. \tag{4.15}$$

Moreover, if $\alpha > -1$, *then the inequality above is sharp, that is,*

$$\sum_{n=1}^{\infty} n^{\alpha} E_{n-1}(f)_{\infty}^{\theta} \asymp \sum_{n=1}^{\infty} n^{\alpha+\theta}(a_n^{\#})^{\theta} \asymp \sum_{n=1}^{\infty} n^{\alpha+\theta}|a_n|^{\theta}, \tag{4.16}$$

where the constants depend only on ν, D, p_0, θ, *and* α.

Remark 4.12. (i) In particular, (4.15) implies that

$$\sum_{n=1}^{\infty}|a_n|^\theta \leqslant C_\theta \sum_{n=1}^{\infty}\left(\frac{E_{n-1}(f)_\infty}{n}\right)^\theta \quad \text{and} \quad \sum_{n=1}^{\infty}|a_n| \leqslant C\sum_{n=1}^{\infty}\frac{E_{n-1}(f)_\infty}{n}.$$

Note that the last two inequalities complement the classical results of Bernstein and Szász [4] that establish the following estimate in the general case:

$$\sum_{n=1}^{\infty}|a_n|^\theta \leqslant C_\theta \sum_{n=1}^{\infty}\frac{E_{n-1}(f)_2^\theta}{n^{\theta/2}}, \tag{4.17}$$

which holds only for $0<\theta\leqslant 2$.

(ii) We point out that (4.15) holds for any function $f\in C(\mathbb{T})$ with Fourier series of the form (1.2). Moreover, for any positive q there exists a positive constant C_q such that for any positive integer n

$$n|a_n| \leqslant C_q n^{-q}\max_{1\leqslant k\leqslant n} k^q E_{k-1}(f)_\infty. \tag{4.18}$$

Proof of Corollary 4.11. By Theorem 3.4, the estimate (3.15) holds. Let $q=\max\{(\alpha+2)/\theta,1\}$. Then $q\geqslant 1$ and $\theta q\geqslant \alpha+2$. It follows from (3.15) that for $N\geqslant 0$ and $2^N\leqslant n\leqslant 2^{N+1}$ we have

$$\begin{aligned}2^{N(q+1)}|a_n| \leqslant n^{q+1}|a_n| &\leqslant C_q \max_{0\leqslant j\leqslant N}\max_{2^j\leqslant n\leqslant 2^{j+1}} k^q\, E_{k-1}(f)_\infty\\ &\leqslant C_q\max_{0\leqslant j\leqslant N} 2^{(j+1)q}E_{2^j-1}(f)_\infty.\end{aligned}$$

Thus, for all $N\geqslant 0$

$$(2^{N(q+1)}M_N)^\theta \leqslant C_q^\theta \max_{0\leqslant j\leqslant N}(2^{(j+1)q}E_{2^j-1}(f)_\infty)^\theta \leqslant C_q^\theta\sum_{j=0}^{N}2^{(j+1)q\theta}E^\theta_{2^j-1}(f)_\infty.$$

Hence, using the fact that $\theta q-\alpha\geqslant 2$, we get that

$$\begin{aligned}\sum_{n=1}^{\infty}n^\alpha(n|a_n|)^\theta &= \sum_{N=0}^{\infty}\sum_{n=2^N}^{2^{N+1}-1}n^{\alpha-\theta q}(n^{q+1}|a_n|)^\theta\\ &\leqslant \sum_{N=0}^{\infty}2^N\,2^{N(\alpha-\theta q)}(2^{(N+1)(q+1)}M_N)^\theta\\ &\leqslant C_q^\theta\sum_{j=0}^{\infty}2^{(q+1)\theta}\,2^{(j+1)q\theta}E^\theta_{2^j-1}(f)_\infty\sum_{N=j}^{\infty}2^{N(\alpha-\theta q+1)}\\ &\leqslant 2C_q^\theta\,2^{(2q+1)\theta}\left(E_0^\theta(f)_\infty+2^{q\theta}\sum_{j=1}^{\infty}\sum_{n=2^{j-1}}^{2^j-1}n^{q\theta}E^\theta_{n-1}(f)_\infty\,2n^{\alpha-\theta q}\right)\\ &\leqslant 4C_q^\theta\,2^{(3q+1)\theta}\sum_{n=1}^{\infty}n^\alpha E^\theta_{n-1}(f)_\infty,\end{aligned}$$

that is, we have (4.15) with $C_{\theta,\alpha}=4C_q^\theta\,2^{(3q+1)\theta}$.

To prove (4.16) we note that the estimate $E_{n-1}(f)_\infty \leqslant \sum_{k=n}^\infty |a_k|$ for all $n \geqslant 1$ (we can assume that $a_0 = 0$) implies that

$$\sum_{n=1}^\infty n^{\alpha+\theta}|a_n|^\theta \leqslant C\sum_{n=1}^\infty n^\alpha E_{n-1}^\theta(f)_\infty \leqslant C\sum_{n=0}^\infty 2^{n(\alpha+1)}\biggl(\sum_{m=n}^\infty \sum_{k=2^m}^{2^{m+1}+1}|a_k|\biggr)^\theta$$
$$\leqslant C\sum_{n=0}^\infty 2^{n(\alpha+1)}\biggl(\sum_{m=n}^\infty 2^m M_m\biggr)^\theta \leqslant C\sum_{n=0}^\infty 2^{n(\alpha+\theta+1)}M_n^\theta,$$

where in the last estimate we have used Hardy's inequality with $\theta > 0$. From Lemma 5.1 and Theorem 2.9 we obtain (4.16). □

4.6. Convergence in L_p, $0 < p < 1$. Note that for $\{a_n\} \in \mathrm{GM}(p_0)$, $p_0 \geqslant 1$, the conditions

$$\sum_{n=1}^\infty \frac{|a_n|}{n} < \infty \quad \text{and} \quad \sum_{m=1}^\infty \frac{1}{m}\biggl(\sum_{n=m}^{2m}\frac{|a_n|^{p_0}}{n}\biggr)^{1/p_0} < \infty \tag{4.19}$$

are equivalent (by Corollary 2.11) and they ensure that the sequence $\{a_n\}$ is of bounded variation. In particular, we have the following result.

Corollary 4.13. *If $\{a_n\}_{n=1}^\infty \in \mathrm{GM}(p_0)$, $p_0 \geqslant 1$, and $\sum_{n=1}^\infty |a_n|/n < \infty$, then $f \in L_p(\mathbb{T})$, $p \in (0,1)$, and*

$$\|f - S_n(f)\|_p^p \leqslant C\sum_{m=n/\gamma}^\infty \frac{1}{m}\biggl(\sum_{k=m}^{2m}\frac{|a_k|^{p_0}}{k}\biggr)^{1/p_0}.$$

This result follows from [4], Chap. X, § 5.

5. Hardy–Littlewood type inequalities

5.1. Inequalities for number sequences.

Lemma 5.1. *If a sequence of complex numbers $\{a_n\}_{n=1}^\infty$ tends to zero, then for all $\alpha > 0$ and $p \in (0,\infty)$ the estimates*

$$\sum_{n=0}^\infty 2^{n\alpha}M_n^p \leqslant \sum_{n=0}^\infty 2^{n\alpha}(a_{2^n}^\#)^p \leqslant 2^{1+\alpha}\sum_{n=1}^\infty n^{\alpha-1}(a_n^\#)^p \tag{5.1}$$

and

$$\sum_{n=1}^\infty n^{\alpha-1}(a_n^\#)^p \leqslant 2^\alpha\sum_{n=0}^\infty 2^{n\alpha}(a_{2^n}^\#)^p \leqslant C_{p,\alpha}\sum_{n=0}^\infty 2^{n\alpha}M_n^p \tag{5.2}$$

hold, where the positive constant $C_{p,\alpha}$ depends only on p and α.

