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О НЕКОТОРЫХ ОБОБЩЕНИЯХ ПЕРЕСТАНОВОЧНОСТИ И *S*-ПЕРЕСТАНОВОЧНОСТИ

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ON SOME GENERALIZATIONS OF PERMUTABILITY AND *S*-PERMUTABILITY

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Пусть H и X – подгруппы конечной группы G. Тогда мы говорим, что: H *X-квазиперестановочна* (соответственно, X_S *-квазиперестановочна*) в G, если G содержит такую подгруппу B, что $G = N_G(H)B$ и H X-перестановочна с B и со всеми подгруппами (соответственно, со всеми силовскими подгруппами) V из B такими, что $(|H|,|V|)=1$; H *X-проперестановочна* (соответственно, X_S *-проперестановочна*) в G, если G содержит такую подгруппу B, что $G = N_G(H)B$ и H X-перестановочна с B и со всеми подгруппами (соответственно, со всеми силовскими подгруппами) из B.

В данной работе мы анализируем влияние X-квазиперестановочных, X_S -квазиперестановочных, X-проперестановочных и X_S -проперестановочных подгрупп на строение группы G.

Ключевые слова: *конечная группа, X-квазиперестановочная подгруппа, X_S -квазиперестановочная подгруппа, X-проперестановочная подгруппа, X_S -проперестановочная подгруппа, силовская подгруппа, холлова подгруппа, p-разрешимая группа, p-сверхразрешимая группа, максимальная подгруппа, насыщенная формация, PST -группа, PT -группа.*

Let H and X be subgroups of a finite group G. Then we say that H is: *X-quasipermutable* (respectively, X_S *-quasipermutable*) in G provided G has a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (respectively, with all Sylow subgroups) V of B such that $(|H|,|V|)=1$; *X-propermutable* (respectively, X_S *-propermutable*) in G provided G has a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (respectively, with all Sylow subgroups) of B.

In this paper we analyze the influence of X-quasipermutable, X_S -quasipermutable, X-propermutable and X_S -propermutable subgroups on the structure of G.

Keywords: *finite group, X-quasipermutable subgroup, X_S -quasipermutable subgroup, X-propermutable subgroup, X_S -propermutable subgroup, Sylow subgroup, Hall subgroup, p-soluble group, p-supersoluble group, maximal subgroup, saturated formation, PST -group, PT -group.*

Introduction

Throughout this paper, all groups are finite and G always denotes a finite group. Moreover p is always supposed to be a prime and π is a subset of the set $\mathbb{P}$ of all primes; $\pi(G)$ denotes the set of all primes dividing $|G|$. The symbol $\pi(n)$ denotes the set of all primes dividing the number n; $\pi(G) = \pi(|G|)$. We say that $x \in G$ is a π *-element* of G provided $\pi(\langle x \rangle) \subseteq \pi$. The symbol $G^{\mathfrak{N}}$ denotes the nilpotent residual of G, that is, the smallest normal subgroup of G with nilpotent quotient.

Let A, B and X be subgroups of G. If $AB = BA$, then A is said to *permute* with B; if $AB^x = B^x A$, for some $x \in X$, then A is said to X *-permute* [1] with B; if $G = AB$, then B is called a *supplement* of A to G.

A subgroup H is said to be *quasinormal* [2] or *permutable* [3] in G if H permutes with all subgroups of G; H is said to be S *-permutable*, S *-quasinormal*, or π *-quasinormal* [4] in G if H permutes with all Sylow subgroups of G. In this paper we study the following generalizations of these concepts.

***Definition* 0.1.** Let H and X be subgroups of G. Then we say that H is X*-quasipermutable* (respectively, X_S*-quasipermutable*) in G provided G has a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (respectively, with all Sylow subgroups) V of B such that $(|H|,|V|) = 1$.

If $X = 1$ and H is X-quasipermutable (respectively, X_S-quasipermutable) in G, then we say that H is *quasipermutable* (respectively, S*-quasipermutable*) [5] in G.

***Definition* 0.2.** Let H and X be subgroups of G. Then we say that H is X*-propermutable* (respectively, X_S*-propermutable*) in G provided G has a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (respectively, with all Sylow subgroups) of B.

If $X = 1$ and H is X-propermutable (respectively, X_S-propermutable) in G, then we say that H is *propermutable* [5] (respectively, S*-propermutable* [6]) in G.

It is clear that every X-propermutable (respectively, X_S-propermutable) subgroup is X-quasipermutable (respectively, X_S-quasipermutable) and the inverse is not true in general. Note, for example, that the subgroup S_3 is quasipermutable, S-propermutable and not propermutable in S_4. If H is the subgroup of order 3 in S_3, then H is S-quasipermutable and not quasipermutable in S_4.

In fact, we meet X-quasipermutable, X_S-quasipermutable, X-propermutable and X_S-propermutable subgroups quite often.

