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NONLINEAR PHYSICS AND MECHANICS

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Nonconservative Cascades in a Shell Model of Turbulence

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Developed turbulent flows in which the intervention of external forces is fundamentally important at scales where the inertial range should exist are quite common. Then the cascade processes are not conservative any more and, therefore, it is necessary to adequately describe the external forces acting in the whole range of scales. If the work of these forces has a power law scaling, then one can assume that the integral of motion changes and the preserving value becomes a quadratic quantity, which includes the dependence on the scale. We develop this idea within the framework of shell models of turbulence. We show that, in terms of nonconservative cascades, one can describe various situations, including (as a particular case) the Obukhov–Bolgiano scaling proposed for turbulence in a stratified medium and for helical turbulence with a helicity injection distributed along the spectrum.

Keywords: turbulence, inertial range, nonconservative cascades, shell models

1. Introduction

According to the fundamental hypothesis of A. N. Kolmogorov for turbulent flows of an incompressible fluid at very high Reynolds numbers, an extended range of scales, called the inertial range, exists. Neither external nor viscous forces affect the flow in the inertial range and all dynamics (statistical properties) of turbulence are determined by the spectral energy flux, which is equal to the dissipation rate of the kinetic energy [1]. The energy cascade in the inertial range is conservative by definition — the absence of energy sources and sinks at these scales is the basis of the ideas about small-scale turbulence, and Kolmogorov's hypotheses, in fact, exhibit the law of energy conservation in the Fourier space.

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The development of the theory of turbulence of an incompressible fluid (we will have in mind only an incompressible fluid) has shown the exceptional role of conservation laws in the behavior of developed turbulent flows [2, 3]. The second positively defined integral of motion, the enstrophy, inherent in two-dimensional Euler equations, fundamentally changes the dynamics of small-scale turbulence, leading to two inertial ranges, the inertial range of an inverse energy cascade and the inertial range of a direct enstrophy cascade. It is interesting that, soon after finding out the role of the second integral of motion in the dynamics of two-dimensional turbulence, the second integral of motion, the helicity, was found for the "classical" three-dimensional turbulence of incompressible fluid [4]. It turned out that ignoring this second integral of motion did not prevent successful progress in understanding the dynamics of small-scale turbulence. This can be explained by the fact that helicity is a quantity having an arbitrary sign and, in most real flows, its average value is close to zero. However, the dynamics of essentially helical turbulence remains a subject of research and discussion up to the present time [5, 6].

In more complex turbulent systems, additional integrals of motion enter the process. For example, in the turbulent flow of conducting fluids, along with the total energy (kinetic energy plus magnetic energy), the magnetic helicity and cross – helicity are the integrals of motion which can significantly affect the cascade processes in the inertial range of scales [7].

In nature, there are developed turbulent flows in which the external forces act over a large range of scales where the inertial range should exist. Then they produce an energy injection distributed along the spectrum. The forces may be of different origin. In particular, the Coriolis force (turbulence in a rotating medium) does not inject energy into the flow, but redistributes the energy of pulsations, violating the isotropy of small-scale turbulence and changing the spectral laws. An important and widely discussed example of turbulence with an impact on the flow in a wide range of scales is the turbulence in a medium of inhomogeneous density, in which the buoyancy force becomes important. The features of turbulence in a stably stratified medium, in which the outflow of kinetic energy into potential energy can occur, were first considered in the papers of A. M. Obukhov [8] and R. Bolgiano [9]. The suggested spectral laws are known as the Obukhov – Bolgiano scaling; the question of the feasibility of this scaling still needs to be answered [10, 11]. Many works have been devoted to the modeling of the Obukhov – Bolgiano scaling, in which the key issue is an adequate description of the Archimedean forces at different scales. More precisely, since statistical characteristics are of interest, the problem concerns the correlation of velocity pulsations and temperature pulsations, because uncorrelated perturbations perform no work on average [12].