Proof. Since $M_n \leqslant a_{2^n}^\#$, the first inequality in (5.1) is obvious. The inequalities

$$\sum_{n=0}^\infty 2^{n\alpha}(a_{2^n}^\#)^p \leqslant (a_1^\#)^p + \sum_{n=0}^\infty 2^{n\alpha+1}\sum_{k=2^{n-1}+1}^{2^n}\frac{(a_k^\#)^p}{k}$$
$$\leqslant (a_1^\#)^p + 2^{\alpha+1}\sum_{n=0}^\infty \sum_{k=2^{n-1}+1}^{2^n} k^\alpha\frac{(a_k^\#)^p}{k} \leqslant 2^{\alpha+1}\sum_{k=1}^\infty k^{\alpha-1}(a_k^\#)^p$$

imply the second inequality in (5.1). Conversely, (5.2) follows from the inequalities

$$\sum_{k=1}^{\infty} k^{\alpha-1}\,(a_k^{\#})^p \leqslant \sum_{n=0}^{\infty} 2^{(n+1)\alpha} \sum_{k=2^n}^{2^{n+1}-1} 2^{-n}(a_{2^n}^{\#})^p \leqslant 2^{\alpha}\sum_{n=0}^{\infty} 2^{n\alpha}(a_{2^n}^{\#})^p$$

and

$$\sum_{n=0}^{\infty} 2^{n\alpha}(a_{2^n}^{\#})^p \leqslant \sum_{n=0}^{\infty} 2^{n\alpha}\Big(\sum_{k=n}^{\infty} M_k\Big)^p \leqslant C_{p,\alpha}\sum_{n=0}^{\infty} 2^{n\alpha} M_n^p,$$

where the last estimate is a corollary of Hardy's inequality. □

5.2. Hardy–Littlewood type theorems. Theorems 3.4 and 3.10 provide sufficient conditions for an estimate of the form (3.14). Therefore, it is of interest to obtain some corollaries of this result.

Theorem 5.2. *Suppose that a sequence of complex numbers* $\{a_n\}_{n=1}^{\infty}$ *is of type* $\mathrm{GM}(\nu, D, p_0)$, *tends to zero, and for some* $p \in (1,\infty)$ *and* $\gamma \in (1-p, 1)$ *satisfies the condition*

$$\sum_{n=1}^{\infty} n^{p-2+\gamma}|a_n|^p < \infty. \tag{5.3}$$

Then the cosine series in (1.2) *is the Fourier series of its sum* $f \in L_1(\mathbb{T})$, *which is such that*

$$\int_0^{\pi} \frac{1}{t^{\gamma}}\,|f(t)|^p\,dt < \infty, \tag{5.4}$$

and the order estimate

$$\int_0^{\pi} \frac{1}{t^{\gamma}}\,|f(t)|^p\,dt \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2+\gamma}(a_n^{\#})^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2+\gamma}|a_n|^p \tag{5.5}$$

holds, where the corresponding positive constants depend only on p, γ, *and the parameters* ν, D, *and* p_0.

Proof. From (2.11) we obtain

$$\sum_{k=2^n}^{2^{n+1}} |a_k - a_{k+1}| \leqslant D_2 a_{2^{n-\nu}}^{\#}$$

for all $n \geqslant \nu$. Hence,

$$\sum_{k=2^n}^{\infty} |a_k - a_{k+1}| \leqslant D_2 \sum_{j=n-\nu}^{\infty} a_{2^j}^{\#},$$

and by Hardy's inequality

$$\sum_{n=\nu}^{\infty} 2^{n(p-1+\gamma)}\Big(\sum_{k=2^n}^{\infty}|a_k - a_{k+1}|\Big)^p \leqslant D_2^p \sum_{n=\nu}^{\infty} 2^{n(p-1+\gamma)}\Big(\sum_{j=n-\nu}^{\infty} a_{2^j}^{\#}\Big)^p$$

$$\leqslant C_{p,\nu,\gamma}\sum_{n=0}^{\infty} 2^{n(p-1+\gamma)}(a_{2^n}^{\#})^p.$$

Applying Lemma 5.1 to the last sum, we arrive at the estimate

$$\sum_{n=\nu}^{\infty} 2^{n(p-1+\gamma)}\Big(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\Big)^p \leqslant C\sum_{n=1}^{\infty} n^{p-2+\gamma}(a_n^{\#})^p, \tag{5.6}$$

where, here and below, a positive constant C depends only on p, γ, and on the parameters ν, D, and p_0. Since the series (5.3) converges,

$$\sum_{n=1}^{\infty}|a_n-a_{n+1}|<\infty. \tag{5.7}$$

Therefore, the series (1.2) converge everywhere on $(0,2\pi)$ to a function $f(x)$. It is well known that if $x\in[\pi 2^{-n-1},\pi 2^{-n}]$, $n\geqslant 0$, then

$$|f(x)|\leqslant |a_0|+\sum_{k=1}^{2^n-1}|a_k|+\frac{\pi}{x}\sum_{k=2^n}^{\infty}|a_k-a_{k+1}| \tag{5.8}$$

and

$$|f(x)|^p\leqslant 3^{p-1}\bigg(|a_0|^p+\Big(\sum_{k=0}^{n-1}2^kM_k\Big)^p+\frac{\pi^p}{x^p}\Big(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\Big)^p\bigg).$$

Hence, it follows that

$$\begin{aligned}\int_0^{\pi}\frac{1}{t^{\gamma}}|f(t)|^p\,dt &\leqslant 3^{p-1}\sum_{n=0}^{\infty}\int_{\pi 2^{-n-1}}^{\pi 2^{-n}}\frac{t^{p-1}}{t^{\gamma+p-1}}\bigg(|a_0|^p+\Big(\sum_{k=0}^{n-1}2^kM_k\Big)^p\\ &\quad+\frac{\pi^p}{t^p}\Big(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\Big)^p\bigg)\,dt\\ &\leqslant 3^{p-1}\sum_{n=0}^{\infty}\pi\, 2^{-n-1}\,2^{\gamma+p-1}\pi^{-\gamma}\,2^{n\gamma}\bigg(|a_0|^p+\Big(\sum_{k=0}^{n-1}2^kM_k\Big)^p\\ &\quad+2^{np+p}\Big(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\Big)^p\bigg)\\ &\leqslant C_{p,\gamma}\sum_{n=0}^{\infty}2^{n(\gamma-1)}\bigg(|a_0|^p+\Big(\sum_{k=0}^{n}2^kM_k\Big)^p\bigg)\\ &\quad+C_{p,\gamma}\sum_{n=0}^{\infty}2^{n(\gamma+p-1)}\Big(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\Big)^p.\end{aligned}$$

From Hardy's inequality and Lemma 5.1 we get the estimates

$$\sum_{n=0}^{\infty}2^{n(\gamma-1)}\Big(\sum_{k=0}^{n}2^kM_k\Big)^p\leqslant C_{p,\gamma}\sum_{n=0}^{\infty}2^{n(p+\gamma-1)}M_n^p\leqslant C_{p,\gamma}\sum_{n=1}^{\infty}n^{p-2+\gamma}(a_n^{\#})^p$$

and

$$\sum_{n=0}^{\nu-1} 2^{n(p+\gamma-1)}\bigg(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\bigg)^p \leqslant 2^{p-1}\sum_{n=0}^{\nu-1} 2^{n(p+\gamma-1)}\bigg(\sum_{k=2^n}^{2^\nu-1}(|a_k|+|a_{k+1}|)\bigg)^p + 2^{p-1}\sum_{n=0}^{\nu-1} 2^{n(\gamma+p-1)}\bigg(\sum_{k=2^\nu}^{\infty}|a_k-a_{k+1}|\bigg)^p \leqslant C_{p,\gamma,\nu}(a_1^{\#})^p + C_{p,\gamma,\nu}\bigg(\sum_{k=2^\nu}^{\infty}|a_k-a_{k+1}|\bigg)^p.$$

This and (5.6) imply that

$$\sum_{n=0}^{\infty} 2^{n(\gamma+p-1)}\bigg(\sum_{k=2^n}^{\infty}|a_k-a_{k+1}|\bigg)^p \leqslant C\sum_{n=1}^{\infty} n^{p-2+\gamma}(a_n^{\#})^p.$$