***Examples*.** (1) A subgroup H of G is called X*-semipermutable* [1] in G provided H X-permutes with all subgroups of some supplement of H to G. Every X-semipermutable subgroup is X-propermutable. In order to prove that the inverse is not true in general, consider the following example. Let $p > q > r$ be primes such that qr divides $p-1$. Let P be a group of order p and $QR \le Aut(P)$, where Q and R are groups of order q and r, respectively. Let $G = P \rtimes (QR)$. Then R is 1-propermutable but not 1-semipermutable in G.

(2) A subgroup H of G is called SS*-quasinormal* [7] in G if H permutes with all Sylow subgroups of some supplement of H in G. Every SS-quasinormal subgroup is S-propermutable. The example in (1) shows that the inverse is not true in general.

(3) A subgroup H of G is called *semipermutable* (respectively, S*-semipermutable*) [8] in G if H permutes with all subgroups (respectively, with all Sylow subgroups) V of G such that $(|H|,|V|) = 1$. Every semipermutable (respectively, S-semipermutable) subgroup is quasipermutable (respectively, S-quasipermutable). The example in (1) shows that the inverse is not true in general.

(4) If $|H| = p^a$ and $H \le Z_\infty(G)$, then $H \le P$, where P is the Sylow p-subgroup of $Z_\infty(G)$. Therefore, since $G/C_G(P)$ is a p-group by [9], we have $G = N_G(H)G_p$ and $H \le P \le G_p$, where G_p is a Sylow p-subgroup of G. Hence H is S-propermutable in G.

(5) If G is metanilpotent, that is $G/F(G)$ is nilpotent, then for every Hall subgroup H of G we have $G = N_G(H)F(G)$. Therefore, in this case, every characteristic subgroup of every Hall subgroup of G is S-propermutable in G. In particular, every Hall subgroup of a supersoluble group is S-propermutable.

(6) If M is a maximal subgroup of a supersoluble group, then M is propermutable in G.

(7) A subgroup H of G is called *semi-normal* [10] in G provided H X-permutes with all subgroups of some supplement of H to G.

In last years, many researches (see, for example [1], [7], [11]–[30]) deal with some interesting subclasses of the classes of all X-quasipermutable, X_S-quasipermutable, X-propermutable, or X_S-propermutable subgroups. In fact, many results of these researches may be developed on the base of the concepts in Definitions 0.1 and 0.2. The results of this paper are partial illustration of this.

1 Base lemma

The first lemma is evident.

***Lemma* 1.1.** *Let A, B be subgroups of G and N, X be normal subgroups of G. Suppose that A X-permutes with B.*

(i) AN/N (XN/N)-permutes with BN/N. Hence AX/X permutes with BX/X.

(ii) If $X \le N_G(A)$, then A permutes with B.

***Lemma* 1.2.** *Let $H \le G$ and N, X be normal subgroups of G.*

(1) If H is X-quasipermutable (X_S-quasipermutable, respectively) in G and either H is a Hall subgroup of G or for every prime p dividing $|H|$ and for every Sylow p-subgroup H_p of H we have $H_p \nleq N$, then HN/N is (XN/N)-quasipermutable ($(XN/N)_S$-quasipermutable, respectively) in G/N.

(2) If H is X-propermutable (X_S-propermutable, respectively) in G, then HN/N is (XN/N)-propermutable ($(XN/N)_S$-propermutable, respectively) in G/N.

(3) *If* H *is* S *-quasipermutable in* G, $\pi = \pi(H)$ *and* G *is* π *-soluble, then* H *permutes with some Hall* π' *-subgroup of* G.

(4) *If* H *is* S *-quasipermutable in* G, *then* H *permutes with some Sylow* p *-subgroup of* G *for every prime* p *such that* $(p, |H|) = 1$.

(5) *If* H *is* S *-propermutable in* G, *then* H *permutes with some Sylow* p *-subgroup of* G *for every prime* p *dividing* $|G|$.

(6) *If* H *is* S *-propermutable in* G *and* G *is* π *-soluble, then* H *permutes with some Hall* π *-subgroup of* G.

(7) *If* H *is* S *-quasipermutable in* G, *then* $|G : N_G(H \cap N)|$ *is a* π *-number, where*

$$\pi = \pi(N) \cup \pi(H).$$

(8) *Suppose that* G *is* π *-soluble. If* H *is a Hall* π *-subgroup of* G *and* H *is quasipermutable* (S *-quasipermutable, respectively*) *in* G, *then* H *is propermutable* (S *-propermutable, respectively*) *in* G.