Here, we restrict ourselves to considering cascade processes within the framework of shell models, which are based on the conservation laws. The shell model approach (for a review, see, e. g., [13]) uses the idea of sampling the spectral density $E(k)$ by a series of variables U_n, which represent the kinetic energy in the vicinity of the corresponding wavenumbers $k_n = \lambda^n$,

$$\frac{U_n^2}{2} = \int\limits_{k_n}^{k_{n+1}} E(k)\, dk, \tag{1.1}$$

where λ is the shell thickness in the logarithmic scale. The shell model equations can be written as

$$\dot{U}_n = W_n(U, U) - \nu k_n^2 U_n + f_n, \tag{1.2}$$

where W_n is the nonlinear term, ν is the viscosity, and the last term f_n describes any external force. The main thing in constructing a particular model is the choice of the form of nonlinear terms which are responsible for the dynamics of the inertial range and, therefore, should provide the conservation laws. The necessity of subjecting the structure of the nonlinear terms to the requirements of the conservation laws was not questioned, because the cascade equations should mimic in some way the initial Navier – Stokes equations. Modeling of any other forces (Coriolis, Archimedes, Lorentz, etc.) was carried out by writing the corresponding forces f_n on the right-hand side of the equation.

In this paper, we propose to look at the problem from a slightly different point of view. If the forces at some scale range affect the turbulent vortices, then this range is no longer inertial. If the work of forces can be described by a power law (on scales), then we assume that the integral of motion changes and the preserving value becomes a quadratic quantity, which includes the dependence on the scale.

The scaling properties of a class of shell models, which preserved energy but had the second integral of motion dependent continuously on a free parameter, were considered in [14]. That second integral could be either like the generalized enstrophy (a positively determined quantity) or like the generalized helicity (a quantity of arbitrary sign), and at certain parameter values it corresponds to the ordinary enstrophy (a two-dimensional turbulence model) or the ordinary helicity (a three-dimensional turbulence model), and in all other cases it is only of academic interest.

In a certain sense, we follow a similar line of behavior, but consider a system that has only one positively defined integral of motion, and a variation of the governing parameter leads to the result that energy is not conserved. That is why we talk about nonconservative cascades. We show that such a model can describe various situations, including (as a particular case) the Obukhov – Bolgiano scaling, proposed for turbulence in a stratified medium. Next, we consider a more complex system in which a second, sign-variable, integral of motion appears. Fixing the first integral (i. e., accepting again the energy conservation), we address the inertial range, which lost the conservatism for the second integral of motion of homogeneous isotropic turbulence, the helicity. Using the proposed approach, we can describe the systems with a helicity injection distributed along the spectrum, in which helicity effects become significant.

2. The Novikov – Desnianskii shell model and its generalizations

The Novikov – Desnianskii shell model was introduced for real variables U_n and shell thickness $\lambda = 2$ [15]. The corresponding nonlinear form in (1.2) reads

$$W_n(U, U) = k_n(U_{n-1}U_{n-1} - bU_nU_{n+1}) \tag{2.1}$$

and includes one parameter b, which controls the conservation law, satisfied by the nonlinear term. The requirement of conservation of the total energy of all shells $E = \sum\limits_n \frac{U_n^2}{2}$ at all interactions leads to the only possible value $b = \lambda$ (we will not restrict ourselves to the value $\lambda = 2$). If the enstrophy transfer (a quadratic quantity conserved in a two-dimensional flow) is modeled, the requirement of enstrophy conservation $\Omega = \sum\limits_n \frac{k_n^2 U_n^2}{2}$ yields $b = \lambda^3$. In the latter case, the energy of the system is not conserved any more. Each value of b provides a power solution of the form $U_n = U_0 k_n^\alpha$ with $\alpha = -\frac{\log_\lambda b}{3}$. If the energy is conserved, this solution corresponds to the Kolmogorov inertial range, i. e., $E(k) \sim k^{-5/3}$ and $U_n = U_0 k_n^{-1/3}$, while, if the enstrophy is conserved, the solution corresponds to the inertial range of enstrophy transfer, known in 2D turbulence, in which $E(k) \sim k^{-3}$ and $U_n = U_0 k_n^{-1}$.