Then

$$\int_0^\pi \frac{1}{t^\gamma}|f(t)|^p\,dt \leqslant C\bigg(|a_0|^p + \sum_{n=1}^{\infty} n^{p-2+\gamma}(a_n^{\#})^p\bigg). \tag{5.9}$$

We now prove a lower bound for the weighted L_p-norm of f. First, we note that it follows from

$$\int_0^\pi |f(t)|\,dt \leqslant C(\gamma,p)\bigg(\int_0^\pi \frac{1}{t^\gamma}|f(t)|^p\,dt\bigg)^{1/p} < \infty$$

that $f \in L_1(\mathbb{T})$, and the series (1.2) is the Fourier series of its sum f. Since

$$|a_0| \leqslant \frac{2}{\pi}\int_0^\pi |f(t)|\,dt,$$

we have

$$|a_0|^p \leqslant C_{p,\gamma}\int_0^\pi \frac{1}{t^\gamma}|f(t)|^p\,dt.$$

Further, we show that if (1.2) is the Fourier series of a function $f \in L_1(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^\infty$ of type $\mathrm{GM}(\nu, D, p_0)$, then for any $p \in (1,\infty)$ and $\gamma \in (1-p, 1+p)$

$$\sum_{n=1}^{\infty} n^{p-2+\gamma}|a_n|^p \leqslant \sum_{n=1}^{\infty} n^{p-2+\gamma}(a_n^{\#})^p \leqslant C\int_0^\pi \frac{|f(t)|^p}{t^\gamma}\,dt, \tag{5.10}$$

where the positive constant C does not depend on f.

Let

$$\beta_n = \int_0^{\pi/n}|f(t)|\,dt + \frac{\pi^2}{n^2}\int_{\pi/n}^{\pi}\frac{|f(t)|}{t^2}\,dt = \int_0^\pi |f(t)|\min\bigg\{1, \frac{\pi^2}{n^2t^2}\bigg\}\,dt$$

for positive integers n. Then $\beta_{n+1} \leqslant \beta_n$ and $(n+1)^2\beta_{n+1} \geqslant n^2\beta_n$ for all positive integers k. In light of (3.14), we have $|a_n| \leqslant C_1\beta_n$, and therefore

$a_n^{\#} = \max_{k\geqslant n}|a_k| \leqslant C_1\beta_n$ for all $n \geqslant 1$. For any positive integer n, if $v \in [\pi/(n+1), \pi/n]$, then

$$\begin{aligned} n^{(p+\gamma)/p}\beta_n &\leqslant \int_0^\pi |f(t)| \min\left\{n^{(p+\gamma)/p}, n^{(p+\gamma)/p}\,\frac{4\pi^2}{(n+1)^2t^2}\right\} dt \\ &\leqslant \int_0^\pi |f(t)| \min\left\{\frac{\pi^{(p+\gamma)/p}}{v^{(p+\gamma)/p}}, \frac{4v^2}{t^2}\,\frac{\pi^{(p+\gamma)/p}}{v^{(p+\gamma)/p}}\right\} dt. \end{aligned}$$

From this,

$$\begin{aligned} n^{p+\gamma}\beta_n^p\,\frac{\pi}{n(n+1)} \leqslant \int_{\pi/(n+1)}^{\pi/n} \biggl(\frac{\pi^{(p+\gamma)/p}}{v^{(p+\gamma)/p}} \int_0^v |f(t)|\,dt \\ + \int_v^\pi |f(t)|\,\frac{4v}{t^2}\,\frac{\pi^{(p+\gamma)/p}}{v^{\gamma/p}}\,dt\biggr)^p dv. \end{aligned}$$

Consequently,

$$\begin{aligned} \sum_{n=1}^\infty n^{p+\gamma}\beta_n^p\,\frac{\pi}{2n^2} &\leqslant \int_0^\pi \biggl(\frac{\pi^{(p+\gamma)/p}}{v^{(p+\gamma)/p}} \int_0^v |f(t)|\,dt + \int_v^\pi |f(t)|\,\frac{4\pi v}{t^2}\,\frac{\pi^{\gamma/p}}{v^{\gamma/p}}\,dt\biggr)^p dv \\ &\leqslant 2^{p-1}\int_0^\pi \biggl(\biggl(\frac{\pi^{(p+\gamma)/p}}{v^{(p+\gamma)/p}} \int_0^v |f(t)|\,dt\biggr)^p \\ &\qquad + \biggl(\int_v^\pi |f(t)|\,\frac{4\pi v}{t^2}\,\frac{\pi^{\gamma/p}}{v^{\gamma/p}}\,dt\biggr)^p\biggr) dv. \end{aligned}$$

Then

$$\begin{aligned} \frac{\pi}{2^p}\sum_{n=1}^\infty n^{p-2+\gamma}\beta_n^p \leqslant \pi^{(p+\gamma)} \int_0^\pi \frac{1}{v^\gamma}\biggl(\frac{1}{v}\int_0^v |f(t)|\,dt\biggr)^p dv \\ + 4^p\pi^{p+\gamma}\int_0^\pi \frac{1}{v^\gamma}\biggl(\int_v^\pi \frac{v|f(t)|}{t^2}\,dt\biggr)^p dv. \end{aligned}$$

But by virtue of Hardy's inequalities

$$\int_0^\pi \frac{1}{v^\gamma}\biggl(\frac{1}{v}\int_0^v |f(t)|\,dt\biggr)^p dv \leqslant \biggl(\frac{p}{\gamma+p-1}\biggr)^p \int_0^\pi \frac{1}{v^\gamma}\,|f(v)|^p\,dv$$

and

$$\int_0^\pi \frac{1}{v^\gamma}\biggl(\int_v^\pi \frac{v|f(t)|}{t^2}\,dt\biggr)^p dv \leqslant \biggl(\frac{p}{1+p-\gamma}\biggr)^p \int_0^\pi \frac{1}{v^\gamma}\,|f(v)|^p\,dv.$$

This immediately gives us the inequality (5.10), because

$$\sum_{n=1}^\infty n^{p-2+\gamma}(a_n^{\#})^p \leqslant (C_1)^p\sum_{n=1}^\infty n^{p-2+\gamma}\beta_n^p \leqslant (C_1)^p C_{p,\gamma}\int_0^{2\pi} \frac{|f(t)|^p}{t^\gamma}\,dt.$$

The order estimate (5.5) follows from (5.9) and (5.10), and Theorem 5.2 is proved. ☐

For odd functions the Hardy–Littlewood theorem holds for a wider class of weights.

Theorem 5.3. *Suppose that a sequence of complex numbers* $\{a_n\}_{n=1}^{\infty}$ *is of type* $\mathrm{GM}(\nu, D, p_0)$, *tends to zero, and for some* $p \in (1,\infty)$ *and* $\gamma \in (1-p, 1+p)$ *satisfies the condition* (5.3). *Then the sine series in* (1.2) *is the Fourier series of its sum* $f \in L_1(\mathbb{T})$, *which is such that* (5.4) *is valid, and the order estimate*

$$\int_0^{\pi} \frac{1}{t^{\gamma}}\, |f(t)|^p\, dt \asymp \sum_{n=1}^{\infty} n^{p-2+\gamma} |a_n|^p \asymp \sum_{n=1}^{\infty} n^{p-2+\gamma} (a_n^{\#})^p \tag{5.11}$$

holds, where the corresponding positive constants depend only on p, γ, *and the parameters* ν, D, *and* p_0.