Proof. By hypothesis, there is a subgroup B such that $G = N_G(H)B$ and H X-permutes with B and with all subgroups (with all Sylow subgroups, respectively) L of B such that

$$(|H|, |L|) = 1.$$

(1) It is clear that

$$G/N = N_{G/N}(HN/N)(BN/N).$$

Let K/N be any subgroup (any Sylow subgroup, respectively) of BN/N such that

$$(|HN/N|, |K/N|) = 1.$$

Then $K = (K \cap B)N$. Let B_0 be a minimal supplement of $K \cap B \cap N$ to $K \cap B$. Then

$$K/N = (K \cap B)N/N =$$
$$= B_0(K \cap B \cap N)N/N = B_0N/N$$

and $K \cap B \cap N \cap B_0 = N \cap B_0 \le \Phi(B_0)$. Therefore

$$\pi(K/N) = \pi(K \cap B / K \cap B \cap N) = \pi(B_0),$$

so $(|HN/N|, |B_0|) = 1$. Suppose that some prime $p \in \pi(B_0)$ divides $|H|$, and let H_p be a Sylow p-subgroup of H. We shall show that $H_p \nleq N$. In fact, we may suppose that H is a Hall subgroup of G. But in this case, H_p is a Sylow p-subgroup of G. Therefore, since $p \in \pi(B_0) \subseteq \pi(G/N)$, $H_p \nleq N$. Hence p divides $|HN/N|$, a contradiction. Thus $(|H|, |B_0|) = 1$, so in the case when H is X-quasipermutable in G, H X-permutes with B_0 and hence HN/N (XN/N)-permutes with $K/N = B_0N/N$ by Lemma 1.1. Thus HN/N is (XN/N)-quasipermutable in G/N.

Finally, suppose that H is X_S-quasipermutable in G. In this case, B_0 is a p-subgroup of B, so for some Sylow p-subgroup B_p of B we have $B_0 \le B_p$ and $(|H|, p) = 1$. Hence

$$K/N = B_0N/N \le B_pN/N,$$

which implies that $K/N = B_pN/N$. But H X-permutes with B_p by hypothesis, so HN/N (XN/N)-permutes with K/N by Lemma 1.1. Therefore HN/N is $(XN/N)_S$-quasipermutable in G/N.

(2) See the proof of (1).

(3) By [31, VI, 4.6], there are Hall π'-subgroups E_1, E_2 and E of $N_G(H)$, B and G, respectively, such that $E = E_1E_2$. Then H permutes with all Sylow subgroups of E_2 by hypothesis, so

$$HE = H(E_1E_2) = (HE_1)E_2 = (E_1H)E_2 =$$
$$E_1(HE_2) = E_1(E_2H) = (E_1E_2)H = EH$$

by [32, A, 1.6].

(4), (5), (6) See the proof of (3).

(7) Let p be a prime such that $p \notin \pi$. Then by (3), there is a Sylow p-subgroup P of G such that $HP = PH$ is a subgroup of G. Hence $HP \cap N = H \cap N$ is a normal subgroup of HP. Thus p does not divide $|G : N_G(H \cap N)|$.

(8) Since G is π-soluble, B is π-soluble. Hence by [31, VI, 1.7], $B = B_\pi B_{\pi'}$, where B_π is a Hall π-subgroup of B and $B_{\pi'}$ is a Hall π'-subgroup of B. By [31, VI, 4.6], there are Hall π-subgroups N_π, B_π and G_π of $N_G(H)$, B and G, respectively, such that $G_\pi = N_\pi B_\pi$. But since $H \le N_\pi$, N_π is a Hall π-subgroup of G. Therefore $G_\pi = N_\pi B_\pi = N_\pi$, so $B_\pi \le N_\pi$. Hence $G = N_G(H)B = N_G(H)B_\pi B_{\pi'} = N_G(H)B_{\pi'}$, so H is propermutable (S-propermutable, respectively) in G.

2 Some new characterizations of PST *-groups and* PT *-groups*

A group G is called a *PT -group* if permutability is a transitive relation on G, that is, every permutable subgroup of a permutable subgroup of G is permutable in G. A group G is called a *PST -group* if S-permutability is a transitive relation on G.

As well as *T*-groups, *PT*-groups and *PST*-groups possess many interesting properties (see Chapter 2 in [33]). The general description of PT-groups and PST-groups was first obtained by Zacher [34] and Agrawal [35], for the soluble case, and by Robinson in [36], for the general case. Nevertheless, in the further publications, authors (see for example the recent papers [15]–[27]) have found out and described many other interesting characterizations of soluble PT and PST-groups. In this section we give new characterizations of such groups.

***Theorem* 2.1** (See Theorem A in [5]). *Let* $D = G^{\mathfrak{N}}$ *and* $\pi = \pi(D)$. *Then the following statements are equivalent:*

(i) D *is a Hall subgroup of* G *and every Hall subgroup of* G *is quasipermutable in* G.

(ii) G *is a soluble* PST *-group.*

(iii) *Every subgroup of* G *is quasipermutable in* G.

(iv) *Every* π *-subgroup of* G *and some minimal supplement of* D *in* G *are quasipermutable in* G.

***Theorem* 2.2** (See Theorem B in [37]). G *is a soluble* PST *-group if and only if all Hall subgroups of* G *and all their maximal subgroups are propermutable.*

We say that a subgroup H is *completely* propermutable in G provided H is propermutable in any subgroup of G containing H.