Among the proposed generalizations of the Novikov – Desnianskii model, two are fundamental. First, an additional pair of terms, symmetric with respect to the shell number n, has been introduced in (2.1) to allow the inverse spectral flux [16]. Second, complex variables U_n were considered [17], which double the number of degrees of freedom and result in an additional integral of motion, which at $b = \lambda$ is $H = i\sum\limits_n k_n \frac{(U_n^*)^2 - U_n^2}{4}$ and has the dimensionality of helicity (m/s^2). This "helical" model has been successfully used subsequently to model the helical turbulence [18, 19]. We write the nonlinear term (2.1) for complex variables in the general form [18]

$$W_n(U, U) = ik_n \left[U_{n-1}^2 + \left(U_{n-1}^*\right)^2 + \lambda U_n^* \left(U_{n+1} - U_{n+1}^*\right) - \lambda^2 U_n \left(U_{n+1} + U_{n+1}^*\right)\right] - \\ - gik_n \left[U_n \left(U_{n-1} + U_{n-1}^*\right) + \lambda U_n^* \left(U_{n-1}^* - U_{n-1}\right) - \lambda^2 \left(U_{n+1}^2 + \left(U_{n+1}^*\right)^2\right)\right], \quad (2.2)$$

where g is a free parameter responsible for the contribution of the inverse cascade. The parameter does not affect the integrals of motion and we will restrict ourselves to the popular choice $g = \lambda^{-5/2}$ [20].

3. Nonconservative energy cascade

Let us consider the simplest nonlinear form (2.1) for the real variables U_n and introduce a quadratic quantity with a free parameter β

$$Q = \sum_n k_n^\beta \frac{U_n^2}{2}, \quad (3.1)$$

for which we write the condition of its conservation at zero viscosity and absence of external forces:

$$\dot{Q} = \sum_n k_n^\beta U_n W_n(U, U) = 0. \quad (3.2)$$

Substituting (2.1) into (3.2) yields that Q is a conserved quantity if

$$b = \lambda^{\beta+1}, \quad \text{i. e.} \quad \beta = \log_\lambda b - 1. \quad (3.3)$$

Then the exponent in the solution $U_n = U_0 k_n^\alpha$ is related to the exponent in Q as $\alpha = -\frac{\beta+1}{3}$. The value $\beta = 0$ $\left(b = \lambda,\ \alpha = -\frac{1}{3}\right)$ provides the energy conservation ($Q = E$) and gives the Kolmogorov spectrum, while $\beta = 2$ ($b = \lambda^3$, $\alpha = -1$) leads to enstrophy conservation ($Q = \Omega$) and the spectrum corresponding to the inertial range of enstrophy transfer.

The spectral flux of Q is defined as

$$\Pi_n^Q = \sum_{m=0}^n k_m^\beta U_m \dot{U}_m = U_0^3 \lambda^{-2\alpha}. \quad (3.4)$$

If $\beta \neq 0$ the spectral flux of energy depends on the wavenumber and is

$$\Pi_n^E = \sum_{m=0}^n U_m \dot{U}_m = U_0^3 k_n^{1+3\alpha} \lambda^{-2\alpha}. \quad (3.5)$$

The value $\beta = -1$ ($b = 1$, $\alpha = 0$) corresponds to the energy equipartition over the shells $\left(E(k) \sim k^{-1}\right)$, which can be maintained only by the energy flux increasing with wavenumber ($\Pi_E \sim k$). Note that $\beta > 0$ corresponds to the energy sink increasing with the wavenumber, and $\beta < 0$ corresponds to the energy input increasing with the wavenumber. Numerical simulations of Eqs. (1.2) and (2.1) show that the solution $U_n = U_0 k_n^\alpha$ is stable for any positive U_0 and $\beta \geqslant -1$.

Returning to the Obukhov – Bolgiano problem, we recall that the Obukhov – Bolgiano scaling $E(k) \sim k^{-11/5}$, $E_T(k) \sim k^{-7/5}$ is suggested for the turbulence in stably stratified media and arises under two hypotheses: the hypothesis of a stable energy outflow due to buoyancy forces (in the language of conservative models this means a strict anticorrelation of velocity and temperature fluctuations) and the hypothesis of a constant spectral flux of energy of temperature fluctuations. The last hypothesis implies that, within the framework of shell models, the equation for the variables T_n characterizing temperature fluctuations at the corresponding scales must be written in such a way as to ensure that the quantity $E^T = \sum\limits_n \frac{T_n^2}{2}$ is conserved in the absence of diffusivity. In our approach, we no longer need the first hypothesis, but we accept the second and write the shell equation for temperature fluctuations as

$$\dot{T}_n = W_n(U, T) - \chi k_n^2 T_n, \tag{3.6}$$

where χ is the thermal diffusivity and W has the form of (2.1) with $b = \lambda$, which provides the conservation of E^T.