Proof. Using for $x \in [\pi 2^{-n-1}, \pi 2^{-n}]$, $n \geqslant 0$, the estimate

$$|f(x)| \leqslant \pi\, 2^{-n} \sum_{k=1}^{2^n-1} k|a_k| + \frac{\pi}{x} \sum_{k=2^n}^{\infty} |a_k - a_{k+1}|$$

in place of (5.8), we repeat the proof of Theorem 5.2. In this case we have

$$|f(x)|^p \leqslant 2^{p-1} \biggl(\biggl(2\pi\, 2^{-n} \sum_{k=0}^{n-1} 2^{2k} M_k \biggr)^p + \frac{\pi^p}{x^p} \biggl(\sum_{k=2^n}^{\infty} |a_k - a_{k+1}| \biggr)^p \biggr).$$

Therefore,

$$\begin{aligned}
\int_0^{\pi} \frac{1}{t^{\gamma}}\, |f(t)|^p\, dt &\leqslant 2^{p-1} \sum_{n=0}^{\infty} \pi\, 2^{-n-1}\, 2^{\gamma+p-1} \pi^{-\gamma}\, 2^{n\gamma} \\
&\quad \times \biggl(\biggl(2\pi\, 2^{-n} \sum_{k=0}^{n-1} 2^{2k} M_k \biggr)^p + 2^{np+p} \biggl(\sum_{k=2^n}^{\infty} |a_k - a_{k+1}| \biggr)^p \biggr) \\
&\leqslant C_{p,\gamma} \sum_{n=0}^{\infty} 2^{n(\gamma-1)} \biggl(2^{-n} \sum_{k=0}^{n} 2^{2k} M_k \biggr)^p \\
&\quad + C_{p,\gamma} \sum_{n=0}^{\infty} 2^{n(\gamma+p-1)} \biggl(\sum_{k=2^n}^{\infty} |a_k - a_{k+1}| \biggr)^p.
\end{aligned}$$

We note that the condition (5.7) holds. Hardy's inequality and Lemma 5.1 imply that

$$\sum_{n=0}^{\infty} 2^{n(\gamma-1-p)} \biggl(\sum_{k=0}^{n} 2^{2k} M_k \biggr)^p \leqslant C_{p,\gamma} \sum_{n=0}^{\infty} 2^{n(p+\gamma-1)} M_n^p \leqslant C_{p,\gamma} \sum_{n=1}^{\infty} n^{p-2+\gamma} (a_n^{\#})^p.$$

Hence,

$$\int_0^{\pi} \frac{1}{t^{\gamma}}\, |f(t)|^p\, dt \leqslant C \sum_{n=1}^{\infty} n^{p-2+\gamma} (a_n^{\#})^p.$$

The rest of the proof of Theorem 5.3 is similar to that of Theorem 5.2. $\square$

From Lemma 5.1 and Theorems 5.2 and 5.3 for $\gamma = 0$ we obtain the following result.

Corollary 5.4. *Suppose that a sequence of complex numbers* $\{a_n\}_{n=1}^{\infty}$ *is of type* $\mathrm{GM}(\nu, D, p_0)$, *tends to zero, and for some* $p \in (1, \infty)$ *satisfies the condition*

$$\sum_{n=1}^{\infty} n^{p-2}|a_n|^p < \infty. \tag{5.12}$$

Then (5.7) *holds, and the series* (1.2) *is the Fourier series of its sum* $f \in L_p(\mathbb{T})$. *Moreover, the order estimate*

$$\|f\|_p^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2}|a_n|^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2}(a_n^{\#})^p \asymp |a_0|^p + \sum_{n=0}^{\infty} 2^{n(p-1)} M_n^p$$

holds, where the positive constants depend only on p *and the parameters* ν, D, *and* p_0.

Proof. It is sufficient to put $\alpha = p - 1$ in Lemma 5.1 and to apply Theorems 5.2 and 5.3 with $\gamma = 0$ and Theorem 2.9 with $\alpha = 0$. □

Interestingly, for sequences with rare sign changes the Hardy–Littlewood theorem for $p \geqslant 2$ is also valid under the condition of weak monotonicity.

Corollary 5.5. *Let* $\{a_n\}_{n=1}^{\infty}$ *be a null sequence of type* SC_ξ *for some* $\xi \in \mathbb{N}$ *(see Definition* 3.9*) and such that the condition* (2.10) *with* $p_0 \geqslant 1$ *and the condition*

$$\sum_{n=1}^{\infty} n^{p-2}\, |a_n|^p < \infty \quad \text{with} \quad p \in [2, \infty)$$

hold. Then the series (1.2) *is the Fourier series of its sum* $f \in L_p(\mathbb{T})$ *and*

$$\|f\|_p^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2}|a_n|^p.$$

In particular, this relation is valid for $p \geqslant 2$, *for positive sequences* $\{a_n\}_{n=1}^{\infty} \in \mathrm{WM}(p_0)$, $p_0 \geqslant 1$.

Proof. An upper bound follows from the Hardy–Littlewood inequality in the general case without additional conditions on $\{a_n\}$. To get a lower bound, we use the inequality (3.42) in Theorem 3.10 and follow the method used in the proof of Theorem 5.2 with $\alpha = 0$. □

We remark that Corollary 5.5 was proved in [62] under more restrictive assumptions than $\mathrm{WM}(p_0)$, $p_0 \geqslant 1$, but without the condition SC_ξ. As the following result shows, for $p > 2$ the condition of positivity or the more general condition $\{a_n\} \in \mathrm{SC}_\xi$ is fundamental in the previous corollary.

Remark 5.6. There exists a continuous function $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$ such that $\{a_n\}_{n=1}^{\infty} \in \mathrm{WM}(p_0)$ for any $p_0 \geqslant 1$. Then for any $p > 2$

$$\sum_{n=1}^{\infty} n^{p-2}|a_n|^p = \infty.$$

Indeed, it is sufficient to consider the series

$$f(x)=\sum_{n=1}^{\infty}2^{-n/2}n^{-2}\sum_{k=2^{n-1}}^{2^n-1}\varepsilon_k e^{ikt}, \tag{5.13}$$

where $\{\varepsilon_k\}_{k=0}^{\infty}$, $\varepsilon_k=\pm1$, $k\geqslant 0$, is a Rudin–Shapiro sequence (see [72], Theorem 1, and [79]).

Using the well-known estimate $\bigl|\sum_{k=0}^{N}\varepsilon_k e^{ikt}\bigr|<5\sqrt{N+1}$ for all $t\in[0,2\pi]$ and $N=0,1,\dots$, we get that

$$\sum_{n=1}^{\infty}\biggl|2^{-n/2}n^{-2}\sum_{k=2^{n-1}}^{2^n-1}\varepsilon_k e^{ikt}\biggr|\leqslant C\sum_{n=1}^{\infty}n^{-2}<\infty.$$

It is clear that for $a_k=2^{-n/2}n^{-2}\varepsilon_k$ the sequence $\{|a_k|\}$ is non-increasing with respect to k. Therefore, $\{a_k\}_{n=1}^{\infty}\in \mathrm{WM}(p_0)$ for any $p_0\geqslant 1$. On the other hand,

$$\sum_{n=1}^{\infty}n^{p-2}|a_n|^p=\sum_{n=1}^{\infty}2^{-np/2}n^{-2p}\sum_{k=2^{n-1}}^{2^n-1}k^{p-2}\asymp\sum_{n=1}^{\infty}2^{n(p/2-1)}n^{-2p}=\infty.$$

6. Order estimates for moduli of smoothness in L_p

6.1. Moduli of smoothness and Fourier coefficients. The next lemma is a well-known result on realization of the K-functional [80], [21]. For completeness we present a simple proof of this result.

Lemma 6.1. *Assume that (1.2) is the Fourier expansion of a function $f\in L_p(\mathbb{T})$, $p\in(1,\infty)$. Then for any positive integer β and any $\delta\geqslant 0$*

$$\omega_\beta(f,\delta)_p^p\asymp\int_0^{2\pi}\biggl|\sum_{n=1}^{\infty}\min\{(n\delta)^\beta,\pi^\beta\}a_n\cos(nx)\biggr|^p\,dx, \tag{6.1}$$

where the corresponding constants depend only on p and β.