Let us note, in passing, that in the group $G = C_3 \times S_3$, where C_3 is a group of order 3 and S_3 is the symmetric group of degree 3, every Hall subgroup is propermutable but G is not a PST -group since $G^{\mathfrak{N}}$ is not a Hall subgroup of G. It is also clear that every semipermutable subgroup is completely propermutable. A Sylow 2 -subgroup of the group G is not semipermutable.

***Theorem* 2.3** (See Theorem C in [37]). *A soluble* G *of odd order is a* PT *-group if and only if all Hall subgroups and all subnormal subgroups of* G *are completely propermutable.*

Recall that G is called an *Iwasawa* group provided all subgroups of G are permutable. We will say that G is a *generalized* Iwasawa group if every subgroup of G is completely propermutable in G.

In fact, in view of [33, 2.1.12], Theorem 2.3 is a corollary of the following

***Theorem* 2.4.** *All Hall subgroups and all subnormal subgroups of* G *are completely propermutable in* G *if and only if* G *is a soluble* PST *-group whose Sylow* 2 *-subgroups are generalized Iwasawa groups and every Sylow* p *-subgroup of* G, *where* p *is odd, is Iwasawa.*

3 Groups with Hall propermutable or quasipermutable subgroups

The proofs of results in Section 2 are based on many quasipermutability and propermutability properties on nilpotent subgroups. Some of them we discuss in the given and in the next sections.

A subgroup S of G is called a *Gaschütz* subgroup of G (L.A. Shemetkov [38, IV, 15.3]) if S is supersoluble and for any subgroups $K \le H$ of G, where $S \le K$, the number $|H:K|$ is not prime.

Every Hall subgroup of every supersoluble group is S -propermutable (see Example (5)). This observation makes natural the following questions:

***Question* 3.1.** *What is the structure of* G *under the hypothesis that every Hall subgroup of* G *is propermutable or, at least, quasipermutable in* G?

***Question* 3.2.** *What is the structure of* G *under the hypothesis that some Hall subgroup of* G *is propermutable or, at least, quasipermutable in* G?

We have proved the following results in this line researches.

***Theorem* 3.3** (See Theorem B in [5]). *The following statements are equivalent:*

(I) G *is soluble, and if* S *is a Gaschütz subgroup of* G, *then every Hall subgroup* H *of* G *satisfying* $\pi(H) \subseteq \pi(S)$ *is quasipermutable in* G.

(II) G *is supersoluble and the following hold:*

(a) $G = DC$, *where* $D = G^{\mathfrak{N}}$ *is an abelian complemented subgroup of* G *and* C *is a Carter subgroup of* G;

(b) $D \cap C$ *is normal in* G *and*

$$(p, |D / D \cap C|) = 1$$

for all prime divisors p *of* $|G|$ *satisfying*

$$(p-1, |G|) = 1.$$

(c) *For any non-empty set* π *of primes, every* π *-element of any Carter subgroup of* G *induces a power automorphism on the Hall* π' *-subgroup of* D.

(III) *Every Hall subgroup of* G *is quasipermutable in* G.

***Theorem* 3.4** (See Theorem A in [37]). *Every Hall subgroup of* G *is propermutable in* G *if and only if* G *is a supersoluble group such that* $D = G^{\mathfrak{N}}$ *is an abelian complemented subgroup of* G, *for any non-empty set* π *of primes, every* π *-element of* G *induces a power automorphism on the Hall* π' *-subgroup of* D *and* D *is a* p' *-group for all primes* p *satisfying* $(p-1, |G|) = 1$.

A group G is said to be π *-separable* if every chief factor of G is either a π -group or a π' -group. Every π -separable group G has a series

$$1 = P_0(G) \le M_0(G) < P_1(G) <$$
$$< M_1(G) < \cdots < P_t(G) \le M_t(G) = G$$

such that

$$M_i(G) / P_i(G) = O_{\pi'}(G / P_i(G))$$

($i = 0, 1, \ldots, t$) and

$$P_{i+1}(G) / M_i(G) = O_{\pi}(G / M_i(G))$$

($i = 1, \ldots, t$).

The number t is called the π *-length* of G and denoted by $l_{\pi}(G)$ (see [39, p. 249]).

***Theorem* 3.5** (See Theorem 3.1 in [5]). *Let* H *be a Hall subgroup of* G *and* $\pi = \pi(H)$. *Suppose that* H *is quasipermutable in* G.

(I) *If* $p > q$ *for all primes* p *and* q *such that* $p \in \pi$ *and* q *divides* $|G : N_G(H)|$, *then* H *is normal in* G.

(II) *If* H *is supersoluble, then* G *is* π *-soluble.*

(III) *If* G *is* π *-separable, then the following holds:*

(i) $H' \le O_{\pi}(G)$. *If, in addition,* $N_G(H)$ *is nilpotent, then* $G' \cap H \le O_{\pi}(G)$.