The Obukhov – Bolgiano scaling presumes that the shell energy scales as $E_n^V \sim k_n^{-6/5}$ and $E_n^T \sim k_n^{-2/5}$. Then $U_n \sim k_n^{-3/5}$, $T_n \sim k_n^{-1/5}$ and the work done by the buoyancy forces decreases with scale as $U_n T_n \sim k_n^{-4/5}$ and at some scale k_B, called the Bolgiano scale, becomes smaller than Kolmogorov's spectral flux, provided by the energy dissipation rate. In the framework of the approach considered, the Bolgiano scale, where the Obukhov – Bolgiano scaling is replaced by the Kolmogorov law $E(k) \sim E^T(k) \sim k^{-5/3}$, corresponds to a change in the integral of motion, that is, at scale k_B, the $\beta = \frac{4}{5}$ $\left(b = \lambda^{9/5}\right)$ should be replaced by $\beta = 0$ ($b = \lambda$). An example of the solution of the system (1.2), (3.6) with such a distribution of the parameter b is shown in Fig. 1.

This numerical solution confirms the existence of a stable power-law solution $U_n \sim k_n^\alpha$ in both ranges of scales $\left(\alpha = -\frac{3}{5} \text{ at } k < k_B \text{ and } \alpha = -\frac{1}{3} \text{ at } k > k_B\right)$. The distribution of temperature variables T_n strictly follows the Kolmogorow law at $k > k_B$, but has some sawtooth perturbation in the Obukhov – Bolgiano part of the spectrum.

4. Complex variables. Nonconservative helicity cascade

Let us turn to the general form of the Novikov – Desnyanskii model for complex variables $U_n = A_n + iB_n$ and write the nonlinear form (2.2) with unspecified coefficients b and c

$$\begin{aligned} W_n(U, U) = ik_n \Big\{ \Big[U_{n-1}^2 + \left(U_{n-1}^*\right)^2 + bU_n^* \left(U_{n+1} - U_{n+1}^*\right) - cU_n \left(U_{n+1} + U_{n+1}^*\right) \Big] - \\ -g \Big[U_n \left(U_{n-1} + U_{n-1}^*\right) + b_1 U_n^* \left(U_{n-1}^* - U_{n-1}\right) - c_1 \left(U_{n+1}^2 + \left(U_{n+1}^*\right)^2\right) \Big] \Big\}. \end{aligned} \tag{4.1}$$

Note that the model (4.1), like the original model (2.1), belongs (according to the classification of review [13]) to the $L1$ (local, first-neighbor) models, which means that only combinations

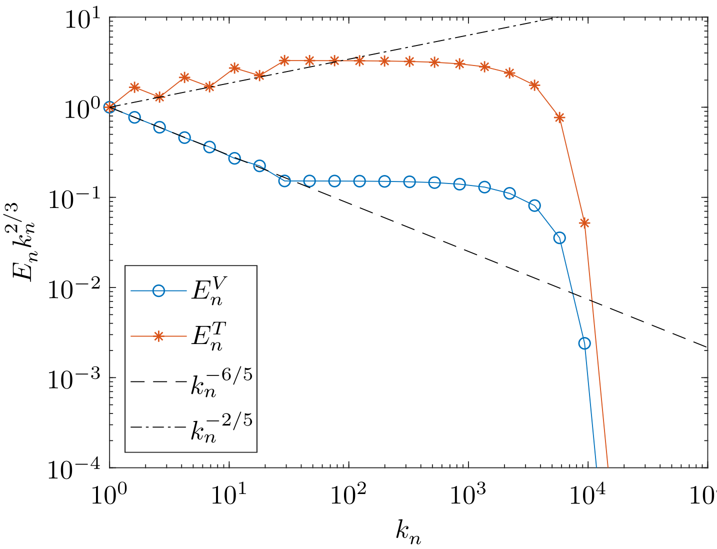

Fig. 1. Shell energy E_n^V and E_n^T, compensated by the Kolmogorov scaling (Kolmogorov's law meets a horizontal line), for the buoyancy affected turbulence with a change of the conservation law at scale $k_B =$ $= 30$. The Obukhov–Bolgiano scaling is traced by the dashed (velocity) and dashed-dot (temperature) lines