Proof. Let $\delta>0$ and $N=[\pi/\delta]$, and define

$$S_N(x)=a_0+\sum_{n=1}^{N}a_n\cos(nx),\qquad f_N(x)=f(x)-S_N(x),$$

$$\varphi_\delta(x)=\sum_{n=1}^{\infty}a_n\min\{(n\delta)^\beta,\pi^\beta\}\cos(nx),$$

$$g_N(x)=\sum_{n=1}^{N}a_n\biggl(2\sin\frac{nh}{2}\biggr)^\beta\cos\biggl(nx+\frac{\pi\beta}{2}\biggr),$$

$$\Delta_h^\beta f(x)=\sum_{k=0}^{\beta}(-1)^k\binom{\beta}{k}f(x+(\beta-k)h)=g_N\biggl(x+\frac{\beta h}{2}\biggr)+\Delta_h^\beta f_N(x).$$

Since the $L_p(\mathbb{T})$-norms of all the partial sums of the function g_N do not exceed $C_p\|g_N\|_p$ and the function $\sin(nh/2)/(nh/2)$ is non-increasing for $n = 1, \dots, N$ and lies in the interval $[2/\pi, 1]$, applying the Abel transformation gives us that

$$C_{p,\beta}\|S_N^{(\beta)}\|_p|h|^\beta \leqslant \|g_N\|_p \leqslant C_p\|S_N^{(\beta)}\|_p|h|^\beta,$$

where

$$S_N^{(\beta)}(x) = \sum_{n=1}^{N} a_n n^\beta \cos\left(nx + \frac{\pi\beta}{2}\right).$$

It is also well known that $\|f_N\|_p \leqslant C_{p,\beta}\omega_\beta(f, \delta)_p$. Thus,

$$\|\Delta_h^\beta f\|_p \leqslant \|g_N\|_p + 2^\beta\|f_N\|_p \leqslant C_p\|S_N^{(\beta)}\|_p\delta^\beta + 2^\beta\|f_N\|_p \leqslant C_{p,\beta}\|\varphi_\delta\|_p.$$

Similarly,

$$\|g_N\|_p \leqslant \|\Delta_h^\beta f\|_p + 2^\beta\|f_N\|_p \leqslant C_{p,\beta}\omega_\beta(f, \delta)_p.$$

This implies that

$$\|S_N^{(\beta)}\|_p|h|^\beta + \|f_N\|_p \leqslant C_{p,\beta}\omega_\beta(f, \delta)_p.$$

Taking $h = \delta$, we arrive at the estimate

$$\|\varphi_\delta\|_p \leqslant \|S_N^{(\beta)}\|_p\delta^\beta + \pi^\beta\|f_N\|_p \leqslant C_{p,\beta}\omega_\beta(f, \delta)_p.$$

These inequalities immediately imply (6.1). □

Corollary 5.4 implies the following result for the transformed series.

Theorem 6.2. *Assume that a sequence of complex numbers* $\{a_n\}_{n=1}^\infty$ *is of type* $\mathrm{GM}(\nu, D, p_0)$, *tends to zero, and for some* $p \in (1, \infty)$ *satisfies the condition* (5.12). *If a sequence of complex numbers* $\{\gamma_n\}_{n=1}^\infty$ *satisfies the condition* GMS *and*

$$|\gamma_n| \leqslant K|\gamma_{2n}| \quad \text{for all } n \geqslant 1, \tag{6.2}$$

then the series

$$a_0 + \sum_{n=1}^{\infty} \gamma_n a_n \cos(nx) \quad \text{or} \quad \sum_{n=1}^{\infty} \gamma_n a_n \sin(nx) \tag{6.3}$$

is the Fourier series of its sum $f_\gamma \in L_p(\mathbb{T})$, *and*

$$\|f_\gamma\|_p^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2}|\gamma_n a_n|^p \asymp |a_0|^p + \sum_{n=0}^{\infty} 2^{n(p-1)}(|\gamma_{2^n}|M_n)^p, \tag{6.4}$$

where the positive constants depend only on p, K, *and the parameters* ν, D, *and* p_0. *Moreover, if* $\sum_{k=1}^n 2^{k(p-1)}|\gamma_{2^k}|^p \leqslant C\, 2^{n(p-1)}|\gamma_{2^n}|^p$, *then*

$$\|f_\gamma\|_p^p \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-2}|\gamma_n|^p(a_n^{\#})^p. \tag{6.5}$$

Remark 6.3. Note that $\{\gamma_n\}_{n=1}^\infty$ can be taken as a positive non-decreasing sequence satisfying the condition

$$\gamma_{2n} \leqslant K\gamma_n \quad \text{for all } n \geqslant 1. \tag{6.6}$$

In this case, the relations (6.4) and (6.5) hold.

Proof of Theorem 6.2. Taking into account Property 2.1, we see that the sequence $\{\gamma_n a_n\}_{n=1}^\infty$ is of type $\mathrm{GM}(\nu, D_\gamma, p_0)$. We note that if $\{\gamma_n\}_{n=1}^\infty$ satisfies the conditions GMS and (6.2), then $\gamma_n \asymp \gamma_k$ for all n and k with $k \leqslant n \leqslant 2k$. Hence, by Theorem 5.2,

$$\|f_\gamma\|_p^p \asymp |a_0|^p + \sum_{n=1}^\infty n^{p-2}|\gamma_n a_n|^p \asymp |a_0|^p + \sum_{n=1}^\infty n^{p-2} \max_{n\leqslant k<\infty} |\gamma_k a_k|^p.$$

By Lemma 5.1 we have

$$\|f_\gamma\|_p^p \asymp |a_0|^p + \sum_{n=0}^\infty 2^{n(p-1)} \max_{2^n\leqslant k<2^{n+1}} |\gamma_k a_k|^p \asymp |a_0|^p + \sum_{n=0}^\infty 2^{n(p-1)}(|\gamma_{2^n}|M_n)^p.$$

An upper bound in (6.5) follows from (6.4). Conversely,

$$\sum_{n=1}^\infty n^{p-2}|\gamma_n|^p(a_n^\#)^p \asymp \sum_{n=0}^\infty 2^{n(p-1)}|\gamma_{2^n}|^p(a_{2^n}^\#)^p \leqslant \sum_{n=0}^\infty 2^{n(p-1)}|\gamma_{2^n}|^p \sum_{k=n}^\infty M_k^p.$$

Using the conditions on γ_n, we see that the right-hand side sum does not exceed

$$C\sum_{n=0}^\infty 2^{n(p-1)}|\gamma_{2^n}|^p M_k^p \asymp \|f_\gamma\|_p^p. \qquad \square$$

Theorem 6.4. *Assume that* $p \in (1,\infty)$ *and* (1.2) *is the Fourier expansion of a function* $f \in L_p(\mathbb{T})$ *with coefficients* $\{a_n\}_{n=1}^\infty$ *of type* $\mathrm{GM}(\nu, D, p_0)$. *Then for any positive integer* β *and any* $\delta > 0$

$$\omega_\beta(f,\delta)_p \asymp \left(\delta^{p\beta}\sum_{n=1}^{[\pi/\delta]} n^{p-2+p\beta}(a_n^\#)^p + \sum_{n=1+[\pi/\delta]}^\infty n^{p-2}(a_n^\#)^p\right)^{1/p} \tag{6.7}$$

$$\asymp \left(\delta^{p\beta}\sum_{n=1}^{[\pi/\delta]} n^{p-2+p\beta}|a_n|^p + \sum_{n=1+[\pi/\delta]}^\infty n^{p-2}|a_n|^p\right)^{1/p}, \tag{6.8}$$

where the corresponding constants depend only on p, β, ν, D, *and* p_0.