(ii) $l_{\pi}(G) \le 2$ *and* $l_{\pi'}(G) \le 2$.

(iii) *If for some prime* $p \in \pi'$ *a Hall* π' *-subgroup* E *of* G *is* p *-supersoluble, then* G *is* p *-supersoluble.*

***Corollary* 3.6** (See [13, Theorem]). *Let* P *be a Sylow* p *-subgroup of* G. *If* P *is semi-normal in* G, *then the following statements hold:*

(i) G *is* p *-soluble and* $P' \le O_p(G)$.

(ii) $l_p(G) \le 2$.

(iii) *If for some prime* $q \in p'$ *a Hall* p' *-subgroup of* G *is* q *-supersoluble, then* G *is* q *-supersoluble.*

***Corollary* 3.7** (See [40, Theorem]). *If a Sylow* p *-subgroup* P *of* G, *where* p *is the largest prime dividing* $|G|$, *is semi-normal in* G, *then* P *is normal in* G.

***Theorem* 3.8** (See Theorem E in [37]). *If every Sylow subgroup* P *of* G *is propermutable in its normal closure* P^G, *then* G *is supersoluble.*

***Corollary* 3.9** (See [40, Theorem 5]). *If every Sylow subgroup of* G *is semi-normal in* G, *then* G *is supersoluble.*

***Theorem* 3.10** (See Theorem F in [37]). *Let* $X = F(G)$ *be the Fitting subgroup of* G *and* H *a Hall* X *-propermutable subgroup of* G. *If* $p > q$ *for all primes* p *and* q *such that* p *divides* $|H|$ *and* q *divides* $|G:H|$, *then* H *is normal in* G.

Theorem 5.4 in [1] is equivalent to the following special case of Theorem 3.10.

***Corollary* 3.11.** *Let* $X = F(G)$ *be the Fitting subgroup of* G *and* H *a Hall* X *-semipermutable subgroup of* G. *If* $p > q$ *for all primes* p *and* q *such that* p *divides* $|H|$ *and* q *divides* $|G:H|$, *then* H *is normal in* G.

4 Groups with S-quasipermutable maximal subgroups of Sylow subgroups

Let $\mathfrak{F}$ be a class of groups. If $1 \in \mathfrak{F}$, then we write $G^{\mathfrak{F}}$ to denote the intersection of all normal subgroups N of G with $G/N \in \mathfrak{F}$. The class $\mathfrak{F}$ is said to be a *formation* if either $\mathfrak{F} = \varnothing$ or $1 \in \mathfrak{F}$ and every homomorphic image of $G/G^{\mathfrak{F}}$ belongs to $\mathfrak{F}$ for any group G. The formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$. A subgroup H of G is said to be an $\mathfrak{F}$ *-covering subgroup* of G provided $H \in \mathfrak{F}$ and $E = E^{\mathfrak{F}}H$ for any subgroup E of G containing H. By the Gaschütz theorem [31, VI, 9.5.4 and 9.5.6], for any saturated formation $\mathfrak{F}$, every soluble group G has an $\mathfrak{F}$-covering subgroup and any two $\mathfrak{F}$-covering subgroups of G are conjugate.

***Theorem* 4.1** (See Theorem C in [5]). *Let* $\mathfrak{F}$ *be a saturated formation containing all nilpotent groups. Suppose that* G *is soluble and let* $\pi = \pi(C) \cap \pi(G^{\mathfrak{F}})$, *where* C *is an* $\mathfrak{F}$ *-covering subgroup of* G. *If every maximal subgroup of every Sylow* p *-subgroup of* G *is* S *-quasipermutable in* G *for all* $p \in \pi$, *then* $G^{\mathfrak{F}}$ *is a Hall subgroup of* G.

***Theorem* 4.2** (See Theorem D in [5]). *Let* $\mathfrak{F}$ *be a saturated formation containing all supersoluble groups and* $\pi = \pi(F^*(G^{\mathfrak{F}}))$. *If* $G^{\mathfrak{F}} \ne 1$, *then for some* $p \in \pi$ *some maximal subgroup of a Sylow* p *-subgroup of* G *is not* S *-quasipermutable in* G.

In this theorem $F^*(G^{\mathfrak{F}})$ denotes the generalized Fitting subgroup of $G^{\mathfrak{F}}$, that is, the product of all normal quasinilpotent subgroups of $G^{\mathfrak{F}}$.

The proofs of Theorems 4.1 and 4.2 consists of many steps and the following result is one of the main stages of it.

***Theorem* 4.3** (See Proposition in [5]). *Let* E *be a normal subgroup of* G *and* P *a Sylow* p *-subgroup of* E *such that* $|P| > p$.