of two neighboring variables ($n-1$ and n, or n and $n+1$) are used in the nonlinear form. This imposes severe restrictions on the possible pair of integrals of motion that can be considered within the model. In the model (4.1) two conserved quantities are allowed: one is a quadratic quantity of the form (3.1), and the other (let us call it a generalized helicity) is

$$S = i \sum_n k_n^\gamma \frac{(U_n^*)^2 - U_n^2}{4} = \sum_n k_n^\gamma A_n B_n, \tag{4.2}$$

which characterizes the coherence (correlation in stochastic modes) of the real and imaginary parts of the shell variables and includes a free parameter γ. It is important to note that such a model cannot require simultaneous conservation of energy and enstrophy (no model for two-dimensional turbulence can be constructed), as it can be done in $L2$-type (local and two-first-neighbor) models, in which the interactions of three neighboring variables ($n-1$, n and $n+1$) are considered [14, 21].

The choice of power indices β and γ in integrals (3.1) and (4.2) determines the values of parameters b, b_1, c and c_1

$$b = \lambda^{\beta+1}, \quad b_1 = \lambda^{\gamma-\beta}, \quad c = c_1 = \lambda^{\gamma+1}. \tag{4.3}$$

The expression for b, as one would expect, coincides with (3.3), and the expression for c at $\gamma =$ $= 1$ gives the dimensionality of the hydrodynamic helicity and leads the nonlinear term to the form (2.2).

Finally, in terms of the real and imaginary parts of the variables, the shell equations read

$$\dot{A}_n = 2k_n \left[\lambda^{\gamma+1} B_n A_{n+1} - \lambda^{\beta+1} A_n B_{n+1} + g\left(A_{n-1}B_n - \lambda^{\gamma-\beta} B_{n-1} A_n\right)\right], \tag{4.4}$$

$$\begin{aligned} \dot{B}_n = 2k_n \Big[&\left(A_{n-1}^2 - B_{n-1}^2\right) + \lambda^{\beta+1} B_n B_{n+1} - \lambda^{\gamma+1} A_n A_{n+1} + \\ &+ g\left(-A_{n-1}A_n + \lambda^{\gamma-\beta} B_{n-1} B_n + \lambda^{\gamma+1}\left(A_{n+1}^2 - B_{n+1}^2\right)\right)\Big]. \end{aligned} \tag{4.5}$$

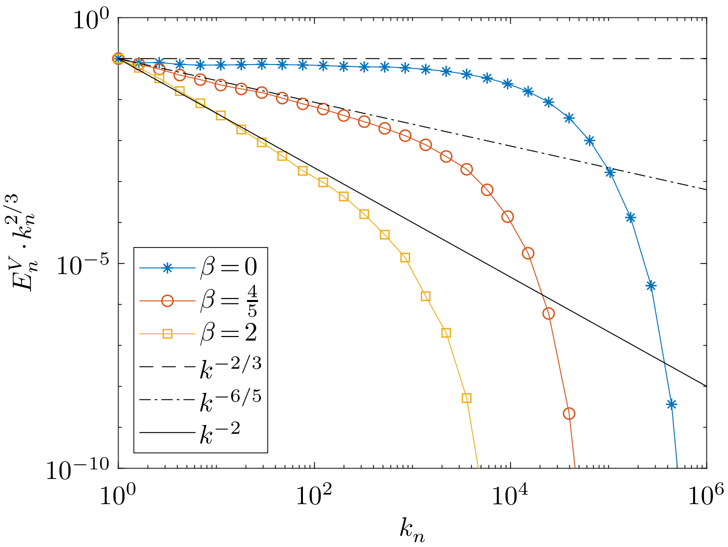

Fig. 2. Simulated energy spectra for the complex shell model for Re $= 10^6$ and different values of the parameter β