Proof. Let $\delta > 0$ and $N = [\pi/\delta]$. For $n \geqslant 1$ we put

$$\gamma_n = \min\{(n\delta)^\beta, \pi^\beta\}.$$

Then

$$\gamma_{2^n} = \min\{(2^n\delta)^\beta, \pi^\beta\} \leqslant 2^\beta\gamma_{2^{n-1}} \quad \text{for all } n \geqslant 1,$$

that is, the condition (6.6) holds for $K = 2^\beta$. Therefore, from Remark 6.3 we get that

$$\int_0^{2\pi}\left|\sum_{n=1}^\infty a_n \min\{(n\delta)^\beta,\pi^\beta\}\cos(nx)\right|^p dx \asymp \sum_{n=1}^\infty n^{p-2}(\gamma_n a_n^{\#})^p \asymp \sum_{n=1}^\infty n^{p-2}\gamma_n^p|a_n|^p,$$

that is, by (6.1) the relations (6.7) and (6.8) hold. □

Proof of Theorem 1.1. A non-increasing sequence of non-negative numbers is a sequence of type $\mathrm{GM}(1, 2^{1/p_0}, p_0)$. Therefore, we also have the relations (5.5), (5.11), and (6.7), where instead of f one can take $f^{\#}$, and the right-hand sides of these relations remain the same. This immediately implies Theorem 1.1. □

6.2. Applications to direct and inverse theorems. The following direct and inverse theorems are well known in approximation theory (see [20], p. 210):

$$\begin{aligned}\frac{1}{n^l}\left(\sum_{\nu=0}^n(\nu+1)^{\tau l-1}E_\nu^\tau(f)_p\right)^{1/\tau} &\lesssim \omega_l\left(f,\frac{1}{n}\right)_p \\ &\lesssim \frac{1}{n^l}\left(\sum_{\nu=0}^n(\nu+1)^{ql-1}E_\nu^q(f)_p\right)^{1/q},\end{aligned} \tag{6.9}$$

where $f\in L_p(\mathbb{T})$, $1<p<\infty$, $l,n\in\mathbb{N}$, $q=\min\{2,p\}$, $\tau=\max\{2,p\}$, and $E_n(f)_p$ is the best approximation of f in L_p by trigonometric polynomials of degree n. Note that the inequalities (6.9) are equivalent (see [16]) to the relations

$$\begin{aligned}t^l\left(\int_t^1 u^{-\tau l-1}\omega_{l+1}^\tau(f,u)_p\,du\right)^{1/\tau} &\lesssim \omega_l(f,t)_p \\ &\lesssim t^l\left(\int_t^1 u^{-ql-1}\omega_{l+1}^q(f,u)_p\,du\right)^{1/q}.\end{aligned}$$

The next theorem gives a more precise connection between the moduli of smoothness $\omega_l(f,t)_p$ and $\omega_{l+1}(f,t)_p$, as well as the relationship between the modulus of smoothness $\omega_l(f,t)_p$ and the best approximation $E_k(f)_p$ for a function f with general monotone Fourier coefficients.

Theorem 6.5. *Let* $1<p<\infty$. *Assume that* (1.2) *is the Fourier expansion of a function* $f\in L_1(\mathbb{T})$ *with coefficients* $\{a_n\}_{n=1}^\infty$ *of type* $\mathrm{GM}(\nu,D,p_0)$ *with* $p_0>1$. *Then*

$$\begin{aligned}\omega_l(f,t)_p &\asymp t^l\left(\int_t^1 u^{-lp}\omega_{l+1}^p(f,u)_p\,\frac{du}{u}\right)^{1/p} \\ &\asymp t^l\left(\sum_{k=0}^{[1/t]}(k+1)^{lp-1}E_k^p(f)_p\right)^{1/p}, \qquad 0<t<\frac{1}{2}.\end{aligned}$$

Theorem 6.5 follows immediately from the relations $\omega_\beta(f,\delta)_p \asymp \omega_\beta(f^{\#},\delta)_p$ and the corresponding results for series with monotone coefficients (see [41]).

7. Characterization of function spaces

7.1. Lorentz spaces. For a measurable function f on $[0,2\pi]$ we define its non-increasing rearrangement f^* by

$$f^*(t)=\inf\{\sigma\colon \mu\{x\in[0,2\pi]\colon |f(x)|>\sigma\}\leqslant t\},$$

where μ is the Lebesgue measure on $[0,2\pi]$. For $0<r,s\leqslant\infty$ we define the Lorentz space $L_{r,s}(\mathbb{T})$ as the set of measurable functions for which the functional

$$\|f\|_{L_{r,s}}:=\begin{cases}\biggl(\displaystyle\int_0^{2\pi}\bigl(t^{1/r-1/s}f^*(t)\bigr)^s\,dt\biggr)^{1/s} & \text{for } 0<r<\infty,\ 0<s<\infty,\\ \displaystyle\sup_{t\in[0,2\pi]} t^{1/r}f^*(t) & \text{for } 0<r\leqslant\infty,\ s=\infty,\end{cases}$$

is finite.

We define the weighted Lebesgue space $L^s_{w(r,s)}(\mathbb{T})$ with weight $w(r,s)(t)\equiv t^{1/r-1/s}$ as the set of measurable functions f for which the functional

$$\|f\|_{L^s_{w(r,s)}}:=\begin{cases}\biggl(\displaystyle\int_0^{2\pi}\bigl|t^{1/r-1/s}f(t)\bigr|^s\,dt\biggr)^{1/s} & \text{for } 0<r<\infty,\ 0<s<\infty,\\ \displaystyle\operatorname*{ess\,sup}_{t\in[0,2\pi]} t^{1/r}|f(t)| & \text{for } 0<r\leqslant\infty,\ s=\infty,\end{cases}$$

is finite.

Let $\{a_n^*\}_{n=1}^\infty$ be the non-increasing rearrangement of a sequence $\{|a_n|\}_{n=1}^\infty$. For $0<r,s\leqslant\infty$ we define the discrete Lorentz space as follows: $a\in l_{r,s}$ if $\|a\|_{l_{r,s}}<\infty$, where

$$\|a\|_{l_{r,s}}=\begin{cases}\biggl(\displaystyle\sum_{n=0}^\infty (n^{1/r-1/s}a_n^*)^s\biggr)^{1/s}, & 0<r,s<\infty,\\ \displaystyle\sup_{n\in\mathbb{N}} n^{1/r}a_n^*, & s=\infty.\end{cases}$$

The discrete spaces $l^s_{w(r,s)}$ are defined similarly with a_n^* in place of $|a_n|$.

Note that $\|f\|_{L_{r,r}}=\|f\|_{L^r_{w(r,r)}}=\|f\|_{L_r}$. Moreover, Hardy's inequality for rearrangements

$$\int_0^{2\pi}|f(x)g(x)|\,dx\leqslant\int_0^{2\pi}f^*(t)g^*(t)\,dt \tag{7.1}$$

(see [8], p. 44) implies that

$$\|f\|_{L_{r,s}}\geqslant\|f\|_{L^s_{w(r,s)}}\quad\text{for } s\leqslant r$$

and

$$\|f\|_{L_{r,s}}\leqslant\|f\|_{L^s_{w(r,s)}}\quad\text{for } s\geqslant r.$$

For an integrable function f with Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^\infty\bigl(a_n\cos(nx)+b_n\sin(nx)\bigr) \tag{7.2}$$

the following fundamental results of Pitt and Hardy–Littlewood–Paley are well known (see [68] and [82]):

$$\|\mathbf{a}\|_{l_{r',s}} + \|\mathbf{b}\|_{l_{r',s}} \lesssim \|f\|_{L_{r,s}(\mathbb{T})} \tag{7.3}$$

and

$$\|\mathbf{a}\|_{l^s_{w(r',s)}} + \|\mathbf{b}\|_{l^s_{w(r',s)}} \lesssim \|f\|_{L^s_{w(r,s)}(\mathbb{T})} \tag{7.4}$$

for

$$1 < r \leqslant s \leqslant r'.$$

The main result of this subsection is the following Hardy–Littlewood–Sagher type theorem for functions with general monotone coefficients.

Theorem 7.1. *Assume that (7.2) is the Fourier expansion of a function $f \in L(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ of type $\mathrm{GM}(\nu, D, p_0)$ with $p_0 > 1$. Then for arbitrary $1 < r, s < \infty$*

$$\|f\|_{L_{r,s}} \asymp \|\mathbf{a}\|_{l_{r',s}} + \|\mathbf{b}\|_{l_{r',s}} \asymp \|\mathbf{a}^\#\|_{l_{r',s}} + \|\mathbf{b}^\#\|_{l_{r',s}}, \tag{7.5}$$

$$\|f\|_{L^s_{w(r,s)}} \asymp \|\mathbf{a}\|_{l^s_{w(r',s)}} + \|\mathbf{b}\|_{l^s_{w(r',s)}} \asymp \|\mathbf{a}^\#\|_{l^s_{w(r',s)}} + \|\mathbf{b}^\#\|_{l^s_{w(r',s)}}, \tag{7.6}$$

$$\|f\|_{L^s_{w(r,s)}} \asymp \|f^\#\|_{L^s_{w(r,s)}} \asymp \|f\|_{L_{r,s}} \asymp \|f^\#\|_{L_{r,s}}, \tag{7.7}$$

where, as in (1.7), $f^\#(x) = a_0 + \sum_{n=1}^\infty \bigl(a_n^\# \cos(nx) + b_n^\# \sin(nx)\bigr)$.