(i) *If every member* V *of some fixed* $\mathcal{M}_{\varphi}(P)$ *is* S *-quasipermutable in* G, *then* E *is* p *-supersoluble.*

(ii) *If every maximal subgroup of* P *is* S *-quasipermutable in* G, *then every chief factor of* G *between* E *and* $O_{p'}(E)$ *is cyclic.*

(iii) *If every maximal subgroup of every Sylow subgroup of* E *is* S *-quasipermutable in* G, *then every chief factor of* G *below* E *is cyclic.*

In this theorem we write $\mathcal{M}_{\varphi}(G)$, by analogy with [7], to denote a set of maximal subgroups of G such that $\Phi(G)$ coincides with the intersection of all subgroups in $\mathcal{M}_{\varphi}(G)$.

Finally, consider some applications of Theorem 4.3.

***Lemma* 4.4** (See Lemma 5.4 in [5]). *Let* E *be a normal subgroup of* G *and* P *a Sylow* p *-subgroup of* E *such that* $(p-1, |G|) = 1$. *If either* P *is cyclic or* G *is* p *-supersoluble, then* E *is* p *-nilpotent and* $E/O_{p'}(E) \le Z_{\infty}(G/O_{p'}(E))$.

The following lemma is well-known (see for example Lemma 2.1.6 in [33]).

***Lemma* 4.5.** *If* G *is* p *-supersoluble and* $O_{p'}(G) = 1$, *then* p *is the largest prime dividing*

$|G|$, G is supersoluble and $F(G)=O_p(G)$ is a Sylow p-subgroup of G.

From Theorem 4.3 and Lemma 4.4 we get

***Corollary* 4.6** (See Theorem 1.1 in [7]). *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every number V of some fixed $\mathcal{M}_\varphi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

***Corollary* 4.7.** *Let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathcal{M}_\varphi(P)$ is S-quasipermutable in G, then G is p-nilpotent.*

Proof. If $|P|=p$, then G is p-nilpotent by Burnside's theorem [31, IV, 2.6]. Otherwise, G is p-supersoluble by Theorem 4.3. The hypothesis holds for $G/O_{p'}(G)$ by Lemma 1.2, so in the case, where $O_{p'}(G)\neq 1$, $G/O_{p'}(G)$ is p-nilpotent by induction. Hence G is p-nilpotent. Therefore we may assume that $O_{p'}(G)=1$. But then, by Lemma 4.5, P is normal in G. Hence G is p-nilpotent by hypothesis.

From Corollary 4.7 we get

***Corollary* 4.8** (See Theorem 1.2 in [7]). *Let P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every number V of some fixed $\mathcal{M}_\varphi(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

***Corollary* 4.9.** *Let P be a Sylow p-subgroup of G. If G is p-soluble and every number V of some fixed $\mathcal{M}_\varphi(P)$ is S-quasipermutable in G, then G is p-supersoluble.*

Proof. In the case, when $|P|=p$, this directly follows from the p-solubility of G. If $|P|>p$, this corollary follows from Theorem 4.3.

The next fact follows from Corollary 4.9.

Corollary 4.10 (See Theorem 1.3 in [7]). *Let P be a Sylow p-subgroup of G. If G is p-soluble and every number V of some fixed $\mathcal{M}_\varphi(P)$ is SS-quasinormal in G, then G is p-supersoluble.*

***Corollary* 4.11.** *If, for every prime p dividing $|G|$ and $P\in Syl_p(G)$, every number V of some fixed $\mathcal{M}_\varphi(P)$ is S-quasipermutable in G, then G is supersoluble.*

Proof. Let p be the smallest prime dividing $|G|$. Then G is p-nilpotent by Corollary 4.6, so G is soluble by Feit-Thompson's theorem. Hence G is supersoluble by Corollary 4.9.

From Corollary 4.11 we get

Corollary 4.12 (See Theorem 1.4 in [7]). *If, for every prime p dividing $|G|$ and $P\in Syl_p(G)$, every number V of some fixed $\mathcal{M}_\varphi(P)$ is SS-quasinormal in G, then G is supersoluble.*

A chief factor H/K of G is called $\mathfrak{F}$*-central* in G provided $(H/K)\rtimes(G/C_G(H/K))\in\mathfrak{F}$. The symbol $Z_{\mathfrak{F}}(G)$ denotes the product of all normal subgroups E of G such that every chief factor of G below E is $\mathfrak{F}$-central.

***Lemma* 4.13** (See Theorem B in [41]). *Let $\mathfrak{F}$ be any formation and E a normal subgroup of G. If $F^*(E)\leq Z_{\mathfrak{F}}(G)$, then $E\leq Z_{\mathfrak{F}}(G)$.*

***Corollary* 4.14** *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and $X\leq E$ normal subgroups of G such that $G/E\in\mathfrak{F}$. Suppose that every maximal subgroup of any non-cyclic Sylow subgroup of X is S-quasipermutable in G. If either $X=E$ or $X=F^*(E)$, then $G\in\mathfrak{F}$.*

The following results are special cases of Corollary 4.14.