The spectral fluxes of quadratic quantities Q and S are

$$\Pi_Q = 2k_n^{\beta+1}\left[\left(A_{n-1}^2 - B_{n-1}^2\right)B_n + gb_1\left(B_n^2 - A_n^2\right)B_{n-1}\right], \tag{4.6}$$

$$\Pi_S = 2k_n^{\gamma+1}\left[A_n\left(A_{n-1}^2 - B_{n-1}^2\right) + gA_{n-1}\left(B_n^2 - A_n^2\right)\right]. \tag{4.7}$$

Already the first numerical simulations based on the models like (2.2) showed that changing to complex variables leads to the chaotization of numerical solutions [20]. Our simulations of Eqs. (4.4)–(4.5) confirmed that no stable solution exists for any of the parameters considered. Scaling laws for the statistical moments of different orders can be determined by averaging over long numerical realizations. Thus, the power law for the kinetic energy distribution has the form $\langle |U_n|^2 \rangle \sim k_n^{\zeta}$, and the relation $2\alpha = \zeta$ is satisfied only approximately (the scaling laws for statistical moments of different orders were studied in detail for a class of shell models in [14]). Note also that the same power laws $|A_n|$, $|B_n| \sim k_n^{\alpha}$ are observed for the real and imaginary components of U_n under all modes considered.

First, we have fixed $\gamma = 1$ and analyzed the numerical solutions of (4.4)–(4.5) for different values of the parameter β. Then the second integral S corresponds to the helicity, $S = H = \sum\limits_n k_n A_n B_n$, and the value of the first one depends on β. For all β considered, the numerical solutions are statistically stable, displaying the pronounced power laws if the averaging time was long enough (the typical time of averaging was above 1000 dimensionless units). An example of the averaged spectra for some β is given in Fig. 2. The spectra in the figure are compensated by the Kolmogorov law $\left(E_n \sim k_n^{-2/3}\right)$, and thus the horizontal part of the spectrum, revealed for $\beta = 0$, corresponds to the classical inertial range of energy cascade. At $\beta = \frac{4}{5}$, the Obukhov–Bolgiano scaling is established in the whole range of scales up to the dissipation scales, because no Bolgiano scale is introduced. The third example is shown for $\beta = 2$ that corresponds to enstrophy conservation and provides the spectrum $E_n \sim k_n^{-2}$ (in terms of the spectral power density, it means $E(k) \sim k^{-3}$). We note that $\beta > 0$ implies that the output of energy increases with k.

Secondly, we have fixed $\beta = 0$ and analyzed the numerical solutions of (4.4)–(4.5) at different values of the parameter γ. Thus, we restore the energy conservation and analyze the cascades under a power–law input of helicity.

The weak influence of helicity on the cascade processes in the inertial range is explained by the fact that the spectral helicity density is related to the spectral energy density as $|H(k)| \leqslant \leqslant kE(k)$. This means that any energy transfer to larger wavenumbers (smaller scales) is accompanied by a decrease in the relative level of helicity $\frac{H(k)}{2kE(k)}$, which determines the fraction of energy attributable to helical motions. That is why a "fully helical cascade" is possible only if the helicity is injected into the inertial range with an injection rate linearly growing with k [19].

Figure 3 presents the results of simulations with a moderate helicity injection, namely, with $\gamma = 0.5$ (the injection rate increases with wavenumber as $\sqrt{k}$). Then the kinetic energy follows the Kolmogorov law, but the helicity deviates from it, decreasing with wave numbers like $H_n \sim k_n^{-1/3}$. The relative helicity, shown in the right panel of the same figure, decreases fast enough at the large-scales part of the spectrum.