Proof. It suffices to consider the case of a cosine series, that is, when $b_n = 0$. The equivalence $\|f\|_{L^s_{w(r,s)}} \asymp \|\mathbf{a}\|_{l^s_{w(r',s)}} \asymp \|\mathbf{a}^\#\|_{l^s_{w(r',s)}}$ in (7.6) follows from Theorem 5.3 with $s = p$ and $\gamma = 1 - s/r$. Further, for the same s and γ we note that the relation $\|f\|_{L^s_{w(r,s)}} \asymp \|f^\#\|_{L^s_{w(r,s)}}$ follows from Theorem 1.1. The equivalences

$$\|\mathbf{a}^\#\|_{l_{r',s}} \asymp \|\mathbf{a}^\#\|_{l^s_{w(r',s)}} \asymp \|f^\#\|_{L^s_{w(r,s)}} \asymp \|f^\#\|_{L_{r,s}}$$

can be obtained from Sagher's well-known results for functions with monotone coefficients [73].

Furthermore, Theorem 2.12 implies that $\|\mathbf{a}\|_{l_{r',s}} \asymp \|\mathbf{a}\|_{l^s_{w(r',s)}}$.

To complete the proof it is sufficient to show that

$$\|f\|_{L_{r,s}} \asymp \|\mathbf{a}^\#\|_{l^s_{w(r',s)}}.$$

By Theorem 3.3,

$$|a_n| \leqslant a_n^\# \leqslant C \int_0^\pi \left(\min\left\{1, \frac{\pi}{nt}\right\}\right)^2 |f(t)|\, dt, \tag{7.8}$$

which, by (7.1), implies that

$$|a_n| \leqslant a_n^\# \leqslant C \int_0^\pi \left(\min\left\{1, \frac{\pi}{nt}\right\}\right)^2 f^*(t)\, dt. \tag{7.9}$$

Then the lower bound $C\|f\|_{L_{r,s}} \geqslant \|\mathbf{a}^\#\|_{l^s_{w(r',s)}}$ can be obtained as in the proof of (5.10) with the help of Hardy's inequalities for averages.

The reverse estimate is obtained as follows. We have

$$\|f\|_{L_{r,s}} \leqslant C\biggl(\sum_{n=0}^{\infty} 2^{-ns/r} \int_{\pi\, 2^{-n-1}}^{\pi\, 2^{-n}} (f^*(t))^s \,\frac{dt}{t}\biggr)^{1/s}$$
$$\leqslant C\biggl(\sum_{n=0}^{\infty} 2^{-ns/r}\biggl(|a_0|^s + \biggl(\sum_{k=0}^{n-1} 2^k M_k\biggr)^s + 2^{ns}\biggl(\sum_{k=2^n}^{\infty} |a_k - a_{k+1}|\biggr)^s\biggr)\biggr)^{1/s},$$

where we have applied the estimate (5.8). Further, the proof of the estimate $\|f\|_{L_{r,s}} \leqslant C\|\mathbf{a}^{\#}\|_{l^s_{w(r',s)}}$ actually repeats the proof of (5.9). $\square$

7.2. Besov spaces.

Definition 7.2. Let $1 \leqslant p \leqslant \infty$ and $\tau, \alpha > 0$. The Besov space $B^{\alpha}_{p,\tau}(\mathbb{T})$ is the set of functions $f \in L_p(\mathbb{T})$ such that

$$\|f\|_{B^{\alpha}_{p,\tau}} := \|f\|_{L_p} + |f|_{B^{\alpha}_{p,\tau}} := \|f\|_{L_p} + \biggl(\int_0^1 \biggl(\frac{\omega_l(f,t)_p}{t^{\alpha}}\biggr)^{\tau} \frac{dt}{t}\biggr)^{1/\tau} < \infty,$$

where $l > \alpha$.

It is well known that the space $B^{\alpha}_{p,\tau}(\mathbb{T})$ does not depend on the choice of l. By $\operatorname{Lip}(\alpha,p)$ we denote the Lipschitz class

$$\operatorname{Lip}(\alpha,p) := \bigl\{f \in L_p(\mathbb{T}) : \omega_l(f,\delta)_p = O(\delta^{\alpha})\bigr\}, \qquad 0 < \alpha < l.$$

Note that $\operatorname{Lip}(\alpha,p) = B^r_{p,\infty}$.

Theorem 7.3. *Let $0 < \tau \leqslant \infty$, $\alpha > 0$, and $1 < p \leqslant \infty$. Assume that (7.2) is the Fourier expansion of a function $f \in L_1(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of type $\mathrm{GM}(\nu, D, p_0)$ with $p_0 > 1$. Then the following conditions are equivalent:*

(i) $f \in B^{\alpha}_{p,\tau}(\mathbb{T})$;

(ii) $f^{\#} \in B^{\alpha}_{p,\tau}(\mathbb{T})$;

(iii) $\displaystyle\sum_{n=1}^{\infty} n^{\alpha\tau+\tau-\tau/p-1}(a_n^{\#} + b_n^{\#})^{\tau} < \infty$ *if* $0 < \tau < \infty$,
$\displaystyle\sup_n n^{\alpha+1-1/p}(a_n^{\#} + b_n^{\#}) < \infty$ *if* $\tau = \infty$;

(iv) $\displaystyle\sum_{n=1}^{\infty} n^{\alpha\tau+\tau-\tau/p-1}(|a_n| + |b_n|)^{\tau} < \infty$ *if* $0 < \tau < \infty$,
$\displaystyle\sup_n n^{\alpha+1-1/p}(|a_n| + |b_n|) < \infty$ *if* $\tau = \infty$.

Remark 7.4. (i) In the case $1 < p < \infty$ Theorem 7.3 is well known for series with monotone coefficients (see [3], [70], [73] and, for some generalizations, see [41], Theorem 7.3, [28], [55], [56], [83]). For series with quasi-monotone coefficients, see [64] and [65]. In the case of continuous functions ($p = \infty$), see [36], [60], [91], [90].

(ii) The relation (i) $\Leftrightarrow$ (iv) is closely related to the following well-known result for general trigonometric series (see [4], [48], [65]):

$$\biggl(\sum_{n=1}^{\infty} n^{\alpha\tau+\tau-\tau/p-1}(|a_n| + |b_n|)^{\tau}\biggr)^{1/\tau} \leqslant C|f|_{B^{\alpha}_{p,\tau}}, \tag{7.10}$$

where $1 \leqslant p \leqslant 2$, $0 < \tau \leqslant p'$, and $\alpha > 0$ (see also (4.17)). From Theorem 7.3 it is clear that for series with p-general monotone coefficients, firstly, the inequality (7.10) is valid for all values of the parameters, and secondly, the reverse inequality is also true.

Proof of Theorem 7.3. Assume first that $\tau < \infty$. In the case $1 < p < \infty$ the relations (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) follow from the equivalence $\omega_\beta(f,\delta)_p \asymp \omega_\beta(f^{\#},\delta)_p$ and (6.7). Theorem 2.9 implies that (iv) $\Leftrightarrow$ (iii).