***Corollary* 4.15** (See Theorem 1.5 in [7]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E\in\mathfrak{F}$. Suppose that every maximal subgroup of every non-cyclic Sylow subgroup of E is SS-quasinormal in G. Then $G\in\mathfrak{F}$.*

***Corollary* 4.16** (See Theorem 3.2 in [11]). *Let E be a normal subgroup of G such that G/E is supersoluble. Suppose that every maximal subgroup of every Sylow subgroup of $F^*(E)$ is SS-quasinormal in G. Then G is supersoluble.*

***Corollary* 4.17** (See Theorem 3.3 in [11]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E\in\mathfrak{F}$. Suppose that every maximal subgroup of every Sylow subgroup of $F^*(E)$ is SS-quasinormal in G. Then $G\in\mathfrak{F}$.*

***Corollary* 4.18** (See Theorem 3.2 in [42]). *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and E a normal subgroup of G such that $G/E\in\mathfrak{F}$. If all maximal subgroups of $F^*(E)$ are S-permutable in G, then $G\in\mathfrak{F}$.*

ЛИТЕРАТУРА

1. *Guo*, *W.* *X*-semipermutable subgroups of finite groups / W. Guo, K.P. Shum, A.N. Skiba // J. Algebra. – 2007. – Vol. 215. – P. 31–41.

2. *Ore*, *O.* Contributions in the theory of groups of finite order / O. Ore // Duke Math. J. – 1939. – Vol. 5. – P. 431–460.

3. *Stonehewer*, *S.E.* Permutable subgroups in Infinite Groups / S.E. Stonehewer // Math. Z. – 1972. – Vol. 125. – P. 1–16.

4. *Kegel*, *O.H.* Sylow-Gruppen and Subnormalteiler endlicher Gruppen / O.H. Kegel // Math. Z. – 1962. – Vol. 78. – P. 205–221.

5. *Yi*, *X.* Some new characterizations of *PST*-groups / X. Yi, A.N. Skiba // J. Algebra. – 2014. – http://dx.doi.org/10.1016/j.jalgebara.2013.10.01.

6. *Yi*, *X.* On *S*-propermutable subgroups of finite groups / X. Yi, A.N. Skiba // Bull. Malays. Math. Sci. Soc. – (to appear).

7. *Li*, *S.* The influence of *SS*-quasinormality of some subgroups on the structure of finite group / S. Li, Z. Shen, J. Liu, X. Liu // J. Algebra. – 2008. – Vol. 319. – P. 4275-4287.

8. *Chen*, *Ch.* Generalization of theorem Srinivasan / Ch. Chen // J. Sothwest China Normal Univ. – 1987. – Vol. I. – P. 1–4.

9. *Gorenstein*, *D.* Finite Groups / D. Gorenstein. – New York, Evanston, London: Harper & Row Publishers, 1968.

10. *Su*, *H.* Semi-normal subgroups of finite groups / H. Su // Math. Mag. – 1988. – Vol. 8. – P. 7–9.

11. *Li*, *S. SS*-quasinormal subgroups of finite group / S. Li, Z. Shen, J. Liu, X. Liu // Comm. Algebra. – 2008. – Vol. 36. – P. 4436–4447

12. *Monakhov*, *V.S.* Finite groups with seminormal Hall subgroup / V.S. Monakhov // Matem. Zametki. – 2006. – Vol. 80, №4. – P. 573–581.

13. *Guo*, *W.* Finite groups with semi-normal Sylow subgroups / W. Guo // Acta Math. Sinica. English Series. – 2008. – Vol. 24, №10. – P. 1751–1758.

14. *Li*, *B.* New characterizations of finite supersoluble groups / B. Li, A.N. Skiba // Science in China. Series A, Mathematics. – 2008. – Vol. 51. – P. 827–841.

15. *Brice*, *R.A.* The Wielandt subgroup of a finite soluble groups / R.A. Brice, J. Cossey // J. London Math. Soc. – 1989. – Vol. 40. – P. 244–256.

16. *Beidleman*, *J.C.* Criteria for permutability to be transitive in finite groups / J.C. Beidleman, B. Brewster, D.J.S. Robinson // J. Algebra. – 1999. – Vol. 222. – P. 400–412.

17. *Ballester-Bolinches*, *A.* Sylow permutable subnormal subgroups / A. Ballester-Bolinches, R. Esteban-Romero // J. Algebra. – 2002. – Vol. 251. – P. 727–738.

18. *Ballester-Bolinches*, *A.* Groups in which Sylow subgroups and subnormal subgroups permute / A. Ballester-Bolinches, J.C. Beidleman, H. Heineken // Illinois J. Math. – 2003. – Vol. 47. – P. 63–69.

19. *Ballester-Bolinches*, *A.* A local approach to certain classes of finite groups / A. Ballester-Bolinches, J.C. Beidleman, H. Heineken // Comm. Algebra. – 2003. – Vol. 31. – P. 5931–5942.