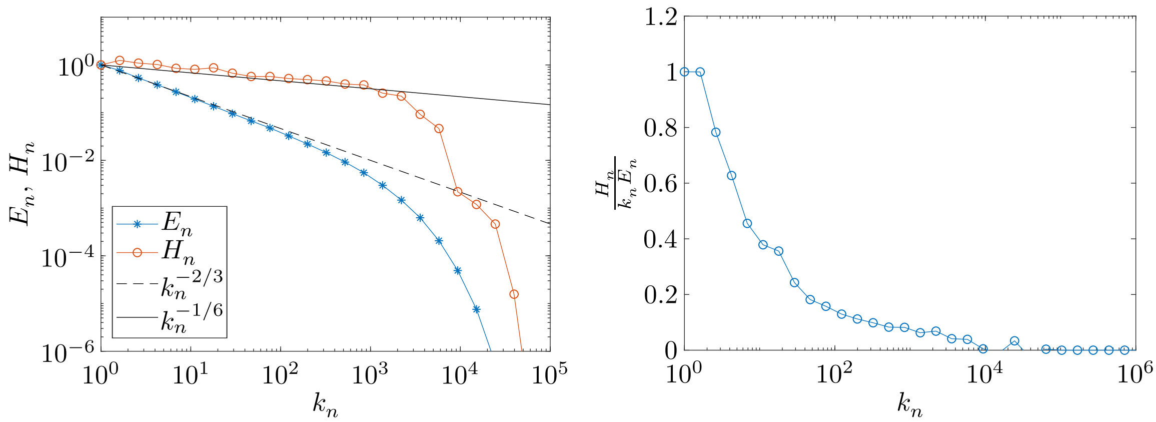

Fig. 3. Simulated energy and helicity spectra by the complex shell model for $\mathrm{Re} = 10^5$, $\beta = 0$, $\gamma = 0.5$. Energy and helicity spectra (left) and relative helicity (right)

The "fully helical cascade" corresponds in our model to $\gamma = 0$, which directly implies the conservation of relative helicity along the spectrum (because $S = \sum A_n B_n$). We have tried to implement this scenario in our model, but we have seen that the numerical solution blows up at $\gamma = 0$. However, a small γ is possible and then the spectrum approaches the spectrum obtained under increasing helicity input [19]. Figure 4 gives the results of the numerical solution for $\beta = 0$, $\gamma = 0.015$ which provides the distributions close to $E_n \sim k_n^{-2/3}$, $H_n \sim k_n^{1/3}$. This means that the energy follows the Kolmogorov law $E(k) \sim k^{-5/3}$, while the helicity $H(k) \sim k^{-2/3}$, and the relative helicity really remains constant along the spectrum (the right panel of Fig. 4).

A constant level of relative helicity along the spectrum can be realized in another way. One can consider the combination $\beta = 1$, $\gamma = 1$, which implies the conservation of helicity, but the output of energy increases with wavenumber. The results of simulations for this combination of parameters is shown in Fig. 5. It is seen that the shell energy E_n and the normalized helicity $\frac{H_n}{k_n}$ indeed follow the same law, but the decrease of shell energy with the shell number is faster $E_n \sim \frac{H_n}{k_n} \sim k_n^{-4/3}$. These distributions correspond to the spectral densities $E(k) \sim k^{-7/3}$, $H(k) \sim k^{-4/3}$, which were suggested for the helical turbulence under the assumption of a direct cascade of helicity and an inverse cascade of energy [22].