Consider the case $p = \infty$. By the well-known characterization of Besov spaces in terms of best approximations [61] and the relation (4.16), Corollary 4.11 implies ($\alpha, \tau > 0$) that

$$\|f\|_{B^{\alpha}_{\infty,\tau}} \asymp \|f\|_{L_\infty} + \biggl(\sum_{n=1}^{\infty} n^{\alpha\tau-1} E_{n-1}(f)_\infty^\tau\biggr)^{1/\tau}$$
$$\asymp \biggl(\sum_{n=1}^{\infty} n^{\alpha\tau+\tau-1}(a_n^{\#}+b_n^{\#})^\tau\biggr)^{1/\tau} \asymp \biggl(\sum_{n=1}^{\infty} n^{\alpha\tau+\tau-1}(|a_n|+|b_n|)^\tau\biggr)^{1/\tau},$$

that is, (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

Now let $\tau = \infty$. For $1 < p < \infty$ it is clear that the condition $|a_n| + |b_n| \leqslant Cn^{-\alpha-1+1/p}$ implies that

$$\omega_l\Bigl(f,\frac{1}{n}\Bigr)_p \asymp \biggl(\sum_{\nu=1}^{\infty} \nu^{p-2}\Bigl(\min\Bigl\{1,\frac{\nu}{n}\Bigr\}\Bigr)^{lp}(|a_\nu|+|b_\nu|)^p\biggr)^{1/p} \leqslant Cn^{-\alpha},$$

that is, $f \in B^{\alpha}_{p,\infty}$. To prove the converse result, we use the relation (6.7) and the monotonicity of $a_n^{\#}$ and $b_n^{\#}$.

For $p = \infty$, we use the estimate (3.38) with $q = \alpha$ (see Corollary 3.5) and get that

$$|a_n|+|b_n| \leqslant Cn^{-\alpha-1} \max_{1/n\leqslant u\leqslant 1} \frac{\omega_{[q]+1}(f,u)_\infty}{u^\alpha} \asymp n^{-\alpha-1} \max_{1/n\leqslant u\leqslant 1} \frac{\omega_l(f,u)_\infty}{u^\alpha} \leqslant Cn^{-\alpha-1}.$$

If $|a_n|+|b_n| \leqslant Cn^{-\alpha-1}$, then using the inverse inequalities in approximation theory, we deduce that

$$\omega_l\Bigl(f,\frac{1}{n}\Bigr)_\infty \leqslant Cn^{-l}\sum_{\nu=1}^{n} \nu^{l-1} E_{\nu-1}(f)_\infty \leqslant Cn^{-l}\sum_{\nu=1}^{n} \nu^{l-1}\|f - S_\nu(f)\|_\infty$$
$$\leqslant Cn^{-l}\sum_{\nu=1}^{n} \nu^{l-1}\sum_{n=\nu}^{\infty}(|a_n|+|b_n|) \leqslant Cn^{-\alpha}. \qquad \square$$

Theorems 7.1 and 7.3 immediately yield the following result.

Corollary 7.5. *Under the conditions of Theorem* 7.1, *if* $1 < p < r$, *then*

$$B^{\theta}_{p,s}(\mathbb{T}) = L_{r,s}(\mathbb{T}), \qquad \theta = \frac{1}{p} - \frac{1}{r}, \quad 1 < s < \infty. \tag{7.11}$$

In particular, $B^{\theta}_{p,s}(\mathbb{T}) = L_s(\mathbb{T})$, $\theta = 1/p - 1/s > 0$.

Note that these embeddings do not only show the sharpness of the known embeddings [95]

$$B^{1/p-1/r}_{p,s}(\mathbb{T}) \hookrightarrow L_{r,s}(\mathbb{T}), \quad p<r, \quad \text{and} \quad B^{1/p-1/s}_{p,s}(\mathbb{T}) \hookrightarrow L_{s}(\mathbb{T}), \quad p<s,$$

but they also describe a class of functions where the corresponding spaces coincide. Such classes of 'boundary' functions are extremely useful in functional analysis (see, for example, [22]–[24]).

Moreover, a comparison of Theorem 7.3, (iv), with Theorems 5.2 and 5.3, enables one to obtain a criterion for a function to belong to a Besov space with low smoothness in terms of the integrability of this function with a weight.

Corollary 7.6. *If* $\{a_n\}_{n=1}^{\infty} \in \mathrm{GM}(\nu, D, p_0)$, $p_0 \geqslant 1$, *and* $p \in (1,\infty)$, *then*

$$f \in B^{\alpha}_{p,p}(\mathbb{T}) \quad \Longleftrightarrow \quad \int_0^{\pi} \frac{|f(t)|^p}{t^{\alpha p}}\,dt < \infty$$

for $\alpha < 1/p$ *in the case of even functions and for* $\alpha < 1+1/p$ *in the case of odd functions.*

Note also that Theorem 5.2 enables us to obtain an analogue of Theorem 7.3 for Calderón spaces [13]:

$$\Lambda_l(L_p;E) = \bigl\{ f\in L_p : \|f\|_p + \|\omega_l(f;\,\cdot\,)_p\|_E < \infty \bigr\}.$$

For example, we define the Besov–Nikolskii class $\mathrm{BN}^{\alpha,\beta}_{p,\tau}(\varphi)$ as follows:

$$\mathrm{BN}^{\alpha,\beta}_{p,\tau}(\varphi) = \biggl\{ f\in L_p \colon \biggl(\int_0^{\delta} \biggl(\frac{\omega_l(f,t)_p}{t^{\alpha}}\biggr)^{\tau} \frac{dt}{t} + \delta^{\beta\tau} \int_{\delta}^{1} \biggl(\frac{\omega_l(f,t)_p}{t^{\alpha+\beta}}\biggr)^{\tau} \frac{dt}{t} \biggr)^{1/\tau} \leqslant C\varphi(\delta) \biggr\},$$

where $0<\theta,\ \alpha,\beta<\infty$, $\alpha<l$, and φ is a continuous almost increasing function on $(0,1)$ satisfying the condition $\varphi(2\delta)\leqslant C\varphi(\delta)$. This is a more general space than a Besov space [86]. Under the conditions of Theorem 7.3 we get that $f\in \mathrm{BN}^{\alpha,\beta}_{p,\theta}(\varphi)$ if and only if

$$\biggl(\sum_{\nu=1}^{\infty} \nu^{(\alpha+1/p')\tau-1} \biggl(\min\biggl\{1,\frac{\nu}{n}\biggr\}\biggr)^{\beta\tau} (|a^{\#}_{\nu}| + |b^{\#}_{\nu}|)^{\tau} \biggr)^{1/\tau} \leqslant C\varphi\biggl(\frac{1}{n}\biggr).$$

In particular, this extends results in [49], [55], [56], [84], [86].

One can also obtain necessary and sufficient conditions for a function to belong to Besov spaces with logarithmic smoothness $B^{\theta,d}_{p,s}(\mathbb{T})$ (see [23]) or to the Lipschitz space $\mathrm{Lip}^{(\alpha,-b)}_{p,q}(\mathbb{T})$ (see [22]).

7.3. Sobolev spaces. As usual, we define the Sobolev space $W^r_p(\mathbb{T})$ as follows:

$$\|f\|_{W^r_p(\mathbb{T})} := \|f\|_{L_p(\mathbb{T})} + \|f^{(r)}\|_{L_p(\mathbb{T})} < \infty.$$

It is easy to extend this definition to the case of positive smoothness $r>0$. Property 2.1 and Remark 6.3 immediately yield the following result.

Theorem 7.7. *Let $r > 0$ and $1 < p < \infty$. Assume that (7.2) is the Fourier expansion of a function $f \in L_1(\mathbb{T})$ with coefficients $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ of type* $\mathrm{GM}(\nu, D, p_0)$ *with $p_0 > 1$. Then*

$$\|f\|_{W_p^r(\mathbb{T})} \asymp \|f^{\#}\|_{W_p^r(\mathbb{T})} \asymp \biggl(\sum_{n=1}^{\infty} n^{rp+p-2}(|a_n| + |b_n|)^p\biggr)^{1/p}.$$

By Theorems 7.7 and 7.3, we see that for $r > 0$

$$B_{p,p}^r(\mathbb{T}) = W_p^r(\mathbb{T}), \qquad 1 < p < \infty.$$

As in the case of the embedding (7.11), this result sharpens the known embeddings

$$B_{p,\min\{2,p\}}^r(\mathbb{T}) \hookrightarrow W_p^r(\mathbb{T}) \hookrightarrow B_{p,\max\{2,p\}}^r(\mathbb{T}), \qquad 1 < p < \infty$$

(see [95], for instance).

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