20. *Asaad*, *M.* Finite groups in which normality or quasinormality is transitive / M. Asaad // Arch. Math. – 2004. – Vol. 83, №4. – P. 289–296.

21. *Ballester-Bolinches*, *A.* Totally permutable products of finite groups satisfying *SC* or *PST* / A. Ballester-Bolinches, J. Cossey // Monatsh. Math. – 2005. – Vol. 145. – P. 89–93.

22. *Lukyanenko*, *V.O.* Finite groups in which τ-quasinormality is a transitive relation / V.O. Lukyanenko, A.N. Skiba // Rend. Semin. Univ. Padova. – 2010. – Vol. 124. – P. 231–246.

23. *Beidleman*, *J.C.* Subnormal, permutable, and embedded subgroups in finite groups / J.C. Beidleman, M.F. Ragland // Central Eur. J. Math. – 2011. – Vol. 9, №4. – P. 915–921.

24. *Finite solvable groups in which seminormality is a transitive relation* / A. Ballester-Bolinches [et al.] // Beitr. Algebra Geom. – DOI 10.1007/s13366-012-0099-1.

25. *Ballester-Bolinches*, *A.* Some new characterizations of solvable *PST*-groups / A. Ballester-Bolinches, J.C. Beidleman, A.D. Feldman // Ricerche mat. – DOI 10.1007/s11587-012-0130-8.

26. *Li*, *Y.* Finite groups in which (*S*)-semipermutability permutability is a transitive relation / Y. Li, L. Wang, Y. Wang // Internat. J. Algebra. – 2008. – Vol. 2, №3. – P. 143–152.

27. *Some Characterizations of Finite Groups in which Semipermutability is a Transitive Relation* / K. Al-Sharo [et al.] // Forum Math. – 2010. – Vol. 22. – P. 855–862.

28. *Guo*, *W.* Criterions of Existence of Hall Subgroups in Non-soluble Finite Groups / W. Guo, A.N. Skiba // Acta Math. Sinica. – 2010. – Vol. 26, №2. – P. 295–304.

29. *Guo*, *W.* New criterions of existence and conjugacy of Hall subgroups of finite groups / W. Guo, A.N. Skiba // Proc. Amer. Math. Soc. – 2011. – Vol. 139. – P. 2327–2336.

30. *Wei*, *X.B.* On *SS*-quasinormal subgroups and the structure of finite groups / X.B. Wei, X.Y. Guo // Science China Mathematics. – 2011. – Vol. 54, №3. – P. 449–456.

31. *Huppert*, *B.* Endliche Gruppen I / B. Huppert. – Berlin, Heidelberg, New York : Springer-Verlag, 1967.

32. *Doerk*, *K.* Finite Soluble Groups / K. Doerk, T. Hawkes. – Berlin, New York : Walter de Gruyter, 1992.

33. *Ballester-Bolinches*, *A.* Products of Finite Groups / A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad. – Berlin, New York : Walter de Gruyter, 2010.

34. *Zacher*, *G.* I gruppi risolubili finiti in cui i sottogruppi di composizione coincidono con i sottogrupi quasi-normali / G. Zacher // Atti Accad. Naz. Lincei Rend. cl. Sci. Fis. Mat. Natur. – 1964. – Vol. 8, №37. – P. 150–154.

35. *Agrawal*, *R.K.* Finite groups whose subnormal subgroups permute with all Sylow subgroups / R.K. Agrawal // Proc. Amer. Math. Soc. – 1975. – Vol. 47. – P. 77–83.

36. *Robinson*, *D.J.S.* The structure of finite groups in which permutability is a transitive relation / D.J.S. Robinson // J. Austral. Math. Soc. – 2001. – Vol. 70. – P. 143–159.

37. *Yi*, *X.* Propermutable characterizations of soluble *PT*-groups and *PST*-groups / X. Yi, A.N. Skiba // Science China. Mathematics. – (submitted).

38. *Shemetkov*, *L.A.* Formations of Finite Groups / L.A. Shemetkov. – Moscow : Nauka, 1978.

39. *Robinson*, *D.J.S.* A Course in the Theory of Groups / D.J.S. Robinson. – New York, Heidelberg, Berlin : Springer-Verlag, 1982.

40. *Podgornaya*, *V.V.* Seminormal subgroups and supersolubility of finite groupds / V.V. Podgornaya // Vesti NAN Belarus. Ser. Phys.-Math. Sciences. – 2000. – Vol. 4. – P. 22–26.

41. *Skiba*, *A.N.* On the $\mathfrak{F}$-hypercentre and the intersection of all $\mathfrak{F}$-maximal subgroups of a finite group / A.N. Skiba // J. Pure Appl. Algebra. – 2012. – Vol. 216, №4. – P. 789–799.

42. *Li*, *Y.* The influence of π-quasinormality of some subgroups of a finite group / Y. Li, Y. Wang // Arch. Math. – 2003. – Vol. 81. – P. 245–252.

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