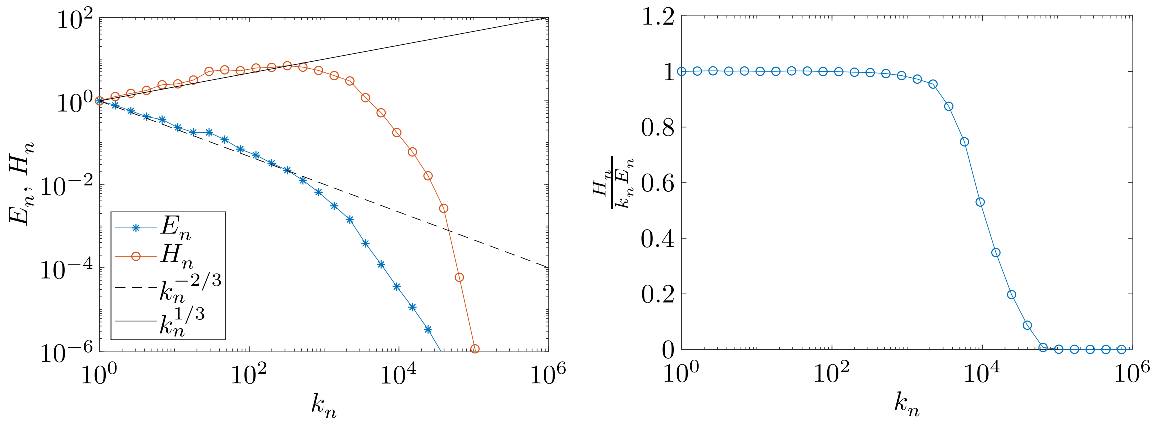

Fig. 4. Results of numerical simulations by the complex shell model for Re $= 10^6$, $\beta = 0$, $\gamma = 0.015$. Energy and helicity spectra (left) and relative helicity (right)

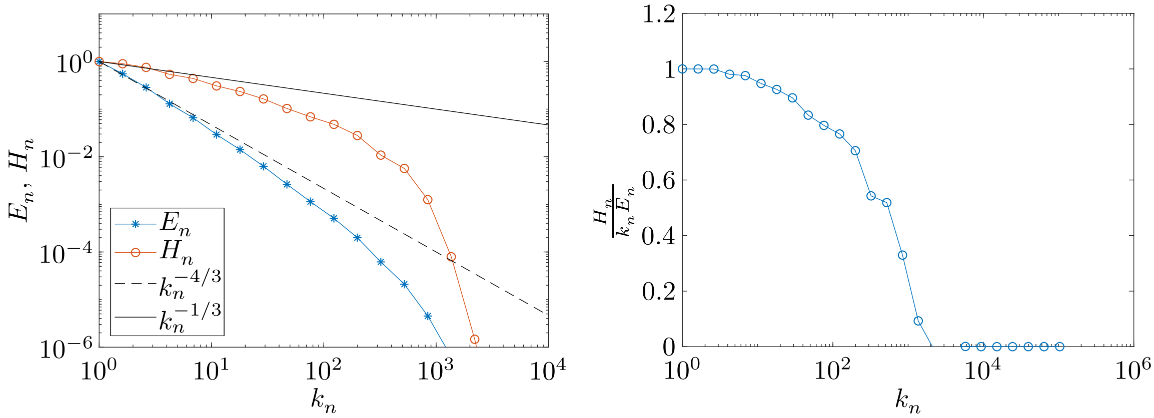

Fig. 5. Simulated energy and helicity spectra by the complex shell model for Re $= 10^5$, $\beta = 1.0$, $\gamma = 1.0$. Energy and helicity spectra (left) and relative helicity (right)

5. Conclusions

We have considered a class of nonconservative shell models of fully developed turbulence in which the conserved quantities are the generalized energy and helicity

$$Q = \int k^\beta E(k)\,dk, \quad S = \int k^{\gamma-1} H(k)\,dk,$$

where $E(k)$ and $H(k)$ are the spectral densities of energy and helicity. The cascade process described by such a model at $\beta \neq 0$ and $\gamma \neq 1$ ceases to be conservative with respect to the known integrals of motion, energy $E = \int E(k)\,dk$ and helicity $H = \int H(k)\,dk$. This approach makes sense when describing cascade processes in turbulence with distributed (along the spectrum) injection of the corresponding quantity. In terms of the model, this means that it is not necessary to describe the source of the corresponding quantity as a separate term, since it is actually included in the nonlinear interactions.

The proposed model allows considering any power law for distributed energy injection, in particular, it reproduces the Obukhov – Bolgiano scaling [8, 9] widely discussed in the context of turbulence in stratified media. Under energy conservation, the spectral distributed helicity injection provides the entire set of solutions predicted for helicity-governed cascades [19].

In general, the proposed approach allows us to consider in a new way the modeling of turbulent flows accompanied by the injection of various quadratic (usually conserved) quantities over a wide range of scales. So, we note that a wide set of problems of nonconservative cascades is provided by the MHD turbulence, which is characterized by the presence of three conservation laws (total energy, cross-helicity and magnetic helicity). However, a simple enumeration of possible cases does not seem to be productive and first a physically interesting situation should be found which can be studied using this approach.

Conflict of interest

The authors declare that they have no conflict of interest.

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